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FACULTY OF MATHEMATICS AND COMPUTER SCIENCE

MATE-INFO UBB COMPETITION and ADMISSIONS EXAM – MODEL 2021
Written test in MATHEMATICS
SOLUTIONS

1. In $\mathbb{R}$ consider the equation

$$2^{x^2+x+\frac{1}{2}} - 4\sqrt{2} = 0\,.$$

The set of solutions of the equation is:

$\boxed{\text{A}}$ $S = \{1\}$; $\boxed{\text{B}}$ $S = \{1, 2\}$; $\boxed{\text{C}}$ $S = \{2\}$; $\boxed{\text{D}}$ $S = \{-2, 1\}$.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **false**; $\boxed{\textbf{C}}$ **false**; $\boxed{\textbf{D}}$ **true**.

Solution: We write the equation in the form

$$2^{x^2+x+\frac{1}{2}} = 2^{\frac{5}{2}},$$

which is equivalent to the equation $x^2 + x - 2 = 0$, with solutions $x_1 = -2$ and $x_2 = 1$. So, the correct answer is $S = \{-2, 1\}$.

2. For the matrix equation

$$\begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} X = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

to have at least one nonzero solution $X \in \mathcal{M}_2(\mathbb{R})$, it is neccesary and sufficient that:

$\boxed{\text{A}}$ $a \in \mathbb{R}^*$; $\boxed{\text{B}}$ $a = 0$; $\boxed{\text{C}}$ $a \in \{-3, 2\}$; $\boxed{\text{D}}$ $a \in \{-1, 1\}$.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **false**; $\boxed{\textbf{C}}$ **false**; $\boxed{\textbf{D}}$ **true**.

Solution: Let $A = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}$ and $X = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$.
If $\det A \neq 0$, then A is invertible and the equation $AX = O_2$ has the unique solution $X = O_2$.
If $\det A = 0$, then the equation leads to the matrix equations

$$\begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

These are two homogeneous indeterminate compatible systems from which the columns of X are found, to get solutions $X \neq O_2$ of the original equation.

So, the initial equation has at least one solution $X \neq O_2$ if and only if $\det A = 0$, i.e. $1 - a^2 = 0$ or, equivalently, $a \in \{-1, 1\}$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + x + 1, \forall x \in \mathbb{R}$. Which of the following statements is true?

$\boxed{\text{A}}$ For any $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $f(x) = y$.
$\boxed{\text{B}}$ For any $y \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $f(x) = y$.
$\boxed{\text{C}}$ If $x_1, x_2 \in \mathbb{R}$ and $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

$\boxed{\text{D}}$ If $x_1, x_2 \in \mathbb{R}$ and $x_1 = x_2$, then $f(x_1) = f(x_2)$.

Answer:

$\boxed{\textbf{A}}$ **true**; $\boxed{\textbf{B}}$ **false**; $\boxed{\textbf{C}}$ **false**; $\boxed{\textbf{D}}$ **true**.

Solution: $\boxed{\text{A}}$ This property is true for any function, since by the definition of a function, a unique element in the codomain corresponds to each element of the domain. In this case, for every $x \in \mathbb{R}$, we compute $y = x^2 + x + 1 \in \mathbb{R}$.

$\boxed{\text{B}}$. The quadratic function from $\mathbb{R}$ to $\mathbb{R}$ is not surjective. (A simple argument would be the fact that $f(x) = x^2 + x + 1$ cannot take negative values.)

$\boxed{\text{C}}$ The quadratic function from $\mathbb{R}$ to $\mathbb{R}$ is not injective (for instance, $0 \neq -1$, but $f(0) = 1 = f(-1)$).

$\boxed{\text{D}}$ This implication is true for any function, because each element in the domain ($x_1 = x_2$ in our case) has a unique corresponding value in the codomain (namely, $f(x_1) = f(x_2)$).

4. If $\alpha \in \mathbb{C}$ and $\alpha^3 = 1$, then the rank of the matrix $\begin{pmatrix} 1 & \alpha & \alpha^2 \\ \alpha & \alpha^2 & 1 \\ \alpha^2 & 1 & \alpha \end{pmatrix}$ is:

$\boxed{\text{A}}$ 1, because $\alpha^3 = 1 \Rightarrow \alpha = 1$. $\boxed{\text{B}}$ 2, because the determinant of the matrix is 0.
$\boxed{\text{C}}$ 1. $\boxed{\text{D}}$ 1 or 2, depending on α.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **false**; $\boxed{\textbf{C}}$ **true**; $\boxed{\textbf{D}}$ **false**.

Solution: Let A be the matrix above. Since

$$\alpha^3 = 1 \Leftrightarrow \alpha^3 - 1 = 0 \Leftrightarrow (\alpha - 1)(\alpha^2 + \alpha + 1) = 0 \Leftrightarrow \alpha = 1 \text{ or } \alpha^2 + \alpha + 1 = 0,$$

there exist 3 distinct complex numbers satisfying the given equality. These are 1 and the complex roots of the equation $\alpha^2 + \alpha + 1 = 0$.

Obviously, $\text{rank}A \geq 1$, and since

$$\begin{vmatrix} 1 & \alpha \\ \alpha & \alpha^2 \end{vmatrix} = \begin{vmatrix} 1 & \alpha^2 \\ \alpha & 1 \end{vmatrix} = \begin{vmatrix} 1 & \alpha \\ \alpha^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & \alpha^2 \\ \alpha^2 & \alpha \end{vmatrix} = 0,$$

$\text{rank}A = 1$ for any third order root of the unity. Thus, the correct answer is $\boxed{\text{C}}$.
$\boxed{\text{A}}$ is false because it ignores the roots in $\mathbb{C} \setminus \mathbb{R}$ of the equation $\alpha^3 = 1$.
$\boxed{\text{B}}$ is false (even if $\det A = 0$) because $\text{rank}A \neq 2$.
$\boxed{\text{D}}$ is false because $\text{rank}A = 1$, independently of the value of α.

5. Let $(a_n)_{n \geq 1}$ be a sequence of real numbers in arithmetic progression. If $a_{101} = 695$ and $a_{1001} = 6995$, determine which of the following statements is true.

$\boxed{\text{A}}$ $a_1 \in [-6, 6]$; $\boxed{\text{B}}$ $a_1 = 5$; $\boxed{\text{C}}$ $a_{2021} = 14135$; $\boxed{\text{D}}$ $\displaystyle\sum_{k=11}^{20} a_k = 965$.

Answer:

$\boxed{\textbf{A}}$ **true**; $\boxed{\textbf{B}}$ **false**; $\boxed{\textbf{C}}$ **true**; $\boxed{\textbf{D}}$ **true**.

Solution: Writing the general term in terms of a_1 and the ratio r, by hypothesis, we obtain the system

$$\begin{cases} a_1 + 100r = 695 \\ a_1 + 1000r = 6995 \end{cases}, \text{ from which we get } a_1 = -5 \text{ and } r = 7.$$

Thus, $\boxed{\text{A}}$ is true and $\boxed{\text{B}}$ is false.

$$a_{2021} = a_1 + 2020r = 14135,$$

so $\boxed{\text{C}}$ is true.

$$\begin{aligned}\sum_{k=11}^{20} a_k =& S_{20} - S_{10} = \frac{(a_1 + a_{20}) \cdot 20}{2} - \frac{(a_1 + a_{10}) \cdot 10}{2} = \\ =&[-5 + (-5 + 19 \cdot 7)] \cdot 10 - [-5 + (-5 + 9 \cdot 7)] \cdot 5 = 1230 - 265 = 965,\end{aligned}$$

hence, $\boxed{\text{D}}$ is also true.

6. Let $f_m : \mathbb{R} \to \mathbb{R}$, $f_m(x) = mx^2 + 2(m+1)x + m - 2, \forall x \in \mathbb{R}$, where $m \in \mathbb{R} \setminus \{0\}$. The vertices of all the parabolas associated with these functions belong to:

$\boxed{\text{A}}$ the line $y = -x + 3$; $\boxed{\text{B}}$ the parabola $y = x^2 - 3$;
$\boxed{\text{C}}$ the line $y = x - 3$; $\boxed{\text{D}}$ the line $y = 4x + 2$.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **false**; $\boxed{\textbf{C}}$ **true**; $\boxed{\textbf{D}}$ **false**.

Solution: The vertex of the parabola associated with a function

$$g : \mathbb{R} \to \mathbb{R},\ g(x) = ax^2 + bx + c\ (a, b, c \in \mathbb{R},\ a \neq 0)$$

is the point $V\left(-\dfrac{b}{2a}, -\dfrac{\Delta}{4a}\right)$.

Let $m \in \mathbb{R}^*$ be fixed. For the function f_m, we have

$$\Delta = 4(m+1)^2 - 4m(m-2) = 4m^2 + 8m + 4 - 4m^2 + 8m = 16m + 4.$$

The abscissa of the vertex of the associated parabola is

$$x_V = -\frac{b}{2a} = -\frac{m+1}{m}, \tag{1}$$

and the ordinate is

$$y_V = -\frac{\Delta}{4a} = -\frac{16m+4}{4m} = -\frac{4m+1}{m}. \tag{2}$$

Since $4m + 1 = (m+1) + 3m$,

$$y_V = -\frac{m+1}{m} - 3 = x_V - 3.$$

Thus, the vertices of the parabolas associated with these functions belong to the line $y = x - 3$.

Alternative solution: (1) can be written as

$$mx_V = -m - 1.$$

It follows that none of the vertices of the parabolas associated with the functions f_m can have abscissa $x_V = -1$ so we can write

$$m = -\frac{1}{x_V + 1}.$$

Substituting this m in (2) yields $y_V = x_V - 3$.

7. Consider the polynomial $P = X^4 + aX^3 - 6X^2 + 15X + b \in \mathbb{R}[X]$. If P is divisible by $Q_1 = X - 1$ and $Q_2 = X + 3$, then:

$\boxed{\text{A}}$ $a = -3$; $\boxed{\text{B}}$ $b = -9$;
$\boxed{\text{C}}$ $a + b = -10$; $\boxed{\text{D}}$ $a = 1$ and $b = 9$.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **true**; $\boxed{\textbf{C}}$ **true**; $\boxed{\textbf{D}}$ **false**.

Solution: Since $Q_1 \mid P$, it follows that $P(1) = 0$, i.e.

$$1 + a - 6 + 15 + b = 0 \Leftrightarrow a + b = -10,$$

which is the equality in $\boxed{\text{C}}$.

From $Q_2 \mid P$ it follows that $P(-3) = 0$, i.e.

$$81 - 27a - 54 - 45 + b = 0 \Leftrightarrow -27a + b = 18.$$

Then we have the system:

$$\begin{cases} a + b = -10 \\ 27a - b = -18 \end{cases}$$

with unique solution $(a, b) = (-1, -9)$. Thus, $\boxed{\text{B}}$ is also true.

8. Let the functions $f : \mathbb{R} \to (0, \infty)$ and $g : (0, \infty) \to \mathbb{R}$ be defined by $f(x) = (2a^2 + 2a - 1)^x, \forall x \in \mathbb{R}$ and $g(y) = \log_{a^2+2} y, \forall y \in (0, \infty)$. Let A be the set of values $a \in \mathbb{R}$ for which f and g are inverses of each other. Which of the following statements is true?

$\boxed{\text{A}}$ $A \subseteq [-1, 4]$; $\boxed{\text{B}}$ $A \subseteq [-4, 1]$;
$\boxed{\text{C}}$ $A \subseteq [-2, 3]$; $\boxed{\text{D}}$ $A \subseteq [-3, 2]$.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **true**; $\boxed{\textbf{C}}$ **false**; $\boxed{\textbf{D}}$ **true**.

Solution: Let us notice that for every $a \in \mathbb{R}$, $2a^2 + 2a - 1 = a^2 + (a - 1)^2 > 0$ and $a^2 + 2 \geq 2$ (hence, positive and not equal to 1). If f and g are inverses of each other, then

$$f \circ g = 1_{(0,\infty)} \text{ and } g \circ f = 1_{\mathbb{R}}.$$

In particular, we have $f(g(a^2 + 2)) = a^2 + 2$, i.e. $f(1) = a^2 + 2$ and so $2a^2 + 2a - 1 = a^2 + 2$. It follows that $a^2 + 2a - 3 = 0$, which has solutions $a = 1$ and $a = -3$. For these values, direct computation shows that f and g are inverses of each other.

9. On the set $\mathbb{R}$ of all real numbers, consider the operation „$*$" defined by

$$x * y = xy - 4x - 4y + 20.$$

Which of the following statements is true?

$\boxed{\text{A}}$ $(1 * 1) * 1 = 3$; $\boxed{\text{B}}$ 4 is the identity element with respect to „$*$";
$\boxed{\text{C}}$ the inverse of 3 is 3; $\boxed{\text{D}}$ $(\mathbb{R}, *)$ is a group.

Answer:

$\boxed{\mathbf{A}}$ **false**; $\boxed{\mathbf{B}}$ **false**; $\boxed{\mathbf{C}}$ **true**; $\boxed{\mathbf{D}}$ **false**.

Solution: $(1*1)*1=-23$, so $\boxed{\text{A}}$ is false.
We have

$$x*e=x \Leftrightarrow x(e-5)-4e+20=0 \Leftrightarrow x(e-5)-4(e-5)=0 \Leftrightarrow (e-5)(x-4)=0.$$

This is true for every $x\in\mathbb{R}$ if and only if $e=5$. So the identity element (which is unique) is 5 and thus, $\boxed{\text{B}}$ is false.

Let $x\in\mathbb{R}$ be arbitrarily fixed. We have

$$x*x'=e \Leftrightarrow xx'-4x-4x'+20=5 \Leftrightarrow xx'-4x-4x'+16=1 \Leftrightarrow (x-4)(x'-4)=1.$$

The equation in x' has a solution $\left(\text{namely, } x'=\dfrac{1}{x-4}+4,\text{ unique solution}\right)$ if and only if $x\neq 4$.

For $x=3$, we get $x'=\dfrac{1}{3-4}+4=3$. It follows that $\boxed{\text{C}}$ is true.

Answer $\boxed{\text{D}}$ is false because 4 is not invertible.

10. Let $A=\{0,1,2,3,4,5,6\}$ and B be the set of three-digit numbers formed with distinct numbers from A. Which of the following statements is true?

$\boxed{\text{A}}$ B has 240 elements; $\boxed{\text{B}}$ B has 210 elements; $\boxed{\text{C}}$ B has 180 elements;
$\boxed{\text{D}}$ exactly 35 of the numbers in B consist of digits written in decreasing order.

Answer:

$\boxed{\mathbf{A}}$ **false**; $\boxed{\mathbf{B}}$ **false**; $\boxed{\mathbf{C}}$ **true**; $\boxed{\mathbf{D}}$ **true**.

Solution: The number of elements of B is the number of triplets (a,b,c) with $a,b,c\in A$, $a\neq b\neq c\neq a\neq 0$. The number of triplets (a,b,c) with $a,b,c\in A$, $a\neq b\neq c\neq a$ is the number of arrangements of 7 elemente — the elements of A — taken 3 at a time, so $\mathbf{A}_7^3=210$. The number of triplets $(0,b,c)$ with $b,c\in A\setminus\{0\}$, $b\neq c$ is $\mathbf{A}_6^2=30$. Thus, the cardinality of B is

$$210-30=180.$$

For the last question, we must count the triplets (a,b,c) with $a,b,c\in A$, $a>b>c$. Obviously, it is no longer possible that $a=0$ and each such triplet corresponds to a subset of 3 elements of A (elements which are ordered decreasingly to construct each triplet). So, the number of such triplets is

$$\mathbf{C}_7^3=35.$$

Thus, $\boxed{\text{A}}$ and $\boxed{\text{B}}$ are false, while $\boxed{\text{C}}$ and $\boxed{\text{D}}$ are true.

11. If the vertices of a triangle ABC have coordinates $A(2,3)$, $B(-1,1)$, $C(-3,4)$, then

$\boxed{\text{A}}$ The aria of ABC is $\frac{13}{2}$. $\boxed{\text{B}}$ ABC is a right triangle.
$\boxed{\text{C}}$ ABC is an isosceles triangle. $\boxed{\text{D}}$ The point C belongs to the line AB.

Answer:

$\boxed{\mathbf{A}}$ **true**; $\boxed{\mathbf{B}}$ **true**; $\boxed{\mathbf{C}}$ **true**; $\boxed{\mathbf{D}}$ **false**.

Solution: Let us notice that $\overrightarrow{BA}(3,2)$, $||\overrightarrow{BA}||=\sqrt{13}$, $\overrightarrow{BC}(-2,3)$, $||\overrightarrow{BC}||=\sqrt{13}$. It follows that ΔABC is isosceles. Moreover, $\overrightarrow{BA}\cdot\overrightarrow{BC}=0\Rightarrow \overrightarrow{BA}\perp\overrightarrow{BC}$ and ABC is a right triangle. Its area is $S=\dfrac{c_1\cdot c_2}{2}=\dfrac{AB\cdot BC}{2}=\dfrac{13}{2}\neq 0$, so it is a nondegenerate triangle, *i.e.* $C\notin AB$.

12. In the xOy coordinate system consider the points $A(-2,3)$ and $B(0,1)$. The distance from the point $M(1,5)$ to the perpendicular bisector of the line segment $[AB]$ is

$\boxed{\text{A}}$ $\frac{1}{\sqrt{2}}$; $\boxed{\text{B}}$ $-\frac{1}{\sqrt{2}}$; $\boxed{\text{C}}$ $\frac{3}{\sqrt{2}}$; $\boxed{\text{D}}$ another answer.

Answer:

$\boxed{\textbf{A}}$ **true**; $\boxed{\textbf{B}}$ **false**; $\boxed{\textbf{C}}$ **false**; $\boxed{\textbf{D}}$ **false**.

Solution: $C(-1,2)$ is the midpoint of $[AB]$. The slope of AB is -1, so the slope of the perpendicular bisector of $[AB]$ is 1, and the equation of the perpendicular bisector is $d: y-2 = 1\cdot(x+1) \Leftrightarrow x-y+3=0$. Hence, $d(M,d) = \frac{|1-5+3|}{\sqrt{1^2+1^2}} = \frac{1}{\sqrt{2}}$.

13. Consider the triangle ABC. Its vertices have coordinates $A(1,1)$, $B(9,1)$, $C(1,5)$, with respect to a Cartesian orthonormal coordinate system in the plane of the triangle.

$\boxed{\text{A}}$ ABC is a right triangle and $m(\widehat{A}) = 90°$;

$\boxed{\text{B}}$ $H = A(1,1)$ is the orthocenter, $G\left(\frac{11}{3}, \frac{7}{3}\right)$ is the centroid and $O(5,3)$ is the circumcenter of the triangle ABC;

$\boxed{\text{C}}$ The points G, H, O are not colinear.

$\boxed{\text{D}}$ The centroid G is equally distanced from $[AB]$ and $[AC]$.

Answer:

$\boxed{\textbf{A}}$ **true**; $\boxed{\textbf{B}}$ **true**; $\boxed{\textbf{C}}$ **false**; $\boxed{\textbf{D}}$ **false**.

Solution: $\boxed{\text{A}}$ **True.** Indeed, AB and AC are parallel to the coordinate axes, since $\overrightarrow{AB}\,(8,0)$ and $\overrightarrow{AC}\,(0,4)$.

$\boxed{\text{B}}$ **True.** Indeed, the orthocenter of a right triangle is the vertex of the right angle. Also, the coordinates of the centroid are the averages of the abscissas and ordinates, respectively, of the vertices. In this case, the coordinates of the centroid are, indeed, $\left(\dfrac{11}{3}, \dfrac{7}{3}\right)$. Finally, the circumcenter of a right triangle is the midpoint of the hypotenuse, since that is the point of intersection of the perpendicular bisectors of the three sides of the triangle. In this case, the coordinates of the circumcenter are, indeed, $(5,3)$ because this is the midpoint of the hypotenuse $[BC]$.

$\boxed{\text{C}}$ **False.** The points G, H, O are colinear in any triangle, and the line they lie on is the well-known Euler line. In this particular case, the colinearity of the three points can be checked directly, by noticing that

$$\overrightarrow{HG}\left(\frac{8}{3}, \frac{4}{3}\right) \text{ and } \overrightarrow{HO}\,(4,2),$$

which means

$$\overrightarrow{HG} = \frac{2}{3}\overrightarrow{HO}\,. \tag{1}$$

Moreover, relation (1) holds in any triangle.

$\boxed{\text{D}}$ **False.** Indeed,

$$\text{dist}(G, AB) = \frac{7}{3} - 1 = \frac{4}{3} \neq \frac{8}{3} = \text{dist}(G, AC).$$

14. In the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ the equation $4 \cdot |\sin(x)| \cdot \cos(x) = 1$

$\boxed{\text{A}}$ has no solution; $\boxed{\text{B}}$ has two solutions; $\boxed{\text{C}}$ has four solutions; $\boxed{\text{D}}$ has infinitely many solutions.

Answer: $\boxed{\textbf{A}}$ **false;** $\boxed{\textbf{B}}$ **false;** $\boxed{\textbf{C}}$ **true;** $\boxed{\textbf{D}}$ **false**.

Solution: Let $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Then $|\sin(x)| \cdot \cos(x) \geq 0$. So, we have:

$4 \cdot |\sin(x)| \cdot \cos(x) = 1 \Leftrightarrow (4 \cdot |\sin(x)| \cdot \cos(x))^2 = 1 \Leftrightarrow 16\sin^2(x)\cos^2(x) = 1 \Leftrightarrow 4\sin^2(2x) = 1 \Leftrightarrow 2(1-\cos(4x)) = 1 \Leftrightarrow \cos(4x) = \frac{1}{2}$.

Thus, the solutions in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ are: $-\frac{5\pi}{12}, \frac{5\pi}{12}, -\frac{\pi}{12}, \frac{\pi}{12}$.

15. If a, b, c are three line segments of lengths $2, 3$ and 4, respectively, then:

| A | a, b, c can form an acute triangle; | B | a, b, c can form an obtuse triangle; |
| C | a, b, c can form an equilateral triangle; | D | a, b, c can form a triangle. |

Answer:

A **false**; **B** **true**; **C** **false**; **D** **true**.

Solution: $2+3 > 4, 3+4 > 2, 4+2 > 3 \implies \exists A, B, C$ vertices of a nondegenerate triangle such that $a = [BC]$, $b = [CA]$, $c = [AB]$. The law of cosines implies

$$\cos C = \frac{2^2 + 3^2 - 4^2}{2 \cdot 2 \cdot 3} = -\frac{1}{4} < 0.$$

Thus, $m(\hat{C}) > 90°$.

16. Consider the vectors $\vec{u} = (m-1)\vec{a} + 2\vec{b}$ and $\vec{v} = 3\vec{a} + m\vec{b}$, where $\vec{a}$ and $\vec{b}$ are not colinear. How many values can the parameter $m \in \mathbb{R}$ take, so that $\vec{u}$ and $\vec{v}$ are colinear?

A 0; B 1; C 2; D 3.

Answer:

A **false**; **B** **false**; **C** **true**; **D** **false**.

Solution: The vectors $\vec{u}$ and $\vec{v}$ are colinear if there exists $k \in \mathbb{R}^*$ such that $\vec{u} = k \cdot \vec{v} \Leftrightarrow (m-1)\vec{a} + 2\vec{b} = k \cdot (3\vec{a} + m\vec{b}) \Leftrightarrow (m-1-3k)\vec{a} = (mk-2)\vec{b}$. Vectors $\vec{a}$ and $\vec{b}$ are not colinear, so

$$\begin{cases} m - 1 - 3k = 0, \\ km - 2 = 0. \end{cases}$$

Thus, $m = 3k+1$ and so $k \cdot (3k+1) = 2$. The solutions of this equation are $k_1 = -1$, $k_2 = \dfrac{2}{3}$, so $m_1 = -2$ and $m_2 = 3$.

17. Let ABC be a right triangle in A. If $AB = 2c$, $BC = 2a$, $AC = 2b$, R and r are the radii of the inscribed and circumscribed circle, respectively, then

A $\mathcal{A}rea(_\triangle ABC) = 2bc$ B $R = \frac{a}{2}$ C $AB + AC = 2(R+r)$ D $r = \frac{2bc}{a+b+c}$.

Answer:

A **true;** **B** **false;** **C** **true;** **D** **true**.

Solution: If ABC is a right triangle in A and $AB = 2c$, $BC = 2a$, $AC = 2b$, then

- $\mathcal{A}rea(_\triangle ABC) = \frac{c_1 \cdot c_2}{2} = \frac{AB \cdot AC}{2} = 2bc$;
- the circumcenter is the midpoint of the hypotenuse, so $R = \frac{BC}{2} = a$;
- $r = \frac{\mathcal{A}_{ABC}}{p} = \frac{2bc}{a+b+c}$ where $p = a+b+c$ is the semiperimeter of the triangle.

Notice that $2(R+r) = 2\left(a+\frac{2bc}{a+b+c}\right) = 2\frac{a^2+ab+ac+2bc}{a+b+c} = 2\frac{(b^2+c^2)+ab+ac+2bc}{a+b+c} = 2\frac{(b^2+c^2+2bc)+ab+ac}{a+b+c} = 2\frac{(b+c)^2+a(b+c)}{a+b+c} = 2\frac{(b+c)(a+b+c)}{a+b+c} = 2b+2c = AB+AC.$

18. If A, B, C, M are distinct points in a plane such that

$$6\overrightarrow{AM} = 3\overrightarrow{AB} + 3\overrightarrow{AC} - 5\overrightarrow{BC},$$

then:

$\boxed{\text{A}}$ B, C, M are colinear; $\boxed{\text{B}}$ B, C, M are not colinear;
$\boxed{\text{C}}$ $\overrightarrow{BM} \cdot \overrightarrow{BC} < 0$; $\boxed{\text{D}}$ $\overrightarrow{BM} \cdot \overrightarrow{BC} > 0$.

Answer:

$\boxed{\textbf{A}}$ **true**; $\boxed{\textbf{B}}$ **false**; $\boxed{\textbf{C}}$ **true**; $\boxed{\textbf{D}}$ **false**.

Solution:
$6\overrightarrow{AM} = 3\overrightarrow{AB}+3\overrightarrow{AC}-5\overrightarrow{BC} \implies 6(\overrightarrow{AB}+\overrightarrow{BM}) = 3\overrightarrow{AB}+3\overrightarrow{AC}-5\overrightarrow{BC} \implies 6\overrightarrow{BM} = -3\overrightarrow{AB}+3\overrightarrow{AC}-5\overrightarrow{BC} = 3\overrightarrow{BC}-5\overrightarrow{BC} = -2\overrightarrow{BC} \implies \overrightarrow{BC} = -3\overrightarrow{BM} \implies B, C, M$ are colinear and $\overrightarrow{BM} \cdot \overrightarrow{BC} = -3|\overrightarrow{BM}|^2 < 0$.

19. Consider the points $A(1,-1)$, $B(3,-1)$, $A'(-4,-2)$ and $B'(0,-2)$. The points C and C' belong to the parabola $\mathcal{P}$ with equation $y = x^2$. Denote by α the area of the triangle ABC and by α' the area of the triangle $A'B'C'$.

$\boxed{\text{A}}$ There exists a point P on the parabola $\mathcal{P}$ such that $C = C' = P$ and $\alpha = \alpha'$.
$\boxed{\text{B}}$ There exists a unique point C such that ABC is a right triangle.
$\boxed{\text{C}}$ There exists at least one point C with integer coordinates such that α is a prime number.
$\boxed{\text{D}}$ There exists at least one point C' such that the line segment $A'C'$ has length $3\sqrt{2}$.

Answer:

$\boxed{\textbf{A}}$ **false;** $\boxed{\textbf{B}}$ **false;** $\boxed{\textbf{C}}$ **true;** $\boxed{\textbf{D}}$ **true**.

Solution: $\boxed{\text{B}}$ is false because the right angle can be both in A and B, and the perpendiculars to AB from A and B, respectively, intersect the parabola in two distinct points.

Let $C = C(x, x^2)$ and $C' = C'(y, y^2)$. We have

$$\alpha = 1 + x^2 \quad \text{and} \quad \alpha' = 2(2 + y^2).$$

So $\boxed{\text{C}}$ is true.

By $1 + x^2 = 2(2 + x^2) \Leftrightarrow x^2 = -3$, it follows that $\boxed{\text{A}}$ is false.

The distance from C' to A' is $\sqrt{y^4 + 5y^2 + 8y + 20} = 3\sqrt{2}$ with solution -1, so $\boxed{\text{D}}$ is true. (Alternatively, from the figure we see that a circle of radius $3\sqrt{2}$ centered in A' intersects the parabola in $(-1, 1)$, so $\boxed{\text{D}}$ is true.)

20. Consider the line with equation $d : x - y = 1$ and the points $A(-1, 0)$ and $B(1, 2)$. For any $M \in d$, denote by s_M the sum of the lengths of the line segments $[AM]$ and $[BM]$. Then:

$\boxed{\text{A}}$ $\forall M \in d : s_M \geq \sqrt{2} + 2$; $\boxed{\text{B}}$ $\forall M \in d : s_M \leq \sqrt{2} + 4$;
$\boxed{\text{C}}$ $\forall M \in d : s_M \geq \sqrt{2} + \sqrt{10}$; $\boxed{\text{D}}$ $\exists M \in d$ such that $s_M = \sqrt{2} + 2$.

Answer:

$\boxed{\textbf{A}}$ **true**; $\boxed{\textbf{B}}$ **false**; $\boxed{\textbf{C}}$ **false**; $\boxed{\textbf{D}}$ **false**.

Solution: Let $A'(1, -2)$ be the symmetrical point of A about d and let $\{M_0(1, 0)\} = BA' \cap d$. Then

$$|AM_0| + |M_0B| = |A'M_0| + |M_0B| = |A'B| \leq |A'M| + |MB| = |AM| + |MB|$$

and $|AM_0| = |M_0B| = 2$. Hence, for every $M \in d$, $s_M \in [4, \infty)$.

21. The limit of the sequence $a_n = n\left(\sqrt[4]{n+1} - \sqrt[4]{n}\right), \forall n \geq 1$ is

$\boxed{\text{A}}$ 4; $\boxed{\text{B}}$ $\frac{1}{4}$; $\boxed{\text{C}}$ 0; $\boxed{\text{D}}$ $+\infty$.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **false**; $\boxed{\textbf{C}}$ **false**; $\boxed{\textbf{D}}$ **true**.

Solution: By $\lim\limits_{x\to 0} \frac{(1+x)^{1/4}-1}{x} = 1/4$, it follows that $\lim\limits_{n\to\infty} n\left(\sqrt[4]{1+1/n}-1\right) = 1/4$, so $\lim\limits_{n\to\infty} n^{3/4}\left(\sqrt[4]{n+1} - \sqrt[4]{n}\right) = 1/4$. Thus, the limit of a_n is $+\infty$.

22. For $n \in \mathbb{N}^*$, let $a_n = \frac{n!}{n^n}$. Which of the following statements is true?

$\boxed{\text{A}}$ The sequence $(a_n)_{n\in\mathbb{N}^*}$ is strictly increasing. $\boxed{\text{B}}$ $\lim\limits_{n\to\infty} a_n = 1$.

$\boxed{\text{C}}$ $a_{2021} \leq \frac{1}{2021}$. $\boxed{\text{D}}$ $\lim\limits_{n\to\infty} a_n = 0$.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **false**; $\boxed{\textbf{C}}$ **true**; $\boxed{\textbf{D}}$ **true**.

Solution: Since $a_1 = 1 > \frac{1}{2} = a_2$, the sequence is not strictly increasing. Using the inequality $n! \leq n^{n-1}$, $\forall n \in \mathbb{N}^*$, it follows that

$$0 \leq a_n \leq \frac{1}{n}, \ \forall n \in \mathbb{N}^*,$$

from which we get $\lim\limits_{n\to\infty} a_n = 0$.

23. Denote by $I = \int\limits_{\pi}^{2\pi} \frac{\cos^2 \frac{x}{2}}{x + \sin x} dx$. The value of I is

$\boxed{\text{A}}$ $\frac{\ln 2}{2}$; $\boxed{\text{B}}$ $\frac{1}{2}$; $\boxed{\text{C}}$ $\ln 2$; $\boxed{\text{D}}$ $\frac{\ln 3}{3}$.

Answer:

$\boxed{\textbf{A}}$ **true**; $\boxed{\textbf{B}}$ **false**; $\boxed{\textbf{C}}$ **false**; $\boxed{\textbf{D}}$ **false**.

Solution: If $t = x + \sin x$, then $dt = (1 + \cos x)dx$, so $dt = 2\cos^2 \frac{x}{2} dx$ and we get

$$2I = \ln(x + \sin x)\Big|_{\pi}^{2\pi} = \ln 2.$$

Hence, $I = \frac{\ln 2}{2}$.

24. Denote by A the set of real numbers a for which the function $f : [0, 1] \to \mathbb{R}$, $f(x) = x^2(x+a), \forall x \in \mathbb{R}$ has exactly two extrema points.

$\boxed{\text{A}}$ $A = (-\infty, -\frac{3}{2}] \cup [0, +\infty)$; $\boxed{\text{B}}$ $A = (-\frac{3}{2}, 0)$; $\boxed{\text{C}}$ $A = \emptyset$; $\boxed{\text{D}}$ $A = \mathbb{R}^*$.

Answer:

$\boxed{\textbf{A}}$ **true**; $\boxed{\textbf{B}}$ **false**; $\boxed{\textbf{C}}$ **false**; $\boxed{\textbf{D}}$ **false**.

Solution: From $f'(x) = 3x^2 + 2ax$, we get the critical points $x_1 = 0$ and $x_2 = -2a/3$. The function has exactly two extrema points if the second critical point does not belong to the interval $(0, 1)$. In this case, the extrema points of the function are only the endpoints of the interval of definition. So $-2a/3 \leq 0$ or $-2a/3 \geq 1$. It follows that $a \in (-\infty, -\frac{3}{2}] \cup [0, +\infty)$.

25. Let $f\colon \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \cos x - 2, & -\frac{\pi}{2} \le x \le 0 \\ \frac{\sin x}{x}, & 0 < x \le \frac{\pi}{2}. \end{cases}$$

Then

$\boxed{\text{A}}$ function $|f|$ is continuous in 0;

$\boxed{\text{B}}$ function f has at least one zero in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, because $f\left(-\frac{\pi}{2}\right) \cdot f\left(\frac{\pi}{2}\right) < 0$;

$\boxed{\text{C}}$ function f has no zeros in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$;

$\boxed{\text{D}}$ function f has no limit in 0.

Answer:

$\boxed{\textbf{A}}$ **true**; $\boxed{\textbf{B}}$ **false**; $\boxed{\textbf{C}}$ **true**; $\boxed{\textbf{D}}$ **true**.

Solution: The equalities

$$\lim_{\substack{x\to 0\\ x<0}} |f(x)| = \lim_{\substack{x\to 0\\ x>0}} |f(x)| = 1 = |f(0)|$$

imply the continuity of the function $|f|$ in 0, while the equalities

$$\lim_{\substack{x\to 0\\ x<0}} f(x) = -1 \text{ and } \lim_{\substack{x\to 0\\ x>0}} f(x) = 1$$

show that f does not have a limit at 0.

By the definition of f, it follows that f has no zeros in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

26. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2^{x^2}$, $\forall x \in \mathbb{R}$. Then:

$\boxed{\text{A}}$ $\lim\limits_{x\to\infty} f(-x) = 0$;

$\boxed{\text{B}}$ function f is strictly increasing on the interval $[1, \infty)$;

$\boxed{\text{C}}$ the inequality $f'(x) \ge x f(x)$ holds for every $x \in \mathbb{R}$;

$\boxed{\text{D}}$ $\lim\limits_{x\to\infty} \dfrac{f'(x)}{(2x+1)f(x)} = 1$.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **true**; $\boxed{\textbf{C}}$ **false**; $\boxed{\textbf{D}}$ **false**.

Solution: A) $\lim\limits_{x\to\infty} f(-x) = 0$: this is false, since $\lim\limits_{x\to\infty} f(-x) = \lim\limits_{x\to\infty} 2^{x^2} = \infty$;

B) function f is strictly increasing on the interval $[1, \infty)$: this is true, because for every $x, y \in [1, \infty)$ with $x \le y$, we have $x^2 \le y^2$ which implies $2^{x^2} \le 2^{y^2}$ (the function $u \in \mathbb{R} \mapsto 2^u$ is an increasing function);

C) the inequality $f'(x) \ge x f(x)$ holds for every $x \in \mathbb{R}$: this is false, because

$$f'(x) = 2x \cdot 2^{x^2} \cdot \ln 2 \text{ for every } x \in \mathbb{R} \text{ and } f'(-1) = -4\ln 2,\ x f(x)\Big|_{x=-1} = -2;$$

we have $-4\ln 2 < -2 \Leftrightarrow 4 > e$; thus, the given inequality does not hold, for instance, for $x = -1$;

D) $\lim\limits_{x\to\infty} \dfrac{f'(x)}{(2x+1)f(x)} = 1$: this is false, because

$$\lim_{x\to\infty} \frac{f'(x)}{(2x+1)f(x)} = \lim_{x\to\infty} \frac{2\ln 2 \cdot x \cdot 2^{x^2}}{(2x+1)2^{x^2}} = \lim_{x\to\infty} \frac{2\ln 2 \cdot x}{2x+1} = \ln 2.$$

27. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function whose graph has horizontal asymptotes at $-\infty$ and $+\infty$. Which of the following statements is true?

$\boxed{\text{A}}$ $\lim\limits_{x\to-\infty} \frac{f(x)}{x} = \lim\limits_{x\to+\infty} \frac{f(x)}{x}$;

$\boxed{\text{B}}$ function f is bounded;

$\boxed{\text{C}}$ the graph of the function f intersects any horizontal line in at least one point;

$\boxed{\text{D}}$ equation $f(x) = x^{2021}$ has at least one solution in $\mathbb{R}$.

Answer:

$\boxed{\textbf{A}}$ **true**; $\boxed{\textbf{B}}$ **true**; $\boxed{\textbf{C}}$ **false**; $\boxed{\textbf{D}}$ **true**.

Solution: Since the graph of f has horizontal asymptotes at $-\infty$ and $+\infty$, the limits $\lim\limits_{x\to-\infty} f(x) =: l_1 \in \mathbb{R}$ and $\lim\limits_{x\to+\infty} f(x) =: l_2 \in \mathbb{R}$ exist and are finite. On the other hand, $\lim\limits_{x\to-\infty} x = -\infty$ and $\lim\limits_{x\to+\infty} x = +\infty$. It follows that $\lim\limits_{x\to-\infty} \frac{f(x)}{x} = 0 = \lim\limits_{x\to+\infty} \frac{f(x)}{x}$. So $\boxed{\text{A}}$ is true.

By the definition of the limits l_1 and l_2, there exist $a_1, a_2 \in \mathbb{R}$ such that $f(x) \in (l_1 - 1, l_1 + 1)\ \forall x \in (-\infty, a_1)$ and $f(x) \in (l_2 - 1, l_2 + 1)$, $\forall x \in (a_2, +\infty)$, respectively. Without loss of generality, we can assume $a_1 < a_2$. On the other hand, since f is continuous on the compact interval $[a_1, a_2]$, it is bounded on that interval, which means that there exist $m_1, m_2 \in \mathbb{R}$, $m_1 < m_2$, so that $f(x) \in [m_1, m_2]$, $\forall x \in [a_1, a_2]$. Denoting by $\min\{l_1 - 1, l_2 - 1, m_1\} =: M_1$ and $\max\{l_1 + 1, l_2 + 1, m_2\} =: M_2$, we have $f(x) \in [M_1, M_2]$, $\forall x \in \mathbb{R}$, so f is bounded. Thus, $\boxed{\text{B}}$ is true.

Let $m \in \mathbb{R} \setminus [M_1, M_2]$. Since $f(x) \neq m$ for every $x \in \mathbb{R}$, it follows that the horizontal line with equation $y = m$ has no common points with the graph of f. Hence, $\boxed{\text{C}}$ is false.

Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = f(x) - x^{2021}$ for every $x \in \mathbb{R}$. This function is continuous, being the difference of two continuous functions on $\mathbb{R}$, so it satisfies Darboux's property. On the other hand, we have $\lim\limits_{x\to-\infty} g(x) = l_1 + \infty = +\infty$ and $\lim\limits_{x\to+\infty} g(x) = l_2 - \infty = -\infty$. It follows that $\operatorname{Im} g = \mathbb{R}$ and, thus, for $y = 0$, there exists $x \in \mathbb{R}$ such that $g(x) = y = 0$, i.e., $f(x) = x^{2021}$. So $\boxed{\text{D}}$ is true.

28. Denote by I the value of the integral $\displaystyle\int_0^1 x \ln(1 + x)\, \mathrm{d}x$. Then

$\boxed{\text{A}}$ $I = \frac{1}{4} + \ln 2$; $\boxed{\text{B}}$ $I \in \mathbb{Q}$;

$\boxed{\text{C}}$ $0 < I < \ln 2$; $\boxed{\text{D}}$ $\frac{1}{2} \ln \frac{3}{2} < I < \frac{1}{2} \ln 2$.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **true**; $\boxed{\textbf{C}}$ **true**; $\boxed{\textbf{D}}$ **true**.

Solution: We have

$$\begin{aligned}
I &= \int_0^1 \left(\frac{x^2}{2}\right)' \ln(1 + x)\, \mathrm{d}x = \frac{x^2}{2} \ln(1 + x)\Big|_0^1 - \frac{1}{2} \int_0^1 \frac{x^2}{1 + x}\, \mathrm{d}x \\
&= \frac{1}{2} \ln 2 - \frac{1}{2} \int_0^1 \left(x - 1 + \frac{1}{x + 1}\right) \mathrm{d}x \\
&= \frac{1}{2} \ln 2 - \frac{1}{2} \left(\frac{x^2}{2} - x + \ln(x + 1)\right)\Big|_0^1 = \frac{1}{4}\,.
\end{aligned}$$

The function $f:[0,1]\to\mathbb{R}$ defined by $f(x):=x\ln(1+x), \forall x\in[0,1]$ is strictly increasing on $[0,1]$, so $0<f(x)<\ln 2$ for all $x\in(0,1)$, hence, $0<I<\ln 2$. We have

$$\begin{aligned}
&\frac{1}{2}\ln\frac{3}{2}<I<\frac{1}{2}\ln 2 \quad\Leftrightarrow\quad \frac{1}{2}\ln\frac{3}{2}<\frac{1}{4}<\frac{1}{2}\ln 2\\
\Leftrightarrow\quad &\ln\frac{3}{2}<\frac{1}{2}=\ln\sqrt{e}<\ln 2 \quad\Leftrightarrow\quad 1.5^2=2.25<e<4=2^2,
\end{aligned}$$

which is true.

29. Denote by A the set of real numbers a for which equation $\sqrt{3-x}-x=a$ has at least one real solution.

$\boxed{\text{A}}$ $(-\infty,-10)\subset A$; $\boxed{\text{B}}$ $\{-10,-9,-8\}\subset A$;
$\boxed{\text{C}}$ $\{-3,-2,-1,0\}\subset A$; $\boxed{\text{D}}$ $\{8,9,10\}\subset A$.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **false**; $\boxed{\textbf{C}}$ **true**; $\boxed{\textbf{D}}$ **true**.

Solution: Define the function $f:(-\infty,3]\to\mathbb{R}$ by $f(x)=\sqrt{3-x}-x, \forall x\le 3$. We have $f'(x)<0$, $\forall x\in(-\infty,3)$. So f is strictly decreasing and hence, injective. $f(-3)=3$ and $\lim\limits_{x\to-\infty}f(x)=+\infty$, so the image of f is $[-3,\infty)$. Thus, equation $f(x)=a$ has at least one solution (in fact, has exactly one solution) if and only if $a\in[-3,\infty)$.

30. Let

$$I=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sin x-x\cos x}{x^2+\sin^2 x}\mathrm{d}x.$$

The value of I is

$\boxed{\text{A}}$ 0; $\boxed{\text{B}}$ $\operatorname{arctg}\left(\frac{2\pi\sqrt{3}}{9}\right)-\operatorname{arctg}\left(\frac{\pi\sqrt{2}}{4}\right)$;
$\boxed{\text{C}}$ $\operatorname{arctg}\left(\frac{2\pi\sqrt{3}}{9}\right)-1$; $\boxed{\text{D}}$ 1.

Answer:

$\boxed{\textbf{A}}$ **false**; $\boxed{\textbf{B}}$ **true**; $\boxed{\textbf{C}}$ **false**; $\boxed{\textbf{D}}$ **false**.

Solution:

$$\begin{aligned}
I&=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sin x-x\cos x}{x^2+\sin^2 x}\mathrm{d}x=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\frac{\sin x-x\cos x}{\sin^2 x}}{\frac{x^2+\sin^2 x}{\sin^2 x}}\mathrm{d}x=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\left(\frac{x}{\sin x}\right)'}{1+\left(\frac{x}{\sin x}\right)^2}\mathrm{d}x\\
&=\operatorname{arctg}\left(\frac{x}{\sin x}\right)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}=\operatorname{arctg}\left(\frac{\pi}{3}\cdot\frac{2}{\sqrt{3}}\right)-\operatorname{arctg}\left(\frac{\pi}{4}\cdot\frac{2}{\sqrt{2}}\right)\\
&=\operatorname{arctg}\left(\frac{2\pi\sqrt{3}}{9}\right)-\operatorname{arctg}\left(\frac{\pi\sqrt{2}}{4}\right)
\end{aligned}$$