First-order methods for the impatient: support identification in finite time with convergent Frank-Wolfe variants

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joint work with:

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Overview

1. **Simplex-constrained smooth optimization**

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3. Global convergence of iterates
4. Support identification in finite time

A simple optimization model ...

... is $\min_{\mathbf{x}\in\Delta} f(\mathbf{x})$ with $\Delta = \left\{ \mathbf{x} \in \mathbb{R}^n_+ : \sum_i x_i = 1 \right\}$

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Holds for all (also non-convex) quadratic f and many more.

Frank-Wolfe and away directions

Point $\mathbf{x} \in \Delta$ is (KKT) stationary if and only if

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$$\mathbf{d}_{FW}(\mathbf{x}) = \mathbf{e}_{\widehat{\imath}} - \mathbf{x}\,, \quad \widehat{\imath} \in Argmin\,\{\nabla_i f(\mathbf{x}) : 1 \leq i \leq n\}$$

where $\mathbf{e}_i$ is ith column of $n \times n$ identity matrix.

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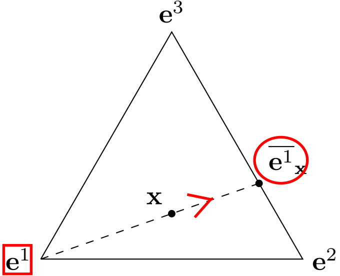
$$\mathbf{d}_A(\mathbf{x}) = \mathbf{x} - \mathbf{e}_{\hat{\jmath}}\,, \quad \hat{\jmath} \in \mathit{Argmax}\{\nabla_j f(\mathbf{x}) : x_j > 0\}\,.$$



e³
e¹ₓ
x
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Both are feasible descent directions unless $\mathbf{x}$ is stationary.

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Both discontinuous in $\mathbf{x}$, as with gradient projection.

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which satisfies, if $0 < \alpha_{exact}(\mathbf{d}, \mathbf{x}) < 1$,

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Happens only in strictly convex case $\dot{\varphi}(0) < 0 < \ddot{\varphi}(0)$.

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For $\mathbf{d} \in \{\mathbf{d}_{AFW}, \mathbf{d}_{PFW}\}$ *and* $\alpha \in \left\{\alpha_{exact}, \alpha_{Armijo}\right\}$, *we have*

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(a) every sequence $\mathbf{x}^k \to \mathbf{x}^*$ as $k \to \infty$, if $\mathbf{x}^0 \in U$;

(b) $f(\mathbf{x}) > f(\mathbf{x}^*)$ for all $\mathbf{x} \in V \setminus \{\mathbf{x}^*\}$ ... contains no KKT point.

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Generic property for objective functions:
only finitely many stationary points.

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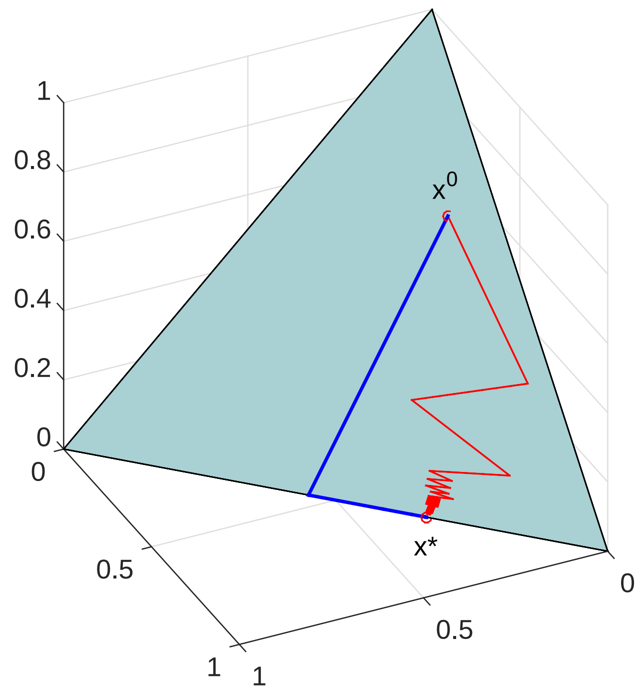
So under strict complementarity $S(\mathbf{x}^k) = S_0^* = S(\mathbf{x}^*)$ if $k \geq \bar{k}$.

Not true for classical Frank-Wolfe !

Example: $f(\mathbf{x}) = \mathbf{x}^\top Q\mathbf{x}$ (convex!) quadratic with α_{Armijo} where

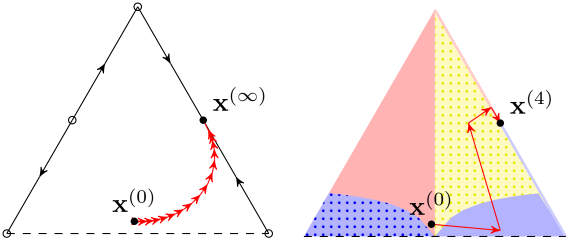


$$Q = \begin{bmatrix} 6 & 0 & 6 \\ 0 & 3 & 3 \\ 6 & 3 & 10 \end{bmatrix}.$$

Cristofari & al. (2017), arXiv:1703.07761

A fresh reference

[B./Rinaldi/Rota Bulò '18] First-order methods for the impatient: support identification in finite time with convergent Frank-Wolfe variants *Optimization online* 2018/07/6694 (3 July 2018).



$\mathbf{x}^{(\infty)}$
$\mathbf{x}^{(0)}$
$\mathbf{x}^{(4)}$
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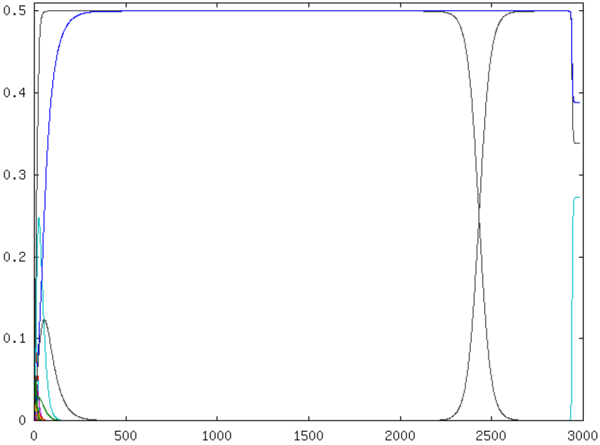
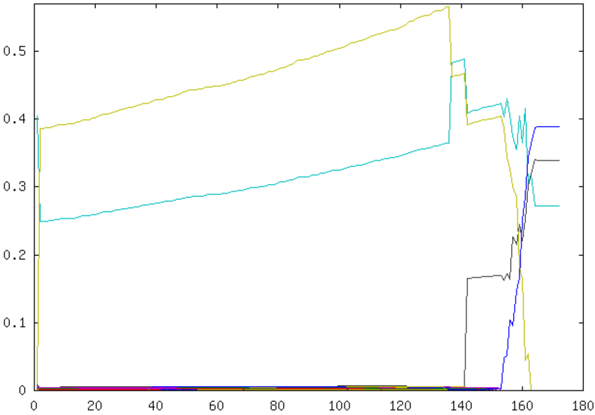


[Rota Bulò/Pelillo '17]



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Computational Results

Experiments: image segmentation

Berkeley database; objects = pixels, Gaussian similarity

$$a_{ij} = \exp(-\|\mathbf{c}(i) - \mathbf{c}(j)\|/\sigma^2)$$

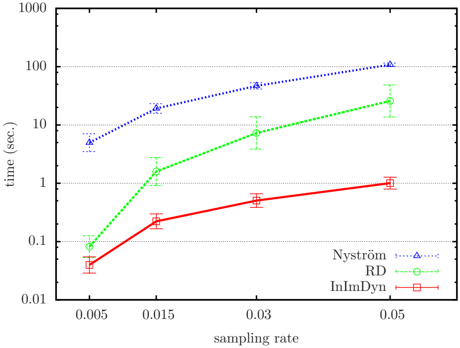
where $\mathbf{c}(i) \in \mathbb{R}^3$ is (Lab) color code and $\sigma > 0$.

Different sampling rates: different $n \approx 200, 600, 1200, 2000$.

Comparison with Nyström method [Fowlkes *et al*:04] (spectral clustering; number of clusters determined by preprocessing) and with RD: stopping criterion: $t \leq t_{\max}$ or Nash error function

$$\varepsilon(\mathbf{x}^t) = \sum_{i \in N} \left[\min\left\{x_i, \left(A\mathbf{x}^t\right)_i - \pi(\mathbf{x}^t)\right\}\right]^2 \leq 10^{-10}.$$

Results: recall/precision identical, but *Inf.Imm.Dyn* much faster.


1000
100
10
1
0.1
0.01
time (sec.)
0.005
0.015
0.03
0.05
sampling rate
Nyström
RD
InImDyn

Experiments: Region-based image matching

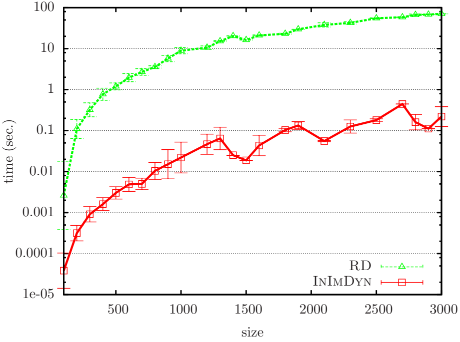
Data from [Todorovic/Ahuja:08] lead to subproblems of tree matching (≥ 100 per image), similarities are adjacencies in *association graph* $G = (V_{\text{ass}}, E_{\text{ass}})$ of two trees $T_i = (V_i, E_i)$, $i = 1, 2$:

$$V_{\text{ass}} = V_1 \times V_2 \,, \quad E_{\text{ass}} = \Big\{ \{(i,h),(j,k)\} : d_{T_1}(i,j) = d_{T_2}(h,k) \Big\} \,,$$

where $d_T(i,j)$ is tree distance of two vertices i, j in T.

This yields instances with n up to 3000, grouped according to their size across images, to obtain error bars.

Compared to *RD*, *Inf.Imm.Dyn* is orders of magnitude faster.


100
10
1
0.1
0.01
0.001
0.0001
1e-05
time (sec.)
500
1000
1500
2000
2500
3000
size
RD
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