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SUMAR – CONTENTS – SOMMAIRE

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GENERALIZED CONTRACTIONS FOR SOLVING RIGHT FOCAL POINT BOUNDARY VALUE PROBLEMS

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Abstract. The main goal of the present paper is to use the generalized contraction mapping principle [4] instead of the classical contraction mapping principle, in order to obtain a more general existence and uniqueness theorem for the nth order ordinary differential equation with deviating arguments (1.1) - (1.3).

1. Introduction

Second order as well as higher order boundary value problems with deviating arguments arise naturally in several engineering applications. In spite of their practical importance, only a few papers are devoted to boundary value problems(see [2] and references therein), even if initial value problems for higher order differential equations with deviating arguments have been studied intensively. Consequently, let us consider, as in [2] (all concepts and notations related to ODE are taken from this paper), the nth order ordinary differential equation with deviating arguments

$$x^{(n)}(t) = f(t, x \circ w(t)), t \in [a, b], \tag{1.1}$$

where $x \circ w(t)$ stands for $(x(w_{0,1}(t)), ..., x(w_{0,p(0)}(t)), ..., x^{(q)}(w_{q,p(q)}(t)))$, $0 \leq q \leq n-1$ (but fixed), and $p(i), 0 \leq i \leq q$, are positive integers.

The function $f(t, < x >)$ is assumed to be continuous on $[a, b] \times \mathbf{R}^N$,where $< x >$ represents $(x_{0,1}, ..., x_{0,p(0)}, ..., x_{q,p(q)})$ and $N = \sum_{i=0}^{q} p(i)$. The functions

$$w_{i,j}, 1 \leq j \leq p(i), 0 \leq i \leq q,$$

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are continuous on $[a, b]$ and $w_{i,j}\ (t) \leq b$ for all $t \in [a, b]$;

Also, they assume the value a at most a finite number of times as t ranges over $[a, b]$.

Let

$$\alpha = \min\{a, \inf_{a \leq t \leq b} w_{i,j}(t), \qquad 1 \leq j \leq p(i), \qquad 0 \leq i \leq q\}.$$

If $\alpha < a$, we assume that a function $\varphi \in C^{(q)}[\alpha, a]$ is given.

Let k be a fixed integer such that $1 \leq k \leq n-1$ and let $r = \min\{q, k-1\}$.

We seek a function

$$x \in B = C^{(r)}[\alpha, b] \cap C^{(q)}[\alpha, a] \cap C^{(q)}[a, b],$$

having at least a piecewise continuous nth derivative on $[a, b]$, and such that:

if

$$\alpha < a \quad and \quad q \geq k-1, then \quad x^{(i)}(t) = \varphi^{(i)}(t), 0 \leq i \leq q, \quad t \in [\alpha, a]; \qquad (1.2)$$

if $\alpha < a$ and $q < k-1$, then

$$x^{(i)}(t) = \varphi^{(i)}(t), \quad 0 \leq i \leq q, \quad t \in [\alpha, a];$$
$$x^{(i)}(a) = A_i, \quad q+1 \leq i \leq k-1;$$

if $\alpha = a$, then

$$x^{(i)}(a) = A_i, \quad 0 \leq i \leq k-1$$

and

$$x^{(i)}(b) = B_i, \quad k \leq i \leq n-1; \qquad (1.3)$$

Also, x is a solution of (1.1) on $[a, b]$.

2. Equivalent integral equation

To obtain an existence and uniqueness theorem for the boundary value problem (1.1)-(1.3) we shall convert it into its equivalent integral equation representation. To this end we need the Green's function expression, $g(t, s)$, for the boundary value problem

$$x^{(n)} = 0, \quad x^{(i)}(a) = 0, \quad 0 \leq i \leq k-1, \quad x^{(i)}(b) = 0, \quad k \leq i \leq n-1. \qquad (2.1)$$

From Lemma 2.1 [2], we have that $g(t,s)$ is given by

$$g(t,s) = \begin{cases} \frac{1}{(n-1)!} \sum_{i=0}^{k-1} \binom{n-1}{i} (t-a)^i (a-s)^{n-i-1}, & if \quad s \le t, \\ -\frac{1}{(n-1)!} \sum_{i=k}^{n-1} \binom{n-1}{i} (t-a)^i (a-s)^{n-i-1}, & if \quad s \ge t. \end{cases}$$

It is known [2] that

$$(-1)^{n-k} g^{(i)}(t,s) \ge 0, \quad 0 \le i \le k, \qquad (t,s) \in [a,b] \times [a,b];$$

$$(-1)^{n-i} g^{(i)}(t,s) \ge 0, \quad k+1 \le i \le n-1, \qquad (t,s) \in [a,b] \times [a,b];$$

$$\sup_{a \le t \le b} \int_a^b | g^{(i)}(t,s) | \, ds \le C_{n,i} (b-a)^{n-i}, \qquad 0 \le i \le n-1,$$

where $g^{(i)}(t,s) = \partial^i g(t,s)/\partial t^i$ and

$$C_{n,i} = \begin{cases} \frac{1}{(n-1)!} \left| \sum_{j=0}^{k-i-1} \binom{n-1}{j} (-1)^{n-j-1} \right|, & 0 \le i \le k-1, \\ \frac{1}{(n-1)!}, & k \le i \le n-1. \end{cases}$$

The boundary value problem (1.1)-(1.3) is equivalent to the integral equation

$$x(t) = \psi(t) + \theta(t) \int_a^b g(t,s) f(s, x \circ w(s)) ds, \tag{2.2}$$

where

$$\theta(t) = \begin{cases} 0, & t \in [\alpha, a] \\ 1, & otherwise, \end{cases}$$

and the function ψ is defined as follows.

If $\alpha < a$ and $q \ge k-1$, then

$$\psi(t) = \begin{cases} \varphi(t), & t \in [\alpha, a], \\ P_{n-1}(t), & t \in [a,b], \end{cases}$$

where $\alpha_i = \varphi^{(i)}(a),\ 0 \le i \le k-1, \quad \beta_i = B_i,\ k \le i \le n-1$,and $p_{n-1}(t)$ is the unique polynomial (see Lemma 2.2,[2]) of degree $n-1$ satisfying

$$P_{n-1}^{(i)}(a) = \alpha_i, \quad 0 \le i \le k-1 \text{ and } P_{n-1}^{(i)}(b) = \beta_i, k \le i \le n-1.$$

If $\alpha < a$ and $q < k-1$,then

$$\psi(t) = \begin{cases} \varphi(t), & t \in [\alpha, a], \\ P_{n-1}(t), & t \in [a, b], \end{cases}$$

where $\alpha_i = \varphi^{(i)}(a), 0 \le i \le q,\ \alpha_i = A_i,\ q+1 \le i \le k-1$, and $\beta_i = B_i,\ k \le i \le n-1$.

If $\alpha = a$, then $\psi(t) = P_{n-1}(t), t \in [\alpha, a]$, where

$$\alpha_i = A_i, 0 \le i \le k-1 \text{ and } \beta_i = B_i, k \le i \le n-1.$$

It is easy to see that $\psi \in \mathcal{B}$, and for all $t \in [a, b]$,with

$$w_{i,j}(t) = a,\ \psi^{(i)}(w_{i,j}(t)) = P_{n-1}^{(i)}(a+0).$$

3. Generalized contraction mapping principle and main result

We shall use a local variant of the generalized contraction mapping principle [4, Theorem 1.5.1.] to state our main result.

Lemma 3.1. (Generalized contraction mapping principle [4]). *Let (X, d) be a complete metric space and let $\mu > 0,\ \mu \in \mathbf{R},\ \overline{S}(u_0, \mu) = \{u \in X : d(u, u_0) \le \mu\}$. Further, let T be an operator which maps $\overline{S}(u_0, \mu)$ into X, and*

(i) for all $u, v \in \overline{S}(u_0, \mu), d(Tu, Tv) \le \phi(d(u, v))$, where ϕ is a (c)-comparison function;

(ii) $\mu_0 = d(Tu_0, u_0) \le \mu - \phi(\mu)$.

Then

(1) T has a fixed point u^* in $\overline{S}(u_0, \mu_0)$;

(2) u^* is the unique fixed point of T in $\overline{S}(u_0, \mu_0)$;

(3) the sequence $\{u_m\}$, where $u_{m+1} = Tu_m, m = 0, 1, \ldots$, converges to u^* with

$$d(u^*, u_m) \le s(\phi^m(d(u_0, u_1)))$$

and

$$d(u^*, u_m) \leq s(d(u_m, u_{m+1}));$$

where $s(t)$ *is the sum of the series* $\sum_{k=0}^{\infty} \phi^k(t)$.

(4) for any $u \in \overline{S}(u_0, \mu_0), u^* = \lim_{m\to\infty} T^m u$.

Remark. For the notion of (c)-comparison function we refer to [4]. A typical comparison function is

$$\phi(t) = \lambda t, \qquad 0 \leq \lambda < 1, \qquad t \in [0, \infty). \tag{3.1}$$

For ϕ given by (3.1), from Lemma 3.1 we obtain Lemma 2.3 [2].

Let $\overline{A}_i$, $0 \leq i \leq k-1$ and $\overline{B}_i$, $k \leq i \leq n-1$,be given fixed numbers and $\psi_2 \in \mathcal{B}$ the function defined in [2], Section 4. Following [2], a function $\overline{x} \in \mathcal{B}$ is called an *approximate solution* of (2.2) if there exist nonnegative constants ϵ and δ such that wherever $\psi^{(i)}(t)$, $\psi_2^{(i)}(t)$ and $\overline{x}^{(i)}(t)$ are defined,

$$\sup_{a\leq t\leq b} \mid \psi_2^{(i)}(t) - \psi^{(i)}(t) \mid \ \leq \ \epsilon C_{n,i}(b-a)^{n-i}, \quad 0 \leq i \leq q, \tag{3.2}$$

$$\sup_{a\leq t\leq b} \mid \overline{x}^{(i)}(t) - \psi_2^{(i)}(t) - \theta(t) \int_a^b g^{(i)}(s,t) f(s, \overline{x} \circ w(s)) ds \mid \leq \delta C_{n,i}(b-a)^{n-i}, 0 \leq i \leq q. \tag{3.3}$$

If we consider the following norm on the space $\mathcal{B}$:

$$\| x \| = \max_{0\leq i\leq q} \left\{ \left(\frac{C_{n,0}(b-a)^i}{C_{n,i}}\right) \sup_{a\leq t\leq b} \mid x^{(i)}(t) \mid wherever \quad x^{(i)}(t) \quad exists \right\}$$

and apply Lemma 3.1 we can prove in a standard way.

Theorem 3.1.. *Suppose that (2.2) has an approximate solution* $\overline{x} \in \mathcal{B}$ *and*

(i) f *satisfies the Lipschitz condition*

$$\mid f(t, <x>) - f(t, <y>) \mid \leq \sum_{i=0}^{q} \sum_{j=1}^{p(i)} L_{i,j} \mid x_{i,j} - y_{i,j} \mid,$$

for all $(t, <x>), (t, <y>) \in [a,b] \times D_1$, *where*

$$D_1=\left\{< x >: \mid x_{i,j} - x^{(i)}(w_{i,j}(t)) \mid \leq \mu \cdot \frac{C_{n,i}}{C_{n,0}(b-a)^i}, \quad 1 \leq j \leq p(i), \quad 0 \leq i \leq q\right\};$$

(ii) ϕ is a (c)-comparison function and

$$(\epsilon + \delta)C_{n,0}(b-a)^n \leq \mu - \phi(\mu). \tag{3.4}$$

Then

(1) There exists a solution $x^*(t)$ of (1.1)-(1.3) in $\overline{S}(\overline{x}, \mu_0)$;

(2) $x^*(t)$ is the unique solution of (1.1)-(1.3) in $\overline{S}(\overline{x}, \mu_0)$;

(3) The sequence $\{x_m(t)\}$ of successive approximations, defined by

$$x_{m+1}(t) = \psi(t) + \theta(t) \int_a^b g(t,s) f(s, x_m \circ w(s)) ds, \quad m = 0, 1, \ldots$$

and $x_0(t) = \overline{x}(t)$, converges to $x^*(t)$ with

$$\| x^* - x_m \| \leq s(\phi^m(\| u_0 - u_1 \|)),$$

$$\| x^* - x_m \| \leq s(\| u_m - u_{m+1} \|);$$

(4) for any $x_0(t) = x(t)$, where $x \in \overline{S}(\overline{x}, \mu_0)$, the iterative process converges to $x^*(t)$.

Remarks

1) For $\phi(t)$ as given by (3.1), from Theorem 3.1 we obtain Theorem 4.1 in [2];

2) If , for instance , we take the comparison function $\phi : \mathbf{R}_+ \to \mathbf{R}_+$, given by :

$$\phi(t) = \begin{cases} \frac{1}{2}t, & 0 \leq t \leq 1 \\ t - \frac{1}{3}, & t > 1, \end{cases}$$

then an operator T, which satisfies all assumptions in Theorem 3.1, will be generally not a contractive operator (with respect to the norm , see [4]), that is, an operator satisfying for all $u, v \in \overline{S}(\overline{x}, \mu_0)$, the classical contraction condition

$$\| Tu - Tv \| \leq \lambda \| u - v \|, \quad 0 < \lambda < 1,$$

but T is a generalized contractive operator. Consequently, Theorem 4.1 from [2] does not apply, while Theorem 3.1 apply to this class of higher order differential equation with deviating argument.

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A NOTE ON THE TRIVIALITY OF THE BOHR-COMPACTIFICATION OF LIE GROUPS

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Abstract. We determine a class of connected Lie groups for which the triviality of the Bohr-compactification is equivalent to the triviality of the Bohr-compactification of the simply connected covering group. We derive from these results some information on the structure of the Bohr-compactification of some class of connected topological groups.

1. Introduction

A topological group G has a trivial Bohr-compactification if $(f, \{1\})$ is the Bohr-compactification of G, where $f\colon G \to \{1\}$ is the trivial homomorphism. One shows quickly that if a topological group G has a simply connected covering group $\widetilde{G}$ and if $\widetilde{G}$ has a trivial Bohr-compactification, then G itself must possess a trivial Bohr-compactification. The converse of this statement is not always true, i.e., that *the triviality of the Bohr-compactification of a topological group G does not imply the triviality of the Bohr-compactification of its simply connected covering group $\widetilde{G}$* (if this covering group exists). In the present paper we look for conditions when this converse is true. The main results are contained in Section 2: We find a class of connected Lie groups for which the triviality of the Bohr-compactification is equivalent to the triviality of the Bohr-compactification of the simply connected covering group (see Theorem 2.10). As we shall see in Section 3, Theorem 2.10 implies statements about the structure of the Bohr-compactification of connected simple Lie groups and of connected semisimple Lie groups. For the sake of completeness we include as a final result of this section the structure theorem for the Bohr-compactification of solvable

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connected topological groups. We mention that Neeb investigates in Proposition X.1 of [5] the structure of the Bohr-compactification of Lie groups, too. But his methods differ essentially from ours. The last section of the paper contains an example for a connected topological group satisfying the property that it has a trivial Bohr-compactification while its simply connected covering group has a non-trivial Bohr-compactification.

We denote by (i_G, G^b) the Bohr-compactification of the topological group G. For the sake of simplicity we shall say that G^b is the Bohr-compactification of G. With this notation, the triviality of the Bohr-compactification of G is equivalent to the fact that $G^b = \{1\}$.

We recall the well-known fact that the Bohr-compactification of a topological group G is also the universal topological group compactification of G.

2. Passing to the universal covering group

It is easy to see that if the simply connected covering group $\widetilde{G}$ of a topological group G has a trivial Bohr-compactification then G has also a trivial Bohr-compactification. This fact follows from the following lemma.

Lemma 2.1. *Let $f\colon H \to K$ be a dense and continuous homomorphism between topological groups. If $H^b = \{1\}$, then $K^b = \{1\}$.*

Proof. The universality of (i_H, H^b) implies the existence of a continuous homomorphism $\phi\colon H^b \to K^b$ such that the diagram

$$\begin{array}{ccc} H & \xrightarrow{\ i_H\ } & H^b \\ {\scriptstyle i_K \circ f}\downarrow & & \downarrow{\scriptstyle \phi} \\ K^b & = & K^b \end{array}$$

commutes. Since $H^b = \{1\}$, we deduce that

$$i_K(f(h)) = 1, \text{ for all } h \in H.$$

The density of f and the continuity of i_K now imply that $K^b = \{1\}$. □

Corollary 2.2. *Let G be a connected topological group and $\widetilde{G}$ its simply connected covering group. If $(\widetilde{G})^b = \{1\}$, then $G^b = \{1\}$.*

Remark. The converse of Corollary 2.2 is not always true. We postpone the presentation of an example (see Proposition 4.1).

In the remainder of this section we look for conditions on a Lie group G which ensure that the converse of Corollary 2.2 is also true. The following well-known lemma on topological groups (whose proof we omit) will be very useful for our purposes.

Lemma 2.3. *Let G, H, and K be topological groups, $q\colon G \to H$ a quotient homomorphism of topological groups, and $f\colon G \to K$ a continuous homomorphism. If $\ker q \subseteq \ker f$, then there exists a unique continuous homomorphism $\bar{f}\colon H \to K$ such that the diagram*

$$\begin{array}{ccc} G & \xrightarrow{\ q\ } & H \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ K & = & K \end{array}$$

is commutative.

We derive from Lemma 2.3 the following isomorphism results for topological groups:

Corollary 2.4. *Let G be a topological group, N a normal subgroup of G, and H an arbitrary subgroup of G. The map $\phi\colon H/(H \cap N) \to HN/N$ defined by $\phi(h(H \cap N)) = hN$, for all $h(H \cap N) \in H/(H \cap N)$, is a continuous algebraic isomorphism.*

Corollary 2.5. *Let G be a topological group, N a normal and closed subgroup of G, and H a compact subgroup of G. Then the map ϕ defined in* Corollary 2.4 *is a homeomorphism.*

The next theorem is basic for what follows.

Theorem 2.6. *Let $Z \xrightarrow{f} H \xrightarrow{p} G$ be a sequence of continuous homomorphisms of topological groups satisfying the following properties:*

(i) $f(Z) = \ker p$.

(ii) *Z is abelian.*

(iii) *p is a quotient map.*

(iv) $G^b = \{1\}$.

Then H^b is abelian.

Proof. The universality of (i_Z, Z^b) implies the existence of a continuous homomorphism $f' \colon Z^b \to H^b$ such that the following diagram

$$\begin{array}{ccc} Z & \xrightarrow{i_Z} & Z^b \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ H & \xrightarrow[i_H]{} & H^b \end{array} \tag{2.1}$$

commutes.

We first prove that $f'(Z^b)$ is a closed normal subgroup of H^b. It is obvious that $f'(Z^b)$ is a closed subgroup of H^b.

The fact that $f(Z) = \ker p$ implies that the subgroup $f(Z)$ is normal in H. Thus for an arbitrary $h \in H$ we have

$$hf(Z)h^{-1} \subseteq f(Z).$$

Applying i_H to both sides of the above inclusion and taking into account (2.1), one obtains that

$$i_H(h)f'(i_Z(Z))(i_H(h))^{-1} \subseteq f'(i_Z(Z)) \subseteq f'(Z^b). \tag{2.2}$$

Since the inner automorphisms of H^b are continuous and since H^b is Hausdorff and compact, the following equality holds

$$\overline{i_H(h)f'(i_Z(Z))(i_H(h))^{-1}} = i_H(h)\overline{f'(i_Z(Z))}(i_H(h))^{-1}. \tag{2.3}$$

Using the continuity of f', the density of i_Z, and the fact that H^b is Hausdorff and compact, one gets the following equalities

$$\overline{f'(i_Z(Z))} = f'(\overline{i_Z(Z)}) = f'(Z^b). \tag{2.4}$$

Relations (2.2), (2.3), and (2.4) imply that

$$i_H(h)f'(Z^b)(i_H(h))^{-1} \subseteq f'(Z^b). \tag{2.5}$$

Taking into account that $\overline{i_H(H)} = H^b$, one concludes from (2.5) that $f'(Z^b)$ is normal in H^b.

Let $K := H^b/f'(Z^b)$ be endowed with the quotient topology. Since H^b is a compact topological group and since $f'(Z^b)$ is a closed normal subgroup of it, K is a compact Hausdorff topological group. Denote by $q\colon H^b \to K$ the canonical quotient map.

We now show that there is a continuous homomorphism $\phi\colon G \to K$ such that the diagram

$$\begin{array}{ccc} H & \xrightarrow{p} & G \\ {\scriptstyle q\circ i_H}\downarrow & & \downarrow{\scriptstyle \phi} \\ K & = & K \end{array} \tag{2.6}$$

commutes. For this we observe that

$$\ker p \subseteq \ker(q \circ i_H). \tag{2.7}$$

Indeed, $\ker p = f(Z)$ and we know by (2.1) that

$$i_H(\ker p) = i_H(f(Z)) = f'(i_Z(Z)) \subseteq f'(Z^b).$$

Since $f'(Z^b) = \ker q$, one obtains from the above relation that

$$(q \circ i_H)(\ker p) \subseteq \{1\},$$

i.e., (2.7) holds. Applying Lemma 2.3, there exists a continuous homomorphism $\phi\colon G \to K$ such that (2.6) is commutative.

Since $G^b = \{1\}$, we must have that

$$\phi(g) = 1, \text{ for all } g \in G.$$

Thus

$$(\phi \circ p)(h) = 1, \text{ for all } h \in H.$$

From the commutative diagram (2.6) we now get that

$$(q \circ i_H)(h) = 1, \text{ for all } h \in H,$$

i.e.,

$$i_H(H) \subseteq \ker q = f'(Z^b).$$

Since $\ker q$ is closed and since i_H is dense, it follows that $H^b = f'(Z^b)$. According to condition (ii) of the hypotheses we know that Z is abelian. Then so are Z^b and $f'(Z^b)$. Thus H^b is abelian. □

We recall that for a group G, the commutator subgroup is denoted by G'.

Lemma 2.7. *Let G be a connected Lie group. Then G^b is abelian if and only if $\ker i_G = \overline{G'}$.*

Proof. First suppose that $\ker i_G = \overline{G'}$. The inclusion $G' \subseteq \ker i_G$ implies that $i_G(G)$ is abelian. Then so is $\overline{i_G(G)} = G^b$.

Now assume that G^b is abelian. This implies the inclusion

$$\overline{G'} \subseteq \ker i_G. \tag{2.8}$$

To prove the converse inclusion, consider $T := G/\overline{G'}$. Thus T is a connected abelian Lie group. According to Korollar III.3.25 of [3] there are natural numbers m and n such that T is both algebraically and topologically isomorphic to the direct product $\mathbb{R}^m \times (\mathbb{R}/\mathbb{Z})^n$. It follows that $i_T\colon T \to T^b$ is injective.

Denote by $q\colon G \to T$ the canonical quotient map. The universality of (i_G, G^b) implies the existence of a continuous homomorphism $\bar{q}\colon G^b \to T^b$ which makes the diagram

$$\begin{array}{ccc} G & \xrightarrow{\ i_G\ } & G^b \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle \bar{q}} \\ T & \xrightarrow[\ i_T\ ]{} & T^b \end{array}$$

commutative. Now consider an arbitrary element $g \in \ker i_G$. Thus $\bar{q} \circ i_G(g) = 1$, or, by the commutativity of the above diagram, $i_T(q(g)) = 1$. Since i_T is injective, it follows that $q(g) = 1$, i.e., $g \in \ker q = \overline{G'}$. Thus

$$\ker i_G \subseteq \overline{G'}. \tag{2.9}$$

By (2.8) and (2.9) one obtains the desired conclusion. □

The following result is a consequence of Theorem 2.6 and Lemma 2.7.

Corollary 2.8. *Add to the hypotheses of* Theorem 2.6 *that* H *is a connected Lie group. Then* $\ker i_H = \overline{H'}$.

We define now a special type of topological groups, which will enable us to give an answer to the problem of the triviality of the Bohr-compactification presented in the previous section.

Definition 2.9. A topological group G is called *topologically perfect* if $G = \overline{G'}$.

The next result gives a class of Lie groups for which the converse of Corollary 2.2 is true.

Theorem 2.10. *Let* G *be a connected Lie group satisfying the property that the simply connected covering group* $\widetilde{G}$ *of it is topologically perfect. Then* $G^b = \{1\}$ *if and only if* $(\widetilde{G})^b = \{1\}$.

Proof. If $(\widetilde{G})^b = \{1\}$, then Corollary 2.2 yields that $G^b = \{1\}$. For the converse statement let $p\colon \widetilde{G} \to G$ be a covering morphism and denote by $Z := \ker p$. It is known that Z is an abelian subgroup of $\widetilde{G}$. Denote by $i\colon Z \to \widetilde{G}$ the inclusion map. The map p is a quotient map since it is a covering morphism. Thus the sequence

$$Z \xrightarrow{i} \widetilde{G} \xrightarrow{p} G$$

satisfies the conditions (i)–(iv) of Theorem 2.6. Applying Corollary 2.8, one obtains that

$$\ker i_{\widetilde{G}} = \overline{(\widetilde{G})'}.$$

Since $\widetilde{G}$ is topologically perfect, one concludes that $\ker i_{\widetilde{G}} = \widetilde{G}$, i.e., $(\widetilde{G})^b = \{1\}$. □

Connected semisimple Lie groups are common examples of topologically perfect groups. Thus the following result is a direct consequence of Theorem 2.10.

Corollary 2.11. *Let* G *be a connected semisimple Lie group and* $\widetilde{G}$ *the simply connected covering group of it. Then* $G^b = \{1\}$ *if and only if* $(\widetilde{G})^b = \{1\}$.

3. The structure of the Bohr-compactification of Lie groups

As we shall see, Corollary 2.11 implies statements about the structure of the Bohr-compactification of connected simple Lie groups and of connected semisimple Lie groups. For this we also need the following theorem, which is a consequence of a deep result by RUPPERT.

Theorem 3.1. *The Bohr-compactification of a non-compact connected simple Lie group G with finite center is trivial.*

Proof. This statement follows from two results of [7], namely from Theorem III.1.19 and assertion (i) of Theorem III.6.3. □

Theorem 3.2. *Let $\mathfrak{g}$ be a simple non-compact Lie algebra and let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then $G^b = \{1\}$.*

Proof. Let $\widetilde{G}$ be a simply connected covering group of G. According to Satz I.5.19 and Satz II.7.1 of [3] there is a connected linear Lie group G^* possessing a Lie algebra isomorphic to $\mathfrak{g}$. Since G^* is a simple linear Lie group, Proposition 5.1 of Chapter 1 of [6] implies that it has finite center. Applying Theorem 3.1, we get that $(G^*)^b = \{1\}$. Now consider a simply connected covering group $\widetilde{G^*}$ of G^*. Corollary 2.11 yields that $(\widetilde{G^*})^b = \{1\}$. On the other hand, the Lie groups $\widetilde{G^*}$ and $\widetilde{G}$ are isomorphic having isomorphic Lie algebras. Thus $(\widetilde{G})^b = \{1\}$ and so the assertion follows from Corollary 2.2. □

Now we turn our attention to connected semisimple Lie groups. A first step in determining the structure of their Bohr-compactification is contained in the following result:

Lemma 3.3. *Let $n \geq 1$ be a natural number and let $\mathfrak{s}$ be a semisimple Lie algebra with the property that*

$$\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n,$$

where $\mathfrak{s}_i$ are simple and non-compact ideals of $\mathfrak{s}$. If S is a connected Lie group with Lie algebra $\mathfrak{s}$, then $S^b = \{1\}$.

Proof. For $i \in \{1, \ldots, n\}$ let $\widetilde{S}_i$ be a simply connected Lie group with Lie algebra $\mathfrak{s}_i$. Put

$$\widetilde{S} := \prod_{i=1}^{n} \widetilde{S}_i.$$

Then $\widetilde{S}$ is a simply connected Lie group whose Lie algebra is isomorphic to $\mathfrak{s}$. Denote by $f_i : \widetilde{S}_i \to \widetilde{S}$ $(i = \overline{1, n})$ the canonical injections. The subgroup $f_i(\widetilde{S}_i)$ $(i \in \{1, \ldots, n\})$ of $\widetilde{S}$ is a connected Lie group whose Lie algebra is isomorphic to $\mathfrak{s}_i$. Thus, according to Theorem 3.2, the group $f_i(\widetilde{S}_i)$ has trivial Bohr-compactification for each $i \in \{1, \ldots, n\}$. It follows that

$$i_{\widetilde{S}}(f_i(\widetilde{S}_i)) = \{1\} \text{ for each } i \in \{1, \ldots, n\}.$$

Since

$$\widetilde{S} = f_1(\widetilde{S}_1) \ldots f_n(\widetilde{S}_n),$$

one obtains that $i_{\widetilde{S}}(\widetilde{S}) = \{1\}$, i.e., $(\widetilde{S})^b = \{1\}$. Since the group $\widetilde{S}$ is a simply connected covering group of S, Corollary 2.2 yields that $S^b = \{1\}$. □

In stating the structure theorem for the Bohr-compactification of connected semisimple Lie groups we need the following result about the structure of connected semisimple Lie groups. In the proof of this result one uses the structure theorem of semisimple Lie algebras (see Satz II.3.7 of [3]).

Lemma 3.4. *Let G be a connected semisimple Lie group and $\mathfrak{g}$ its Lie algebra. Then the following statements hold:*

(1) There are two finite sets I and J and simple ideals $\mathfrak{k}_i$ $(i \in I)$ and $\mathfrak{s}_j$ $(j \in J)$ of $\mathfrak{g}$ satisfying the properties that $\mathfrak{k}_i$ is compact for each $i \in I$, $\mathfrak{s}_j$ is non-compact for each $j \in J$, and

$$\mathfrak{g} = \bigoplus_{i \in I} \mathfrak{k}_i \oplus \bigoplus_{j \in J} \mathfrak{s}_j.$$

(By definition, if $I = \emptyset$, then $\oplus_{i \in I} \mathfrak{k}_i := \{0\}$ and similarly, if $J = \emptyset$, then $\oplus_{j \in J} \mathfrak{s}_j := \{0\}$.)

(2) *There is a compact connected normal subgroup K of G and there is a closed connected normal subgroup S of G such that*

$$\mathbf{L}(K) = \bigoplus_{i \in I} \mathfrak{k}_i, \quad \mathbf{L}(S) = \bigoplus_{j \in J} \mathfrak{s}_j, \text{ and } G = KS.$$

Moreover, $K \cap S$ is a discrete subgroup of G.

We are now prepared for the structure theorem of the Bohr-compactification of a connected semisimple Lie group.

Theorem 3.5. *Let G be a connected semisimple Lie group and $\mathfrak{g}$ its Lie algebra. There is a compact connected normal subgroup K of G and there is a closed connected normal subgroup S of G such that the following assertions hold:*

(i) $G = KS$.

(ii) *The groups G/S and $K/K \cap S$ are algebraically and topologically isomorphic.*

(iii) *If $q\colon G \to G/S$ denotes the canonical quotient map, then $(q, G/S)$ is the Bohr-compactification of G.*

Proof. Let K and S be the subgroups of assertion (2) of Lemma 3.4. Then (i) obviously holds.

(ii) This assertion follows from (i) and Corollary 2.5.

(ii) According to (ii) the pair $(q, G/S)$ is a topological group compactification of G. Consider an arbitrary continuous homomorphism $f\colon G \to T$ of G into a compact Hausdorff topological group T. We know by Lemma 3.3 that $S^b = \{1\}$, hence $f(S) = \{1\}$, i.e., $\ker q = S \subseteq \ker f$. Applying Lemma 2.3, we find a continuous homomorphism $\bar{f}\colon G/S \to T$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{q} & G/S \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \bar{f}} \\ T & = & T \end{array}$$

is commutative. This means that $(q, G/S)$ is the universal topological group compactification of G, hence also the Bohr-compactification of G. □

For the sake of completeness we finish this section with some considerations on the Bohr-compactification of another important class of Lie groups, namely the solvable Lie groups. We determine even the structure of the Bohr-compactification of solvable connected topological groups. For this we state first the following useful result:

Proposition 3.6. *A compact connected Hausdorff topological group G which is solvable is abelian.*

Proof. Proposition 9.4 of [4] implies that $G'' = G'$. Since G is solvable, it follows that $G' = \{1\}$. Thus G is abelian. □

We are now able to give the structure of the Bohr-compactification of solvable topological groups. For a topological group G denote by T the quotient group $G/\overline{G'}$ and by $q\colon G \to T$ the canonical quotient map.

Theorem 3.7. *Let G be a solvable connected topological group. Then $(i_T \circ q, T^b)$ is the Bohr-compactification of G.*

Proof. It is clear that $(i_T \circ q, T^b)$ is a topological group compactification of G. Now consider an arbitrary continuous and dense homomorphism $f\colon G \to K$ of G into a compact Hausdorff topological group K. Since G is connected and solvable, so is $f(G)$. Thus the group $\overline{f(G)} = K$ is also connected and solvable. Hence Proposition 3.6 yields that K is abelian. It follows that

$$\ker q = \overline{G'} \subseteq \ker f.$$

In view of Lemma 2.3 there exists a continuous homomorphism $f'\colon T \to K$ such that the diagram

(3.1)
$$\begin{array}{ccc} G & \xrightarrow{\ q\ } & T \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ K & = & K \end{array}$$

commutes. The universality of (i_T, T^b) implies the existence of a continuous homomorphism $\bar{f}\colon T^b \to K$ such that the diagram

$$\begin{array}{ccc} T & \xrightarrow{i_T} & T^b \\ f'\big\downarrow & & \big\downarrow \bar{f} \\ K & = & K \end{array} \tag{3.2}$$

commutes. The diagrams (3.1) and (3.2) yield

$$\bar{f} \circ i_T \circ q = f' \circ q = f,$$

i.e., the following diagram

$$\begin{array}{ccc} G & \xrightarrow{i_T \circ q} & T^b \\ f\big\downarrow & & \big\downarrow \bar{f} \\ K & = & K \end{array}$$

is commutative. This shows that $(i_T \circ q, T^b)$ is the universal topological group compactification of G, hence also the Bohr-compactification of G. □

Remark. Suppose in addition to the hypotheses of Theorem 3.7 that G is locally compact. In view of assertion (iii) of Theorem 7.57 of [4] the connected locally compact abelian group T is both algebraically and topologically isomorphic to the direct product $\mathbb{R}^n \times C$ with a compact connected group C. Since the Bohr-compactification of this direct product is known, it is now clear what the Bohr-compactification of a solvable connected locally compact topological group looks like.

4. An example

We give now the example promised in the remark after Corollary 2.2 for a connected topological group satisfying the conditions that it has a trivial Bohr-compactification and the simply connected covering group of it has a non-trivial Bohr-compactification. Let $\widetilde{G} = \mathbb{R} \times \widetilde{\mathrm{Sl}}(2,\mathbb{R})$ be the direct product of the additive group of real numbers (endowed with the usual topology) and the simply connected covering group of the special linear group. Let $T = \mathbb{Z} + \sqrt{2}\mathbb{Z}$. We know by Lemma I.3.14 of [3] that T is a dense subgroup of $\mathbb{R}$. It is known (see, for example, Theorem V.4.37 of [2]) that $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ has a discrete center which is isomorphic to $\mathbb{Z}$. Denote by $z \in \widetilde{\mathrm{Sl}}(2,\mathbb{R})$

the generator of this center. Now consider the subgroup Z of $\widetilde{G}$ generated by the elements $(1,1)$ and $(\sqrt{2}, z)$. Then

$$Z = \{(m + n\sqrt{2}, z^n) \mid m, n \in \mathbb{Z}\}.$$

The subgroup Z of $\widetilde{G}$ is discrete and normal in $\widetilde{G}$. We consider the quotient group $G := \widetilde{G}/Z$. For this group we can state the following proposition:

Proposition 4.1. *The topological group G has a trivial Bohr-compactification. The Bohr-compactification of its simply connected covering group $\widetilde{G}$ satisfies $(\widetilde{G})^b \simeq \mathbb{R}^b$.*

Proof. Denote by $q\colon \widetilde{G} \to G$ the canonical quotient map. Since q is a homomorphism, we have

$$q\left((\widetilde{G})'\right) = G'.$$

Thus $G' = (\widetilde{G})'Z/Z$. Applying Corollary 2.4, there is a continuous isomorphism $\phi\colon (\widetilde{G})'/((\widetilde{G})' \cap Z) \to (\widetilde{G})'Z/Z$. Hence $\phi\colon (\widetilde{G})'/((\widetilde{G})' \cap Z) \to G'$ is a continuous isomorphism. On the other hand, since $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ is simple, we have the following equality

$$(\widetilde{G})' = \{0\} \times \widetilde{\mathrm{Sl}}(2,\mathbb{R}).$$

Thus $(\widetilde{G})' \cap Z = \{(0,1)\}$. It follows that $(\widetilde{G})'/((\widetilde{G})' \cap Z)$ is both algebraically and topologically isomorphic to $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$. Thus $(\widetilde{G})'/((\widetilde{G})' \cap Z)$ has a trivial Bohr-compactification by Theorem 3.2. Applying Lemma 2.1 to the map ϕ, we conclude that $(G')^b = \{1\}$. Let us observe that

$$G' = q\left((\widetilde{G})'Z\right).$$

Since

$$T \times \widetilde{\mathrm{Sl}}(2,\mathbb{R}) \subseteq (\widetilde{G})'Z$$

and since $\overline{T} = \mathbb{R}$, we deduce that

$$\overline{(\widetilde{G})'Z} = \widetilde{G}.$$

The continuity of q and the above relations yield that

$$G = q(\widetilde{G}) \subseteq \overline{q((\widetilde{G})'Z)} = \overline{G'}.$$

Thus $\overline{G'} = G$. Taking into account that $(G')^b = \{1\}$, it follows that $i_G(G') = \{1\}$. Since $\overline{G'} = G$, the continuity of i_G implies that $i_G(G) = \{1\}$, i.e., $G^b = \{1\}$. Since q is open and since $\ker q = Z$ is discrete, it follows that q is a covering morphism. Since $\widetilde{G}$ is simply connected, it is a simply connected covering group of G. On the other hand, since $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ has a trivial Bohr-compactification, one has that $(\widetilde{G})^b \simeq \mathbb{R}^b$. □

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SCHUNCK CLASSES OF π-SOLVABLE GROUPS

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Abstract. The paper deals with some properties of $\underline{X}$-maximal subgroups, $\underline{X}$-pro-
jectors and $\underline{X}$-covering subgroups in finite π-solvable groups related to a π-closed Schunck class $\underline{X}$, where π is an arbitrary set of primes. The main results are: 1) an existence and conjugacy theorem for $\underline{X}$-maximal subgroups; 2) the proof of a property of covering subgroups in the more general case of projectors and some important corollaries if π is the set of all primes.

1. Preliminaries

The aim of this paper is to study in the case of finite π-solvable groups some special subgroups introduced by W. Gaschütz in [6] and [7].
All groups considered in the paper are finite. We denote by π an arbitrary set of primes and by π' the complement to π in the set of all primes.
The notions in the paper are resumed in the following definitions.

Definition 1.1. a) ([7]) We call $\underline{X}$ a *class* of groups if the members of $\underline{X}$ are finite groups and $\underline{X}$ has the properties:
(1) $1 \in \underline{X}$;
(2) if $G \in \underline{X}$ and f is an isomorphism of G then $f(G) \in \underline{X}$.
b) ([8]) A class $\underline{X}$ of groups is a *homomorph* if $\underline{X}$ is closed under homomorphisms, i.e. if $G \in \underline{X}$ and N is a normal subgroup of G imply $G/N \in \underline{X}$.
c) A group G is *primitive* if there is a maximal subgroup W of G with $\text{core}_G W = 1$, where

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$\text{core}_G W = \cap \{ W^g / g \in G \}$.

d) ([8]) A homomorph $\underline{X}$ is a *Schunck class* if $\underline{X}$ is *primitively closed*, i.e. if any group G , all of whose primitive factor groups are in $\underline{X}$, is itself in $\underline{X}$.

Definition 1.2. Let $\underline{X}$ be a class of groups, G a group and H a subgroup of G.

a) ([7]) H is *$\underline{X}$-maximal* in G if:

(1) $H \in \underline{X}$;

(2) $H \leq K \leq G, K \in \underline{X} \Rightarrow H = K$.

b) ([7]) H is an *$\underline{X}$-projector* of G if for any normal subgroup N of G, HN/N is $\underline{X}$-maximal in G/N. c) ([6]) H is an *$\underline{X}$-covering subgroup* of G if:

(1) $H \in \underline{X}$;

(2) $H \leq K \leq G, K_0 < K, K/K_0 \in \underline{X} \Rightarrow K = HK_0$.

Definition 1.3. a) ([5]) A group is *π-solvable* if every chief factor is either a solvable π-group or a π'-group. If π is the set of all primes, we obtain the notion of *solvable group*.

b) A class $\underline{X}$ of groups is *π-closed* if:

$G/O\pi'(G) \in \underline{X} \Rightarrow G \in \underline{X}$,

where $O\pi'(G)$ denotes the largest normal π'-subgroup of G. We shall call ***π-homomorph*** (*π-Schunck class*) a π-closed homomorph (Schunck class).

We shall use in the paper the following result given by R. Baer in [1]:

Theorem 1.4. *A solvable minimal normal subgroup of a group is abelian.*

2. Basic properties of special subgroups

We remind here some basic properties of special subgroups defined in 1.2.

Theorem 2.1. *([6]; [8]) Let $\underline{X}$ be a homomorph, G a group and H a subgroup of G.*

a) If H is an $\underline{X}$-covering subgroup of G, then:

(1) for any $x \in G$, H^x is an $\underline{X}$-covering subgroup of G;

(2) for any normal subgroup N of G, HN/N is an $\underline{X}$-covering subgroup of G/N;

(3) for any subgroup K with $H \leq K \leq G$, it follows that H is an $\underline{X}$-covering subgroup of

K.

b) If N is a normal subgroup of G and $H \leq H^* \leq G$ such that $N \subseteq H^*$, H is an $\underline{X}$-covering subgroup of H^* and H^*/N is an $\underline{X}$-covering subgroup of G/N, then H is an

$\underline{X}$-covering subgroup of G.

Theorem 2.2. *([7]) Let $\underline{X}$ be a class of groups, G a group and H a subgroup of G.*

a) If H is an $\underline{X}$-projector of G and $x \in G$, then H^x is an $\underline{X}$-projector of G.

b) H is an $\underline{X}$-projector of G if and only if:

(1) H is $\underline{X}$-maximal in G;

(2) HM/M is an $\underline{X}$-projector of G/M for all minimal normal subgroups M of G.

c) If H is an $\underline{X}$-projector of G and N is a normal subgroup of G, then HN/N is an $\underline{X}$-projector of G/N.

Theorem 2.3. *Let $\underline{X}$ be a class of groups, G a group and H an $\underline{X}$-maximal subgroup of G. Then:*

a) for any $x \in G$, H^x is an $\underline{X}$-maximal subgroup of G;

b) for any subgroup K with $H \leq K \leq G$, it follows that H is $\underline{X}$-maximal in K.

Concerning to the connection between $\underline{X}$-maximal subgroups, $\underline{X}$-projectors and

$\underline{X}$-covering subgroups in finite groups we give:

Theorem 2.4. *([4]) Let $\underline{X}$ be a class of groups, G a group and H a subgroup of G.*

a) If H is an $\underline{X}$-covering subgroup or an $\underline{X}$-projector of G, then H is $\underline{X}$-maximal in G.

b) If further $\underline{X}$ is a homomorph, then: H is an $\underline{X}$-covering subgroup of G if and only if H is an $\underline{X}$-projector in any subgroup K with $H \leq K \leq G$. Particularly, any $\underline{X}$-covering subgroup of G is an $\underline{X}$-projector of G.

Remaark. The converse of the last assertion does not hold, as the following example shows: Let $\underline{A}$ be the homomorph of all finite abelian groups. Any subgroup of order

4 which is not normal in the symmetric group S_4 is an $\underline{A}$-projector, but is not an $\underline{A}$-covering subgroup in S_4.

3. Existence and conjugacy theorems

The fundamental problem on the special subgroups defined in 1.2. is to prove the existence and conjugacy theorems. We give below such theorems for finite π-solvable groups.

All groups in this section are finite π-solvable.

Theorem 3.1. *([2]) Let $\underline{X}$ be a π-homomorph.*

a) $\underline{X}$ is a Schunck class if and only if any π-solvable group has $\underline{X}$-covering subgroups.

b) Any two $\underline{X}$-covering subgroups of a π-solvable group G are conjugate in G.

Theorem 3.2. *([3]; [4]) Let $\underline{X}$ be a π-homomorph. Then: $\underline{X}$ is a Schunck class if and only if any π-solvable group has $\underline{X}$-projectors.*

Corollary 3.3. *Let $\underline{X}$ be a π-homomorph. The following conditions are equivalent:*

(1) $\underline{X}$ is a Schunck class;

(2) any π-solvable group has $\underline{X}$-covering subgroups;

(3) any π-solvable group has $\underline{X}$-projectors.

Theorem 3.4. *([3]) If $\underline{X}$ is a π-Schunck class, then any two $\underline{X}$-projectors of a π-solvable group G are conjugate in G.*

In the proof of 3.4. given in [3], we use a lemma, important in itself, because it can be considered as an existence and conjugacy theorem for $\underline{X}$-maximal subgroups in finite π-solvable groups.

Theorem 3.5. *([3]) Let $\underline{X}$ be a π-Schunck class, G a π-solvable group and A an abelian normal subgroup of G with $G/A \in \underline{X}$. Then:*

a) there is a subgroup S of G with $S \in \underline{X}$ and $AS = G$ (which imply that there is an $\underline{X}$-maximal subgroup S of G such that $AS = G$);

b) if S_1 and S_2 are $\underline{X}$-maximal subgroups of G with $AS_1 = G = AS_2$, then S_1 and S_2 are conjugate in G.

4. New results on projectors

In our intention to study some properties of special subgroups in finite π-solvable groups we raised the following question: Does an analogous property of 2.1.b) hold for projectors? The answer is affirmative in finite π-solvable groups, as the result below shows.

Theorem 4.1. *Let $\underline{X}$ be a π-Schunck class, G a π-solvable group such that for any minimal normal subgroup M of G which is a π'-group we have $G/M \in \underline{X}$ and let B be a normal abelian subgroup of G such that:*
(1) S is $\underline{X}$-maximal in BS;
(2) BS/B is an $\underline{X}$-projector of G/B.
Then S is an $\underline{X}$-projector of G.

Proof. We consider two cases:
1) $B = 1$. Then $BS/B \cong S$ and $G/B \cong G$. By (2), S is an $\underline{X}$-projector of G.
2) $B \neq 1$. To prove that S is an $\underline{X}$-projector of G we use 2.2.b).
(1) S is $\underline{X}$-maximal in G. Indeed, if we put $S^* = BS$, our assumptions (1) and (2) imply that S is $\underline{X}$-maximal in S^* and S^*/B is an $\underline{X}$-projector of G/B. Then $S \in \underline{X}$. Let $S \leq T \leq G$ and $T \in \underline{X}$. We show that $S = T$. From $BT/B \cong T/B \cap T$ and $\underline{X}$ being a homomorph we obtain $BT/B \in \underline{X}$. By 2.4.a), S^*/B is $\underline{X}$-maximal in G/B. This and $BS/B \leq BT/B$, where $BT/B \in \underline{X}$, imply $BS/B = BT/B$, hence $S^* = BS = BT$ and $T \leq S^*$. But $S \leq T \leq S^*$, $T \in \underline{X}$ and S $\underline{X}$-maximal in S^* imply $S = T$.
(2) For any minimal normal subgroup M of G, MS/M is an $\underline{X}$-projector of G/M. Indeed, M being a minimal normal subgroup of the π-solvable group G, two cases are possible:
a) *M is a solvable π-group.* Then, by 1.4., M is abelian. $\underline{X}$ being a π-Schunck class, 3.2. shows that the π-solvable group G/M has an $\underline{X}$-projector T^*/M. We shall prove that MS/M and T^*/M are conjugate in G/M, hence, by 2.2.a), MS/M is an $\underline{X}$-projector of G/M.
We are in the hypotheses of 3.5. because T^* is a π-solvable group and M is an abelian normal subgroup of T^* with $T^*/M \in \underline{X}$. By 3.5.a), there is an $\underline{X}$-maximal subgroup T

of T^* such that $MT = T^*$. We shall prove that T is $\underline{X}$-maximal in G. Indeed, $T \in \underline{X}$. Further, let $T \leq T' \leq G$ with $T' \in \underline{X}$. We show that $T = T'$. Since $T^* = MT \leq MT'$ it follows that

$T^*/M \leq MT'/M \cong T'/M \cap T' \in \underline{X}$.

Using that T^*/M is an $\underline{X}$-projector of G/M, that means that T^*/M is $\underline{X}$-maximal in G/M, we obtain $T^*/M = MT'/M$, hence $MT = T^* = MT'$. So $T \leq T' \leq T^*$. But T is an

$\underline{X}$-maximal subgroup of T^* and $T' \in \underline{X}$. Then $T = T'$. So T is $\underline{X}$-maximal in G.

Let $A = BM$. Clearly A is a normal abelian subgroup of G. Further AS/A and AT/A are $\underline{X}$-projectors of the π-solvable group G/A. By 3.4., AS/A and AT/A are conjugate in G/A. It follows that $AS^g = AT$ for some $g \in G$. But S and T are $\underline{X}$-maximal in G. By 2.3.b), S^g and T are $\underline{X}$-maximal in $AT = AS^g$. Applying now 3.5.b) to the π-solvable group AT and its abelian normal subgroup A with $AT/A \in \underline{X}$, it follows that S^g and T are conjugate in AT. Hence $M\ S^g/M$ and $MT/M = T^*/M$ are conjugate in G/M. Then MS/M and T^*/M are conjugate in G/M and so MS/M is an $\underline{X}$-projector of G/M.

B) *M is a π'-group.* Then $M \leq O\pi'(G)$ and

$G/\ O\pi'(G) \cong (G/M)/(\ O\pi'(G)/M)$.

But M being a minimal normal subgroup of G which is a π'-group, we have $G/M \in \underline{X}$. So, $\underline{X}$ being a homomorph, we also have $G/\ O\pi'(G) \in \underline{X}$. It follows, by the π-closure of $\underline{X}$, that $G \in \underline{X}$. But S is $\underline{X}$-maximal in G. Then $S = G$ is its own $\underline{X}$-projector, which means also that $MS/M = G/M$ is its own $\underline{X}$-projector. □

From now on let π be the set of all primes, i.e. all groups we consider are finite solvable groups. Theorem 4.1. has in this particular case the following immediate corollaries (given also in [7]).

Corollary 4.2. *Let $\underline{X}$ be a Schunck class, G a solvable group, S a subgroup of G and* $G = G_0 \geq G_1 \geq \ldots \geq G_r = 1$

such that for any i, $G_i < G$ and G_i/G_{i+1} is abelian. Then S is an $\underline{X}$-projector of G if and only if for any i, G_iS/G_i is $\underline{X}$-maximal in G/G_i.

Proof. By induction on $|G|$. If S is an $\underline{X}$-projector of G, then, by 1.2.b), for any i, G_iS/G_i is $\underline{X}$-maximal in G/G_i. Conversely, let, for any i, G_iS/G_i be $\underline{X}$-maximal in G/G_i.

By the induction, $G_{r-1}S/G_{r-1}$ is an $\underline{X}$-projector of G/G_{r-1}. Then putting in 4.1. B $= G_{r-1}$, we obtain that S is an $\underline{X}$-projector of G. □

Corollary 4.3. *Let $\underline{X}$ be a Schunck class, G a solvable group, H a subgroup of G and S an $\underline{X}$-projector of G such that $S \subseteq H$. Then S is an $\underline{X}$-projector of H.*

Proof. G being solvable, there is a chain

$G = G_0 \geq G_1 \geq \ldots \geq G_r = 1$

such that for any i, $G_i < G$ and G_i/G_{i+1} is abelian. We denote for any i, $H_i = H \cap G_i$. Then

$H = H_0 \geq H_1 \geq \ldots \geq H_r = 1$

is a chain with $H_i < H$ and H_i/H_{i+1} abelian for any i. Applying 4.2. for the $\underline{X}$-projector S of G, we obtain that for any i, G_iS/G_i is $\underline{X}$-maximal in G/G_i. But, for any i, we also have:

$H_iS/H_i \cong S/S\cap H_i = S/S\cap(H\cap G_i) = S/(S\cap H) \cap G_i = S/S\cap G_i \cong G_iS/G_i$

and

$H/H_i = H/H\cap G_i \cong HG_i/G_i \leq G/G_i$.

It follows that for any i, H_iS/H_i is $\underline{X}$-maximal in H/H_i, hence, by 4.2., S is an $\underline{X}$-projector of H. □

From 2.4.b) and 4.3. follows:

Corollary 4.4. *Let $\underline{X}$ be a Schunck class, G a solvable group and S a subgroup of G. Then S is an $\underline{X}$-covering subgroup of G if and only if S is an $\underline{X}$-projector of G.*

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LE PROBLÈME EXTERIEUR DE DIRICHLET EN DEUX DIMENSIONS POUR LES CHEMINEMENTS ALÉATOIRES SUR DES DÉMIGROUPS DISCRETS

OCTAVIAN MUSTAFA

Abstract. The paper examines some problems concerning the exterior Dirichlet problem for random walks on discrete semigroups.

1. Introduction. Périodicité et récurrence. La fonction caracteristique

Soit $(Z, \dot{+})$ un démigroupe commutatif, dénombrable et dans lequel toute équation $a + x = b$, où $a, b \in Z$, admet au plus une solution. Supplémentaire, $0 \in Z$.

Exemple 1.1. Toutes les parties stables d'un groupe lesquelles le zéro du groupe est contenu.

Soit $(\Omega, \mathcal{F}, P)$ un espace de probabolité, $f_n : \Omega \to Z$, $n \geq 1$ des variables aléatoires independentes et identiquement repatisées et $S_0 = 0$, $S_n \doteq f_1 + \cdots + f_n$, $n \geq 1$, un cheminement aléatoire sur Z.

Proposition 1.2. *Soit $z, z' \in Z$. Si l'équation $z + x = z'$ n'a pas de solution et $P(S_n = z) \neq 0$, alors*

$$P(S_{n+1} = z' | \, S_n = z) = 0.$$

On défini l'application $T : Z \times Z \to [0,1]$ par

$$T(z, z') = \begin{cases} T(0, l), & \text{pour } z + l = z' \\ 0, & \text{en rest,} \end{cases}$$

où $T(0, l) = P(f_n = l)$, $n \geq 1$. Évidemment, $\sum_{z \in Z} T(0, z) = 1$. T est la fonction de transition du cheminement aléatoire considéré.

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Proposition 1.3. *Si* $P(S_n = z) \neq 0$, *alors*

$$T(z,z') = P(S_{n+1} = z' | \, S_n = z).$$

Supplémentaire,

$$T(z+l, z'+l) = T(z,z'), \quad pour\ z, z', l \in Z.$$

Et, si $z + z' = 0$, *alors*

$$T(z,0) = T(0,z').$$

En effet, le chaine de Markov homogène $\{S_n | \, n \geq 0\}$ a la matrice des probabilités de transition

$$\mathbf{P} = \|T(z,z')\|_{z,z' \in Z}.$$

Pour $z, z' \in Z$, on défini

$$T_0 = \delta, \quad T_1 = T,$$

$$T_n(z,z') = \sum_{z_1, z_2, \ldots, z_{n-1} \in Z} T(z,z_1)\ldots T(z_{n-1},z'), \quad n \geq 2$$

où δ est le symbole de Kronecker. Biensûr, si $P(S_m = z) \neq 0$, alors

$$T_n(z,z') = P(S_{m+n} = z' | \, S_m = z), \quad n \geq 1.$$

Soit

$$F_0 = 0, \quad F_1 = T,$$

$$F_n(z,z') = \sum_{z_1, \ldots, z_{n-1} \neq z'} T(z,z_1)\ldots T(z_{n-1},z').$$

Finalement,

$$G_n(z,z') = \sum_{k=0}^{n} T_k(z,z'), \quad n \geq 0.$$

Proposition 1.4. *a)* $F_n(z,z) = F_n(0,0)$,

b) $\sum_{k=1}^{\infty} F_k(z,z') \leq 1$,

c) $G_n(z,z') \leq G_n(0,0)$,

d) $T_n(z+l, z'+l) = T_n(z,z')$,

e) $G_n(z,z) = G_n(0,0)$,

f) $T_n(z,z') = \sum_{k=1}^{n} F_k(z,z')T_{n-k}(z',z'), \quad n \geq 1, \quad z,z',l \in Z.$

Un état $z \in Z$, avec $F(z,z) = 1$, où $F = \sum_{n=1}^{\infty} F_n$, est récurrent. Du b) on obtient que $0 \leq F(z,z') \leq 1$, pour $z,z' \in Z$. Si $F(z,z) < 1$, l'état z est transitoire (à voit [4.3], I-er tome, pag. 353 et [4.4], pag.19).

Observation 1.5. Parce que $F(z,z) = F(0,0)$, $(\forall)$ $z \in Z$, si un état z du cheminement aléatoire est récurrent (transitoire), alors tous ses états sont récurrents (transitoires), le cheminement etant récurrent (transitoire).

On définit

$$\Sigma := \{z \in Z |\ T(z,0) > 0\}, \quad Z^+ := \{z \in Z |\ (\exists)\ n \geq 0,\ t_n(z,0) > 0\},$$

$$\overline{Z} := \{z \in Z |\ (\exists)\ z', z'' \in Z^+,\ z + z' = z''\}.$$

Proposition 1.6. *L'ensemble Z^+ contient toutes les sommes finies d'élèments de Σ et $(Z^+,+)$ est le plus petit démigroupe de Z pour lequel $\Sigma \subseteq Z^+$. $(Z^+,+)$ est le plus petit sousgroupe aditif de Z avec $Z^+ \subseteq \overline{Z}$.*

Proposition 1.7. *Si le cheminement est réccurent, alors $Z^+ = \overline{Z}$ et*

$$F(0,z) = 1, \quad z \in \overline{Z}, \quad F(0,z) = 0, \quad z \in Z \setminus \overline{Z}.$$

Observation 1.8. Les ensembles Σ et Z^+ sont les inverses des ensembles Σ et Z^+ definis en [4.2].

On dit que le cheminement aléatoire T est apériodique si et seulement si $Z = \overline{Z}$. Particulierement, Z doit être groupe.

Soit $Z \subseteq \mathbf{R}^d$, $d \geq 1$. Pour $x = (x_1, \ldots, x_d) \in Z$ et $\theta = (\theta_1, \ldots, \theta_d) \in \mathbf{R}^d$ on utilisera les notations suivantes

$$|x|^2 = \sum_{i=1}^{d} |x^i|^2, \quad |\theta|^2 = \sum_{i-1}^{d} |\theta_i|^2, \quad x\theta = \sum_{i=1}^{d} x^i\theta_i.$$

La fonction caracteristique asociée au cheminement aléatoire T est

$$\Phi(\alpha) := \sum_{x \in Z} T(0,x)e^{ix\alpha}, \text{ pour } \alpha \in \mathbf{R}^d.$$

Soit $C = \{\theta | \; |\theta_i| \leq 2\pi, \; 1 \leq i \leq d\}$. Alors,

$$a) \; [\Phi(\theta)]^n = \sum_{x \in Z} T_n(0,x)e^{ix\theta}, \quad \theta \in \mathbf{R}^d,$$

$$b) \; T_n(0,y) = (2\pi)^{-d} \int_C e^{-iy\Theta}[\Phi(\theta)]^n d\theta, \quad y \in Z.$$

Ici, $[\Phi]^n$ sera remplacée par Φ^n, en concordance qvec [4.2].

Si $d = 1$, soit

$$\mu_k = \sum_{x \in Z} x^k T(0,x), \quad k = 1, 2, \quad m_1 = \sum_{x \in Z} |x| T(0,x).$$

Proposition 1.9. *Si $m_1 < \infty$, alors Φ est du classe C^1 et $\Phi'(0) = i\mu_1$.*

La réciproque n'est pas valable. Pourtant,

Proposition 1.10. *Si $T(0,x) = 0$ pour $x < 0$ et si*

$$0 \leq -i\Phi'(0) = \lim_{\theta \to 0} i \frac{1 - \Phi(\theta)}{\theta} = \alpha < \infty,$$

alors $\mu_1 = m_1 = \alpha$.

Finalement,

Proposition 1.11. *Si $\mu_1 = 0$ et $\sigma^2 = \mu_2 - \mu_1^2 < \infty$, alors*

$$\lim_{n \to \infty} \sum_{x < \sqrt{n} \cdot \sigma \cdot t} T_n(0,x) = \frac{1}{2\pi} \int_{-\infty}^{t} e^{-\frac{y^2}{2}} dy, \quad t \in \mathbf{R}.$$

2. Démigroupes suffisants. La caracterisation des cheminements aléatoires sur les démigroupes suffisants

On dit qu'un sousgroupe G de $\mathbf{R}^d$, $d \geq 1$, est suffisant si et seulement si:

$$\forall \; (A_k \subseteq G | \; \theta_k > 0, \; \theta \in A_k) \; \exists \; \left(\theta^{(k)} \in A_k | \; \theta_k^{(k)} = \min_{\theta \in A_k} \theta_k\right), \quad 1 \leq k \leq d.$$

Proposition 2.1. *Pour tous les groupes suffisants $G \subseteq \mathbf{R}^d$ il existe $1 \leq k \leq d$ et des vecteurs $x_1, \ldots, x_k$ en G independents sur $\mathbf{R}$ tels que*

$$G = \mathbf{Z}z_1 \oplus \times \oplus \mathbf{Z}x_k$$

(G est un $\mathbf{Z}$-module de dimension k).

Démonstration. À voir [4.1], les pages 24-25. □

Un démigroupe (groupe) aditif Z, denombrable, inclus en $\mathbf{R}^d$ est consideré comme suffisant par raport au cheminement $T : Z \times Z \to [0,1]$ si et seulement si $\overline{Z}$ est un groupe suffisant de $\mathbf{R}^d$.

Exemple 2.2. Tous les cheminements sur le démigroupe suffisant $\mathbf{N}^d \subseteq (\mathbf{R}^d, +)$ sont transitoires ($F = 0$, $G = 1$, où $G = G(0,0)$, $G = \lim\limits_{n \to \infty} G_n$). Ce qui donne un reponse au celebre problème du vol des oiseaux!

Proposition 2.3. *Soit $Z \subseteq (\mathbf{R}^d, +)$ un groupe suffisant par raport au cheminement T. Alors, le cheminement est apériodique si et seulement si*

$$\{\theta \in \mathbf{R}^d |\ \Phi(\theta) = 1\} = \{\theta \in \mathbf{R}^d |\ (2\pi)^{-1} x\theta \in \mathbf{Z},\ x \in Z\}.$$

Démonstration. Biensûr, $X := \{\theta \in \mathbf{R}^d |\ (2\pi)^{-1} x\theta \in \mathbf{Z},\ z \in Z\} \subseteq E := \{\theta \in \mathbf{R}^d |\ \Phi(\theta) = 1\} = \left\{\theta \in \mathbf{R}^d |\ \Phi(\theta) = \sum\limits_{x \in Z} T(0,x)\cos(x\theta) = 1\right\}$.

Si $Z = \overline{Z}$ et $\Phi(\theta) = 1$, alors $\sum\limits_{x \in Z} T(0,x)\cos(x\theta) = \sum\limits_{x \in Z} T(-x,0)\cos(x\theta) = \sum\limits_{x \in Z} T(x,0)\cos(x\theta) = \sum\limits_{x \in \Sigma} T(x,0)\cos(x\theta) = 1$, donc $(2\pi)^{-1}x\theta \in \mathbf{Z}$, $x \in \Sigma$. Et $\Phi^n(\theta) = 1 = \sum\limits_{y \in Z} T_n(y,0)\cos(y\theta) = \sum\limits_{T_n(y,0)>0} T_n(y,0)\cos(y\theta) = 1$, donc $(2\pi)^{-1}x\theta \in \mathbf{Z}$, $x \in Z^+$ et $(2\pi)^{-1}x\theta = (2\pi)^{-1}(x_1 - x_2)\theta \in ZZ$, pour $x = x_1 - x_2$, où $x_1, x_2 \in Z^+$, d'où $(2\pi)^{-1}x\theta \in \mathbf{Z}$, $x \in \overline{Z} = Z$.

Reciproquement, si $\overline{Z}$ est un sousgroupe propre de Z, il existe des vecteurs independentes sur $\mathbf{R}$: $a_1, \ldots, a_k \in \overline{Z}$, $1 \leq k \leq d$, avec $\overline{Z} = \mathbf{Z}a_1 \oplus \cdots \oplus \mathbf{Z}a_k$. Si $k \leq d-1$, soit $\{a_1, \ldots, a_k, a_{k+1}, \ldots, a_d\}$ unde base algébrique en $\mathbf{R}^d$ et β un vecteur

3. Moment d'arrêt. Moment d'impact. La fonction H_A. Le problème exterieur de Dirichlet en deux dimensions

On considère l'espace de probabilité $(\Omega_z, \mathcal{F}_z, P_z)$, $z \in Z$, où

1) $\Omega_z = \{z\} \times \mathbf{Z}^{\mathbf{N}^*}$, i.e. $\omega \in \Omega_z$ si et seulement si $\omega_0 = z$, où $\mathbf{N}^* = \mathbf{N} \setminus \{0\}$.

2) $\mathcal{F}_z$ est la plus petite des sous σ-algébres de Ω_z qui contient les cilindres

$$A_n = \{\omega \in \Omega_z | \, \omega_i = a_i, \; i = 1, \ldots, n\}.$$

3) $P_z : \mathcal{F}_z \to [0,1]$ est l'unique mesure de probabilité avec

$$P_z(A_n) = T(z, a_1)T(a_1, a_2)\ldots T(a_{n-1}, a_n)$$

Évidemment,

$$P_z(\omega \in \Omega_z | \, \omega_{i_1} = a_{i_1}, \ldots, \omega_{i_m} = a_{i_m}) = T_{i_1}(z, a_{i_1})T_{i_2-i_1}(a_{i_1}, a_{i_2})\ldots T_{i_m-i_{m-1}}(a_{i_{m-1}}, a_{i_m}).$$

Pour $X_k"\Omega_z \to Z$, $X_k(\omega) = \omega_k$, $k \geq 0$, on note avec S_0^*, S_n^* les fonctions $0, X_1 + \cdots + X_n$ définies sur Ω_z. Dans ce cas-là, $\{S_n^* | \, n \geq 0\}$ est un cheminement aléatoire avec la fonction de transition T.

Proposition 3.1. *a)* $P(z + S_k = y_k; \; k = 1, \ldots, n) = P_z(X_k = y_k; \; k = 1, \ldots, n)$;

b) $P_z(X_{k+1} = X_k + z') = T(0, z')$, $k \geq 0$;

c) $P_z(X_{m+1} = X_m + v; \; X_{n+1} = X_n + t) = T(0, v)T(0, t)$, $m \neq n \geq 0$.

$\{X_n | \, n \geq 0\}$ signifie la translation de vecteur x du cheminement $\{S_n | \, n \geq 0\}$.

Soit $\mathcal{F}_{k,z}$, $k \geq 0$, la plus petite des sous σ-algébres de $\mathcal{F}_z$ qui contient tous les ensembles $\{\omega \in \Omega_z | \, \omega_n = y\}$, $y \in Z$, $k \geq n \geq 0$.

Une application $T : Z \times Z^{\mathbf{N}^*} \to \overline{\mathbf{N}}$, où $\overline{\mathbf{N}} = \mathbf{N} \cup \{+\infty\}$, s'appelle moment d'arrêt si et seulement si

$$\{\omega \in \Omega_z | \, T|_{\Omega_z}(\omega) = k\} \in \mathcal{F}_{k,z}, \; (\forall) \; k \geq 0, \; z \in Z.$$

Pour $A \subseteq Z$, soit $T_A : Z \times Z^{\mathbf{N}^*} \to \overline{\mathbf{N}}$ le moment d'arrêt:

$$T_A(\omega) = \min\{k | \, 1 \leq k \leq \infty : \; X_k(\omega) \in A\}, \quad \omega \in Z \times Z^{\mathbf{N}^*},$$

où $P_z[T_A = +\infty] := 1 - \sum_{k=1}^{\infty} P_z[X_k \in A]$, $z \in Z$, s'appelle le moment d'impact avec A. Pour $A = \{z\}$, on utilise la notation $T_A = T_z$.

Soit $H_A^{(n)} : Z \times A \to [0, \infty)$,

$$H_A^{(n)}(z, z') = \begin{cases} P_z[X_{T_A} = z';\ T_A = n],\ z \in Z \setminus A \\ 0,\ z \in A,\ n \geq 1 \\ \delta(z, z'),\ z \in A,\ n = 0. \end{cases}$$

Soit $H_A : Z \times A \to [0, \infty)$,

$$H_A(z, z') = \begin{cases} P_z[X_{T_A} = z';\ T_A < \infty],\ z \in Z \setminus A \\ \delta(z, z'),\ z \in A. \end{cases}$$

Proposition 3.2. *a)* $H_A^{(n)}(z, z') = \sum_{z_1, \ldots, z_{n-1} \in Z \setminus A} T(z, z_1) \ldots T(z_{n-1}, z')$, $z \in Z \setminus A$, $n \geq 1$;

b) $H_A(z, z') = \sum_{n=1}^{\infty} H_A^{(n)}(z, z')$, $z \in Z \setminus A$.

Proposition 3.3. *a)* $P_x[T_y = n] = F_n(x, y)$;

b) $P_z[T_z < +\infty] = F(z, z)$;

c) $P_z[T_{A+z} = n] = P_0[T_A = n]$, $x, y, z \in Z$.

Démonstration. c)

$$P_z[T_{A+z} = n] = \sum_{\omega_1, \ldots, \omega_{n-1} \notin A+z} T(z, \omega_1) \ldots T(\omega_{n-2}, \omega_{n-1}) \sum_{\omega_n \in A+z} T(\omega_{n-1}, \omega_n)$$

Si $\omega_{k+1} \neq \omega_k + \omega'_{k+1}$, $(\forall)$ $\omega'_{k+1} \in Z$, pour $0 \leq k \leq n-1$, où $\omega_0 = z$, alors $T(\omega_k, \omega_{k+1}) = 0$. Donc

$$P_z[T_{A+z} = n] =$$

$$= \sum_{\left\{\sum_{k=1}^{j} t_k \notin A, 1 \leq j \leq n-1; \sum_{k=1}^{n} t_k \in A\right\}} T(z, z+t_1) \ldots T(z+t_1+\cdots+t_{n-1}, z+t_1+\cdots+t_n) =$$

$$= \sum_{\{\omega_1, \ldots, \omega_{n-1} \notin A; \omega_n \in A\}} T(0, \omega_1) \ldots T(\omega_{n-1}, \omega_n) = P_0[T_A = n], \quad z \in Z.$$

□

Soit $d = 2$, $(Z, +)$ un démigroupe par raport à T et A un sousensemble de Z. On pose le problème:

$$\text{(ProbDirich)} \quad \begin{cases} \sum_{y \in Z} T(x,y) f(y) = f(x), \ x \in Z \setminus A \\ f(x) = \varphi(x), \ x \in A \\ |f(x)| < \infty. \end{cases}$$

Si A est infinie, on ajoute la condition suivante:

$$\text{(infini)} \quad |\varphi(x)| < \infty.$$

Proposition 3.4. *Si T est la fonction de transition d'un cheminement aléatoire arbitraire (récurrent ou transitoire), alors le problème bidimensional exterieur de Dirichlet admet la solution*

$$f(z) = \sum_{y \in B} H_A(z,y) \varphi(y).$$

La solution est unique si:

a) le cheminement est récurrent et apériodique;

b) il y a $y \in A$, avec $Z \setminus A \subset y + \overline{Z}$ et T est récurrent (donc Z n'est pas necessairement un groupe!).

Démonstration. Pour $x \notin Z \setminus A$,

$$\sum_{t \in Z} T(x,t) H_A(t,y) = P_x[X_{T_A} = y;\ T_A = 1] + \sum_{t \in Z \setminus A} \sum_{k=1}^{\infty} T(x,t) P_x[X_{T_A} = y;\ T_A = k] =$$

$$= P_x[X_{T_A} = y;\ T_A = 1] + \sum_{t \in Z \setminus A} \sum_{k=1}^{\infty} P_x[X_1 = t;\ X_{T_A} = y;\ T_A = k+1] =$$

$$= \sum_{k=1}^{\infty} P_x[X_{T_A} = y;\ T_A = k] = P_x[X_{T_A} = y;\ T_A < +\infty] = H_A(x,y), \quad y \in A.$$

Évidemment, $|f(x)| \leq \sup_{y \in A} |\varphi(y)| < +\infty$, donc f est majorée. Si f_1, f_2 sont des solutions du problème de Dirichlet, $h = f_1 - f_2$ le serait aussi pour $\varphi|_A = 0$.

Soit $M > 0$, avec $|h(x)| \leq M$, $x \in Z$, d'où

$$|h(x)| \leq MP_x[T_A > n] \leq MP_x[T_y > n], \quad x \in Z \setminus A,\ y \in A,\ n \geq 1.$$

Á voir [4.2]. Enfin,

$$|h(x)| \leq \lim_{n\to\infty} MP_x[T_y > n] = \lim_{n\to\infty} M\left(F(x,y) - \sum_{k=1}^{n} F_k(x,y) + P_x[T_y = +\infty]\right) =$$
$$= MP_x[T_y = +\infty],$$

en concordance avec b) de la dernière proposition.

Mais $1 = P_x[T_y \in \overline{\mathbf{N}}] = F(x,y) + P_x[T_y = +\infty]$. D'après la condition b),

$$F(x,y) = F(z+y,y) = F(z,0),\ (\forall)\ x \in Z \setminus A,\ x = z+y,\ z \in \overline{Z}.$$

Maintenant,

1) la condition de récurrence a comme consequence

$$F(z,0) = 1,\ z \in \overline{Z},\ \text{d'où } P_x[T_y = +\infty] = 0,\ x \in Z \setminus A.$$

2) la condition d'apériodicité et récurrence implique

$$P_x[T_y = +\infty] = 0 = 1 - P_x[T_y < +\infty] = 1 - F(x,y) = 1 - F(0, y-x) = 0,$$

pour tous les x, y de Z. □

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ON DARBOUX LINES

NEBI ÖNDER AND ABDÜLKADIR ÖZDEĞER

Abstract. In this paper, we first prove that the only surface (other then a sphere) on which the two families of Darboux lines form a Tschebycheff net and the third family of Darboux lines is transversal to one of the two families of Darboux lines is a cylinder of revolution. We next show that the surface (other than a sphere or a developable surface) on which the Darboux lines correspond to those on its parallel surface is a surface of constant mean curvature. Moreover, if the two families of Darboux lines on a surface of constant mean curvature form a Tschebyscheff net, then a surface becomes a cylinder of revolution or a plane. Furthermore, we prove that surfaces (other than a sphere or a plane) whose Darboux lines are preserved under inversion are Dupin's cyclides or, in particular, a pipe surface of revolution. Finally, we show that Molure surface on which the two families of Darboux lines which are different from the lines of curvature form two semi-Tschebyscheff nets together with a family of lines of curvature are either a surface of revolution or a pipe surface.

1. Introduction

Let S be a surface of class C^4 in Euclidean 3-space and let C be a line on S. C is said to be a Darboux line if the relation

$$\mathcal{D} = \frac{d\tau_g}{ds} + (\rho - \overline{\rho}_n)\rho_g = 0 \tag{1.1}$$

holds all along C, where $\mathcal{D}$ is the Darboux's direction function and ρ_n, ρ_g, τ_g and s are, respectively, the normal curvature, the geodesic curvature, the geodesic torsion and the arc-length of C, $\overline{\rho}_n$ being the normal curvature of the orthogonal trajectories of C.

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Let a Darboux line make angles γ^* and γ^{**} ($\gamma^*+\gamma^{**}=\delta$) with the parametric lines $v=$ const. and $u=$ const. respectively. Then, by using the respective generalised Euler, Ossian-Bonnet and Liouville formulae

$$\rho_m = \{r^* \cos\gamma^* \sin\gamma^{**} + r^{**} \sin\gamma^* \cos\gamma^{**} + (t^* - t^{**}) \sin\gamma^* \sin\gamma^{**}\}/\sin\delta$$

$$\tau_g = \{t^* \cos\gamma^* \sin\gamma^{**} + t^{**} \sin\gamma^* \cos\gamma^{**} + (r^{**} - r^*) \sin\gamma^* \sin\gamma^{**}\}/\sin\delta$$

$$\rho_g = \{(g^* + \gamma_1^*) \sin\gamma^{**} + (g^{**} - \gamma_2^{**}) \sin\gamma^*\}/\sin\delta$$

the equation (1.1) can be expressed in the form

$$\begin{aligned}\mathcal{D} = &\left(\frac{t_1^* \sin\gamma^{**} + t_2^*\gamma^*}{\sin\delta}\right)\left(\frac{\cos\gamma^* \sin\gamma^{**}}{\sin\delta}\right) + \\ &+\left(\frac{t_1^{**} \sin\gamma^{**} + t_2^{**} \sin\gamma^*}{\sin\delta}\right)\left(\frac{\sin\gamma^* \cos\gamma^{**}}{\sin\delta}\right) + \\ &+\left[\frac{(r^{**} - r^*)_1 \sin\gamma^{**} + (r^{**} - r^*)_2 \sin\gamma^*}{\sin\delta}\right]\left(\frac{\sin\gamma^* \sin\gamma^{**}}{\sin\delta}\right) + \\ &+\frac{1}{\sin^3\delta}\{(\gamma_1^* \sin\gamma^{**} + \gamma_2^* \sin\gamma^*)[(r^{**} - r^*)\sin^2\gamma^{**} - \\ &-(t^* - t^{**})\cos\gamma^{**}\sin\gamma^{**}] + (\gamma_1^{**} \sin\gamma^{**} + \gamma_2^{**} \sin\gamma^*)[(r^{**} - r^*)\sin^2\gamma^* + \\ &+(t^* - t^{**})\cos\gamma^* \sin\gamma^*]\} + \frac{1}{\sin^2\delta}\{(r^* - r^{**})\sin(\delta - 2\gamma^*) + \\ &+(t^* - t^{**})\cos(\delta - 2\gamma^*)\}\{(g^* + \gamma_1^*)\sin\gamma^{**} + (g^{**} - \gamma_2^{**})\sin\gamma^*\} = 0\end{aligned} \tag{1.2}$$

where $r^*, r^{**}; g^*, g^{**}; t^*, t^{**}$ are, respectively, the normal curvatures, geodesic curvatures and geodesic torsions of the parametric lines and 1, 2 are the indices of the first order invariant derivatives.

The derivative of the differentiable function $f(u,v)$ in the direction of the curve C is

$$\frac{df}{ds} = [(f)_1 \sin\psi + (f)_2 \sin\phi]/\sin\delta,$$

where ψ, ϕ are the angles between the tangent of the curve C and the parametric lines $u=$const., $v=$const., respectively, and s is the arc-length of C; $(f)_1$ and $(f)_2$ being the invariant derivatives of f in the direction of the parametric lines $v=$const., $u=$const.

If the lines of curvature of S are taken as parametric lines, then (1.2) becomes

$$(1.2)' \quad -\overline{r}_1 \sin^3 \varphi + (\overline{r}_2 - 2r_2) \sin^2 \varphi \cos \varphi + (2\overline{r}_1 - r_1) \sin \varphi \cos^2 \varphi + r_2 \cos^3 \varphi = 0$$

where φ is the angle between a Darboux line and the line v =const., and $r, \overline{r}$ are the principal curvatures of S.

A vector field $\vec{u}$ making an angle φ with the unit tangent vector field $\vec{t}$ of the curve C on S will undergo a parallel displacement along C in the sense of Levi-Civita if

$$\frac{d\varphi}{ds} = -\rho_g, \tag{1.3}$$

where ρ_g and s are, respectively, the geodesic curvature and the arc-length of C [1]. We assume that the angle φ is measured in the positive sense around the normal of S from $\vec{t}$ towards $\vec{u}$.

A Tschebyscheff net is by definition a set of parameter curves p =const., q =const., on a surface in terms of which the linear element takes the form

$$ds^2 = dp^2 + 2\cos\varphi dp dq + dq^2$$

or, equivalently, a net is a Tschebyschedd net if and only if the tangent vector field of either family of the net undergoes a parallel displacement in the sense of Levi-Civita along each curve of the other family [2].

A curve C is called a transversal of a vector field, if the vector field undergoes a parallel displacement along C.

The couple $(\mathcal{D}_1, \mathcal{D}_2)$ of two families of curves on S is called a 2-net. A 2-net is said to be a semi-Tschebyscheff net if one of the two families of this net is the transversal of the tangent vector field of the other family.

2. Tschebyscheff nets formed by Darboux lines

In this section surfaces on which the two families of Darboux lines form a Tschebyscheff net and the third family of Darboux lines is transversal to one of the two families of the Tschebyscheff net will be determined.

Let the three families of Darboux lines on the surface S be denoted by $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$. Withput loss of generality, we can take the two families $\mathcal{D}_1, \mathcal{D}2$ as the families of parametric lines v =const., u =const., respectively. In this case, from (1.2) we obtain

$$t_1^* + [(r^* - r^{**}) + (t^* - t^{**})\text{cotg}\delta]g^* = 0 \tag{2.1}$$

$$t_2^{**} + [-(r^* - r^{**}) + (t^* - t^{**})\text{cotg}\delta]g^{**} = 0. \tag{2.2}$$

The conditions for the 2-net $\Delta \equiv (\mathcal{D}_1, \mathcal{D}_2)$ to be a Tschebyscheff net are, according to (1.3)

$$g^{**} - \delta_2 = 0, \quad g^* + \delta_1 = 0. \tag{2.3}$$

On the other hand, the condition for the third family $\mathcal{D}_3$ to be transversal of the tangent vector field to one of the two families $\mathcal{D}_1$ and $\mathcal{D}_2$, to $\mathcal{D}_1$ say, is, according to (1.3)

$$\frac{d\gamma^*}{ds} - (\rho_g)_{\mathcal{D}_3} = 0 \tag{2.4}$$

where $(\rho_g)_{\mathcal{D}_3}$ is the geodesic curvature of the curves belonging to $\mathcal{D}_3$, s being the arc-length of $\mathcal{D}_3$.

The conditions (2.3) and (2.4) give

$$g^* = 0, \quad \delta = \delta(v). \tag{2.5}$$

Then, by using the Gauss equation [3]

$$K = r^*\{r^{**} + t^{**} - t^*)\text{cotg}\delta - t^{*2} =$$

$$= [g_2^* - g_1^* + qg^* - \overline{q}g^{**} + q\delta_1 + \delta_{12}]/\sin\delta$$

where

$$q = \frac{(g^{**} - \delta_2)\cos\delta - (g^* + \delta_1)}{\sin\delta}, \quad \overline{q} = \frac{(g^{**} - \delta_2) - (g^* + \delta_1)\cos\delta}{\sin\delta}$$

we have

$$K = r^*[r^{**} + (t^{**} - t^*)\text{cotg}\delta] - t^{*2} = 0 \tag{2.6}$$

so that S is a developable surface.

Under the conditions (2.5), from (2.1) it follows that

$$t^* = h(v) \tag{2.7}$$

where $h(v)$ is an arbitrary differentiable function of its argument.

Differentiating (2.2) with respect to u and making use of the relation

$$(r^{**} - r^*)\text{cotg}\delta = t^* + t^{**}, \tag{2.8}$$

we obtain

$$t^{**}_{uv} = (2\delta_v \text{cotg} 2\delta)t^{**}_u \tag{2.9}$$

where we have assumed that

(2.8)′ $$\delta \neq \frac{\pi}{2}.$$

For

$$t^{**}_u \neq 0, \tag{2.10}$$

integration of (2.9) gives

(2.10)′ $$t^{**}_u = f(u) \sin 2\delta(v)$$

where $f(u)$ is an arbitrary differentiable function of its argument.

Under the conditions (2.3), (2.4), (2.7), (2.10)', the Mainardi-Codazzi equations [3]

$$r^*_2 = [(t^* - t^{**})(g^{**} - \delta_2)/\sin\delta + \{r^* - r^{**} - (t^* - t^{**})\text{cotg}\delta\}(g^* + \delta_1) + \tag{2.11}$$
$$+ t^*_2 \cos\delta - t^{**}_1]/\sin\delta$$

$$r^{**}_1 = [(t^{**} - t^*)(g^* + \delta_1)/\sin\delta + \{r^* - r^{**} - (t^{**} - t^*)\text{cotg}\delta\})g^{**} - \delta_2) + \tag{2.12}$$
$$+ t^*_2 - t^{**}_1 \cos\delta]/\sin\delta$$

take the respective forms

$$r^*_v = h'(v)\text{cotg}\delta(v) - \frac{f(u)\sin 2\delta(v)}{\sin\delta(v)} \tag{2.13}$$

$$r_u^{**} = \frac{h'(v)}{\sin\delta(v)} - f(u)\sin 2\delta(v)\mathrm{cotg}\delta(v). \tag{2.14}$$

Differentiating (2.8) with respect to u and v and using (2.13), (2.14), (2.10)' and (2.7) we have

$$r_u^* = \frac{h'(v)}{\sin\delta(v)} - 2f(u), \quad r_v^{**} = \frac{2h'(v)}{\sin 2\delta(v)} - 2f(u)\cos\delta(v) + 2t^{**}g^{**}. \tag{2.15}$$

With the help of (2.13) and (2.15), the integrability condition $r_{uv}^* = r_{vu}^*$ for r^* gives

$$\frac{1}{\cos\delta(v)}\frac{d}{dv}\left[\frac{h'(v)}{\sin\delta(v)}\right] = -2f'(u) = A = const. \tag{2.16}$$

Differentiating (2.6) with respect to u and using (2.7), (2.8), (2.10)' and the first equation of (2.15) we get

$$h^2(v)\left[\frac{h'(v)}{\sin\delta(v)} - 2f(u)\right] + r^{*2}\frac{h'(v)}{\sin\delta(v)} = 0 \tag{2.17}$$

or

$$(2.17)' \qquad r^{*2} = -h^2(v)\left[1 - \frac{2f(u)}{h'(v)/\sin\delta(v)}\right]$$

with

$$(2.17)'' \qquad r^* \neq 0, \quad h'(v) \neq 0.$$

Differentiating (2.17)' with respect to u and using (2.15) and (2.16) we obtain

$$r^* = \frac{-(A/2)h^2(v)}{l(v)[l(v) - 2f(u)]}, \quad l(v) = h'(v)/\sin\delta(v) \tag{2.18}$$

where

$$(2.18)' \qquad l(v) - 2f(u) \neq 0.$$

Substituting of r^* in (2.17)' gives

$$\left[\frac{(A/2)^2h^2(v)}{l(v)}\right]^{1/3} - l(v) = -2f(u) = B = const. \tag{2.19}$$

From (2.16) and (2.19) it follows that $r^* = 0$ which contradicts the condition $r^* \neq 0$ in (2.17)".

Now we consider the cases where the conditions (2.18)', (2.17)", (2.10), (2.8)' are not satisfied, namely the cases

I. $h'(v)/\sin\delta(v) - 2f(u) = 0$, II. $h'(v) = 0$, III. $r^* = 0$, IV. $t_u^{**} = 0$, V. $\delta(v) = \pi/2$.

It is easy to see that the cases I and III cannot hold.

In case II, from (2.17) it follows that

$$h^2(v)f(u) = 0. \tag{2.20}$$

If, in (2.20) $f(u) = 0$, then (2.10)' gives $t_u^{**} = 0$ which contradicts (2.10).

If, in (2.20) $h(v) = 0$, then from (2.16) and (2.7) we get $A = 0$, $f(u) = C_1 = const.$ and $t^* = 0$. Substituting of $f(u), h(v)$ and t^* in (2.15), (2.14), (2.13), (2.10)', (2.2) yields

$$r_u^* = -2C_1, \quad r_v^* = -2C_1\cos\delta(v) \tag{2.21}$$

$$r_u^{**} = -2C_1\cos^2\delta(v), \quad r_v^{**} = -2C_1\cos\delta(v) + 2t^{**}g^{**} \tag{2.22}$$

$$t_u^{**} = C_1\sin 2\delta(v), \quad t_v^{**} = 2t^{**}g^{**}\text{cotg}2\delta(v) \tag{2.23}$$

With the help of the condition (2.3), (2.23) gives

$$t^{**} = (C_1 u + C_2)\sin 2\delta(v), \quad (C_1, C_2) = const. \tag{2.24}$$

Then, from (2.24) and (2.22), we obtain

$$r^{**} = -2(C_1 u + C_2)\cos^2\delta(v) - 2C_1\int\cos\delta(v)dv + C_4 \quad (C_4 = const.) \tag{2.25}$$

Substituting the values of r^{**}, t^*, t^{**} in (2.6) and remembering that $r^* \neq 0$, we get

$$2C_1\int\cos\delta(v)dv = C_4$$

from which it follows that $C_1 = C_4 = 0$. In this case, from the first equation of (2.23) we find that $t^{**} = 0$ which contradicts (2.10). Therefore, the case II can not hold.

In case IV, we have

$$t^{**} = k(v) \tag{2.26}$$

where $k(v)$ is an arbitrary differentiable function of its argument. Then, Mainardi-Codazzi equations become

$$r_v^* = h'(v)\text{cotg}\delta(v), \quad r_u^{**} = \frac{h'(v)}{\sin\delta(v)}. \tag{2.27}$$

Differentiating (2.8) with respect to u and v and using (2.27), (2.28), we get

$$r_u^* = \frac{h'(v)}{\sin\delta(v)}, \quad r_v^{**} = \frac{2h'(v)}{\sin 2\delta(v)} + 2k(v)g^{**}. \tag{2.28}$$

By means of (2.28) and (2.29) it follows that

$$r^* = A_1 \int \cos\delta(v)dv + A_1 u + A_2 \quad \left(\frac{h'(v)}{\sin\delta(v)} = A_1 = const.,\ A_2 = const.\right),$$

$$r^{**} = A_1 u + A_3 + \int \left[\frac{A_1}{\cos\delta(v)} + 2k(v)g^{**}(v)\right] dv \quad (A_3 = const.). \tag{2.29}$$

If (2.30), (2.27) and (2.7) are taken into consideration, the Gauss equation (2.6) reduces to

$$A_1^2 u^2 + A_1[\alpha(v) + \beta(v)]u + \alpha(v)\beta(v) - h^2(v) = 0$$

where we have put

$$\alpha(v) = A_2 + A_1 \int \cos\delta(v),$$

$$\beta(v) = A_2 + \int \left[\frac{A_1}{\cos\delta(v)} + 2k(v)g^{**}(v)\right] dv + (k(v) - h(v))\text{cotg}\delta(v),$$

from which it follows that

$$A_1 = 0, \quad \alpha(v)\beta(v) - h^2(v) = 0. \tag{2.30}$$

Combining (2.31) with (2.30) we get

$$h(v) = const., \quad \int 2k(v)g^{**}(v)dv + (k(v) - h(v))\text{cotg}\delta(v) = \frac{B^2}{A_2} - A_3 \tag{2.31}$$

where we have assumed that

$$A_2 \neq 0. \tag{2.32}$$

Differentiating (2.32) with respect to v, we obtain

$$-k(v)g^{**}\cos 2\delta(v) + \frac{1}{2}k'(v)\sin 2\delta(v) + Bg^{**} = 0, \quad (g^{**} = \delta_v)$$

which, by the condition $g^{**} - \delta_v = 0$, gives

$$k(v) = B\cos 2\delta(v) + D\sin 2\delta(v), \quad (D = const.) \tag{2.33}$$

On the other hand, (2.34) and (2.32) give

$$k(v) = B\cos 2\delta(v) + \left(\frac{B^2}{A_2} - A_3\right)\sin 2\delta(v). \tag{2.34)'$$

With the use of (2.34)', (2.32) and (2.31), the equations (2.30), (2.27) and (2.7) become respectively

$$t^* = B, \quad t^{**} = B\cos 2\delta(v) + \left(\frac{B^2}{A_2} - A_3\right)\sin 2\delta(v)$$

$$r^* = A_2, \quad r^{**} = A_3 + B\sin 2\delta(v) - \left(\frac{B^2}{A_2} - A_3\right)\cos 2\delta(v).$$

Then it can be easily seen that one of the principal curvatures is zero while the other one is a constant which means that S is a cylinder of revolution.

We next consider the case where the condition (2.33) is not satisfied. Namely, the case of $A_2 = 0$. Then, from (2.31) we get $B = 0$ by which the equations (2.30), (2.27) and (2.7) reduce to

$$t^* = 0, \quad t^{**} = k(v), \quad r^* = 0, \quad r^{**} = 2\int k(v)g^{**}dv + A_3. \tag{2.34}$$

If these values of t^*, t^{**}, r^*, r^{**} are substituted in (2.8) we obtain

$$\left(2\int k(v)g^{**}dv + A_3\right)\mathrm{cotg}\delta(v) = k(v). \tag{2.35}$$

Differentiating (2.36) with respect to v we get

$$\frac{k'(v)}{k(v)} = 2\delta'(v)\mathrm{cotg}2\delta(v)$$

where $k(v) \neq 0$. (In case of $k(v) = 0$, S is a plane), from which it follows that

$$k(v) = C_1 \sin 2\delta(v) \quad (C_1 = const.)$$

Then, from (2.36) we find that $A_3 = 0$, by which (2.35) becomes

$$t^* = 0, \quad t^{**} = C_1 \sin 2\delta(v), \quad r^* = 0, \quad r^{**} = 2C_1 \sin^2 \delta(v)$$

which means that, S is a cylinder of revolution.

In case V, from (2.1), (2.2), (2.3), (2.5) we obtain

$$g^* = g^{**} = 0, \quad t^* = -t^{**} = A \tag{2.36}$$

by which Mainardi-Codazzi equations reduce to

$$r^* = l(u), \quad r^{**} = m(v), \tag{2.37}$$

where $l(u)$ and $m(v)$ are arbitrary functions of their arguments.

By the use of (2.37) and (2.38), the Gauss equation yields

$$m(v) = \frac{A^2}{l(u)} = B = const. \tag{2.38}$$

where we have assumed that

$$l(u) \neq 0. \tag{2.39}$$

By (2.39), the equations (2.38) become

$$r^* = l(u), \quad r^{**} = B$$

stating that S is a cylinder of revolution or a plane.

Finally, we consider the case where the condition (2.41) is not satisfied. Namely, the case of $l(u) = 0$. In this case, we have $A = 0$. Then the equations (2.38) and (2.37) become

$$t^* = t^{**} = 0, \quad r^* = 0, \quad r^{**} = m(v).$$

With these values of t^*, t^{**}, r^* and r^{**}, (1.2) reduces to

$$m'(v) \sin^2 \gamma^* \sin \gamma^{**} = 0$$

from which it follows that

$$m(v) = const.$$

Therefore, S is a cylinder of revolution or a plane.

Summing up what we have found above we obtain the

Theorem 2.1. *The only surface (other than a plane) on which the two families of Darboux lines form a Tschebycheff net and the third family of Darboux lines is* *transversal to one of the two families of the Tschebycheff net is a cylinder of revolution.*

3. Darboux lines on parallel and inverse surfaces

3.1. Surfaces on which the Darboux lines correspond to those on their parallel surfaces. Let S be a real surface in Euclidean 3-space with vector equation $\vec{x} = \vec{x}(u, v)$. A surface $\overline{S}$ parallel to S is defined by $\vec{\overline{x}} = \vec{x} + C\vec{n}$ ($C = const.$) where $\vec{n}$ is the unit normal vector of S. If the lines of curvature of S are taken as the parametric lines, the coefficients of the first fundamental form of $\overline{S}$ denoted by $\overline{E}, \overline{F}, \overline{G}$ are given by

$$\overline{E} = \left(\frac{C-\alpha}{\alpha}\right)^2 E, \quad \overline{F} = 0, \quad \overline{G} = \left(\frac{C-\beta}{\beta}\right)^2 G \tag{3.1}$$

where E and G are the coefficients of the first fundamental form of S, $\alpha = 1/r$ and $\beta = 1/\overline{r}$ being its the radii of principal curvatures. The principal radii of curvatures, denoted by R and $\overline{R}$, of $\overline{S}$ are given by

$$R = \frac{\varepsilon}{(\alpha - C)}, \quad \overline{R} = \frac{\varepsilon}{(\beta - C)} \quad (\varepsilon = \pm 1) \tag{3.2}$$

If the lines of curvatures of S are taken as the parametric lines, the equation (1.2) becomes

$$-\overline{r}_1 \tan^3\varphi + (\overline{r}_2 - 2r_2)\tan^2\varphi + (2\overline{r}_1 - r_1)\tan\varphi + r_2 = 0. \tag{3.3}$$

Using the fact that $\tan\varphi = (G/E)^{1/2}\dfrac{dv}{du}$, (3.3) takes the form

$$-\overline{r}_u G^2(dv)^3 + (\overline{r}_v - 2r_v)EG(dv)^2 du + (2\overline{r}_u - r_u)EGdv(du)^2 + r_v E^2(du)^3 = 0. \tag{3.4}$$

Using this equation and remembering that the lines of curvature on S and $\overline{S}$ correspond, the differential equation of the Darboux lines of $\overline{S}$ is obtained in the form

$$-\overline{R}_u\overline{G}^2(dv)^3 + (\overline{R}_v - 2R_v)\overline{E}\,\overline{G}(dv)^2 du + (2\overline{R}_u - R_u)\overline{E}\,\overline{G}dv(du)^2 + R_v\overline{E}^2(du)^3 = 0. \tag{3.5}$$

The Darboux lines of S and $\overline{S}$ will correspond to each other, if and only if the respective coefficients of the equations (3.4) and (3.5) are proportional. Then, with the help of (3.1) and (3.2), from (3.4) and (3.5) it follows that

$$\frac{\beta^2}{(\beta-C)^2}=\frac{-\beta_v\alpha^2+2\alpha_v\beta^2}{-\beta_v(\alpha-C)^2+2\alpha_v(\beta-C)^2}=\frac{-2\beta_u\alpha^2+\alpha_u\beta_2}{-2\beta_u(\alpha-C)^2+\alpha_u(\beta-C)^2}=\frac{\alpha^2}{(\alpha-C)^2} \tag{3.6}$$

where we have assumed that

$$(3.6)' \qquad \alpha_v\neq 0,\quad \beta_u\neq 0.$$

From (3.6) we get

$$\beta_v(\beta-\alpha)(1-CW)=0,\quad (\beta-\alpha)(1-CW)(\alpha_u\beta_v-4\alpha_v\beta_u)=0 \tag{3.7}$$

$$(\beta-\alpha)(1-CW)=0,\quad \alpha_u(\beta-\alpha)(1-CW)=0 \tag{3.8}$$

where W is the mean curvature of S and $K\neq 0$.

In the case of $K=0$, the Darboux lines of S and $\overline{S}$ do correspond.

If S is neither a sphere nor a developable surface, from (3.7) and (3.8), we get $1-CW=0$ which means that S is a surface of constant mean curvature.

Now we consider the cases where the conditions in (3.6)' are not satisfied, namely

$$I.\quad \beta_u=0,\qquad II.\quad \alpha_v=0.$$

In case I, by (3.1) and (3.2), equations (3.4) and (3.5) take the respective forms

$$Edu\left\{\left[-\frac{\beta_v}{\beta^2}+2\frac{\alpha_v}{\alpha^2}\right]G(dv)^2+\frac{\alpha_u}{\alpha^2}Gdvdu-\frac{\alpha_v}{\alpha^2}E(du)^2\right\}=0 \tag{3.9}$$

$$\overline{E}du\left\{\left[-\frac{\beta_v}{(\beta-C)^2}+\frac{2\alpha_v}{(\alpha-C)^2}\right]\overline{G}(dv)^2+\frac{\alpha_u}{(\alpha-C)^2}\overline{G}dvdu-\frac{\alpha_v}{(\alpha-C)^2}\overline{E}(du)^2\right\}=0 \tag{3.10}$$

showing that one of the families of Darboux lines on S and $\overline{S}$ coincides with the family of lines of curvature $u = const$. The other two families of Darboux lines on S and $\overline{S}$ are given by

$$\left[-\frac{\beta_v}{\beta^2} + 2\frac{\alpha_v}{\alpha^2}\right] G(dv)^2 + \frac{\alpha_u}{\alpha^2} Gdvdu - \frac{\alpha_v}{\alpha^2} E(du)^2 = 0,$$

$$\left[-\frac{\beta_v}{(\beta - C)^2} + \frac{2\alpha_v}{(\alpha - C)^2}\right] \overline{G}(dv)^2 + \frac{\alpha_u}{(\alpha - C)^2} \overline{G}dvdu - \frac{\alpha_v}{(\alpha - C)^2} \overline{E}(du)^2 = 0.$$

These lines will be in correspondence provided that

$$\beta_v(\beta - \alpha)(1 - CW) = 0 \tag{3.11}$$

$$(\beta - \alpha)(1 - CW) = 0 \tag{3.12}$$

$$\alpha_u \neq 0, \quad \alpha_v \neq 0 \tag{3.13}$$

By (3.14) we get $1 - CW = 0$ from which it follows that $\alpha_u = 0$ contradicting with (3.15).

Next we consider the cases where the conditions in (3.15) are not satisfied. Namely,

$$I'. \quad \alpha_i = 0, \qquad II''. \quad \alpha_v = 0.$$

In case I', using (3.1) and (3.2), from (3.4) and (3.5) we find that

$$\left[-\frac{\beta_v}{\beta^2} + 2\frac{\alpha_v}{\alpha^2}\right] G(dv)^2 - \frac{\alpha_v}{\alpha^2} E(du)^2 = 0, \tag{3.14}$$

$$\left[-\frac{\beta_v}{(\beta - C)^2} + \frac{2\alpha_v}{(\alpha - C)^2}\right] \overline{G}(dv)^2 - \frac{\alpha_v}{(\alpha - C)^2} \overline{E}(du)^2 = 0. \tag{3.15}$$

Since the coefficients of (3.16) and (3.17) must be proportional, we obtain

$$(\beta - \alpha)(1 - CW) = 0. \tag{3.16}$$

(3.18) gives $1 - CW = 0$ for $\alpha \neq \beta$ showing that S is a surface of revolution of constant mean curvature other then a sphere.

In case II", we have $\beta_u = 0, \ \alpha_v = 0$ implying that S is a cylinder of revolution or a plane.

In case II, we apply the same reasoning as above and obtain the same results included in case I.

We therefore obtain the

Theorem 3.1. *Let $\overline{S}$ be a surface parallel to S and suppose that S is neither a sphere nor a developable surface. If the Darboux lines of S and $\overline{S}$ correspond to each other, then S is a surface of constant mean curvature.*

3.1.1. *Surfaces of constant mean curvature on which the two families of Darboux lines form a Tschebycheff net.* It is well known that [4] the Darboux lines of a surface of constant mean curvature other than a sphere, a cylinder of revolution or a plane, cut each other under an angle of 120°.

Suppose that the lines of curvature on S are taken as the parametric lines. Let the lines of the two families $\mathcal{D}_1$ and $\mathcal{D}_2$ of Darboux lines make, respectively, the angles φ and ϕ with the parametric line v =const. Then $\varphi - \phi = 120°$.

The conditions for the 2-net $\Delta \equiv (\mathcal{D}_1, \mathcal{D}_2)$ to be a Tschebycheff net are, according to (1.3),

$$\frac{d(\varphi - \phi)}{ds_2} = B - (\rho_g)_2, \quad \frac{d(\phi - \varphi)}{ds_1} = -(\rho_g)_1 \tag{3.17}$$

where $s_1, s_2; (\rho_g)_1, (\rho_g)_2$ are the arc-lengths and the geodesic curvatures of $\mathcal{D}_1$ and $\mathcal{D}_2$ given by

$$(\rho_g)_1 = \varphi_1 \cos\varphi + \varphi_2 \sin\varphi + g\cos\varphi + \overline{g}\sin\varphi = 0, \tag{3.18}$$

$$(\rho_g)_2 = \phi_1 \cos\phi + \phi_2 \sin\phi + g\cos\phi + \overline{g}\sin\phi = 0. \tag{3.19}$$

Using the fact that $\varphi - \phi = 120°$, by (3.20) and (3.21), (3.19) becomes

$$[\varphi_1 + g]\cos\varphi + [\varphi_2 + \overline{g}]\sin\varphi = 0,$$

$$[\varphi_1 + g]\cos(\varphi + 120°) + [\varphi_2 + \overline{g}]\sin(\varphi + 120°) = 0$$

from which it follows that

$$\varphi_1 + g = 0, \quad \varphi_2 + \overline{g} = 0. \tag{3.20}$$

With the help of (3.22), the integrability condition $\varphi_{12} - \varphi_{21} = g\varphi_1 + \overline{g}\varphi_2$ for φ gives

$$g_2 - \overline{g}_1 - g^2 - \overline{g}^2 = K = 0$$

which means that S is a developable surface. On the other hand, since $2W = \dfrac{1}{\alpha} + \dfrac{1}{\beta} = r + \overline{r} =$const., S is a cylinder of revolution or a plane.

Thus, we obtain the

Theorem 3.2. *A surface of constant mean curvature on which the two families of Darboux lines form a Tschebycheff net is a cylinder of revolution or a plane.*

3.2. **Surfaces on which the Darboux lines are preserved by inversion.** Let the surface S be given by the vector equation $\vec{x} = \vec{x}(u,v)$ and let S^* be its inverse. Then S^* is defined by the equation $\vec{x}^* = \frac{c^2}{x^2}\vec{x}$, c being the radius of inversion. Denoting by E, F, G and E^*, F^*, G^* the coefficients of the first fundamental forms of S and S^* respectively, we have

$$E^* = \frac{c^4}{x^4}E, \quad F^* = \frac{c^4}{x^4}F, \quad G^* = \frac{c^4}{x^4}G. \tag{3.21}$$

If the lines of curvatures on S are taken as parametric lines, the principal curvatures, denoted by r^* and $\overline{r}^*$ of S^* are given by

$$r^* = -\frac{1}{c^2}(rx^2 + 2p), \quad \overline{r}^* = -\frac{1}{c^2}(\overline{r}x^2 + 2p) \tag{3.22}$$

where p is the perpendicular distance, measured in the sense of the unit normal vector of S, from the centre of inversion to the tangent plane of S at the point considered. The quantities $r, \overline{r}, p, x$ are related by

(3.24)′
$$2p_u = -(x^2)_u r, \quad 2p_v = -(x^2)_v \overline{r}.$$

Using the fact that $\tan\varphi = (G/E)^{1/2}\dfrac{dv}{du}$, (1.2) takes the form

$$-\overline{r}_u G^2(dv)^3 + (\overline{r}_v - 2r_v)EG(dv)^2du + (2\overline{r}_u - r_u)EGdv(du)^2 + r_v E^2(du)^3 = 0. \tag{3.23}$$

Since the lines of curvature on S and S^* correspond, the differential equation of the Darboux lines of S^* is obtained in the form

$$-\overline{r}_u^* G^{*2}(dv)^3 + (\overline{r}_v^* - 2r_v^*)E^*G^*(dv)^2 du + \tag{3.24}$$

$$+(2\overline{r}_u^* - r_u^*)E^*G^* dv(du)^2 + r_v^* E^{*2}(du)^3 = 0.$$

The Darboux lines of S and S^* will correspond, if and only if the respective coefficients of the equations (3.25) and (3.26) are proportional. Consequently, by using (3.23), (3.24), (3.24)', from (3.25) and (3.26) we get

$$2\overline{r}_u(x^2)_v - (\overline{r}_v - 2r_v)(x^2)_u = 0, \tag{3.25}$$

$$r_u(x^2)_u = 0, \quad \overline{r}_v(x^2)_v = 0, \tag{3.26}$$

$$\overline{r}_u(x^2)_v + r_v(x^2)_u = 0, \tag{3.27}$$

$$(\overline{r}_v - 2r_v)(x^2)_u - (x^2)_v(2\overline{r}_u - r_u) = 0, \tag{3.28}$$

$$2r_v(x^2)_u + (2\overline{r}_u - r_u)(x^2)_v = 0 \tag{3.29}$$

from which it follows that

$$r_u = \overline{r}_v = 0, \tag{3.30}$$

$$\overline{r}_u(x^2)_v + r_v(x^2)_u = 0 \tag{3.31}$$

where we have assumed that

$$(x^2)_u \neq 0, \quad (x^2)_v \neq 0. \tag{3.32}$$

It is well known that the conditions (3.32) characterize the Dupin's Cyclides [5].

The general solution of (3.33) is

$$x^2 = \phi(r - \overline{r})$$

which means that the curves $r - \overline{r}$ =const. are spherical.

If, in (3.34), $(x^2)_u = 0$, $(x^2)_v = 0$ it is clear that S is a sphere.

Now suppose that one of the quantities $(x^2)_u$ and $(x^2)_v$, say $(x^2)_u$ is zero. Then, we find that

$$\bar{r} = const., \quad r_u = 0, \quad \bar{g} = 0$$

showing that S is a pipe surface of revolution.

We therefore obtain the

Theorem 3.3. *If the Darboux lines of the surfaces S (other then a sphere) and S^* correspond to each other, then S belongs to one of the following two classes:*

1. S is a Dupin's cyclide and the curves $r - \bar{r}$ =const., are spherical,

2. S is a pipe surface of revolution.

3.3. **Molure surfaces admitting two semi-Tschebycheff nets formed by Darboux lines.** In this section, we consider Molure surfaces which include pipe surfaces as a special case and are characterized by the condition $\bar{r}_1 = 0$ ($\bar{r} = f(v)$, $f(v)$ being a differentiable function of its argument).

It is easy to see from (3.4) that one of the families of Darboux lines on a Molure surface coincides with the family of lines of curvature u =const. Then the other two families of Darboux lines are given, according to (3.3), by

$$\tan^2\varphi - \frac{r_1}{\bar{r}_2 - 2r_2}\tan\varphi + \frac{r_2}{\bar{r}_2 - 2r_2} = 0 \tag{3.33}$$

where we have assumed that

$$\bar{r}_2 - 2r_2 \neq 0. \tag{3.35)'$$

Let the two families of Darboux lines make, respectively the angles ϕ and ψ with the parametric line v =const.

The conditions for the family $\mathcal{D}_1$: u =const. to be transversal of the tangent vector fields of the other two families are, according to (1.3),

$$\left(\frac{\pi}{2} - \phi_1\right)_2 = \bar{g}, \quad \left(\frac{\pi}{2} - \psi_2\right)_2 = \bar{g}. \tag{3.34}$$

Since the geodesic curvature $\bar{g}$ of the lines of curvature u =const. is zero, by a suitable choice of the parameter v, we can make $G = 1$. Then, (3.36) becomes

$$\phi_2 = \psi_2 = 0. \tag{3.36)'$$

Taking the invariant derivative of (3.35) in the direction of u =const., and using (3.36)' and replacing φ by ϕ and ψ we respectively get

$$-\left(\frac{r_1}{f'(v)-2r_2}\right)_2 \tan\phi+\left(\frac{r_2}{f'(v)-2r_2}\right)_2=0, \quad -\left(\frac{r_1}{f'(v)-2r_2}\right)_2 \tan\psi+\left(\frac{r_2}{f'(v)-2r_2}\right)_2$$

from which it follows that

$$\frac{r_1}{f'(v)-2r_2}=a(u), \quad \frac{r_2}{f'(v)-2r_2}=b(u) \tag{3.35}$$

where $a(u), b(u)$ are arbitrary differentiable functions of their arguments.

From (3.37), it follows that

$$r_1=\frac{f'(v)a(u)}{1+2b(u)}, \quad r_2=\frac{f'(v)b(u)}{1+2b(u)} \tag{3.36}$$

where we have assumed that

$$(3.38)' \qquad 1+2b(u)\neq 0.$$

Integration of the second equation in (3.38) we obtain

$$r=\frac{f(v)b(u)}{1+2b(u)}+C(u) \tag{3.37}$$

$C(u)$ being an arbitrary differentiable function of its argument.

Substituting r in (3.38) and assuming that

$$f'(v)\neq 0, \quad a(u)\neq 0 \tag{3.38}$$

we get

$$(3.40)' \qquad \sqrt{E}=\left(\frac{f(v)}{f'(v)}\right)U(u)+\left(\frac{1}{f'(v)}\right)\overline{U}(u)$$

where

$$(3.40)'' \qquad U(u)=\frac{b'(u)}{a(u)[1+2b(u)]}, \quad \overline{U}(u)=\frac{C'(u)[1+2b(u)]}{a(u)}.$$

With the help of (3.39) and (3.38) the Mainardi-Codazzi equation $r_2=(r-\overline{r})g$ gives

$$[f(v)R(u)-C(u)]\left[\left(\frac{f(v)}{f'(v)}\right)U(u)+\left(\frac{1}{f'(v)}\right)'\overline{U}(u)\right]= \tag{3.39}$$

$$=[1-R(u)][f(v)U(u)-\overline{U}(u)]$$

where

$$R(u) = \frac{1+b(u)}{1+2b(u)}. \tag{3.41)'$$

Differentiating (3.41) with respect to v, we obtain

$$[X(v)R(u) - Y(v)C(u)]U(u) + [Z(v)R(u) - T(v)C(u)]\overline{U}(u) = [1 - R(u)]U(u) \tag{3.40}$$

where we have put

$$X(v) = \frac{\left[f(v)\left(\frac{f(v)}{f'(v)}\right)'\right]'}{f'(v)}, \quad Y(v) = \frac{\left(\frac{f(v)}{f'(v)}\right)''}{f'(v)}, \tag{3.42)'$$

$$Z(v) = \frac{\left[f(v)\left(\frac{1}{f'(v)}\right)'\right]'}{f'(v)}, \quad T(v) = \frac{\left(\frac{1}{f'(v)}\right)''}{f'(v)}.$$

Differentiation of (3.42) with respect to v and division through by

$$T'(v) \neq 0 \tag{3.41}$$

gives the equation

$$[\overline{X}(v)R(u) - \overline{Y}(v)C(u)]U(u) + [\overline{z}(v)R(u) - C(u)]\overline{U}(u) = 0 \tag{3.43)'$$

where

$$\overline{X}(v) = \frac{X'(v)}{T'(v)}, \quad \overline{Y}(v) = \frac{Y'(v)}{T'(v)}, \quad \overline{Z}(v) = \frac{Z'(v)}{T'(v)} \quad (T'(v) \neq 0). \tag{3.43)''$$

Differentiating (3.43)' with respect to v and dividing the resulting equation by

$$\overline{Z}' \neq 0 \tag{3.42}$$

we get

$$\frac{\overline{X}'(v)}{\overline{Z}'(v)}U(u)R(u) - \frac{\overline{Y}'(v)}{\overline{Z}'(v)}U(u)C(u) + \overline{U}(u)R(u) = 0. \tag{3.44)'$$

Finally, differentiation of (3.44)' with respect to v we have

$$U(u)\left[\left(\frac{\overline{X}'(v)}{\overline{Z}'(v)}\right)' R(u) - \left(\frac{\overline{Y}'(v)}{\overline{Z}'(v)}\right)' C(u)\right] = 0. \tag{3.43}$$

If, in (3.45) $U(u) = 0$, by (3.44)', (3.43)' and (3.40), it follows that $E = 0$ which is not possible.

For $U(u) \neq 0$, (3.45) becomes

$$(3.45)' \qquad \frac{\left(\dfrac{\overline{X}'(v)}{\overline{Z}'(v)}\right)'}{\left(\dfrac{\overline{Y}'(v)}{\overline{Z}'(v)}\right)'} = \frac{C(u)}{R(u)} = d = const.$$

where we have assumed that

$$(3.45)'' \qquad \left(\frac{\overline{Y}'(v)}{\overline{Z}'(v)}\right)' \neq 0, \quad R(u) \neq 0.$$

By (3.43)", (3.45)' gives

$$X(v) = dY(v) + eZ(v) + kT(v) + l, \quad C(u) = dR(u)$$

where e, k, l are arbitrary constants. Substitution of $X(v)$ and $C(u)$ in (3.44)' and (3.43)', by (3.42), we obtain

$$eU + \overline{U} = 0, \quad k + ed = 0, \quad (1+l)\frac{1+b(u)}{1+2b(u)} = 1$$

from which it follows that $b(u)$ =const. Then, from (3.40)" we get $U(u) = 0$ which can not be the case.

Now, we consider the case where the conditions (3.45) are not satisfied. Here, we distinguish two cases:

$$1a. \quad \left(\frac{\overline{Y}'(v)}{\overline{Z}'(v)}\right)' = 0, \quad R(u) \neq 0; \qquad 1b. \quad R(u) = 0, \quad \left(\frac{\overline{Y}'(v)}{\overline{Z}'(v)}\right)' \neq 0$$

In case 1a, by (3.45) and (3.43)' we obtain

$$Y(v) = C_1 Z(v) + C_2 T(v) + C_3, \tag{3.44}$$

$$X(v) = C_4 Z(v) + C_5 T(v) + C_6 \tag{3.45}$$

where $C_1, C_2, C_3, C_4, C_5, C_6$ are arbitrary constants. Substitution of $X(v)$, $Y(v)$ in (3.44)', (3.43)' and (3.42) we have

$$[C)4R(u) - C_1C(u)]U(u) + \overline{U}(u)R(u) = 0, \tag{3.46}$$

$$[C_5R(u) - C_2C(u)]U(u) - \overline{U}(u)C(u) = 0, \tag{3.47}$$

$$(C_6 + 1)R(u) - C_3C(u) = 1. \tag{3.48}$$

Using (3.42)', from (3.46) and (3.47) we get

$$f(v)\left(\frac{f(v)}{f'(v)}\right)' = C_4f(v)\left(\frac{1}{f'(v)}\right)' + C_5\left(\frac{1}{f'(v)}\right)' + C_6f(v) + C_7, \tag{3.49}$$

$$\left(\frac{f(v)}{f'(v)}\right)' = C_1f(v)\left(\frac{1}{f'(v)}\right)' + C_2\left(\frac{1}{f'(v)}\right)' + C_3f(v) + C_8 \tag{3.50}$$

C_7 and C_8 being arbitrary constants. Combining (3.51) and (3.52), for

(3.52)′ $$(C_1 - 1)f(v) + C_2 \neq 0,$$

we obtain

$$C_3f^3(v) - [(1 - C_8) - (C_1 - 1)(C_6 - 1) + C_3C_4]f^2(v) -$$
$$-[C_3C_5 - C_7(C_1 - 1) - C_2(C_6 - 1) - C_4(1 - C_8)]f(v) +$$
$$+[C_5(1 - C_8) - C_2(1 - C_6) + C_2C_7] = 0.$$

Since $f(v) \neq$const., this equation gives

$$C_3 = 0, \quad C_7(C_1 - 1) + C_2(C_6 - 1) + C_4(1 - C_8) = 0,$$
$$(1 - C_8) - (C_1 - 1)(C_6 - 1) = 0,$$
$$C_5(1 - C_8) - C_2(1 - C_6) + C_2C_7 = 0.$$

With the help of these relations, from (3.50), (3.41)' and (3.40)" it follows that $U(u) = 0$ which is not possible.

Suppose now that the condition (3.52)' is not satisfied, i.e. $(C_1 - 1)f(v) + C_2 = 0$. We then have $C_1 = 1$, $C_2 = 0$ so that the equation (3.52) reduces to $(1 - C_8) - C_3f(v) = 0$. But this gives $C_8 = 1$, $C_3 = 0$. Under these conditions $U(u)$ becomes zero which cannot be the case.

In case 1b, by (3.41)' and (3.40) we find that $U(u) = 0$ which is impossible. Therefore the cases 1a and 1b can not hold.

If the condition (3.43) or (3.44) is not satisfied considerations similar to that given above show that these two cases cannot hold.

We next consider the case where (3.40) is not satisfied. Then, either 1°. $f'(v) = 0,\ a(u) = 0$ or 2°. $f'(v) \neq 0,\ a(u) = 0$. In case 1°, from (3.38) it follows that r =const. from which we obtain $\overline{r}_2 - r_2 = 0$ which contradicts (3.35)'. In case 2°, from (3.38) and (3.39) we get

$$f(v) = \frac{C'(u)(1+2b(u))^2}{b'(u)} = \overline{D}_1 = const., \quad b'(u) \neq 0 \tag{3.51}$$

Then, by (3.38), we obtain $\overline{r}_2 = r_2 = 0$ which contradicts (3.35)'.

If, in (3.53) $b'(u) = 0$, then from (3.38) and (3.39) we have $C(u)$ =const., so that

$$r = r(v), \quad \overline{r} = \overline{r}(v)$$

which shows that S is a surface of revolution.

Finally, suppose that the condition $1 + 2b(u) \neq 0$ in (3.38)' is not satisfied. In this case, from (3.37), it follows that $f'(v) = 0$ which means that S is a pipe surface [5].

We therefore obtain the

Theorem 3.4. *If the two families of Darboux lines different from the lines of curvature on a Molure surface form two semi-Tschebycheff nets together a family of lines of curvature, then such a surface is either a surface of revoluion or a pipe surface.*

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ON WIRTINGER AND OPIAL TYPE INEQUALITIES IN THREE INDEPENDENT VARIABLES

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Abstract. In this paper we establish some new integral and discrete inequalities of Wirtinger and Opial type involving functions of three independent variables. The analysis used in the proofs is elementary and our results provide new estimates on inequalities of this type.

1. Introduction

The inequalities of Wirtinger and Opial type and their variants have played a vital role in the study of many qualitative as well as quantitative properties of solutions of differential equations. Because of their usefulness and importance these inequalities have received a wide attention and a large number of papers have appeared in the literature. During the past few years, various investigators have discovered many useful and new Wirtinger and Opial type inequalities involving functions of more than one independent variables, see [1-16] and the references given therein. The main purpose of the present paper is to establish some new integral and discrete inequalities of the Wirtinger and Opial type involving functions of three independent variables. An important feature of the inequalities established in this paper is that the analysis used in their proofs is elementary and our results provide new estimates on this type of inequalities.

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2. Integral inequalities

In what follows R denotes the set of real numbers. We use the notation $E = [a,x] \times [b,y] \times [c,z]$ for $a,b,c,x,y,z \in R$. If $f(r,s,t)$ is a differentiable function defined on E, then its partial derivatives are denoted by $D_1 f(r,s,t) = \frac{\partial}{\partial r} f(r,s,t)$, $D_2 f(r,s,t) = \frac{\partial}{\partial s} f(r,s,t)$, $D_3 f(r,s,t) = \frac{\partial}{\partial t} f(r,s,t)$, and

$$D_3 D_2 D_1 f(r,s,t) = \frac{\partial^3}{\partial t \partial s \partial r} f(r,s,t).$$

We denote by $F(E)$ the class of continuous functions $f : E \to R$ for which

$$D_1 f(r,s,t), D_2 f(r,s,t), D_3 f(r,s,t), D_3 D_2 D_1 f(r,s,t)$$

exist and continuous on E such that $f(a,s,t) = f(x,s,t) = f(r,b,t) = f(r,y,t) = f(r,s,c) = f(r,s,z) = 0$ for $a \le r \le x,\ b \le s \le y,\ c \le t \le z$.

Our main result on Wirtinger type integral inequality involving functions of three independent variables is given in the following theorem.

Theorem 1. *Let $p(r,s,t)$ be a real-valued nonnegative continuous function defined on E. Suppose that $f_i \in F(E)$ for $i = 1,2,\ldots,n$, and let $m_i \ge 1$ for $i = 1,2,\ldots,n$ are constants. Then*

$$\int_a^x \int_b^y \int_c^z p(r,s,t) \left[\prod_{i=1}^n |f_i(r,s,t)|^{m_i} \right]^{2/n} dtdsdr \le \tag{2.1}$$

$$\le \frac{1}{n} K(a,b,c,x,y,z,n,m_1,\ldots,m_n) \left(\int_a^x \int_b^y \int_c^z p(r,s,t) dtdsdr \right) \times$$

$$\times \int_a^x \int_b^y \int_c^z \sum_{i=1}^n |D_3 D_2 D_1 f_i(r,s,t)|^{2m_i} dtdsdr,$$

where

$$K(a,b,c,x,y,z,n,m_1,\ldots,m_n) = \left(\frac{1}{8}\right)^{\frac{2}{n}\sum_{i=1}^n m_i} [(x-a)(y-b)(z-c)]^{1+\frac{2}{n}\sum_{i=1}^n (m_i - 1)}, \tag{2.2}$$

is a constant depending on $a,b,c,x,y,z,n,m_1,\ldots,m_n$.

Proof. From the hypothesis, it is easy to observe that the following identities hold for $i = 1, 2, \ldots, n$ and $(r, s, t) \in E$:

$$f_i(r,s,t) = \int_a^r \int_b^s \int_c^t D_3D_2D_1 f_i(u,v,w)\,dw\,dv\,du, \tag{2.3}$$

$$f_i(r,s,t) = -\int_a^r \int_b^s \int_t^z D_3D_2D_1 f_i(u,v,w)\,dw\,dv\,du, \tag{2.4}$$

$$f_i(r,s,t) = -\int_a^r \int_s^y \int_c^t D_3D_2D_1 f_i(u,v,w)\,dw\,dv\,du, \tag{2.5}$$

$$f_i(r,s,t) = -\int_r^x \int_b^s \int_c^t D_3D_2D_1 f_i(u,v,w)\,dw\,dv\,du, \tag{2.6}$$

$$f_i(r,s,t) = \int_a^r \int_s^y \int_t^z D_3D_2D_1 f_i(u,v,w)\,dw\,dv\,du, \tag{2.7}$$

$$f_i(r,s,t) = \int_r^x \int_s^y \int_c^t D_3D_2D_1 f_i(u,v,w)\,dw\,dv\,du, \tag{2.8}$$

$$f_i(r,s,t) = \int_r^x \int_b^s \int_t^z D_3D_2D_1 f_i(u,v,w)\,dw\,dv\,du, \tag{2.9}$$

$$f_i(r,s,t) = -\int_r^x \int_s^y \int_t^z D_3D_2D_1 f_i(u,v,w)\,dw\,dv\,du. \tag{2.10}$$

From (2.3)-(2.10) it is easy to observe that

$$|f_i(r,s,t)| \le \frac{1}{8} \int_a^x \int_b^y \int_c^z |D_3D_2D_1 f_i(u,v,w)|\,dw\,dv\,du, \tag{2.11}$$

for $i = 1, 2, \ldots, n$ and $(r, s, t) \in E$. From (2.11) and using the Hölder's integral inequality in three dimensions with indices m_i and $m_i/(m_i - 1)$ for $i = 1, 2, \ldots, n$ we obtain

$$|f_i(r,s,t)|^{m_i} \le \left(\frac{1}{8}\right)^{m_i} [(x-a)(y-b)(z-c)]^{m_i - 1} \times \tag{2.12}$$

$$\times \int_a^x \int_b^y \int_c^z |D_3D_2D_1 f_i(u,v,w)|^{m_i}\,dw\,dv\,du.$$

From (2.12) and using the elementary inequalities (see [4])

$$\left(\prod_{i=1}^{n} b_i\right)^{1/n} \leq \frac{1}{n}\sum_{i=1}^{n} b_i, \tag{2.13}$$

(for $b_1, b_2, \ldots, b_n$ nonnegative reals and $n \geq 1$) and

$$\left(\sum_{i=1}^{n} b_i\right)^2 \leq n\sum_{i=1}^{n} b_i^2, \tag{2.14}$$

(for $b_1, b_2, \ldots, b_n$ reals) and Schwarz integral inequality in three dimensions we observe that

$$\left[\prod_{i=1}^{n} |f_i(r,s,t)|^{m_i}\right]^{2/n} \leq \tag{2.15}$$

$$\leq \left[\prod_{i=1}^{n} \left(\frac{1}{8}\right)^{m_i} [(x-a)(y-b)(z-c)]^{m_i-1} \times \right.$$

$$\left. \times \int_a^x \int_b^y \int_c^z |D_3D_2D_1 f_i(u,v,w)|^{m_i} dw dv du\right]^{2/n} =$$

$$= \left(\frac{1}{8}\right)^{\frac{2}{n}\sum_{i=1}^{n} m_i} [(x-a)(y-b)(z-c)]^{\frac{2}{n}\sum_{i=1}^{n}(m_i-1)} \times$$

$$\times \left[\prod_{i=1}^{n} \int_a^x \int_b^y \int_c^z |D_3D_2D_1 f_i(u,v,w)|^{m_i} dw dv du^{1/n}\right]^2 \leq$$

$$\leq \left(\frac{1}{8}\right)^{\frac{2}{n}\sum_{i=1}^{n} m_i} [(x-a)(y-b)(z-c)]^{\frac{2}{n}\sum_{i=1}^{n}(m_i-1)} \times$$

$$\times \left[\frac{1}{n}\sum_{i=1}^{n} \int_a^x \int_b^y \int_c^z |D_3D_2D_1 f_i(u,v,w)|^{m_i} dw dv du\right]^2 \leq$$

$$\leq \left(\frac{1}{8}\right)^{\frac{2}{n}\sum_{i=1}^{n} m_i} [(x-a)(y-b)(z-c)]^{\frac{2}{n}\sum_{i=1}^{n}(m_i-1)} \times$$

$$\times \frac{1}{n^2} n \sum_{i=1}^{n} \left[\int_a^x \int_b^y \int_c^z |D_3D_2D_1 f_i(u,v,w)|^{m_i} dw dv du\right]^2 \leq$$

$$\leq \left(\frac{1}{8}\right)^{\frac{2}{n}\sum_{i=1}^{n} m_i} [(x-a)(y-b)(z-c)]^{\frac{2}{n}\sum_{i=1}^{n}(m_i-1)} \times$$

$$\times \frac{1}{n}[(x-a)(y-b)(z-c)] \sum_{i=1}^{n} \int_a^x \int_b^y \int_c^z |D_3D_2D_1 f_i(u,v,w)|^{2m_i} dw dv du =$$

$$= \frac{1}{n} K(a,b,c,x,y,z,n,m_1,\dots,m_n) \times \int_a^x \int_b^y \int_c^z \sum_{i=1}^n |D_3 D_2 D_1 f_i(u,v,w)|^{2m_i} dw dv du.$$

Multiplying both sides of (2.15) by $p(r,s,t)$ and integrating the resulting inequality over E we get the desired inequality in (2.1). The proof is complete.

Remark 1. We note that in the special cases when (i) $m_i = 1$ for $i = 1,2,\dots,n$, (ii) $n = 2$, (iii) $n = 1$, (iv) $n = 2$ and $m_1 = m_2 = 1$, and (v) $n = 1$ and $m_1 = 1$, the inequality established in (2.1) reduces respectively to the following inequalities

$$\int_a^x \int_b^y \int_c^z p(r,s,t) \left[\prod_{i=1}^n |f_i(r,s,t)| \right]^{2/n} dt ds dr \leq \tag{2.16}$$

$$\leq \frac{1}{n} K(a,b,c,x,y,z,n,1,\dots,1) \left(\int_a^x \int)b^y \int_c^z p(r,s,t) dt ds dr \right) \times$$

$$\times \int_a^x \int_b^y \int_c^z \sum_{i=1}^n |D_3 D_2 D_1 f_i(r,s,t)|^2 dt ds dr,$$

$$\int_a^x \int_b^y \int_c^z p(r,s,t) |f_1(r,s,t)|^{m_1} |f_2(r,s,t)|^{m_2} dt ds dr \tag{2.17}$$

$$\leq \frac{1}{2} K(a,b,c,x,y,z,2,m_1,m_2) \left(\int_a^x \int_b^y \int_c^z p(r,s,t) dt ds dr \right) \times$$

$$\times \int_a^x \int_b^y \int_c^z \left[|D_3 D_2 D_1 f_1(r,s,t)|^{2m_i} + |D_3 D_2 D_1 f_2(r,s,t)|^{2m_2} \right] dt ds dr,$$

$$\int_a^x \int_b^y \int_c^z p(r,s,t) |f_1(r,s,t)|^{2m_1} dt ds dr \leq \tag{2.18}$$

$$\leq K(a,b,c,x,y,z,l,m_1) \left(\int_a^x \int)b^y \int_c^z p(r,s,t) dt ds dr \right) \times$$

$$\times \int_a^x \int_b^y \int_c^z |D_3 D_2 D_1 f_1(r,s,t)|^{2m_1} dt ds dr,$$

$$\int_a^x \int_b^y \int_c^z p(r,s,t) |f_1(r,s,t)| |f_2(r,s,t)| dt ds dr \leq \tag{2.19}$$

$$\leq \frac{1}{2} K(a,b,c,x,y,z,2,1,1) \left(\int_a^x \int_b^y \int_c^z p(r,s,t) dt ds dr \right) \times$$

$$\times \int_a^x \int_b^y \int_c^z [|D_3 D_2 D_1 f_1(r,s,t)|^2 + |D_3 D_2 D_1 f_2(r,s,t)|^2] dt ds dr,$$

$$\int_a^x \int_b^y \int_c^z p(r,s,t) |f_1(r,s,t)|^2 dt ds dr \leq \tag{2.20}$$

$$\leq K(a,b,c,xy,z,1,1)\left(\int_a^x\int_b^y\int_c^z p(r,s,t)dtdsdr\right)\times$$

$$\times\int_a^x\int_b^y\int_c^z |D_3D_2D_1f_1(r,s,t)|^2 dtdsdr.$$

We note that the inequalities obtained in (2.17) and (2.19) are the three independent variable analogues of the Wirtinger type inequalities established by the present author in [10] and the inequality obtained in (2.20) is a three independent variable analog of the Wirtinger type inequality established by Traple in [15, p.160].

The following theorem deals with an integral inequality of Opial type involving functions of three independent variables.

Theorem 2. *Let the functions $p(r,s,t)$, $f_i(r,s,t)$ and the constants m_i for $i = 1,2,\ldots,n$ be as in Theorem 1. Then*

$$\int_a^x\int_b^y\int_c^z \sqrt{p(r,s,t)}\left[\prod_{i=1}^n |f_i(r,s,t)|^{m_i}\right]^{1/n}\times \tag{2.21}$$

$$\times\sum_{i=1}^n |D_3D_2D_1f_i(r,s,t)|^{m_i}dtdsdr\leq$$

$$\leq\left[K(a,b,c,x,y,z,n,m_1,\ldots,m_n)\int_a^x\int_b^y\int_c^z p(r,s,t)dtdsdr\right]^{1/2}\times$$

$$\times\int_a^x\int_b^y\int_c^z\sum_{i=1}^n |D_3D_2D_1f_i(r,s,t)|^{2m_i}dtdsdr,$$

where the constant $K(a,b,c,x,y,z,n,m_1,\ldots,m_n)$ is defined by (2.2).

Proof. By using the Schwarz integral inequality in three dimensions and the inequalities (2.1) and (2.14) we observe that

$$\int_a^x\int_b^y\int_c^z \sqrt{p(r,s,t)}\left[\prod_{i=1}^n |f_i(r,s,t)|^{m_i}\right]^{1/n}\times\sum_{i=1}^n |D_3D_2D_1f_i(r,s,t)|^{m_i}dtdsdr\leq$$

$$\leq\left[\int_a^x\int_b^y\int_c^z p(r,s,t)\left[\prod_{i=1}^n |f_i(r,s,t)|\right]^{2/n}dtdsdr\right]^{1/2}\times$$

$$\times\left[\int_a^x\int_b^y\int_c^z\left(\sum_{i=1}^n |D_3D_2D_1f_i(r,s,t)|^{m_i}\right)^2 dtdsdr\right]^{1/2}\leq$$

$$\leq \left[\frac{1}{n}K(a,b,c,x,y,z,m_1,\ldots,m_n)\left(\int_a^x\int_b^y\int_c^z p(r,s,t)dtdsdr\right)\times\right.$$

$$\left.\times\int_a^x\int_b^y\int_c^z\sum_{i=1}^n|D_3D_2D_1f_i(r,s,t)|^{2m_i}dtdsdr\right]^{1/2}\times$$

$$\times\left[n\int_a^x\int_b^y\int_c^z\sum_{i=1}^n|D_3D_2D_1f_i(r,s,t)|^{2m_i}dtdsdr\right]^{1/2}=$$

$$=\left[K(a,b,c,x,y,z,n,m_1,\ldots,m_n)\left(\int_a^x\int_b^y\int_c^z p(r,s,t)dtdsdr\right)\right]^{1/2}\times$$

$$\times\int_a^x\int_b^y\int_c^z\sum_{i=1}^n|D_3D_2D_1f_i(r,s,t)|^{2m_i}dtdsdr.$$

This is the desired inequality in (2.21) and hence the proof is complete.

Remark 2. If we take

(i) $m_i = 1$ for $i = 1, 2, \ldots, n$,

(ii) $n = 1$,

(iii) $n = 1$ and $m_1 = 1$ in (2.21),

then we get respectively the following inequalities

$$\int_a^x\int_b^y\int_c^z\sqrt{p(r,s,t)}\left[\prod_{i=1}^n|f_i(r,s,t)|\right]^{1/n}\times\sum_{i=1}^n|D_3D_2D_1f_i(r,s,t)|dtdsdr\leq \tag{2.22}$$

$$\leq\left[K(a,b,c,x,y,z,n,1,\ldots,1)\int_a^x\int_b^y\int_c^z p(r,s,t)dtdsdr\right]^{1/2}\times$$

$$\times\int_a^x\int_b^y\int_c^z\sum_{i=1}^n|D_3D_2D_1f_i(r,s,t)|^2dtdsdr,$$

$$\int_a^x\int_b^y\int_c^z\sqrt{p(r,s,t)}|f_1(r,s,t)|^{m_1}|D_3D_2D_1f_1(r,s,t)|^{m_1}dtdsdr\leq \tag{2.23}$$

$$\leq\left[K(a,b,c,x,y,z,1,m_1)\int_a^x\int_b^y\int_c^z p(r,s,t)dtdsdr\right]^{1/2}\times$$

$$\times\int_a^x\int_b^y\int_c^z|D_3D_2D_1f_1(r,s,t)|^{2m_1}dtdsdr,$$

$$\int_a^x\int_b^y\int_c^z\sqrt{p(r,s,t)}|f_1(r,s,t)||D_3D_2D_1f_1(r,s,t)|dtdsdr\leq \tag{2.24}$$

$$\leq \left[K(a,b,c,x,y,z,1,1)\int_a^x\int_b^y\int_c^z p(r,s,t)dtdsdr\right]^{1/2}\times$$
$$\times \int_a^x\int_b^y\int_c^z |D_3D_2D_1f_1(r,s,t)|^2 dtdsdr.$$

We note that the inequality obtained in (2.24) is a three independent variable analog of the Opial inequality established by Traple in [15, p.160]. In the special case when $p(r,s,t)$ is constant, then from (2.24) we have the following Opial type inequality

$$\int_a^x\int_b^y\int_c^z |f_1(r,s,t)||D_3D_2D_1f(r,s,t)|dtdsdr \leq \tag{2.25}$$
$$\leq [K(a,b,c,x,y,z,1,1)(x-a)(y-b)(z-c)]^{1/2}\times$$
$$\times \int_a^x\int_b^y\int_c^z |D_3D_2D_1f_1(r,s,t)|^2 dtdsdr.$$

For similar inequalities involving functions of two independent variables, see [6,8,10,14].

3. Discrete inequalities

Let $N = \{1,2,\dots\}$ and for x,y,z in N, we define

$$A = \{1,2,\dots,x+1\},\quad B = \{1,2,\dots,y+1\},\quad C = \{1,2,\dots,z+1\}$$

and $Q = A\times B\times C$. For a function $f : N^3 \to R$, we define the difference operators

$$\Delta_1 f(r,s,t) = f(r+1,s,t) - f(r,s,t),$$
$$\Delta_2 f(r,s,t) = f(r,s+1,t) - f(r,s,t),$$
$$\Delta_3 f(r,s,t) = f(r,s,t+1) - f(r,s,t),$$
$$\Delta_2\Delta_1 f(r,s,t) = \Delta_2[\Delta_1 f(r,s,t)]$$

and

$$\Delta_3\Delta_2\Delta_1 f(r,s,t) = \Delta_3[\Delta_2\Delta_1 f(r,s,t)].$$

We denote by $G(Q)$ the class of functions $f : Q \to R$ such that

$$f(1,s,r) = f(x+1,s,t) = f(r,1,t) = f(r,y+1,t) = f(r,s,1) = f(r,s,z+1) = 0.$$

The discrete analogue of the inequality given in Theorem 1 is embodied in the following theorem.

Theorem 3. *Let $p(r,s,t)$ be a real-valued nonnegative function defined on Q. Suppose that $f_i \in G(Q)$ for $i = 1,2,\ldots,n$ and let $m_i \geq 1$ for $i = 1,2,\ldots,n$ are constants. Then*

$$\sum_{r=1}^{x}\sum_{s=1}^{y}\sum_{t=1}^{z} p(r,s,t)\left[\prod_{i=1}^{n} |f_i(r,s,t)|^{m_i}\right]^{2/n} \leq \tag{3.1}$$

$$\leq \frac{1}{n} M(x,y,z,n,m_1,\ldots,m_n)\left(\sum_{r=1}^{x}\sum_{s=1}^{y}\sum_{t=1}^{z} p(r,s,t)\right)\times$$

$$\times \sum_{r=1}^{x}\sum_{s=1}^{y}\sum_{t=1}^{z}\left[\sum_{i=1}^{n} |\Delta_3\Delta_2\Delta_1 f_i(r,s,t)|^{2m_i}\right],$$

where

$$M(x,y,z,n,m_1,\ldots,m_n) = \left(\frac{1}{8}\right)^{\frac{2}{n}\sum_{i=1}^{n} m_i} (xyz)^{1+\frac{2}{n}\sum_{i=1}^{n}(m_i-1)}, \tag{3.2}$$

is a constant depending on $x,y,z,n,m_1,\ldots,m_n$.

Proof. From the hypotheses, it is easy to observe that the following identities hold for $i = 1,2,\ldots,n$ and $(r,s,t) \in Q$:

$$f_i(r,s,t) = \sum_{u=1}^{r-1}\sum_{v=1}^{s-1}\sum_{w=1}^{t-1} \Delta_e\Delta_2\Delta_1 f_i(u,v,w), \tag{3.3}$$

$$f_i(r,s,t) = -\sum_{u=1}^{r-1}\sum_{v=1}^{s-1}\sum_{w=t}^{z} \Delta_e\Delta_2\Delta_1 f_i(u,v,w), \tag{3.4}$$

$$f_i(r,s,t) = -\sum_{u=1}^{r-1}\sum_{v=s}^{y}\sum_{w=1}^{t-1} \Delta_e\Delta_2\Delta_1 f_i(u,v,w), \tag{3.5}$$

$$f_i(r,s,t) = -\sum_{u=r}^{x}\sum_{v=1}^{s-1}\sum_{w=1}^{t-1} \Delta_e\Delta_2\Delta_1 f_i(u,v,w), \tag{3.6}$$

$$f_i(r,s,t) = \sum_{u=1}^{r-1}\sum_{v=s}^{y}\sum_{w=t}^{z} \Delta_e\Delta_2\Delta_1 f_i(u,v,w), \tag{3.7}$$

$$f_i(r,s,t) = \sum_{u=r}^{x}\sum_{v=s}^{y}\sum_{w=1}^{t-1} \Delta_e\Delta_2\Delta_1 f_i(u,v,w), \tag{3.8}$$

$$f_i(r,s,t) = \sum_{u=r}^{x}\sum_{v=1}^{s-1}\sum_{w=t}^{z} \Delta_e\Delta_2\Delta_1 f_i(u,v,w), \tag{3.9}$$

$$f_i(r,s,t) = -\sum_{u=r}^{x}\sum_{v=s}^{y}\sum_{w=t}^{z} \Delta_e\Delta_2\Delta_1 f_i(u,v,w), \tag{3.10}$$

From (3.3)-(3.10) it is easy to observe that

$$|f_i(r,s,t)| \le \frac{1}{8}\sum_{u=1}^{x}\sum_{v=1}^{y}\sum_{w=1}^{z} |\Delta_3\Delta_2\Delta_1 f_i(u,v,w)|, \tag{3.11}$$

for $(r,s,t) \in Q$ and $i = 1,2,\ldots,n$. From (3.9) and using the Hölder's inequality for summations in three dimensions with indices m_i, $m_i/(m_i-1)$ for $i = 1,2,\ldots,n$ we obtain

$$|f_i(r,s,t)|^{m_i} \le \left(\frac{1}{8}\right)^{m_i} (xyz)^{m_i-1} \times \sum_{u=1}^{x}\sum_{v=1}^{y}\sum_{w=1}^{z} |\Delta_3\Delta_2\Delta_1 f_i(u,v,w)|^{m_i}. \tag{3.12}$$

From (3.12) and using the elementary inequalities (2.13) and (2.14) and Schwarz inequality for summation in three dimensions we observe that

$$\left[\prod_{i=1}^{n} |f_i(r,s,t)|^{m_i}\right]^{2/n} \le \tag{3.13}$$

$$\le \left[\prod_{i=1}^{n} \left(\frac{1}{8}\right)^{m_i} (xyz)^{m_i-1} \times \sum_{u=1}^{x}\sum_{v=1}^{y}\sum_{w=1}^{z} |\Delta_3\Delta_2\Delta_1 f_i(u,v,w)|^{m_i}\right]^{2/n} =$$

$$= \left(\frac{1}{8}\right)^{\frac{2}{n}\sum_{i=1}^{n} m_i} (xyz)^{\frac{2}{n}\sum_{i=1}^{n}(m_i-1)} \times \left[\left[\prod_{i=1}^{n}\sum_{u=1}^{x}\sum_{v=1}^{y}\sum_{w=1}^{z} |\Delta_3\Delta_2\Delta_1 f_i(u,v,w)|^{m_i}\right]^{1/n}\right]^2 \le$$

$$\le \left(\frac{1}{8}\right)^{\frac{2}{n}\sum_{i=1}^{n} m_i} (xyz)^{\frac{2}{n}\sum_{i=1}^{n}(m_i-1)} \times \left[\frac{1}{n}\sum_{i=1}^{n}\left[\sum_{u=1}^{x}\sum_{v=1}^{y}\sum_{w=1}^{z} |\Delta_3\Delta_2\Delta_1 f_i(u,v,w)|^{m_i}\right]\right]^2 \le$$

$$\le \left(\frac{1}{8}\right)^{\frac{2}{n}\sum_{i=1}^{n} m_i} (xyz)^{\frac{2}{n}\sum_{i=1}^{n}(m_i-1)} \times \frac{1}{n^2} n \sum_{i=1}^{n}\left[\sum_{u=1}^{x}\sum_{v=1}^{y}\sum_{w=1}^{z} |\Delta_3\Delta_2\Delta_1 f_i(u,v,w)|^{m_i}\right]^2 \le$$

$$\le \frac{1}{n}\left(\frac{1}{8}\right)^{\frac{2}{n}\sum_{i=1}^{n} m_i} (xyz)^{\frac{2}{n}\sum_{i=1}^{n}(m_i-1)} \times \sum_{i=1}^{n}(xyz)\sum_{u=1}^{x}\sum_{v=1}^{y}\sum_{w=1}^{z} |\Delta_3\Delta_2\Delta_1 f_i(u,v,w)|^{2m_i} =$$

$$= \frac{1}{n} M(x,y,z,n,m_1,\ldots,m_n) \times \sum_{u=1}^{x}\sum_{v=1}^{y}\sum_{w=1}^{z}\left[\sum_{i=1}^{n} |\Delta_3\Delta_2\Delta_1 f_i(u,v,w)|^{2m_i}\right].$$

Multiplying both sides of (3.13) by $p(r,s,t)$ and taking the sum over t,s,r from 1 to z, 1 to y, 1 to x respectively we get the required inequality in (3.1). The proof is complete.

Remark 3. If we take (i) $m_i = 1$ for $i = 1,2,\ldots,n$, (ii) $n = 2$, (iii) $n = 1$, (iv) $n = 2$ and $m_1 = m_2 = 1$ and (v) $n = 1$ and $m_1 = 1$, then the inequality established in (3.1) reduces to the various new inequalities which can be used in certain applications.

The following theorem deals with the discrete analogue of the inequality given in Theorem 2.

Theorem 4. *Let the functions $p(r,s,t), f_i(r,s,t)$ and the constants m_i for $i = 1,2,\ldots,n$ be as in Theorem 3. Then*

$$\sum_{r=1}^{x}\sum_{s=1}^{y}\sum_{t=1}^{z}\sqrt{p(r,s,t)}\left[\prod_{i=1}^{n}|f_i(r,s,t)|^{m_i}\right]^{1/n}\times\sum_{i=1}^{n}|\Delta_3\Delta_2\Delta_1 f_i(r,s,t)|^{m_i}\le \tag{3.14}$$

$$\le\left[M(x,y,z,n,m_1,\ldots,m_n)\sum_{r=1}^{x}\sum_{s=1}^{y}\sum_{t=1}^{z}p(r,s,t)\right]^{1/2}\times$$

$$\times\sum_{r=1}^{x}\sum_{s=1}^{y}\sum_{t=1}^{z}\left[\sum_{i=1}^{n}|\Delta_3\Delta_2\Delta_1 f_i(r,s,t)|^{2m_i}\right],$$

where the constant $M(x,y,z,n,m_1,\ldots,m_n)$ is defined by (3.2).

Proof. By using Schwarz inequality for summation in three dimensions and the inequalities (3.1) and (2.14) we observe that

$$\sum_{r=1}^{x}\sum_{s=1}^{y}\sum_{t=1}^{z}\sqrt{p(r,s,t)}\left[\prod_{i=1}^{n}|f_i(r,s,t)|^{m_i}\right]^{1/n}\times\sum_{i=1}^{n}|\Delta_3\Delta_2\Delta_1 f_i(r,s,t)|^{m_i}\le$$

$$\le\left[\sum_{r=1}^{x}\sum_{s=1}^{y}\sum_{t=1}^{z}p(r,s,t)\left[\prod_{i=1}^{n}|f_i(r,s,t)|^{m_i}\right]^{2/n}\right]^{1/2}\times$$

$$\times\left[\sum_{r=1}^{x}\sum_{s=1}^{y}\sum_{t=1}^{z}\left[\sum_{i=1}^{n}|\Delta_3\Delta_2\Delta_1 f_i(r,s,t)|^{m_i}\right]^{2}\right]^{1/2}\le$$

$$\le\left[\frac{1}{n}M(x,y,z,n,m_1,\ldots,m_n)\left(\sum_{r=1}^{x}\sum_{s=1}^{y}\sum_{t=1}^{z}p(r,s,t)\right)\times\right.$$

$$\left.\times\sum_{r=1}^{x}\sum_{s=1}^{y}\sum_{t=1}^{z}\left[\sum_{i=1}^{n}|\Delta_3\Delta_2\Delta_1 f_i(r,s,t)|^{2m_i}\right]\right]^{1/2}\times$$

$$\times \left[n \sum_{r=1}^{x} \sum_{s=1}^{y} \sum_{t=1}^{z} \left[\sum_{i=1}^{n} |\Delta_3 \Delta_2 \Delta_1 f_i(r,s,t)|^{2m_i} \right] \right]^{1/2} \times$$

$$\times \left[n \sum_{r=1}^{x} \sum_{s=1}^{y} \sum_{t=1}^{z} \left[\sum_{i=1}^{n} |\Delta_3 \Delta_2 \Delta_1 f_i(r,s,t)|^{2m_i} \right] \right]^{1/2} =$$

$$= \left[M(x,y,z,n,m_1,\ldots,m_n) \sum_{r=1}^{x} \sum_{s=1}^{y} \sum_{t=1}^{z} p(r,s,t) \right]^{1/2} \times$$

$$\times \sum_{r=1}^{x} \sum_{s=1}^{y} \sum_{t=1}^{z} \left[\sum_{i=1}^{n} |\Delta_3 \Delta_2 \Delta_1 f_i(r,s,t)|^{2m_i} \right].$$

This is the desired inequality in (3.14) and hence the proof is complete.

Remark 4. In the special cases, if we take (i) $m_i = 1$ for $i = 1, 2, \ldots, n$, (ii) $n = 1$, (iii) $n = 1$ and $m_1 = 1$ in (3.14), then we get the new inequalities which may be useful in certain situations. For similar inequalities, see [7,9,11,12,13] and the references given therein.

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AN EXTENSION OF RUSCHEWEYH'S UNIVALENCE CONDITION

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Abstract. We obtain a new sufficient univalence condition, generalizing the univalence criterion of S.Ruscheweyh.

1. Introduction

We denote by U_r the disk of z−plane, $U_r = \{ z \in C : |z| < r \}$, where $r \in (0,1]$, $U_1 = U$ and $I = [0, \infty)$.
Let A be the class of functions f which are analytic in U with $f(0) = 0$ and $f'(0) = 1$.

Theorem 1.1. ([4]). *Let $s = \alpha + i\beta$, $\alpha > 0$ and $f \in A$. Assume that for a certain $c \in C$ and all $z \in U$,*

$$\left| c|z|^2 + s - \alpha(1-|z|^2)\left[s\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-s)\frac{zf'(z)}{f(z)}\right]\right| \leq M \,, \tag{1}$$

where

$$M = \begin{cases} \alpha|s| + |s+c|(\alpha - 1) \,, & 0 < \alpha < 1, \\ |s| \,, & \alpha \geq 1. \end{cases} \tag{2}$$

Then the function f is univalent in U.

We will need Loewner's parametric method to prove our results.

2. Preliminaries

Theorem 2.1. ([3]. *Let r be a real number, $r \in (0,1]$. Let $L(z,t) = a_1(t)z + a_2(t)z^2 + \ldots$, $a_1(t) \neq 0$, be analytic in U_r, for all $t \in I$, locally absolutely continuous in I and*

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locally uniform with respect to U_r . *For almost all* $t \in I$ *suppose*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t} \qquad (\forall) z \in U_r \,,$$

where $p(z,t)$ *is analytic in* U *and satisfies* $Re p(z,t) > 0,\ z \in U,\ t \in I$.

If $|a_1(t)| \to \infty$ *for* $t \to \infty$ *and* $\{ L(z,t)/a_1(t) \}$ *forms a normal family in* U_r , *then for each* $t \in I$, $L(z,t)$ *has an analytic and univalent extension to the whole disk* U.

3. Main results

Theorem 3.1. *Let* $f \in A$ *and let* s, c *be complex numbers,* $s = \alpha + i\beta$, $\alpha > 0$, $c \neq 0$, $|s + c| \leq |s|$. *If there exists an analytic function in* U, $p(z) = 1 + c_1 z + \ldots$, *such that*

$$\left|\frac{c}{p(z)} + s\right| \leq |s| \,, \tag{3}$$

$$\left|\frac{c}{p(z)}|z|^{2/\alpha} + s - \alpha(1 - |z|^{2/\alpha})\left[s\left(1 + \frac{zf''(z)}{f'(z)}\right) + \right.\right. \tag{4}$$

$$\left.\left. +(1-s)\frac{zf'(z)}{f(z)} + s\frac{zp'(z)}{p(z)}\right]\right| \leq |s| \,,$$

for all $z \in U$, *then the function* f *is univalent in* U.

Proof. The conditions (3) and (4) implies that $p(z) \neq 0$ and $f(z)f'(z)/z \neq 0$ in U. If $c \neq 0$ let

$$f(z,t) = f(e^{-st}z)\left[1 - \frac{\alpha}{c}(e^{2t} - 1)p(e^{-st}z)e^{-st}z\frac{f'(e^{-st}z)}{f(e^{-st}z)}\right]^s \tag{5}$$

The inequalities $|c+s| \leq |s|$ and Re $s > 0$ imply $\alpha/c \notin [0,\infty)$. It follows that there exists $r \in (0,1]$ such that

$$1 - \frac{\alpha}{c}(e^{2t} - 1)p(e^{-st}z)e^{-st}zf'(e^{-st}z)/f(e^{-st}z) \neq 0$$

for all $z \in U_r$ and $t \geq 0$, and hence the function $f(z,t)$ is analytic in U_r for all $t \geq 0$. Furthermore

$$\left|\frac{\partial f(0,t)}{\partial z}\right| = \left|\left[\left(1 + \frac{\alpha}{c}\right)e^{-t} - \frac{\alpha}{c}e^t\right]\right|^s \neq 0$$

in I, and $\lim_{t\to\infty}\left|\frac{\partial f(0,t)}{\partial z}\right| = \infty$ (we have chosen a fixed branch for $\frac{\partial f(0,t)}{\partial z}$). It follows that $\{ f(z,t)/\frac{\partial f(0,t)}{\partial z} \}$ forms a normal family in U_{r_0}, $r_0 < r$.

A simple calculation yields

$$\frac{\partial f(z,t)}{\partial t}/z \cdot \frac{\partial f(z,t)}{\partial z} = s\frac{1+P(e^{-st}z,t)}{1-P(e^{-st}z,t)},$$

where

$$P(z,t) = \frac{c}{\alpha}e^{-2t}\frac{1}{p(z,t)} + 1 - (1-e^{-2t})H_s(e^{-st}z) \quad \%; and \tag{6}$$

$$H_s(z) = s\left(1+\frac{zf''(z)}{f'(z)}\right) + (1-s)\frac{zf'(z)}{f(z)} + s\frac{zp'(z)}{p(z)}.$$

If $h(z,t) = \frac{\partial f(z,t)}{\partial t}/(z\frac{\partial f(z,t)}{\partial z})$ then the inequality $Re\ h(z,t) > 0$ for all $z \in U$ and $t \in I$ is equivalent to

$$|\ \alpha P(e^{-st}z,t) + i\beta\ | \le |s|, \quad z \in U,\ t \in I. \tag{7}$$

Replacing the function $P(z,t)$ defined from (6) in (7) we obtain

$$\left| e^{-2t}\left(\frac{c}{p(e^{-st}z)} + s\right) + (1-e^{-2t})[\ -\alpha H_s(e^{-st}z) + s] \right| \le |z|. \tag{8}$$

In order to prove the inequality (8) we consider the function

$$Q(z,t) = e^{-2t}\left(\frac{c}{p(e^{-st}z)} + s\right) + (1-e^{-2t})[\ -\alpha H_s(e^{-st}z) + s]$$

which for all $t \in I \setminus \{0\}$ is analytic in $\bar{U}$ and hence

$$\max_{|z|\le 1} |\ Q(z,t)\ | = |\ Q(e^{i\theta},t)\ |, \quad \theta \in R. \tag{9}$$

If $\xi = e^{-st}e^{i\theta}$, then $|\xi| = e^{-\alpha t}$, $e^{-t} = |\xi|^{1/\alpha}$ and by (8), (9) and (4) it results

$$|\ Q(z,t)\ | < |\ Q(e^{i\theta},t)\ | = \left| |\xi|^{2/\alpha}\left(\frac{c}{p(\xi)} + s\right) + \right.$$

$$\left. + \left(1 - |\xi|^{\frac{2}{\alpha}}\right)[\ -\alpha H_s(\xi) + s\ ] \right| \le |s|,$$

for all $z \in U$ and $t \in I \setminus \{0\}$.

If $t = 0$, then $Q(z,0) = c/p(z) + s$ and by (3) it results that $|Q(z,0)| \le |s|$ for all $z \in U$ and hence the inequality (8) holds true for all $z \in U$ and $t \in I$.

Theorem 3.2. *Let $f \in A$ and let s, c be complex numbers, $s = \alpha + i\beta$, $\alpha \geq 1$, $c \neq 0$, $|s + c| \leq |s|$. If there exists an analytic function in U, $p(z) = 1 + c_1(z) + \ldots$, such that*

$$\left| \frac{c}{p(z)} + s \right| \leq |s| \tag{10}$$

$$\left| \frac{c}{p(z)}|z|^2 + s - \alpha(1-|z|^2)\left[s\left(1 + \frac{zf''(z)}{f'(z)}\right) + \right.\right. \tag{11}$$

$$\left.\left. +(1-s)\frac{zf'(z)}{f(z)} + s\frac{zp'(z)}{p(z)} \right]\right| \leq |s|,$$

for all $z \in U$, then the function f is univalent in U.

Proof. The function

$$w(z, \lambda) = \lambda\left(\frac{c}{p(z)} + s\right) + (1-\lambda)[\,-\alpha H_s(z) + s\,]$$

is analytic in U for all $\lambda \in [0,1]$. From (10) and (11) it results that

$$|\, w(z,\ |z|^2)\,| \leq |s| \qquad (\forall) z \in U\,; \tag{12}$$

$$|\, w(z,\ 1)\,| \leq |s| \qquad (\forall) z \in U. \tag{13}$$

If λ increases from $\lambda_1 = |z|^2$ to $\lambda_2 = |z|^{2/\alpha}$, then the point $w(z, \lambda)$ moves on the segment whose endpoints are $A = w(z,\ |z|^2)$ and $B = w(z,\ 1)$, and hence from (12) and (13) it results that

$$|\, w(z,\ |z|^{2/\alpha})\,| \leq |s| \tag{14}$$

for all $z \in U$. Because

$$w(z,\ |z|^{2/\alpha}) = \frac{c}{p(z)}|z|^{2/\alpha} + s - \alpha(1-|z|^{2/\alpha})\left[\left(1 + \frac{zf''(z)}{f'(z)}\right) + \right. \tag{15}$$

$$\left. +(1-s)\frac{zf'(z)}{f(z)} + s\frac{zp'(z)}{p(z)}\right]$$

from (14) and (15) it results that (4) holds true for all $z \in U$ and from Theorem 3.1 it results that the function f is univalent in U. *Remark.* For $\alpha \geq 1$ and $p(z) \equiv 1$, from Theorem 3.2 we obtain Theorem 1.1 .

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SOME PROPERTIES OF THE ω-LIMIT POINTS SET OF AN OPERATOR

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Abstract. In this paper we study the ω -limit points set of an operator, in the terms of the fixed points set, the periodic points set and the recurrent points set.

1. Introduction

Let (X, d) be a metric space and $A : X \to X$ an operator. In this paper we shall use the following notations and notions:

$P(x) := \{Y \subset X | Y \neq \emptyset\}$,

$I(A) := \{Y \in P(X) | A(Y) \subset Y\}$,

$F_A := \{x \in X | A(x) = x\}$ - the fixed points set of A,

$P_A := \bigcup_{n \in N^*} F_{A^n}$ - the periodic points set of A,

$P_A^n := \{x \in X | A^k(x) \neq x, k = \overline{1, n-1}, A^n(x) = x\}$-the n-order periodic points set of A,

$\omega_A(x) := \{y \in X | \ \exists \ n_k \to \infty$, such that $f^{n_k}(x) \to y$ as $n \to \infty\}$ - the ω - limit points set of A,

$\omega_A(X) := \bigcup_{x \in X} \omega_A(x)$,

$R_A := \{x \in X | x \in \omega_A(x)\}$ - the recurrent points set of A,

$O_A(x) := \{x, A(x), \ldots, A^n(x), \ldots\}$.

The purpose of this paper is to study the ω - limit points set of an operator A in the terms of F_A, P_A, R_A.

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2. F_A, P_A, P_A^n and ω_A. Examples. Basic problems

Let (X,d) be a metric space and $A : X \to X$ an operator. It is clear that

$$F_A \subset P_A \subset R_A \subset \omega_A$$

In what follow we give some examples and counterexamples to these notions.

Example 2.1 (see[20], [21] and [23]). Let (X,d) be a metric space and $A : X \to X$ a weakly Picard operator. Then

$$F_A = P_A = R_A = \omega_A(X).$$

Example 2.2 (see [1], [2]). $X = \{z \in \mathbf{C} | |z| = 1\}$ and $A(z) := e^{i\alpha} z$. If $\alpha = 1$, then

$$F_A = \emptyset,\ P_A = R_A = \omega_A(X) = X.$$

If α/π is an irrational real number, then

$$F_A = P_A = \emptyset,\ R_A = \omega_A(X) = X.$$

Example 2.3 (see [3], [5], [7]). Let $A \in C([0,1],[0,1])$. If $P_A^3 \neq \emptyset$, then $P_A^n \neq \emptyset$, for all $n \in N^*$ (Sarkovskii's theorem).

Example 2.4 (see [19]). Let $A \in C(R,R)$ such that $A^2 = 1_R$ (an involution). Then $F_A = \{x^*\}$ and $P_A = R$.

Example 2.5 Let (X,d) be a metric space and $A : X \to X$ an operator. We suppose that

(i)$X = \bigcup_{i \in I} X_i,\ X_i \neq \emptyset,\ X_i \bigcap X_j \neq \emptyset,\ i \neq j;$

(ii)$X_i \in I(A)$,

(iii)$cl(X_i) = int(X_i), i \in I.$

Then

$$\omega_A(X_i) \bigcap \omega_A(X_j) = \emptyset,$$

for all $i, j \in I, i \neq j$.

Example 2.6 (see [4]). Let $A \in C([0,1],[0,1])$. Then,

$$\overline{P_A} = \overline{R_A}.$$

Example 2.7 (see [26]). Consider the nonlinear Cauchy problem

$$\frac{du}{dt} = -u - u^3, \; u(0) = U \in \mathbf{R},$$

where $\omega(U) = \{0\}$ for all $U \in \mathbf{R}$. Application of the forward Euler numerical method gives

$$U_{n+1} = U_n - \Delta t(U_n + U_n^3), \; U_0 = U,$$

where $U_n \approx u(n\Delta t)$, $n = 0, 1, \ldots$ and Δt is the time step. If $A(u) = u - \Delta t(u + u^3)$, it may be shown that

$$\begin{array}{lll} \omega_A(U) = 0 & \textit{for} & \Delta t(1+U^2) \in (0,2) \\ \omega_A(U) = \{-U, U\} & \textit{for} & \Delta t(1+U^2) = 2 \\ |U_n| \to \infty \textit{ as } n \to \infty & \textit{for} & \Delta t(1+U^2) \in (2, +\infty) \end{array}$$

Thus, if $\Delta t < \frac{2}{1+U^2}$ we obtain the correct asymptotic behaviour of the differential equation. If $U = \sqrt{\frac{2}{\Delta t} - 1}$ we obtain a spurious period two solution $U_n = (-1)^n\sqrt{\frac{2}{\Delta t} - 1}$, i.e. $F_A \neq \emptyset$, $P_A^2 \neq \emptyset$ for all $\Delta t \in [0,2]$.

The following problems arise:

Let (X,d) be a metric space and $A : X \to X$ an operator.

Problem 2.1 (see [4], [24]). Establish conditions on X and A which imply that

a) $P_A \neq \emptyset$;

b) $P_A^n \neq \emptyset$.

Problem 2.2 (see [11]). Which are the operators with the following property:

$$P_A \neq \emptyset \;\Rightarrow\; F_A \neq \emptyset?$$

Problem 2.3 ([4]). Which are the metric spaces, X, with the following property

$$A \in C(X,X) \;\Rightarrow\; \overline{P_A} = \overline{R_A}$$

Problem 2.4 Let $n \in N$. In which conditions on X, A and n we have:

$$F_A = P_A^k = \emptyset,\ k = \overline{2, n-1} \text{ and } P_A^k \neq \emptyset,\ k \geq n.$$

Problem 2.5 (see [3], [4], [6], [9], [12], [16], [21], [25]).

Establish conditions on X and A which imply that:

a)$\omega_A(x) \neq \emptyset,\ \forall x \in X$;

b)$R_A \neq \emptyset$;

c)$\omega_A(X) = F_A$;

d)$\omega_A(X) = P_A$;

e)$\omega_A(X) = R_A$;

f) there exists $x \in X : \overline{\omega_A(x)} = X$.

For other examples and countraexamples to the above problems and for some results see [2], [5], [7], [8], [10] and [25].

3. Periodic points

Theorem 3.1. *Let $(X, \leq)$ be a complete lattice and $A : X \to X$ a monoton operator. Then $P_A \neq \emptyset$.*

Proof. If the operator A is monoton increasing, then by the fixed point theorem of Tarski we have that $F_A \neq \emptyset$. If the operator A is monoton decreasing then A^2 is monoton increasing, so, $F_{A^2} \neq \emptyset$.

Theorem 3.2. *Let (X, S, M) be a fixed point structure (see [22]) and $A : X \to X$ an operator. We suppose that there exists $k \in N^*$ such that*

(i)$A^k \in M(X)$;

(ii) there exists $Y \in S(X)$ such that $A^k(X) \subset Y$.

Then $P_A \neq \emptyset$.

Proof. From $A^k : X \to X$ and $A^k(X) \subset Y \subset X$ we have that $Y \in I(A^K)$. On the other hand, $A^k \in M(X)$ implies that $A^k|_Y \in M(Y)$, so, $F_{A^k} \neq \emptyset$.

If in the Theorem 3.2 we consider the fixed point structure of Schauder (X - Banach space, $S(X) = P_{cp,cv}(X)$ and $M(Y) = C(Y, Y)$) we have

Theorem 3.3. *Let X be a Banach space and $A : X \to X$ a continous operator such that there exists $k \in N^*$ such that $A^k(X)$ is relatively compact. Then, $P_A \neq \emptyset$.*

Proof. If we take $Y = \overline{co}A^k(X)$, then $Y \in P_{cp,cv}(X)$ and we are in the conditions of the Theorem 3.2.

Theorem 3.4. *Let $X = [-a, a] \subset \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$ such that $A(u) = u + \Delta t f(u)$ is a contraction on X, $A : X \to X$ and $A(0) = 0$, where $\Delta t > 0$. Then the numerical method $U_{n+1} = A(U_n)$, $U_0 = U \in X$ (forward Euler method for the Cauchy problem $\frac{du}{dt} = f(u), u(0) = U \in X$) has no spurious period two solutions in X.*

Proof. By the above conditions, A^2 is a contraction on X and by theorem 3.2. for the fixed point structure of Banach, A^2 has a unique fixed point in X and this point is 0.

Example 3.1. Let $f(u) = -u - u^3$ and $A(u) = u - \Delta t(u + u^3)$ for $\Delta t \in (0, 1)$. Let $X = [-\sqrt{\frac{1-\Delta t}{3\Delta t}}, \sqrt{\frac{1-\Delta t}{3\Delta t}}]$. It may be shown that $A : X \to X$ and A is a contraction on X. Thus we have no spurious period two solutions.

Note that all Runge-Kutta methods retain all the equilibria of $\frac{du}{dt} = f(u)$ (see [26], Th. 5.3.3.). Consequently, the forward Euler method gives the correct asymptotic behaviour of this differential equation on X.

4. Recurrent points

Lemma 4.1 (see [9], [18]). *Let (X, d) be a compact metric space and $A : X \to X$ a continuous operator. Then $R_A \neq \emptyset$.*

Theorem 4.1. *Let (X, d) be a metric space and $A : X \to X$ such that*

(i)A is continous;

(ii) there exists $k \in N^*$ such that $A^k(X)$ is relatively compact.

Then, $R_A \neq \emptyset$.

Proof. It is clear that $clA^k(X) \in I(f)$. So, the operator $A : \overline{A^k(X)} \to \overline{A^k(X)}$ satisfies the conditions in the Lemma 4.1.

Theorem 4.2. *Let (X, d) be a complete metric space, $\alpha : P_b(X) \to R_+$ a measure of noncompactness (see [23]) on X and $A : X \to X$ an operator. We suppose that:*

(i)A is continuous;

(ii)A is a (α, a) - contraction.

Then, $R_A \neq \emptyset$.

Proof. Let $Y_1 := \overline{A(X)}, \ldots, Y_{n+1} := \overline{A(Y_n)}, n \in \mathbf{N}^*$. We remark that

$$Y_n \in P_{b,cl}(X) \bigcap I(A), n \in \mathbf{N}^*$$

From the condition (ii) we have that

$$\alpha(Y_n) \leq a\alpha(Y_{n-1}) \leq \cdots \leq a^n \alpha(Y) \to 0 \textit{ as } n \to \infty.$$

But α is a measure of noncompactness on X, i.e, α satisfies the following conditions (see [23]):

(a)$\alpha(A) = 0 \;\Rightarrow\; \overline{A} \in P_{cp}(X)$,

(b)$\alpha(A) = \alpha(\overline{A})$, for all $A \in P_b(X)$,

(c)$A \subset B \;\Rightarrow\; \alpha(A) \leq \alpha(B)$, for all $A, B \in P_b(X)$,

(d) If $A_n \in P_{b,cl}(X),\; A_{n+1} \subset A_n,\; n \in \mathbf{N}$, and $\alpha(A_n) \to 0$ *as* $n \to \infty$, *then*

$$A_\infty := \bigcap_{n \in N} A_n \neq \emptyset \text{ and } \alpha(A_\infty) = 0.$$

From the condition (d) and (a) we have that

$$Y_\infty := \bigcap_{n \in N} Y_n \in I_{cp}(A).$$

Now the theorem follows from the Lemma 4.1.

Theorem 4.3. *Let (X, d) be a bounded metric space, α_{DP} a Danes-Pasicki measure of noncompactness (see [23]) and $A : X \to X$ an operator. We suppose that*

(i) the operator A is continous,

(ii) the operator A is α_{DP} - condensing.

Then, $R_A \neq \emptyset$.

Proof. Let $x_0 \in X$. By Lemma 3.1. in [22], there exists $A_0 \subset X$ such that

$$cl(f(A_0) \bigcup \{x_0\}) = A_0$$

This implies that $\alpha_{DP}(A_0) = 0$. Thus

$$A_0 \in P_{cp}(X) \bigcap I(A).$$

Now the proof follows from Lemma 4.1.

5. The set ω_A

In what follow we consider operators on ordered metric space (for the ordered Banach spaces see [6], [10], [11], [25]). We have

Theorem 5.1. *Let $(X, d, \leq)$ be an ordered metric space and $A : X \to X$ an increasing operator. Then*

(i) $x \leq Ax \Rightarrow x \leq \omega_A(x)$

and $y \geq A(y) \Rightarrow y \geq \omega_A(y)$;

(ii) $\omega_A(x) \leq y \Rightarrow \omega_A(x) \leq \omega_A(y)$ *and*

$x \leq \omega_A(y) \Rightarrow \omega_A(x) \leq \omega_A(y)$;

Proof. (i) Let, for example, $x \leq Ax$. Then $x \leq A^n(x)$ for all $n \in \mathbf{N}$. This implies that $x \leq \omega_A(x)$.

(ii) Let, for example, $x, y \in X$, such that $\omega_A(x) \leq y$. Since $\omega_A(x) \in I(A)$, it follows that $\omega_A(x) \leq T^n y$ for all $n \in \mathbf{N}$. Hence we have $\omega_A(x) \leq \omega_A(y)$.

Remark 5.1. The above results improve some results given by E.N. Dancer in [6].

Remark 5.2. From the Theorem 5.1. we have the following results given in [23]:

Theorem 5.2. *Let $(X, d, \leq)$ be an ordered metric space and $A : X \to X$ an operator and $x, y \in X$ such that $x \leq y$, $x \leq A(x)$, $y \geq A(y)$. We suppose that*

(i) A is weakly Picard operator;

(ii) A is monoton increasing.

Then

(a) $x \leq A^\infty(x) \leq A^\infty(y) \leq y$;

(b) $A^\infty(x)$ *is the minimal fixed point of A in* $[x, y]$ *and* $A^\infty(y)$ *is the maximal fixed point of A in* $[x, y]$.

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INEQUALITIES FOR GENERALIZED CONVEX FUNCTIONS WITH APPLICATIONS, I.

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Abstract. The Theory of Inequalities has a majore role in Mathematical Analysis, and in almost all areas of Mathematics, too. In this theory, the convex functions and the generalized convexity plays a special role. The author has published a series of papers with applications of convexity inequalities in various fields of Mathematics. We quote applications in geometry (see e.g. [16], [22]), special functions ([19], [18], [23]); number theory (see many articles collected in the monograph [34]); the theory of means ([24], [25], [31], [33]), etc.

The aim of this series of papers (planned to have 4 parts) is to survey the most important ideas and results of the author in the theory of convex inequalities. In the course of this survey, many new results and applications will be obtained. In most cases only the new results will be presented with a proof; the other results will be stated only, with connections and/or applications to known theorems. All material is centered around three most important inequalities, namely: Jensen's inequality, Jensen-Hadamard's (or Hermite-Hadamard's) inequality and Jessen's inequality.

1. Jensen's inequality

One of the most important inequalities is Jensen's inequality either in its discrete or in its integral form. In what follows we will discuss various generalizations, extensions, special cases, or refinements.

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A. Let start with $f : [a,b] \to \mathbf{R}$, a convex function in the classical sense, and let $L : C[a,b] \to \mathbf{R}$ be a linear and positive functional defined on the space $C[a,b]$ of all continuous functions on $[a,b]$. Put $e_k(x) = x^k$, $x \in [a,b]$, $(k \in \mathbf{N})$.

Theorem 1.1. (see [9] and [22]) *If the above conditions are satisfied and the functional L has the property $L(e_0) = 1$, then the following double-inequality holds true:*

$$f(L(e_1)) \le L(f) \le L(e_1)\left[\frac{f(b)-f(a)}{b-a}\right] + \frac{bf(a)-af(b)}{b-a}. \tag{1}$$

Proof. See [22].

Corollary 1.1. Let $L(f) = \dfrac{1}{b-a}\displaystyle\int_a^b f(t)dt$. Then $L(e_0) = 1$ and the relation (1) gives us the classical Jensen-Hadamard inequality

$$(b-a)f\left(\frac{a+b}{2}\right) \le \int_a^b f(x)dx \le (b-a)\left[\frac{f(a)+f(b)}{2}\right] \tag{2}$$

which will be considered later.

Corollary 1.2. Let $w_i \ge 0$ $(i = \overline{1,n})$ with $\displaystyle\sum_{i=1}^n w_i = 1$, and let $a_i \in [a,b]$, $(i = \overline{1,n})$. Let us define the functional $L(f) = \displaystyle\sum_{i=1}^n w_i f(a_i)$, which is linear and positive. From (1) we can deduce the double relation

$$f\left(\sum_{i=1}^n w_i a_i\right) \le \sum_{i=1}^n w_i f(a_i) \le \left(\sum_{i=1}^n w_i a_i\right)\left[\frac{f(b)-f(a)}{b-a}\right] + \frac{bf(a)-af(b)}{b-a}. \tag{3}$$

The left side of this relation is the famous discrete (pondered) inequality by Jensen ([5], [8], [10]).

Corollary 1.3. Let $p : [a,b] \to \mathbf{R}$ be a strictly positive, integrable function, and let $g : [a,b] \to \mathbf{R}$ continuous, strictly monotone on $[a,b]$. Define

$$L_g(f) = \frac{\displaystyle\int_a^b p(x)f[g(x)]dx}{\displaystyle\int_a^b p(x)dx}.$$

We can deduce from (1) the important integral inequality by Jensen:

$$f\left[\frac{\int_a^b p(x)g(x)dx}{\int_a^b p(x)dx}\right] \leq \frac{\int_a^b p(x)f[g(x)]dx}{\int_a^b p(x)dx}, \tag{4}$$

with various applications in different branches of Mathematics. We will see later, how can be applied (4) in the theory of means.

Remark. From the proof of the left side of (1) one can see that in place of convex functions one can consider **invex functions related to** $\eta : [a, b] \times [a, b] \to [a, b]$ (see [32]). This gives the following result:

Theorem 1.2. *If the function $f : [a, b] \to \mathbf{R}$ in invex related to a given function η, and the following condition is satisfied:*

$$L(\eta(e_1, L(e_1))) = 0, \tag{5}$$

then one has

$$f(L(e_1)) \leq L(f). \tag{6}$$

Corollary 1.4. Under the above conditions, as well as the conditions of Corollary 1.2, if in addition we assume that

$$\sum_{i=1}^{n} \eta\left(a_i, \sum_{i=1}^{n} w_i a_i\right) = 0,$$

then

$$f\left(\sum_{i=1}^{n} w_i a_i\right) \leq \sum_{i=1}^{n} w_i f(a_i). \tag{7}$$

We note that this relation holds true (with the analogous proof) for invex functions $f : S \to \mathbf{R}$ with $S \subset \mathbf{R}^n$ (see [32]).

B. Let $f : [a, b] \to \mathbf{R}$ and put $a = (a_1, \ldots, a_n) \in ([a, b])^n$. Let us consider the following expression

$$A_{k,n} = A_{k,n}(a) = \frac{1}{C_n^k} \sum_{1 \leq i < \cdots < i_k \leq n} f\left[\frac{a_{i_1} + \cdots + a_{i_k}}{k}\right] \tag{8}$$

(where $C_n^k = \binom{n}{k}$). Clearly

$$A_{n,n} = f\left(\frac{a_1 + \cdots + a_n}{a}\right); \quad A_{1,n} = \frac{f(a_1) + \cdots + f(a_n)}{n}.$$

This expression was considered for the first time by S. Gabler [7]. A more general (pondered) form is given by

$$A_{k,n}(a,w) = \frac{1}{C_{n-1}^{k-1} W_n} \sum_{1 \le i_1 < \cdots < i_k \le n} (w_{i_1} + \cdots + w_{i_k}) f\left(\frac{w_{i_1} a_{i_1} + \cdots + w_{i_k} a_{i_k}}{w_{i_1} + \cdots + w_{i_k}}\right) \tag{9}$$

with $W_n = \sum_{i=1}^{n} w_i$. The following refinement of the Jensen inequality holds true:

Theorem 1.3. ([29]) *One has*

$$f\left(\frac{\sum_{i=1}^{n} w_i a_i}{W_n}\right) = A_{n,n} \le \cdots \le A_{k+1,n} \le A_{k,n} \le \ldots A_{1,n} =$$

$$= \frac{\sum_{i=1}^{n} w_i f(a_i)}{\sum_{i=1}^{n} w_i}. \tag{10}$$

Corollary 1.5.

$$\frac{1}{n-1} \sum (w_1 + \cdots + \widehat{w}_i + \cdots + w_n) f\left(\frac{w_1 a_1 + \cdots + \widehat{w}_i \widehat{a}_i + \cdots + w_n a_n}{w_1 + \cdots + \widehat{w}_i + \cdots + w_n}\right) \le$$

$$\le \frac{1}{n} \frac{\sum \widehat{w}_i f(\widehat{a}_i)}{\sum \widehat{w}_i} \tag{11}$$

where $\widehat{w}_i$ denotes the fact that the term w_i is missing in the summation with $n-1$ terms (between n terms).

Proof. Apply (10) for $k = n - 1$.

Another refinement of Jensen's inequality is contained in

Theorem 1.4. ([29]) *Let* $w_i \geq 0$, $\sum_{i=1}^{n} w_i = W_n > 0$, $a_i \in [a,b]$ $(i = \overline{1,n})$. *If* $f : [a,b] \to \mathbf{R}$ *is convex, then for all* $u, v \geq 0$ *with* $u + v > 0$ *one has the inequality*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i a_i\right) \leq \left(\frac{1}{W_n}\right)^2 \sum_{i,j=1}^{n} w_i w_j f\left(\frac{u a_i + v a_j}{u+v}\right) \leq$$

$$\leq \frac{1}{W_n}\sum_{i=1}^{n} w_i f(a_i). \tag{12}$$

C. The above theorems still hold in arbitrary linear spaces, by considering the elements a_i $(i = \overline{1,n})$ to be contained in a convex subset.

Let now X be a real prehilbertian space with scalar product $(,)$ and norm $\|\cdot\|$. Let $S \subset X$ be a convex subset of X. The function $f : S \to \mathbf{R}$ will be called **uniformly-convex** on S if

$$\lambda f(x) + (1-\lambda) f(y) - f[\lambda x + (1-\lambda) y] \geq \lambda(1-\lambda)\|x-y\|^2 \tag{13}$$

for all $x, y \in S$, $\lambda \in [0,1]$.

Holds true the following characterization of uniformly-convex functions:

Proposition 1.1. ([27]) Let $f : S \to \mathbf{R}$ defined on the convex subset $S \subset X$. Then the following assertions are equivalent:

(i) f is uniformly-convex on S

(ii) $f - \|\cdot\|^2$ is convex on S.

Examples. 1) Let $A : \mathcal{D}(A) \subset X \to X$ let be a linear, symmetric operator on the subspace $\mathcal{D}(A)$ of X, which is coerciv, i.e. satisfying the relation

$$(Ax, x) \geq \gamma \|x\|^2, \ \forall\, x \in \mathcal{D}(A)\ (\gamma > 0).$$

Then the function $f_A : \mathcal{D}(A) \to \mathbf{R}$, $f_A(x) = \frac{1}{\gamma}(Ax, x)$ is uniformly-convex on $\mathcal{D}(A)$.

2) Let $f : (a,b) \subseteq \mathbf{R} \to \mathbf{R}$ be a twice differentiable functions satisfying $f''(x) \geq 0 > 0$, $x \in (a,b)$. Let $g(x) = \frac{2}{m} f(x)$, $x \in (a,b)$. Then g is uniformly-convex.

The following theorem gives also a refinement of Jensen's inequality, in case of uniformly-convex functions:

Theorem 1.6. ([27]) *Let $f : S \subset X \to \mathbf{R}$ be uniformly-convex functions on the convex set S; let $w_i \geq 0$, $W_n > 0$, (where $W_n = \sum_{i=1}^{n} w_i$) and let $a_i \in S$ ($i = \overline{1,n}$). Then*

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i f(a_i) - f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i a_i\right) \geq$$

$$\geq \frac{1}{W_n}\sum_{i=1}^{n} w_i ||a_i||^2 - \left\|\frac{1}{W_n}\sum_{i=1}^{n} w_i a_i\right\|^2 \geq 0. \tag{14}$$

Corollary 1.6. Let $A : \mathcal{D}(A) \subset X \to X$ be an operator defined as in Example 1. Then for all $a_i \in \mathcal{D}(A)$, $w_i \geq 0$, $W_n > 0$ ($i = \overline{1,n}$), holds true the following inequality:

$$W_n \sum_{i=1}^{n} w_i (Aa_i, a_i) - \left(A\left(\sum_{i=1}^{n} a_i w_i\right), \sum_{i=1}^{n} a_i w_i\right) \geq$$

$$\geq \gamma\left(W_n \sum_{i=1}^{n} w_i ||a_i||^2 - \left\|\sum_{i=1}^{n} w_i a_i\right\|\right)^2 \geq 0. \tag{15}$$

Corollary 1.7. Let $f : (a,b) \to \mathbf{R}$ be defined as in Example 2. Then for all $a_i \in (a,b)$, $w_i \geq 0$ with $W_n > 0$ ($i = \overline{1,n}$), we have

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i f(a_i) - f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i a_i\right) \geq$$

$$\geq \frac{m}{2}\left[\frac{1}{W_n}\sum_{i=1}^{n} w_i a_i^2 - \frac{1}{W_n^2}\left(\sum_{i=1}^{n} w_i x_i\right)^2\right] \geq 0.$$

D. The **convex functions of order** n were introduced in the science by Tiberiu Popoviciu [11]. The following result is related to the discrete inequality by Jensen:

Theorem 1.7. *Let $f : (a,b) \to \mathbf{R}$ be a concave and 3-convex function. Let (a_i), (b_i) ($i = \overline{1,n}$) two sequences in (a,b) having the properties*

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq b_n \leq \cdots \leq b_2 \leq b_1$$

$$a_{i+1} - a_i \geq b_i - b_{i+1} \quad (i = 1,2,\ldots,n-1)\ (n \geq 2).$$

Then

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i f(a_i) - f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i a_i\right) \leq$$

$$\leq \frac{1}{W_n}\sum_{i=1}^{n} w_i f(b_i) - f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i b_i\right). \tag{16}$$

Proof. We will use induction with respect to n. Let $n = 2$. For simplicity, let us assume that $W_2 = w_1 + w_2 = 1$. Let $a_0 = w_1a_1 + w_2a_2$, $b_0 = w_1b_1 + w_2b_2$. If $b_1 = b_2$, by concavity of f it results $w_1f(a_1) + w_2f(a_2) - f(w_1a_1 + w_2a_2) \leq 0$, which shows that (16) is true in this case. If $b_1 \neq b_2$, then $a_1 < a_0 < a_2 \leq b_2 < b_0 < b_1$, so f being 3-convex, we can write:

$$\frac{f(a_1)}{(a_1-a_2)(a_1-a_0)} + \frac{f(a_2)}{(a_2-a_1)(a_2-a_0)} + \frac{f(a_0)}{(a_0-a_1)(a_0-a_2)} \leq$$

$$\leq \frac{f(b_1)}{(b_1-b_2)(b_1-b_0)} + \frac{f(b_2)}{(b_2-b_1)(b_2-b_0)} + \frac{f(b_0)}{(b_0-b_1)(b_0-b_2)}. \tag{$*$}$$

By definition, $a_0 - a_1 = w_2(a_2 - a_1)$; $a_2 - a_0 = w_1(a_2 - a_1)$, so by multiplying both sides of $(*)$ with $w_1w_2(a_2 - a_1)^2$, one can deduce

$$w_1f(a_1) + w_2f(a_2) - f(a_0) \leq [w_1f(b_1) + w_2f(b_2) - f(b_0)]\frac{(a_1-a_2)^2}{(b_1-b_2)^2}.$$

By concavity of f it results $w_1f(b_1) + w_2f(b_2) - f(b_0) \leq 0$. From $a_2 - a_1 \geq b_1 - b_2 > 0$ we get $w_1f(a_1) + w_2f(a_2) - f(a_0) \leq w_1f(b_1) + w_2f(b_2) - f(b_0)$, proving (16) for $n = 2$.

Let us assume now that (16) holds true for all arguments from 2 to $n - 1$. Then

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i f(a_i) - \frac{1}{W_n}\sum_{i=1}^{n} w_i f(b_i) =$$

$$= \frac{W_{n-1}}{W_n}\left\{\frac{1}{W_{n-1}}\sum_{i=1}^{n-1} w_i f(a_i) - \frac{1}{W_{n-1}}\sum_{i=1}^{n-1} w_i f(b_i)\right\} +$$

$$+\frac{w_n}{W_n}[f(a_n) - f(b_n)] \leq \frac{W_{n-1}}{W_n}\left\{f\left(\frac{1}{W_{n-1}}\sum_{i=1}^{n-1} w_i a_i\right) -\right.$$

$$\left.-f\left(\frac{1}{W_{n-1}}\sum_{i=1}^{n-1} w_i b_i\right)\right\} + \frac{w_n}{W_n}[f(a_n) - f(b_n)].$$

Let

$$c_1 = \frac{1}{W_{n-1}}\sum_{i=1}^{n-1} w_i a_i, \quad c_2 = a_n,$$

$$d_1 = \sum_{i=1}^{n-1} w_i b_i / W_{n-1}, \quad d_2 = b_n.$$

Then the sequences $\{c_1, c_2\}$ and $\{d_1, d_2\}$ satisfy the conditions of the theorem. Applying the above proved case $n = 2$ with W_{n-1} and w_n in place of w_1, w_2, we obtain the desired inequality.

Corollary 1.8. Let $b > 0$ and $a_i \in (0, b]$ $(i = \overline{1,n})$. Let $f : (0, 2b] \to \mathbf{R}$ have a negative second derivative and a nonnegative third derivative. Then

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i f(a_i) - f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i a_i\right) \le$$

$$\le \frac{1}{W_n}\sum_{i=1}^{n} w_i f(2b - a_i) - f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i (2b - a_i)\right). \tag{17}$$

Proof. Put $b_i = 2b - a_i$, where $a_1 \le a_2 \le \cdots \le a_n$. Then the conditions of Theorem 1.7 are satisfied, and we get relation (17). This inequality has been obtain by N. Levinson (see [5]) as a generalization of the famous inequality of Ky Fan ([5], [4], [13]).

Let $a_i \in \left(0, \frac{1}{2}\right]$, $(i = \overline{1,n})$,

$$A_n(a) = \frac{1}{W_n}\sum_{i=1}^{n} w_i a_i, \quad G_n(a) = \prod_{i=1}^{n} a_i^{w_i/W_n},$$

where $a = (a_1, \ldots, a_n)$. Put $A\prime_n(a) = A_n(1-a)$, $G\prime_n(a) = G_n(1-a)$. Then

$$\frac{G_n}{G\prime_n} \le \frac{A_n}{A\prime_n}. \tag{18}$$

Proof. Apply (17) with $b = \frac{1}{2}$ to $f(x) := \ln x$. Then $f''(x) = -\frac{1}{x^2} < 0$, $f'''(x) = \frac{2}{x^3} > 0$, and after certain elementary computations we obtain Ky Fan's inequality (18).

Let now, for simplicity, $w_i \equiv 1$, $(i = \overline{1,n})$. Then relation (17) can be written also as

$$\frac{1}{n}\sum_{i=1}^{n} f(a_i) - f(A_n) \le \frac{1}{n}\sum_{i=1}^{n} f(1 - a_i) - f(A\prime_n) \tag{19}$$

where A_n is the (unweighted) arithmetic mean of (a), and A'_n is the (unweighted) arithmetic mean of $(1-a)$.

Let us introduce also

$$H_n = H_n(a) = n/\sum_{i=1}^{n} \frac{1}{a_i}, \quad H'_n = H_n(a') = H_n(1-a),$$

the corresponding harmonic means. Let $f(x) = -\frac{1}{x}$ in (19). Then we can deduce the following "additive variant" of the Ky Fan inequality:

$$\frac{1}{A_n} - \frac{1}{H_n} \le \frac{1}{A'_n} - \frac{1}{H'_n}. \tag{20}$$

For other variants and refinements we quote the author's papers [13], [14]. See also [4].

E. Inequalities for nondifferentiable η-invex functions generally are fairly difficult to obtain. More precisely, either we must assume that the function f satisfies certain complicated functional equations (see [32]), or if we do not admit such relations, the informations contained in these inequalities are more restrictive.

Let us remind that the function $f : S \to \mathbf{R}$ is called η-invex on the η-invex domain S, if one has

$$f(u + \lambda\eta(x,u)) \le \lambda f(x) + (1-\lambda)f(u) \text{ for all } x, u \in S,\ \lambda \in [0,1]. \tag{21}$$

Let $\lambda = \frac{p}{p+q}$ $(p, q > 0)$. From (21) it follows

$$f\left[\frac{(p+q)u + p\eta(x,u)}{p+q}\right] \le \frac{pf(x) + qf(u)}{p+q}. \tag{22}$$

Let now $S \subset \mathbf{R}_+ = [0,\infty)$ and apply relation (22) to $p := x_1$, $q := x_2$, $x := x_1 + x_2$, yielding:

$$f\left[\frac{(x_1+x_2)u + x_1\eta(x_1+x_2,u)}{x_1+x_2}\right] \le \frac{x_1 f(x_1+x_2) + x_2 f(u)}{x_1+x_2}.$$

By interchanging x_1 with x_2 we can write

$$f\left[\frac{(x_1+x_2)u + x_2\eta(x_1+x_2,u)}{x_1+x_2}\right] \le \frac{x_2 f(x_1+x_2) + x_1 f(u)}{x_1+x_2}.$$

By addition we get

$$f(x_1+x_2)+f(u)\geq f[u+\alpha_1\eta(x_1+x_2,u)]+f[u+\alpha_2\eta(x_1+x_2,u)] \tag{23}$$

where $\alpha_1+\alpha_2=1$, $\alpha_1>0$, $\alpha_2>0$.

Put $u:=0$ in (23) and assume that f satisfies

$$f(a\theta(b))\geq f(ab) \text{ cu } a,b>0 \tag{24}$$

where $\theta(x)=\eta(x,0)$. By taking into account of

$$f\left[\frac{x_1}{x_1+x_2}\theta(x_1+x_2)\right]\geq f(x_1) \quad \text{şi} \quad f\left[\frac{x_2}{x_1+x_2}\theta(x_1+x_2)\right]\geq f(x_2),$$

one gets the inequality

$$f(x_1)+f(x_2)\leq f(x_1+x_2)+f(0), \quad \text{with} \quad x_1>0,\ x_2>0. \tag{25}$$

By mathematical induction it easily follows now that

$$f(x_1)+\cdots+f(x_n)\leq f(x_1+\cdots+x_n)+(n-1)f(x_0), \quad x_i>0\ (i=\overline{1,n})\ (n\geq 1) \tag{26}$$

So, we have proved the following result:

Theorem 1.8. Let $f:[0,\infty)\to\mathbf{R}$ be η-invex function and let $\theta(x)=\eta(x,0)$ with $x>0$. Let us assume that for $a,b>0$ one has the inequality $f(a\theta(b))\geq f(ab)$. Then, for all $x_i>0$ $(i=\overline{1,n})$, $(n\geq 1)$ we have the inequality (26).

Remark. For convex f and $\theta(x)=x$ we can reobtain from (26) the known inequality by M. Petrović ([10]).

In what follows we shall introduce the notion of **invex combination**. Let X be a linear space and let $S\subset X$ be an invex subset of X. We say that z is an invex combination of x_1 and x_2, in notation $z\in \operatorname{inv}(x_1,x_2)$ if there exists $\lambda\in[0,1]$ such that $z=x_2+\lambda\eta(x_1,x_2)$. Let $x_1,\ldots,x_n\in S$. Then $z\in\operatorname{inv}(x_1,x_2,\ldots,x_n)$ (invex combination of n elements) if there exist $y\in\operatorname{inv}(x_1,\ldots,x_{n-1})$ and there exists $\lambda\in[0,1]$ such that

$$z=y+\lambda\eta(x_n,y)\in\operatorname{inv}(y,x_n).$$

We can prove the following analogue of Jensen's inequality:

Theorem 1.9. Let $f : S \to \mathbf{R}$ be η-invex function. Then for all $n \geq 2$ and $x_1, x_2, \ldots, x_n \in S$ and $z \in \operatorname{inv}(x_1, x_2, \ldots, x_n)$ there exists $Z \in \operatorname{conv}(f(x_1), \ldots, f(x_n))$ with the property

$$f(z) \leq Z \tag{27}$$

where conv *is the convex combination.*

Proof. We shall proceed by mathematical induction. For $n = 2$ we have $z \in \operatorname{inv}(x_1, x_2) \in S$, so $z = x_2 + \lambda\eta(x_1, x_2)$ and from (21) we can deduce $f(z) \leq \lambda f(x_1) + (1-\lambda) f(x_2) = Z \in \operatorname{conv} f(x_1), f(x_2))$. Let us assume that relation (27) holds for n elements, and let $z\prime \in \operatorname{inv}(x_1, x_2, \ldots, x_{n+1})$, where $z \in \operatorname{inv}(x_1, x_2, \ldots, x_n)$. Then $z\prime$ has a form $z\prime = z + \lambda\eta(x_{n+1}, z)$ so we can write $f(z\prime) \leq \lambda f(x_{n+1}) + (1-\lambda) f(z) = \lambda f(x_{n+1}) + (1-\lambda)[\overline{\lambda}_1 f(x_1) + \overline{\lambda}_2 f(x_2) + \cdots + \overline{\lambda}_n f(x_n)]$ where $\overline{\lambda}_1 + \cdots + \overline{\lambda}_n = 1$. Therefore, $f(\overline{z}) \leq \overline{\lambda}_1(1-\lambda) f(x_1) + \overline{\lambda}_2(1-\lambda) f(x_2) + \cdots + \overline{\lambda}_n(1-\lambda) f(x_n) + \lambda f(x_{n+1})$. Remarking that $\overline{\lambda}_1(1-\lambda) + \cdots + \overline{\lambda}_n(1-\lambda) + \lambda = 1$ we get $f(z\prime) \leq Z\prime \in \operatorname{conv}(f(x_1), \ldots, f(x_{n+1}))$, finishing the proof of Theorem 1.9.

F. In this final subsection on Jensen's inequality we mention certain applications. First we reobtain the classical inequality of weighted means. This inequality plays a central role in information theory (Shannon's theory of entropy) [1], in the theory of codes (Kraft's inequality), in the theory of functional equations and rational group decision [2], etc. (See e.g. [3] for applications and economics, and [6] for geometric programming).

Theorem 1.10. (Theorem of means) *Let $a_j > 0$, $q_j > 0$ $(j = \overline{1,n})$ with $\sum_{j=1}^{n} q_j = 1$. Then we have*

$$\prod_{j=1}^{n} a_j^{q_j} \leq \sum_{j=1}^{n} q_j a_j \tag{28}$$

Proof. Select $b_j := \log a_j$ and the convex function $f(t) = e^t$ $(t \in (-\infty, \infty))$ and apply Jensen's discrete inequality.

By letting $n = 2$, $q_1 = \frac{1}{p}$, $q_2 = \frac{1}{q}$; $a_1 = x^p$, $a_2 = y^q$ with $\frac{1}{p} + \frac{1}{q} = 1$, we obtain:

Corollary 1.9. a) (Young's inequality)

$$xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q, \text{ where } \frac{1}{p} + \frac{1}{q} = 1,\ p > 1 \tag{29}$$

b) (Hölder's inequality)

$$\sum_{j=1}^{n} x_j y_j \leq \left(\sum_{j=1}^{n} x_j^p\right)^{1/p} \left(\sum_{j=1}^{n} y_j^q\right)^{1/q} \quad (x_j, y_j > 0) \tag{30}$$

Proof. It is sufficient to consider

$$u := \left(\sum_{j=1}^{n} x_j^p\right)^{1/p}, \quad v = \left(\sum_{j=1}^{n} y_j^q\right)^{1/q}$$

and apply (29) for $x := x_j/u$, $y := y_j/v$. After summation we get (30).

The following little known refinement of (23) is due to the author [15]:

Theorem 1.11. *Let* $\lambda > 0$, $p > 0$ *and let*

$$J(a_i, q_i, p, \lambda) = \left\{ p \int_0^\infty \left[\frac{\prod_{j=1}^{n} (1 + \lambda a_j + \lambda x)^{q_j} - 1}{\lambda} \right]^{-p-1} dx \right\}^{1/p}$$

and

$$J(a_i, q_i, p) = \left\{ p \int_0^\infty \left[\prod_{j=1}^{n} (x + a_j)^{q_j} \right]^{-p-1} dx \right\}^{-1/p}.$$

Then we have the following inequalities:

$$\prod_{j=1}^{n} a_j^{q_j} \leq J(a_i, q_i, p) \leq J(a_i, q_i, p, \lambda) \leq \sum_{j=1}^{n} q_j a_j. \tag{31}$$

Proof. Since this result has been published in a journal with reduced circulation, we give here the proof of (31). First we prove that

$$\prod_{j=1}^{n} a_j^{q_j} \leq \frac{1}{\lambda} \left[\prod_{j=1}^{n} (1 + \lambda a_j)^{q_j} - 1 \right] \leq \sum_{j=1}^{n} a_j q_j. \tag{32}$$

Indeed, let $f(x)=\ln(1+\lambda e^x)$, $x\in\mathbf{R}$, which is strictly convex since $f''(x)=\lambda e^x/(1+\lambda e^x)^2>0$. By Jensen's inequality we have

$$\ln\left(1+\lambda e^{\sum\limits_{j=1}^{n} a_j q_j}\right)\le\sum_{j=1}^{n} q_j\ln(1+\lambda e^{a_j}).$$

By the substitution $e^{a_j}\to a_j$ we obtain

$$1+\lambda\prod_{j=1}^{n} a_j^{q_j}\le\prod_{j=1}^{n}(1+\lambda a_j)^{q_j}.$$

On the other hand, from the inequality of means we can write

$$\prod_{j=1}^{n}(1+\lambda a_j)^{q_j}\le 1+\lambda\sum_{j=1}^{n} a_j q_j,$$

which combined with the above inequality gives (32). Apply now this inequality to a_j+x in place of a_j and integrate the obtained relation. We can successively deduce

$$\prod_{j=1}^{n}(a_j+x)^{q_j}\le\frac{1}{\lambda}\left[\prod_{j=1}^{n}(1+\lambda a_j+\lambda x)^{q_j}-1\right]\le\sum_{j=1}^{n} q_j a_j+x$$

and since $p>0$, we have

$$\int_0^\infty\left[\sum_{j=1}^{n}(x+a_j)^{q_j}\right]^{-p-1}dx\ge\int_0^\infty\left[\frac{\prod\limits_{j=1}^{n}(x+\lambda a_j+\lambda x)^{q_j}-1}{\lambda}\right]^{-p-1}dx\ge$$

$$\ge\int_0^\infty\left[x+\sum_{j=1}^{n} q_j a_j\right]^{-p-1}dx=\frac{1}{p}\left(\sum_{j=1}^{n} q_j a_j\right)^{-p}. \tag{33}$$

By Hölder's integral inequality for n functions (which for 2 functions is in fact a consequence of (30), while for n functions follows by mathematical induction, see e.g. [8]) we can write

$$\int_0^\infty\left[\prod_{j=1}^{n}(x+a_j)^{q_j}\right]^{-p-1}dx=\int_0^\infty\prod_{j=1}^{n}[(x+a_j)^{-p-1}]^{q_j}dx\le$$

$$\le\prod_{j=1}^{n}\left[\int_0^\infty(x+a_j)^{-p-1}dx\right]^{q_j}=\prod_{j=1}^{n}\frac{1}{p}a_j^{-pq_j},$$

which combined with (33) gives us

$$\prod_{j=1}^{n} \frac{1}{p}(a_j)^{-pq_j} \geq \int_0^\infty \left[\sum_{j=1}^{n}(x+a_j)^{q_j}\right]^{-p-1} dx \geq$$

$$\geq \int_0^\infty \left[\frac{\prod_{j=1}^{n}(1+\lambda a_j+\lambda x)^{q_j}-1}{\lambda}\right]^{-p-1} dx \geq \frac{1}{p}\left(\sum_{j=1}^{n} q_j a_j\right)^{-p},$$

finishing the proof of theorem.

Corollary 1.10. $(a_1 a_2 \dots a_n)^{1/n} \leq \left\{p\int_0^\infty [(x+a_1)\dots(x+a_n)]^{-(p+1)/n}dx\right\}^{1/p} \leq$

$$\leq \left\{p\int_0^\infty \left[\frac{(1+\lambda a_1+\lambda x)^{1/n}\dots(1+\lambda a_n+\lambda x)^{1/n}-1}{\lambda}\right]^{-p-1} dx\right\}^{-1/p} \leq$$

$$\leq \frac{1}{n}(a_1+\dots+a_n). \tag{34}$$

(Put $q_1 = \dots = q_n = \frac{1}{n}$ in (31)).

Application. Let $n = 3$ in (34). We shall apply this relation in the theory of **geometric inequalities**. Let ABC be a triangle of sides a, b, c; with r as the inscribed circle radius, R as the circumscribed circle radius. Then (see [16]) it is known that

$$R \geq \frac{a+b+c}{3\sqrt{3}} \quad \text{and} \quad 2r \leq \frac{(abc)^{1/3}}{\sqrt{3}}.$$

From the above inequality for $n = 3$ we can obtain the following refinements

$$2r\sqrt{3} \leq (abc)^{1/3} \leq J(a,b,c,p) \leq J(a,b,c,p,\lambda) \leq \frac{1}{3}(a+b+c) \leq R\sqrt{3},$$

implying in fact infinitely many refinements of the classical Euler inequality $2r \leq R$.

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