Metric Regularity - Directional Metric Regularity - Relative Metric Regularity of Multifunctions

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Outline

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Metric Regularity

Consider an equation of the form

$$F(x) = y, \tag{1}$$

where $F : X \to Y$ is a function, X, Y are metric spaces.

The distance $d(y, F(x))$ is used to judge approximate solutions. The error of some approximate solution x is

$$d(x, F^{-1}(y)) = \inf\{d(x, u) : \ F(u) = y\}.$$

One seeks so an error bound of the form

$$d(x, F^{-1}(y)) \leq Kd(y, F(x)) \tag{2}$$

for all (x, y) globally, or locally, that is, (x, y) near a given $(\bar{x}, \bar{y})$ with $\bar{y} = F(\bar{x})$, and F is said to metrically regular at $\bar{x}$. The infimum of such K is the regular modulus: $\operatorname{reg} F(\bar{x})$.

Banach-Schauder open mapping theorem

X, Y : Banach spaces; $A \in \mathcal{L}(X, Y)$

If $\mathrm{Im}\ A = Y$ then A is open: $\exists r > 0$ such that

$$rB_Y \subseteq A(B_X).$$

The upper bound of such r is the Banach constant of A :

$$C(A) = \inf\{\|A^* y^*\| : \quad \|y^*\| = 1\}.$$

Moreover,

$$d(x, A^{-1}(y)) \leq C(A)^{-1} \|Ax - y\| \quad \text{for all} \ \ (x, y) \in X \times Y.$$

Lusternik-Graves theorem

X, Y : Banach spaces; $F : X \to Y$ continuously differentiable at $\bar{x}$; $F(\bar{x}) := \bar{y}$.

When $\mathrm{Im}\ F'(\bar{x}) = Y$ *then* $\exists r > 0$, $\exists \varepsilon > 0$:

$$B(\bar{y}, rt) \subseteq F(B(\bar{x}, t)) \ \ \forall t \in (0, \varepsilon).$$

The upper bound of such r is $C(F'(\bar{x}))$, *is the Banach constant of* $F'(\bar{x})$. *Moreover,*

$$d(x, F^{-1}(y)) \leq r^{-1} d(y, F(x)) \quad \textit{for all } (x, y) \textit{ near } (\bar{x}, \bar{y}).$$

Robinson and Mangasarian-Fromovitz constraint qualifications

$F := g - C$, $g : X \to Y$ is $\mathcal{C}^1$; $C \subseteq Y$ is a nonempty closed convex subset. Given $(\bar{x}, 0) \in \operatorname{gph} F$,

- F is metrically regular at $(\bar{x}, 0)$ $\iff$ Robinson constraint qualification (RCQ):

$$0 \in \operatorname{int}[g(\bar{x}) + \nabla g(\bar{x})X - C].$$

- System of equality and inequality: $(\mathrm{RCQ}) \Leftrightarrow (MFCQ)$ (Mangasarian-Fromovitz constraint qualification)

When Y is a m-dimensional space, we often deal with a system of inequalities:

$$F_i(x) \leq y_i, i = 1, ..., m. \tag{3}$$

Such inequalities systems are used in optimization for problems with inequalities constraints. This system of inequalities can be studied via the generalized equation : $y \in F(x)$, where,

$$F(x) := (F_i(x))_{i=1,\ldots,m} + \mathbb{R}^m_+; \quad y = (y_i)_{i=1,\ldots,m}, \tag{4}$$

then $F : X \rightrightarrows \mathbb{R}^m$ is a multifunction.

A multifunction (set-valued mapping) is regular at $(\bar{x}, \bar{y})$ $(\bar{y} \in F(\bar{x}))$ if

$$d(x, F^{-1}(y)) \leq Kd(y, F(x)) \quad \text{for all } (x, y) \text{ near } (\bar{x}, \bar{y}).$$

Definitions of regularity

- **metric regularity**: $\exists K > 0, \varepsilon > 0$ s.t.

$$d(x, F^{-1}(y)) \leq Kd(y, F(x)), \quad \forall (x, y) \in B((\bar{x}, \bar{y}), \varepsilon),$$

$\operatorname{reg} F(\bar{x}, \bar{y}) :=$ infimum of such K : *the rate of metric regularity.*

- **openness at a linear rate**: $\exists r, \varepsilon > 0$ s.t.

$$B(y, tr) \subseteq F(B(x, t)), \quad \forall (x, y) \in B((\bar{x}, \bar{y}), \varepsilon) \cap \operatorname{gph} F,$$

$\operatorname{sur} F(\bar{x}, \bar{y}) :=$ supremum of such r : *rate of openness (or surjection).*

- **Lipschitz-like (or Aubin) property**: $\exists K, \varepsilon > 0$ s.t.

$$d(y, F(x)) \leq Kd(x, u), \quad \forall x \in B(\bar{x}, \varepsilon), \ (u, y) \in B((\bar{x}, \bar{y}), \varepsilon) \cap \operatorname{gph} F,$$

$\operatorname{lip} F(\bar{x}, \bar{y}) :=$ supremum of such K : *Lipschitz rate.*

Equivalence.

Under the convention $1/\infty = 0$, *one has*

$$\operatorname{reg} F(\bar{x}, \bar{y}) = \operatorname{lip} F^{-1}(\bar{y}, \bar{x}) = \frac{1}{\operatorname{sur} F(\bar{x}, \bar{y})}.$$

(cf. Borwein & Zuang 1988, Kruger 1988, Penot 1989, also Ioffe 1981)

- F is said to be *regular* at $(\bar{x}, \bar{y})$ if (one of) the three properties hold.

Characterization of regularity

- Set
$$\varphi_y(x) = \varphi(x, y) = \liminf_{u \to x} d(y, F(u)),$$
the *lower semicontinuous envelope* of the distance function $d(y, F(\cdot))$.

- **Characterization of regularity.** (Ngai-Théra, SIOPT 2008) *Suppose that* $\mathrm{gph}\ F$ *is closed. Then* F *is regular at* $(\bar{x}, \bar{y}) \in \mathrm{gph}\ F$ *with* $\mathrm{sur}\, F(\bar{x}, \bar{y}) > r > 0$ *iff for any* (x, y) *in a neighborhood of* $(\bar{x}, \bar{y})$ *with* $y \notin F(x)$*, we can find* $u \in X$ *s.t.*
$$rd(u, x) < \varphi(x, y) - \varphi(u, y).$$

Infinitesimal characterization: slopes

Given $a \in \mathbb{R}$, we set $a_+ = \max\{a, 0\}$. Recall that for an extended real-valued function $f : X \to \mathbb{R} \cup \{+\infty\}$ and a point $x \in X$ with $f(x) < +\infty$, the local and the global strong slope $|\nabla f|(x)$ and $|\Gamma f|(x)$ of f at x are defined by

$$|\nabla f|(x) = \limsup_{x \neq y \to x} \frac{[f(x) - f(y)]_+}{d(x, y)} \quad \text{and} \quad |\Gamma f|(x) = \sup_{y \neq x} \frac{[f(x) - f(y)]}{d(x, y)}.$$

If $f(x) = +\infty$, then we set $|\nabla f|(x) = |\Gamma f|(x) = +\infty$.

- **Example.** X, Y are normed spaces, $f \in C^1 : |\nabla f|(x) = \|f'(x)\|$.

Let P denote a topological space.

Theorem (Theorem 2, Corollary 1 - NTT)

Let $f : X \times P \to [0, +\infty]$ be a function. For each $p \in P$, set

$$S(p) = \{x \in X : \quad f(x, p) = 0\}.$$

Suppose that $(\bar{x}, \bar{p}) \in X \times P$ is such that $\bar{x} \in S(\bar{p})$, and that, for any p near $\bar{p}$, the function $f(\cdot, p)$ is lower semicontinuous at $\bar{x}$, and $f(\bar{x}, \cdot)$ is continuous at $\bar{p}$. Let $\tau > 0$ be given and consider the following statements:

1. *(i) There exist $\gamma > 0$ and a neighborhood $\mathcal{V} \times \mathcal{W}$ of $(\bar{x}, \bar{p})$ in $X \times P$ such that for any $p \in \mathcal{W}$, we have $\mathcal{V} \cap S(p) \neq \emptyset$ and*

$$d(x, S(p)) \leq \tau f(x, p) \quad \text{for all} \quad (x, p) \in \mathcal{V} \times \mathcal{W} \quad \text{with } f(x, p) \in (0, \gamma); \tag{5}$$

2. *(ii) There exist a neighborhood $\mathcal{V} \times \mathcal{W}$ of $(\bar{x}, \bar{p})$ in $X \times P$ and $\gamma > 0$ such that for each $(x, p) \in \mathcal{V} \times \mathcal{W}$ with $f(x, p) \in (0, \gamma)$ and for any $\varepsilon > 0$, there exists $z \in X$ such that*

$$0 < d(x, z) < (\tau + \varepsilon)(f(x, p) - f(z, p)); \tag{6}$$

3. *(iii) There exists a neighborhood $\mathcal{V} \times \mathcal{W}$ of $(\bar{x}, \bar{p})$ in $X \times P$ along with positive reals γ and τ such that $|\nabla f(\cdot, p)|(x) \geq 1/\tau$ for all $(x, p) \in \mathcal{V} \times \mathcal{W}$ with $f(x, p) \in (0, \gamma)$.*

Then (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iii).

Subdifferentials

$f : X \to \mathbb{R} \cup \{+\infty\}$ extended real-valued function defined on a Banach space X.

- *Fréchet subdifferential* of f at $\bar{x} \in \mathrm{Dom}\, f$ is given as

$$\partial_{\mathcal{F}} f(\bar{x}) = \left\{ x^* \in X^* : \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0 \right\}.$$

- *Limiting subdifferential* (also known as the Mordukhovich subdifferential) is defined as

$$\partial_{\mathcal{M}} f(\bar{x}) = \left\{ x^* \in X^* : \ \exists x_k \to \bar{x}, \ f(x_k) \to f(\bar{x}), \ \text{ and } \exists x_k^* \in \partial f(x_k), \ x_k^* \xrightarrow{*} x^* \right\}.$$

The graph of the limiting Fréchet subdifferential is the sequential closure of the graph of the Fréchet subdifferential in the product of the norm topology on X with the weak*- topology on X^*.

Abstract Subdifferentials

Let X be a Banach space. We use the symbol ∂ to denote any abstract subdifferential, that is a correspondance $(X, f, x) \to \partial f(x) \subset X^\star$ in such a way that

- If f attains a local minimum at x then $0 \in \partial f(x)$ (Fermat rule);
- If $f : X \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous convex function, then ∂f coincides with the subdifferential in the sense of convex analysis:

$$\partial f(x) := \{x^* \in X^* : \langle x^*, y - x \rangle \leq f(y) - f(x) \quad \forall y \in X\};$$

- $\partial f(x) = \partial g(x)$ if $f(y) = g(y)$ for all y in a neighborhood of x.
- *Approximate Fermat rule for sums:* Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and $g : X \to \mathbb{R}$ be convex and Lipschitz. If $f + g$ attains a *local minimum* at x_0, then for any $\varepsilon > 0$, there are $x_1, x_2 \in x_0 + \varepsilon B_X$, $x_1^* \in \partial f(x_i)$, $x_2^* \in \partial g(x_2)$, such that

$$|f(x_i) - f(x_0)| < \varepsilon,\ i = 1, 2,\ \|x_1^* + x_2^*\| < \varepsilon..$$

Abstract subdifferentials-examples

Examples. Abstract subdifferential includes Fréchet subdifferentials in Asplund spaces, viscosity subdifferentials in smooth Banach spaces, Dini-Hadamard subdifferential on Gateaux smooth spaces as well as the Ioffe and the Clarke-Rockafellar subdifferentials in Banach spaces.

Asplund spaces are those Banach spaces such that every convex continuous function is generically Fréchet differentiable. Any space with Fréchet smooth renorming and hence any reflexive Banach space) is Asplund,

Normal cones-Coderatives

Normal cone to a closed subset C of X, with respect to a subdifferential operator ∂ at $x \in C$

$$N_{\partial}(C, x) = \partial\delta_C(x).$$

We assume here that $\partial\delta_C(x)$ is *a cone* for any closed subset C of X.

Coderivative. Let X, Y be Banach spaces, and ∂ be a subdifferential on $X \times Y$. Let $F : X \rightrightarrows Y$ be a closed multifunction (graph-closed) and let $(\bar{x}, \bar{y}) \in \mathrm{gph} F$.

The coderivative of F at $(\bar{x}, \bar{y})$ is the multifunction $D^*F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$ defined by

$$D^*F(\bar{x}, \bar{y})(y^*) = \{x^* \in X^* : (x^*, -y^*) \in N_{\partial}(\mathrm{gph} F, (\bar{x}, \bar{y}))\}.$$

Limiting normal cone - Coderivatives

Given a normal cone $\mathbb{N}$, we can associate with a set-valued mapping $F : X \rightrightarrows Y$ a coderivative $D^*_{\mathbb{N}} : Y^\star \rightrightarrows X^*$.

- When $\mathbb{N}$ is the Fréchet (regular) normal cone, the coderivative of F is denoted by $D^*_{\mathcal{F}}F$;
- When $\mathbb{N}$ is the limiting normal cone, we use the notation by $D^*_{\mathcal{M}}F$;
- When $\mathbb{N}$ s the normal cone to a convex set C, all the coderivatives coincide and are simply denoted by D^*.

Examples

- $A \in \mathcal{L}(X, Y) : D^*A(y^*) = A^*y^*.$
- $F : X \to Y$ is C^1 : $D^*F(x)(y^*) = (F'(x))^*y^* = y^* \circ F'(x).$

Estimation for slopes

$\varphi_y(x) = \liminf_{u \to x} d(y, F(u))$ • F:X$\rightrightarrows$ Y • $(\bar{x}, \bar{y}) \in \mathrm{gph}\ F$

Theorem (Ngai-Tron-Théra, Math. Programming, 2011)

For any subdifferential on $X \times Y$, one has

$$\liminf_{(x,y) \to (\bar{x},\bar{y}), y \notin F(x)} |\nabla \varphi_y|(x) \geq$$
$$\lim_{\varepsilon \to 0} \inf\{\|x^*\| : \ x^* \in D^*F(u,v)(y^*), \ (u,v) \in B((\bar{x},\bar{y}),\varepsilon), \ \|y^*\| = 1\}.$$

Equality holds when X, Y are Asplund spaces and ∂ is the Fréchet subdifferential.

Subdifferential regularity criterion

As a result, (Ioffe 1987)

$$\mathrm{sur}\, F(\bar{x}, \bar{y}) \geq$$
$$\geq \lim_{\varepsilon \to 0} \inf\{\|x^*\| : \; x^* \in D^* F(u, v)(y^*), \; (u, v) \in B((\bar{x}, \bar{y}), \varepsilon), \; \|y^*\| = 1\};$$

Equality holds when X, Y are Fréchet smooth or more generally, Asplund spaces and ∂ is the Fréchet subdifferential (Kruger 1988, Mordukhovich-Shao 1997).

Relative Metric Regularity

Given a subset V of $X \times Y$ and a point $(x, y) \in X \times Y$, we set

$$V_x := \{z \in Y : \quad (x, z) \in V\} \quad \text{and} \quad V_y := \{u \in X : \quad (u, y) \in V\}.$$

Definition (Ioffe)

Let X, Y be metric spaces, and let $V \subset X \times Y$. We say that a set-valued mapping $F : X \rightrightarrows Y$ is *metrically regular relatively to V at* $(\bar{x}, \bar{y}) \in V \cap \operatorname{gph}\ T$ *with a modulus* $\tau > 0$, if there exist $\varepsilon > 0$ such that

$$d\left(x, F^{-1}(y) \cap \operatorname{cl} V_y\right) \leq \tau d(y, F(x)) \tag{7}$$

whenever $(x, y) \in \big(B(\bar{x}, \varepsilon) \times B(\bar{y}, \varepsilon)\big) \cap V$ and $d(y, F(x)) < \varepsilon$. The infinum of $\tau > 0$ such that (7) holds for some $\varepsilon > 0$ is called the *exact modulus of the metric regularity relative to V for F at* $(\bar{x}, \bar{y})$ and is denoted by $\operatorname{reg}_V F(\bar{x}, \bar{y})$.

Subcase of this concept: directional metric regularity introduced by Arutuynov & Izmailov and extensively studied by Gfrerer, Ioffe and Ngai & Théra.

The idea behind relative metric regularity is that the values of ambient and image variables are not arbitrary points or neighborhood of a certain nominal point of the product space, but are taken from a certain set V.

Thus, by choosing among possible V, one may obtain various versions of metric regularity models existing in the literature and central for the analysis of sensitivity and controllability in optimization and control.

For instance when $V = X \times Y$, this model subsumes the usual local metric regularity.

Some other examples are described in the recent book by Ioffe.

Relative lower semicontinuous envelope

Given a subset V of $X \times Y$ and a point $(x, y) \in X \times Y$, a multifunction $F : X \rightrightarrows Y$, set

$$V_x := \{z \in Y : \ (x, z) \in V\} \quad \text{and} \quad V_y := \{u \in X : \ (u, y) \in V\}.$$

- *The lower semicontinuous envelope of $d(y, F(\cdot))$ relative to $V \subseteq X \times Y$:*

$$\varphi_{F,V}(x, y) := \begin{cases} \liminf\limits_{\mathrm{cl}\, V_y \ni u \to x} d(y, F(u)) & \text{if } x \in \mathrm{cl}\, V_y \\ +\infty & \text{otherwise.} \end{cases} \tag{8}$$

Lemma

Suppose that the multifunction $F : X \rightrightarrows Y$ has a closed graph. Then

(i)

$$F^{-1}(y) \cap \mathrm{cl}\, V_y = \{x \in X :\ \varphi_{F,V}(x,y) = 0\} \quad \textit{whenever} \quad y \in Y;$$

(ii) *F is metrically regular relative to V at (x_0, y_0) with a modulus $\tau > 0$, if and only if there exists $\varepsilon > 0$ such that*

$$d(x, F^{-1}(y) \cap \mathrm{cl}\, V_y) \leq \tau \varphi_{F,V}(x,y)$$

for all $(x,y) \in (\mathcal{B}(x_0,\varepsilon) \times \mathcal{B}(y_0,\varepsilon)) \cap V$ *with* $d(y, F(x)) < \varepsilon$.

Theorem

Let X and Y be metric spaces with X complete and $F : X \rightrightarrows Y$ be a set-valued mapping with closed graph. Let $(\bar{x}, \bar{y}) \in \operatorname{gph} F \cap V$, $V \subset X \times Y$; $\tau \in (0, +\infty)$ be given. Then, among the following statements,

1. *(i) F is metrically regular relative to V at $(\bar{x}, \bar{y})$ with modulus τ;*
2. *(ii) There exist $\delta, \gamma > 0$ such that*

$$|\Gamma\varphi_{F,V}(\cdot, y)|(x) \geq \tau^{-1} \quad \text{for all } (x, y) \in \mathcal{B}(\bar{x}, \delta) \times \mathcal{B}(\bar{y}, \delta) \text{ with } \varphi_{F,V}(x, y) \in (0, \gamma);$$

one has $(i) \Leftrightarrow (ii)$

Slope Characterization

Corollary

If there exist $\delta, \gamma > 0$ such that

$$|\nabla \varphi_{F,V}(\cdot, y)|(x) \geq \tau^{-1}, \tag{9}$$

for all $(x, y) \in \big(B(\bar{x}, \delta) \times B(\bar{y}, \delta)\big)$ with $\varphi_{F,V}(x, y) \in (0, \gamma)$, then F is metrically regular relative to V at $(\bar{x}, \bar{y})$.

Proof. This condition implies (2) from the preceding theorem.

Metric Regularity relative to a cone

$C \subseteq Y$: a nonempty cone;

$$C(\delta) := \{y \in Y : \quad d(y, C) \leq \delta \|y\|\}, \quad \delta > 0.$$

For a given multifunction $F : X \rightrightarrows Y$ from a complete metric space X to a normed linear space Y, a cone $C \subseteq Y$, and a positive real δ, we remind that

$$V_F(C, \delta) = \{(x, y) : \quad y \in F(x) + C(\delta)\}.$$

For $y \in Y$, set

$$V_{F,y}(C, \delta) := \{x \in X : \quad y \in F(x) + C(\delta)\}.$$

We note $\varphi_{V_F(C,\delta)}(x, y)$, the lower semicontinuous envelope relative to $V_F(C, \delta)$ of $d(y, F(\cdot))$.

Definition

F is metrically regular relative to C if F is metrically regular relatively to $V_F(C, \delta) = \{(x, y) : \quad y \in F(x) + C(\delta)\}$, for some $\delta > 0$.

Robustness

Theorem

Let X be a complete metric space and Y be a normed space. Let $C \subseteq Y$ be a nonempty cone in Y. Let $F : X \rightrightarrows Y$ be a closed multifunction and $(x_0, y_0) \in \operatorname{gph} F$. Suppose that F is metrically regular with a modulus $\tau > 0$ relative to C, i.e., there exist reals $\varepsilon > 0$ and $\delta > 0$ such that for all $(x, y) \in \mathcal{B}((x_0, y_0), \varepsilon) \cap V_F(C, \delta)$ with $d(y, F(x)) < \varepsilon$. we have:

$$d(x, F^{-1}(y) \cap \operatorname{cl} V_{F,y}(C, \delta)) \leq \tau d(y, F(x)). \tag{10}$$

Let $g : X \to Y$ be locally Lipschitz around x_0 with a Lipschitz constant $L > 0$. Then $F + g$ is metrically regular relative to C at $(x_0, y_0 + g(x_0))$ with modulus

$$\operatorname{reg}_C(F + g)(x_0, y_0 + g(x_0)) \leq \left(\frac{1 - \alpha}{\tau(1 + \alpha)} - L \right)^{-1},$$

provided

$$\alpha \in (0, 1), \text{ and } L < \frac{\delta(1 - \alpha)\alpha}{\tau(1 + \alpha)(1 + \delta(1 - \alpha))}.$$

Coderivative characterizations

Denote by S_{Y^*} the unit sphere in the continuous dual Y^* of Y, and by d_* the metric associated with the dual norm on X^*. For given $\bar{y} \in Y$ and $\delta > 0$, let us define the set

$$T(C,\delta) := \left\{ \begin{array}{ll} (y_1^*, y_2^*) & \in Y^* \times Y^* : \ \exists a \in C \cap S_{Y^*}, \\ & \max\{\langle y_1^*, a\rangle, |\langle y_2^*, a\rangle|\} \leq \delta, \ \|y_1^* + y_2^*\| = 1 \end{array} \right\}. \tag{11}$$

- To a given multifunction $F : X \rightrightarrows Y$, we associate the multifunction $G : X \rightrightarrows Y \times Y$ defined by

$$G(x) = F(x) \times F(x), \quad x \in X.$$

- $D^*_{\mathcal{F}} G$: *the coderivative of G with respect to the Fréchet subdifferential.*

Theorem

Let X, Y be Asplund spaces and let $F : X \rightrightarrows Y$ be a closed multifunction. Let $(x_0, y_0) \in \operatorname{gph} F$ and a nonempty cone $C \subseteq Y$ be given. Assume that F has convex values around x_0, i.e, $F(x)$ is convex for all x near x_0. If

$$\liminf_{\substack{(x,y_1,y_2)\overset{G}{\to}(x_0,y_0,y_0)\\ \delta\downarrow 0^+}} d_*(0, D^*_{\mathcal{F}}G(x, y_1, y_2)(T(C, \delta))) > m > 0, \qquad (12)$$

then F is metrically regular relative to C with modulus $\tau \leq m^{-1}$ at (x_0, y_0).

The notation $(x, y_1, y_2) \overset{G}{\to} (x_0, y_0, y_0)$ means that $(x, y_1, y_2) \to (x_0, y_0, y_0)$ with $(x, y_1, y_2) \in \operatorname{gph} G$.

The following proposition shows that condition (12) is also a necessary condition for metric regularity relative to a cone in Banach spaces when F is either a multifunction with a convex graph or $F : X \to Y$ is a continuous single-valued mapping.

Theorem (Necessary Condition)

Let X, Y be Banach spaces and $C \subseteq Y$ a nonempty cone. Suppose that $F : X \rightrightarrows Y$ is either a closed convex multifunction or $F : X \to Y$ is a continuous single-valued mapping. For a given $(x_0, y_0) \in \mathrm{gph}\; F$, if F is metrically regular relative to C at (x_0, y_0), then

$$\liminf_{\substack{(x,y_1,y_2)\xrightarrow{G}(x_0,y_0,y_0)\\ \delta\downarrow 0^+}} d_*(0, D^*_{\mathcal{F}}G(x, y_1, y_2)(T(C,\delta))) > 0.$$

Convex multifunctions

Corollary (Ioffe-08)

Let X, Y be Banach spaces and $F : X \rightrightarrows Y$ be a closed convex multifunction and let $(x_0, y_0) \in \operatorname{gph} F$ and $v \in Y$. F is directionally metrically regular in direction v at (x_0, y_0) if and only if

$$\operatorname{cone}\{v\} \cap \operatorname{int}(F(X) - y_0) \neq \emptyset. \tag{13}$$

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Thank you!

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