Relaxation of mincut and normalized mincut

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1 Mincut

Finding the minimum cut (mincut) in a graph is a discrete optimization problem formalized as

$$\min_{A,\overline{A}} \sum_{i\in A, j\in\overline{A}} W_{ij}$$

where W_{ij} represents the similarity (edge weight) between the i-th and j-th point (node). If sets A and $\overline{A}$ are represented by a vector $\boldsymbol{y} \in \{-1, +1\}^N$, with $y_i = 1$ if $i \in A$ and $y_i = -1$ if $i \in \overline{A}$, then the function to be minimized can be written as

$$\begin{aligned}
\frac{(\boldsymbol{y}+\mathbf{1})'}{2}\boldsymbol{W}\frac{(-\boldsymbol{y}+\mathbf{1})}{2} &= -\frac{1}{4}\boldsymbol{y}'\boldsymbol{W}\boldsymbol{y} + \frac{1}{4}\mathbf{1}'\boldsymbol{W}\mathbf{1} \\
&= \frac{1}{4}\left(\boldsymbol{y}'\boldsymbol{D}\boldsymbol{y} - \boldsymbol{y}'\boldsymbol{W}\boldsymbol{y}\right) = \frac{1}{4}\boldsymbol{y}'(\boldsymbol{D}-\boldsymbol{W})\boldsymbol{y} \\
&= \frac{1}{4}\boldsymbol{y}'\boldsymbol{L}\boldsymbol{y}
\end{aligned}$$

where $\boldsymbol{W}$ is the symmetric matrix of similarities and $\boldsymbol{D}$ is the diagonal degree matrix defined as $\boldsymbol{D} = \text{diag}(\boldsymbol{W}\mathbf{1})$. $\boldsymbol{L}$ is the graph Laplacian.

In the above derivation we used the following identities:

$$\mathbf{1}'\boldsymbol{W}\mathbf{1} = \mathbf{1}'\boldsymbol{D}\mathbf{1} = \boldsymbol{y}'\boldsymbol{D}\boldsymbol{y}$$

The discrete optimization problem can now be relaxed to numerically minimizing $\boldsymbol{y}'\boldsymbol{L}\boldsymbol{y}$, allowing any real value for y_i. The cut is obtained by thresholding the values of $\boldsymbol{y}$ at zero.

2 Normalized mincut

Normalized mincut is an *extended* variant of mincut defined as

$$\min_{A,\overline{A}} \frac{\sum_{i\in A, j\in\overline{A}} W_{ij}}{\sum_{i\in A, v\in V} W_{iv}} + \frac{\sum_{i\in A, j\in\overline{A}} W_{ij}}{\sum_{i\in \overline{A}, v\in V} W_{iv}}$$

where V denotes the whole set (all nodes of the graph).

Using the same notation as above, normalized cut can written as

$$\frac{1}{2}\frac{\boldsymbol{y}'(\boldsymbol{D}-\boldsymbol{W})\boldsymbol{y}}{\boldsymbol{y}'\boldsymbol{D}\boldsymbol{y}+\boldsymbol{y}'\boldsymbol{D}\mathbf{1}}+\frac{1}{2}\frac{\boldsymbol{y}'(\boldsymbol{D}-\boldsymbol{W})\boldsymbol{y}}{\boldsymbol{y}'\boldsymbol{D}\boldsymbol{y}-\boldsymbol{y}'\boldsymbol{D}\mathbf{1}}$$

This, however, can be written as minimizing

$$\frac{1}{2}\frac{\boldsymbol{y}'\boldsymbol{L}\boldsymbol{y}}{\boldsymbol{y}'\boldsymbol{D}\boldsymbol{y}}$$

requiring $\boldsymbol{y}'\boldsymbol{D}\mathbf{1}=0$.

Using $\boldsymbol{z}=\boldsymbol{D}^{1/2}\boldsymbol{y}$ (from which $\boldsymbol{y}=\boldsymbol{D}^{-1/2}\boldsymbol{z}$) we obtain the form of normalized mincut used in the literature:

$$\begin{aligned}\min_{\boldsymbol{y}} \quad & \frac{\boldsymbol{z}'\boldsymbol{D}^{-1/2}(\boldsymbol{D}-\boldsymbol{W})\boldsymbol{D}^{-1/2}\boldsymbol{z}}{\boldsymbol{z}'\boldsymbol{z}}\\ \text{s.t.} \quad & \boldsymbol{z}'\boldsymbol{D}^{1/2}\mathbf{1}=0\end{aligned}$$

The vector $\boldsymbol{z}$ is relaxed to take any real value, and the cut is obtained by thresholding the resulting vector at zero.

The matrix $\boldsymbol{D}^{-1/2}(\boldsymbol{D}-\boldsymbol{W})\boldsymbol{D}^{-1/2}$ is called the normalized graph Laplacian.