Games, Graphs, and Dynamics

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$n \times n$ matrix games

Let A be an $n \times n$ matrix.
a_{ij} payoff for i against j symmetric 2 person game
$\sum_j a_{ij} x_j = iAx$ payoff for i against $x \in \Delta_n$

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Game Dynamics: ODE on the simplex Δ_n

1. Replicator dynamics

$$\dot{x}_i = x_i\,(iAx - xAx)\,, \quad i = 1, \dots, n \qquad \text{(REP)}$$

2. Best response dynamics

$$\dot{x} \in \mathrm{BR}(x) - x \qquad \text{(BR)}$$

with $\mathrm{BR}(x) = \{y \in \Delta_n : yAx = \max_i iAx\}$

Special case $A = A^{\mathsf{T}}$
optimization problem
xAx increases along solutions of (REP) and (BR)

For general A the dynamics of (REP) and (BR) can be complicated (oscillations, chaos).

Can we predict the behaviour somehow?

Equilibria and Supports

$\mathcal{E}$ the set of equilibria of the replicator dynamics and $\mathcal{S}$ be the set of their supports.

$x \in \mathcal{E} \Leftrightarrow iAx = jAx$ for $i, j \in I = \mathrm{supp}(x)$ and
x is a NE if $x \in \mathcal{E}$ and $iAx \geq jAx$ for $i \in I, j \notin I$

$\mathcal{E}$ includes all unit vectors of the standard basis in $\mathbb{R}^n$ (the corners of Δ_n), and $\mathcal{S}$ contains all one element sets $\{i\}$, with $i \in [n]$.

Regular games

Assumption (R):

The game A is regular, i.e., all equilibria in $\mathcal{E}$ are regular equilibria of (REP).

(R) implies (R1): for each support $I \in \mathcal{S}$ there is a unique equilibrium $p_I \in \mathcal{E}$ with supp $p_I = I$.

Let

$$r_j(I) = jAp_I - p_IAp_I \tag{1}$$

be the invasion rate/excess payoff of strategy j at the equilibrium $p_I \in \mathcal{E}$ with $\mathrm{supp}(p_I) = I \in \mathcal{S}$. Note that $r_i(I) = 0$ for all $i \in I$.

(R) implies (R2): $r_j(I) \neq 0$ whenever $j \notin I$.
Note that (R) is equivalent to (R1) $\cap$ (R2).

The invasion graph

We define the associated digraph $\mathcal{G}$ as the directed graph with vertex set $\mathcal{S}$ and a directed edge $I \to J$ if $I \neq J$ (no loops) and

- $r_j(I) > 0$ for all $j \in J \setminus I$, and
- $r_i(J) < 0$ for all $i \in I \setminus J$.

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The first condition implies that all strategies in J missing from I are better replies to p_I, while the second condition implies that all strategies in I missing from J are worse against p_J, i.e., p_J is a NE in the game restricted to $I \cup J$.

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Examples/simple observations.

For pure strategies, $i \to j$ holds iff $a_{ji} > a_{ii}$ and $a_{ij} < a_{jj}$, i.e., iff j strictly dominates i in the game reduced to the two strategies i, j.

E2

Assume $J \subset I$. Then $I \to J$ holds iff $r_i(J) < 0$ holds for all $i \in I \setminus J$ iff p_J is a NE in the game restricted to I. This implies that for the game restricted to I, for (BR) and (REP) there are orbits starting in $\Delta^\circ(I)$ converging to p_J.

E3

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Dynamics of 2×2 games is captured by the digraph
$i \to ij \leftarrow j$
$i \leftarrow ij \to j$

Lemma

If I is a terminal node (absorbing state) of $\mathcal{G}$ then p_I is a NE with index $+1$.

Proof. Suppose p_I is not a NE. Then there is a $j \notin I$ with $r_j(I) > 0$. Consider the game restricted to the strategies in $I \cup \{j\}$. Let p_J be a NE of this restricted game: $iAp_J \leq p_J A p_J$ for all $i \in I$ and by regularity $iAp_J < p_J A p_J$ for all $i \in I \setminus J$. If $J \subset I$ then by (E2), there is an arrow $I \to J$, so I is not terminal, a contradiction. Hence $j \in J$, and we have again the contradiction $I \to J$.
Now consider any subset $J \subset I$. Since there is no arrow $I \to J$, by (E2), p_J is not a NE in the game restricted to I. Hence p_I is the unique NE of the game restricted to I and therefore its index is $+1$. $\square$

Replicator dynamics

$$\dot{x}_i = x_i\left[iAx - xAx\right] \qquad \text{(REP)}$$

Lemma. Let $I, J \in \mathcal{S}$ with $I \neq J$. If there exists a connecting orbit $x \in \Delta_n$ such that $\lim_{t\to-\infty} x(t) = p_I$ and $\lim_{t\to+\infty} x(t) = p_J$ then $I \to J$ in the invasion graph $\mathcal{G}$.

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Theorem: Assume that $\mathcal{G}$ is acyclic, and $[n]$ is the only absorbing state in $\mathcal{G}$.
Then (REP) is permanent:
$\exists \delta > 0$ s.t. $\liminf_{t \to \infty} x_i(t) > \delta$ for all positive solutions.

Best response dynamics

$$\dot{x} \in \mathrm{BR}(x) - x \qquad \text{(BR)}$$

Lemma. If along a (piecewise linear) BR path $x(t)$, for some times $t_0 < t_1 < t_2$, $p_I \in BR(x(t))$ for $t_0 < t < t_1$ and $p_J \in BR(x(t))$ for $t_1 < t < t_2$ $(I \neq J)$ then $I \to J$ in the digraph $\mathcal{G}$.

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Proof. At the turning point $x(t_1)$ we have

$$x(t_1) = (1 - \varepsilon)x(t_0) + \varepsilon p_I$$

with $\varepsilon = 1 - e^{t_0 - t_1} \in (0, 1)$. And
$iAx(t_1) = \max_{i \in [n]} iAx(t_1) = p_I Ax(t_1) = p_J Ax(t_1)$ for all $i \in I \cup J$.
Hence

$$iAx(t_1) = (1 - \varepsilon)iAx(t_0) + \varepsilon iAp_I$$

is the same for $i \in I \cup J$.

Best response dynamics

$iAx(t_0) = \max_{i\in[n]} iAx(t_0)$ for $i \in I$ and $jAx(t_0) \leq \max_{i\in[n]} iAx(t_0)$ for $j \notin I$. Hence $jAp_I \geq iAp_I = p_IAp_I$ for $j \in J \setminus I$ and $i \in I$. By regularity (R2), $jAp_I > p_IAp_I$ for $j \in J \setminus I$ which show the first claim. By construction of BR paths, p_J is a NE of the game restricted to the pure best replies at $x(t_1)$, which contains $I \cup J$ as a subset. Hence $p_JAp_J \geq iAp_J$ for all $i \in I \setminus J$ and because of (R2): $p_JAp_J > iAp_J$ for all $i \in I \setminus J$, i.e., the second claim. $\square$

Result. If the graph $\mathcal{G}$ is acyclic, then all orbits of (BR) converge to a NE.

Proof. Let $x(t)$ be a solution of (BR). Since $\mathcal{G}$ has no cycles, by the Lemma $x(t)$ has only finitely many turning points. Let J be the final node along $x(t)$, i.e., $x(t)$ approaches p_J in a straight way. Then $p_J \in \mathrm{BR}(x(t))$ for all large t, hence $p_J \in \mathrm{BR}(p_J)$ and hence p_J is a NE. $\square$

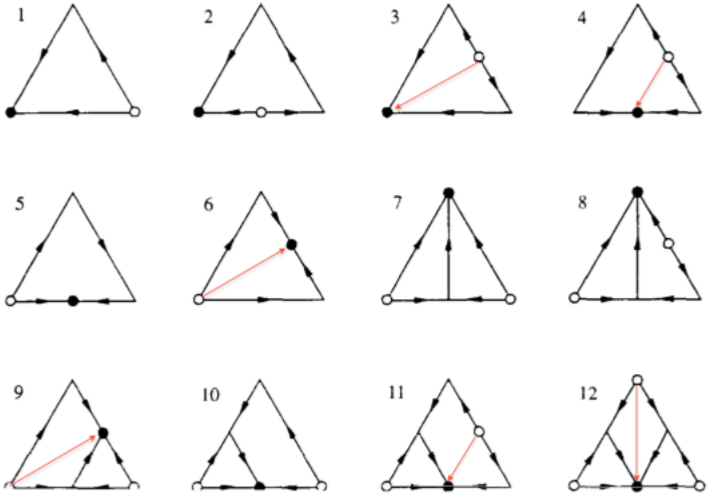
Examples: 3×3 games

How many different graphs modulo symmetry?

33 graphs:
see Mary Lou Zeeman (1989, 1993), based on E.C. Zeeman's classification (1980) of (robust) phase portraits of the replicator dynamics

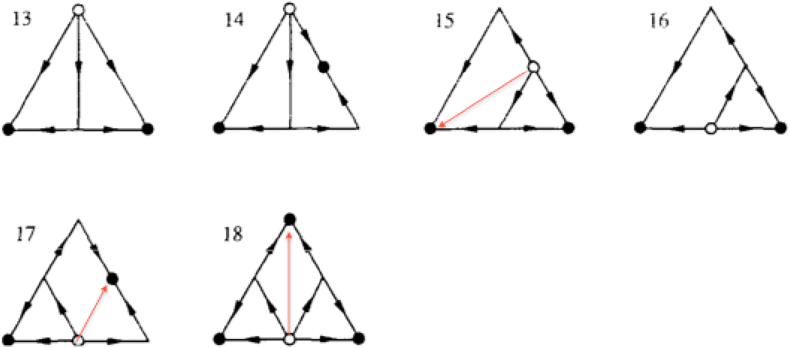
3×3 games: I

no interior equilibrium, a unique NE on the boundary



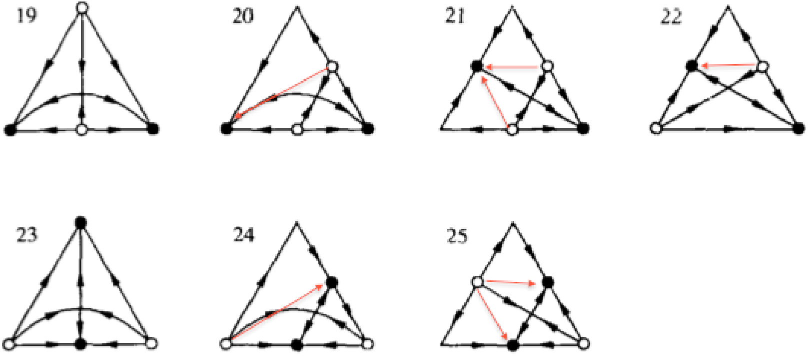
3 × 3 games: II

no interior equilibrium, several NE on the boundary



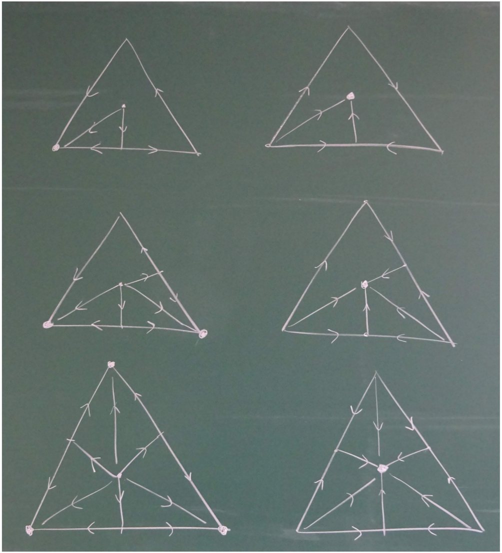
3×3 games: III

an interior equilibrium with index -1 (saddle), hence at least 2 NE on the boundary



3×3 games: IV

a unique interior equilibrium with index $+1$

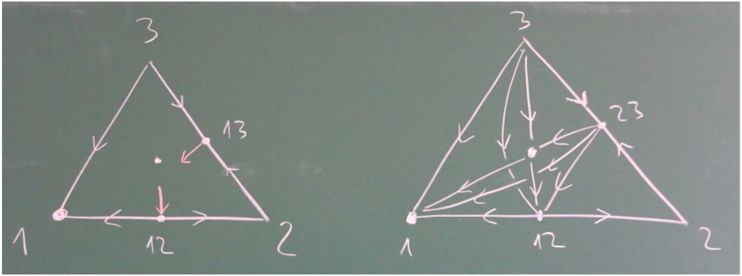


3×3 games

so far 31 graphs, acyclic, describe the dynamics (phase portrait) of (REP) and (BR) well.

2 more cases, with a cyclic graph:

3×3 games: Zeeman (1980)



$\mathcal{G}$ has three strongly connected classes:
the terminal node 1 (corresponding to a strict NE),
a nonabsorbing class $C : 123 \to 12 \to 2 \to 23 \to 12, 123$,
and the node 3 (a repeller).

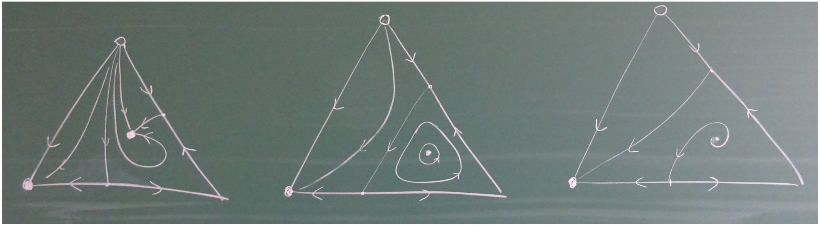
3×3 games: Zeeman (1980)

3 possible phase portraits for (REP)

a) p_{123} is an attractor

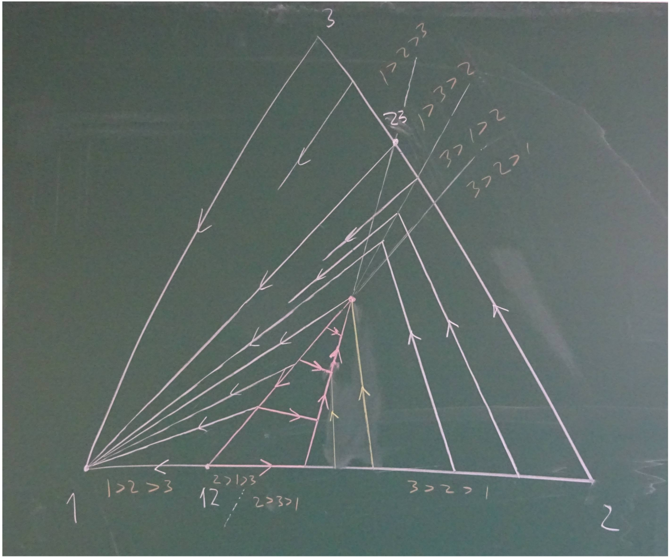
b) p_{123} is a center

c) p_{123} is a repeller, almost all orbits go to 1



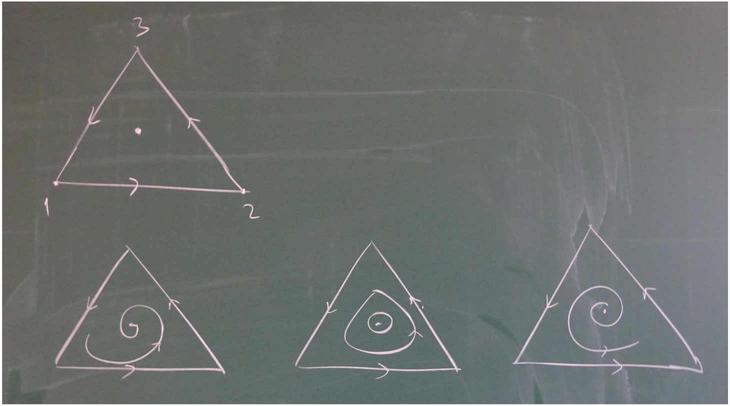
3 × 3 games: Zeeman (1980)

The class C gives rise to a transitive region in the BR dynamics.



3×3 games: rock–paper–scissors (RPS)

the digraph is disconnected, it consists of two absorbing strong components: $1 \to 2 \to 3 \to 1$ and 123.



4×4 games: ROCK–SCISSORS–PAPER–DUMB

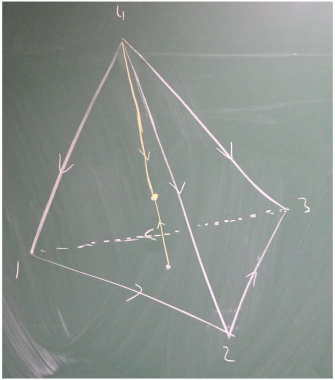
$$A = \begin{pmatrix} a & c & b & \gamma \\ b & a & c & \gamma \\ c & b & a & \gamma \\ a-\beta & a-\beta & a-\beta & 0 \end{pmatrix} \qquad (c < a < b, \beta > 0, \gamma > 0) \tag{2}$$

$p_{123} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$
$p_{1234} = (\bar{x}, \bar{x}, \bar{x}, \bar{x}_4)$ exists if $\gamma > 0$ and

$$\frac{a+b+c}{3} < a - \beta.$$

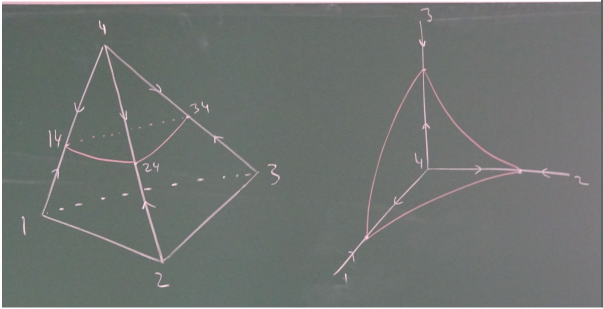
$\bar{x} = \frac{\gamma}{2a-b-c-3\beta+\gamma}$ and $\bar{x}_4 = \frac{2a-b-c-3\beta}{2a-b-c-3\beta+\gamma}$

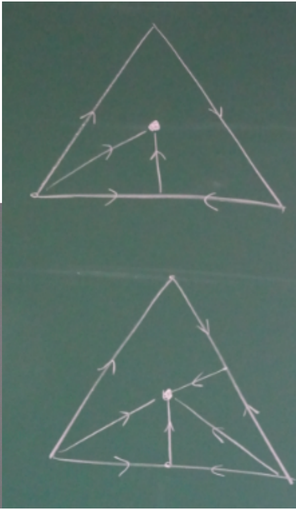
4×4 games: ROCK–PAPER–SCISSORS–DUMB



1234 is an absorbing state, and the cycle $1 \to 2 \to 3 \to 1$ is an absorbing strong component. Along almost all orbits of (REP) and (BR), the DUMB strategy is eliminated: $x_4 \to 0$. p_{1234} is a NE with index $+1$ in agreement with Lemma 1. But it is unstable. There is no NE with supp $\subseteq \{1, 2, 3\}$.

4 × 4 games: via 3d competitive LV systems

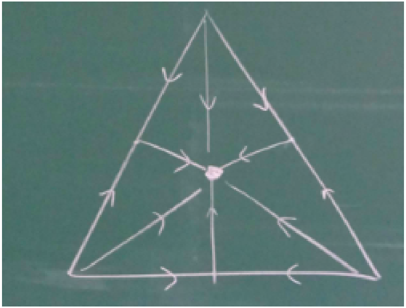




MaryLou Zeeman (1993): in these two acyclic classes there are Hopf bifurcations and hence periodic orbits, even several periodic orbits. The unique NE (unique absorbing state of $\mathcal{G}$) is not stable under (REP).

Examples: anti-coordination games

nodes of $\mathcal{G} : \{I \subseteq [n] : I \neq \emptyset\}$
$I \to J$ iff $I \subset J$
graph is acyclic, $[n]$ is the unique absorbing state
the positive equilibrium is global attractor for (BR)



Example: 5×5 anti-coordination game

$$\begin{pmatrix} 0 & 1 & 2 & 2 & 10 \\ 10 & 0 & 1 & 2 & 2 \\ 2 & 10 & 0 & 1 & 2 \\ 2 & 2 & 10 & 0 & 1 \\ 1 & 2 & 2 & 10 & 0 \end{pmatrix}$$

The positive equilibrium $\frac{1}{5}\mathbf{1}$ is unstable for (REP): 4 complex eigenvalues, 2 with positive real part.

stable limit cycle