BABEŞ-BOLYAI UNIVERSITY CLUJ-NAPOCA
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE

MATE-INFO UBB COMPETITION 2022
Written test in MATHEMATICS

IMPORTANT NOTE: Problems can have one or more correct answers, which the candidate should indicate in the electronic system. The grading system of the multiple choice exam can be found in the set of rules of the competition.

1. Let $x = \sin \dfrac{12133}{6}\pi$. Then

A $x = \dfrac{\sqrt{3}}{2}$; B $x = \dfrac{1}{2}$; C $x > 0$; D $x < 0$.

2. If in a Cartesian coordinate system the vertices of a triangle ABC have coordinates $A(2,3)$, $B(-1,1)$, $C(-3,4)$, and G is the center of gravity of the triangle ABC, then the midpoint F of the line segment AG has coordinates

A $F(0,0)$; B $F(\frac{2}{3}, \frac{17}{6})$; C $F(-\frac{2}{3}, \frac{8}{3})$; D some other value.

3. The number of solutions of the equation $3\sin x - 2 = 0$ in the interval $[0, \pi]$ is

A 0; B 1; C 2; D infinite.

4. Consider in $\mathbb{R}$ the equation $\sqrt{x^2 - 3} = x^2 - 5$. Which of the following statements are true?

A The equation has no solutions.
B The equation has exactly two solutions.
C The equation has exactly four solutions.
D The equation has only positive solutions.

5. The number of rational terms in the expansion $\left(\sqrt{2} + \sqrt[3]{5}\right)^{300}$ is

A 50; B 51; C 52; D 150.

6. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$. The sum of the elements of the matrix A^5 is:

A 19; B 20; C 21; D 22.

7. Let $(x_n)_{n\geq 1}$ be a sequence of positive real numbers satisfying $(n+1)x_{n+1} - nx_n < 0$, for every $n \geq 1$. Then the limit of the sequence is:

A 1; B ∞; C does not exist; D 0.

8. The function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x < 0 \\ x^3 + x + \alpha & \text{if } x \geq 0, \end{cases}$$

is continuous if:

A $\alpha \in \mathbb{R}$;
B $\alpha = 1$;
C $\alpha = 0$;
D there is no $\alpha \in \mathbb{R}$ for which the function is continuous.

9. The equation of the tangent line to the graph of the function $f(x) = \sqrt[3]{x-1}$ at the point $x = 9$ is:

$\boxed{\text{A}}$ $-12y + x - 15 = 0$; $\boxed{\text{B}}$ $12y - x - 15 = 0$; $\boxed{\text{C}}$ $y - 12x - 15 = 0$; $\boxed{\text{D}}$ $y + 12x + 15 = 0$.

10. The set of solutions of the equation

$$4 \cdot \sin x \cdot \cos^3 x - 4 \cdot \sin^3 x \cdot \cos x = 1$$

is

$\boxed{\text{A}}$ $\{\frac{\pi}{8} + \frac{k\pi}{4} \mid k \in \mathbb{Z}\}$; $\boxed{\text{B}}$ $\{\frac{\pi}{8} + \frac{k\pi}{2} \mid k \in \mathbb{Z}\}$; $\boxed{\text{C}}$ $\{\frac{\pi}{8} - k\pi \mid k \in \mathbb{Z}\}$; $\boxed{\text{D}}$ $\{\frac{\pi}{4} + \frac{k\pi}{8} \mid k \in \mathbb{Z}\}$.

11. The vertices A and B of the parallelogram $ABCD$ belong to the line of equation $3x - y - 4 = 0$ and the point of intersection O of the diagonals AC and BD has coordinates $(3, 4)$. If the coordinates of A are $(0, -4)$, then the equation of the line CD is:

$\boxed{\text{A}}$ $x + 3y - 42 = 0$; $\boxed{\text{B}}$ $x - 3y - 6 = 0$; $\boxed{\text{C}}$ $3x - y - 6 = 0$; $\boxed{\text{D}}$ $y = 3x + 6$.

12. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x - [2x]$, where $[a]$ denotes the integer part of $a \in \mathbb{R}$. Which of the following statements are true?

$\boxed{\text{A}}$ f has period $\frac{1}{2}$; $\boxed{\text{B}}$ f is injective; $\boxed{\text{C}}$ f is surjective; $\boxed{\text{D}}$ f is even.

13. Let $S_n = i + 2i^2 + 3i^3 + \cdots + ni^n, n \in \mathbb{N}^*$, where i is the imaginary unit ($i^2 = -1$). Which of the following statements are true?

$\boxed{\text{A}}$ S_{2020} is a real number; $\boxed{\text{B}}$ $|S_{2020}|$ is an irrational number;
$\boxed{\text{C}}$ the imaginary part of S_{2022} is 1011; $\boxed{\text{D}}$ $|S_{2022}| = 1011$.

14. Let $(x, y) \in \mathbb{R}_+^2$ be the solution of the system of equations

$$\begin{cases} \log_{225} x + \log_{64} y = 0 \\ \log_x 225 - \log_y 64 = 1. \end{cases}$$

The value of the expression $\log_{30}\left(x^3\right) - \log_{30} y$ is:

$\boxed{\text{A}}$ 0; $\boxed{\text{B}}$ 12; $\boxed{\text{C}}$ 1; $\boxed{\text{D}}$ 10.

15. The sum of the solutions of the equation $6^{x+1} - 4^x = 3^{2x}$ is

$\boxed{\text{A}}$ -1; $\boxed{\text{B}}$ 0; $\boxed{\text{C}}$ 1; $\boxed{\text{D}}$ 2.

16. The point $A(3, 1)$ is the vertex of a square, for which one of the diagonals has equation $y - x = 0$.

$\boxed{\text{A}}$ the distance from the point A to the diagonal is 2;
$\boxed{\text{B}}$ the equation of the other diagonal is $x + y + 2 = 0$;
$\boxed{\text{C}}$ the aria of the square is 4;
$\boxed{\text{D}}$ the point $C(1, 3)$ is also a vertex of the square.

17. Consider the triangle ABC, with the notations $BC = a, AC = b, AB = c$. Assume that the length of the median AM is equal to c. Then:

$\boxed{\text{A}}$ $a^2 + 2c^2 = 3b^2$; $\boxed{\text{B}}$ $a^2 + 2c^2 = 2b^2$; $\boxed{\text{C}}$ $\cos C = \frac{4a}{3b}$; $\boxed{\text{D}}$ $\cos C = \frac{3a}{4b}$.

18. The value of the limit $\lim\limits_{x \to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{\sin^2 x} \right)$ is:

$\boxed{\text{A}}$ $-\dfrac{1}{3}$; $\boxed{\text{B}}$ -1; $\boxed{\text{C}}$ 0; $\boxed{\text{D}}$ $\dfrac{1}{2}$.

19. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \operatorname{arctg} x + \operatorname{arcctg} x$ for every $x \in \mathbb{R}$. Which of the following statements are true?

$\boxed{\text{A}}$ $f(-1) = -\dfrac{\pi}{2}$;

$\boxed{\text{B}}$ $f(x) = \dfrac{\pi}{2}$, for every $x \in (0, \infty)$;

$\boxed{\text{C}}$ the function f is odd;

$\boxed{\text{D}}$ $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to-\infty} f(x)$.

20. The number of real solutions of the equation $x\mathrm{e}^x = -\dfrac{1}{3}$ is:

$\boxed{\text{A}}$ 0; $\boxed{\text{B}}$ 1; $\boxed{\text{C}}$ 2; $\boxed{\text{D}}$ 3.

21. Let ABC be a triangle and $A' \in [BC]$, $B' \in [CA]$, $C' \in [AB]$ so that $\frac{BA'}{BC} = \frac{CB'}{CA} = \frac{AC'}{AB} = \alpha$. If $\mathcal{A}_{ABC}$ is the aria of the triangle ABC and $\mathcal{A}_{A'B'C'}$ is the aria of the triangle $A'B'C'$, then

$\boxed{\text{A}}$ $\dfrac{\mathcal{A}_{A'B'C'}}{\mathcal{A}_{ABC}} = 1 - 3\alpha(1-\alpha)$; $\boxed{\text{B}}$ $\dfrac{\mathcal{A}_{A'B'C'}}{\mathcal{A}_{ABC}} \in \left[\frac{1}{4}, 1\right]$;

$\boxed{\text{C}}$ $\dfrac{\mathcal{A}_{A'B'C'}}{\mathcal{A}_{ABC}} = 1 - 12\alpha^2(1-\alpha)^2$; $\boxed{\text{D}}$ $\dfrac{\mathcal{A}_{A'B'C'}}{\mathcal{A}_{ABC}} \in \left[\frac{1}{2}, 1\right]$.

22. The value of the limit $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{\int_1^{\operatorname{tg} x} \mathrm{e}^{t^2}\mathrm{d}t}{\int_1^{\operatorname{ctg} x} \mathrm{e}^{t^2}\mathrm{d}t}$ is:

$\boxed{\text{A}}$ 1; $\boxed{\text{B}}$ π; $\boxed{\text{C}}$ 0; $\boxed{\text{D}}$ -1.

23. A triangle in which $\sin(B) + \cos(B) = \sin(C) + \cos(C)$ is:
$\boxed{\text{A}}$ right-angled $\boxed{\text{B}}$ isosceles $\boxed{\text{C}}$ equilateral $\boxed{\text{D}}$ right-angled or isosceles.

24. Let $(a_n)_{n\in\mathbb{N}^*}$ be the sequence defined by

$$a_n = \sqrt{\frac{1}{n^2} + \frac{1}{n^3}} + \sqrt{\frac{1}{n^2} + \frac{2}{n^3}} + \cdots + \sqrt{\frac{1}{n^2} + \frac{n}{n^3}}, \text{ for every } n \in \mathbb{N}^*.$$

Denote by $\ell = \lim\limits_{n\to\infty} a_n$. Which of the following statements are true?

$\boxed{\text{A}}$ $\ell = 0$; $\boxed{\text{B}}$ $\ell = \frac{\sqrt{2}}{3}$; $\boxed{\text{C}}$ $\ell \in \overline{\mathbb{R}} \setminus \mathbb{Q}$; $\boxed{\text{D}}$ $\ell = \infty$.

25. Two sides of a rectangle have equations:

$$(d_1) : 2x - 3y + 5 = 0$$

$$(d_2) : 3x + 2y - 7 = 0$$

and one of its vertices is $A(2, -3)$. The equations of the other two sides of the rectangle are:

$\boxed{\text{A}}$ $2x - 3y - 13 = 0$ and $3x + 2y = 0$; $\boxed{\text{B}}$ $y + 3 = \frac{2}{3}(x-2)$ and $y + 3 = -\frac{3}{2}(x-2)$;

$\boxed{\text{C}}$ $2x - 3y + 13 = 0$ and $3x - 2y = 0$; $\boxed{\text{D}}$ $y - 3 = \frac{2}{3}(x-2)$ and $y - 3 = -\frac{3}{2}(x-2)$.

26. Consider a parameter $\alpha \in \mathbb{C}$ and the linear system with 3 unknowns

$$\begin{cases} 2x + \alpha y + 2z &= 1 \\ 4x - y + 5z &= 1 \\ 2x + 10y + z &= 1. \end{cases}$$

Determine the truth value of the following statements:

- A The matrix of the system has rank 3 for every value of α.
- B The extended matrix of the system has rank 3 for every value of α.
- C The system is incompatible if and only if $\alpha \neq 3$.
- D The system is compatible if and only if $\alpha \neq 3$.

27. Let $G \subseteq \mathbb{R}$ be a set such that the relation

$$x * y = \frac{xy}{2xy - x - y + 1}, \forall x, y \in G$$

defines a composition law on G. Which of the following statements are true?

- A G can be the interval $(0, 2)$.
- B G can be the interval $(0, 1)$.
- C If $G = (0, 1)$, then ,,$*$'' has an identity element.
- D If $G = (0, 1)$, then the inverse of $\frac{1}{3}$ is $\frac{2}{3}$.

28. The value of the integral

$$\int_{\frac{1}{2022}}^{2022} \frac{\ln x}{1 + x^2} \, \mathrm{d}x$$

is:

- A 0;
- B 1;
- C 2;
- D 3.

29. Let $x, y, z \in \mathbb{Z}^\star$ be numbers with the property that xy, yz, zx are in a geometric progression with ratio an integer number not equal to 1.

- A If y is a perfect square, then z is also a perfect square.
- B If z is a perfect square, then y is also a perfect square.
- C If y is a perfect square, then x is also a perfect square.
- D If z is a perfect square, then x is also a perfect square.

30. Let $(x_n)_{n \in \mathbb{N}^*}$ be the sequence defined by $x_n = \displaystyle\int_0^2 \frac{(2-x)^{2n-1}}{(2+x)^{2n+1}} \, \mathrm{d}x$, for every $n \in \mathbb{N}^*$. Which of the following statements are true?

- A $x_{23} = \frac{1}{184}$.
- B $\lim\limits_{n \to \infty} n^2 x_n = 1$.
- C $\lim\limits_{n \to \infty} n x_n = \dfrac{1}{8}$.
- D $\lim\limits_{n \to \infty} n x_n = 0$.

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Correct Answers

BBU Math-CS Contest 2022
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1. $\boxed{\textbf{B}}$, $\boxed{\textbf{C}}$
2. $\boxed{\textbf{B}}$
3. $\boxed{\textbf{C}}$
4. $\boxed{\textbf{B}}$
5. $\boxed{\textbf{B}}$
6. $\boxed{\textbf{C}}$
7. $\boxed{\textbf{D}}$
8. $\boxed{\textbf{C}}$
9. $\boxed{\textbf{B}}$
10. $\boxed{\textbf{B}}$
11. $\boxed{\textbf{C}}$
12. $\boxed{\textbf{A}}$
13. $\boxed{\textbf{B}}$, $\boxed{\textbf{C}}$
14. $\boxed{\textbf{B}}$
15. $\boxed{\textbf{B}}$
16. $\boxed{\textbf{C}}$, $\boxed{\textbf{D}}$
17. $\boxed{\textbf{B}}$, $\boxed{\textbf{D}}$
18. $\boxed{\textbf{A}}$
19. $\boxed{\textbf{B}}$, $\boxed{\textbf{D}}$
20. $\boxed{\textbf{C}}$
21. $\boxed{\textbf{A}}$, $\boxed{\textbf{B}}$
22. $\boxed{\textbf{D}}$
23. $\boxed{\textbf{D}}$
24. $\boxed{\textbf{C}}$
25. $\boxed{\textbf{A}}$, $\boxed{\textbf{B}}$
26. $\boxed{\textbf{B}}$, $\boxed{\textbf{D}}$
27. $\boxed{\textbf{B}}$, $\boxed{\textbf{C}}$, $\boxed{\textbf{D}}$
28. $\boxed{\textbf{A}}$
29. $\boxed{\textbf{A}}$, $\boxed{\textbf{B}}$
30. $\boxed{\textbf{A}}$, $\boxed{\textbf{C}}$

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MATE-INFO UBB COMPETITION 2022
Written test in MATHEMATICS
SOLUTIONS

1. Let $x = \sin \dfrac{12133}{6}\pi$. Then

$\boxed{\text{A}}$ $x = \dfrac{\sqrt{3}}{2}$; $\boxed{\text{B}}$ $x = \dfrac{1}{2}$; $\boxed{\text{C}}$ $x > 0$; $\boxed{\text{D}}$ $x < 0$.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **true**; $\boxed{\textbf{C}}$ **true**; $\boxed{\textbf{D}}$ **false**.

Solution: $\dfrac{12133}{6} = 2022 + \dfrac{1}{6}$, so $\dfrac{12133}{6}\pi = 2022\pi + \dfrac{\pi}{6}$.
$\sin\left(2022\pi + \dfrac{\pi}{6}\right) = \sin\dfrac{\pi}{6} = \dfrac{1}{2}$.

2. If in a Cartesian coordinate system the vertices of a triangle ABC have coordinates $A(2,3)$, $B(-1,1)$, $C(-3,4)$, and G is the center of gravity of the triangle ABC, then the midpoint F of the line segment AG has coordinates

$\boxed{\text{A}}$ $F(0,0)$; $\boxed{\text{B}}$ $F(\frac{2}{3}, \frac{17}{6})$; $\boxed{\text{C}}$ $F(-\frac{2}{3}, \frac{8}{3})$; $\boxed{\text{D}}$ some other value.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **true**; $\boxed{\textbf{C}}$ **false**; $\boxed{\textbf{D}}$ **false**.

Solution: The coordinates of the center of gravity are: $G\left(\frac{2+(-1)+(-3)}{3}, \frac{3+1+4}{3}\right) = G\left(-\frac{2}{3}, \frac{8}{3}\right)$. The midpoint of the line segment AG has coordinates: $F\left(\frac{2+(-2/3)}{2}, \frac{3+8/3}{2}\right) = F\left(\frac{2}{3}, \frac{17}{6}\right)$.

3. The number of solutions of the equation $3\sin x - 2 = 0$ in the interval $[0,\pi]$ is

$\boxed{\text{A}}$ 0; $\boxed{\text{B}}$ 1; $\boxed{\text{C}}$ 2; $\boxed{\text{D}}$ infinite.

Answer:

$\boxed{\textbf{A}}$ **false;** $\boxed{\textbf{B}}$ **false;** $\boxed{\textbf{C}}$ **true**; $\boxed{\textbf{D}}$ **false**.

Solution: The set of solutions is: $\{\arcsin\frac{2}{3}, \pi - \arcsin\frac{2}{3}\}$, a set with two elements.

4. Consider in $\mathbb{R}$ the equation $\sqrt{x^2-3} = x^2 - 5$. Which of the following statements are true?

$\boxed{\text{A}}$ The equation has no solutions.
$\boxed{\text{B}}$ The equation has exactly two solutions.
$\boxed{\text{C}}$ The equation has exactly four solutions.
$\boxed{\text{D}}$ The equation has only positive solutions.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **true**; $\boxed{\textbf{C}}$ **false**; $\boxed{\textbf{D}}$ **false**.

Solution: Because of the square root, we must have $x^2-3 \geq 0$ and $x^2-5 \geq 0$, thus, $x \in (-\infty, -\sqrt{5}] \cup [\sqrt{5}, +\infty)$. Taking the square of the equation, we get

$$x^2 - 3 = (x^2-5)^2 \iff x^2 - 3 = x^4 - 10x^2 + 25 \iff x^4 - 11x^2 + 28 = 0.$$

Denoting by $y = x^2$, the previous equation can be written as $y^2 - 11y + 28 = 0$. This quadratic equation has solutions $y_1 = 4$ and $y_2 = 7$. Thus, the solutions of the equation $x^4 - 11x^2 + 28 = 0$ are $x_1 = -2$, $x_2 = 2$, $x_3 = -\sqrt{7}$ and $x_4 = \sqrt{7}$. Of these, only x_3 and x_4 satisfy the condition $x^2 - 5 \geq 0$, so equation $\sqrt{x^2-3} = x^2 - 5$ has two solutions and $\boxed{\text{A}}$, $\boxed{\text{C}}$ are false, while $\boxed{\text{B}}$ is true. One solution is negative, so $\boxed{\text{D}}$ is false.

5. The number of rational terms in the expansion $\left(\sqrt{2} + \sqrt[3]{5}\right)^{300}$ is

$\boxed{\text{A}}$ 50; $\boxed{\text{B}}$ 51; $\boxed{\text{C}}$ 52; $\boxed{\text{D}}$ 150.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **true**; $\boxed{\textbf{C}}$ **false**; $\boxed{\textbf{D}}$ **false**.

Solution: $T_{k+1} = C_{300}^{k} 2^{150-\frac{k}{2}} \cdot 5^{\frac{k}{3}}$, $0 \leq k \leq 300$.

In order for a term to be rational, k must be divisible by both 2 and 3, so k must be divisible by 6. Thus, $k \in \{0, 6, 12, \ldots, 300\}$. There are 51 rational terms.

6. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$. The sum of the elements of the matrix A^5 is:

$\boxed{\text{A}}$ 19; $\boxed{\text{B}}$ 20; $\boxed{\text{C}}$ 21; $\boxed{\text{D}}$ 22.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **false**; $\boxed{\textbf{C}}$ **true**; $\boxed{\textbf{D}}$ **false**.

Solution: Notice that $A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$, where $(F_n)_{n\geq 0}$ is the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$, $\forall n \geq 1$. The sum of the elements of A^n is

$$S_n = (F_{n+1} + F_n) + (F_n + F_{n-1}) = F_{n+2} + F_{n+1} = F_{n+3}.$$

Hence, $S_5 = F_8 = 21$.

The result can be obtained by direct computation of A^2, A^4 and A^5.

7. Let $(x_n)_{n\geq 1}$ be a sequence of positive real numbers satisfying $(n+1)x_{n+1} - nx_n < 0$, for every $n \geq 1$. Then the limit of the sequence is:

$\boxed{\text{A}}$ 1; $\boxed{\text{B}}$ ∞; $\boxed{\text{C}}$ does not exist; $\boxed{\text{D}}$ 0.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **false**; $\boxed{\textbf{C}}$ **false**; $\boxed{\textbf{D}}$ **true**.

Solution: We have $x_1 > 2x_2 > 3x_3 > \ldots > nx_n \quad \Rightarrow \quad 0 < x_n < \dfrac{x_1}{n} \quad \Rightarrow \quad \lim\limits_{n\to\infty} x_n = 0.$

8. The function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} \mathrm{e}^{-\frac{1}{x^2}} & \text{if } x < 0 \\ x^3 + x + \alpha & \text{if } x \geq 0, \end{cases}$$

is continuous if:

$\boxed{\text{A}}$ $\alpha \in \mathbb{R}$; $\boxed{\text{B}}$ $\alpha = 1$;
$\boxed{\text{C}}$ $\alpha = 0$; $\boxed{\text{D}}$ there is no $\alpha \in \mathbb{R}$ for which the function is continuous.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **false**; $\boxed{\textbf{C}}$ **true**; $\boxed{\textbf{D}}$ **false**.

Solution: We have

$$\lim_{x \nearrow 0} f(x) = \lim_{x \nearrow 0} \mathrm{e}^{-\frac{1}{x^2}} = 0, \qquad \lim_{x \searrow 0} f(x) = \lim_{x \searrow 0} (x^3 + x + \alpha) = \alpha = f(0).$$

By the definition of continuity using left and right limits, we get $\alpha = 0$.

9. The equation of the tangent line to the graph of the function $f(x) = \sqrt[3]{x-1}$ at the point $x = 9$ is:

$\boxed{\text{A}}$ $-12y + x - 15 = 0$; $\boxed{\text{B}}$ $12y - x - 15 = 0$; $\boxed{\text{C}}$ $y - 12x - 15 = 0$; $\boxed{\text{D}}$ $y + 12x + 15 = 0$.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **true**; $\boxed{\textbf{C}}$ **false**; $\boxed{\textbf{D}}$ **false**.

Solution: Since $f'(x) = \dfrac{1}{3\sqrt[3]{(x-1)^2}}$, the equation of the tangent line is

$$y - f(9) = f'(9)(x-9) \qquad \Leftrightarrow \qquad y - 2 = \frac{1}{12}(x-9) \qquad \Leftrightarrow \qquad 12y - x - 15 = 0.$$

10. The set of solutions of the equation

$$4 \cdot \sin x \cdot \cos^3 x - 4 \cdot \sin^3 x \cdot \cos x = 1$$

is

$\boxed{\text{A}}$ $\{\frac{\pi}{8} + \frac{k\pi}{4} \mid k \in \mathbb{Z}\}$; $\boxed{\text{B}}$ $\{\frac{\pi}{8} + \frac{k\pi}{2} \mid k \in \mathbb{Z}\}$; $\boxed{\text{C}}$ $\{\frac{\pi}{8} - k\pi \mid k \in \mathbb{Z}\}$; $\boxed{\text{D}}$ $\{\frac{\pi}{4} + \frac{k\pi}{8} \mid k \in \mathbb{Z}\}$.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **true**; $\boxed{\textbf{C}}$ **false**; $\boxed{\textbf{D}}$ **false**.

Solution: $4 \cdot \sin x \cdot \cos^3 x - 4 \cdot \sin^3 x \cdot \cos x = 1 \Leftrightarrow 4 \cdot \sin x \cdot \cos x \cdot (\cos^2 x - \sin^2 x) = 1 \Leftrightarrow$
$2 \cdot \sin 2x \cdot \cos 2x = 1 \Leftrightarrow \sin 4x = 1 \Leftrightarrow 4x \in \{\frac{\pi}{2} + 2k\pi \mid k \in \mathbb{Z}\} \Leftrightarrow x \in \{\frac{\pi}{8} + \frac{k\pi}{2} \mid k \in \mathbb{Z}\}$.
Thus, $\boxed{\text{B}}$ is true, while $\boxed{\text{A}}$, $\boxed{\text{C}}$ and $\boxed{\text{D}}$ are false.

11. The vertices A and B of the parallelogram $ABCD$ belong to the line of equation $3x - y - 4 = 0$ and the point of intersection O of the diagonals AC and BD has coordinates $(3, 4)$. If the coordinates of A are $(0, -4)$, then the equation of the line CD is:

$\boxed{\text{A}}$ $x + 3y - 42 = 0$; $\boxed{\text{B}}$ $x - 3y - 6 = 0$; $\boxed{\text{C}}$ $3x - y - 6 = 0$; $\boxed{\text{D}}$ $y = 3x + 6$.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **false**; $\boxed{\textbf{C}}$ **true**; $\boxed{\textbf{D}}$ **false**.

Solution: The point C is the symmetrical image of A about O, so we have

$$x_O = \frac{x_A + x_C}{2} \Leftrightarrow 3 = \frac{0 + x_C}{2} \Leftrightarrow x_C = 6$$

and

$$y_O = \frac{y_A + y_C}{2} \Leftrightarrow 4 = \frac{-4 + y_C}{2} \Leftrightarrow y_C = 12.$$

The slope of the line CD is equal to the slope of the line AB, so 3. Hence, the equation of the line CD is

$$y - 12 = 3(x - 6) \Leftrightarrow 3x - y - 6 = 0.$$

12. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x - [2x]$, where $[a]$ denotes the integer part of $a \in \mathbb{R}$. Which of the following statements are true?

$\boxed{\text{A}}$ f has period $\frac{1}{2}$; $\boxed{\text{B}}$ f is injective; $\boxed{\text{C}}$ f is surjective; $\boxed{\text{D}}$ f is even.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **false**; $\boxed{\textbf{C}}$ **false**; $\boxed{\textbf{D}}$ **true**.

Solution: Function f has period $\frac{1}{2}$, since for every $x \in \mathbb{R}$, we have

$$f\left(x + \frac{1}{2}\right) = 2\left(x + \frac{1}{2}\right) - \left[2\left(x + \frac{1}{2}\right)\right] = 2x + 1 - [2x + 1] = 2x - [2x] = f(x).$$

This also shows that f is neither injective, nor surjective, because the image of f is the same as the image of the restriction of f to the interval $[0, \frac{1}{2}]$, so $2 \notin \operatorname{Im} f$. Since $f(\frac{1}{8}) = \frac{1}{4}$ and $f(-\frac{1}{8}) = \frac{3}{4}$, the function f is not even.

13. Let $S_n = i + 2i^2 + 3i^3 + \cdots + ni^n, n \in \mathbb{N}^*$, where i is the imaginary unit ($i^2 = -1$). Which of the following statements are true?

$\boxed{\text{A}}$ S_{2020} is a real number; $\boxed{\text{B}}$ $|S_{2020}|$ is an irrational number;
$\boxed{\text{C}}$ the imaginary part of S_{2022} is 1011; $\boxed{\text{D}}$ $|S_{2022}| = 1011$.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **true**; $\boxed{\textbf{C}}$ **true**; $\boxed{\textbf{D}}$ **false**.

Solution: We know that $i^{4k} = 1, i^{4k+1} = i, i^{4k+2} = -1, i^{4k+3} = -i$ for every $k \in \mathbb{N}$. Consider the intermediate sums:
$s_1 = i + 2i^2 + 3i^3 + 4i^4 = 2 - 2i\,,$
$s_2 = 5i^5 + 6i^6 + 7i^7 + 8i^8 = 2 - 2i\,,$
...
$s_{505} = 2017i^{2017} + 2018i^{2018} + 2019i^{2019} + 2020i^{2020} = 2 - 2i.$
We have

$$S_{2020} = s_1 + s_2 + \ldots + s_{505} = 505(2 - 2i) = 1010(1 - i),$$

so S_{2020} is not a real number and $|S_{2020}| = 1010\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$, i.e. $|S_{2020}|$ is an irrational number.
We write

$$S_{2022} = S_{2020} + 2021i^{2021} + 2022i^{2022} = 1010(1 - i) + 2021i - 2022 = -1012 + 1011i,$$

so the imaginary part of S_{2022} is 1011 and $|S_{2022}| = \sqrt{(-1012)^2 + 1011^2} \neq 1011$.

14. Let $(x, y) \in \mathbb{R}_+^2$ be the solution of the system of equations

$$\begin{cases} \log_{225} x + \log_{64} y = 0 \\ \log_x 225 - \log_y 64 = 1. \end{cases}$$

The value of the expression $\log_{30}\left(x^3\right) - \log_{30} y$ is:

$\boxed{\text{A}}$ 0; $\boxed{\text{B}}$ 12; $\boxed{\text{C}}$ 1; $\boxed{\text{D}}$ 10.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **true**; $\boxed{\textbf{C}}$ **false**; $\boxed{\textbf{D}}$ **false**.

Solution: Let $(x, y) \in \mathbb{R}^2$ be o solution of this system. If we denote by $A = \log_{225} x$ and by $B = \log_{64} y$, then from the first equation we get $A + B = 0 \Rightarrow B = -A$. From the second equation we have $\frac{1}{A} - \frac{1}{B} = \frac{1}{A} + \frac{1}{A} = 1$, so $A = 2$ and $B = -2$. Thus, $\log_{225} x = 2$, so $x = 225^2 = 15^4$. Similarly, $\log_{64} y = -2$, so $y = 64^{-2} = 2^{-12}$.

Then the value of the expression is $\log_{30}\left(x^3\right) - \log_{30} y = \log_{30}(15^{12} \cdot 2^{12}) = 12$, and the correct answer is $\boxed{B}$.

15. The sum of the solutions of the equation $6^{x+1} - 4^x = 3^{2x}$ is

$\boxed{\text{A}}$ -1; $\boxed{\text{B}}$ 0; $\boxed{\text{C}}$ 1; $\boxed{\text{D}}$ 2.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **true**; $\boxed{\textbf{C}}$ **false**; $\boxed{\textbf{D}}$ **false**.

Solution: We can rewrite the equation as

$$6 \cdot \left(\frac{3}{2}\right)^x - 1 = \left(\frac{3}{2}\right)^{2x}.$$

Denoting by $t = \left(\frac{3}{2}\right)^x$, we get the quadratic equation $t^2 - 6t + 1 = 0$, with solutions $t_{1,2} = 3 \pm 2\sqrt{2}$. If $t_1 = \left(\frac{3}{2}\right)^{x_1}$ and $t_2 = \left(\frac{3}{2}\right)^{x_2}$, then

$$\left(\frac{3}{2}\right)^{x_1+x_2} = t_1 \cdot t_2 = 9 - 8 = 1,$$

so $x_1 + x_2 = 0$.

16. The point $A(3, 1)$ is the vertex of a square, for which one of the diagonals has equation $y - x = 0$.

$\boxed{\text{A}}$ the distance from the point A to the diagonal is 2;
$\boxed{\text{B}}$ the equation of the other diagonal is $x + y + 2 = 0$;
$\boxed{\text{C}}$ the aria of the square is 4;
$\boxed{\text{D}}$ the point $C(1, 3)$ is also a vertex of the square.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **false**; $\boxed{\textbf{C}}$ **true**; $\boxed{\textbf{D}}$ **true**.

Solution: The distance from A to the diagonal is

$$d = \frac{|1 - 3|}{\sqrt{1^2 + (-1)^2}} = \sqrt{2}.$$

The second diagonal of the square passes through the point A and is perpendicular to the first diagonal. Hence, $d_2 : y - 1 = (-1)(x - 3)$, so $d_2 : x + y - 4 = 0$. The aria of the square is equal to half of the square of the diagonal length, so $\frac{(2\sqrt{2})^2}{2} = 4$. The point $C(1, 3)$ is the symmetrical image of A about the given diagonal.

17. Consider the triangle ABC, with the notations $BC = a, AC = b, AB = c$. Assume that the length of the median AM is equal to c. Then:

$\boxed{\text{A}}$ $a^2 + 2c^2 = 3b^2$; $\boxed{\text{B}}$ $a^2 + 2c^2 = 2b^2$; $\boxed{\text{C}}$ $\cos C = \frac{4a}{3b}$; $\boxed{\text{D}}$ $\cos C = \frac{3a}{4b}$.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **true**; $\boxed{\textbf{C}}$ **false**; $\boxed{\textbf{D}}$ **true**.

Solution: By the median theorem, we have $AM^2 = \frac{b^2+c^2}{2} - \frac{a^2}{4}$. Since $AM = c$, we get $a^2 + 2c^2 = 2b^2$. Hence, answer $\boxed{\mathbf{A}}$ is false and answer $\boxed{\mathbf{B}}$ is true. Using the law of cosines for c in the previous relation, we obtain $3a = 4b\cos C$. Thus, statement $\boxed{\mathbf{D}}$ is true and statement $\boxed{\mathbf{C}}$ is false.

18. The value of the limit $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2} - \dfrac{1}{\sin^2 x}\right)$ is:

$\boxed{\text{A}}$ $-\dfrac{1}{3}$; $\boxed{\text{B}}$ -1; $\boxed{\text{C}}$ 0; $\boxed{\text{D}}$ $\dfrac{1}{2}$.

Answer:

$\boxed{\mathbf{A}}$ **true**; $\boxed{\mathbf{B}}$ **false**; $\boxed{\mathbf{C}}$ **false**; $\boxed{\mathbf{D}}$ **false**.

Solution: By direct computation, we get:

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right) &= \lim_{x\to 0}\frac{\sin^2 x - x^2}{x^2\sin^2 x} = \lim_{x\to 0}\frac{\sin x - x}{x^2 \sin x}\lim_{x\to 0}\frac{\sin x + x}{\sin x}\\
&= 2\lim_{x\to 0}\frac{\sin x - x}{x^2\sin x} = 2\lim_{x\to 0}\frac{\cos x - 1}{2x\sin x + x^2\cos x}\\
&= 2\lim_{x\to 0}\frac{-\sin x}{2\sin x + 4x\cos x - x^2\sin x}\\
&= 2\lim_{x\to 0}\frac{-\cos x}{6\cos x - 6x\sin x - x^2\cos x}\\
&= -\frac{1}{3}.
\end{aligned}$$

19. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \operatorname{arctg} x + \operatorname{arcctg} x$ for every $x \in \mathbb{R}$. Which of the following statements are true?

$\boxed{\text{A}}$ $f(-1) = -\dfrac{\pi}{2}$;

$\boxed{\text{B}}$ $f(x) = \dfrac{\pi}{2}$, for every $x \in (0, \infty)$;

$\boxed{\text{C}}$ the function f is odd;

$\boxed{\text{D}}$ $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to-\infty} f(x)$.

Answer:

$\boxed{\mathbf{A}}$ **false**; $\boxed{\mathbf{B}}$ **true**; $\boxed{\mathbf{C}}$ **false**; $\boxed{\mathbf{D}}$ **true**.

Solution: Since

$$f'(x) = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0 \quad \text{for every } x \in \mathbb{R},$$

f is constant on $\mathbb{R}$. Hence, $f(x) = f(0) = \dfrac{\pi}{2}$ for every $x \in \mathbb{R}$ and so it follows immediately that only statements $\boxed{\mathbf{B}}$ and $\boxed{\mathbf{D}}$ are true.

20. The number of real solutions of the equation $xe^x = -\dfrac{1}{3}$ is:

$\boxed{\text{A}}$ 0; $\boxed{\text{B}}$ 1; $\boxed{\text{C}}$ 2; $\boxed{\text{D}}$ 3.

Answer:

$\boxed{\mathbf{A}}$ **false**; $\boxed{\mathbf{B}}$ **false**; $\boxed{\mathbf{C}}$ **true**; $\boxed{\mathbf{D}}$ **false**.

Solution: Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = xe^x$, $\forall x \in \mathbb{R}$. We have $f'(x) = (x+1)e^x$ for every $x \in \mathbb{R}$. The variation of the function f is given in the table below.

x	$-\infty$			-1			∞
$f'(x)$		$-$	$-$	0	$+$	$+$	
$f(x)$	0	$\searrow$	$\searrow$	$-\dfrac{1}{\mathrm{e}}$	$\nearrow$	$\nearrow$	∞

Since $-\dfrac{1}{\mathrm{e}} < -\dfrac{1}{3} < 0$, from the table we see that the equation $x\mathrm{e}^x = -\dfrac{1}{3}$ has exactly two solutions.

21. Let ABC be a triangle and $A' \in [BC]$, $B' \in [CA]$, $C' \in [AB]$ so that $\frac{BA'}{BC} = \frac{CB'}{CA} = \frac{AC'}{AB} = \alpha$. If $\mathcal{A}_{ABC}$ is the aria of the triangle ABC and $\mathcal{A}_{A'B'C'}$ is the aria of the triangle $A'B'C'$, then

$\boxed{\text{A}}$ $\frac{\mathcal{A}_{A'B'C'}}{\mathcal{A}_{ABC}} = 1 - 3\alpha(1-\alpha)$; $\boxed{\text{B}}$ $\frac{\mathcal{A}_{A'B'C'}}{\mathcal{A}_{ABC}} \in \left[\frac{1}{4}, 1\right]$;
$\boxed{\text{C}}$ $\frac{\mathcal{A}_{A'B'C'}}{\mathcal{A}_{ABC}} = 1 - 12\alpha^2(1-\alpha)^2$; $\boxed{\text{D}}$ $\frac{\mathcal{A}_{A'B'C'}}{\mathcal{A}_{ABC}} \in \left[\frac{1}{2}, 1\right]$.

Answer: $\boxed{\text{A}}$ **true**; $\boxed{\text{B}}$ **true**; $\boxed{\text{C}}$ **false**; $\boxed{\text{D}}$ **false**.

Solution: $\frac{BA'}{BC} = \frac{CB'}{CA} = \frac{AC'}{AB} = \alpha \in [0,1]$. Writing the aria with sines, the arias of the triangles $AB'C'$, $A'BC'$ and $A'B'C$ are equal to $\alpha(1-\alpha)\mathcal{A}_{ABC}$. It then follows that $\mathcal{A}_{A'B'C'} = \mathcal{A}_{ABC} - 3\alpha(1-\alpha)\mathcal{A}_{ABC}$, so

$$\frac{\mathcal{A}_{A'B'C'}}{\mathcal{A}_{ABC}} = 1 - 3\alpha(1-\alpha) = 3\alpha^2 - 3\alpha + 1 = 3\left(\alpha - \frac{1}{2}\right)^2 + \frac{1}{4} \in \left[\frac{1}{4}, 1\right].$$

22. The value of the limit $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{\int_1^{\operatorname{tg} x} \mathrm{e}^{t^2}\mathrm{d}t}{\int_1^{\operatorname{ctg} x} \mathrm{e}^{t^2}\mathrm{d}t}$ is:

$\boxed{\text{A}}$ 1; $\boxed{\text{B}}$ π; $\boxed{\text{C}}$ 0; $\boxed{\text{D}}$ -1.

Answer:
$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **false**; $\boxed{\textbf{C}}$ **false**; $\boxed{\textbf{D}}$ **true**.

Solution: The function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(t) = \mathrm{e}^{t^2}$ is continuous, so it has primitives. Let F be a primitive of f. We have

$$\lim_{x\to\frac{\pi}{4}} \frac{\int_1^{\operatorname{tg} x} \mathrm{e}^{t^2}\mathrm{d}t}{\int_1^{\operatorname{ctg} x} \mathrm{e}^{t^2}\mathrm{d}t} = \lim_{x\to\frac{\pi}{4}} \frac{F(\operatorname{tg} x) - F(1)}{F(\operatorname{ctg} x) - F(1)} = \lim_{x\to\frac{\pi}{4}} \frac{f(\operatorname{tg} x)(1 + \operatorname{tg}^2 x)}{f(\operatorname{ctg} x)(-1 - \operatorname{ctg}^2 x)} = -1.$$

23. A triangle in which $\sin(B) + \cos(B) = \sin(C) + \cos(C)$ is:
$\boxed{\text{A}}$ right-angled $\boxed{\text{B}}$ isosceles $\boxed{\text{C}}$ equilateral $\boxed{\text{D}}$ right-angled or isosceles.

Answer: $\boxed{\text{A}}$ **false**; $\boxed{\text{B}}$ **false**; $\boxed{\text{C}}$ **false**; $\boxed{\text{D}}$ **true**.

Solution: We rewrite the given relation as: $\sin(B) - \sin(C) = \cos(C) - \cos(B)$. Converting differences into products, we have:

$$2\sin\frac{B-C}{2}\cos\frac{B+C}{2} = 2\sin\frac{B-C}{2}\sin\frac{B+C}{2}$$

or

$$\sin\frac{B-C}{2}\left(\cos\frac{B+C}{2} - \sin\frac{B+C}{2}\right) = 0.$$

If $\sin\frac{B-C}{2} = 0$, we get $B - C = 0$, so the triangle is isosceles. The relation $\cos\frac{B+C}{2} - \sin\frac{B+C}{2} = 0$ can be rewritten as $\cos\frac{B+C}{2} = \sin\left(\frac{\pi}{2} - \frac{B+C}{2}\right)$, from which we have $B + C = \frac{\pi}{2}$, so in this case the triangle is right-angled.

24. Let $(a_n)_{n\in\mathbb{N}^*}$ be the sequence defined by

$$a_n = \sqrt{\frac{1}{n^2} + \frac{1}{n^3}} + \sqrt{\frac{1}{n^2} + \frac{2}{n^3}} + \cdots + \sqrt{\frac{1}{n^2} + \frac{n}{n^3}}, \text{ for every } n \in \mathbb{N}^*.$$

Denote by $\ell = \lim_{n\to\infty} a_n$. Which of the following statements are true?

$\boxed{\text{A}}$ $\ell = 0$; $\boxed{\text{B}}$ $\ell = \frac{\sqrt{2}}{3}$; $\boxed{\text{C}}$ $\ell \in \overline{\mathbb{R}} \setminus \mathbb{Q}$; $\boxed{\text{D}}$ $\ell = \infty$.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **false**; $\boxed{\textbf{C}}$ **true**; $\boxed{\textbf{D}}$ **false**.

Solution: Notice that

$$a_n = \frac{1}{n}\left(\sqrt{1+\frac{1}{n}} + \sqrt{1+\frac{2}{n}} + \cdots + \sqrt{1+\frac{n}{n}}\right) \quad \text{for every } n \in \mathbb{N}^*.$$

Let $f\colon [0,1] \to \mathbb{R}$, $f(x) = \sqrt{1+x}$. For each $n \in \mathbb{N}^*$, consider the partition $\Delta_n = (0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1)$ of the interval $[0,1]$ and let $\xi_n = (\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1)$ be the system of intermediate points associated with Δ_n. For every $n \in \mathbb{N}^*$ the term a_n is the Riemann sum of f with partition Δ_n and intermediate points ξ_n, i.e. $a_n = \sigma_{\Delta_n}(f, \xi_n)$. Since $\lim_{n\to\infty} ||\Delta_n|| = \lim_{n\to\infty} \dfrac{1}{n} = 0$, the integrability of the function f implies

$$\ell = \lim_{n\to\infty} \sigma_{\Delta_n}(f, \xi_n) = \int_0^1 f(x)\mathrm{d}x = \int_0^1 \sqrt{1+x}\,\mathrm{d}x = \frac{2}{3}(1+x)^{\frac{3}{2}}\Big|_0^1 = \frac{2}{3}(2\sqrt{2}-1).$$

Thus, only statement $\boxed{\textbf{C}}$ is true.

25. Two sides of a rectangle have equations:

$$(d_1) : 2x - 3y + 5 = 0$$

$$(d_2) : 3x + 2y - 7 = 0$$

and one of its vertices is $A(2, -3)$. The equations of the other two sides of the rectangle are:

$\boxed{\text{A}}$ $2x - 3y - 13 = 0$ and $3x + 2y = 0$; $\boxed{\text{B}}$ $y + 3 = \frac{2}{3}(x-2)$ and $y + 3 = -\frac{3}{2}(x-2)$;

$\boxed{\text{C}}$ $2x - 3y + 13 = 0$ and $3x - 2y = 0$; $\boxed{\text{D}}$ $y - 3 = \frac{2}{3}(x-2)$ and $y - 3 = -\frac{3}{2}(x-2)$.

Answer:

$\boxed{\textbf{A}}$ **true**; $\boxed{\textbf{B}}$ **true**; $\boxed{\textbf{C}}$ **false**; $\boxed{\textbf{D}}$ **false**.

Solution: Notice that lines d_1 and d_2 are perpendicular, having slopes $m_1 = \frac{2}{3}$ and $m_2 = -\frac{3}{2}$, and they do not pass through the point A. Hence, the other sides of the rectangle do pass through A and are parallel to d_1 and d_2, respectively. The line through A parallel to d_1 has equation: $2x - 3y - 13 = 0$ or $y + 3 = \frac{2}{3}(x-2)$. The line through A parallel to d_2 has equation: $y + 3 = -\frac{3}{2}(x-2)$ or $3x + 2y = 0$.

26. Consider a parameter $\alpha \in \mathbb{C}$ and the linear system with 4 unknowns

$$\begin{cases} 2x + \alpha y + 2z & = 1 \\ 4x - y + 5z & = 1 \\ 2x + 10y + z & = 1. \end{cases}$$

Determine the truth value of the following statements:

$\boxed{\text{A}}$ The matrix of the system has rank 3 for every value of α.
$\boxed{\text{B}}$ The extended matrix of the system has rank 3 for every value of α.
$\boxed{\text{C}}$ The system is incompatible if and only if $\alpha \neq 3$.
$\boxed{\text{D}}$ The system is compatible if and only if $\alpha \neq 3$.

Answer:

A **false**; **B** **true**; **C** **false**; **D** **true**.

Solution:
The matrix and the extended matrix of the system are

$$A = \begin{pmatrix} 2 & \alpha & 2 \\ 4 & -1 & 5 \\ 2 & 10 & 1 \end{pmatrix}, \ \overline{A} = \begin{pmatrix} 2 & \alpha & 2 & 1 \\ 4 & -1 & 5 & 1 \\ 2 & 10 & 1 & 1 \end{pmatrix}, \text{ respectively.}$$

We have $\det(A) = 6\alpha - 18 = 0$ if and only if $\alpha = 3$, so the system is compatible if $\alpha \neq 3$. When $\alpha = 3$, we have $\operatorname{rank}(A) = 2$ and $\operatorname{rank}(\overline{A}) = 3$, so by the Kronecker-Capelli theorem it follows that the system is incompatible for $\alpha = 3$.

27. Let $G \subseteq \mathbb{R}$ be a set such that the relation

$$x * y = \frac{xy}{2xy - x - y + 1}, \forall x, y \in G$$

defines a composition law on G. Which of the following statements are true?

A G can be the interval $(0, 2)$.

B G can be the interval $(0, 1)$.

C If $G = (0, 1)$, then „$*$" has an identity element.

D If $G = (0, 1)$, then the inverse of $\frac{1}{3}$ is $\frac{2}{3}$.

Answer:

A **false**; **B** **true**; **C** **true**; **D** **true**.

Solution: For $x = \frac{1}{4}$ and $y = \frac{3}{2}$ the denominator is 0, so **A** is false. If $0 < x, y < 1$, then $xy > 0$ and $(1 - x)(1 - y) > 0$; adding the two, we get $2xy - x - y + 1 > 0$; from here it follows easily that $0 < x * y < 1$, so **B** is true. From the condition $x * e = x$ for every $x \in (0, 1)$, we find the identity element to be $e = \frac{1}{2}$, so **C** is true. From $x * x' = \frac{1}{2}$, we get $x' = 1 - x$, so **D** is true.

28. The value of the integral

$$\int_{\frac{1}{2022}}^{2022} \frac{\ln x}{1 + x^2} \, \mathrm{d}x$$

is:

A 0; B 1; C 2; D 3.

Answer:

A **true**; **B** **false**; **C** **false**; **D** **false**.

Solution: With the change of variables $x = \frac{1}{t}$, we get $\mathrm{d}x = -\frac{1}{t^2} \, \mathrm{d}t$ and

$$\begin{aligned} \int_{\frac{1}{2022}}^{2022} \frac{\ln x}{1 + x^2} \, \mathrm{d}x &= \int_{2022}^{\frac{1}{2022}} \frac{\ln \frac{1}{t}}{1 + \frac{1}{t^2}} \cdot \left(-\frac{1}{t^2} \right) \mathrm{d}t = \int_{\frac{1}{2022}}^{2022} \frac{\ln \frac{1}{t}}{t^2 + 1} \mathrm{d}t = \int_{\frac{1}{2022}}^{2022} \frac{-\ln t}{t^2 + 1} \mathrm{d}t = \\ &= -\int_{\frac{1}{2022}}^{2022} \frac{\ln t}{t^2 + 1} \mathrm{d}t = -\int_{\frac{1}{2022}}^{2022} \frac{\ln x}{1 + x^2} \mathrm{d}x. \end{aligned}$$

Thus, we have $\displaystyle\int_{\frac{1}{2022}}^{2022} \frac{\ln x}{1+x^2}\, dx = 0.$

29. Let $x, y, z \in \mathbb{Z}^\star$ be numbers with the property that xy, yz, zx are in a geometric progression with ratio an integer number not equal to 1.

$\boxed{\text{A}}$ If y is a perfect square, then z is also a perfect square.
$\boxed{\text{B}}$ If z is a perfect square, then y is also a perfect square.
$\boxed{\text{C}}$ If y is a perfect square, then x is also a perfect square.
$\boxed{\text{D}}$ If z is a perfect square, then x is also a perfect square.

Answer:

$\boxed{\textbf{A}}$ **true**; $\boxed{\textbf{B}}$ **true**; $\boxed{\textbf{C}}$ **false**; $\boxed{\textbf{D}}$ **false**.

Solution:

Let q be the ratio of the geometric progression. Then we have $qxy = yz$ and $qyz = zx$, i.e. $q^2xy = zx$. Hence, $z = q^2y$, so $\boxed{\textbf{A}}$ and $\boxed{\textbf{B}}$ are true.

For $y = 1$, $x = 2$ and $z = 4$, the conditions are satisfied, y and z are perfect squares, but x is not, so answers $\boxed{\text{C}}$ and $\boxed{\text{D}}$ are false.

30. Let $(x_n)_{n\in\mathbb{N}^*}$ be the sequence defined by $x_n = \displaystyle\int_0^2 \frac{(2-x)^{2n-1}}{(2+x)^{2n+1}}\,\mathrm{d}x$, for every $n \in \mathbb{N}^*$. Which of the following statements are true?

$\boxed{\text{A}}$ $x_{23} = \frac{1}{184}$. $\boxed{\text{B}}$ $\displaystyle\lim_{n\to\infty} n^2x_n = 1$. $\boxed{\text{C}}$ $\displaystyle\lim_{n\to\infty} nx_n = \frac{1}{8}$. $\boxed{\text{D}}$ $\displaystyle\lim_{n\to\infty} nx_n = 0$.

Answer:

$\boxed{\textbf{A}}$ **true**; $\boxed{\textbf{B}}$ **false**; $\boxed{\textbf{C}}$ **true**; $\boxed{\textbf{D}}$ **false**.

Solution: Let $n \in \mathbb{N}^*$. Then

$$x_n = \int_0^2 \left(\frac{2-x}{2+x}\right)^{2n-1} \cdot \frac{1}{(2+x)^2}\,\mathrm{d}x = -\frac{1}{4}\int_0^2 \left(\frac{2-x}{2+x}\right)^{2n-1} \cdot \left(\frac{2-x}{2+x}\right)'\,\mathrm{d}x = -\frac{1}{8n}\left(\frac{2-x}{2+x}\right)^{2n}\Bigg|_0^2 = \frac{1}{8n}.$$

Since $184 = 8 \cdot 23$, it follows that only statements $\boxed{\textbf{A}}$ and $\boxed{\textbf{C}}$ are true.