NO-REGRET CRITERIA IN LEARNING, GAMES AND CONVEX OPTIMIZATION

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Games, Dynamics and Optimization
Babes-Bolyai University, Cluj
April 9-11, 2019

Abstract

The purpose of this talk is to underline links between no-regret algorithms used in learning, games and convex optimization.

In particular we will study continuous and discrete time versions and their connections.

We will comment on recent advances on:

- Euclidean and non-euclidean approaches
- speed of convergence of the evaluation
- convergence of the trajectories

Model

V normed vector space, finite dimensional
dual V^* and duality map $\langle . | . \rangle$
$X \subset V$ compact convex
The aim is to study properties of algorithms that associate to a process of observations $\{u_t \in V^*, t \geq 0\}$, a process of choices $\{x_t \in X, t \geq 0\}$, where x_t is function of $\{(x_s, u_s), 0 \leq s < t\}$, satisfying:

$$R_t(y) = \int_0^t \langle u_s | y - x_s \rangle ds \leq o(t), \qquad t \geq 0, \forall y \in X \tag{1}$$

or in discrete time $\{x_m\}$ depending on $\{x_1, u_1, ..., x_{m-1}, u_{m-1}\}$ with:

$$R_n(y) = \sum_{m=1}^{n} \langle u_m | y - x_m \rangle \leq o(n), \qquad \forall y \in X. \tag{2}$$

This means that the average regret vanishes.

Basic properties

Case 1 : general bounded process $\{u_t\}$ or $\{u_n\}$
no-regret learning

Case 2 : vector field $g : X \to V^*$
$u_t = g(x_t)$ or $u_n = g(x_n)$
Variational inequalities or game framework
Consider a game with a finite set of players I where equilibria are solution of variational inequalities:

$$\langle g^i(x) | x^i - y^i \rangle \geq 0, \qquad \forall y^i \in X^i, \forall i \in I$$

$X^i \subset V^i$ is the strategy set of player i, $X = \prod_i X^i$, and $g^i : X \to V^{i*}$ is his evaluation function.
Examples include:
- finite games
- continuous games with payoff G^i $\mathscr{C}^1$ and concave wrt x^i, $\forall i \in I$
- population games

Basic properties

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- population games

At stage n each player i chooses x_n^i, this defines a profile $x_n \in X$ and the reference process for player i is $u_n^i = g^i(x_n)$. Let:

$$\langle g(x)|y-x\rangle = \sum_i \langle g^i(x)|y^i - x^i\rangle$$

S' is the set of solutions of the variational inequality:

$$\langle g(x)|y-x\rangle \leq 0, \qquad \forall y \in X \tag{3}$$

Lemma

If g is continuous and $x_s \to x$ then $x \in S'$.

S is the set of solutions of the variational inequality:

$$\langle g(y) | y - x \rangle \leq 0, \qquad \forall y \in X. \tag{4}$$

g *dissipative* ($\langle g(x) - g(y) | x - y \rangle \leq 0, \forall x, y \in X$) implies $S' \subset S$ and g continuous implies: $S \subset S'$.

Let $\bar{x}_t = \frac{1}{t} \int_0^t x_s ds$, and $\bar{x}_n = \frac{1}{n} \sum_1^n x_m$.

Lemma

If g is dissipative the accumulation points of $\{\bar{x}_t\}$ or $\{\bar{x}_n\}$ are in S.

Proof:

$$\frac{R_t(y)}{t} = \frac{1}{t} \int_0^t \langle g(x_s) | y - x_s \rangle \geq \frac{1}{t} \int_0^t \langle g(y) | y - x_s \rangle = \langle g(y) | y - \bar{x}_t \rangle$$

∎

Case 3 : $u_t = -\nabla f(x_t)$, f convex $\mathscr{C}^1$
convex optimization

$$\langle \nabla f(x_t) | y - x_t \rangle \leq f(y) - f(x_t)$$

gives:

$$\int_0^t [f(x_s) - f(y)]dt \leq \int_0^t \langle -\nabla f(x_s) | y - x_s \rangle ds = R_t(y)$$

which implies by Jensen's inequality:

$$f(\overline{x}_t) - f(y) \leq \frac{1}{t} \int_0^t [f(x_s) - f(y)]ds \leq \frac{R_t(y)}{t} \tag{5}$$

Lemma
The accumulation points of $\{\overline{x}_t\}$ or $\{\overline{x}_n\}$ belong to $S = argmin_X f$.

Continuous time

Potential function $P(t;y) \geq 0$ satisfying:

$$\langle u_t, y - x_t \rangle \leq -\frac{d}{dt} P(t;y), \qquad \text{hence}$$

$$R_t(y) = \int_0^t \langle u_s | y - x_s \rangle ds \leq P(0;y) - P(t;y)$$

(1) rate of convergence $1/t$.
(2) Assume $y^* \in S$, then $P(t;y^*)$ is decreasing:

$$\frac{d}{dt} P(t;y^*) \leq \langle g(x_t), x_t - y^* \rangle \leq 0$$

(3) If $\{x_t\}$ is a descent procedure ($\frac{d}{dt} f(x_t) \leq 0$),

$$E(t;y) = t(f(x_t) - f(y)) + P(t;y)$$

is decreasing, for all $y \in X$.

$$\begin{aligned} \frac{d}{dt} E(t;y) &= f(x_t) - f(y) + t \frac{d}{dt} f(x_t) + \frac{d}{dt} P(t;y) \\ &\leq f(x_t) - f(y) + \langle \nabla f(x_t), y - x_t \rangle \leq 0 \end{aligned}$$

Accumulation points of $\{x_t\}$ are in S.

A. Projected gradient

V Hilbert, $X \subset V$, convex closed.

Dynamics

(Projected) gradient descent is defined by:

$$\langle u_t - \dot{x}_t, y - x_t \rangle \leq 0, \forall y \in X. \tag{6}$$

which is:

$$\dot{x}_t = \Pi_{T_X(x_t)}(u_t) \tag{7}$$

where Π_C is the projection on the closed convex set C and $T_C(x)$ is the tangent cône to C at x.

Potential

Let:

$$V(t;y) = \frac{1}{2}\|x_t - y\|^2, \quad y \in X. \tag{8}$$

$$\langle u_t, y - x_t \rangle \leq \langle \dot{x}_t, y - x_t \rangle = -\frac{d}{dt}V(t;y)$$

Trajectories

Lemma

Assume $S \neq \emptyset$ and g dissipative.
$\{\bar{x}_t\}$ converges weakly to a point in S.

Proof:
- $\{\bar{x}_t\}$ is bounded hence has weak accumulation points.
- The weak limit points of $\{\bar{x}_t\}$ are in S
- $\|x_t - y^*\|$ converges when $y^* \in S$

Hence by Opial's lemma, $\bar{x}_t$ converges weakly to a point in S. ■

Descent properties

Consider case 3: $u_t = -\nabla f(x_t)$.

Lemma

$f(x_t)$ *is decreasing*

Proof:

$$\frac{d}{dt} f(x_t) = \langle \nabla f(x_t), \dot{x}_t \rangle = -\|\dot{x}_t\|^2$$

since $\langle u_t - \dot{x}_t, \dot{x}_t \rangle = 0$ (Moreau's decomposition). ■

Lemma

$\{x_t\}$ *weakly converges to a point in* S

Proof:

Weak accumulation points of $\{x_t\}$ are in S. Then Opial's lemma applies. ■

to summarize :
- R_t is bounded
- in addition in case 2, for g dissipative, $\{\bar{x}_t\}$ weakly converges to a point in S
- in case 3, $f(x_t)$ is decreasing thus $f(x_t)$ converges to $f^* = \min_X f$ at speed $1/t$ and $\{x_t\}$ weakly converges to a point in S.

B. Mirror descent

Continuous version of “Mirror descent algorithm”’
Nemirovski and Yudin [49], Beck and Teboulle [12]
Alvarez, Bolte and Brahic, Attouch and Teboulle, Bolte and Teboulle ...

Dynamics

H strictly convex, $\mathscr{C}^1$
X, compact, convex $\subset domH$.
The continuous time process satisfies:

$$\langle u_t - \frac{d}{dt}\nabla H(x_t) | y - x_t \rangle \leq 0, \forall y \in X. \tag{9}$$

The previous analysis corresponds to the case: $H(x) = \frac{1}{2}\|x\|^2$.

Potential

Bregman distance associated to H

$$D_H(y,x) = H(y) - H(x) - \langle \nabla H(x) | y - x \rangle \, (\geq 0).$$

$$\frac{d}{dt} D_H(y,x_t) = \langle -\frac{d}{dt} \nabla H(x_t) | y - x_t \rangle \tag{10}$$

so that (9) implies

$$\langle u_t | y - x_t \rangle \leq -\frac{d}{dt} D_H(y,x_t)$$

and the potential is $P(t;y) = D_H(y,x_t)$.

The use of a special functions H adapted to X allows to get rid of the normal cône and to produce a trajectory that remains in $intX$.
This leads to:

$$\frac{d}{dt}\nabla H(x_t) = u_t \tag{11}$$

$$\dot{x}_t = \nabla^2 H(x_t)^{-1} u_t. \tag{12}$$

which corresponds to a Riemannian metric.
In this case one has a descent algorithm for the gradient since:

$$\langle \nabla f(x_t) | \dot{x}_t \rangle = -\langle \nabla f(x_t) | \nabla^2 H(x_t)^{-1} \nabla f(x_t) \rangle \leq 0$$

To prove convergence of the trajectory $\{x_t\}$ the steps are:
1) $\{x_t\}$ has accumulation points (sublevels of $D_H(x^*,.)$ bounded)
2) If $x_{t_k} \to \bar{x}$ then $\bar{x} \in S$
3) H1 if $z^k \to y$ then $D_H(y,z^k) \to 0$
For example H L-smooth and then:

$$0 \leq D_H(x,y) \leq \frac{L}{2}\|x-y\|^2$$

4) H2 if $D_H(y,z^k) \to 0$ then $z^k \to y$
For example H β-strongly convex and then:

$$D_H(x,y) \geq \frac{\beta}{2}\|x-y\|^2$$

C. Dual averaging

Continuous version of dual averaging Nesterov [51], “Lazy gradient mirror descent ”, Kwon and Mertikopoulos [37].

Dynamics

Assume h bounded strictly convex sci with $dom\, h = X \subset V$ convex compact.

Let $U_t = \int_0^t u_s ds$ and x_t be the argmax of:

$$\langle U_t | x \rangle - h(x).$$

Let $h^*(w) = \sup_{x \in V} \langle w | x \rangle - h(x)$ be the Fenchel conjugate. h^* is differentiable.

The dynamics is given by:

$$x_t = \nabla h^*(U_t) \in X \qquad (13)$$

Potential

Define, for $y \in X$:

$$W(t;y) = h^*(U_t) - \langle U_t | y \rangle + h(y) \qquad (\geq 0). \tag{14}$$

$$\frac{d}{dt} h^*(U_t) = \langle u_t | \nabla h^*(U_t) \rangle = \langle u_t | x_t \rangle \tag{15}$$

thus:

$$\frac{d}{dt} W(t;y) = \langle u_t | x_t - y \rangle$$

and $P = W$.

Trajectories

Lemma

$f(x_t)$ is decreasing.

Proof:

$$\frac{d}{dt}f(x_t) = \langle \nabla f(x_t) | \nabla^2 h^*(U_t)(u_t) \rangle$$

with $u_t = -\nabla f(x_t)$. ∎

Hence the accumulation points of x_t are in S.

Discrete time

A. Projected gradient

Dynamics

Levitin and Polyak [41], Polyak [58]

$$x_{m+1} = argmin_X\{\langle \nabla f(x_m), x\rangle + (1/2\eta_m)\|x - x_m\|^2\}, \tag{16}$$

(η_m decreasing) which corresponds to:

$$x_{m+1} = \Pi_X[x_m + \eta_m u_m], \tag{17}$$

or with variational characterization:

$$\langle x_m + \eta_m u_m - x_{m+1}, y - x_{m+1}\rangle \leq 0, \forall y \in X. \tag{18}$$

Values

Let $m(X)$ be the diameter of X. Assume $\|u_m\|_* \leq M$.

Proposition

$$R_n(x) \leq \frac{1}{2\eta_n} m(X)^2 + \frac{M^2}{2} \sum_{m=1}^{n} \eta_m$$

hence with $\eta_n = 1/\sqrt{n}$, $R_n(x) \leq O(\sqrt{n})$.

Trajectories

Assume $S \neq \emptyset$.

Lemma

For $x^* \in S$, $\|x_m - x^*\|$ converges if $\eta_n \in \ell^2$.

Lemma

If $\eta_n \in \ell^2$ and g is dissipative, $\{\bar{x}_n\}$ converges to a point in S.

B. Mirror descent

Assumption: H *L-strongly convex for some norm* $\|.\|$ *on* $V = I\!R^n$.
$\|u_n\|_* \leq M$.

Dynamics

Nemirovski and Yudin [49], Beck and Teboulle [12]

The usual MD algorithm is given by :

$$x_{m+1} = argmin_X\{\langle \nabla f(x_m)|x\rangle + (1/\eta_m)D_H(x, x_m)\}, \tag{19}$$

General formulation:

$$\langle \nabla H(x_m) + \eta_m u_m - \nabla H(x_{m+1})|x - x_{m+1}\rangle \leq 0, \forall x \in X. \tag{20}$$

Values

Proposition

$$R_n(x) \leq \frac{D_H(x, x_1)}{\eta} + n\eta \frac{M^2}{2L}.$$

Then $\eta = 1/\sqrt{n}$ and $R_n(x) \leq O(\sqrt{n})$.

Trajectories

Assume $S \neq \emptyset$.

Lemma

For $x^* \in S$, $D_H(x^*, x_n)$ converges if $\{\eta_n\} \in \ell^2$.

C: Dual averaging

Assumption: h L-strongly convex for some norm $\|.\|$ on $V = I\!R^n$.

Dynamics

Nesterov [51]

The algorithm is given by:

$$x_{m+1} = \nabla h^*(\eta_m U_m).$$

and $\{\eta_m\}$ is decreasing.

Values

Nesterov [51] or discrete approximation of (13) Kwon and Mertikopoulos [37]:

Proposition

$$R_n(x) = \sum_{m=1}^{n} \langle u_m | x - x_m \rangle \leq \frac{r_X(h)}{\eta_n} + \frac{\sum_{m=1}^{n} \eta_{m-1} \|u_m\|_*^2}{2L} \tag{21}$$

Assume: $\|u_m\|_* \leq M$.

Hence the convergence rate $O(\sqrt{n})$ with time varying parameters $\eta_m = 1/\sqrt{m}$.

Smooth case

Assume that f is β smooth:

$$|f(y)-f(x)-\langle\nabla f(x)|y-x\rangle| \leq \frac{\beta}{2}\|x-y\|^2 \tag{22}$$

A: Projected gradient

Let $x_{m+1}=\Pi_X(y_{m+1})$, $y_{m+1}=x_m+\eta u_m$ and $u_m=-\nabla f(x_m)$.
Take $\eta=1/\beta$ and define $v_n=\beta(x_{n+1}-x_n)$

$$f(x_{n+1})-f(y) \leq \langle v_n, y-x_n\rangle - \frac{1}{2\beta}\|v_n\|^2$$

in particular $f(x_n)$ decreasing and $\{\|v_n\|\} \in \ell^2$.

Values

$$n[f(x_{n+1})-f(y)] \leq R_n^v(y) - \frac{1}{2\beta}\|\sum_{m=1}^{n}\|v_m\|^2 = \frac{\beta}{2}\|y-x_1\|^2$$

Hence convergence rate of the order $\frac{1}{n}$.

Trajectories

Lemma

Let $y^* \in S$. Then $\|x_n - y^*\|$ decreases.

Lemma

$\{x_n\}$ weakly converge to a point in S.

B: Mirror descent

We follow Bauschke, Bolte and Teboulle [11].

$$\langle \nabla H(x_n) - \lambda \nabla f(x_n) - \nabla H(x_{n+1}) | x - x_{n+1} \rangle \le 0, \forall x \in X$$

Hypothesis 1:

$$L D_H - D_f \ge 0$$

($LH - f$ convex) If H is strongly convex and f is smooth, there exist L such that this holds.

Values

One has, by H1:

$$f(x) \le f(y) + \langle \nabla f(z) | x - y \rangle + L D_h(x,z) - D_f(y,z)$$

(the last term is ≤ 0 when f is convex). Take $2\lambda L = 1$

Theorem

Assume f convex, lower bounded.

1) $f(x_n)$ is decreasing.

2) $\sum D_H(x_{n+1}, x_n) < +\infty$.

$$f(x_n) - f(y) \le \frac{2L}{n} D_H(y, x_1)$$

Trajectories

Theorem

Assume f convex, S compact $\neq \emptyset$.
1) $y^* \in S$ implies $D_H(y^*, x_n)$ decreases.
2) Assume
$H2 : x^k \to x^* \in S \Rightarrow D_H(x^*, x^k) \to 0$
$H3 : x^* \in S, D_H(x^*, x^k) \to 0 \Rightarrow x^k \to x^*$
Then $\{x_n\}$ converges to a point in S.

C: Dual averaging

Similar results for the values in case 3.

Lu, Freund and Nesterov (2018)

D: Mirror prox

Nemirovski (2004)

Assume g to be β Lipschitz.

Dynamics

x_n gives y_{n+1} via usual MD i.e. $v_n = g(x_n)$

$$\langle \nabla H(x_n) + \lambda g(x_n) - \nabla H(y_{n+1}) - |x - y_{n+1}\rangle \leq 0, \forall x \in X$$

x_n gives x_{n+1} via translated MD i.e. $u_n = g(y_{n+1})$

$$\langle \nabla H(x_n) + \lambda g(y_{n+1}) - \nabla H(x_{n+1}) | x - x_{n+1}\rangle \leq 0, \forall x \in X$$

Values

If H is α strongly convex and $\alpha \geq \lambda\beta$

$$\lambda \sum_{m=1}^{n} \langle g(y_m) | u - y_m\rangle \leq D_H(u, x_1) - D_H(u, x_n)$$

Acceleration: from discrete to continuous

Nesterov (1983)

$$x_{k+1} = y_k - s\nabla f(y_k)$$
$$y_{k+1} = x_{k+1} + \frac{k}{k+3}(x_{k+1} - x_k)$$

f with Lip gradient L and $s \leq 1/L$
convergence of $f(x_k)$ of the order $O(1/k^2)$ (best bound)

Su, Boyd, Candes (NIPS 2014, JMLR 2016)

$$\ddot{x}_t + \frac{r}{t}\dot{x}_t + \nabla f(x_t) = 0,$$

$r = 3$: continuous version of Nesterov discrete algorithm.
Lyapounov function

$$E(t;y) = \frac{t^2}{r-1}[f(x_t) - f(y)] + \frac{r-1}{2}\|x_t + \frac{t}{r-1}\dot{x}_t - y\|^2$$

For $r = 3$, $E(t;y)$ is decreasing for all y.
If $r > 3$, $E(t;y^*)$ is decreasing for $y^* \in S$. In particular

$$f(x_t) - f^* \leq O(\frac{1}{t^2})$$

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For $r = 3$, $E(t;y)$ is decreasing for all y.
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$$f(x_t) - f^* \leq O(\frac{1}{t^2})$$

Attouch, Chbani, Peypouquet, Redont (Math Pro 2018) extend the analysis
$r \geq 3$ Hilbert space H + L^1 perturbation
same speed of convergence for the values (with the same Lyapunov function)
if $r > 3$ weak convergence of the trajectory x_t using energy functions of the form (with real parameters a, b)

$$F(t) = \frac{t^2}{r-1}[f(x_t) - f^*] + \frac{r-1}{2}\|a(x_t - x^*) + \frac{t}{r-1}\dot{x}_t\|^2 + b\|x_t - x^*\|^2$$

leading (for some specific b) to

$$F'(t) \leq (2-a)t[f(x_t) - f^*] - (r-a-1)t\|\dot{x}(t)\|^2$$

in fact for $r > 3$ speed of cv $o(\frac{1}{t^2})$ (May, 2017)

Extension non euclidean
Krichene, Bayen, Bartlett (NIPS 2015)

$$F(t;y) = \frac{t^2}{q}[f(x_t) - f(y)] + q[h^*(z_t) - \langle y, z_t \rangle + h(y)]$$

$$F'(t;y) = \frac{2t}{q}[f(x_t) - f(y)] + \frac{t^2}{q}\langle \nabla f(x_t), \dot{x}_t \rangle + q\langle \nabla h^*(z_t) - y, \dot{z}_t \rangle$$

choose

$$\dot{z}_t = -\frac{t}{q}\nabla f(x_t), \qquad x_t + \frac{t}{q}\dot{x}_t = \nabla h^*(z_t)$$

$$F'(t;y) = \frac{2t}{q}[f(x_t) - f(y)] - t\langle \nabla f(x_t), -\frac{t}{q}\dot{x}_t + \nabla h^*(z_t) - y \rangle$$

$$= \frac{2t}{q}[f(x_t) - f(y)] - t\langle \nabla f(x_t), x_t - y \rangle \leq \frac{2t}{q}[f(x_t) - f(y)] - t[f(x_t) - f(y)]$$

which is non positive if $q = 2$ or $q > 2$ and $y = y^* \in S$.
Note: no condition on ∇f.

For the euclidean unconstrained case take $h(x) = \frac{1}{2}\|x\|^2$ so that $\nabla h^* = Id$ and one has

$$\frac{d}{dt}[x_t + \frac{t}{q}\dot{x}_t] = -\frac{t}{q}\nabla f(x_t)$$

which is the SBC equation with $r = q+1$.

The second equation can be written

$$t^q x_t = q\int_0^t s^{q-1}\nabla h^*(z_s)ds$$

so that x_t is an average of the previous $\nabla h^*(z_s)$.

Alternative approach : Wibisono, Wilson, Jordan (PNAS 2016)

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Alternative approach : Wibisono, Wilson, Jordan (PNAS 2016)

More extension KBB, (NIPS 2016)

$$\dot{z}_t = -\eta_t \nabla f(x_t), \qquad x_t = \frac{1}{W_t}\int_0^t w_s \nabla h^*(z_s) ds$$

with η and w positive.
new Lyapounov function is of the form

$$E(t) = a_t[f(x_t) - f(y)] + [h^*(z_t) - \langle y, z_t \rangle]$$

and speed of cv $1/a_t$, with compatibility conditions between η, w and a (standard case $a_t = t^2$)

$$E'(t) \leq [f(x_t) - f(y)](a'_t - \eta_t) + \langle \nabla f(x_t), \dot{x}_t \rangle (a_t - \frac{\eta_t W_t}{w_t})$$

Discrete properties

no natural discretization
2 first order equations: choice of coefficients

$$x_{k+1} = y_k - s\nabla f(y_k)$$

$$y_{k+1} = x_{k+1} + \frac{k}{k+r}(x_{k+1} - x_k)$$

discrete Lyapounov function (SBC)

$$E(k) = \frac{2(k+r-2)^2 s}{r-1}[f(x_k) - f^*] + (r-1)\|w_k - x^*\|^2$$

with

$$w_k = \frac{k+r-1}{r-1}y_k - \frac{k}{r-1}x_k$$

satisfies

$$E(k) + \frac{2s[(r-3)(k+r-2)+1]}{r-1}[f(x_k) - f^*] \leq E(k-1)$$

Similar computations in BBK and WWJ

In addition for $r > 3$:
weak convergence of x_n, Chambolle and Dossal (2015), ACPR (2018)
Attouch and Peypouquet (2016) cv of the value with rate $o(\frac{1}{n^2})$
The property of f is used trough

$$f(y - s\nabla f(y)) \leq f(x) + \langle \nabla f(y), y - x \rangle - \frac{s}{2}\|\nabla f(y)\|^2$$

Note: this allows for a simpler proof for the (weak) cv of x_n compared to the continuous case (cv of x_t) where f is not assumed to have Lipschitz gradient.

Open pb:
- link between continuous and discrete:
property of the curve
property of the approximation
- cv of the trajectory in the non euclidean setting
- similar procedure for smooth learning ??
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