## MATE-INFO UBB COMPETITION and ADMISSIONS EXAM - MODEL 2021 Written test in MATHEMATICS

IMPORTANT NOTE: Problems can have one or more correct answers. Answers should be indicated on the special form on the exam sheet. The partial grading system of the multiple choice exam can be found in the set of rules of the competition.

1. In $\mathbb{R}$ consider the equation

$$
2^{x^{2}+x+\frac{1}{2}}-4 \sqrt{2}=0 .
$$

The set of solutions of the equation is:
A $S=\{1\}$;
B $S=\{1,2\}$;
C $S=\{2\}$;
D $S=\{-2,1\}$.
2. For the matrix equation

$$
\left(\begin{array}{ll}
1 & a \\
a & 1
\end{array}\right) X=\left(\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right)
$$

to have at least one nonzero solution $X \in \mathcal{M}_{2}(\mathbb{R})$, it is neccesary and sufficient that:
A $a \in \mathbb{R}^{*}$;
B $a=0$;
C $a \in\{-3,2\}$;
D $a \in\{-1,1\}$.
3. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x)=x^{2}+x+1, \forall x \in \mathbb{R}$. Which of the following statements is true?

A For any $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $f(x)=y$.
B For any $y \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $f(x)=y$.
C If $x_{1}, x_{2} \in \mathbb{R}$ and $x_{1} \neq x_{2}$, then $f\left(x_{1}\right) \neq f\left(x_{2}\right)$.
D If $x_{1}, x_{2} \in \mathbb{R}$ and $x_{1}=x_{2}$, then $f\left(x_{1}\right)=f\left(x_{2}\right)$.
4. If $\alpha \in \mathbb{C}$ and $\alpha^{3}=1$, then the rank of the matrix $\left(\begin{array}{ccc}1 & \alpha & \alpha^{2} \\ \alpha & \alpha^{2} & 1 \\ \alpha^{2} & 1 & \alpha\end{array}\right)$ is:

$$
\begin{array}{|ll}
\hline \text { A } 1, \text { because } \alpha^{3}=1 \Rightarrow \alpha=1 . & \text { B } 2 \text {, because the determinant of the matrix is } 0 . \\
\hline \hline \text { C } 1 . & \text { D } 1 \text { or } 2 \text {, depending on } \alpha .
\end{array}
$$

5. Let $\left(a_{n}\right)_{n \geq 1}$ be a sequence of real numbers in arithmetic progression. If $a_{101}=695$ and $a_{1001}=6995$, determine which of the following statements is true.

$$
\mathrm{A} a_{1} \in[-6,6] ; \quad \mathrm{B} a_{1}=5 ; \quad \mathrm{C} a_{2021}=14135 ; \quad \mathrm{D} \sum_{k=11}^{20} a_{k}=965 .
$$

6. Let $f_{m}: \mathbb{R} \rightarrow \mathbb{R}, f_{m}(x)=m x^{2}+2(m+1) x+m-2, \forall x \in \mathbb{R}$, where $m \in \mathbb{R} \backslash\{0\}$. The vertices of all the parabolas associated with these functions belong to:

$$
\begin{array}{|l|l|}
\hline \text { A the line } y=-x+3 ; & \text { B the parabola } y=x^{2}-3 ; \\
\hline \mathrm{C} & \text { the line } y=x-3 ;
\end{array}
$$

7. Consider the polynomial $P=X^{4}+a X^{3}-6 X^{2}+15 X+b \in \mathbb{R}[X]$. If $P$ is divisible by $Q_{1}=X-1$ and $Q_{2}=X+3$, then:
$\begin{array}{rll}\mathrm{A} & a=-3 ; \\ \mathrm{y} & a+b=-10 ;\end{array}$
$\mathrm{B} \quad b=-9 ;$
$\mathrm{D} \quad$
$a=1$ and $b=9$.
8. Let the functions $f: \mathbb{R} \rightarrow(0, \infty)$ and $g:(0, \infty) \rightarrow \mathbb{R}$ be defined by $f(x)=\left(2 a^{2}+2 a-1\right)^{x}, \forall x \in \mathbb{R}$ and $g(y)=\log _{a^{2}+2} y, \forall y \in(0, \infty)$. Let $A$ be the set of values $a \in \mathbb{R}$ for which $f$ and $g$ are inverses of each other. Which of the following statements is true?

$$
\begin{array}{|l|l}
\mathrm{A} & A \subseteq[-1,4] ;
\end{array} \quad \mathrm{B} A \subseteq[-4,1] ;
$$

9. On the set $\mathbb{R}$ of all real numbers, consider the operation ,,*" defined by

$$
x * y=x y-4 x-4 y+20 .
$$

Which of the following statements is true?

$$
\begin{array}{|l|l}
\hline \mathrm{A} & (1 * 1) * 1=3 ; \\
\hline \hline \mathrm{C} \text { the inverse of } 3 \text { is } 3 ; & \mathrm{B} 4 \text { is the identity element with respect to },, * " ; \\
\mathrm{D} & (\mathbb{R}, *) \text { is a group. }
\end{array}
$$

10. Let $A=\{0,1,2,3,4,5,6\}$ and $B$ be the set of three-digit numbers formed with distinct numbers from $A$. Which of the following statements is true?

A $B$ has 240 elements; $\quad \mathrm{B} B$ has 210 elements; $\mathrm{C} B$ has 180 elements;
D exactly 35 of the numbers in $B$ consist of digits written in decreasing order.
11. If the vertices of a triangle $A B C$ have coordinates $A(2,3), B(-1,1), C(-3,4)$, then

| A The aria of $A B C$ is $\frac{13}{2}$. | B $A B C$ is a right triangle. |
| :--- | :--- | :--- |
| C $A B C$ is an isosceles triangle. | D The point $C$ belongs to the line $A B$. |

12. In the $x O y$ coordinate system consider the points $A(-2,3)$ and $B(0,1)$. The distance from the point $M(1,5)$ to the perpendicular bisector of the line segment $[A B]$ is
A $\frac{1}{\sqrt{2}} ;$
B $-\frac{1}{\sqrt{2}}$;
(C $\frac{3}{\sqrt{2}} ;$
D another answer.
13. Consider the triangle $A B C$. Its vertices have coordinates $A(1,1), B(9,1), C(1,5)$, with respect to a Cartesian orthonormal coordinate system in the plane of the triangle.

A $A B C$ is a right triangle and $m(\widehat{A})=90^{\circ}$;
B $H=A(1,1)$ is the orthocenter, $G\left(\frac{11}{3}, \frac{7}{3}\right)$ is the centroid and $O(5,3)$ is the circumcenter of the triangle $A B C$;

C The points $G, H, O$ are not colinear.
D The centroid $G$ is equally distanced from $[A B]$ and $[A C]$.
14. In the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ the equation $4 \cdot|\sin (x)| \cdot \cos (x)=1$

A has no solution; B has two solutions; C has four solutions; D has infinitely many solutions.
15. If $a, b, c$ are three line segments of lengths 2,3 and 4 , respectively, then:

| A | $a, b, c$ can form an acute triangle; | B $a, b, c$ can form an obtuse triangle; |
| :--- | :--- | :--- |
| C | $a, b, c$ can form an equilateral triangle; | $\mathrm{D} a, b, c$ can form a triangle. |

16. Consider the vectors $\vec{u}=(m-1) \vec{a}+2 \vec{b}$ and $\vec{v}=3 \vec{a}+m \vec{b}$, where $\vec{a}$ and $\vec{b}$ are not colinear. How many values can the parameter $m \in \mathbb{R}$ take, so that $\vec{u}$ and $\vec{v}$ are colinear?
A 0 ;
B 1;
C 2;
D 3 .
17. Let $A B C$ be a right triangle in $A$. If $A B=2 c, B C=2 a, A C=2 b, R$ and $r$ are the radii of the inscribed and circumscribed circle, respectively, then
A $\mathcal{A r e a}(\triangle A B C)=2 b c$
B $R=\frac{a}{2}$
C $A B+A C=2(R+r)$
D $r=\frac{2 b c}{a+b+c}$.
18. If $A, B, C, M$ are distinct points in a plane such that

$$
6 \overrightarrow{A M}=3 \overrightarrow{A B}+3 \overrightarrow{A C}-5 \overrightarrow{B C},
$$

then:

$$
\begin{array}{|l|l}
\hline \mathrm{A} & B, C, M \text { are colinear; } \\
\hline \mathrm{C} \overrightarrow{B M} \cdot \overrightarrow{B C}<0 ; & \mathrm{B} B, C, M \text { are not colinear; } \\
\hline \mathrm{D} \overrightarrow{B M} \cdot \overrightarrow{B C}>0 .
\end{array}
$$

19. Consider the points $A(1,-1), B(3,-1), A^{\prime}(-4,-2)$ and $B^{\prime}(0,-2)$. The points $C$ and $C^{\prime}$ belong to the parabola $\mathcal{P}$ with equation $y=x^{2}$. Denote by $\alpha$ the area of the triangle $A B C$ and by $\alpha^{\prime}$ the area of the triangle $A^{\prime} B^{\prime} C^{\prime}$.

A There exists a point $P$ on the parabola $\mathcal{P}$ such that $C=C^{\prime}=P$ and $\alpha=\alpha^{\prime}$.
B There exists a unique point $C$ such that $A B C$ is a right triangle.
C There exists at least one point $C$ with integer coordinates such that $\alpha$ is a prime number.
$\overline{\mathrm{D}}$ There exists at least one point $C^{\prime}$ such that the line segment $A^{\prime} C^{\prime}$ has length $3 \sqrt{2}$.
20. Consider the line with equation $d: x-y=1$ and the points $A(-1,0)$ and $B(1,2)$. For any $M \in d$, denote by $s_{M}$ the sum of the lengths of the line segments $[A M]$ and $[B M]$. Then:

$$
\begin{array}{ll}
\hline \text { A } \forall M \in d: s_{M} \geq \sqrt{2}+2 ; & \\
\hline \hline \text { C } \forall M \in d: s_{M} \geq \sqrt{2}+\sqrt{10} ; & \forall M \in d: s_{M} \leq \sqrt{2}+4 ; \\
\hline \text { D } \exists M \in d \text { such that } s_{M}=\sqrt{2}+2 .
\end{array}
$$

21. The limit of the sequence $a_{n}=n(\sqrt[4]{n+1}-\sqrt[4]{n}), \forall n \geq 1$ is
A 4;
(B) $\frac{1}{4}$;
C 0 ;
D $+\infty$.
22. For $n \in \mathbb{N}^{*}$, let $a_{n}=\frac{n!}{n^{n}}$. Which of the following statements is true?
A The sequence $\left(a_{n}\right)_{n \in \mathbb{N}^{*}}$ is strictly increasing.
(B) $\lim _{n \rightarrow \infty} a_{n}=1$.
C $a_{2021} \leq \frac{1}{2021}$.
D $\lim _{n \rightarrow \infty} a_{n}=0$.
23. Denote by $I=\int_{\pi}^{2 \pi} \frac{\cos ^{2} \frac{x}{2}}{x+\sin x} d x$. The value of $I$ is
A $\frac{\ln 2}{2}$;
B $\frac{1}{2}$;
C $\ln 2$;
D $\frac{\ln 3}{3}$.
24. Denote by $A$ the set of real numbers $a$ for which the function $f:[0,1] \rightarrow \mathbb{R}, f(x)=x^{2}(x+a), \forall x \in \mathbb{R}$ has exactly two extrema points.

$$
\mathrm{A} A=\left(-\infty,-\frac{3}{2}\right] \cup[0,+\infty) ; \quad \mathrm{B} A=\left(-\frac{3}{2}, 0\right) ; \quad \mathrm{C} A=\emptyset ; \quad \mathrm{D} A=\mathbb{R}^{*} .
$$

25. Let $f:\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow \mathbb{R}$ be defined by

$$
f(x)=\left\{\begin{array}{l}
\cos x-2,-\frac{\pi}{2} \leq x \leq 0 \\
\frac{\sin x}{x}, 0<x \leq \frac{\pi}{2} .
\end{array}\right.
$$

Then
A function $|f|$ is continuous in 0 ;
B function $f$ has at least one zero in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, because $f\left(-\frac{\pi}{2}\right) \cdot f\left(\frac{\pi}{2}\right)<0$;
C function $f$ has no zeros in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$;
D function $f$ has no limit in 0 .
26. Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x)=2^{x^{2}}, \forall x \in \mathbb{R}$. Then:

A $\lim _{x \rightarrow \infty} f(-x)=0$;
B function $f$ is strictly increasing on the interval $[1, \infty)$;
C the inequality $f^{\prime}(x) \geq x f(x)$ holds for every $x \in \mathbb{R}$;
D $\lim _{x \rightarrow \infty} \frac{f^{\prime}(x)}{(2 x+1) f(x)}=1$.
27. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function whose graph has horizontal asymptotes at $-\infty$ and $+\infty$. Which of the following statements is true?

A $\lim _{x \rightarrow-\infty} \frac{f(x)}{x}=\lim _{x \rightarrow+\infty} \frac{f(x)}{x}$;
B function $f$ is bounded;
C the graph of the function $f$ intersects any horizontal line in at least one point;
D equation $f(x)=x^{2021}$ has at least one solution in $\mathbb{R}$.
28. Denote by $I$ the value of the integral $\int_{0}^{1} x \ln (1+x) \mathrm{d} x$. Then

$$
\begin{array}{ll}
\text { A } I=\frac{1}{4}+\ln 2 ; & \text { B } I \in \mathbb{Q} ; \\
\text { C } 0<I<\ln 2 ; & \text { D } \frac{1}{2} \ln \frac{3}{2}<I<\frac{1}{2} \ln 2 .
\end{array}
$$

29. Denote by $A$ the set of real numbers $a$ for which equation $\sqrt{3-x}-x=a$ has at least one real solution.
$\mathrm{A}(-\infty,-10) \subset A$;
С $\{-3,-2,-1,0\} \subset A$;
B
D D $\{-10,-9,-8\} \subset A ;$
D
30. Let

$$
I=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin x-x \cos x}{x^{2}+\sin ^{2} x} \mathrm{~d} x
$$

The value of $I$ is

$$
\begin{array}{ll}
\mathrm{A} 0 ; & \mathrm{B} \operatorname{arctg}\left(\frac{2 \pi \sqrt{3}}{9}\right)-\operatorname{arctg}\left(\frac{\pi \sqrt{2}}{4}\right) ; \\
\mathrm{C} \operatorname{arctg}\left(\frac{2 \pi \sqrt{3}}{9}\right)-1 ; & \mathrm{D} 1 .
\end{array}
$$

