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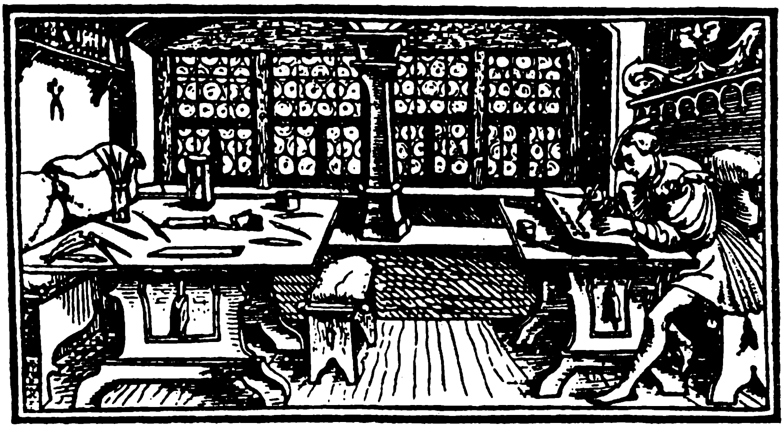

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# STUDIA

## UNIVERSITATIS "BABEŞ-BOLYAI"

## MATHEMATICA

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# RIGIDITY OF HARMONIC MEASURE OF TOTALLY DISCONNECTED FRACTALS

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**Abstract.** Let $f : V \to U$ be a generalized polynomial-like map. Suppose that harmonic measure $\omega = \omega(\cdot, \infty)$ on the Julia set $J_f$ is equal to measure of maximal entropy $\mu$ for $f : J_f \hookleftarrow$. Then the dynamics $(f, V, U)$ is called maximal. We are going to give a necessary condition for the dynamics to be conformally equivalent to a maximal one, that is to be conformally maximal. Namely the purpose of the paper is to prove that if the Julia set is totally disconnected then $\omega \approx \mu$ implies that the system $(f, U, V)$ is conformally maximal. This shows that maximal systems are natural substitutes for polynomials in the class of genereralized polynomial-like mappings.

## 0. Introduction

Let $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ be a polynomial of degree $d$. A result of H. Brolin (see [Br]) says that the backward orbits of $f$ are equidistributed with respect to the measure $\omega$ : the harmonic measure on the boundary of the domain of attraction to $\infty$ and evaluated at $\infty$.

Almost twenty years later the ergodic theory of rational maps has started by the works of M. Lyubich ([Ly1], [Ly2]) and independently by A. Freire, A. Lopes and R. Mañé ([FLM], [Ma1]). It was established that for any rational map $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ there is a unique $f$ -invariant probability measure $\mu$ on the Borel $\sigma$-algebra such that:

$$\mu(f(E)) = d \cdot \mu(E) \tag{0.1}$$

for any Borel set $E$ such that $f|_E$ is injective. The measure $\mu$ is the unique $f$-invariant probability measure that maximizes the entropy i.e. $h_\mu(f) = \log d$. In the light of these works Brolin's result can be interpreted as the fact that for polynomials we have $\omega = \mu$.

Conversely A. Lopes proved in [Lo] that if we have a rational map $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ such that $\omega = \mu$ (where $\omega$ is the harmonic measure on the boundary of an attracting forward invariant component of the Fatou set) then $f$ is conjugated to a polynomial by a Möbius transformation. A simpler proof of this result was given in [MR].

The purpose of this paper is to extend the result of Lopes to the class of generelized polynomial-like mappings. Let us recall this definition from [BPV], [BV1], [BV2].

We will be considering triples $(f, V, U)$, where $U$ is a topological disc and $V = V_1 \cup \cdots \cup V_k$ is the union of topological discs whose closures are disjoint and are contained in $U$. Also $f_i := f|V_i$ is a regular or branched covering $V_i \to U$ of degree $d_i$ (so "regular" means that $d_i = 1$). By $d = d_1 + \cdots + d_k$ we denote the degree of the map $f : V \to U$. These dynamical systems will be called generalized polynomial-like systems or GPL. The limit set ($\equiv$ *Julia set*) is $J_f = \partial K_f$, where $K_f = \bigcap_{n \geq 0} f^{-n}(U)$ is the filled Julia set.

If $k = 1$, $d \geq 2$, we come to a class of polynomial-like systems (PL) introduced in [DH] and playing an important role in classification of polynomial dynamics.

Being GPL means to be quasiconformally equivalent to a polynomial:

$$f \in \text{GPL} \Rightarrow \exists\, h \in qc(U) : f = h^{-1} \circ \text{poly} \circ h.$$

The starting point is to see whether the result of Lopes is true under the weaker assumption $\omega \approx \mu$. Here "$\approx$" means that $\omega$ and $\mu$ are mutually absolutely continuous and in addition to that we assume that there exists $M > 0$ such that for any $x \in J$ and $r > 0$:

$$\frac{1}{M} \leq \frac{\mu(B(x,r))}{\omega(B(x,r))} \leq M. \tag{0.2}$$

In other words we ask the following question: is it true that if for a GPL $(f, U, V)$ we have $\omega \approx \mu$ does it follow that $(f, U, V)$ is conformally conjugated to a polynomial?

This is a rigidity-type question aiming to rule out quasiconformal deformations.

The converse is obviously true: if our GPL is conformally conjugated to a polynomial then by Brolin's result and Harnack's inequality we obtain $\omega \approx \mu$.

It is quite a suprise to see that the answer to this question is generally negative. This was shown by M. Lyubich and A. Volberg. Namely a GPL $(f, U, V)$ was constructed in [LyV] where $\mu = \omega$ but $f$ is not conformally conjugated to a polynomial.

To formulate the appropriate question for the class of GPL we call a GPL system $(f, U, V)$ *maximal* if $\mu_f = \omega_f$. Maximal GPL systems have been introduced in [BPV] as natural substitues for polynomials.

Next we call a GPL system $(f, U, V)$ *conformally maximal* if it is conformally equivalent to a maximal system; that is:

$$f = H^{-1} \circ g \circ H,$$

where $H : U_f \to U_g$ is a conformal map and $(g, U_g, V_g)$ is a maximal system.

In this paper we are going to prove the following rigidity result:

**Theorem 4.1** *Let $(f, U, V)$ be a GPL with totally disconnected Julia set. Then $\omega_f \approx \mu_f$ implies that $(f, U, V)$ is conformally maximal.*

The converse of this result follows immediately by Harnack's inequality.

A weaker form of Theorem 4.1 under the condition of semihyperbolicity of $f$ was proved in [BPV]. Also in [BPV] it was explained that this result is an analog of a theorem of Shub and Sullivan (see [SS]) on "wild" (i.e. totally disconnected) $J_f$ and nonexpanding $f$.

## 1. Idea of proof.

Let us start with the following criterion of conformal maximality proven in [BPV]:

**Theorem A** *Let $(f, V, U)$ be a GPL system. Two assertions are equivalent:*

1) *$(f, V, U)$ is conformally maximal;*
2) *there exists a non-negative subharmonic function $\tau$ on $U$, which is positive and harmonic in $U \setminus K_f$, vanishes on $K_f$ and satisfies*

$$\tau(fz) = d\tau(z). \qquad (Aut)$$

If $\nu$ is an arbitrary probability measure on $J_f$ then a general theorem [Pa], says that there exists *Jacobian $J_\nu = J_\nu(f)$ on a set of full measure $\nu$.* It means that there exists $Y \subset J$, $\nu(J \setminus Y) = 0$, and a-$\nu$ integrable function $J_\nu$ such that for every $E \subset Y$ on which $f$ is 1-to-1 onto $f(E)$ we have $\nu(f(E)) = \int_E J_\nu d\nu$. By (0.1) the Jacobian of $\mu$ is $J_\mu = d$. We denote the Jacobian of the harmonic measure by $J_\omega$.

For a better exposition I would like to sketch the strategy of the approach in [BPV]:

-if the dynamics $f$ is semihyperbolic and $J$ is totally disconnected the function $\varphi(x) = \log J_\omega(x)$ is Hölder continuous on $J$,

-Hölder continuity of $\varphi$ together with $\omega \approx \mu$ is used to prove that there exists a Hölder continuous function $u : J \to \mathbf{R}$ satisfying the homologous equation:

$$\varphi(x) - \log d = u(fx) - u(x) \;\; \forall x \in J, \qquad (1.1)$$

-starting from (1.1) we can build an automorphic function $\tau$ as required by Theorem A to prove conformal maximality of $(f, U, V)$.

In our more general case we do not have the Hölder continuity of $\varphi = \log J_\omega$. Therefore the above approach based on thermodynamic formalism is not applicable. Still we have a modified strategy as follows:

-Pesin theory gives a certain regularity property of $\varphi$,

-we consider the function $\phi = \varphi - \log d$ and the sequence of random variables $\{\phi \circ f^k\}_k$ on the probability space $(J, \mu)$,

-$\omega \approx \mu$ implies that $\int_J \phi \, d\mu = 0$,

-using a technique from [DPU] it follows that the sequence $\{\phi \circ f^k\}_k$ obeys the law of Central Limit Theorem or CLT

- applying CLT we obtain a function $u \in L^2(\mu)$ satisfying the homologous equation:

$$\varphi(x) - \log d = u(fx) - u(x) \text{ for } \mu \text{ a.e. } x \in J, \tag{1.2}$$

-staring from (1.2) we can construct again the automorphic function $\tau$ required by Theorem A.

## 2. Jacobian of the harmonic measure.

In this section we study regularity properties of the function $\varphi = \log J_\omega$. Our first ingredient is a result of F. Grishin (see [Gr]):

**Lemma B** *Let $\infty \notin K \subset \hat{\mathbf{C}}$ be a compact set and denote by $\omega$ the harmonic measure in $\hat{\mathbf{C}} \setminus K$ evaluated at $\infty$. Let $\mathcal{O}$ be an open set containing $K$ and let $u$ , $v \geq 0$ be two continuous subharmonic function; positive and harmonic in $\mathcal{O} \setminus K$ and vanishing on $K$. Let us suppose that the limit:*

$$\rho(x) = \lim_{\substack{z \to x \\ z \in \mathcal{O} \setminus K}} \frac{u(z)}{v(z)} \quad \textit{exists for } \omega \textit{ a.e. } x \in \partial K.$$

*Then we have that $d\mu_u = \rho d\mu_v$ where $\mu_u$ and $\mu_v$ denote the Riesz measures of $u$ and $v$.*

In our applications we put $K = J$, $\mathcal{O} = U$ , $u = G \circ f$ and $v = G$, where G is Green's function of $\hat{\mathbf{C}} \setminus J$ with pole at $\infty$.

Let us suppose for the moment that the limit $\lim_{\substack{z\to x \\ z\in U\setminus J}} \frac{G(fz)}{G(z)}$ exists for $\omega$ a.e. $x \in J$. Using that $\omega = \Delta G$, $\omega \circ f = \Delta(G \circ f)$ Lemma B gives :

$$J_\omega(x) = \lim_{\substack{z\to x \\ z\in U\setminus J}} \frac{G(fz)}{G(z)} \text{ for } \omega \text{ a.e. } x \in J \tag{2.1}$$

To establish the existence of the above limit we introduce:

**Definition 2.1** *Let $K \subset \hat{\mathbb{C}}$ and $\mathcal{O}$ be as in Lemma B and let us fix a number $\beta > 0$. We say that a set $E \subset \mathcal{O}$ is k-nested if there exist annuli $\{A_i\}_{i=1}^k$ with the properties:*

(1) $modA_i > \beta$ ,

(2) $A_i \subset \mathcal{O} \setminus K$ ,

(3) $E \subset inA_k \subset inA_{k-1} \subset \ldots \subset inA_1$ ,

*where $inA_i$ denotes the component of $\hat{\mathbb{C}} \setminus A_i$ containing $E$.*

The existence of the limit in (2.1) will be based on the following result called Boundary Harnack Principle (see [MaV] or [BV2] for the proof):

**Lemma C** *Let $K$, $\mathcal{O}$, $u$, $v$ be as in Lemma B and $\beta > 0$ be fixed. There exist $C > 0$ , $0 < q < 1$ depending only on $K$ and $\beta > 0$ such that if $E \subset \mathcal{O}$ is k-nested we have :*

$$\left|\log \frac{u(x)}{v(x)} - \log \frac{u(y)}{v(y)}\right| \le C \cdot q^k \;\; \forall x, y \in E \setminus K \tag{2.2}$$

If we choose as before $K = J$, $\mathcal{O} = U$, $u = G \circ f$, $v = G$ (2.2) becomes:

$$\left|\log \frac{G(fx)}{G(x)} - \log \frac{G(fy)}{G(y)}\right| \le C \cdot q^k, \tag{2.3}$$

for any $x, y \in U \setminus J$ such that $\{x, y\}$ is k-nested. We also mention that as $\beta > 0$ will be fixed we have $C > 0$, $0 < q < 1$ fixed throughout the paper.

**Definition 2.2** *A point $x \in J$ is called a good point if it is $\infty$-tely nested.*

We will see that the limit in (2.1) exists for the good points but first we prove:

**Lemma 2.3** *Suppose that $\omega \approx \mu$ then there exists $\beta > 0$ such that $\omega$ a.e. point is a good point.*

*Proof.* We are going to show that $\mu$ a.e. point is a good point.

We consider the natural extension $(\tilde{f}, \tilde{J}, \tilde{\mu})$ of the dynamical system $(f, J, \mu)$; that is :

$$\tilde{f} : \tilde{J} := \{(x_k)_{k \in -\mathbf{N}} : f(x_k) = x_{k+1} \ (k \leq -1\} \to \tilde{J},$$

where

$$\tilde{f}((x_k)_{k \in -\mathbf{N}}) = (f(x_k))_{k \in -\mathbf{N}}.$$

Then the Borel $\sigma$-field in $J$ defines a $\sigma$-field $M_0$ in $\tilde{J}$ by $M_0 := \pi^{-1}(B)$ where $\pi$ denotes the projection of $\tilde{J}$ onto the first coordinate. It is clear that $(\tilde{f})^{-1} M_0 \subset M_0$. Finally denote by $\tilde{\mu}$ the natural extension of $\mu$ to $\tilde{J}$.

A standard fact in Pesin theory (see [Pr1] pp.16) shows that for $\tilde{\mu}$ a.e. $\tilde{x} \in \tilde{J}$ there exists $r = r(\tilde{x}) > 0$ such that univalent branches $f_n$ of $f^{-n}$ on $B(\pi(\tilde{x}), r(\tilde{x}))$ for $n = 1, 2, \ldots$ such that $f_n(\pi(\tilde{x})) = \pi((\tilde{f})^{-n}(\tilde{x}))$ exist.

Moreover for an arbitrary $\lambda$ : $1/d < \lambda < 1$ (not depending on $\tilde{x}$) and a constant $C = C(\tilde{x}) > 0$:

$$\left| f_n'(\pi(\tilde{x})) \right| \leq C \cdot \lambda^n \quad \text{and} \quad \frac{|f_n'(\pi(\tilde{x}))|}{|f_n'(z)|} \leq C \,, \tag{2.4}$$

for every $z \in B(\pi(\tilde{x}, r), \ n > 0$.

Furthermore $r$ and $C$ are measurable functions of $\tilde{x}$.

To use this fact observe that there are $C, r > 0$ and a set $\tilde{Y} \subset \tilde{J}$ with $\tilde{\mu}(\tilde{Y}) > 0$ such that the above properties hold for $\tilde{x} \in \tilde{Y}$ and for these $C$ and $r$. As $\tilde{\mu}$ is ergodic, by Birkhoff's ergodic theorem, there exists a set $\tilde{X} \subset \tilde{J}, \ \tilde{\mu}(\tilde{X}) = 1$ such that:

$$\lim_{n \to \infty} \frac{\#\{k \leq n : \tilde{f}^k(\tilde{x}) \in \tilde{Y}\}}{n} = \tilde{\mu}(\tilde{Y}) > 0, \ \ \forall \tilde{x} \in \tilde{X}. \tag{2.5}$$

Let us put $X := \pi(\tilde{X}) \subset J$. Then $\mu(X) = 1$ and our goal is now to prove that there exists $\chi > 0$ such that $\forall x \in X$ there is $N = N(x)$ with the property that for any $n \geq N$ we have that $x$ is $\chi \cdot n$-nested. Once this is proved we are done.

Let us pick $x \in X$ and consider $\tilde{x} \in \tilde{X}$ such that $\pi(\tilde{x}) = x$. By (2.5) there exists $N = N(\tilde{x})$ such that if $n \geq N$:

$$\#\{k \leq n : \tilde{f}^k(\tilde{x}) \in \tilde{Y}\} \geq \frac{\mu(\tilde{Y})}{2} \cdot n. \tag{2.6}$$

Let $A(n,x) := \{k \leq n : \tilde{f}^k(\tilde{x}) \in \tilde{Y}\}$. We denote by $B_k(x)$ the component of $f^{-k}(B(f^k(x), r))$ which contains $x$. If $k \in A(n,x)$ we have that the mapping $f^k : B_k(x) \to B(f^k(x), r)$ is univalent.

We are going to pull annuli from $B(f^k(x), r)$ to $B_k(x)$ using the univalency of $f^k$. First we have the following:

**Claim:** There exist numbers $\beta > 0,\ r' > 0$ such that for any $y \in J$ there exists an annulus $A(y,r) \subset B(y,r) \setminus J$ such that $y \in inA(y,r)$ with the properties:

(a) $mod\ A(y,r) > \beta$

(b) $dist\ (A(y,r),\ y) > r'$

To prove the claim consider an annulus $A_0 \subset U$ such that $J \subset inA_0,\ \ mod(A_0) = \beta_0$. Since $J$ is totally disconnected and $J = \cap f^{-n}(U)$ there exists $N_0 > 0$ such that $diam\ B_{N_0} < r$ for any component $B_{N_0}$ of $f^{-N_0}(U)$.

For $y \in J$ let us denote by $A_{N_0}(y) \subset B_{N_0}(y) \subset B(y,r) \setminus J$ the component of $f^{-N_0}(A_0)$ such that $y \in inA_{N_0}(y)$. Because $N_0$ is fixed properties (a) and (b) follow for $A(y,r) := A_{N_0}(y)$. This proves the claim.

• Let us put now $A_k(x) := f^{-k}(A(f^k(x), r))$ for any $k \in A(n,x)$. It is clear that $x \in inA_k(x),\ \ A_k(x) \subset B_k(x) \setminus J,$ and $mod\ A_k(x) = mod\ A(f^k(x), r) > \beta_0$ by the univalency of $f^k|_{B_k(x)}$.

Our annuli will be selected from $A_k(x),\ k \in A(n,x)$; however we need to exclude some of them to make sure that they are nested inside each other.

To do that we use (2.4) to see that there exists $L > 0$ (independent of $n$ and $x$) such that if $k_1, k_2 \in A(n,x),\ \ k_2 \geq k_1 + L$ we have $B_{k_2-k_1}(f^{k_1}(x)) \subset B(f^{k_1}(x), r')$. This implies that $x \in inA_{k_2}(x) \subset inA_{k_1}(x)$.

By the above consideration we obtain for $n \geq N(x)$ at least $\frac{1}{L} \cdot \#A(n,x)$ annuli nested inside each other, containing $x$ and with modulus greater than $\beta_0 > 0$. If we put now $\chi := \frac{\tilde{\mu}(\tilde{Y})}{2L}$ we are done by (2.6).

Lemma 2.3 has the following useful:

**Corolarry 2.4** *There exists a set $X \subset J$ such that $\mu(X) = 1$ and the function $\varphi : X \to \mathbf{R}$ by:*

$$\varphi(x) = \lim_{\substack{z \to x \\ z \in U \setminus J}} \log \frac{G(fz)}{G(z)}$$

*is well defined and continuous on $X$.*

*Proof* Let $X$ be the full measure set given by Lemma 2.3 and let $x \in X$. If $y, z \in U \setminus J$ are close to $x$ it follows that $\{x, y, z\}$ is $k$-nested for $k = k(x, y, z)$. Furthermore $k \to \infty$ as $y, z \to x$. By Lemma C we have:

$$\left| \log \frac{G(fz)}{G(z)} - \log \frac{G(fy)}{G(y)} \right| \leq C \cdot q^k. \tag{2.7}$$

Now (2.7) implies the existence of $\varphi(x)$ for $x \in X$. Furthermore there exists $C_1 > 0$ such that if $x \in X$ and $z \in U \setminus J$ are so that $\{x, z\}$ is $k = k(x, z)$-nested we have:

$$\left| \varphi(x) - \log \frac{G(fz)}{G(z)} \right| \leq C_1 \cdot q^k. \tag{2.8}$$

To see the continuity at $x \in X$ let $y \in X$, $y \to x$. Then $\{x, y\}$ is contained in a topological disc $D(x, y)$ which is $n(x, y)$-nested with $n(x, y) \to \infty$ as $y \to x$. By (2.8) we have

$$|\varphi(x) - \varphi(y)| \leq 2C_1 \cdot q^{n(x,y)}, \tag{2.9}$$

which proves the continuity of $\varphi$.

As we do not have control on the locations and sizes of the nests we cannot extend the definition and continuity of $\varphi$ to the whole $J$.

We would like to mention that whenever we obtain a full measure set $X$ with a certain property we can assume that it is f-invariant. Indeed , if $X$ is not invariant we just replace it by:

$$\hat{X} = J \setminus (\bigcup_{n\geq 0} f^{-n}(\bigcup_{m\geq 0} f^{m}(J\setminus X))).$$

It is clear that $\hat{X} \subset X$, $\mu(\hat{X}) = 1$ and $f^{-1}(\hat{X}) = \hat{X}$.

We finish this section with a technical result which is based on Lemma 2.3 and will be useful in the next section:

**Lemma 2.5** *There exists a set* $X_0 \subset J$ , $\mu(X_0) \geq 1 - 1/20$ *and numbers* $1 > \delta > 0$, $K_1 > 0$, $K_2 > 0$, $N_0 \in \mathbf{N}$ *such that for any* $n \geq N_0$, $x \in X_0$ *the ball* $B(x, \delta^n)$ *is* $\frac{1}{K_1} \cdot n$ *nested by annuli contained in* $B(x, \delta^{\frac{n}{K_2}})$.

*Proof.* Let $\tilde{X}$, $\tilde{Y}$ be the sets considered in the proof of Lemma 2.3 and let us introduce:

$$\tilde{X}_N := \{\tilde{x} \in \tilde{X} : \#\{k \leq n : \tilde{f}^k(\tilde{x}) \in \tilde{Y}\} \leq \frac{\tilde{\mu}(\tilde{Y})}{2} \cdot n, \ \forall\, n \geq N\}.$$

It is clear that

$$\tilde{X}_N \subset \tilde{X}_{N+1} \ , \ \ \tilde{X} = \cup_{N\geq N_1} \tilde{X}_N \ \ \forall \ N_1 > 0$$

and hence $\lim_{N\to\infty} \tilde{\mu}(\tilde{X}_N) = 1$.

If we choose $N_0$ such that $\tilde{\mu}(\tilde{X}_{N_0}) \geq 1 - 1/20$ and put $X_0 := \pi(\tilde{X}_{N_0})$ then $\mu(X_0) \geq 1 - 1/20$.

For $x \in X_0$ and $n \geq N_0$ there are $\chi \cdot n$ annuli nesting $x$ obtained in Lemma 2.3. Let us consider the ones obtained as preimages using univalent branches $f_k$ of $f^{-k}_{|B(f^k(x),r)}$ for $\frac{\chi \cdot n}{2} \leq k \leq n$. Their number is at least $\frac{\chi \cdot n}{2}$. By (2.4) these annuli are contained in $B(x, C_1 \cdot \lambda^{\frac{\chi \cdot n}{2}})$. Furthermore it is easy to see that they are nesting the ball $B(x, c_2 \cdot \frac{1}{L^n})$ where $L := \sup|f'(z)|$ and $C_1, c_2$ are two fixed constants.

Without loss of generality we can assume that $C_1 = c_2 = 1$. Let us put $\delta = \frac{1}{L}$ and choose $K_2 > 0$ such that $\lambda^{\frac{\chi \cdot n}{2}} \leq \delta^{\frac{n}{K_2}}$. Finally choosing $K_1 = \frac{2}{\chi}$ we are done.

## 3. **Homologous equation.**

The purpose of this section is to prove:

**Proposition 3.1** *Suppose that $(f, U, V)$ is a GPL with totally disconnected Julia set. If $\mu \approx \omega$ there exists a function $u \in L^2(\mu)$ satisfying the homologous equation:*

$$\varphi(x) - \log d = u(fx) - u(x) \quad \textit{for } \mu \textit{ a.e. } x \in J.$$

The proof of Proposition 3.1 will be done in several steps. Let us introduce the function $\phi = \varphi - \log d$. The first step is to prove:

**Lemma 3.2** *Under the assumptions of Proposition 3.1 we have $\int_J \phi d\mu = 0$.*

*Proof.* As in the proof of Lemma 2.3 consider the natural extension $(\tilde{f}, \tilde{J}, \tilde{\mu})$ of the system $(f, J, \mu)$. Let $B$ be a ball in $\hat{\mathbf{C}}$. We consider the "good" branches of $f^{-n}$ defined in $B$. Following [Z] we say that a branch $f_\nu^{-n}$ is "good" ( or $\delta$-good for $\delta > 0$) if:

$$f_\nu^{-n} \text{ is well defined and univalent in } 2B \tag{3.1}$$

$$diam f_\nu^{-n}(B) < K \cdot e^{-n\delta}. \tag{3.2}$$

In [Z] and [PUZ] it was proved that there exists $\delta > 0$ such that for every $\tilde{\epsilon} > 0$ there exists $M \in \mathbf{N}$ such that if there are no critical values up to order $M$ in $B$ then one can find a subset $\tilde{K}_B \subset \tilde{B} = \pi^{-1}(B) \subset \tilde{J}$ with $\tilde{\mu}(\tilde{K}_B) > (1 - \tilde{\epsilon})\mu(B)$ and consisting of "good" trajectories. (The trajectory $\tilde{x} = (x_0 x_{-1} \ldots x_k \ldots)$ is "good" if $x_k$ is the image of some "good" branch of $f^{-k}$ defined on B.)

We are going to apply this fact in a similar way as in [Z]: let $p_1, \ldots, p_s$ be critical values up to order $M$. Take a small $r > 0$ and let $\epsilon > 0$. Let $B_1, \ldots, B_s$ be centered at $p_i$-s with radius $r$. Let $\mathcal{B}$ be cover of $\hat{\mathbf{C}} \setminus \bigcup B_i$ by balls of radius $r/4$. If $r > 0$ is small enough then:

$$\tilde{\mu}(\cup_{B \in \mathcal{B}} \tilde{K}_B > 1 - \epsilon. \tag{3.3}$$

Introduce the function $\tilde{\phi} = \phi \circ \pi$ and suppose by contradiction that

$$\int_J \phi \, d\mu = \int_{\tilde{J}} \tilde{\phi} \, d\tilde{\mu} = \chi > 0.$$

Let us consider the partial sums:

$$\tilde{S}_n(\tilde{x}) = \sum_{i=0}^{n-1} \tilde{\phi}(\tilde{f}^i(\tilde{x})) = \sum_{i=0}^{n-1} \phi(f^i(\pi(\tilde{x}))) = S_n(\pi(\tilde{x})),$$

$$\tilde{S}^1_n(\tilde{x}) = \sum_{i=0}^{n-1} \tilde{\varphi}(\tilde{f}^i(\tilde{x})) = \sum_{i=0}^{n-1} \varphi(f^i(\pi(\tilde{x}))) = S^1_n(\pi(\tilde{x})).$$

By Birkhoff's ergodic theorem there exists $\tilde{X} \subset \tilde{J}$, $\tilde{\mu}(\tilde{X}) = 1$ such that for $\tilde{x} \in \tilde{X}$

$$\lim_{n\to\infty} \frac{\tilde{S}_n(\tilde{x})}{n} = \chi > 0.$$

Let us denote by

$$\tilde{X}_n = \{\tilde{x} \in \tilde{X} : \frac{\tilde{S}_k(\tilde{x})}{k} > \frac{\chi}{2}, \forall k \geq n \}.$$

It is clear that $\tilde{X}_n \subset \tilde{X}_{n+1}$, $\forall n > 0$ and $\tilde{X} \subset \cup_{n\geq N} \tilde{X}_n$, $\forall N > 0$. It follows that for $\epsilon > 0$ there exists $N = N(\epsilon)$ such that

$$\tilde{\mu}(\tilde{X}_n) > 1 - \epsilon , \forall n \geq N.$$

By (3.3) it follows that

$$\tilde{\mu}(\tilde{X}_n \cap \tilde{f}^{-n}(\bigcup_{B\in\mathcal{B}} \tilde{K}_B)) > 1 - 2\epsilon , \forall n \geq N.$$

Consequently there exists $B \in \mathcal{B}$ and $\beta > 0$ such that

$$\tilde{\mu}(\tilde{X}_n \cap \tilde{f}^{-n}(\tilde{K}_B)) > \beta \text{ for infinitely many } n \in \mathbf{N}$$

Let us denote by $X^n := \pi(\tilde{X}_n \cap \tilde{f}^{-n}(\tilde{K}_B))$; then $\mu(X^n) > \beta$.

If $x \in X^n$ then $x = \pi(\tilde{x})$ for some $\tilde{x} \in \tilde{X}_n \cap \tilde{f}^{-n}(\tilde{K}_B)$, and thus $x$ is a preimage of $f^n(x) \in B$ under some univalent branch $f_\nu^{-n}|_{2B}$. Let us denote the set of univalent branches of $f^{-n}|_{2B}$ by $\mathcal{G}_n$. By the above consideration we have

$$X^n \subset \bigcup_{\nu\in\mathcal{G}_n} f_\nu^{-n}(2B). \tag{3.4}$$

Our goal is to show that while $\mu(X^n) > \beta$ we have that $\omega(X^n) \to 0$ as $n \to \infty$. By (3.4) we write

$$\omega(X^n) \leq \sum_{\nu\in\mathcal{G}_n} \omega(f_\nu^{-n}(2B) \cap X^n). \tag{3.5}$$

On the other hand $x \in X^n$ implies that

$$\frac{S_n(x)}{n} > \frac{\chi}{2} \text{ for } n \geq N,$$

which means that

$$S_n^1(x) > \log d^n + n \cdot \frac{\chi}{2} \text{ for } n \geq N,\ x \in X^n. \tag{3.6}$$

Using (3.6) , for any $\nu \in \mathcal{G}_n$ we can estimate:

$$1 \geq \omega(2B \cap f^n(X^n)) = \int_{f_\nu^{-n}(2B) \cap X^n} e^{S_n^1(x)}\, d\omega(x) \geq$$
$$\geq d^n \cdot e^{\frac{\chi}{2} \cdot n} \cdot \omega(f_\nu^{-n}(2B) \cap X^n).$$

Consequently we obtain:

$$\omega(f_\nu^{-n}(2B) \cap X^n) \leq e^{-\frac{\chi}{2} \cdot n} \cdot d^{-n}. \tag{3.7}$$

Relations (3.5) and (3.7) now give:

$$\omega(X^n) \leq \sum_{\nu \in \mathcal{G}_n} \omega(f_\nu^{-n}(2B) \cap X^n) \leq e^{-\frac{\chi}{2} \cdot n} \cdot \sum_{\nu \in \mathcal{G}_n} d^{-n} =$$
$$= e^{-\frac{\chi}{2} \cdot n} \cdot \frac{1}{\mu(2B)} \cdot \sum_{\nu \in \mathcal{G}_n} \mu(f_\nu^{-n}(2B)) \leq e^{-\frac{\chi}{2} \cdot n}.$$

This shows that $\omega(X^n) \leq e^{-\frac{\chi}{2} \cdot n}$ and as $n$ can be chosen arbitrarily large we obtain that $\mu$ and $\omega$ are singular. This contradiction shows that $\int_J \phi \, d\mu = 0$.

Let us consider now the sequence $\{\phi \circ f^k\}_k$ of random variables. Our next step is

**Lemma 3.3** *There exists a finite asymptotic variance:*

$$\sigma^2 := \sigma_\mu^2(\phi) := \lim_{n \to \infty} \frac{\int_J (\sum_{i=0}^{n-1} \phi \circ f^i)^2 d\mu}{n} =$$
$$= \int_J \phi^2 d\mu + 2 \cdot \sum_{i=0}^{\infty} \int_J \phi \cdot \phi \circ f^i d\mu.$$

*Moreover, the sequence $\{\phi \circ f^k\}_k$ obeys the law of Central Limit Theorem (CLT).*

*Proof* We need to investigate the behaviour of the Perron-Frobenius-Ruelle operator $L : L^2(\mu) \to L^2(\mu)$ defined by:

$$Lu(x) = \frac{1}{d} \sum_{y \in f^{-1}(x)} u(y). \tag{3.8}$$

If $x$ is a critical value we count the preimages in the above sum together with their multiplicities.

The formula (3.8) is correct if $u \in \mathcal{C}(J)$ and in this way $L : \mathcal{C}(J) \to \mathcal{C}(J)$ is a well defined linear operator with $\| L \|_{\mathcal{C}(J)} \leq 1$. We can extend $L$ to $L^2(\mu)$ by continuity or by formula (3.8) on an invariant set $X \subset J$ with $\mu(X) = 1$ on which $u \in L^2(\mu)$ is defined. We have by Jensen's inequality that $\| L \|_{L^2(\mu)} \leq 1$. It is well known ( [Ly1], [FLM]) that $L^* \mu = \mu$ and thus $\int_J u \cdot v \circ f \, d\mu = \int_J v \cdot Lu \, d\mu$. In other words $L : L^2(\mu) \to L^2(\mu)$ is the adjoint operator of

$$A : L^2(\mu) \to L^2(\mu), \quad Au = u \circ f$$

Our goal is to prove the following decay property of $L^k \phi$: for any $p > 0$ there exist $C = C(p)$ and $K = K(p)$ such that for $k \geq K$ we have:

$$\| L^k \phi \|_\infty < C \cdot \frac{1}{k^p}. \tag{3.9}$$

Estimate (3.9) gives the first statement immediately. For the second statement we apply a theorem of Gordin (see [Go] or [D] Theorem 1.1.2). Following exactly the same arguments as in [DPU] - Theorem 5.3 we obtain that the estimate (3.9) together with Gordin's theorem imply the second statement.

To prove (3.9) we are going to use a similar idea as in Section 4 from [DPU]. The lack of the uniform Hölder continuity of $\{L^k \phi\}_k$ is compensated by Lemma 2.5 and a result of F. Przytycki ([Pr2]).

Let us start by reminding the following fact proven in [DU]: there exists a measurable Markov partition $\alpha$ of $J$ and numbers $0 < \lambda < 1, \ C > 0$ such that for

$A \in \alpha,\ f(A) = J\ \mu$ a.e. and for all $n \geq 1$:

$$\mu(\cup\{A \in \vee_{j=0}^{n-1} f^{-j}(\alpha) : \operatorname{diam} f^k(A) > C_1 \cdot \lambda^{n-k} \text{ for some } k = 0, 1, \ldots, n\}) < 1/20. \tag{3.10}$$

For $n \geq 0$ let $\alpha_b^n$ be the collection of the elements of the partition $\alpha^n = \vee_{j=0}^{n-1} f^{-j}(\alpha)$ defined in (3.10) and let $\alpha_g^n = \alpha^n \setminus \alpha_b^n$.

We also are going to use the fact ( see [DPU] Lemma 4.3) that if $\psi \in L^2(\mu)$, $\int_J \psi\, d\mu = 0$ and $\Delta \geq \| \psi \|_\infty$ then

$$\mu(\{x : \psi(x) \leq \Delta/4 \}) \geq 1/5. \tag{3.11}$$

The crucial estimate (3.9) follows immediately from the following:

**Claim** *For any $b > 1$ there exists an integer $j(b)$ such that if $j \geq j(b)$ and $[b^j] \leq k \leq [b^{j+1}]$ we have the estimate:*

$$\| L^k \phi \|_\infty \leq (39/40)^j. \tag{3.12}$$

In fact the closer we choose $b$ to 1 the greater value of $p$ can be obtained in (3.9).

To prove prove (3.12) we introduce the sequence $\{n_j\}_j,\ n_j = [b^j]$ and observe that since $\| L\psi \|_\infty \leq \| \psi \|_\infty$ it is enough to show that

$$\| L^{n_j} \phi \|_\infty \leq (39/40)^j. \tag{3.13}$$

We are going to use induction over $j$: let us assume (3.13) for $j$. By (3.11) we obtain

$$\mu(\{x : L^{n_j}\phi(x) \leq 1/4 \cdot (39/40)^j \}) \geq 1/5.$$

Let us introduce the set

$$G_j := \{x : L^{n_j}\phi(x) \leq 1/4 \cdot (39/40)^j \} \bigcap X_0,$$

where $X_0$ is the set from Lemma 2.5. Then it is clear that $\mu(G_j) \geq 1/10$.

Let us denote by $k_j = n_{j+1} - n_j$ and define:

$$\alpha_G^j := \{A \in \alpha_g^{k_j} : A \cap G_j \neq \emptyset\}$$

Relation (3.10) then yields: $\mu(\alpha_G^j) \geq 1/20$ .

First of all we are going to show that for $y \in \cup \alpha_G^j$

$$L^{n_j}\phi(y) \leq 1/2 \cdot (39/40)^j. \tag{3.14}$$

Observe that if $y \in \cup\alpha_G^j$ there exists $x \in G_j$ such that $y \in B(x, C \cdot \lambda^{k_j})$. Consequently there is a constant $K_3 > 0$ such that $y \in B(x, \delta^{\frac{n_j}{K_3}})$.

On the other hand by Lemma 2.5 there is a number of $\frac{1}{K_4} \cdot n_j$ annuli nesting $\{x, y\}$ and contained in $B(x, \delta^{\frac{n_j}{K_5}})$.

Since $x \in G_j$ gives

$$L^{n_j}\phi(x) \leq 1/4 \cdot (39/40)^j,$$

we intend to estimate the difference

$$| L^{n_j}\phi(x) - L^{n_j}\phi(y) | = \left| \sum_{z_x \in f^{-n_j}x} 1/d^{n_j}\phi(z_x) - \sum_{z_y \in f^{-n_j}y} 1/d^{n_j}\phi(z_y) \right|.$$

Let us denote by $\{C_i^j\}_{i \in I}$ the collection of the components of $f^{-n_j}(B(x, \delta^{\frac{n_j}{K_5}}))$.

¿From Przytycki's finiteness lemma (see [Pr2] Lemma 2) it follows that there exists an integer $M = M(K_5, \delta)$ such that the degree of the maps: $f^{n_j} : C_i^j \to B(x, \delta^{\frac{n_j}{K_5}})$ is at most $M$.

Using Lemma 2.5 we obtain in $C_i^j$ a number of at least $\frac{1}{K_4} \cdot n_j$ annuli with moduli bounded below by a fixed constant $\beta_1 = \beta_1(\beta, M)$ nesting $\{z_x, z_y\}$. Here we have used that the modulus of preimages under bounded degree mappings is distorted by a bounded amount.

Consequently we can use (2.9) to obtain:

$$|\phi(z_x) - \phi(z_y)| \leq C_2 \cdot q_1^{n_j},$$

where $0 < q_1 < 1$ , $q_1 = q_1(\beta_1, K_4)$.

This implies that for $j$ large enough:

$$|L^{n_j}\phi(x) - L^{n_j}\phi(y)| \le C_2 \cdot q_1^{n_j} \le 1/4 \cdot (39/40)^j,$$

and (3.14) follows.

For $x \in J$ we define $G_j(x) := f^{-k_j}(x) \cap \cup\alpha_G^j$, and $B_j(x) := f^{-k_j}(x) \setminus G_j(x)$. We are now ready to estimate:

$$L^{n_{j+1}}\phi(x) = L^{k_j}(L^{n_j}\phi(x) = \sum_{y \in G_j(x)} 1/d^{k_j} \cdot L^{n_j}\phi(y) + \sum_{y \in B_j(x)} 1/d^{k_j} \cdot L^{n_j}\phi(y) \le$$

$$\le (39/40)^j \cdot 1/d^{k_j} \cdot (1/2 \cdot \#G_j(x) + \#B_j(x)) = (39/40)^j \cdot 1/d^{k_j} \cdot (d^{k_j} - 1/2 \cdot \#G_j(x)).$$

Finally we use that

$$\#G_j(x) \cdot 1/d^{k_j} = \mu(\cup\alpha_G^j) \ge 1/20$$

and obtain

$$L^{n_{j+1}}\phi(x) \le (39/40)^{j+1}. \tag{3.15}$$

Changing $\phi$ to $-\phi$ we obtain the counterpart of (3.15):

$$L^{n_{j+1}}\phi(x) \ge -(39/40)^{j+1}.$$

The above estimates yield (3.13) for $j+1$. This finishes the proof of the Claim and we are done.

The last step toward the proof of Proposition 3.1 is:

**Lemma 3.4** *Under the conditions of Proposition 3.1 we have* $\sigma^2 = 0$.

*Proof* Let us suppose by contradiction that $\sigma^2 > 0$. Let us consider the function $\phi_1 = -\phi = \log d - \varphi$ and apply CLT for the sequence of random variables $\{\phi_1 \circ f^k\}_k$. As in the proof of Lemma 3.2 we consider the corresponding partial sums but now for the function $\phi_1$. Instead of Birkhoff's ergodic theorem we apply now CLT: for any $A > 0$ we have:

$$\tilde{\mu}(\{\tilde{x} \in \tilde{J} : \tilde{S}_n(\tilde{x}) < -A \cdot \sigma \cdot n^{1/2}\}) \to \psi(-A) ,$$

where $\psi(-A) = \int_{-\infty}^{-A} e^{-\frac{t^2}{2}} dt$.

Exactly as in the proof of Lemma 3.2 we consider the cover $\mathcal{B}$ satisfying the relation (3.3). Choosing $\epsilon > 0$ in (3.3) to be small we can find $\beta > 0$ and a ball $B \in \mathcal{B}$ such that:

$$\tilde{\mu}(\{\tilde{x} \in \tilde{J} \ : \ \tilde{S}_n(\tilde{x}) \ < \ -A \cdot \sigma \cdot n^{1/2}\} \ \cap \ \tilde{f}^{-n}(\tilde{K}_B)) > \beta > 0\,, \tag{3.16}$$

for infinitely many $n$-s.

Let us denote by $X^n := \pi(\{\tilde{x} \in \tilde{J} \ : \ \tilde{S}_n(\tilde{x}) \ < \ -A \cdot \sigma \cdot n^{1/2}\} \ \cap \ \tilde{f}^{-n}(\tilde{K}_B))$. Then $\mu(X^n) > \beta$ and

$$X^n \subset \bigcup_{\nu \in \mathcal{G}_n} f_\nu^{-n}(2B), \tag{3.17}$$

where $\mathcal{G}_n$ denotes the set of univalent branches $f_\nu^{-n}|_{2B}$.

Our goal again is to show that $\omega(X^n) \to 0$ as $n \to \infty$. As before we have

$$\omega(X^n) \leq \sum_{\nu \in \mathcal{G}_n} \omega(f_\nu^{-n}(2B) \cap X^n). \tag{3.18}$$

If $x \in X^n$ we have

$$S_n(x) < -A \cdot \sigma \cdot n^{1/2},$$

or equivalently

$$S_n^1(x) > \log d^n + A \cdot \sigma \cdot n^{1/2}. \tag{3.19}$$

Using (3.19) we can estimate for any $\nu \in \mathcal{G}_n$:

$$1 \ \geq \ \omega(2B \cap f^n(X^n)) \ = \ \int_{f_\nu^{-n}(2B) \cap X^n} e^{S_n^1(x)}\, d\omega(x) \ \geq$$

$$\geq d^n \cdot e^{A\sigma n^{1/2}} \omega(f_\nu^{-n}(2B) \cap X^n).$$

As a consequence we obtain

$$\omega(f_\nu^{-n}(2B) \cap X^n) \leq e^{-A\sigma n^{1/2}} \cdot d^{-n}. \tag{3.20}$$

Finally (3.18) and (3.20) give:

$$\omega(X^n) \leq e^{-A\sigma n^{1/2}}.$$

As $n$ can be chosen to be arbitrarily large we obtain that the two measures are singular which is a contradiction proving the lemma.

Based on the above three Lemmas the proof of Proposition 3.1 follows immediately as seen e.g. in [PUZ] Lemma 1.1.

## 4. Conformal maximality.

In this final section we are going to give the proof of:

**Theorem 4.1** *Let $(f, U, V)$ be a GPL with totally disconnected Julia set. Then $\omega \approx \mu$ implies that $(f, U, V)$ is conformally maximal.*

*Proof.* The proof is based on the homologous equation given by Proposition 3.1

$$\varphi(x) - \log d = u(fx) - u(x)\ ,\ \mu \text{ a.e. } x \in J. \tag{4.1}$$

Starting from (4.1) we are going to construct an automorphic function $\tau$ that is required by Theorem A for conformal maximality.

Different kind of homologous equations appear naturally when investigating the relations between two measures on the Julia set as seen e.g. in [Z], [Vo], [LyV], [BPV], [BV2]. Also the techniques to handle these equations are different accordingly. Our approach is based on the main idea in [BPV] and [BV2]; however we have here the difficulty due to lack of regularity of $\varphi$ and $u$.

Let us notice first that we can assume that the invariant set $X$, $\mu(X) = 1$ on which (4.1) holds consists of "good" points (in the sense of Definition 2.2).

¿From the proof it will be clear that there is no loss of generality to assume that there exists a repelling fixed point $p \in J$ of $f$ which is not a critical value (i.e. $p \neq f^n(c)$ for all $n \geq 0$ and all critical points of $f$). Let us consider such $p \in J$. Notice that $p \in J$ is a good point and thus a point of continuity of $\phi = \varphi - \log d$. Our first step is to show that

$$\varphi(p)\ -\ \log d\ =\ 0. \tag{4.2}$$

(Warning: (4.2) does not follow from (4.1) since we do not know *apriori* that $p \in X$.)

Let us denote by $B$ a small disc centered at $p$ such that all components $B_n$ of $f^{-n}B$ containing $p$ are included in $B$ and $B$ is free from critical points of $f$.

It is clear that $\mu(B_n) = \frac{1}{d^n} \cdot \mu(B)$ and

$$\omega(B) = \int_{B_n} e^{S_n^1(x)}\, d\omega(x) \text{ where } S_n^1(x) = \sum_{i=0}^{n-1} \varphi(f^i(x)).$$

Furthermore observe that for any $x \in B_n \bigcap J$ and $i = 0, \dots, n-1$ we have that $\{f^i(x), p\}$ is $(n-i)$-nested. Consequently by (2.9) the inequality

$$|\varphi(f^i(x)) - \varphi(p)| \leq 2C_1 \cdot q^{n-i}$$

holds. This implies that

$$|S_n^1(x) - S_n^1(p)| \leq C_2 \; \forall x \in B_n$$

and therefore there exists $K > 0$ such that:

$$\frac{1}{K} \cdot e^{-n \cdot \varphi(p)} \cdot \omega(B) \leq \omega(B_n) \leq K \cdot e^{-n \cdot \varphi(p)} \cdot \omega(B).$$

Consequently $\frac{\omega(B_n)}{\mu(B_n)} \sim e^{n \cdot (\log d - \varphi(p))}$.

On the other hand $\omega \approx \mu$ and thus (0.2) together with the above relation imply that $\varphi(p) - \log d = 0$.

Now we can start the construction of $\tau$. This will be done in three steps:

**Step I : construction of $\tau$ on $B$.**

Let us denote by $g$ the inverse branch of $f^{-1} : B \to B_1 \subset B$. Notice that $\{p, g^n z, g^{n-1} z\}$ is $n-1$-nested for any $z \in B$, $n \in \mathbf{N}$. By (2.8) it follows that

$$\left|\log \frac{G(g^{n-1}z)}{G(g^n z)} - \varphi(p)\right| \leq C_1 \cdot q^{n-1}.$$

Using (4.2) this gives:

$$\left|\frac{G(g^{n-1}z)}{d \cdot G(g^n z)} - 1\right| \leq C_1 \cdot q^{n-1}. \tag{4.3}$$

Now (4.3) implies that the following limit:

$$\tau^1(z) := \lim_{n\to\infty} d^n \cdot G(g^n z)\,,\ z \in B \tag{4.4}$$

represents a subharmonic function which is harmonic in $B \setminus J$ and vanishing on J. Notice also that $\tau^1$ is automorphic on $B_1$: $\tau^1(fz) = d \cdot \tau^1(z)$.

Our next objective is to show that there exists a function $u^1 \in L^2_{|B}(\mu)$ such that for any $x \in X \bigcap B_1$:

$$\lim_{\substack{z\to x\\ z\in B\setminus J}} \frac{G(z)}{\tau^1(z)} = e^{u(x)-u^1(x)}. \tag{4.5}$$

To define $u^1 \in L^2_{|B}(\mu)$ observe that for any $x \in B \bigcap X$ the sequence $\{u(g^n x)\}_n$ is a Cauchy sequence.

To see this we use (4.1),(4.2) and the inequality:

$$|\varphi(p) - \varphi(g^n x)| \leq 2C_1 \cdot q^n.$$

It follows that $|u(g^n x) - u(g^{n-1}x)| \leq 2C_1 \cdot q^n$.

Now for $x \in B \bigcap X$ we denote by $u^1(x) := \lim_{n\to\infty} u(g^n x)$. It is clear that $u^1 \in L^2_{|B}(\mu)$.

In order to prove (4.5) notice that as $x \in X$ is a good point and $z \to x$, there exists $N(x,z)$ such that $\{x,z\}$ is $N(x,z)$ -nested and $N(x,z) \to \infty$ as $z \to x$.

Let us suppose that $x, z \in B_1$; $z$ is close to $x$, thus $\{x,z\}$ is $N(x,z)$-nested for some large $N(x,z)$. By the definition of $\tau^1$:

$$\frac{G(z)}{\tau^1(z)} = \lim_{n\to\infty} \frac{G(z)}{d^n \cdot G(g^n z)} = \prod_{n=0}^{\infty} \frac{G(g^n z)}{d \cdot G(g^{n+1}z)}. \tag{4.6}$$

Notice also that $\{g^n x, g^n z\}$ is $(N(x,z) - N_0)$-nested for some fixed $N_0$. Without loss of generality we can assume that $N_0 = 0$ and hence $\{g^n x, g^n z\}$ is $N(x,z)$-nested.

Let us put $N := N(x,z)$ and consider $i \leq 2N$. Because $\{g^{i-1}x, g^{i-1}z\}$ is $N$-nested (4.1) and (2.8) give:

$$\left|\log \frac{G(g^{i-1}z)}{d \cdot G(g^i z)} - (u(g^{i-1}x) - u(g^i x))\right| \leq C_1 \cdot q^N,$$

which means

$$e^{u(g^{i-1}x)-u(g^ix)-C_1\cdot q^N} \leq \frac{G(g^{i-1}z)}{d\cdot G(g^iz)} \leq e^{u(g^{i-1}x)-u(g^ix)+C_1\cdot q^N}.$$

This implies:

$$e^{u(x)-u(g^{2N}x)-2C_1N\cdot q^N} \leq \prod_{i=0}^{2N}\frac{G(g^{i-1}z)}{d\cdot G(g^iz)} \leq e^{u(x)-u(g^{2N}x)+2C_1N\cdot q^N}. \tag{4.7}$$

On the other hand for $i > 2N$ we are going to use that $\{g^{i-1}x, g^{i-1}z\}$ is $i$-nested and so

$$e^{u(g^{i-1}x)-u(g^ix)-C_1\cdot q^i} \leq \frac{G(g^{i-1}z)}{d\cdot G(g^iz)} \leq e^{u(g^{i-1}x)-u(g^ix)+C_1\cdot q^i}.$$

For $n > 2N$ this implies:

$$e^{u(g^{2N}x)-u(g^nx)-C_2\cdot q^N} \leq \prod_{i=2N}^{n}\frac{G(g^{i-1}z)}{d\cdot G(g^iz)} \leq e^{u(g^{2N}x)-u(g^nx)+C_2\cdot q^N}. \tag{4.8}$$

Now (4.7) and (4.8) imply:

$$e^{u(x)-u(g^nx)-C_3\cdot q^{N/2}} \leq \prod_{i=0}^{n}\frac{G(g^{i-1}z)}{d\cdot G(g^iz)} \leq e^{u(x)-u(g^nx)+C_3\cdot q^{N/2}}. \tag{4.9}$$

Consequently if $\{x, z\}$ is $N$-nested (4.6) and (4.9) give:

$$e^{u(x)-u^1(x)-C_3\cdot q^{N/2}} \leq \frac{G(z)}{\tau^1(z)} \leq e^{u(x)-u^1(x)+C_3\cdot q^{N/2}}. \tag{4.10}$$

Recalling that $N = N(x, z) \to \infty$ as $z \to x$ the estimate (4.10) gives (4.5).

Let us consider now the union of backward orbits of $B$ : $\mathcal{O} := \bigcup_{n\geq 0} f^{-n}B$.

**Step II: extension of $\tau$ to $\mathcal{O}$**

Let $B_\theta$ be a component of $f^{-n}B$ for some $n > 0$. We define a function $\tau_\theta^2$ on $B_\theta$ by:

$$\tau_\theta^2(z) = \frac{1}{d^n}\tau_1(f^nz), \quad z \in B_\theta.$$

We would like to prove that $\tau_\theta^2$ (or a symmetrized version of it) does not depend on $\theta$ (and $n$).

We are going to calculate first the limit:

$$\lim_{\substack{z\to x\\ z\in B_\theta\setminus J}} \frac{G(z)}{\tau_\theta^2(z)} \text{ for } x\in f^{-n}(X\cap B)\,\cap\, B_\theta.$$

To do that we write:

$$\frac{G(z)}{\tau_\theta^2(z)} = \frac{G(z)}{\frac{1}{d^n}\cdot\tau^1(f^n z)} = \frac{G(f^n z)}{\tau^1(f^n z)}\cdot\frac{d\cdot G(z)}{G(fz)}\cdots\frac{d\cdot G(f^{n-1}z)}{G(f^n z)}.$$

As $n$ is being fixed we use (4.5) to obtain:

$$\lim_{\substack{z\to x\\ z\in B_\theta\setminus J}} \frac{G(z)}{\tau_\theta^2(z)} = e^{u(x)-u^1(f^n x)}. \tag{4.11}$$

Let us take now $n_2 > n_1$ and two corresponding branches $f_{\theta_2}^{-n_2}$, $f_{\theta_1}^{-n_1}$. If $x\in B_{\theta_1}\cap B_{\theta_2}\cap f^{-n_1}(X)\cap f^{-n_2}(X)$ by (4.11) we can write:

$$\lim_{\substack{z\to x\\ z\in B_{\theta_1}\cap B_{\theta_2}\setminus J}} \frac{\tau_{\theta_1}^2(z)}{\tau_{\theta_2}^2(z)} = e^{u^1(f^{n_2}(x))-u^1(f^{n_1}(x))}. \tag{4.12}$$

let us denote by $x_2 := f^{n_2}(x)$, $x_1 = f^{n_1}(x)$. Then $x_1, x_2 \in B\cap X$ and $f^{n_2-n_1}(x_1) = x_2$. By the definition of $u^1$ it follows that $u^1(x_1) = u^1(x_2)$. Consequently (4.12) becomes:

$$\lim_{\substack{z\to x\\ z\in B_\theta\setminus J}} \frac{\tau_{\theta_1}^2(z)}{\tau_{\theta_2}^2(z)} = 1 \text{ for } \omega \text{ a.e. } x\in B_{\theta_1}\cap B_{\theta_2}\cap J.$$

Now we can apply Grishin's lemma (Lemma B) to obtain that $\Delta\tau_{\theta_1}^2 = \Delta\tau_{\theta_2}^2$ on $B_{\theta_1}\cap B_{\theta_2}$ and hence the function $\tau_{\theta_1}^2 - \tau_{\theta_1}^2$ is harmonic in $B_{\theta_1}\cap B_{\theta_2}$ and it vanishes on $B_{\theta_1}\cap B_{\theta_2}\cap J$.

Now , either $\tau_{\theta_1}^2 \equiv \tau_{\theta_1}^2$ or $B_{\theta_1}\cap B_{\theta_2}\cap J$ is covered by a finite number of real analytic curves. It's not hard to see that if the latter happens the whole $J$ can be covered by a finite number of real analytic curves so this will be the case for any pair of $\theta_1, \theta_2$ for which $B_{\theta_1}\cap B_{\theta_2}$ is not empty.

Furthermore, without loss of generality (see [BPV],[LyV] or [Vo]) we can consider the situation when the curves are disjoint. Let $*$ be a holomorphic symmetry with respect to these curves. Instead of $\tau_\theta^2$ we are going to work with

$$\tau_\theta^3(z) \stackrel{\text{def}}{=} \tau_\theta^2(z) + \tau_\theta^2(z^*).$$

The advantage is that now $\tau_{\theta_1}^3 \equiv \tau_{\theta_2}^3$ in $B_{\theta_1}\cap B_{\theta_2}$.

In any case we obtain a function $\tau^4$ on $\mathcal{O}$ such that $\tau^4_{|B_\theta} = \tau^3_\theta$ (or $\tau^2_\theta$ if the first possibility "$\tau^2_{\theta_1} = \tau^2_{\theta_2}$" always occurs).

It is clear that our function $\tau^4$ has the automorphic property on $f^{-1}\mathcal{O}$.

Since $J$ is totally disconnected there is a number $N > 0$ such that $f^{-N}U \subset B$. In the last step we are going to extend $\tau^4$ to the whole $U$.

**Step III: extension of $\tau$ to $U$**

Consider $z \in U$ which is not a critical value of $f^N$. Choose a topological disc free from critical values of $f^N$ and containing both $z$ and $p$.

Let $V_N$, be the component of of $f^{-N}V$, containing the point $p$ and contained in $B$. Then the map $f^{-N} : V \to V_N$ is univalent and we can define:

$$\tau^5(z) = d^N \tau^4(f^{-N}(z)), \quad z \in V$$

It is clear that $\tau^5$ does not depend on $V$ since $\tau^5_{|V \cap B} = \tau^4$. We extend $\tau^5$ to the critical values of $f^N$ by continuity.

Because $\tau^5$ is a positive subharmonic function, harmonic on $U \setminus J$ and vanishing on $J$ we only need to check the automorphic property.

To do that let us denote by $B_1$ an arbitrary component of $f^{-1}B$. We are going to show that $\tau^5_{|B_1} = \tau^4$. Since $\tau^4$ was automorphic on $B_1$ this proves that $\tau^5$ is automorphic on $V_i$ where $V_i$ contains $B_1$.

Let us pick $z \in B_1$ and an appropriate univalent branch $f_\theta^{-N}$ of $f^{-N}$. Put $z_1 = f_\theta^{-N} z$ and by our definition we have $\tau^5(z) = d^N \tau^4(z_1)$. On the other hand observe that $f^{N+1} z_1 = fz \in B$.

By the automorphic property of $\tau^4$ we have:

$$\tau^4(z_1) = \frac{1}{d^{N+1}} \tau^4(f^{N+1} z_1) = \frac{1}{d^{N+1}} \tau^4(fz).$$

It follows that $\tau^5(z) = \frac{1}{d}\tau^4(fz)$ and consequently, by the definition of $\tau^4$ in Step II we have: $\tau^5(z) = \tau^4(z)$ for $z \in B_1$. This shows that $\tau^5_{|V_i}$ is automorphic. As $B_1$ was arbitrary we obtain $\tau^5_{|V_i}$ is automorphic for any $i$.

This concludes our construction and proves the theorem.

## Final Remarks:

Our first remark is that our result holds true for general GPL where the Julia set is disconnected (whithout being totally disconnected). The reason for this is that an invariant, ergodic measure with positive entropy is supported on the 'totally disconnected' part of the the Julia set. This fact follows from arguments used in [PUZ] or [Z].

After this paper has been written the author has found out about the work of Anna Zdunik [Zd] where similar problems are discussed in the setting of polynomial-like maps. The approach in [Zd] is different. It is based on a very elegant idea (similar to the one in [MR] to apply the Perron-Frobenius operator to subharmonic functions. In this way the author constructs an invariant measure absolutely continuous with respect to harmonic measure and then relate this invariant measure to the measure of maximal entropy. This can be applied also in the case of generalized polynomial-like maps and our result follows by the method in [Zd]. We think however that our approach might be useful in treating similar problems and is worthy of future developpement.

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## References

[Bo] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lect. Notes Math., 1975, v. 470.

[BPV] Z. Balogh, I. Popovici and A. Volberg, *Conformally maximal polynomial - like dynamics and invariant harmonic measure*, Ergod. Th. & Dynam. Sys. (1997), **17**, 1-27.

[Br] H. Brolin, *Invariant sets under iterations of rational functions*, Ark. Mat. 1967, v. 6, 103-149.

[BV1] Z. Balogh, A. Volberg, *Principe de Harnack à frontière pour les repulseurs holomorphes non-recurrents*, C.R. Acad. Sci. Paris, 1994, t. 319, p. 351-354.

[BV2] Z. Balogh, A. Volberg, *Boundary Harnack principle for separated semihyperbolic repellers. Harmonic measure applications* Rev. Mat. Iberoamericana (1996) Vol. 12, No 2, 299-336.

[D] M. Denker, *The central limit theorem for dynamical systems.* Dynamical Systems and Ergodic Theory, ed. K. Krzyzewsi. Banach Center Publ. 23,33-62. Polish Scientific Publ., Warszawa 1989.

[DH] A. Douady, F. Hubbard, *On the dynamics of polynomial like mappings*, Ann. Sci. Ec. Norm. Sup. 1985, v. 18, No. 2, pp. 287-345.

[DPU] M. Denker, F. Przytycki, M. Urbanski, *On the transfer operator for rational functions on the Riemann sphere*, to appear in Ergodic Theory & Dynamical Systems.

[DU] M. Denker, M. Urbanski, *Ergodic theory of equilibrium states for parabolic rational maps*, Nonlinearity, 4 (1991), 103-134.

[FLR] A. Freire, A. Lopes, R. Mañé, *An invariant measure for rational maps*, Bol. Soc. Bras. Mat. (1983), v. 14, No. 1, 45-62.

[Go] M.I. Gordin, *On the central limit theorem for stationary processes*, Soviet. Math. Dokl. (10) ,1969,1174-1176.

[Gr] A.F. Grishin, *On sets of regular growth of entire functions*, Teor. Funct. Func. Anal. i ich prilozh, 1983, v. 40 (in Russian).

[Lo] A. Lopes, *Equilibrium measure for ratational functions,* Erg. Th. & Dyn. Syst., 1986, 6, 393-399.

[Ly1] M. Lyubich, *Entropy of the analytic endomorphisms of the Riemann sphere*, Funk. An. and Appl. 1981, 15:4, 83-84.

[Ly2] M. Lyubich,*Entropy properties of rational endomorphisms of the Riemann sphere*, Erg. Theory & Dyn. Syst., 1983, 3, 351-386.

[LyV] M. Lyubich, A. Volberg,*A comparison of harmonic and balanced measures on Cantor repellers*, to appear in Asterisque.

[Ma1] R. Mañé,*On the uniqueness of maximizing measure for rational maps*, Bol. Soc. Mat. Bras. 14(1983),27-43.

[MR] R. Mañé, L.F. da Rocha,*Julia sets are uniformly perfect*, Proc. AMS, 1992, v. 116, 251-257.

[MV] N.G. Makarov, A. Volberg *On the harmonic measure of discontinuous fractals*, Preprint, LOMI E-6-86, Leningrad,(1986).

[Pa] W. Parry, *Entropy and Generators in Ergodic Theory*, W.A. Benjamin Inc., New York, 1969.

[PP] W. Parry, M. Pollicott,*Zeta functions and the periodic orbit structure of hyperbolic dynamics*, Asterisque 187-188,1990

[Pr1] F. Przytycki, *Accessibility of typical points for inariant measure of positive Lyapunov exponent for iterations of holomorphic maps,* Inst. Math. Sci., SUNY Stony Brook, 1993, 3, 1-19.

[Pr2] F. Przytycki, *Lyapunov characteristic exponents are non-negative* Proc. Amer. Math. Soc. 119.1 (1993), 309-317.

[PUZ] F. Przytycki, M. Urbanski, A. Zdunik, *Harmonic, Gibbs and Hausdorff measures on repellers for holomorphic maps, Part I* Annals of Math. 130, (1989),1-40.

[SS] M. Shub, D. Sullivan, *Expanding endomorphisms of the circle revisited*, Ergodic Theory and Dynamical Systems, 1985, v. 5, 285-289.

[Vo] A. Volberg, *On the dimension of harmonic measure of Cantor repellers*, Mich. Math. J. 40(1993), 239-258.

[Z] A. Zdunik, *Parabolic orbifolds and the dimension of the maximal measure for rational maps*, Invent. Mat. 99 (1990), 627-649.

[Zd] A. Zdunik, *Harmonic measure on the Julia set for polynomial-like maps*, Invent. Mat. 128 (1997), 303-327.

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# THE CONTINUITY OF THE METRIC PROJECTION OF A FIXED POINT ONTO MOVING CLOSED-CONVEX SETS IN UNIFORMLY-CONVEX BANACH SPACES

ANDRÁS DOMOKOS

We will show that a result similar to Hölder continuity in Hilbert spaces of the metric projections of a fixed point onto a pseudo-Lipschitz continuous family of closed convex sets [6] holds for uniformly-convex Banach spaces. The continuity of the metric projections with respect to perturbations play an important role in the sensitivity analysis of variational inequalities in Hilbert spaces [1, 3, 4, 6, 7] and hence in a wide range of nonlinear optimization, evolution and boundary value problems. The results from this paper offer us the possibility of extending the studies involving the metric projections in a larger class of spaces.

We denote by $(\Lambda, d)$ a metric space and by $X$ a uniformly-convex Banach space. We suppose $X^*$ locally-uniformly-convex. Let $\omega_0, x_0 \in X$, $\lambda_0 \in \Lambda$, and their neighborhoods $\Omega_0 = B(\omega_0, r)$ (the closed ball centered at $\omega_0$ and radius $r$) of $\omega_0$, $\Lambda_0$ of $\lambda_0$. Let $C : \Lambda_0 \rightsquigarrow X$ be a set-valued mapping with nonempty, closed, convex values. Let us consider the following problem:

- for $\lambda \in \Lambda_0$ and $\omega \in \Omega_0$ find $x(\omega, \lambda) = P_{C(\lambda)}(\omega) \in C(\lambda)$ such that

$$||\omega - x(\omega, \lambda)|| = \min_{x \in C(\lambda)} ||\omega - x|| . \tag{1}$$

In our context such an element exists for all $\omega \in \Omega_0$ and $\lambda \in \Lambda_0$ and satisfies

$$\langle J(\omega - x(\omega, \lambda)) , x - x(\omega, \lambda) \rangle \le 0 , \quad \forall \ x \in C(\lambda) , \tag{2}$$

where $J$ is the normalized duality mapping.

(2) is equivalent with

$$0 \in -J(\omega - x(\omega, \lambda)) + N_{C(\lambda)}(x(\omega, \lambda)) , \tag{3}$$

where

$$N_{C(\lambda)}(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq 0, \ \forall \ y \in C(\lambda)\}$$

is the normal cone to the set $C(\lambda)$ at the point $x$.

Hence we need to study the sensitivity with respect to $\lambda$ of the following generalized equation:

$$0 \in -J(\omega - x) + N_{C(\lambda)}(x)\,. \tag{4}$$

For Theorem 1 it is enough to consider $(\Omega, d)$ be a metric space, $\omega_0 \in \Omega$ and $\Omega_0$ be a neighborhood of $\omega_0$. Let $f : X_0 \times \Omega_0 \to X^*$ be a single-valued mapping .

**Definition 1.** *The mappings $f(\cdot,\omega)$ are $\varphi$-monotone on $X_0$ for all $\omega \in \Omega_0$, if there exists an increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, with $\varphi(r) > 0$ when $r > 0$, such that*

$$\langle f(x_1,\omega) - f(x_2,\omega)\,,\, x_1 - x_2\rangle \geq \varphi(\|x_1 - x_2\|)\|x_1 - x_2\|\,,$$

*for all $x_1, x_2 \in X_0$ and $\omega \in \Omega_0$.*

The following proposition shows that the $\varphi$-monotonicity assumption is a natural one in uniformly-convex Banach spaces.

**Proposition 1.** [5] *A Banach space $X$ is uniformly-convex if and only if for each $R > 0$ there exists an increasing function $\varphi_R : \mathbf{R}_+ \to \mathbf{R}_+$, with $\varphi_R(r) > 0$ when $r > 0$, such that the normalized duality mapping $J : X \rightsquigarrow X^*$, defined by*

$$J(x) = \{\, x^* \in X^* : \langle x^*, x\rangle = \|x\|^2, \|x\| = \|x^*\|\,\}\,,$$

*is $\varphi_R$-monotone in $B(0,R)$.*

**Definition 2.** *$C$ is pseudo-continuous around $(\lambda_0, x_0) \in Graph\,C$ if there exist neighborhoods $V \subset \Lambda_0$ of $\lambda_0$, $W \subset X_0$ of $x_0$ and there exists a function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ continuous at 0, with $\beta(0) = 0$, such that*

$$C(\lambda_1) \cap W \subset C(\lambda_2) + \beta\,(d(\lambda_1, \lambda_2))\; B(0,1) \tag{5}$$

*for all* $\lambda_1, \lambda_2 \in V$.

*If the function* $\beta$ *is defined as* $\beta(r) = Lr$, *with* $L \geq 0$, *([2]) then we say that* $C$ *is pseudo-Lipschitz continuous around* $(\lambda_0, x_0)$.

**Theorem 1.** *Let us suppose that:*

*a)* $0 \in f(x_0, \omega_0) + N_{C(\lambda_0)}(x_0)$;

*b)* $f$ *is continous on* $X_0 \times \Omega_0$;

*c) the mappings* $f(\cdot, \omega)$ *are* $\varphi$*-monotone in* $X_0$ *for all* $\omega \in \Omega_0$;

*d)* $C$ *is pseudo-continuous around* $(\lambda_0, x_0)$.

*Then there exist neighborhoods* $\Lambda_1$ *of* $\lambda_0$, $\Omega_1$ *of* $\omega_0$ *and a unique continuous mapping* $x : \Omega_1 \times \Lambda_1 \to X_0$, *such that* $x(\omega_0, \lambda_0) = x_0$ *and* $x(\omega, \lambda)$ *is a solution of the variational inequality*

$$0 \in f(x, \omega) + N_{C(\lambda)}(x) ,$$

*for all* $(\omega, \lambda) \in \Omega_1 \times \Lambda_1$.

*Proof.* Let us note that assumptions *b), c)* imply that $\varphi(r) \to 0$, $\varphi(r)r \to 0$ iff $r \to 0$. We choose positive constants $s, r, \varepsilon$ such that $B(x_0, s) \subset X_0$, $B(\lambda_0, \varepsilon) \subset \Lambda_0$, $B(\omega_0, r) \subset \Omega_0$, $\beta(d(\lambda, \lambda_0)) \leq s$ for all $\lambda \in B(\lambda_0, \varepsilon)$ and the pseudo-continuity of $C$ to be written as:

$$C(\lambda_1) \cap B(x_0, s) \subset C(\lambda_2) + \beta(d(\lambda_1, \lambda_2))\, B(0, 1)$$

for all $\lambda_1, \lambda_2 \in B(\lambda_0, \varepsilon)$.

Let $\lambda \in B(\lambda_0, \varepsilon)$ and $\omega \in B(\omega_0, r)$ be arbitrarily choosen. Then the inclusion

$$x_0 \in C(\lambda_0) \cap B(x_0, s) \subset C(\lambda) + Ld(\lambda, \lambda_0)\, B(0, 1)$$

implies the existence of an $u_\lambda \in C(\lambda)$ such that

$$||x_0 - u_\lambda|| \leq \beta(d(\lambda, \lambda_0)) \leq s .$$

This means that $C(\lambda) \cap B(x_0, s)$ is nonempty for all $\lambda \in B(\lambda_0, \varepsilon)$. Corollary 32.35 from [8] shows that the variational inequality

$$0 \in f(x,\omega) + N_{C(\lambda)\cap B(x_0,s)}(x)$$

has a unique solution $x(\omega,\lambda) \in C(\lambda) \cap B(x_0, s)$. So

$$\langle f(x(\omega,\lambda), \omega), u - x(\omega,\lambda)\rangle \geq 0$$

for all $u \in C(\lambda) \cap B(x_0, s)$.

The pseudo-Lipschitz continuity of the set-valued mapping $C$ implies that for $x(\omega,\lambda)$ there exists an element $u_0 \in C(\lambda_0)$ such that $\|x(\omega,\lambda) - u_0\| \leq \beta(d(\lambda,\lambda_0))$.

Using the $\varphi$-monotonicity of $f(\cdot,\omega)$ we obtain

$$\varphi(\|x(\omega,\lambda) - x_0\|)\, \|x(\omega,\lambda) - x_0\| \leq$$

$$\leq \langle f(x(\omega,\lambda), \omega) - f(x_0,\omega), x(\omega,\lambda) - x_0\rangle \leq$$

$$\leq \langle f(x(\omega,\lambda), \omega) - f(x_0,\omega), x(\omega,\lambda) - x_0\rangle + \langle f(x_0,\omega_0), u_0 - x_0\rangle +$$

$$+ \langle f(x(\omega,\lambda), \omega), u_\lambda - x(\omega,\lambda)\rangle =$$

$$= \langle f(x(\omega,\lambda), \omega), u_\lambda - x_0\rangle + \langle f(x_0,\omega), u_0 - x(\omega,\lambda)\rangle +$$

$$+ \langle f(x_0,\omega_0) - f(x_0,\omega), u_0 - x_0\rangle \leq$$

$$\leq \|f(x(\omega,\lambda), \omega)\|\, \|u_\lambda - x_0\| + \|f(x_0,\omega)\|\, \|u_0 - x(\omega,\lambda)\| +$$

$$+ \|f(x_0,\omega_0) - f(x_0,\omega)\|\, \|u_0 - x_0\| .$$

Assumption *a)* implies that $\|f(x_0,\omega_0)\| < \infty$, and hence using the continuity of $f$, we can suppose that $\|f(x,\omega)\| \leq M < \infty$, for all $x \in B(x_0, s)$ and $\omega \in B(\omega_0, r)$.

We know also that

$$\|u_0 - x_0\| \leq \|u_0 - x(\omega,\lambda)\| + \|x(\omega,\lambda) - x_0\| \leq$$

$$\leq \beta(d(\lambda,\lambda_0)) + s .$$

So,

$$\varphi(\|x(\omega,\lambda) - x_0\|)\, \|x(\omega,\lambda) - x_0\| \leq$$

$$\leq 2M\beta(d(\lambda,\lambda_0)) + \|f(x_0,\omega_0) - f(x_0,\omega)\|(\beta(d(\lambda,\lambda_0)) + s) .$$

This means that $x(\omega,\lambda) \to x_0$, when $(\omega,\lambda) \to (\omega_0,\lambda_0)$. Thus we can choose neighborhoods $\Omega_1 \subset B(\omega_0,r)$ of $\omega_0$ and $\Lambda_1 \subset B(\lambda_0,\varepsilon)$ of $\lambda_0$ such that $x(\omega,\lambda) \in intB(x_0,s)$, for all $(\omega,\lambda) \in \Omega_1 \times \Lambda_1$. Hence

$$0 \in f(x(\omega,\lambda)\,,\,\omega) + N_{C(\lambda)}(x(\omega,\lambda))\;,$$

because

$$N_{C(\lambda)}(x(\omega,\lambda)) = N_{C(\lambda)\cap B(x_0,s)}(x(\omega,\lambda))\;,$$

for all $(\omega,\lambda) \in \Omega_1 \times \Lambda_1$.

Let us choose $\lambda_1,\lambda_2 \in \Lambda_1$ and $\omega_1,\omega_2 \in \Omega_1$.

For $x(\omega_1,\lambda_1) \in C(\lambda_1) \cap B(x_0,s)$ there exists $u_2 \in C(\lambda_2)$, such that

$$\|x(\omega_1,\lambda_1) - u_2\| \leq \beta(d(\lambda_1,\lambda_2))\;.$$

For $x(\omega_1,\lambda_2) \in C(\lambda_2) \cap B(x_0,s)$ there exists $u_1 \in C(\lambda_1)$ such that

$$\|x(\omega_1,\lambda_2) - u_1\| \leq \beta(d(\lambda_1,\lambda_2))\;.$$

Then

$$\begin{aligned}
&\varphi\left(\|x(\omega_1,\lambda_1) - x(\omega_1,\lambda_2)\|\right)\|x(\omega_1,\lambda_1) - x(\omega_1,\lambda_2)\| \leq \\
&\leq \langle f(x(\omega_1,\lambda_1)\,,\,\omega_1) - f(x(\omega_1,\lambda_2)\,,\,\omega_1)\,,\, x(\omega_1,\lambda_1) - x(\omega_1,\lambda_2)\rangle + \\
&+ \langle f(x(\omega_1,\lambda_1)\,,\,\omega_1)\,,\, u_1 - x(\omega_1,\lambda_1)\rangle + \\
&+ \langle f(x(\omega_1,\lambda_2)\,,\,\omega_1)\,,\, u_2 - x(\omega_1,\lambda_2)\rangle = \\
&= \langle f(x(\omega_1,\lambda_1)\,,\,\omega_1)\,,\, u_1 - x(\omega_1,\lambda_2)\rangle + \\
&+ \langle f(x(\omega_1,\lambda_2)\,,\,\omega_1)\,,\, u_2 - x(\omega_1,\lambda_1)\rangle \leq \\
&\leq 2M\beta(d(\lambda_1,\lambda_2))\,.
\end{aligned}$$

Hence we obtain that $x(\omega_1,\lambda_1) \to x(\omega_1,\lambda_2)$, when $\lambda_1 \to \lambda_2$, uniformly for all $\omega_1 \in \Omega_1$.

We have also that

$$\begin{aligned}
&\varphi\left(\|x(\omega_1,\lambda_2) - x(\omega_2,\lambda_2)\|\right)\|x(\omega_1,\lambda_2) - x(\omega_2,\lambda_2)\| \leq \\
&\leq \langle f(x(\omega_1,\lambda_2)\,,\,\omega_1) - f(x(\omega_2,\lambda_2)\,,\,\omega_1)\,,\, x(\omega_1,\lambda_2) - x(\omega_2,\lambda_2)\rangle + \\
&+ \langle f(x(\omega_1,\lambda_2)\,,\,\omega_1)\,,\, x(\omega_2,\lambda_2) - x(\omega_1,\lambda_2)\rangle +
\end{aligned}$$

$$+ \langle f(x(\omega_2,\lambda_2),\omega_2)\, ,\, x(\omega_1,\lambda_2) - x(\omega_2,\lambda_2)\rangle =$$

$$= \langle f(x(\omega_2,\lambda_2),\omega_2) - f(x(\omega_2,\lambda_2),\omega_1)\, ,\, x(\omega_1,\lambda_2) - x(\omega_2,\lambda_2)\rangle \le$$

$$\le \|f(x(\omega_2,\lambda_2),\omega_2) - f(x(\omega_2,\lambda_2),\omega_1)\| \|x(\omega_1,\lambda_2) - x(\omega_2,\lambda_2)\| .$$

Thus $x(\omega_1,\lambda_2) \to x(\omega_2,\lambda_2)$, when $\omega_1 \to \omega_2$.

The two convergence imply the continuity of $x(\cdot,\cdot)$ at $(\omega_2,\lambda_2)$. This point being choosed arbitrarily the continuity hold in $\Omega_1 \times \Lambda_1$.

As a corollary of the previous theorem we can prove the continuity of the metric projection with respect to perturbations.

Let $\Omega = X$ and $\omega_0 \in X$.

**Corollary 1.** *Let us suppose that:*

*i)* $x_0 = P_{C(\lambda)}(\omega_0)$;

*ii)* $C$ *is pseudo-continuous around* $(\lambda_0, x_0)$.

*Then there exists neighborhoods* $\Omega_0'$ *of* $\omega_0$, $\Lambda_0'$ *of* $\lambda_0$, *such that* $x(\cdot,\cdot) = P_{C(\cdot)}(\cdot)$ *is continuous on* $\Omega_0' \times \Lambda_0'$ *and hence* $x(\omega,\cdot) = P_{C(\cdot)}(\omega)$ *is continuous on* $\Lambda_0'$ *for all* $\omega \in \Omega_0'$.

*Proof.* In the case of a uniformly-convex Banach space with locally-uniformly convex dual the normalized duality mapping is single-valued, $\varphi$-monotone on each closed-ball and continuous from the strong topology of $X$ to the strong topology of $X^*$.

So, we can define the mapping $f(x,\omega) = -J(\omega - x)$ and we can use Theorem 1 to prove the continuity of $x(\cdot,\cdot)$ on $\Omega_0' \times \Lambda_0'$.

Hence for all $\omega \in \Omega_0'$ the metric projections $P_{C(\lambda)}(\omega)$ vary continuously with respect to $\lambda$ on $\Lambda_0'$.

As we have seen, even when $C$ is pseudo-Lipschitz continuous, this continuity is not the same $\frac{1}{2}$-Hölder type as in [6], because the normalized duality mapping is not strongly-monotone in a general uniformly-convex Banach spaces.

In the case of a Hilbert space, the $\frac{1}{2}$-Hölder-continuity with respect to $\lambda$ is a consequence of Theorem 1 and Corollary 1.

## References

[1] W. Alt and I. Kolumbán, Implicit function theorems for monotone mappings, *Kybernetika* **29** (1993), 210-221.
[2] J. P. Aubin and H. Frankowska, "Set-valued analysis", Birkhäuser, 1990.
[3] S. Dafermos, Sensitivity analysis in variational inequalities, *Math. Oper. Res.* **13** (1988), 421-434.
[4] R. N. Mukherjee and H. L. Verma, Sensitivity analysis of generalized variational inequalities, *J. Math. Anal. Appl.* **167** (1992), 299-304.
[5] J. Prüß, A characterization of uniform-convexity and application to accretive operators, *Hiroshima Math. J.* **11** (1981), 229-234.
[6] N. D. Yen, Hölder continuity of solutions to a parametric variational inequality, *Appl. Math. Optim.* **31** (1995), 245-255.
[7] N. D. Yen and G. M. Lee, Solution sensitivity of a class of variational inequalities, *J. Math. Anal. Appl.* **215** (1997), 48-55.
[8] E. Zeidler, "Nonlinear Functional Analysis and its Applications", II/b, Springer-Verlag, 1990.

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# A NUMERICAL METHOD FOR APPROXIMATINGTHE SOLUTION OF AN INTEGRAL EQUATION FROM BIOMATHEMATICS

C. IANCU

**Abstract.** 'In lucrare se dă o metodă numerică unei ecuaţii integrale cu argument modificat, care modelează procesul de răspândire a unei infecţii. Rezultatul principal al lucrării este enunţat sub forma teoremei 5.1

## 1. Introduction

In the study of the problem which appears in the population dynamics, where certain periodical phenomena occur, the following integral equation holds:

$$x(t) = \int_{t-\tau}^{t} f(u, x(u))du, \quad t \in \mathbf{R} \tag{1.1}$$

where $f \in C(\mathbf{R} \times \mathbf{R}_+)$ fulfils the condition of periodicity with respect to $t$, that is

$$f(t+\omega, x) = f(t, x), \text{ for } t \in \mathbf{R},\ x \in \mathbf{R}_+,\ \omega > 0. \tag{1.2}$$

If we suppose that $\tau \in \mathbf{R}_+$ and

$$0 \leq f(t, x) \leq M, \text{ for } t \in \mathbf{R} \text{ and } x \in \mathbf{R}_+ \tag{1.3}$$

then the problem of finding periodical solutions of equation (1.1) can be considered.

The equation (1.1) can be a mathematical model for the spreading of certain infectious diseases with a contact rate that varies seasonally. In this case $x(t)$ represents the proportion of the infectives in population at the time $t$, $\tau$ is the time interval an individual remains infectious and $f(t, x(t))$ represents the proportion of new infectives per unit time. In the papers [1]-[4], it is tackled this important problem and are given sufficient conditions for existence of non-trivial periodic nonnegative and continuous solutions of equation (1.1).

On the basis of these results, the aim of this paper is to present a numerical method for obtaining the solutions of equation (1.1).

## 2. The existence and uniqueness of solution

In [5] the following mapping is attached to equation (1.1):

$$A : X_+ \to C(\mathbf{R}),$$

which is defined by the right-hand side of (1.1), where

$$X_+ = \{x \in X |\ x(t) > 0,\ (\forall)\ t \in \mathbf{R}\},$$

and

$$X = \{x \in C(\mathbf{R}) |\ x(t+\omega) = x(t),\ (\forall)\ t \in \mathbf{R}\}.$$

Because we have

$$(Ax)(t+\omega) = \int_{t+\omega-\tau}^{t+\omega} f(s, x(s))ds = \int_{t-\tau}^{t} f(s+\omega, x(s+\omega))ds =$$
$$= \int_{t-\tau}^{t} f(s, x(s))ds = (Ax)(t)$$

and

$$t - \tau < t, \quad f \geq 0,$$

it results that $X_+$ is a invariant subset of $A$.

If we suppose that

$$|f(t,x) - f(t,y)| \leq a(t)|x-y|,\ (\forall)\ t \in \mathbf{R} \text{ and } x, y \in \mathbf{R}_+ \tag{2.1}$$

$$\int_{t-\tau}^{t} a(s)ds \leq q \leq 1 \text{ for all } t \in \mathbf{R} \tag{2.2}$$

then $A$ is a contraction mapping.

The following result is given in [5]:

**Theorem 2.1.** *If the conditions (1.2), (1.3), (2.1) and (2.2) are satisfied, then in* $C(\mathbf{R}, \mathbf{R}_+)$ *the equation (1.1) has a unique periodic continuous nonnegative solution which can be obtained by the method of successive approximations.*

Also, in [3], is proved the following theorem.

**Theorem 2.2.** *If the following assumptions are satisfied: (i) $f(t,x)$ is nonnegative and continuous for*

$$-\tau \le t \le T \text{ and } x \ge 0,\ T > 0$$

*(ii) $\phi(t)$ is continuous and $0 < a \le \phi(t)$ foe $-\tau \le t \le 0$ where the proportion $\phi(t)$ of infectives in population is known for $-\tau \le t \le 0$, i.e.*

$$x(t) = \phi(t), \text{ for } -\tau \le t \le 0$$

*and*

$$\phi(0) = b = \int_{-\tau}^{0} f(s,\phi(s))ds$$

*(iii) there exists an integrable function $g(t)$ such that $f(t,x) \ge g(t)$ for $-\tau \le t \le T$, $x \ge a$ and*

$$\int_{t-\tau}^{t} g(s)ds \ge a \text{ for } 0 \le t \le T$$

*(iv) there exists $L > 0$ such that*

$$|f(t,x) - f(t,y)| \le L|x-y|$$

*for all $t \in [-\tau, T]$ and $x, y \in [a,\infty)$, then equation (1.1) has a unique continuous solution $x(t)$, $x(t) \ge a$, for $-\tau \le t \le T$, which satisfies the condition $x(t) = \phi(t)$, for $-\tau \le t \le 0$; moreover,*

$$\max_{0 \le t \le T} |x_n(t) - x(t)| \to 0 \text{ as } n \to \infty$$

*where $x_n(t) = \phi(t)$ for $-\tau \le t \le 0$ $(n = 0,1,2,\ldots)$, $x_0(t) = b$ and $x_n(t) = \int_{t-\tau}^{t} f(s, x_{n-1}(s))ds$, $0 \le t \le T$ $(n = 1,2,\ldots)$.*

### 3. The statement of the problem

We consider the nonlinear integral equation (1.1) and we suppose that the hypotheses of the Theorems 2.1 and 2.2 are satisfied. Then this equation has a unique solution on the interval $[-\tau, T]$. Let $\varphi$ be the solution, which, by virtue of the theorem 2.2, can be obtained by successive approximation method. So, we have

$$\begin{cases} \varphi(t) = \phi(t), \text{ for } t \in [-\tau, 0) \text{ and for } t \in [0, T] \\ \qquad \text{we have :} \\ \varphi_0(t) = \phi(0) = b = \int_{-\tau}^{0} (s, \phi(s))ds \\ \varphi_1(t) = \int_{t-\tau}^{t} f(s, \varphi_0(s))ds \\ \varphi_2(t) = \int_{t-\tau}^{t} f(s, \varphi_1(s))ds \\ \dots \\ \varphi_m(t) = \int_{t-\tau}^{t} f(s, \varphi_{m-1}(s))ds, \\ \dots \end{cases} \tag{3.1}$$

To obtain the sequence of successive approximations (3.1), it is necessary to calculate the integrals which appear in the right-hand side. In general, this problem is difficult. We shall use the trapezoidal rule.

Let an interval $[a, b] \subseteq \mathbf{R}$ be given, and the function $f \in C^2[a, b]$.

Divide the interval $[a, b]$ by the points

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b \tag{3.2}$$

into $n$ equal parts of length $\Delta x = \dfrac{b-a}{n}$.

Then we have the trapezoidal formula:

$$\int_a^b f(x)dx = \frac{b-a}{2n}\left[f(a) + f(b) + 2\sum_{i=1}^{n-1} f(x_i)\right] + r_n(f) \tag{3.3}$$

where $r_n(f)$ is the remainder of the formula.

To evaluate the approximation error of the trapezoidal formula there exists the following result.

**Theorem 4.1.** *For every function $f \in C^2[a,b]$, the remainder $r_n(f)$ from the trapezoidal formula (3.3), satisfies the inequality:*

$$|r_n(f)| \leq \frac{(b-a)^3}{12n^2} \max_{x\in[a,b]} |f''(x)|. \tag{3.4}$$

## 4. The calculation of the integrals which appear in the successive approximations methods, (3.3) [6]

Now we suppose that $f \in C^2([0,T] \times \mathbf{R}_+)$, and in order to calculate the integral $\varphi_m$ from (3.1), we apply the formula (3.3). Then we divide the interval $[0,T]$ by the points:

$$0 = t_0 < t_1 < \cdots < t_n = T \tag{4.1}$$

where: $t_i = t_{i-1} + h$, $h = \frac{\tau}{2^v}$, $v = 0,1,2,\ldots$, $i = \overline{1,n}$, $n = \left[\frac{T}{h}\right]$ ($[\cdot]$ is integer part).

Thus we have

$$\varphi_m(t_k) = \int_{t_k-\tau}^{t_k} f(s,\varphi_{m-1}(s))ds = \tag{4.2}$$

$$= \frac{\tau}{2n}\left[f(t_k-\tau,\varphi_{m-1}(t_k-\tau)) + f(t_k,\varphi_{m-1}(t_k)) + 2\sum_{i=1}^{n-1} f(t_i,\varphi_{m-1}(t_i))\right] + r_{m,k}(f),$$

where, for the remainder $r_{m,k}(f)$, we have the estimation:

$$|r_{m,k}(f)| \leq \frac{\tau^3}{12n^2} \max_{s\in[0,T]} |[f(s,\varphi_{m-1}(s))]''_s|, \quad k = \overline{0,n},\ m \in \mathbf{N}.$$

Taking into account the fact that:

$$[f(s,\varphi_{m-1}(s))]''_s = \frac{\partial^2 f(s,\varphi_{m-1}(s))}{\partial s^2} + 2\frac{\partial^2 f(s,\varphi_{m-1}(s))}{\partial s\partial\varphi}\varphi'_{m-1}(s) +$$

$$+\frac{\partial^2 f(s,\varphi_{m-1}(s))}{\partial\varphi^2}(\varphi'_{m-1}(s))^2 + \frac{\partial f(s,\varphi_{m-1}(s))}{\partial\varphi}\varphi''_{m-1}(s)$$

and:

$$\varphi_{m-1}(t) = \int_{t-\tau}^{t} f(s,\varphi_{m-2}(s))ds$$

$$\varphi'_{m-1}(t) = \int_{t-\tau}^{t} \frac{\partial f(s,\varphi_{m-2}(s))}{\partial s}ds$$

$$\varphi''_{m-1}(t) = \int_{t-\tau}^{t} \frac{\partial^2 f(s,\varphi_{m-2}(s))}{\partial s^2} ds$$

and denoting by

$$M_1 = \max_{\substack{|\alpha|\leq 2\\ s\in[0,T]\\ |u|\leq R}} \left| \frac{\partial^\alpha f(s,u)}{\partial s^{\alpha_1} \partial u^{\alpha_2}} \right|,$$

we obtain

$$|\varphi_{m-1}(t)| \leq \tau M_1; \quad |\varphi'_{m-1}(t)| \leq \tau M_1; \quad |\varphi''_{m-1}(t)| \leq \tau M_1$$

again from here we have:

$$|[f(s,\varphi_{m-1}(s))]''_s| \leq M_0 \tag{4.3}$$

where $M_0 = M_1 + 3\tau M_1^2 + \tau^2 M_1^3$ and $M_0$ does not depend on $m$ and $k$.

For the remainder $r_{m,k}(f)$, from the formula (4.2) we have:

$$|r_{m,k}(f)| \leq \frac{\tau^3}{12n^2} M_0, \quad m = 0,1,2,\ldots, \ k = \overline{0,n}. \tag{4.4}$$

In this way we have obtained a formula for the approximate calculation of the integrals from (3.1).

## 5. The approximate calculation of the terms of the successive approximations sequence

Using the approximation (3.1) and the formula (4.2) with the remainder estimation (4.4), we shall present further down an algorithm for the approximate solution of equation (1.1).

So, we have:

$$\varphi_1(t_k) = \int_{t_{k-\tau}}^{t_k} f(s,\varphi_0(s))ds =$$

$$= \frac{\tau}{2n}\left[ f(t_k - \tau, \varphi_0(t_k-\tau)) + 2\sum_{i=1}^{n-1} f(t_i,\varphi(t_i)) + f(t_k,\varphi_0(t_k)) \right] + r_{1,k}(f) =$$

$$= \tilde{\varphi}_1(t_k) + r_{1,k}(f), \quad k = \overline{0,n}$$

$$\varphi_2(t_k) = \int_{t_{k-\tau}}^{t_k} f(s,\varphi_1(s))ds = \frac{\tau}{2n}\Big[ f(t_k - \tau, \tilde{\varphi}_1(t_k-\tau) + r_{1,0}(f)) +$$

$$+2\sum_{i=1}^{n-1} f(t_i,\tilde\varphi_1(t_i)+r_{1,i}(f))+f(t_k,\tilde\varphi_1(t_k)+r_{1,n}(f))\Bigg]+r_{2,k}(f)=$$

$$=\frac{\tau}{2n}\left[f(t_k-\tau,\tilde\varphi_1(t_k-\tau))+2\sum_{i=1}^{n-1} f(t_i,\tilde\varphi_1(t_i))+f(t_k,\tilde\varphi_1(t_k))\right]+\tilde r_{2,k}(f)=$$

$$=\tilde\varphi_2(t_k)+\tilde r_{2,k}(f).$$

Observe that $\tilde r_{2,k}(f)=\varphi_2(t_k)-\tilde\varphi_2(t_k)$.

Taking into account Theorem 2.2,(iv), and the remainder estimation given by (4.4), we have:

$$|\tilde r_{2,k}(f)|\le\frac{\tau}{2n}L\left[|r_{1,0}(f)|+\sum_{i=1}^{n-1}|r_{1,i}(f)|+|r_{1,n}(f)|\right]+|r_{2,k}(f)|\le$$

$$\le\frac{\tau}{2n}L\left(\frac{\tau^3}{12n^2}M_0+(n-1)\frac{\tau^3}{12n^2}M_0+\frac{\tau^3}{12n^2}M_0\right)+\frac{\tau^3}{12n^2}M_0=$$

$$=\frac{\tau}{2n}L\cdot\frac{\tau^3}{12n^2}M_0(1+n-1+1)+\frac{\tau^3}{12n^2}M_0=$$

$$=\frac{\tau^3}{12n^2}M_0\left[\frac{(n+1)\tau}{2n}L+1\right]\le\frac{\tau^3}{12n^2}M_0(\tau L+1).$$

We continue in this manner, for $m=3,\ldots,$ by induction, and obtain:

$$\varphi_m(t_k)=\frac{\tau}{2n}\Bigg[f(t_k-\tau,\tilde\varphi_{m-1}(t_k-\tau)+\tilde r_{m-1,0}(f))+$$

$$+2\sum_{i=1}^{n-1} f(t_i,\tilde\varphi_{m-1}(t_i)+\tilde r_{m-1,i}(f))+$$

$$+f(t_k,\tilde\varphi_{m-1}(t_k)+\tilde r_{m-1,n}(f))\Bigg]+r_{m,k}(f)=$$

$$=\frac{\tau}{2n}\Bigg[f(t_k-\tau,\tilde\varphi_{m-1}(t_k-\tau))+2\sum_{i=1}^{n-1} f(t_i,\tilde\varphi_{m-1}(t_i))+$$

$$+f(t_k,\tilde\varphi_{m-1}(t_k))\Bigg]+\tilde r_{m,k}(f)=\tilde\varphi_m(t_k)+\tilde r_{m,k}(f),\quad k=\overline{0,n}$$

where

$$|\tilde r_{m,k}(f)|=|\varphi_m(t_k)-\tilde\varphi_m(t_k)|\le$$

$$\le\frac{\tau^3}{12n^2}M_0[\tau^{m-1}L^{m-1}+\cdots+1],\quad k=\overline{0,n}$$

or

$$|\tilde{r}_{m,k}(f)| \leq \frac{\tau^3}{12n^2} M_0 \frac{1-\tau^m L^m}{1-\tau L} \leq \frac{\tau^3 M_0}{12n^2(1-\tau L)}.$$

In this way we got the sequence

$$(\tilde{\varphi}_m(t_k))_{m\in\mathbf{N}}, \quad k=\overline{0,n}$$

which approximates the sequence of successive approximation (3.1) on the knots (4.1), with the error

$$|\varphi_m(t_k) - \tilde{\varphi}_m(t_k)| \leq \frac{\tau^3 M_0}{12n^2(1-\tau L)}. \tag{5.1}$$

By Picard's theorem, [6], we have the following estimation

$$|\varphi(t_k) - \varphi_m(t_k)| \leq \frac{\tau^m L^m}{1-\tau L} ||\varphi_0 - \varphi_1||_{C[0,T]}. \tag{5.2}$$

In this way there was obtained the main result of our paper:

**Theorem 5.1.** *Consider the integral equation (1.1) under the conditions of Theorems 2.1 and 2.2. If the exact solution* $\varphi$ *is approximated by the sequence* $(\tilde{\varphi}_m(t_k))_{m\in\mathbf{N}}$, $k=\overline{0,n}$, *on the knots (4.1), by the successive approximations method (3.1), combined with the trapezoidal rule (3.3), then the following error estimation holds:*

$$|\varphi(t_k) - \tilde{\varphi}_m(t_k)| \leq \frac{\tau^3}{1-\tau L}\left[\tau^{m-3} L^m ||\varphi_0 - \varphi_1||_{C[0,T]} + \frac{M_0}{12n^2}\right], \tag{5.3}$$

$$m=1,2,\ldots, \quad k=\overline{0,n}.$$

*Proof.* We have

$$|\varphi(t_k) - \tilde{\varphi}_m(t_k)| = |\varphi(t_k) - \varphi_m(t_k) + \varphi_m(t_k) - \tilde{\varphi}_m(t_k)| \leq$$

$$\leq |\varphi(t_k) - \varphi_m(t_k)| + |\varphi_m(t_k) - \tilde{\varphi}_m(t_k)|$$

which, by virtue of formulae (5.1) and (5.2), can also be written

$$|\varphi(t_k) - \tilde{\varphi}_m(t_k)| \leq \frac{\tau^m L^m}{1-\tau L} ||\varphi_0 - \varphi_1||_{C[0,T]} + \frac{\tau^3 M_0}{12n^2(1-\tau L)}$$

and, from here, it results immediately (5.3). The theorem is proved. □

## References

[1] K.L. Cooke, J.L. Kaplan, *A periodicity threshold theorem for epidemics and population growth*, Math. Biosc. 31(1976), 87-104.

[2] D. Guo, V. Lakshmikanthan, *Positive solutions of nonlinear integral equations arising in infectious deseases*, J. Math. Anal. Appl. 134(1988), 1-8.

[3] R. Precup, *Positive solutions of the initial value problem for an integral equation modelling infectious disease*, Seminar on Fixed Point Theory, Preprint Nr.3(1991) (Ed. by Ioan A. Rus), 25-30.

[4] Ioan A. Rus, *A delay integral equation from biomathematics*, Seminar on diff. eq., Preprint Nr.3(1989), (ed. by Ioan A. Rus), 87-91.

[5] Ioan A. Rus, *Principii şi aplicaţii ale teoriei punctului fix*, Editura Dacia, Cluj-Napoca, 1979.

[6] Gh. Coman, Garofiţa Pavel, Ileana Rus, Ioan A. Rus, *Introducere în teoria ecuaţiilor operatoriale*, Editura Dacia, Cluj-Napoca, 1976.

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# SOME ANALYTIC INTEGRAL OPERATORS AND HARDY CLASSES

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## 1. Introduction

Let $A$ denote the set of functions $f(x) = z + a_2 z^2 + \dots$ that are analytic in the unit disk $U$ and $S$ denote the subset of $A$ consisting of univalent functions. In [4] the authors show that the integral operator

$$I_{\phi,\varphi}(f)(z) = \left[\frac{\beta+\gamma}{z^\gamma \phi(z)} \int_0^z f^\alpha(t)\varphi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta}} \tag{1}$$

maps certain subsets of $A$ into $S$.

In [2] and [3] were obtained Hardy classes for integral operator (1) and

$$I[f](z= \left[\frac{\beta+\gamma}{z^\gamma} \int_0^z f^\beta(t)t^{\gamma-1}dt\right]^{\frac{1}{\beta}}, \quad z \in U. \tag{2}$$

In this paper we obtain Hardy classes for these operators using the "open door" function [4], a special mapping from $U$ onto a slit domain.

## 2. Preliminaries

**Definition 1.** Let $c$ be a complex number such that Re $c > 0$ and let

$$N = N(c) = \frac{1}{\text{Re } c}\left[|c|(1+2\text{Re } c)^{\frac{1}{2}} + \text{Im } c\right].$$

If $h$ is the univalent function $h(z) = \dfrac{2Nz}{1-z^2}$ and $b = h^{-1}(c)$ then we define the "open door" function $Q_c$ as

$$Q_c(z) = h\left(\frac{z+b}{1+\bar{b}z}\right), \quad z \in U.$$

From its definition we see that $Q_c$ is univalent, $Q_c(0) = c$ and $Q_o(U) = h(U)$ is the complex plane slit along the half-lines Re $w = 0$, Im $w \geq N$ and Re $w = 0$, Im $w \leq -N$.

**Definition 2.** Let $f$ and $g$ be analytic in $U$. The function $f$ is subordinate to $g$, written $f \prec g$ or $f(z) \prec g(z)$ if $g$ is univalent, $f(0) = g(0)$ and $f(U) \subset g(U)$.

For $f$ analytic in $U$ and $z = re^{i\theta}$ we denote

$$M_p(r,f) = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}}, & \text{for } 0 < p < \infty \\ \sup\limits_{0\leq\theta<2\pi} |f(re^{i\theta})|, & \text{for } p = \infty. \end{cases}$$

A function is said to be of Hardy class $H^p$, $0 < p \leq \infty$ if $M_p(r, f)$ remains bounded as $r \to 1^-$, $H^\infty$ is the class of bounded analitic functions in the unit disk.

We shall need the following lemmas:

**Lemma 1.** [5] *Let $Q_c$ be the function given by Definition 1 and let $B(z)$ be analytic function in $U$ satisfying $B(z) \prec Q_c(z)$.*

*If $p$ is analytic in $U$, $p(0) = \frac{1}{c}$ and $p$ satisfies the differential equation $zp'(z)+ B(z)p(z) = I$ then* Re $p(z) > 0$, $z \in U$.

**Lemma 2.** [4] *Let $\alpha, \delta \in \mathbf{C}$,* Re $(\alpha + \delta) > 0$ *and let $\varphi$ be analytic function in $U$ with $\varphi(0) = I$, $\varphi(z) \neq 0$ in $U$. If $f \in A$ satisfies*

$$\alpha\frac{zf'(z)}{f(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta \prec Q_{\alpha+\delta}(z) \tag{3}$$

*and $F$ is defined by*

$$F(z) = I[f](z) = \left[(\alpha+\delta)\int_0^z f^\alpha(t)t^{\delta-1}\varphi(t)dt\right]^{\frac{1}{\alpha+\delta}} \tag{4}$$

*then*

$$\text{Re } \left[(\alpha+\delta)\frac{zF'(z)}{F(z)}\right] > 0 \circ i \; F \in S.$$

*Moreover, if $\alpha + \delta > 0$ then $F \in S^*$ (starlike functions).*

**Lemma 3.** (Prawitz, 1927) *If $f \in S$, then $f \in H^p$, $p < \frac{1}{2}$.*

## 3. Main results

**Theorem 1.** *If $\beta > 0$, $\gamma > 0$, $f \in A$ and* $\frac{\beta z f'(z)}{f(z)} \prec Q_c(z)$ *then*

$$I(f) \in H^{\lambda}, \quad \lambda = \frac{q\beta}{\beta+\gamma}, \quad q < \frac{1}{2}$$

*where $I$ is defined by (2).*

*Proof.* The operator $I$ as given by (2), can be written as $I : I = G_1 \circ G_2$ where

$$G_1[f](z) = \left(\frac{f^{\beta+\gamma}(z)}{z^{\gamma}}\right)^{\frac{1}{\beta}}$$

$$G_2[f](z) = \left((\beta+\gamma)\int_0^z f^{\beta}(t)t^{\gamma-1}dt\right)^{\frac{1}{\beta+\gamma}}.$$

If in Lemma 1 $\varphi(z) \equiv 1$ then condition (3) to maps $\frac{\beta z f'(z)}{f(z)} \prec Q_c(z)$ and the operator $F$ defined by (4) to maps in $G_2$. Hence $G_2(z) \in S^*$ and $G_2 \in H^q$, $q < \frac{1}{2}$. On the other hand if $f \in H^p$ then $f^{\beta+\gamma} \in H^{\frac{p}{\beta+\gamma}}$ and $\frac{f^{\beta+\gamma}(z)}{z^{\gamma}} \in H^{\frac{p}{\beta+\gamma}}$. Hence $\left(\frac{f^{\beta+\gamma}(z)}{z^{\gamma}}\right)^{\frac{1}{\beta}} \in H^{\frac{\beta p}{\beta+\gamma}}$ and we obtain $G_1 \in H^{\frac{\beta p}{\beta+\gamma}}$ if $f \in H^p$. Since $G_2 \in H^q$, $q < \frac{1}{2}$ we obtain $G_1 \circ G_2 \in H^{\frac{q\beta}{\beta+\gamma}}$, $q < \frac{1}{2}$. □

**Corollary 2.** *Let $\alpha, \delta$ complex numbers with* Re $(\alpha+\delta) > 0$ *and $\varphi$ analytic in $U$, $\varphi(0) = 1$, $\varphi(z) \neq 0$, $z \in U$. If $f \in A$ and satisfying (3) then $F \in H^{\lambda}$, $\lambda < \frac{1}{2}$ ($F$ defined by (4)). If $\alpha + \delta > 0$ then $F \in H^{\frac{1}{2}}$ and $F' \in H^{\frac{1}{3}}$.*

**Theorem 3.** *Let $Q_c$ "open door" function and $B(z)$ analytic in $U$ satisfying $B(z) \prec Q_c(z)$. Let $\phi$ an analytic function in $U$, $\phi(0) = \frac{1}{c}$, and $z\phi'(z) + B(z)\phi(z) = 1$. Let $\alpha, \beta, \delta$ be real numbers $\beta > 0$, $\alpha\delta > 0$ and $\varphi$ analytic in $U$ with $\varphi(0) = 1$, $\varphi(z) \neq 0$, $z \in U$. If $f \in A$ satisfies (3) then*

$$I_{\phi,\varphi}[f] \in H^p, \quad p = \frac{q\lambda\beta}{\lambda(\alpha+\delta)+q}, \quad \lambda < 1, \quad q < \frac{1}{2}.$$

*Proof.* From Lemma 1 we have Re $\phi(z) > 0$. Hence Re $\frac{1}{\phi(z)} > 0$ and $\frac{1}{\phi(z)} \in H^\lambda$, $\lambda < 1$. The integral operator $I_{\phi,\varphi}$ as given by (1) can be written as: $I_{\phi,\varphi} = G \circ F$ where $G(x) = \left(\frac{f^{\alpha+\delta}(z)}{z^\gamma \phi(z)}\right)^{\frac{1}{\beta}}$ and $F$ is defined by (4).

From Lemma 2, $F \in S^*$ and from Lemma 3, $F \in H^q$, $q < \frac{1}{2}$. Since $\frac{1}{\phi(z)} \in H^\lambda$ applying Hölder's inequality we obtain $\frac{1}{z^\gamma \phi(z)} F^{\alpha+\delta}(z) \in H^\mu$ where

$$\mu = \frac{\lambda \frac{q}{\alpha+\delta}}{\lambda + \frac{q}{\alpha+\delta}} = \frac{q\lambda}{\lambda(\alpha+\delta)+q}, \quad q < \frac{1}{2}.$$

Hence

$$G(F) \in H^p, \quad p = \frac{q\lambda\beta}{\lambda(\alpha+\delta)+q}, \quad q < \frac{1}{2}, \quad \lambda < 1.$$

□

## References


[1] Duren, P.L., *Theory of $H^p$ Spaces*, Academic Press, New York and London, 1970.
[2] Miclăuş, Gh., *Integral Operator of Singh and Hardy Classes*, Studia Univ. Babeş-Bolyai, Mathematica, 42, 2(1997), 71-77.
[3] Miclăuş, Gh., *Some Integral Operators and Hardy Classes*, Studia Univ. Babeş-Bolyai, Mathematica, 41, 3(1996), 57-64.
[4] Miller, S.S., Mocanu, P.T., *Classes of Univalent Integral Operators*, J. of Math. Anal. Appl., New York and London, 151, 1(1991), 147-165.
[5] Mocanu, P.T., *Some Integral and Starlike Functions*, Rev. Roumaine Math. Pures Appl., 31(1986), 231-235.


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# DIFFERENTIAL AND INTEGRAL OPERATORS PRESERVING FUNCTIONS WITH POSITIVE REAL PART AND HARDY CLASSES

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## 1. Introduction

Let $\mathcal{H}(U)$ set of denote the functions analytic in the unit disk $U = \{z : |z| < 1\}$. In [4] the authors develop differential and integral operators preserving functions with positive real part.

In [2] and [3] sharp results concerning the boundary behaviour of $I(f)$, $I_g(f)$ and $I_{\phi,\varphi}(f)$ when $f$ belongs to the Hardy spaces $H^p$, $0 < p \leq \infty$, where $I(f)$, $I_g(f)$ and $I_{\phi,\varphi}(f)$ is the integral operator defined by:

$$I[f](z) = \left[\frac{\beta+\gamma}{z^\gamma}\int_0^z f^\beta(t)t^{\gamma-1}dt\right]^{\frac{1}{\beta}}, \quad z \in U, \text{ (Singh, 1973)} \tag{1}$$

$$I_g(f)(z) = \left[\frac{\beta+\gamma}{z^\gamma}\int_0^z \left[\frac{f(t)}{t}\right]^\alpha \left[\frac{g(t)}{t}\right]^\delta t^{\alpha+\delta-1}dt\right]^{\frac{1}{\beta}}, \quad z \in U \tag{2}$$

$$I_{\phi,\varphi}(f)(z) = \left[\frac{\beta+\gamma}{z^\gamma\phi(z)}\int_0^z f^\alpha(t)\varphi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta}}, \quad z \in U \text{ (Miller, Mocanu, 1991)} \tag{3}$$

In this paper we obtain results for the Hardy classes of these integral operators when $f$ satisfy some differential conditions.

## 2. Preliminaries

For $f \in \mathcal{H}(U)$ and $z = re^{i\theta}$ we denote

$$M(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}}, \text{ for } 0 < p < \infty$$

and

$$M_\infty(r,f) = \sup_{0\le\theta\le 2\pi} |f(re^{i\theta})| \text{ for } p=\infty.$$

A function is said to be of Hardy class $H^p$, $0 < p \le \infty$ if $M(r,f)$ remains bounded as $r \to 1^-$, $H^\infty$ is the class of bounded analytic functions in the unit disk.

We shall need the following lemmas:

**Lemma 1.** *Let $A \ge 0$ and $B, C, D : U \to \mathbf{C}$ with*

$$\begin{aligned} &\text{Re } B(z) \ge A \\ &[\text{Im } C(z)]^2 \le [\text{Re } B(z) - A]\text{Re } [B(z) - A - 2D(z)]. \end{aligned} \tag{4}$$

*If $f$ is analytic in $U$ with $f(0) = 1$ and*

$$\text{Re } [Az^2 f''(z) + B(z)zf'(z) + C(z)f(z) + D(z)] > 0 \tag{5}$$

*then* $\text{Re } f(z) > 0$.

**Lemma 2.** *Let $\eta \ne 0$, $\eta \in \mathbf{C}$, $\text{Re } \eta \ge 0$ and $\varphi, \phi$ analytic functions in $U$, $\varphi(z)\phi(z) \ne 0$, $\varphi(0) = \phi(0)$, and*

$$\left|\text{Im } \frac{\eta\phi(z) + z\phi'(z)}{\eta\varphi(z)}\right| \le \text{Re } \frac{\phi(z)}{\eta\varphi(z)}. \tag{6}$$

*Let $f$ be analytic in $U$ with $f(0) = 1$ and* $\text{Re } f(z) > 0$, $z \in U$.

*If $f$ is defined by:*

$$F(z) = \frac{\eta}{z^\eta \phi(z)} \int_0^z f(t)t^{\eta-1}\varphi(t)dt \tag{7}$$

*then $F$ is analytic in $U$, $F(0) = 1$ and* $\text{Re } F(z) > 0$, $z \in U$.

**Lemma 3.** *Let $\beta$ and $\gamma$ be complex numbers with $\beta\gamma > 0$, $\text{Re } \beta \ge 0$, $\text{Re } \gamma \ge 0$, $\varphi$ and $\phi$ be analytic in $U$ with $\varphi(z)\phi(z) \ne 0$, $\varphi(0) = \phi(0)$ and $w$ be analytic in $U$ with $w(0) = 0$. Suppose that (4) holds with*

$$\begin{cases} A = \dfrac{1}{B\gamma}, \quad D(z) = -w(z) \\ B(z) = \dfrac{1}{\beta}\left[\beta + \gamma + 1 + z\dfrac{\varphi'(z)}{\varphi(z)} + \dfrac{z\phi'(z)}{\phi(z)}\right] \\ C(z) = \dfrac{1}{\beta\gamma}\left[\left(\beta + \dfrac{z\phi'(z)}{\phi(z)}\right)\left(\gamma + \dfrac{z\varphi'(z)}{\varphi(z)} + z\left(\dfrac{z\varphi'(z)}{\varphi(z)}\right)'\right)\right] \end{cases} \tag{8}$$

*Let $f$ be analytic in $U$ with $f(0) = 1$ and* $\text{Re } f(z) > 0$, $z \in U$.

*If $f$ is defined by:*

$$F(z) = \frac{\beta}{z^{\gamma}\varphi(z)} \int_0^z \frac{\varphi(t)}{\phi(t)} t^{\gamma-\beta-1} \int_0^t [f(s) + w(s)]\phi(s)s^{\beta-1} ds dt \tag{9}$$

*then $F$ is analytic in $U$, $F(0) = 1$ and* Re $F(z) > 0$, $z \in U$.

Lemma 1-3 was proved in [4].

**Lemma 4.** [1] *If $f \in \mathcal{H}(U)$ and* Re $f(z) > 0$, $z \in U$ *then $f \in H^p$, $p < 1$.*

**Lemma 5.** *If $f \in H^p$, $b > 0$ and $I$ is the integral operator of Singh (3) then*

*(i) if $\beta > p$ then $I[f] \in H^{\frac{\beta p}{\beta - p}}$;*

*(ii) if $\beta \leq p$ then $I[f] \in H^{\infty}$.*

**Lemma 6.** *If $f \in H^p$, $g \in H^q$, $p, q, \alpha, \beta, \delta \in \mathbf{R}_+^*$ then*

*(i) if $\dfrac{pq}{\delta p + \alpha q} < 1$ then $I_g(f) \in H^{\frac{\beta pq}{\delta p + \alpha q - pq}}$;*

*(ii) if $\dfrac{pq}{\delta p + \alpha q} \geq 1$ then $I_g(f) \in H^{\infty}$.*

**Lemma 7.** *If $f \in H^p$, $\varphi \in H^q$, $\dfrac{1}{\phi} \in H^r$, $\alpha, \beta > 0$ then*

*(i) if $pq < p + \alpha q$ then $I_{\phi,\varphi}[f] \in H^{\frac{\beta pqr}{pq + pr + \alpha qr - pqr}}$;*

*(ii) if $pq > p + \alpha q$ then $I_{\phi,\varphi}[f] \in H^r$.*

Lemma 5 was proved in [2] and Lemma 6 and Lemma 7 was proved in [3].

## 3. Main results

**Theorem 1.** *Let be $A \geq 0$ and $B, C, D : U \to \mathbf{C}$ satisfying condition (4). If $f$ analytic in $U$, $f(0) = 1$ and*

$$\text{Re } [Az^2 f''(z) + B(z)zf'(z) + C(z)f(z) + D(z)] > 0, \quad A_i \in \mathbf{C}$$

*then*

$$F(z) = A_0 + A_1 f(z) + A_2 f^2(z) + \cdots + A_n f^n(z) \in H^{\frac{\lambda}{n}}, \quad \lambda < 1.$$

*Proof.* From Lemma 1 and Lemma 4 we have $f(z) \in H^{\lambda}$, $\lambda < 1$. Hence we deduce $f^n(z) \in H^{\frac{\lambda}{n}}$. By applying Minkowski's inequality we obtain the result. □

**Theorem 2.** *Let $A \geq 0$ and $B, C, D : U \to \mathbf{C}$ satisfying conditions (4) and $f$ analytic in $U$, $f(0) = 1$ and (5). If $\beta > 0$, $\gamma \in \mathbf{C}$ and $I$ is integral operator of Singh (1) then*

(i) *if* $\beta > 1$ *then* $I[f] \in H^{\frac{\beta}{\beta-\lambda}}$, $\lambda < 1$;

(ii) *if* $\beta \leq 1$ *then* $I[f] \in H^{\infty}$.

*Proof.* From Lemma 1 and Lemma 4 we obtain $f(z) \in H^{\lambda}$, $\lambda < 1$. From Lemma 5 we obtain the result. □

**Theorem 3.** *Let be* $\delta \neq 0$, Re $\delta \geq 0$ *and* $\varphi, \phi$ *analytic functions in* $U$, *with* $\varphi(z)\phi(z) \neq 0$, $\varphi(0) = \phi(0)$ *satisfying condition (6),* $f$ *analytic in* $U$ *with* $f(0) = 1$ *and* Re $f(z) > 0$, $z \in U$ *and* $F$ *defined by (7). If* $\beta > 0$, $\gamma \in \mathbf{C}$ *and* $I$ *is defined by (1) then:*

(i) *if* $\beta > 1$ *then* $I[F] \in H^{\frac{\beta}{\beta-\lambda}}$, $\lambda < 1$;

(ii) *if* $\beta \leq 1$ *then* $I[F] \in H^{\infty}$.

*Proof.* From Lemma 2 and Lemma 4 we deduce $F \in H^{\lambda}$, $\lambda < 1$, and from Lemma 5 we obtain the result. □

**Theorem 4.** *Let be* $\eta \neq 0$, $\eta \in \mathbf{C}$, Re $\eta \geq 0$ *and* $\varphi, \phi$ *analytic functions in* $U$, *with* $\varphi(z)\phi(z) \neq 0$, $\varphi(0) = \phi(0)$ *and satisfying (6).*

*If* $f$ *analytic in* $U$, $f(0) = 1$, Re $f(z) > 0$, $z \in U$, $F$ *defined by (7) and* $I_g$ *defined by (2) then:*

(i) *if* $\dfrac{\lambda\mu}{\delta\lambda + \alpha\mu} < 1$ *then* $I_F[f] \in H^{\frac{\beta\lambda\mu}{\delta\lambda+\alpha\mu-\lambda\mu}}$, $0 < \lambda < 1$, $0 < \mu < 1$;

(ii) *if* $\dfrac{\lambda\mu}{\delta\lambda + \alpha\mu} < 1$ *then* $I_F[f] \in H^{\infty}$, $0 < \lambda < 1$, $0 < \mu < 1$.

*Proof.* From Lemma 2 we obtain Re $F(z) > 0$ and from Lemma 4 we deduce $F(z) \in H^{\mu}$, $\mu < 1$. Since Re $f(z) > 0$ we have $f \in H^{\lambda}$, $\lambda < 1$. Applying again Lemma 6 we obtain the result. □

*Remark 1.* An analog result we can obtain for $F$ defined by (9).

**Theorem 5.** *Let* $\alpha = 1$, $\beta > 0$, $\gamma \in \mathbf{C}$, Re $\gamma \geq 0$, $\delta = \gamma$, $\phi$ *and* $\gamma$ *analytic functions in* $U$, *with* $\varphi(z)\phi(z) \neq 0$, $\varphi(0) = \phi(0)$ *satisfying (6) and* $f$ *analytic in* $U$, $f(0) = 1$, Re $f(z) \geq 0$, $z \in U$, *then*

$$I_{\phi,\varphi}[f] \in H^{p\beta}, \quad p < 1.$$

*Proof.* From Lemma 2, for $\eta \in \mathbf{C}^*$, Re $\eta \geq 0$,

$$F(z) = \frac{\eta}{z^\eta \phi(z)} \int_0^z f(t)\varphi(t)t^{\eta-1}dt,$$

we have Re $F(z) > 0$. Hence $F \in H^p$, $p < 1$.

For $\eta = \gamma$ we obtain

$$\frac{\gamma}{\beta+\gamma} \cdot \frac{\beta+\gamma}{z^\gamma \phi(z)} \int_0^z f(t)\varphi(t)t^{\gamma-1}dt \in H^p$$

and

$$\frac{\gamma}{\beta+\gamma} I_{\phi,\varphi} \in H^{p\beta}$$

and

$$I_{\phi,\varphi} \in H^{p\beta}.$$

□

**Theorem 6.** *Let $\eta \neq 0$, $\eta \in \mathbf{C}$,* Re $\eta \geq 0$ *and $\varphi, \phi$ analytic functions in $U$, with $\varphi(z)\phi(z) \neq 0$, $\varphi(0) = \phi(0)$ satisfying (6). Let be $f$ analytic in $U$, $f(0) = 1$,* Re $f(z) > 0$*, $z \in U$ and $F$ defined by (7). If $i_{\phi,\varphi}$ is defined by (3), $\alpha, \beta, \gamma, \delta > 0$ and $g \in H^p$, $0 < \lambda < 1$, $0 < \mu < 1$ then*

*(i) if $p\lambda < p + \alpha\lambda$ then $U_{F,f}(g) \in H^{\frac{\beta p\lambda\mu}{p\lambda+p\mu+\alpha\lambda\mu-p\lambda\mu}}$;*

*(ii) if $p\lambda \geq p + \alpha\lambda$ then $I_{F,f}(g) \in H^\infty$.*

*Proof.* From Lemma 4 we have $f \in H^\lambda$, $\lambda < 1$. From Lemma 2 we obtain Re $F(z) > 0$ and Re $\dfrac{1}{F(z)} > 0$. From Lemma 4 we have $\dfrac{1}{F(z)} \in H^\mu$, $\mu < 1$.

Applying again Lemma 7 replacing $\phi$ with $F$, $\varphi$ with $f$ and $f$ with $g$ we obtain the result. □

*Remark 2.* An analog result we can obtain for $F$ defined by (9) or $g$ is defined by (9).

## References


[1] Duren, P.L., *Theory of $H^p$ Spaces*, Academic Press, New York and London, 1970.

[2] Miclăuş, Gh., *Integral Operator of Singh and Hardy Classes*, Studia Univ. Babeş-Bolyai, Mathematica, 42, 2(1997), 71-77.

[3] Miclăuş, Gh., *Some Integral Operators and Hardy Classes*, Studia Univ. Babeş-Bolyai, Mathematica, 41, 3(1996), 57-64.

[4] Miller, S.S., Mocanu, P.T., *The theory and Applications of Second-Order Differential Subordinations*, Studia Univ. Babeş-Bolyai Math., 34, 4(1989), 3-33.


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# AVERAGING INTEGRAL OPERATORS AND HARDY CLASSES

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## 1. Introduction

Let $\mathcal{H}(U)$ denote the spaces of analytic functions in the unit disk $U = \{z : |z| > 1\}$ and $H_0 = \{f \in \mathcal{H}(U) : f(0) = 0\}$. If $K \subset \mathcal{H}(U)$ then an operator $A : K \to \mathcal{H}(U)$ is said to be an averaging operator on $K$ if $A(f(0)) = f(0)$ and $A(f)[U] \subset co\, f(U)$, for all $f \in K$, where $co\, f(U)$ is the convex hull of $f(U)$. In [4] was obtained the integral averaging operator:

$$A[f](z) = \frac{1}{z^{\gamma}\phi(z)} \int_0^z f(t)t^{\gamma-1}\varphi(t)dt \tag{1}$$

and in [6] was obtained the second-order averaging integral operator

$$F(z) = \frac{1}{\alpha z^{\gamma}\varphi(z)} \int_0^z \frac{\varphi(t)}{\phi(t)} t^{\gamma-\beta-1} \int_0^t f(s)s^{\beta-1}\phi(s)dsdt. \tag{2}$$

In this paper we obtain Hardy classes for these operators and we obtain result for a more general operator $A_{\varphi}$, $\varphi \in H_0$ defined by $A_{\varphi}(f) = A(f) + f'(0)A(\varphi)$, $\varphi \in H_0$.

In [2] and [3] were obtained Hardy classes for integral operators

$$I[f](z) = \left[\frac{\beta+\gamma}{z^{\gamma}} \int_0^z f^{\beta}(t)t^{\gamma-1}dt\right]^{\frac{1}{\beta}}, \quad z \in U \text{ (Singh, 1973)} \tag{3}$$

$$I_{\phi,\varphi}(f)(z) = \left[\frac{\beta+\gamma}{z^{\gamma}\phi(z)} \int_0^z f^{\alpha}(t)\varphi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta}}, \quad z \in U. \tag{4}$$

In this paper we obtain Hardy classes for these operators, using averaging operators.

## 2. Preliminaries

For $f$ analytic in $U$ and $z = re^{i\theta}$ we denote

$$M_p(r,f) = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}}, & \text{for } 0 < p < \infty \\ \sup\limits_{0\le\theta<2\pi} |f(re^{i\theta})|, & \text{for } p = \infty. \end{cases}$$

A function is said to be of Hardy class $H^p$, $0 < p \le \infty$ if $M_p(r,f)$ remains bounded as $r \to 1^-$, $H^\infty$ is the class of bounded analytic functions in the unit disk.

If $f, g$ analytic in $U$, then function $f$ is subordinate to $g$, written $f \prec g$ or $f(z) \prec g(z)$, if $g$ is univalent, $f(0) = g(0)$ and $f(U) \subset g(U)$.

A function $h$ is said to be convex if $h$ is univalent and $h(U)$ is a convex domain.

It is easy to show that an operator $A : K \to \mathcal{H}(U)$ is an averaging operator on $K$ if and only if $[f \in K,\ h$ convex and $f \prec h] \Rightarrow A(f) \prec h$.

We shall need the following lemmas.

**Lemma 1.** *Let $h \in H_0$, convex and let $A \ge 0$. Suppose that $k > \frac{4}{|h'(0)|}$ and $B(z), C(z), D(z)$ are analytic in $U$ and satisfy*

$$\text{Re } B(z) \ge A + |C(z) - 1| - \text{Re } [C(z) - 1] + k|D(z)|, \quad z \in U. \tag{5}$$

*If $p \in H_0$ satisfies the differential subordination*

$$Az^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z) \tag{6}$$

*then $p(z) \prec h(z)$.*

**Lemma 2.** *Let $\delta \in \mathbf{C}$, $\delta \ne -1, -2, \ldots$ and let $\varphi, \phi \in \mathcal{H}(U)$ analytic functions with $\varphi(z)\phi(z) \ne 0$, $z \in U$. If*

$$\text{Re } B(z) \ge |C(z) - 1| - \text{Re } [C(z) - 1], \quad z \in U, \tag{7}$$

*where $B(z) = \frac{\phi(z)}{\varphi(z)}$ and $C(z) = \frac{\gamma\phi(z) + z\phi'(z)}{\varphi(z)}$, then the integral operator $A$ defined by (1) is an averaging operator on $H_0$.*

**Lemma 3.** *Let $\alpha \geq 0$, $\beta, \gamma \in \mathbf{C}$, with* $\operatorname{Re}\beta > -1$ *and* $\operatorname{Re}\gamma > -1$ *and let $\varphi, \phi$ analytic functions with $\varphi(z)\phi(z) \neq 0$, $z \in U$. Let*

$$\begin{cases} B(z) = \alpha\left[\beta + \gamma + 1 + \dfrac{z\phi'(z)}{\phi(z)} + \dfrac{z\phi'(z)}{\varphi(z)}\right] \\ C(z) = \alpha\left[\left(\beta + \dfrac{z\phi'(z)}{\phi(z)}\right)\left(\gamma + \dfrac{z\varphi'(z)}{\varphi(z)} + z\left(\dfrac{z\varphi'(z)}{\varphi(z)}\right)'\right)\right]. \end{cases} \tag{8}$$

*If $\theta \in H_0$ and* $\operatorname{Re} B(z) \geq \alpha - |C(z) - 1| - \operatorname{Re}[C(z) - 1] + 4|\theta(z)|$, $z \in U$, *then the operator*

$$F_\theta[f] = F[f] + f'(0)F[\theta], \quad f \in H_0$$

*where $F$ is defined by (2), is an averaging operator on $H_0$.*

These lemmas were proced in [5].

**Lemma 4.** *If $f \in H^p$, $\beta > 0$ and $I$ is the integral operator of Singh (3) then*

*(i) if $\beta > p$ then $I[f] \in H^{\frac{\beta p}{\beta - p}}$*

*(ii) if $\beta \leq p$ then $I[f] \in H^\infty$.*

**Lemma 5.** *If $f \in H^p$, $\varphi \in H^q$, $\dfrac{1}{\phi} \in H^r$, $\alpha, \beta > 0$ then*

*(i) if $pq < p + \alpha q$ then $I_{\phi,\varphi}[f] \in H^{\frac{\beta pqr}{pq + pr + \alpha qr - pqr}}$*

*(ii) if $pq \geq p + \alpha q$ then $I_{\phi,\varphi}[f] \in H^r$.*

Lemma 4 was proved in [2] and Lemma 5 in [3].

## 3. Main Results

**Theorem 1.** *Let $h \in H_0$, be convex and $A \geq 0$, $k > \dfrac{4}{|h'(0)|}$, $B(z), C(z), D(z)$ analytic in $U$ satisfies (5) and $p \in H_0$ satisfies the differential subordination (6) then $p(z) \in H^\lambda$, $\lambda < 1$.*

*Proof.* From Lemma 1 we obtain $p(z) \prec h(z)$. From subordination theorem of Littlewood [1] we deduce $M_\lambda(r, p) \leq M_\lambda(r, h)$. Since $h$ is convex is very know that $h \in H^\lambda$, $\lambda < 1$. Hence $p(z) \in H^\lambda$, $\lambda < 1$. □

**Theorem 2.** *Let $\delta \in \mathbf{C}$ with $\delta \neq -1, -2, \ldots$ and $\varphi, \phi \in \mathcal{H}(U)$ with $\varphi(z)\phi(z) \neq 0$, $z \in U$, satisfying conditions (7) and $A$ is operator defined by (1) then $A(f) \in H^p$, $p < 1$, for all $f$, $f \in H_0$.*

*Proof.* From Lemma 2 the integral operator $A$ is averaging operator on $H_0$. Hence $I[f](U) \subset co\, f(U)$ for all $f \in H_0$. Since $co\, f(U)$ is convex domain, from conformal mappings's theorem (Riemann) there is a function $g$ analytic in $U$ such that $g(U) = co\, f(U)$. Since $g(U)$ is convex domain we deduce that $g$ is convex. Since $M_\lambda(r, I[f]) \leq M_\lambda(r, g)$ we obtain $I(f) \in H^p$, $p < 1$. □

**Theorem 3.** *Let* $\gamma \in \mathbf{C}$ *with* $\text{Re}\, \gamma > 0$ *and let* $g \in H_0$ *with* $\text{Re}\, \dfrac{\gamma z g'(z)}{g(z)} > 0$ *in* $U$. *If* $A$ *is defined by*

$$A[f](z) = \frac{\gamma}{g(z)} \int_0^z f(t) g(t)^{\gamma-1} g'(t) dt$$

*then* $A[f] \in H^p$, $p < 1$ *for all* $f \in H_0$.

*Proof.* If in $A$ $\varphi(z) = [g(z)]^{\gamma-1} g'(z) z^{1-\gamma}$ and $\phi(z) = [g(z)]^\gamma z^{-\gamma} \gamma^{-1}$ then is satisfying condition (7) and from Lemma 2, $A$ is averaging operator in $H_0$ and from Theorem 2 we obtain the result.

Hence we obtain some particular results. For $\gamma = 1$, $\alpha = 1$ and $g(z) = z$ we have the operator

$$\frac{1}{2} \int_0^z f(t) dt \in H^p, \quad p < 1.$$

Since, the Libera's operator is

$$\frac{2}{z} \int_0^z f(t) dt$$

we obtain Hardy classes for that operator:

$$\frac{2}{z} \int_0^z f(t) dt \in H^p, \quad p < 1.$$

For $\gamma = 0$, $\phi(z) = \dfrac{1}{2}$ and $\varphi(z) = 1$ we obtain

$$A[f](z) = \frac{1}{2} \int_0^z \frac{f(t)}{t} dt \in H^p, \quad p < 1.$$

Hence, we have Alexander's operator

$$\int_0^z f(t) t^{-1} dt \in H^p, \quad p < 1, \text{ for all } f,\ f \in H_0.$$

□

**Theorem 4.** *Let be $\gamma \in \mathbf{C}$, $\gamma \neq -1, -2, \dots$ and let $\varphi, \phi$ analytic functions with $\varphi(z)\phi(z) \neq 0$, $z \in U$ and satisfying conditions (7). If $I_{\phi,\varphi}$ is defined by (4) and $\alpha = 1$, $\delta = \gamma$ then $I_{\phi,\varphi}[f] \in H^{\beta\lambda}$, $\beta > 0$, $0 < \lambda < 1$, for all $f \in H_0$.*

*Proof.* The operator $I_{\phi,\varphi}$ can be written as: $I_{\phi,\varphi} = B \circ A$ where $B(f) = (\beta+\gamma)[f(z)]^{\frac{1}{\beta}}$ and $A$ is defined by (1). Since $f \in H^p$ applying Hölder's inequality we obtain $B(f) \in H^{\beta p}$. From Theorem 2 we have $A(f) \in H^\lambda$, $\lambda < 1$ for all $f \in H_0$. Hence $B(A(f)) \in H^{\beta\lambda}$ and $I_{\phi,\varphi}(f) \in H^{\beta\lambda}$ for all $f \in H_0$. □

**Theorem 5.** *If $\delta \in \mathbf{C}$, $\delta \neq -1, -2, \dots$, $\varphi, \phi$ analytic functions and $\varphi(z)\phi(z) \neq 0$, $z \in U$ satisfies conditions (7), and $A$ is the integral operator defined by (1) substituting $\gamma$ with $\delta$ and $I$ is the integral operator of Singh (3) then:*

*(i) if $\beta > 1$ then $I(A) \in H^{\frac{\beta\lambda}{\beta-1}}$, $\lambda < 1$*

*(ii) if $0 < \beta \leq 1$ then $I(A) \in H^\infty$, for all $f \in H_0$.*

*Proof.* From Theorem 2 we deduce $A \in H^\lambda$, $\lambda < 1$. From Lemma 4 we obtain the result. □

**Theorem 6.** *Let $\alpha \geq 0$, $\beta, \gamma \in \mathbf{C}$, with $\mathrm{Re}\,\beta > -1$ and $\mathrm{Re}\,\gamma > -1$ and let $\varphi, \phi$ analytic function with $\varphi(z)\phi(z) \neq 0$, $z \in U$. Let $F : H_0 \to H_0$ defined by (2) and suppose that are satisfying (8).*

*If $\theta \in H_0$ and $\mathrm{Re}\, B(z) \geq \alpha - |C(z) - 1| - \mathrm{Re}\,[C(z) - 1] + 4|\theta(z)|$, $z \in U$, then the operator $J_\theta[f] = F[f] + f'(0)F[\theta]$ we have $J_\theta(f) \in H^\lambda$, $\lambda < 1$, for all $f \in H_0$.*

*Proof.* From Lemma 3 we obtain that $J_\theta$ is averaging integral operator. Hence $J_\theta[f](U) \subset co\, f(U)$, where $co\, f(U)$ is convex domain. From Riemann's theorem exists a convex function $g$ such that $g(U) = co\, f(U)$. Hence we deduce $M_\lambda(r, J_\theta) \leq M_\lambda(r, g)$, and we obtain the results. □

*Remark 1.* Analog with Theorem 4 we can obtain results for Hardy classes for $I[J_\theta]$ where $I$ is the integral operator of Singh.

*Remark 2.* Analog with Theorem 7 [2] we can obtain results for Hardy classes for the $n$-order integral operator of Singh.

## References


[1] Duren, P.L., *Theory of $H^p$ Spaces*, Academic Press, New York and London, 1970.

[2] Miclăuş, Gh., *Integral Operator of Singh and Hardy Classes*, Studia Univ. Babeş-Bolyai, Mathematica, 42, 2(1997), 71-77.

[3] Miclăuş, Gh., *Some Integral Operators and Hardy Classes*, Studia Univ. Babeş-Bolyai, Mathematica, 41, 3(1996), 57-64.

[4] Miller, S.S., Mocanu, P.T., *Mean-value Theorems in the Complex Plane*, Babeş-Bolyai Univ., Fac. of Math. Reasearch Seminars, Seminar in Geometric Functions Theory, Preprint 5(1987), 199-211.

[5] Miller, S.S., Mocanu, P.T., *The theory and Applications of Second-Order Differential Subordinations*, Studia Univ. Babeş-Bolyai, Math., 34, 4(1989), 3-33.

[6] Mocanu, P.T., *Second-order Averaging Operators for Analytic Functions*, Rev. Roumaine Math. Pures Appl., 33(1988), 10, 875-881.


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# SUR LES OSCILLATIONS ENTRETENUES PAAR UNE FORCE PRESQUE PÉRIODIQUE DANS LE SENS DE BOHR

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**Abstract.** Dans cet article nous allons présenter le phénomène physique des oscillations entretenues par une fonction prériodique et nous allons insister sur une modélation plus fidèle des oscillations par les fonctions presque périodiques dans le sens de H. Bohr [3,6]. Ensuite, nous allons pérturber la force extérieure jusqu'àune force presque périodique et nous étudierons sur un cas particulier, àl'aide d'un algorithme de Fox Goodwin, de divers évaluations numériques de la solution de l'équation d'état [2,4].

## 1. Oscillations entretenues

Souvent, dans l'étude des oscillations harmoniques, on ignore l'influence de la force de frottement, mais si l'on tient compte de l'influence du milieu, l'importance de la fonction de frottement augumente en déterminant toutefois un amortissement du mouvement oscillatoire. Le sens de cette force s'oppose au sens de la vitesse du point materiel et, en plusieurs cas, elle est proportionelle à celle-ci: $F_f = -\gamma v$, où $\gamma$ est un réel positif nommé coefficent d'amortissement.

Dans plusieurs problèmes pratiques nous sentons le besoin d'anéantir l'influence de la force de frottement pour obtenir des variations controlables des oscillations, nous désirons donc d'entretenir les oscillations. Pour réaliser cette chose nous devons imprimer à l'oscillateur une énergie de l'extérieur [2], [4], [7].

**2.** *Le modèle physique*

**2.1.** *La première approximation*

On suppose que notre système est soumis à l'action des trois forces: la force élastique $F_e = -kx$, où $k$ est la constante d'élasticité, la force de frottement $F_f = \gamma v$

et une force parallèle à ox, dont l'intensité est une fonction périodique de temps

$$F : [0, +\infty[\to R, \quad \text{décrite de} \quad F(t) = F_0 \sin \omega t, \quad F_0, \omega \in R.$$

L'équation de mouvement prend la forme suivante:

$$mx'' + \gamma x' + kx = F_0 \sin \omega t$$

où bien,

$$x'' + 2\delta x' + \omega^2 x = \frac{F_0}{m} \sin \omega t, \tag{1}$$

où $\delta = \delta(\gamma) > 0$, m>0 et $\omega_0$, $\omega \in$ R sont les pulsations du mode de travail et de la force d'entretien. Si on regarde la relation (1) on observe facillement qu'une solution de l'équation linéaire homogène est

$$x_1(t) = A_0 \exp(-\delta t) \sin(\sqrt{\omega_0^2 - \delta^2 t} + \varphi_0), \quad \text{où} \quad A, \varphi_0 \in R$$

et une solution particulière pour la même équation (1) a la forme:

$$x_2(t) = A \sin(\omega t + \varphi), \quad \text{où } A \text{ et } \varphi \text{ sont des réels inconnus.}$$

Le mouvement décrit par la solution (2) de l'équation (1) est en régime stationaire si les oscillations du système ont lieu à une fréquence égale à celle de la force d'entretien est qui est bien differente de la fréquence propre.

Les réels A et $\varphi$ on les déterminent en remplaçant la rélation (2) dans l'équation (1) de la façon suivante:

$$-A\omega^2 \sin(\omega t + \varphi) + 2\delta\omega A \cos(\omega t + \varphi) + \omega_0^2 A \sin(\omega t + \varphi) = \frac{F_0}{m} \sin(\omega t) \tag{2}$$

En développant sin($\omega$t+$\varphi$) et cos($\omega$t+$\varphi$) nous obtenons, par une simple identification des coefficents, le système:

$$\begin{cases} A(\omega_0^2 - \omega^2) \sin \varphi + 2\delta\omega A \cos \varphi = 0 \\ A(\omega_0^2 - \omega^2) \cos \varphi + 2\delta\omega A \sin \varphi = \frac{F_0}{m} \end{cases} \tag{3}$$

La résolution du système (4) nous montre que:

$$A = \frac{F_0}{m\sqrt{(\omega_0^2 - \omega^2) + 4\delta^2}} \text{ et } \mathrm{tg}\varphi = \frac{2\delta\omega}{\omega_0^2 - \omega^2}, \text{ quand } \omega^2 \neq \omega_0^2.$$

En général, les oscillations ne sont pas en phase avec la force de l'entretien, la valeur de l'amplitude A este une fonction de la pulsation $\omega$ de la force d'entretien, l'amplitude éxtrème $A_{max}$ étant obtenue quand

$$\frac{dA}{d\omega}(\omega) = 0 \text{ et } \frac{d^2A}{d\omega^2}(\omega) < 0.$$

De cette manière nous avons obtenu maintenant une pulsation $\omega_r = \sqrt{\omega_0^2 - 2\delta^2} > 0$ pour laquelle l'amplitude du mouvement prend la valeur maxime. Dans ce cas $\omega_r$ s'appelle pulsation de résonance.

$$A_{max} = A(\omega_r) = \frac{F_0}{2m\delta\omega_r}.$$

2.2. *Les fonctions presque périodiques dans le sens de Bohr*

Un sousensemble $S \subseteq R$ est relatif dense s'il existe un numeró positif l, de manière que $[a,a+l] \cap S \neq \emptyset$ pour tous les $a \in R$.

Soit f: $R \rightarrow R$ une fonction bornée, et soit $\varepsilon > 0$ fixé arbitrairement. Nous définissons l'ensemble

$$T(f,\varepsilon) = \{\tau \in R : |f(t+\tau) - f(t)| < \varepsilon \text{ pour tous les } t \in R\}.$$

Une fonction f: $R \rightarrow R$ est presque périodique dans le sens de Bohr si pour tout $\varepsilon > 0$, l'ensemble $T(f,\varepsilon)$ est R-relatif dense.
On note cela par $f \in AP(R)$.

Intuitivement, on remarque que les fonctions presque périodiques ne s'éloignent trop des fonctions périodiques.

**2.2.1.** *Théorème.* Soit f: $R \rightarrow R$ une fonction. Les suivantes proprietés ont toujours lieu:

a) Si $f \in AP(R)$, alors f est uniformément continue sur R;

b) Si $f \in AP(R)$, $g \in AP(R)$, alors nous avons $f+g \in AP(R)$ et $f \cdot g \in AP(R)$;

c) Si $f \in AP(R)$ et si f' est définie sur la doite réele, alors nous avons $f' \in AP(R)$ si et seulement si f' est uniformèment continue.

La démonstration de cette théorème ainsi que la théorie de ces fonctions, se trouvent dans le travail [3].

**2.3.** *La deuxième approximation*

Nous nous trouvons maintenant dans la même situation d'un système oscillatoire harmonique mais cette fois dans un milieu à frottement. Cette fois-ci la force éxterieure ne sera plus d'une intensité périodique de temps, elle sera d'une intensité presque périodique dans le sens de Bohr, phénomène beaucoup plus approché de la réalité comparé à *la stricte périodicité pratiquement inéxistente.*

Soit donc la fonction F: $[0,+\infty]\to R$ définie par $F(t) = F_0(t)\cos(\omega t)$, où $\omega \in R$ et $F_0 \in AP(R)$.

Avec le support de la théorème 2.2.1. on déduit que $F \in AP(R)$, l'équation (1) devenant cette fois

$$x'' + 2\delta x' + \omega_0^2 x = F(t), \text{ pour } x \in R^R \text{ et } t \in [0, +\infty[.$$

Nous comptons sur l'aide des moyens de l'analyse numérique pour la résolution de cette dernière équation.

3. *La méthode de Fox Goodwin*

Fox Goodwin est une méthode spécialisée dans la résolution des équations differentielles linéaires de la forme suivante:

$$y'' + a(x)y' + b(x)y + c(x) = 0, \text{ où } x \in [a, b]$$

Pour formuler le problème de Cauchy, nous formulons les conditions initiales

$$y(a) = y_0,\ y'(a) = y'_0.$$

L' algorithme va calculer les valeurs de y en differents noeuds. Nous nous fixerons sur les n+1 noeuds construits de la manière:

$$x_k = a + k\frac{b-a}{n} = a + kh, \text{ quand } k \in \{0, \ldots, n\} \text{ et } h = \frac{b-a}{n},\ n > 0.$$

Pour un k fixé, on note $\bar{y}_k, a_k, b_k, c_k$ les valeurs de $y, a, b$ et $c$, calculés dans le noeud $x_k$.

L'algorithme est:

$$\bar{y}_{k+2} = \left(1 + \frac{h}{2}a_{k+1}\right)^{-1}\left[-\left(1 + \frac{h}{2}a_{k+1}\right)\bar{y}_k + (2 - h^2 b_{k+1})\bar{y}_{k+1} - h^2 c_{k+1}\right],$$

dont l'erreur de la méthode est:

$$\eta_{k+2} \cong \frac{1}{12} h^4 y^{(4)}(x_{k+1}), \text{ où } k \in \{1, \ldots, n\}.$$

On observe que pour le bon fonctionnement de l'algorithme nous avons besoin de deux valeurs de départ $\bar{y}_0$ et $\bar{y}_1$ . La valeur $\bar{y}_0 = y(a) = y_0$ nous l'avons et la valeur $\bar{y}_1$ se calcule en dévéloppant en série de Taylor [5].

4. *Un exemple numèrique*

Nous nous fixons maintenant sur le suivant problème de Cauchy:

$$\begin{cases} x'' + 0.5x' + 10x = (\cos 314.16t + \cos 17.72t) \cos 314.16t \\ x(0) = 0 \\ x'(0) = 0 \end{cases}, \text{ où } t \in [0, 10]. \quad (4)$$

C'est bien claire que la fonction $F_0(t) = \cos 314.16t + \cos 17.72t$ est presque périodique dans le sens de H. Bohr et $F_0$ n'est pas périodique, chose qui souligne une fois de plus le fait que l'éspace de fonctions périodiques n'est pas un éspace dans le sens de Banach. Le théorème 2.2.1.b) justifie que $F_0 \in AP(R)$. Pour montrer que $F_0$ n'est pas périodique nous reprenons la formule générale

$$F_0(t) = \cos \omega t + \cos \sqrt{\omega} t, \text{ où } t \in [0, +\infty[.$$

Supposons que $F_0$ soit périodique, donc il existe $T > 0$ avec la proprieté

$$F_0(t + T) = F_0(t), \text{ pour } t \geq 0. \quad (5)$$

En posons t=0 dans la relation (6) nous obtenons $\cos \omega T + \cos \sqrt{2} \omega T = 2$, et nous avons

$$\begin{cases} \cos \omega T = 1 \Rightarrow T = \frac{2n\pi}{\omega}, n \in N \\ \cos \sqrt{2} \omega T = 1 \Rightarrow \sqrt{2} T = \frac{2m\pi}{\omega}, m \in N \end{cases}$$

d'oùon remarque que $\sqrt{2} = \frac{m}{n}$ ce qui est absurde. Donc notre supposition est fausse. Il résulte que $F_0(t) \cos \omega t \in AP(R)$ et aussi la fonction

$$F(t) = (\cos 314.16t + \cos 17.72t) cos 314.16t \in AP(R), \text{ pour } t \in [0, 3].$$

Ce résultat peut etre facilement généralisé en prenant a la place de F une somme finie ou infinie des fonctions presque périodiques dans le sens de Bohr, en tenant compte de la structure d'éspace vectoriel topologique de AP(R) .

En appliquant la méthode de Fox Goodwin nous obtenons les suivantes evaluations pour la solution presque périodique du problème de Cauchy (5). On a:

| Tableau de variation de l'oscillation $x$ | | | | | |
|---|---|---|---|---|---|
| *nr. crt.* | $t$ | $x(t)$ | *nr. crt.* | $t$ | $x(t)$ |
| 0 | 0.0 | 0 | 15 | 1.5 | 1.72321463 |
| 1 | 0.1 | -1.22521782 | 16 | 1.6 | 1.19096839 |
| 2 | 0.2 | -2.22468138 | 17 | 1.7 | 0.55492908 |
| 3 | 0.3 | -3.88867235 | 18 | 1.8 | -0.11387726 |
| 4 | 0.4 | -3.17627549 | 19 | 1.9 | -0.739344 |
| 5 | 0.5 | -3.08133364 | 20 | 2 | -1.23563397 |
| 6 | 0.6 | -2.62782693 | 21 | 2.1 | -1.55284929 |
| 7 | 0.7 | -1.90001404 | 22 | 2.2 | -1.68048322 |
| 8 | 0.8 | -1.01691794 | 23 | 2.3 | -1.61115265 |
| 9 | 0.9 | -0.08499254 | 24 | 2.4 | -1.35509455 |
| 10 | 1.0 | 0.79002186 | 25 | 2.5 | -0.96647346 |
| 11 | 1.1 | 1.49523771 | 26 | 2.6 | -0.50920916 |
| 12 | 1.2 | 1.95475125 | 27 | 2.7 | -0.0287871 |
| 13 | 1.3 | 2.14632511 | 28 | 2.8 | 0.41838172 |
| 14 | 1.4 | 2.06395769 | 29 | 2.9 | 0.76669407 |

Nous remarquons dans le tableau antérieur , que l'oscillation x(t) est la petite pérturbation d'une fonction périodique, mais qui est entretenue tout de même dans un interval temporel bien détérminé.

Les valeurs de ce tableau ont été obtenues à l'aide d'un programme en C++ dont l'erreur de méthode est celle précisée auparavent . Le programme va aussi pour d'autres fonctions F(t). Pour le même probleme on peut utiliser aussi la méthode d'Adams-Störmer on obtenant résultats semblables avec une erreur peu differente de la notre.

On regardant les donées obtenues on remarque un aplatissement de l'oscillation, à cause de la force de friction [5]. Cet aplatissement est assez fort dans des milieux fluides compressibles et il est fort brusque dans des millieux fluides incompressibles. Le modele mathématique pour l'entretien des oscillations presque périodiques est identique a celui qui étudie les phénomenes de résonance et de battements dans les circuits RLC, plus exactement, l'étude de regime tranzitoire se fait d'habitude avec un modele mathématique qui introduit les distributions presque périodiques et aussi une transformée de Fourier tout a fait spéciale. Mais tout cela sera realisé dans un prochain travail.

## Bibliographie


[1] Bogolioubov, N., Mitropolski, Y.: *Les méthodes asymptotiques en théorie des oscillations non linéaires*, Ed. Gauthier- Vilars, Paris, 1962

[2] Ciubotariu, C., Păduraru, A.: *Fizică Generală*, Editura Didactică şi Pedagogică, Bucureşti, 1981

[3] Fink, A., M.: *Almost Periodic Differential Equations,* Springer Verlag, Berlin-Heidelberg-New York, 1974

[4] Iacob, C.: *Mecanică Teoretică,* Editura Didactică şi Pedagogică, Bucureşti, 1980

[5] Ixaru, L.,G.: *Metode numerice pentru ecuaţii diferenţiale cu aplicaţii,* Editura Academiei, Bucureşti, 1979

[6] Muntean, I.: *Capitole speciale de analiză funcţională*, lito. Univ. Babeş-Bolyai, Cluj-Napoca, 1993

[7] Smith, P., Smith, R.C.: *Mechanics*, John-Wiley & Sons Editions, Chichester, New-York, Brisbane, Toronto, Singapore, 1990.



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# A CHEBYSHEV SYSTEM APPROACH TO THE BOUNDARY BEHAVIOUR OF THE SUBLINEAR FUNCTIONS

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**Abstract.** The aim of this note is to show that the problem of the augmentation of a function system consisting of the coordinate functions of a parametrization of a convex surface in $R^{n-1}$ with 0 in its interior, to a Chebyshev system of order $n-3$ [9] has its natural interpretation in the contex of the boundary behaviour of a strictly sublinear or a strictly superlinear function.

A strictly convex or strictly concave real function can be defined by the condition that its graph intersects every straight line in at most two distinct points. In this definition we have to do in fact with the two dimensional subspace of the affine functions which augmented by the strictly convex (or strictly concave) function in question to a three dimensional space, becomes a space having the property that each nonzero element of its vanishes in at most two points. A possible generalization of the convexity notion introduced this way is the following: Consider an arbitrary two dimensional subspace $P_2$ of the space $C(Q)$ of continuous real functions defined on the connected Hausdorff space $Q$. Then $f \in C(Q)$ is convex with respect to $P_2$ if every member of $P_2$ can agree with $f$ in at most two distinct points. Suprisingly this generalization goes not far from the real function case: the notion is consistent for the case of $Q$ compact if and only if this is homeomorphic with a (compact, connected) subset of the circle $S^1$ [5].

Starting with the above generalized convexity notion (a two dimensional underlying vector space and a convex function with respect to it) and trying to get a natural extension, we can follow two lines. To consider for instance an $n-1$ dimensional subspace $P_{n-1}$ in $C(Q)$ and to call $f \in C(Q)$ convex with respect to it, if this $f$ agrees with each member of $P_{n-1}$ in at most $n-1$ distinct points. Again, this generalization is consistent for connected compact $Q$ if and only if $Q$ can be imbedded into $S^1$ and the imbedding can be surjective only when $n$ is odd [5].

A second way is the following: to consider an $n-1$ dimensional subspace $P_{n-1}$ of $C(Q)$ ($n \geq 3$) and to consider $f \in C(Q)$ convex with respect to $P_{n-1}$ if it agrees with $n-2$ linearly independent elements of $P_{n-1}$ in at most two distinct common points.

In the case $n = 3$ the two convexity notions coincide.

The first generalization has an old history. It goes back to Popoviciu (see [8] and [11]) and is used in the constructive function theory (see also [3] and [4]). The second one is not explored explicitely, but it corresponds to a natural geometrical picture (this is emphasised also by the content of our note). We note that this second generalization can be consistent also for rather strong topological conditions on $Q$. This follows from some results in topological setting in [6].

Both the above two generalized convexities can be interpreted as augmentation of a given system of functions by a function to another system with prescribed properties.

The aim of this note is to show that the problem of the augmentation of a function system consisting of the coordinate functions of a parametrization of a convex surface in $R^{n-1}$ with 0 in its interior, to a Chebyshev system of order $n-3$ [9] has its natural interpretation in the contex of the boundary behaviour of a strictly sublinear or a strictly superlinear function. Our geometric approach as well as the method used in proofs are prolific in both the convex analysis and the theory of Chebyshev systems. They emphasise the strong relation existing between these two fields.

## 1. Parametrized convex surfaces in $R^{n-1}$

We say that $S^{n-2}$ is the *standard* $n-2$ *sphere* if it is the subspace of the Euclidean space $R^{n-1}$ consisting of the set of points with the distance 1 from the origin of a Cartesian system in $R^{n-1}$. We say that the set $C$ in $R^{n-1}$ is a *topological* $n-2$ *sphere* or a *closed surface* if it is the homeomorphic image of $S^{n-2}$. Denote by

*$\phi$ a homeomorphism from $S^{n-2}$ to $C$. Then $\phi$ will be called a parametrization of $C$.* If $\phi = (\varphi_1, \ldots, \varphi_{n-1})$, then $\varphi_j$, $j = 1, \ldots, n-1$ will be called the *coordinate functions of a parametrization of the surface $C$.*

We are particularly interested in the case when the topological $n-2$ sphere $C$ in $R^{n-1}$ is a convex (or a strictly convex) surface in the sense that it is the boundary of a convex (or respectively, of a strictly convex) body in $R^{n-1}$. A body $B$ in $R^{n-1}$ is a closed, connected and bounded set with non empty interior. The body $B$ is convex if and only if every straight line containing an interior point of its, meets its boundary $C$ in exactly two points. This follows from basic properties of convex sets (see e.g. [10]). Therefore a straight line can meet a convex surface $C$ in a set having at most two connected components. If the convex surface $C$ contains no line segment, then it is called a strictly convex surface and the set $B$ it bounds, a strictly convex body. Thus the closed surface $C$ is strictly convex if and only if any straight line in $R^{n-1}$ can have an intersection with $C$ consisting of at most two points.

A straight line in $R^{n-1}$ is the intersection of $n-2$ hyperplanes, i.e., it is the locus of the points $x = (x^1, \ldots, x^{n-1}) \in R^{n-1}$ satisfying a system of the form

$$c_0^j + c_1^j x^1 + \cdots + c_{n-1}^j x^{n-1} = 0, \quad j = 1, \ldots, n-2 \tag{1}$$

with

$$(c_1^j, \ldots, c_{n-1}^j), \quad j = 1, \ldots, n-2 \tag{2}$$

linearly independent vectors (the normal vectors of the mentioned hyperplanes).

According to our above observations the surface $C = \phi(S^{n-2})$ with the parametrization $\phi = (\varphi_1, \ldots, \varphi_{n-1})$ is convex (respectively, it is strictly convex) if and only if the equations

$$c_0^j + c_1^j \varphi_1(q) + \cdots + c_{n-1}^j \varphi_{n-1}(q) = 0, \quad j = 1, \ldots, n-2 \tag{3}$$

possess a set of solutions $q \in S^{n-2}$ having at most two connected components (possess at most two distinct solutions $q \in S^{n-2}$) for every set (2) of $n-2$ linearly independent vectors.

Let us consider now instead of the vectors (2) the vectors of the form

$$c_j = (c_0^j, c_1^j, \ldots, c_{n-1}^j), \quad j = 1, \ldots, n-2. \tag{4}$$

If the vectors $c_1, \ldots, c_{n-2}$ were linearly independent but the vectors (2) were not, then the system (1) were incompatible and the equations (3) could not have any solution $q \in S^{n-2}$.

By gathering the above observations we arive to the following statement:

**1.1.** *The surface $C$ in $R^{n-1}$ with the parametrization $\phi = (\varphi_1, \ldots, \varphi_{n-1})$ is convex (respectively, it is strictly convex) if and only if for each set (4) of $n-2$ linearly independent vectors the system (3) can have a set of solutions $q \in S^{n-2}$ consisting of at most two connected components (respectively, this set of solutions can have at most two distinct points).*

## 2. Chebyshev systems

Denote by $C(Q)$ the vector space of the real valued continuous functions defined on the connected topological space $Q$. The set $\{\varphi_0, \varphi_1, \ldots, \varphi_{n-1}\} \subset C(Q)$ is called an $(n, k)$ *system* (or a Chebyshev system of order $k-1$ [9]), if it is linearly independent and any $k$ linearly independent elements in $sp\{\varphi_0, \varphi_1, \ldots, \varphi_{n-1}\}$ possess at most $n-k$ common zeros in $Q$. An $(n, 1)$ system is a so called Chebyshev or Haar system ([3], [4]). By a *weak* $(n, k)$ *system* we mean a set of functions of the above form relaxing the last requirement in the above definition to the following one: any $k$ linearly independent elements in $sp\{\varphi_0, \varphi_1, \ldots, \varphi_{n-1}\}$ can have a set of common solutions having at most $n-k$ connected components. A weak $(n, 1)$ system is called a weak Chebyshev system. Weak Chebyshev systems have been defined in [2] for the case $Q$ an interval in $R$ by an oscillation condition. For this particular case the notion agrees with ours (other equivalent conditions were considered in [1]).

We are especially interested in the case when $k = n-2$. An $(n, n-2)$ system (a weak $(n, n-2)$ system) is for $n = 3$ a Chebyshev system (respectively a weak Chebyshev system).

We have the following relation of $(n, n-2)$ systems with the surface parametrizations:

**2.1.** *Let $C$ be a topological $n-2$ sphere in $R^{n-1}$ wuth the parametrization $\phi = (\varphi_1, \ldots, \varphi_{n-1})$. Then $C$ is a convex surface (respectively, it is a strictly convex surface) if and only if the set of functions $\{1, \varphi_1, \ldots, \varphi_{n-1}\}$, where 1 is the constant 1 function on $S^{n-2}$, is a weak $(n, n-2)$ system (respectively, it is an $(n, n-2)$ system).*

To verify this statement we note that by 1.1 $C$ is a convex surface (respectively, it is a strictly convex surface) if and only if for every set of $n-2$ linearly independent vectors (4) the system of equations (3) can have a set of solutions $q$ in $S^{n-2}$ with at most two connected components (respectively, this set consists of at most two points). This is nothing but the requirement that any $n-2$ linearly independent elements in $sp\{1, \varphi_1, \ldots, \varphi_{n-1}\}$ have a set of common zeros possessing at most two connected components (respectively, this set consists of at most two points). That is, the convexity of $C$ (the strict convexity of $C$) is equivalent with the fact that $\{1, \varphi_1, \ldots, \varphi_{n-1}\}$ is a weak $(n, n-2)$ system (respectively, it is an $(n, n-2)$ system).

## 3. Wedges and cones

A non empty subset $W$ in $R^n$ is called a *wedge* if $W + W \subset W$ and if $tW \subset W$ for each non negative real number $t$. A wedge is obviously a convex set which contains the vector 0. The wedge $K$ is called a *cone* if $K \cap (-K) = \{0\}$. Thus the wedge $K$ is a cone if and only if from $u$, $-u \in K$ it follows that $u = 0$.

The subset $F$ of the wedge $W$ is called a *face* of $W$, if it is a wedge and if the conditions $u \in F$, $v \in W$ and $u - v \in W$ imply that $v \in F$.

Any wedge contained in a cone is itself a cone, hence the face of a cone is a cone.

The face $F$ of the wedge $W$ is called *proper face* if $\{0\} \neq F \neq W$.

We gather next some results which we shall use later. Most of them are easy consequences of the definitions or are standard results of the theory of convex sets (see e.g. [10]).

**3.1.** If $W$ is a wedge and $intW \neq \emptyset$, then $W + intW \subset intW$.

**3.2.** No proper face of a wedge $W$ can contain points of $intW$.

**3.3.** If a subspace $L$ of dimension 2 of $R^n$ contains three affinely independent points of the boundary $\partial W$ of the wedge $W$, then $L \cap intW = \emptyset$.

**3.4.** If $W$ is a wedge with $intW \neq \emptyset$ and if $L$ is a subspace of $R^n$ with $L \cap intW = \emptyset$, then there exists an $n-1$ dimensional subspace $H$ of $R^n$ with $L \subset H$ and $H \cap intW = \emptyset$.

**3.5.** If $W$ is a wedge in $R^n$ with $intW \neq \emptyset$ and if $H$ is a hyperplane through 0 in $R^n$ with $H \cap intW = \emptyset$, then $F = H \cap W$ is a face of $W$. If $F \neq 0$, it is a proper face.

The closed cone $K$ in $R$ is called *strictly convex cone* if it possesses only one dimensional proper faces. The condition $\dim K \geq 2$ is here intrinsic.

**3.6.** The intersection of two wedges is a wedge. The intersection of two strictly convex cones is a strictly convex cone if the dimension of the intersection is $\geq 2$.

## 4. Sublinear and superlinear functions

Consider the function $f : R^{n-1} \to R$ $(n \geq 2)$. The graph, epigraph and hypograph of $f$ are the sets

$$grf = \{(x,t) \in R^{n-1} \times R : f(x) = t\},$$

$$epif = \{(x,t) \in R^{n-1} \times R : f(x) \leq t\},$$

$$hypof = \{(x,t) \in R^{n-1} \times R : f(x) \geq t\}$$

respectively. If $f$ is continuous then these sets are closed and $int(epif) \neq \emptyset$, $int(hypof) \neq \emptyset$.

The function $f : R^{n-1} \to R$ is called *positively homogeneous* if $f(tx) = tf(x)$ for each $x \in R^{n-1}$ and each $t \in R^{n-1}$ and each $t \in R_+ = [0, +\infty)$. The function $f$ is called *subadditive (superadditive)* if $f(x+y) \leq f(x) + f(y)$ $(f(x+y) \geq f(x) + f(y))$ for any $x, y \in R^{n-1}$. If $f$ is both positively homogeneous and subadditive (positively homogeneous and superadditive) then it is called *sublinear* (respectively, *superlinear*). The function $f$ is superlinear if and only if $-f$ is sublinear.

The sublinear (superlinear) function $f : R^{n-1} \to R$ is called *strictly sublinear (strictly superlinear)* if the equality $f(x+y) = f(x) + f(y)$ for non zero $x$ and $y$ implies that $x$ and $y$ are positive multiple of each other.

The property of a function of being sublinear, superlinear, strictly sublinear or strictly superlinear can be expressed geometrically using the notions of wedges and cones:

**4.1.** *The positively homogeneous continuous function $f : R^{n-1} \to R$ is*

*(a) sublinear (superlinear) if and only if epif (hypof) is a wedge;*

*(b) strictly sublinear (strictly superlinear) if and only if epig (hypof) is a strictly convex one.*

We prove the statement (b) for the sublinear case. The other cases can be similarly handled.

Suppose that $f$ is strictly sublinear and denote $K = epif$. If $(x,s) \in K$ (that is, if $f(x) \leq s$) and $t \in R_+$ then $tf(x) \leq ts$ and by the positive homogeneity of $f$ we get $f(tx)ts$, that is, $(tx, ts) = t(x,s) \in K$ which shows that $tK \subset K$, $\forall\, t \in R_+$.

Let be $(x,s), (y,t) \in K$. Then $f(x+y) \leq f(x) + f(y) \leq s + t$ and hence $(x+y, s+t) \in epif = K$. That is, $K + K \subset K$.

We have proved that $K$ is a wedge. Suppose that $F$ is a proper face of $K$. Then $F \subset \partial K$ since by 3.2, $F \cap intK = \emptyset$. By the continuity of $f$, $grf = \partial K$. Hence $F \subset grf$. Suppose now that $(x,s), (y,t) \in F$. Then $(x+y, s+t) \in F$ since $F$ is a wedge. But then $F(x+y) = s+t = f(x)+f(y)$. According to the strict sublinearity of $f$, if $x$ and $y$ are non zero vectors, it follows that $y = rx$ for some $r > 0$. But then $f(y) = f(rx) = rf(x) = rs$. That is, $(y,t) = r(x,s)$. In conclusion, $\dim F = 1$.

If $(x,s), -(x,s) \in K$, then $f(x) \leq s$ and $f(-x) \leq -s$. By the sublinearity of $f$ we have $-f(x) \leq f(-x)$. These relations give $f(x) = s$ and $f(-x) = -s$. Thus $0 = f(x-x) = f(x) + f(-x)$. From the strict sublinearity of $f$ it follows then that $x = 0$, Hence $s = 0$ and we conclude that $K$ is a cone.

Suppose now that $f$ is positively homogeneous and $K = epif$ is a strictly convex cone. Assume that there exist some linearly independent vectors $x, y \in R^{n-1}$ such that $f(x+y) = f(x) + f(y)$. Put $s = f(x)$, $t = f(y)$ and consider the space $L$ in

$R^{n-1} \times R$ engendered by the vectors $(x, s), (y, t)$. From the definition of $K$ we have $intK \neq \emptyset$. The vectors $(x, s), (y, t)$ and $(x + y, s + t)$ are affinely independent and are contained in $grf = \partial K$. Hence $L \cap intK = \emptyset$ by 3.3. According 3.4 there exists a hyperplane $H$ with $L \subset H$ and $H \cap intK = \emptyset$. Then $F = K \cap H$ is a face of $K$ by 3.5. But $\dim K \geq 2$ and we get a contradiction with the hypothesis of strict convexity of $K$. Thus $f$ must be strictly sublinear.

## 5. Traces of sublinear and superlinear functions on convex surface

Let $f : R^{n-1} \to R$ $(n \geq 3)$ be a sublinear function and let $D \subset R^{n-1}$ be a convex body with $0 \in intD$. Denote $C = \partial D$. We shall in this case say that $C$ is a convex surface with 0 in its interior. It is standard question in the global optimization to search the maximum of $f$ on $D$. Obviously, it suffices to get its maximum on the boundary $C$ of $D$. It is also immediate from the position of $C$ that the values of $f$ on $C$ determine this function. This motivates the investigation of the sublinear function $f$ on convex surfaces like $C$.

Let $\phi : S^{n-2} \to C$, $\phi = (\varphi_1, \dots, \varphi_{n-1})$ be a parametrization of the closed surface $C$. We can describe the behaviour of a function $g : R^{n-1} \to R$ on $C$ by the function $g \circ \phi$ called the *trace of* $g$ *on* $C$.

The geometrical approach we have outlined enables us to answer (using the notions introduced in section 2) the question whether or not a real function $\varphi : S^{n-2} \to R$ can be the trace on a convex surface with 0 in its interior of a sublinear (or strictly sublinear) function. We have in this contex the following result:

**5.1.** *Let $C$ be a convex surface in $R^{n-1}$ with 0 in its interior having the parametrization $\phi : S^{n-2} \to C$, $\phi = (\varphi_1, \dots, \varphi_{n-1})$. Then the continuous function*

$$\varphi : S^{n-2} \to R$$

*(a) is the trace on $C$ of a continuous sublinear or superlinear function if and only if the set of functions*

$$\{\varphi, \varphi_1, \dots, \varphi_{n-1}\} \tag{5}$$

*is a weak $(n, n-2)$ system;*

*(b) is the trace on $C$ of a continuous strictly sublinear or strictly superlinear function if and only if (5) is an $(n, n-2)$ system.*

We shall prove the assertion (b). The proof of (a) is similar.

Suppose that $f$ is strictly sublinear and $\varphi = f \circ \phi$. We have to show that (5) is an $(n, n-2)$ system. To this end, let us take $n-2$ linearly independent elements in $sp\{\varphi, \varphi_1, \ldots, \varphi_{n-1}\}$:

$$c_0^j \varphi + c_1^j \varphi_1 + \cdots + c_{n-1}^j \varphi_{n-1}, \quad j = 1, \ldots, n-2. \tag{6}$$

The vectors $c_j = (c_0^j, c_1^j, \ldots, c_{n-1}^j)$, $j = 1, \ldots, n-2$ are linearly independent. Hence the set of solutions of the system

$$c_1^j u^1 + \cdots + c_{n-1}^j u^{n-1} + c_0^j u^n = 0, \quad j = 1, \ldots, n-2 \tag{7}$$

(in $u = (u^1, \ldots, u^n)$) is a two dimensional subspace $L$ of $R^n$. Let us identify the domain of $f$ with the subspace $u^n = 0$ in $R^n$. Then $epi f$ is by 4.1(b) a strictly convex cone $K$ with $gr f$ being its boundary. There exist three possibilities: $L \cap \partial K = \{0\}$, $L \cap \partial K$ consists of a ray from 0 on $gr f$ or $L \cap \partial K$ consists of two distinct rays on $gr f$ from 0.

Consider the surface $C_1 = \Psi(S^{n-2}) \subset R^n$ having the parametrization $\Psi = (\varphi_1, \ldots, \varphi_{n-1}, \varphi)$. Then geometrically $C_1$ is the intersection of $gr f$ with the cylinder with the generator parallel with the axis $Ou^n$ and the base $C$ in $R^{n-1}$. From the configuration of $C$ each ray from 0 on $gr f$ intersects $C_1$ once. Hence the plane $L$ of dimension 2 consisting of the set of solutions of the system (7) has an intersection with $C_1$ which is the empty set if $L \cap \partial K = \{0\}$, it contains a single point if $L \cap \partial K$ is a single ray and it consists of two points if $L \cap \partial K$ consists of two rays. The intersections of $C_1$ with $L$ are given by the solutions in $q \in S^{n-2}$ of the system

$$c_0^j \varphi(q) + c_1^j \varphi_1(q) + \cdots + c_{n-1}^j \varphi_{n-1}(q) = 0, \quad j = 1, \ldots, n-2. \tag{8}$$

Since $\Psi = (\varphi_1, \ldots, \varphi_{n-1}, \varphi)$ is one to one, to each intersection point of $L$ with $C_1$ corresponds exactly one solution $q \in S^{n-2}$ of the system (8). By the above observations on the intersection of $L$ with $C_1$ we conclude that (8) can have at most two distinct solutions $q \in S^{n-2}$ which is nothing but the condition for (5) to form an $(n, n-2)$ system.

Conversely, let us suppose that $\varphi \in C(S^{n-2})$ is a function with the property that (5) is an $(n, n-2)$ system. Consider $C_1$ as being the set $\Psi(S^{n-2})$ with $\Psi = (\varphi_1, \ldots, \varphi_{n-1}, \varphi)$. Then $\Psi$ is a parametrization of the surface $C_1$. the set $C = \phi(S^{n-2}) \subset R^{n-1}$ with $R^{n-1}$ the subspace of vectors in $R^n$ with the last component 0, is a closed convex surface containing 0 in its interior. Hence every ray from 0 in $R^{n-1}$ intersects $C$ in exactly one point.

Let $\psi$ be the Minkowski functional with respect to 0 of $D = coC$. We define the function $f : R^{n-1} \to R$ by putting

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \psi(x)\varphi(\phi^{-1}(x/\psi(x))) & \text{if } x \neq 0 \end{cases}.$$

The trace of $f$ on $C$ is $\varphi$. Indeed, if $x \in C$, then $\psi(x) = 1$ and $x/\psi(x) = x$. Hence $f(x) = \varphi(\phi^{-1}(x))$ and denoting $q = \phi^{-1}(x)$, we have that

$$(f \circ \phi)(q) = \varphi(q).$$

The function $f$ is positively homogeneous since $\psi$ is so. Hence $grf$ is engendered by a moving ray with center 0, running on $C_1$.

The function $f$ is strictly sublinear or strictly superlinear. If none, then no $epif$, no $hypof$ can be a strictly convex cone by 4.1. This means that there exists a straight line $d$ in $R^n$, not passing through 0, which meets $grf$, the boundary of $epif$ (and of $hypof$) in at least three distinct points $u_1, u_2, u_3$.

Consider the two dimensional plane $L$ through 0 engendered by $d$. This plane can be represented as the set of solutions of a system of the form (7) with the vectors $(c_1^j, \ldots, c_{n-1}^j)$, $j = 1, \ldots, n-2$ being linearly independent.

The plane $L$ will meet $C$ in the points $u_1, u_2, u_3$. Let be $q_j = \Psi^{-1}(u_j)$, $j = 1, 2, 3$. Then $q_1, q_2, q_3$ will be distinct solutions of the system (8), where the vectors

$$c_j = (c_0^j, c_1^j, \ldots, c_{n-1}^j), \quad j = 1, \ldots, n-2$$

are linearly independent. This means that (5) cannot be an $(n, n-2)$ system, contradiction which completes the proof.

## 6. Augmentation of a parametrization to $(n, n-2)$ systems

Let the set of functions

$$\{\varphi_1, \ldots, \varphi_{n-1}\} \subset C(S^{n-2}) \tag{9}$$

be the coordinate function of a parametrization $\phi$ of a closed surface $C$ in $R^{n-1}$. We say that a continuous function $\varphi : S^{n-2} \to R$ is an *augmentation to a weak* $(n, n-2)$ *system (to an* $(n, n-2)$ *system)* of (9) if

$$\{\varphi, \varphi_1, \ldots, \varphi_{n-1}\} \tag{10}$$

is a weak $(n, n-2)$ system (is an $(n, n-2)$ system).

We have seen (section 2) that if (9) are the coordinate functions of a parametrization of a convex (strictly convex) surface in $R^{n-1}$, then the function $\varphi = 1$ is an augmentation to a weak $(n, n-2)$ system (to an $(n, n-2)$ system) of (9). Conversely, if any constant nonzero function augmentation (9) to a weak $(n, n-2)$ system (to an $(n, n-2)$ system), then the functions (9) must be the coordinate functions of the parametrization of a closed convex (a closed strictly convex) surface.

We are prepared to consider the augmentation to a weak $(n, n-2)$ system (to an $(n, n-2)$ system) of the set of coordinate functions of the parametrization of a convex surface in $R^{n-1}$ with 0 in its interior.

Let (9) be the set of coordinate functions of the parametrization of a convex surface $C$ in $R^{n-1}$ $(n \geq 3)$ with 0 in its interior. The augmentation $\varphi \in C(S^{n-2})$ to a weak $(n, n-2)$ system (to an $(n, n-2)$ system) (10) will be called *sublinear* (**strictly sublinear**) if $\varphi$ is the trace $C$ of a sublinear (of a strictly sublinear) function (see 5.1). The superlinear (strictly superlinear) augmentation is defined similarly.

Using this terminology we have the following result:

**6.1.** *Let (9) be the set of the coordinate functions of a parametrization of a convex surface.*

*(a) The set of the sublinear (superlinear) augmentations $\varphi$ of (9) to (10)*

*(b) The set of strictly sublinear (strictly superlinear) augmentations $\varphi$ of (9) to (10)*
*is invariant with respect to the multiplication with positive scalars and its invariant with respect to taking the pointwise maximum (the pointwise minimum) of two elements.*

We prove (a) for the sublinear case. The invariance of the set of augmentations with respect to the multiplication with positive scalars is obvious.

Let $\varphi$ and $\psi$ be two sublinear augmentations. Then $\varphi$ and $\psi$ are traces of the sublinear functions $f$ and $g$ respectively. The function $\max\{f, g\}$ is sublinear and possesses as trace on $C$ the function $\max\{\varphi, \psi\}$. Thus by 5.1 $\max\{\varphi, \psi\}$ will be a sublinear augmentation.

(b) Suppose that $\varphi$ and $\psi$ are strictly sublinear augmentations of (9). If $f, g : R^{n-1} \to R$ are the strictly sublinear functions with traces $\varphi$ and $\psi$ respectively, then $epif$ and $epig$ are strictly convex cones. Since the relation

$$epi(\max\{f, g\}) = (epif) \cap (epig),$$

and since the set on the right hand side is a strictly convex cone (see 3.6), $\max\{f, g\}$ is a strictly sublinear function. The trace of this function on $C$ is $\max\{\varphi, \psi\}$. Hence

$$\{\max\{\varphi, \psi\}, \varphi_1, \dots, \varphi_{n-1}\}$$

is an $(n, n-2)$ system.

## References


[1] Deutsch, F., Nürnberger, G., Singer, I., *Weak Chebyshev systems and alternation*, Pacific J. Math., 89(1980), 9-31.

[2] Jones, R.C., Karlovitz, L.A., *Equioscillation under nonuniqueness in the approximation of continuous functions*, J. Approx. Theory, 3(1970), 138-145.

[3] Karlin, S., Studden, W.J., *Tchebysheff Systems with Applications in Analysis and Statistics*, Interscience, New York, 1966.

[4] Kreĭn, M.G., Nudel'man, A.A., *The Moment Problem of Markov and Extremal Problems*, (Russian), Nauka, Moskow, 1973.

[5] Mairhuber, J.C., *On Haar's theorem concerning Chebyshev approximation problems having unique solutions*, Proc. Amer. Math. Soc., 7(1956), 609-615.

[6] Németh, A.B., *About an imbedding conjecture for k-independent sets*, Fundamenta Math., 67(1970), 203-207.

[7] Németh, A.B., *On two notions of generalized convexity and the boundary behaviour of bivariate convex functions*, Pure Math. Appl. 6(1995), 251-271.

[8] Popoviciu, T., *Notes sur les généralisation des fonctions convexes d'ordre supérieur*, I, Disquisit. Math. Phys., 1(1940), 35-42.
[9] Rubinstein, G.S., *On a method of investigation of convex sets*, (Russian), Doklady SSSR, 102(1955), 451-454.
[10] Stoer, J., Witzgall, Ch., *Convexity and Optimization in Finite Dimensions*, Springer-Verlag, Heidelberg - New York, 1970.
[11] Tornheim, L., *On n parameter families of functions and associated convex functions*, Trans. Amer. Math. Soc., 69(1950), 457-467.

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# ON CERTAIN NEW CONDITIONS FOR STARLIKENESS AND STRONGLY-STARLIKENESS

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**Abstract.** In this paper we will obtain conditions for starlikeness and strongly-starlikeness starting from the differential subordination of the form:

$$\alpha z p'(z) + p^2(z) \prec h(z), \text{ where } \alpha \geq 0,$$
$$h(z) = \alpha n z q'(z) + q^2(z),$$

and $q$ is a convex function in the unit disc $U$, with $q(0) = 1$ and Re $q(z) > 0$, $n \geq 1$. We will obtain our results by using the differential subordination method developed in [1], [2], [3].

## 1. Introduction and preliminaries

Let $A_n$ denote the class of function of the form:

$$f(z) = z + a_{n+1} z^{n+1} + a_{n+2} a^{n+2} + \ldots,$$

which are analytic in the unit disc $U = \{z|\ |z| < 1\}$ and let $A = A_1$.

We let $\mathcal{H}[a, n]$ denote the class of analytic functions in $U$ of the form:

$$f(z) = z + a_n z^n + a_{n+1} z^{n+1} + \ldots, \quad z \in U$$

and let

$$S^* = \left\{ f \in A,\ \text{Re}\, \frac{zf'(z)}{f(z)} > 0,\ z \in U \right\}$$

be the class of starlike functions in the unit disc $U$.

For $\lambda \in (0, 1]$ let

$$S^*[\lambda] = \left\{ f \in A|\ \left| \arg \frac{zf'(z)}{f(z)} \right| < \lambda \frac{\pi}{2},\ z \in U \right\}$$

denote the class of strongly-starlike functions.

We will need the following notions and lemmas to prove our main results.

If $f$ and $F$ are analytic functions in $U$, then we say that $f$ is subordinate to $F$, written $f \prec F$ or $f(z) \prec F(z)$, if there is a function $w$ analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$, for $z \in U$ and if $f(z) = F(w(z))$, $z \in U$. If $F$ is univalent then $f \prec F$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

**Lemma A** ([1], [2], [3]). *Let $q$ be univalent in $\overline{U}$ with $q'(\zeta) \neq 0$, $|\zeta| = 1$, $q(0) = a$ and let $p(z) = a + p_n z + \ldots$ be analytic in $U$, $p(z) \not\equiv a$ and $n \geq 1$.*

*If $p \not\prec q$ then there exist $z_0 \in U$, $\zeta_0 \in \partial U$ and $m \geq n$ such that:*

*(i) $p(z_0) = q(\zeta_0)$*

*(ii) $z_0(p'(z_0) = m\zeta_0 q'(\zeta_0)$.*

*The function $L(z,t)$, $z \in U$, $t \geq 0$ is a subordination chain if $L(z,t) = a_1(t)z + a_2(t)z^2 + \ldots$ is analytic and univalent in $U$ for any $t \geq 0$ and if $L(z,t_1) \prec L(z,t_2)$ where $0 \leq t_1 \leq t_2$.*

**Lemma B** ([7]). *The function $L(z,t) = a_1(t)z + a_2(t)z^2 + \ldots$, with $a_1(t) \neq 0$ for $t \geq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$ is a subordination chain if and only if there are the constants $r \in (0,1]$ and $M > 0$ such that:*

*(i) $L(z,t)$ is analytic in $|z| < r$ for any $t \geq 0$, locally absolute continuous in $t \geq 0$ for every $|z| < r$ and satisfies $|L(z,t)| \leq M|a_1(t)|$ for $|z| < r$ and $t \geq 0$.*

*(ii) There is a function $p(z,t)$ analytic in $U$ for any $t \geq 0$ measurable in $[0,\infty)$ for any $z \in U$ with $\operatorname{Re} p(z,t) > 0$ for $z \in U$, $t \geq 0$ so that*

$$\frac{\partial L(z,t)}{\partial t} = z\frac{\partial L(z,t)}{\partial z}p(z,t) \text{ for } |z| < r$$

*and for almost any $t \geq 0$.*

## 2. Main results

**Theorem 1.** *Let $\alpha \geq 0$ and let $q$ be a convex function in the unit disc $U$, with $q(0) = 1$, $\operatorname{Re} q(z) > 0$ and let*

$$h(z) = \alpha n z q'(z) + q^2(z), \tag{1}$$

*where $n$ is a positive integer.*

*If $p \in \mathcal{H}[1,n]$, satisfies the condition:*

$$\alpha z p'(z) + p^2(z) \prec h(z) \tag{2}$$

*where $h$ is given by (1) then $p(z) \prec q(z)$, and $q$ is the best dominant.*

*Proof.* Let

$$L(z,t) = \alpha(n+t)zq'(z) + q^2(z) = \psi(q(z),(n+t)zq'(z)). \tag{3}$$

If $t = 0$ we have

$$L(z,0) = \alpha n z q'(z) + q^2(z) = h(z).$$

We will show that condition (2) implies $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

From (3) we easily deduce:

$$\frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}} = (n+t)\left[1 + \frac{zq''(z)}{q'(z)}\right] + \frac{2}{\alpha}q(z)$$

and by using the convexity of $q$ and condition Re $q(z) > 0$, from Theorem 1 we obtain:

$$\text{Re}\,\frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}} > 0.$$

Hence by Lemma B we deduce that $L(z,t)$ is a subordination chain. In particular, the function $h(z) = L(z,0)$ is univalent and $h(z) \prec L(z,0)$, for $t \geq 0$.

If we suppose that $p(z)$ is not subordinate to $q(z)$, then, from Lemma A, there exist $z_0 \in U$, and $\zeta_0 \in \partial U$ such that $p(z_0) = q(\zeta_0)$ with $|\zeta_0| = 1$, and $z_0 p'(z_0) = (n+t)\zeta_0 q'(\zeta_0)$, with $t \geq 0$.

Hence

$$\psi_0 = \psi(p(z_0), z_0 p'(z_0)) = \psi(q(\zeta_0),(n+t)\zeta_0 q'(\zeta_0)) = L(\zeta_0,t),\ t \geq 0.$$

Since $h(z_0) = L(z_0,0)$, we deduce that $\psi_0 \notin h(U)$, which contradicts condition (2). Therefore, we have $p(z) \prec q(z)$ and $q(z)$ is the best dominant. □

If we let $p(z) = \frac{zf'(z)}{f(z)}$, (where $f \in A_n$), then Theorem 1 can be written in the following equivalent form:

**Theorem 2.** *Let $\alpha \geq 0$, $q$ be a convex function in the unit disc $U$, with $q(0) = 1$, Re $q(z) > 0$, $n \geq 1$.*

*If $f \in A_n$, with $\frac{f(z)}{z} \neq 0$, $z \in U$, satisfies the condition*

$$\frac{zf'(z)}{f(z)} J(\alpha, f; z) \prec h(z),$$

*where $h$ is given by (1) then*

$$\frac{zf'(z)}{f(z)} \prec q(z)$$

*and $q$ is the best dominant.*

In the case $\alpha = 1$ this result was obtained in [4].

## 3. Particular cases

1) If we let $q(z) = 1 + z$, then

$$h(z) = 1 + (\alpha n + 2)z + z^2, \quad n \geq 1$$

and Theorem 1 can be written as:

**Theorem 3.** *Let $\alpha \geq 0$, and let $n$ be a positive integer.*

*If $p \in \mathcal{H}[1, n]$, satisfies the condition:*

$$\alpha z p'(z) + p^2(z) \prec 1 + (\alpha n + 2)z + z^2,$$

*then*

$$p(z) \prec 1 + z$$

*and this result is sharp.*

In this case Theorem 2 becomes:

**Theorem 4.** *Let $\alpha \geq 0$, and let $n$ be a positive integer.*

*If $f \in A_n$, with $\frac{f(z)}{z} \neq 0$, satisfies the condition*

$$\frac{zf'(z)}{f(z)} J(\alpha, f; z) \prec 1 + (\alpha n + 2)z + z^2$$

*then*

$$\frac{zf'(z)}{f(z)} \prec 1+z,$$

*and this result is sharp.*

2) If we let $q(z) = \frac{1+z}{1-z}$, then

$$h(z) = \frac{1+2(1+\alpha n)z+z^2}{(1-z)^2}.$$

and Theorem 1 becomes:

**Theorem 5.** *Let $\alpha \geq 0$ and let $n$ be a positive integer.*

*If $p \in \mathcal{H}[1,n]$, satisfies the condition*

$$\alpha zp'(z) + p^2(z) \prec \frac{1+2(1+\alpha n)z+z^2}{(1-z)^2},$$

*then*

$$p(z) \prec \frac{1+z}{1-z}$$

*and this result is sharp.*

Theorem 2 becomes the following criterion for starlikeness:

**Theorem 6.** *Let $\alpha \geq 0$, and let $n$ be a positive integer.*

*If $f \in A_n$, with $\frac{f(z)}{z} \neq 0$, satisfies the condition*

$$\frac{zf'(z)}{f(z)} J(\alpha, f; z) \prec \frac{1+2(1+\alpha n)z+z^2}{(1-z)^2}$$

*then $f \in S^*$.*

3) If we let $q(z) = \left(\frac{1+z}{1-z}\right)^\lambda$, where $0 < \lambda < 1$, then

$$h(z) = \left(\frac{1+z}{1-z}\right)^\lambda \left[\frac{2\alpha n\lambda z}{1-z^2} + \left(\frac{1+z}{1-z}\right)^\lambda\right]$$

and Theorem 1 becomes:

**Theorem 7.** *Let $\alpha \geq 0$, $0 < \lambda < 1$, let $n$ be a positive integer and let*

$$h(z) = \left(\frac{1+z}{1-z}\right)^\lambda \left[\frac{2\alpha n\lambda z}{1-z^2} + \left(\frac{1+z}{1-z}\right)^\lambda\right] = \left(\frac{1+z}{1-z}\right)^{\lambda-1} \left[\frac{2\alpha n\lambda z}{(1-z)^2} + \left(\frac{1+z}{1-z}\right)^{\lambda+1}\right]. \tag{4}$$

*If $p \in \mathcal{H}[1,n]$, satisfies the condition:*

$$\alpha zp'(z) + p^2(z) \prec h(z),$$

*where h is given by (4), then*

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\lambda}$$

*and this result is sharp.*

From Theorem 2 we deduce the following criterion for strongly-starlikeness.

**Theorem 8.** *Let $\alpha \geq 0$, $0 < \lambda < 1$, and let $n$ be a positive integer.*

*If $f \in A_n$, with $\frac{f(z)}{z} \neq 0$, satisfies the condition*

$$\frac{zf'(z)}{f(z)} J(\alpha, f; z) \prec h(z)$$

*where h is given by (4), then*

$$f \in S^*[\lambda].$$

By choosing certain subdomains of $h(U)$ we can deduce the following particular criteria for strongly-starlikeness.

**Corollary 1.** *Let $0 < \lambda < 1$, $n \geq 1$, $\alpha \geq 0$.*

*If $f \in A_n$, with $\frac{f(z)}{z} \neq 0$, satisfies the condition*

$$\left|\arg\left\{\frac{zf'(z)}{f(z)} J(\alpha, f; z)\right\}\right| < \phi_0(n, \alpha, \lambda), \tag{5}$$

*where*

$$\phi_0(n, \alpha, \lambda) = \frac{\lambda\pi}{2} + \arctan \frac{\frac{n\alpha\lambda}{1-\lambda} + \left(\frac{1+\lambda}{1-\lambda}\right)^{\frac{1+\lambda}{2}} \sin\frac{\lambda\pi}{2}}{\left(\frac{1+\lambda}{1-\lambda}\right)^{\frac{1+\lambda}{2}} \cos\frac{\lambda\pi}{2}} \tag{6}$$

*then $f \in S^*[\lambda]$.*

*Proof.* The domain $h(U)$, where $h$ is given by (4) is symmetric with respect to the real axis. Therefore, if $z = e^{i\theta}$, then in order to obtain the boundary of $h(U)$ it is sufficient to suppose $0 \leq \theta \leq \pi$.

Letting $\cot\frac{\theta}{2} = t$ and $h(e^{i\theta}) = u + iv$, we find:

$$\begin{cases} u(t) = t^{\lambda}\left[-\frac{\alpha n\lambda a}{2t}(1+t^2) + (b^2 - a^2)t^{\lambda}\right] \\ v(t) = t^{\lambda}\left[\frac{\alpha n\lambda b}{2t}(1+t^2) + 2abt^2\right] \end{cases} \tag{7}$$

where $a = \sin\frac{\lambda\pi}{2}$ and $b = \cos\frac{\lambda\pi}{2}$.

We also have:

$$\phi = \phi(t) = \arg h(e^{i\theta}) = \frac{\lambda\pi}{2} + \arctan \frac{\frac{\alpha n\lambda}{2}(1+t^2) + t^{\lambda+1}\sin\frac{\lambda\pi}{2}}{t^{\lambda+1}\cos\frac{\lambda\pi}{2}}. \tag{8}$$

From (8) it is easy to show that the equation $\phi'(t) = 0$, has the root:

$$t_0 = \sqrt{\frac{1+\lambda}{1-\lambda}}$$

and

$$\min_{t\geq 0} \phi(t) = \phi(t_0) = \phi_0(n, \lambda)$$

where $\phi_0(n, \lambda)$ is given by (6).

We deduce that the sector $\{w : \ |\arg w| < \phi_0(n, \alpha, \lambda\}$ is the largest sector which lies in $h(U)$. Hence (5) implies

$$\frac{zf'(z)}{f(z)} J(\alpha, f; z) \prec h(z)$$

where $h$ is given by (4) and Corollary 1 follows from Theorem 2. □

**Corollary 2.** *Let* $0 < \lambda < 1$, $n \geq 1$, $\alpha \geq 0$.

*If* $f \in A_n$, *with* $\frac{f(z)}{z} \neq 0$, *satisfies the condition*

$$\left| \operatorname{Im} \frac{zf'(z)}{f(z)} J(\alpha, f; z) \right| < V(n, \alpha, \lambda), \tag{9}$$

*where* $V(n, \alpha, \lambda) = v(t_0)$, *with* $v$ *given by (7) and* $t_0$ *is the root of the equation:*

$$4t^{\lambda+1} \sin \lambda\pi + \alpha n(\lambda+1)t^2 \cos\frac{\lambda\pi}{2} + \alpha n(\lambda-1)\cos\frac{\lambda\pi}{2} = 0 \tag{10}$$

*then* $f \in S^*[\lambda]$.

*Proof.* From (7) we deduce:

$$v' = \lambda t^{\lambda-2} \left[ \frac{\alpha n(\lambda-1)b}{2} + \frac{\alpha n(\lambda+1)b}{2} t^2 + 4abt^{\lambda+1} \right]$$

and the equation $v'(t) = 0$ becomes (10).

Hence

$$V(n, \alpha, \lambda) = v(t_0) = \min_{t\geq 0} v(t)$$

and we deduce that the strip $|v| < V(n, \alpha, \lambda)$ lies in $h(U)$. Therefore (9) implies

$$\frac{zf'(z)}{f(z)} J(\alpha, f; z) \prec h(z)$$

and Corollary 2 follows from Thereom 2. □

**Corollary 3.** *Let* $0 < \lambda < 1$, $n \geq 1$, $\alpha \geq 0$.

*If* $f \in A_n$, *with* $\frac{f(z)}{z} \neq 0$, $z \in U$, *satisfies the condition:*

$$\text{Re}\left[\frac{zf'(z)}{f(z)}J(\alpha, f; z)\right] > U(n, \alpha, \lambda) \tag{11}$$

*where* $U(n, \alpha, \lambda) = u(t_0)$, *with* $u$ *given by (7) and* $t_0$ *is the root of the equation:*

$$4t^{\lambda+1}\cos\lambda\pi - n\alpha(\lambda+1)t^2\cos\frac{\lambda\pi}{2} - n\alpha(\lambda-1)\sin\frac{\lambda\pi}{2} = 0 \tag{12}$$

*then* $f \in S^*[\lambda]$.

*Proof.* From (7) we deduce:

$$u' = \lambda t^{\lambda-2}\left[-\frac{\alpha na(\lambda-1)}{2} - \frac{\alpha na(\lambda+1)}{2}t^2 + 2(b^2 - a^2)t^{\lambda+1}\right]$$

and the equation $u'(t) = 0$ becomes (10).

Hence

$$U(n, \lambda) = u(t_0) = \max_{t \geq 0} u(t)$$

and we deduce that the half-plane $\{w : \text{ Re } w > U(n, \alpha, \lambda)\}$ lies in $h(U)$. Therefore (11) implies

$$\frac{zf'(z)}{f(z)}J(\alpha, f; z) \prec h(z)$$

and Corollary 3 follows from Theorem 2. □

## 4. Examples

1) If we let $n = 1$, $\alpha = \frac{1}{2}$, $\lambda = \frac{1}{2}$, then from (6) we deduce

$$\phi_0\left(1, \frac{1}{2}, \frac{1}{2}\right) = \frac{\pi}{4} + \text{arctg}\left(1 + \frac{1}{3^{\frac{3}{4}}\sqrt{2}}\right) = 1.7027\ldots$$

and by Corollary 1 we have the following result:

If $f \in A$, with $\frac{f(z)}{z} \neq 0$, $z \in U$ and:

$$\left|\arg\left[\frac{zf'(z)}{f(z)}\left(\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} + 1\right)\right]\right| < 1.7027\ldots$$

then $f \in S^*\left[\frac{1}{2}\right]$, i.e.

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{4}.$$

2) If we let $n = 2$, $\alpha = \frac{1}{2}$, $\lambda = \frac{1}{2}$, then from (6) we deduce

$$\phi_0\left(2, \frac{1}{2}, \frac{1}{2}\right) = \frac{\pi}{4} + \operatorname{arctg}\left(1 + \frac{\sqrt{2}}{3^{\frac{3}{4}}}\right) = 1.863\ldots$$

and by Corollary 1 we have the following result:

If $f \in A_2$, with $\frac{f(z)}{z} \neq 0$, $z \in U$ and:

$$\left|\arg\left[\frac{zf'(z)}{f(z)}\left(\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} + 1\right)\right]\right| < 1.863\ldots$$

then $f \in S^*\left[\frac{1}{2}\right]$, i.e.

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{4}.$$

3) If we let $n = 2$, $\alpha = \frac{1}{2}$, $\lambda = \frac{2}{3}$, then from (6) we deduce

$$\phi_0\left(2, \frac{1}{2}, \frac{2}{3}\right) = \frac{\pi}{3} + \operatorname{arctg}\frac{2 + 5^{\frac{5}{6}} \cdot \frac{\sqrt{3}}{2}}{5^{\frac{5}{6}} \cdot \frac{1}{2}} = 2.2725\ldots$$

and by Corollary 1 we have the following result:

If $f \in A_2$, with $\frac{f(z)}{z} \neq 0$, $z \in U$ and:

$$\left|\arg\left[\frac{zf'(z)}{f(z)}\left(\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} + 1\right)\right]\right| < 2.2725\ldots$$

then $f \in S^*\left[\frac{2}{3}\right]$ i.e.

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{3}.$$

4) If we let $n = 2$, $\alpha = \frac{1}{2}$, $\lambda = \frac{1}{3}$, then from (6) we deduce

$$\phi_0\left(2, \frac{1}{2}, \frac{1}{3}\right) = \frac{\pi}{6} + \operatorname{arctg}\frac{1 + 2^{\frac{2}{3}}}{2^{\frac{2}{3}} \cdot \sqrt{3}} = 1.2792\ldots$$

and by Corollary 1 we have the following result:

If $f \in A_2$, with $\frac{f(z)}{z} \neq 0$, $z \in U$ and

$$\left|\arg\left[\frac{zf'(z)}{f(z)}\left(\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} + 1\right)\right]\right| < 1.2792\ldots$$

then $f \in S^*\left[\frac{1}{3}\right]$ i.e.

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{6}.$$

5) If we let $n = 2$, $\alpha = \frac{1}{2}$, $\lambda = \frac{1}{3}$, then equation (10) becomes:

$$16t^{\frac{3}{2}} + 3\sqrt{2}t^2 - \sqrt{2} = 0$$

which has the root $t_0 = 0.1846\ldots$ Hence by Corollary 2 we deduce the following result:

If $f \in A_2$, with $\frac{f(z)}{z} \neq 0$, $z \in U$ and:

$$\left|\mathrm{Im}\ \left[\frac{zf'(z)}{f(z)}\left(\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} + 1\right)\right]\right| < 1.220\ldots$$

then $f \in S^*\left[\frac{1}{2}\right]$.

6) If we let $n = 2$, $\alpha = \frac{1}{2}$, $\lambda = \frac{1}{2}$, then equation (12) becomes: $3t^2 - 1 = 0$ and from Corollary 3 we deduce the following result:

If $f \in A_2$, with $\frac{f(z)}{z} \neq 0$, $z \in U$, and:

$$\left|\mathrm{Re}\ \left[\frac{zf'(z)}{f(z)}\left(\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} + 1\right)\right]\right| > -0.610\ldots$$

then $f \in S^*\left[\frac{1}{2}\right]$.

## References


[1] S.S. Miller and P.T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl., 65(1978), 289-305.

[2] S.S. Miller and P.T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. I., 28(1981), 157-171.

[3] S.S. Miller and P.T. Mocanu, *The theory and applications of second order differential subordinations*, Studia Univ. Babeş-Bolyai, Math., 34, 4(1989), 3-33.

[4] P.T. Mocanu and Gh. Oros, *Certain sufficient conditions for strongly-starlikeness*, (to appear).

[5] Ch. Pommerenke, *Univalent Function*, Vandenhoeck Ruprecht in Götingen, 1975.


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# ON THE UNIVALENCE OF AN INTEGRAL OPERATOR

VIRGIL PESCAR

**Abstract.** In this paper we investigate the conditions of univalence for the analicity and univalence in the unit disc of the integral $\int_0^z \left[\frac{g(u)}{u}\right]^\gamma du$.

## 1. INTRODUCTION

Let A be the class of analytic functions f in the unit disc $U = \{z \in C; |z| < 1\}$, $f(0) = f'(0) - 1 = 0$ and S be subclass of univalent functions in the class A.

Kim and Merkes [3] investigated the univalence of the integral $\int_0^z \left[\frac{g(\xi)}{\xi}\right]^\gamma d\xi$.

**THEOREM A [3].** If the function f belongs to the class S then for any complex number $\gamma$, $|\gamma| \leq \frac{1}{4}$ the function

$$F_\gamma(z) = \int_0^z \left[\frac{f(\xi)}{\xi}\right]^\gamma d\xi \tag{1}$$

is in S.

## 2. PRELIMINARIES

We will need the following theorem and lemma for proving our main result.

**THEOREM B [1].** If the function f is regular in the unit disc, $f(z) = z + a_2 z^2 + \ldots$ and

$$\left(1 - |z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{2}$$

for all $z \in U$, then the function f is univalent in U.

**LEMMA SCHWARZ [2].** If the function g is regular in U, $g(0) = 0$ and $|g(z)| \leq 1$ for all $z \in U$, then the following inequalities hold

$$|g(z)| \leq |z| \tag{3}$$

for all $z \in U$, and $|g'(0)| \leq 1$, the equalities (in inequality (3) for $z \neq 0$) hold only in the case $g(z) = \epsilon z$, where $|\epsilon| = 1$.

## 3. MAIN RESULT

**THEOREM 1.** Let $\gamma$ be a complex number and the function $h \in S$, $h(z) = z + a_2 z^2 + \cdots$. If

$$\left| \frac{zh'(z)}{h(z)} - 1 \right| \leq 1 \tag{4}$$

for all $z \in U$ and

$$|\gamma| \leq \frac{3\sqrt{3}}{2} \tag{5}$$

then the function

$$F_\gamma(z) = \int_0^z \left[ \frac{h(u)}{u} \right]^\gamma du \tag{6}$$

is in S.

*Proof.* Let us consider the function

$$f(z) = \int_0^z \left[ \frac{h(u)}{u} \right]^\gamma du. \tag{7}$$

The function

$$p(z) = \frac{1}{\gamma} \frac{zf''(z)}{f'(z)} \tag{8}$$

where the constant $\gamma$ satisfies the inequality (5) is regular in U. From (8) and (7) it follows that

$$p(z) = \frac{\gamma}{|\gamma|} \left[ \frac{zh'(z)}{h(z)} - 1 \right]. \tag{9}$$

Using (9) and (4) we have

$$|p(z)| \leq 1 \tag{10}$$

for all $z \in U$. From (9) we obtain $p(0) = 0$ and applying Schwarz-Lemma we have

$$\frac{1}{|\gamma|} \left| \frac{zf''(z)}{f'(z)} \right| \leq |z| \tag{11}$$

for all $z \in U$, and hence, we obtain

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \le |\gamma|\left(1-|z|^2\right)|z|. \tag{12}$$

Because $\max_{|z|\le 1}\left(1-|z|^2\right)|z| = \frac{2}{3\sqrt{3}}$,from (12) and (5) we obtain

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \le 1. \tag{13}$$

for all $z \in U$. From (13), (7), (6) and Theorem B it follows that $F_\gamma$ is in the class S. □

## References


[1] J. Becker, Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funcktionen, J.Reine Angew. Math., 255(1972),23-43.

[2] G. M. Goluzin, Gheometriceskaia teoria funktii Kompleksnogo peremennogo, ed. a II-a, Nauka, Moscova, 1966.

[3] Y. J. Kim, E.P.Merkes, On an integral of powers of a spirallike function, Kyungpook Math. J., Vol. 12, No. 2, December 1972,249-253.


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# ON AN INTEGRAL OPERATOR

VIRGIL PESCAR

**Abstract.** In this paper we investigate the conditions of univalence for the analicity and univalence in the unit disc of the integral $\int_0^z \left[\frac{g(u)}{u}\right]^\gamma du$.

## 1. INTRODUCTION

Let A be the class of the functions f which are analytic in the unit disc $U = \{z \in C; |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$

We denote by S the class of the function $f \in A$ which are univalent in U.

The integral operators which transform the class S into S are prezented in the theoremes A and B, which follow.

**THEOREM A [2].** If the function f belongs to the class S then for any complex number $\gamma$, $|\gamma| \leq \frac{1}{4}$ the function

$$F_\gamma(z) = \int_0^z \left[\frac{f(\xi)}{\xi}\right]^\gamma d\xi \tag{1}$$

is in S.

**THEOREM B[4].** Let $\alpha, \beta, \gamma$ be complex numbers and $h(z) = z + a_2 z^2 + \dots$ a regular and univalent function in U. If

(i) $\quad \text{Re}\beta \geq \text{Re}\alpha > 0$

and

(ii) $\quad |\gamma| \leq \frac{Re\alpha}{2}$ for $\text{Re}\alpha \in \left(0, \frac{1}{2}\right)$

or

$|\gamma| \leq \frac{1}{4}$ for $Re\alpha \in \left[\frac{1}{2}, \infty\right)$.

then the function

$$G_{\beta,\gamma}(z) = \left[\beta \int_0^z u^{\beta-1} \left(\frac{h(u)}{u}\right)^\gamma du\right]^{\frac{1}{\beta}} \tag{2}$$

belongs to the class S.

## 2. PRELIMINARIES

We will need the following theorem and lemma for proving our main result.

**THEOREM C [3].** Let $\alpha$ be a complex number, $Re\alpha > 0$ and $f(z) = z + a_2 z^2 + \cdots$ be a regular function in U. If

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{3}$$

for all $z \in U$, then for any complex number $\beta$, $Re\beta \geq Re\alpha$ the function

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u) du\right]^{\frac{1}{\beta}} \tag{4}$$

is in the class S.

**LEMMA SCHWARZ [1].** If the function g is regular in U, $g(0) = 0$ and $|g(z)| \leq 1$ for all $z \in U$, then the following inequalities hold

$$|g(z)| \leq |z| \tag{5}$$

for all $z \in U$, and $|g'(0)| \leq 1$, the equalities (in inequality (5) for $z \neq 0$) hold only in the case $g(z) = \epsilon z$, where $|\epsilon| = 1$.

## 3. MAIN RESULT

**THEOREM 1.** Let $\alpha, \gamma$ be complex numbers, $\text{Re}\alpha = a > 0$ and the function $h \in S$, $h(z) = z + a_2 z^2 + \ldots$. If

$$\left|\frac{zh'(z)}{h(z)} - 1\right| \leq 1 \tag{6}$$

for all $z \in U$ and

$$|\gamma| \leq \frac{(2a+1)^{\frac{(2a+1)}{2a}}}{2}, \tag{7}$$

then for any complex number $\beta$, Re $\beta \geq a$ the function

$$G_{\beta,\gamma}(z) = \left[\beta \int_0^z \left(\frac{h(u)}{u}\right)^\gamma du\right]^{\frac{1}{\beta}} \tag{8}$$

is in the class S.

*Proof.* Let us consider the function

$$f(z) = \int_0^z \left[\frac{h(u)}{u}\right]^\gamma du. \tag{9}$$

The function

$$p(z) = \frac{1}{\gamma}\frac{zf''(z)}{f'(z)} \tag{10}$$

where the constant $\gamma$ satisfies the inequality (7) is regular in U. From (10) and (9) it follows that

$$p(z) = \frac{\gamma}{|\gamma|}\left[\frac{zh'(z)}{h(z)} - 1\right]. \tag{11}$$

Using (11) and (6) we obtain

$$|p(z)| \le 1 \tag{12}$$

for all $z \in U$. From (11) we have $p(0) = 0$ and applying Schwarz-Lemma we obtain

$$\frac{1}{|\gamma|}\left|\frac{zf''(z)}{f'(z)}\right| \le |z| \tag{13}$$

for all $z \in U$, and hence, we have

$$\frac{1-|z|^{2a}}{a}\left|\frac{zf''(z)}{f'(z)}\right| \le |\gamma|\frac{1-|z|^{2a}}{a}|z|. \tag{14}$$

Because $\max\limits_{|z|\le 1}\frac{1-|z|^{2a}}{a}|z| = \frac{2}{(2a+1)^{\frac{2a+1}{2a}}}$,from (14) and (7) we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zf''(z)}{f'(z)}\right| \le 1 \tag{15}$$

for all $z \in U$. From (15), (9) and Theorem C it follows that $G_{\beta,\gamma}$ is in the class S. □

## References

[1] G.M. Goluzin, Gheometriceskaia teoria funktii Kompleksnogo peremennogo, ed. a II-a, Nauka, Moscova, 1966.

[2] Y.J. Kim, E.P. Merkes, On an integral of powers of a spirallike function, Kyungpook Math. J., Vol. 12, No. 2, December 1972,249-253.

[3] N.N. Pascu, An improvement of Becker's univalence criterion, Proceedings of the Commemorative Session Simion Stoilow, Braşov, (1987), 43-48.

[4] N.N. Pascu, V. Pescar, On the integral operators of Kim-Merkes and Pfaltzgraff, Studia (Mathematica), Univ. Babeş-Bolyai, Cluj-Napoca, 32, 2(1990), 185-192.

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# ON BROWDER'S FIXED POINT THEOREM

ADRIAN PETRUŞEL AND AUREL MUNTEAN

## 1. Introduction

In 1968 F.E. Browder stated the following fixed point principle in topological vector spaces:

**Theorem 1.1.** ([3]) *Let $X$ be a Hausdorff topological vector space and $K$ be a nonempty, compact, convex subset of $X$. Let $F$ be a multivalued operator such that $F : K \to P_{cv}(K)$ and for each $y \in K$ the set $F^{-1}(y) := \{x \in K|\ y \in F(x)\}$ is open. Then there exists $x_0$ in $K$ such that $x_0 \in F(x_0)$.*

The key tool in his proof is the compactness of the set $K$, which is used to construct a continuous selection for $T$ and, in the same time, permett to apply Schauder's fixed point theorem.

The first purpose of this note is to give another proof for this theorem, using the notion of locally selectionable multivalued operator. The virtue of this proof is that one use the compactness of $K$ only for the application of Schauder's theorem.

On the other side, using the property of decomposability as substitute for convexity in Theorem 1.1, we get the second main result of the paper: a selection principle for multivalued operators with decomposable values.

We follow the notations and symbols from [7].

## 2. Main results

The concept of locally selectionable multivalued operator has been introduced because these set-valued maps do posses continuous selection on paracompact topological spaces.

**Definition 2.1.** Let $X, Y$ be two nonempty sets and $F : X \to P(Y)$ a multivalued operator. Then a singlevalued operator $f : X \to Y$ is a selection for $F$ iff $f(x) \in F(x)$, for each $x \in X$.

**Definition 2.2.** Let $X$ be a nonempty set and $F : X \to P(X)$ a multivalued operator. Then $x_0 \in X$ is called a fixed point for $F$ iff $x_0 \in F(x_0)$.

The fixed points set for $F$ will be denoted by $FixF$.

**Definition 2.3.** ([1]) Let $X, Y$ be two topological spaces. We say that $F : X \to P(Y)$ is locally selectionable at $x_0 \in X$ iff for all $y_0 \in F(x_0)$ there exist an open neighborhood $N(x_0)$ of $x_0$ and a continuous map $f : N(x_0) \to Y$ such that $f(x_0) = y_0$ and $f(x) \in F(x)$, for all $x \in N(x_0)$. $F$ is said to be locally selectionable if it is locally selectionable at every $x_0 \in X$.

*Remark 2.4.* ([1]) Any locally selectionable multivalued operator is lower semicontinuous.

The main tools in our proof of the Browder fixed point theorem are:

**Lemma 2.5.** ([1]) *Let $X, Y$ be two topological spaces and $F : X \to P(Y)$ a multivalued operator. If $F^{-1}(y)$ is open for each $y \in Y$ then $F$ is locally selectionable.*

**Lemma 2.6** ([1]) *Let $X$ be a paracompact space and $F$ be a locally selectionable operator with nonempty, convex values from $X$ to a Hausdorff topological vector space $Y$. Then $F$ has a continuous selection.*

The first result of this note is the following:

**Theorem 2.7.** *Let $X$ be a paracompact vector space, $K$ a nonempty, compact, convex subset of $X$ and $F : K \to P_{cv}(K)$ a multivalued operator such that for each $y \in K$, $F^{-1}(y)$ is open. Then $FixF \neq \emptyset$.*

*Proof.* From Lemma 2.5, $F$ is locally selectionable. Lemma 2.6 implies the existence of a continuous selection $f : K \to K$ of $F$. A simple application of the Schauder's fixed point theorem concludes the proof. □

For the second part of the paper, consider $(T, \mathcal{A}, \mu)$ a complete $\sigma$-finite and nonatomic measure space and $E$ a Banach space. Let $L^1(T, E)$ be the Banach space of all measurable functions $u : T \to E$ which are Bochner $\mu$-integrable. We call a set $K \subset L^1(T, E)$ decomposable iff for all $u, v \in K$ and each $t \in \mathcal{A}$ we have that

$u\chi_A + v\chi_{T\setminus A} \in K$, where $\chi_A$ stands for the characteristic function of the set $A$ (see also [6]).

An userful result is:

**Theorem 2.8.** ([4]) *Let $K$ be a bounded, decomposable set of $L^1(T, E)$. Then the Kuratowski's index of the set $K$ is the diameter of $K$.*

The second main result of this paper is:

**Theorem 2.9.** *Let $E$ be a Banach space such that $L^1(T, E)$ is separable. Let $K$ be a nonempty, paracompact, decomposable subset of $L^1(T, E)$ and $F : K \to P_{dec}(K)$ be a multivalued operator such that $F^{-1}(y)$ is open, for each $y \in K$. Then $F$ has a continuous selection.*

*Proof.* For each $y \in K$, $F^{-1}(y)$ is an open subset of $K$. Since $K$ is compact, the open covering $(F^{-1}(y))_{y\in K}$ admits a locally finite, open refinement, so $K = \bigcup_{j\in J} F^{-1}(y_j)$, $y_i \in K$ for $J \subset \mathbf{N}$. Let $\{\psi_j\}_{j\in J}$ be a continuous partition of unity subordinate to $(F^{-1}(y_j))_{j\in J}$. Using the same construction as in the proof of Lemma 3.1 from [7] (see also Proposition 1.1 - Proposition 1.3 in [5]), we get a continuous function $f : K \to K$, $f(x) = \sum_{j\in J} f_j(x)\chi_j(x)$, where $f_j(x) \in F(x)$ for each $x \in K$. This function is a continuous selection for $F$. □

*Remark 2.10.* A "decomposable" version of the Browder's fixed point theorem is an open problem. It is well known that a compact, decomposable subset of $L^1(T, E)$ consists of only one point (see Theorem 2.8). On the other hand, each closed, decomposable subset of $L^1(T, E)$ has the compact fixed point property (see [2]). The problem is if there exists a continuous, compact selection for $F$.

## References

[1] J.P. Aubin, A. Cellina, *Differential Inclusions*, Springer, Berlin, 1984.
[2] A. Bressan, G. Colombo, *Extensions and selections of maps with decomposable values*, Studia Math., 90(1988), 69-86.
[3] F.E. Browder, *The fixed point theory of multivalued mappings in topological spaces*, Math. Annalen, 177(1968), 283-301.
[4] A. Cellina, C. Mariconda, *Kuratowski's index of a decomposable set*, Bull. Pol. Acad. Sci. Math. 37(1989), 679-685.
[5] A. Fryszkowski, *Continuous selections for a class of non-convex multivalued maps*, Studia Math., 76(1983), 163-174.
[6] C. Olech, *Decomposability as substitute for convexity*, Lect. Notes in Math., 1091, Springer, Berlin, 1984.

[7] A. Petruşel, *Continuous selections for multivalued operators with decomposable values*, Studia Univ. Babeş-Bolyai, seria Mathematica, 41(1996), 97-100.

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# SOME OBSERVATIONS ON CONFORMAL METRICAL $N$-LINEAR CONNECTIONS IN THE BUNDLE OF ACCELERATIONS

MONICA PURCARU

**Abstract.** In the present paper we treat some special classes of conformal metrical $N$-linear connections on $E = Osc^2M$, which preserve the nonlinear connection N. We analyze the role of the torsion $d$-tensor fields $T_{(0)}$, $S_{(1)}$ and $S_{(2)}$ in this theory and we study the semi-symmetric conformal metrical $N$-linear connections, which preserve the nonlinear connection N.

## 1. Introduction

The geometry of k-osculator spaces presents not only a special theoretical interest, but also an applicative one. Motivated by concrete problems in variational calculation, higher order Lagrange geometry has witnessed a wide acknowledgment due to the papers $[7-11]$ published by Acad.dr.R.Miron and Prof.dr.Gh.Atanasiu.

The various applications of the Lagrange geometry of order $k$ in Physics and Mechanics are considerable [14].

In the present paper we introduce the conformal metrical $d$-structure notion on $E = Osc^2M$, we define the conformal metrical $N$-linear connection notion (§2), we analyze the role of the torsion $d$-tensor fields $T_{(0)}$, $S_{(1)}$ and $S_{(2)}$ in this theory, and we study the semi-symmetric conformal metrical $N$-linear cconnections, which preserve the nonlinear connection $N$(§3). As to the terminology and notations we use those from [12], which are essentially based on M.Matsumoto's book [4].

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*Key words and phrases.* osculator bundle, $N$-linear connection, conformal metrical structure, torsion.

## 2. The notion of conformal metrical $N$-linear connection

Let $M$ be a real $n$-dimensional $C^{\infty}$-differentiable manifold and $(Osc^2M, \pi, M)$ its 2-osculator bundle, or the bundle of accelerations. The local coordinates on $E = Osc^2M$ are denoted by $(x^i, y^{(1)i}, y^{(2)i})$. If $N$ is a nonlinear connection on $E$, with the coefficients $N_{(1)}{}^i{}_j$, $N_{(2)}{}^i{}_j$, then let $D\Gamma(N) = (L^i_{jk}, C_{(1)}{}^i_{jk}, C_{(2)}{}^i_{jk})$ be an $N$- linear connection on $E$. We consider a metric $d$-structure on E, defined by a $d$-tensor field of the type $(0,2)$, let us say $g_{ij}(x^i, y^{(1)i}, y^{(2)i})$, symmetric and nondegenerate.

We associate to this $d$-structure Obata's operators:

$$\Omega^{ir}_{sj} = \frac{1}{2}(\delta^i_s\delta^r_j - g_{sj}g^{ir}), \ \Omega^{*ir}_{sj} = \frac{1}{2}(\delta^i_s\delta^r_j + g_{sj}g^{ir}), \tag{2.1}$$

where $(g^{ij})$ is the inverse matrix of $(g_{ij})$.

Obata's operators have the same properties as ones associated with the Finsler space [12]. Let $\mathcal{S}_2(E)$ be the set of all symmetric $d$-tensor fields of the type $(0,2)$ on $E$. As is easily shown, the relation for $a_{ij}$, $b_{ij} \in \mathcal{S}_2(E)$ defined by:

$$a_{ij} \sim b_{ij} \Leftrightarrow \exists \rho(x, y^{(1)}, y^{(2)}) \in \mathcal{F}(E) \mid a_{ij} = e^{2\rho}b_{ij}, \tag{2.2}$$

is an equivalent relation on $\mathcal{S}_2(E)$.

**Definition 2.1.** [14] *The equivalence class $\hat{g}$ of $\mathcal{S}_2(E)/_{\sim}$, to which the metric d-structure $g_{ij}$ belongs, is called conformal metrical d-structure on E.*

**Definition 2.2.** *An $N$-linear connection $D\Gamma(N) = (L^i_{jk}, C_{(1)}{}^i_{jk}, C_{(2)}{}^i_{jk})$ on E, is said to be compatible with the conformal metrical d-structure $\hat{g}$, or a conformal metrical $N$-linear connection, if the following relations are verified:*

$$g_{ij|k} = 2\omega_k g_{ij}, \ g_{ij} \overset{(\alpha)}{|}_k = 2\lambda_{(\alpha)k} g_{ij}, \ (\alpha = 1, 2), \tag{2.3}$$

where $\omega_k = \omega_k(x, y^{(1)}, y^{(2)})$, $\lambda_{(\alpha)k} = \lambda_{(\alpha)k}(x, y^{(1)}, y^{(2)}), (\alpha = 1, 2)$ are covariant $d$-vector fields and $|$ and $\overset{(\alpha)}{|}$ denote the $h$-and $v_\alpha$-covariant derivatives $(\alpha = 1, 2)$ with respect to $D\Gamma(N)$.

**Theorem 2.1.** [14] *The set of all conformal metrical $N$-linear connections on $E$, which preserve the nonlinear connection $N$, $D\Gamma(N) = (L^i_{jk}, C_{(1)}{}^i_{jk}, C_{(2)}{}^i_{jk})$ is given by:*

$$L^i_{jk} = \overset{0}{L^i_{jk}} + \Omega^{ir}_{sj} X^s_{rk}, \quad C_{(\alpha)}{}^i_{jk} = \overset{0}{C}_{(\alpha)}{}^i_{jk} + \Omega^{ir}_{sj} Y_{(\alpha)}{}^s_{rk}, \ (\alpha = 1, 2), \tag{2.4}$$

*where $X^i_{jk}$, $Y_{(1)}{}^i_{jk}$, $Y_{(2)}{}^i_{jk}$ are arbitrary tensor fields of the type $(1,2)$ and $D\overset{0}{\Gamma}(N) = (\overset{0}{L^i_{jk}}, \overset{0}{C}_{(1)}{}^i_{jk}, \overset{0}{C}_{(2)}{}^i_{jk})$ are the coefficients of an arbitrary fixed conformal metrical $N$-linear connection on $E$.*

## 3. Some special classes of conformal metrical $N$-linear connections

We shall try to replace the arbitrary tensor fields $X^i_{jk}$, $Y_{(1)}{}^i_{jk}$, $Y_{(2)}{}^i_{jk}$ in Theorem 2.1 by the torsion $d$-tensor fields $T_{(0)}{}^i{}_{jk}$, $S_{(1)}{}^i{}_{jk}$, $S_{(2)}{}^i{}_{jk}$. We put:

$$\begin{cases} T^*_{(0)}{}^i{}_{jk} = \frac{1}{2} g^{il}(g_{lh} T_{(0)}{}^h{}_{jk} - g_{jh} T_{(0)}{}^h{}_{lk} + g_{kh} T_{(0)}{}^h{}_{jl}), \\ S^*_{(\alpha)}{}^i{}_{jk} = \frac{1}{2} g^{il}(g_{lh} S_{(\alpha)}{}^h{}_{jk} - g_{jh} S_{(\alpha)}{}^h{}_{lk} + g_{kh} S_{(\alpha)}{}^h{}_{jl}), \ (\alpha = 1, 2). \end{cases} \tag{3.1}$$

**Theorem 3.1.** *Let $T_{(0)}{}^i{}_{jk}$, $S_{(1)}{}^i{}_{jk}$, $S_{(2)}{}^i{}_{jk}$ be given alternate d-tensor fields. Then there exists a unique conformal metrical $N$-linear connection $D\Gamma(N) =$ $= (L^i_{jk}, C_{(1)}{}^i_{jk}, C_{(2)}{}^i_{jk})$ with respect to $\hat{g}$, having $T_{(0)}{}^i{}_{jk}$, $S_{(1)}{}^i{}_{jk}$, $S_{(2)}{}^i{}_{jk}$ as the torsion d-tensor fields. It is given by:*

$$L^i_{jk} = \overset{0}{L^i_{jk}} + T^*_{(0)}{}^i{}_{jk}, \quad C_{(\alpha)}{}^i_{jk} = \overset{0}{C}_{(\alpha)}{}^i_{jk} + S^*_{(\alpha)}{}^i{}_{jk}, \ (\alpha = 1, 2), \tag{3.2}$$

*where $D\overset{0}{\Gamma}(N) = (\overset{0}{L^i_{jk}}, \overset{0}{C}_{(1)}{}^i_{jk}, \overset{0}{C}_{(2)}{}^i_{jk})$ is an arbitrary fixed conformal metrical $N$-linear connection on $E$.*

**Theorem 3.2.** *There exists a unique conformal metrical $N$-linear connection $D\Gamma(N)$ on $E$, whose torsion d-tensor fields $T_{(0)}$, $S_{(\alpha)}$, $(\alpha = 1, 2)$ vanish.*

**Definition 3.1.** [15] *An N-linear connection* $D\Gamma(N) = (L^i_{jk}, C_{(1)}{}^i_{jk}, C_{(2)}{}^i_{jk})$ *is called semi-symmetric if the torsion d-tensor fields* $T_{(0)}{}^i{}_{jk}$, $S_{(1)}{}^i{}_{jk}$, $S_{(2)}{}^i{}_{jk}$ *have the form:*

$$\begin{cases} T_{(0)}{}^i{}_{jk} = \frac{1}{n-1}(T_{(0)j}\delta^i_k - T_{(0)k}\delta^i_j), \\ \\ S_{(\alpha)}{}^i{}_{jk} = \frac{1}{n-1}(S_{(\alpha)j}\delta^i_k - S_{(\alpha)k}\delta^i_j), \ (\alpha = 1,2), \end{cases} \tag{3.3}$$

*where* $T_{(0)j} = T_{(0)}{}^i{}_{ji}$, $S_{(\alpha)j} = S_{(\alpha)}{}^i{}_{ji}$, $(\alpha = 1,2)$.

Putting $\sigma_j = \frac{1}{n-1}T_{(0)j}$, $\tau_{(\alpha)j} = \frac{1}{n-1}S_{(\alpha)j}$, $T^*_{(0)}{}^i{}_{jk}$, $S^*_{(\alpha)}{}^i{}_{jk}$, $(\alpha = 1,2)$ given by (3.1) become:

$$T^*_{(0)}{}^i{}_{jk} = 2\Omega^{ir}_{kj}\sigma_r, \ S^*_{(\alpha)}{}^i{}_{jk} = 2\Omega^{ir}_{kj}\tau_{(\alpha)r}, \ (\alpha = 1,2). \tag{3.4}$$

From Theorem 3.1 we have:

**Theorem 3.3.** *The set of all semi-symmetric conformal metrical N-linear connections on E, which preserve the nonlinear connection N,* $D\Gamma(N) =$
$= (L^i_{jk}, C_{(1)}{}^i_{jk}, C_{(2)}{}^i_{jk})$, *is given by:*

$$\begin{cases} L^i_{jk} = \overset{0}{L}{}^i_{jk} + 2\Omega^{ir}_{kj}\sigma_r, \\ \\ C_{(\alpha)}{}^i_{jk} = \overset{0}{C}_{(\alpha)}{}^i_{jk} + 2\Omega^{ir}_{kj}\tau_{(\alpha)r}, \ (\alpha = 1,2), \end{cases} \tag{3.5}$$

*where* $D\overset{0}{\Gamma}(N) = (\overset{0}{L}{}^i_{jk}, \overset{0}{C}_{(1)}{}^i_{jk}, \overset{0}{C}_{(2)}{}^i_{jk})$ *is an arbitrary fixed conformal metrical N-linear connection on E.*

## References


[1] Atanasiu Gh., *The Equations of Structure of an N- linear Connection in the Bundle of Accelerations* , Balkan Journal of Geometry and Its Applications,1, 1 (1996), 11-19.

[2] Ghinea I., *Conexiuni Finsler compatibile cu anumite structuri geome-trice,* Teză de doctorat, Univ.Cluj, 1978.

[3] Hashiguchi M., *On conformal transformations of Finsler metrics,* J.Math.Kyoto Univ., 16(1976),25-50.

[4] Matsumoto M., *The Theory of Finsler Connections,* Publ.Study Group Geom.5, Depart.Math., Okayama Univ., 1970.

[5] Miron R.,*On Transformation Groups of Finsler Connections,* Tensor, N.S.35(1981), 235-240.

[6] Miron R.,*The Geometry of Highero Order Lagrange Spaces. Applications to Mechanics and Physics,*Kluwer Acad.Publ. FTPH 82, 1997.

[7] Miron R. and Atanasiu Gh.,*Compendium sur les espaces Lagrange d'ordre supérieur: La géométrie du fibré, k-osculateur; Le prolongement des structures Riemanniennes, Finsleriennes et Lagrangiennes; Les espaces Lagrange* $L^{(k)n}$.,Univ.Timişoara, seminarul de Mecanică, no.40, 1994, 1-27.

[8] Miron R. and Atanasiu Gh.,*Lagrange Geometry of Second Order, Math. Comput. Modelling,*vol.20, 4, (1994), 41-56.

[9] Miron R. and Atanasiu Gh.,*Differential Geometry of the k-Osculator Bundle,*Rev.Roumaine Math.Pures Appl., 41,3/4(1996),205-236.

[10] Miron R. and Atanasiu Gh.,*Higher-order Lagrange spaces,* Rev.Roumaine Math.Pures Appl., 41, 3/4(1996), 251-262.

[11] Miron R. and Atanasiu Gh.,*Prolongation of Riemannian, Finslerian and Lagrangian Structures,*Rev.Roumaine Math.Pures Appl., 41, 3/4(1996), 237-249.

[12] Miron R. and Hashiguchi M.,*Metrical Finsler Connections,* Fac.Sci. Kagoshima Univ.(Math., Phys.& Chem.), 12, 1979, 21-35.

[13] Miron R. and Hashiguchi M., *Conformal Finsler Connections,* Rev.Roumaine Math.Pures Appl., 26, 6, 1981, 861-878.

[14] Purcaru M.,*Conformal Metrical Structures in the Bundle off Accelerations,*Proc.of the International Conference on Lagrange and Finsler Geometry with Application to Diffusion in Phisics and Biology, Univ.Braşov, January, 1994(to appear).

[15] Purcaru M.,*Semi-Symmetric Metrical N-Linear Connections in the bundle of Accelerations,*Balkan Journ.of Geometry and Its applications, vol.2, no.2, 1997, 113-118.

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# ON THE STABILITY OF THE ALTERNATIVE METHOD

DAMIAN TRIF

**Abstract.** The stability of the alternative method is investigated. An optimization of the volume of computation for the numerical approximation of a solution of the equation $Lu = Nu$ is also given.

## 1. Introduction

The stability of the fixed point iteration procedures has been investigated by A.M. Harder and T.L. Hicks [1]:

**Definition 1.** *Let $(X, d)$ be a metric space, $T : X \to X$, $x_0 \in X$ and the iteration procedure $x_{n+1} = f(T, x_n)$. If $x_n \to p$, where $p$ is a fixed point of $T$, let $y_n \in X$ and $\varepsilon_n = d(y_{n+1}, f(T, y_n))$. If $\varepsilon_n \to 0$ implies $y_n \to p$ then the iteration procedure $f$ is $T$-stable relating to $T$.*

If $T$ is a contraction, a theorem of Ostrowski [1] shows that the iteration procedure $f(T, x_n) = Tx_n$ is $T$-stable:

**Theorem 1.** *Let $T : X \to X$ be a contraction on the complete metric space $(X, d)$. Let $p$ a fixed point of $T$, $x_0 \in X$, $x_{n+1} = Tx_n$, $n = 0, 1, \ldots$ be. Let $y_n \in X$ and $\varepsilon_n = d(y_{n+1}, Ty_n)$, $n = 0, 1, \ldots$ . Then*

1. $d(p, y_{n+1}) \leq (1 - k)^{-1}(\varepsilon_n + kd(y_n, y_{n+1}))$
2. $d(p, y_{n+1}) \leq d(p, x_{n+1}) + k^{n+1}d(x_0, y_0) + \sum_{i=0}^{n} k^{n-i}\varepsilon_i$
3. $y_n \to p$ *if and only if* $\varepsilon_n \to 0$.

If $X$ is a Banach space, $E : X_E \to X$ is a linear operator and $N : X_N \to X$ is a nonlinear operator, let us consider the equation $Eu = Nu$ , $u \in X_E \cap X_N$.

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If $E$ is an invertible operator, this equation is equivalent to $u = E^{-1}Nu$, a fixed point problem for $T = E^{-1}N$. If $T$ is a contraction, Theorem 2 applies. If $T$ is not a contraction or $E$ is not invertible, the equation $Eu = Nu$ is studied by the alternative (Lyapunov-Schmidt) method. Using an idea of Sanchez [2] it is easy to conclude that the alternative method is $T$-stable. An optimization of the volume of computation for the numerical approximation of the solution of the equation $Eu = Nu$ by the alternative method is also given.

## 2. The stability of the alternative method

Let $X$ be a Banach space, $E : X_E \to X$ a linear operator, $N : X_N \to X$ a nonlinear operator and we suppose that

**a):** there exists a projection $P : X \to X$ such that $X = R(P) \oplus R(I-P)$ and $PE = EP$

**b):** there exists $H : R(I-P) \to R(I-P)$, a linear operator such that

$$H(I-P)Eu = (I-P)u \text{ for all } u \in X_E$$
$$EH(I-P)Nu = (I-P)Nu \text{ for all } u \in X_N$$

**c):** all the fixed points of $P + H(I-P)N$ are in $X_E$.

**Theorem 2.** *$Eu = Nu$ if and only if*

$$(I-P)u = H(I-P)Nu$$
$$P(EPu - Nu) = 0$$

Let $D : R(P) \to R(P)$ be a linear, invertible and with bounded inverse operator. For $a, b > 0$ we define

$$C = \{(v,w) \,|\, v \in R(P), ||v - v_0|| \le a, w \in R(I-P), ||w|| \le b\}$$

where $v_0 \in R(P)$ is an approximation of the solution of the equation $Eu = Nu$. On $C$ we define $||(v,w)|| = ||v|| + ||w||$. Let $p \in \mathbb{N}, u = v + w$ where $(v,w) \in C$, $w^0 = w, w^i = H(I-P)N(v + w^{i-1})$, for $i = 1, 2, ..., p+1$ and $W = w^{p+1}$. Let $V = v - D^{-1}P(Ev - N(v + W))$. We define an operator on $C$ by $T(v,w) = (V,W)$. We remark that in the paper of Sanchez [2], $p = 0$.

**Theorem 3.** *If*

1. *there exists* $\eta \geq 0$ *such that* $(v, w) \in C$ *implies* $\|N(v+w)\| \leq \eta$
2. $H(I-P)$ *is a bounded operator and* $\|H(I-P)\| \leq b/\eta$
3. *there exists* $\sigma \geq 0$ *such that* $\|D^{-1}\| \sigma < 1$ *and if* $(v_1, w), (v_2, w) \in C$ *then* $\|D(v_1 - v_2) - P(Ev_1 - N(v_1 + w) - Ev_2 + N(v_2 + w))\| \leq \sigma \|v_1 - v_2\|$
4. *there exists* $\gamma \geq 0$ *such that* $(v_0, w) \in C$ *implies*

$$\|D^{-1}P(Ev_0 - N(v_0 + w))\| \leq \gamma \leq (1 - \|D^{-1}\| \sigma)a,$$

*then* $T$ *applies* $C$ *into* $C$.

*Proof.* From $(v, w) \in C$ we have $(v, w^k) \in C$ for all $k$ thus

$$\|W\| = \|H(I-P)N(v + w^p)\| \leq b/\eta \cdot \eta = b$$

We have also

$$\|V - v_0\| \leq \|D^{-1}\| \|D(v - v_0) - P(Ev - N(v + W)) - P(Ev_0 - N(v_0 + W))\| \leq$$
$$\leq \|D^{-1}\| \sigma \|v - v_0\| + (1 - \|D^{-1}\| \sigma)a \leq a$$

**Theorem 4.** *If the conditions 1-4 of theorem 4 hold and*

5) *there exists* $L > 0$ *such that if* $u_i = v_i + w_i$, $(v_i, w_i) \in C, i = 1, 2$ *then* $\|Nu_1 - Nu_2\| \leq L \|u_1 - u_2\|$

6) $\mu = \|D^{-1}\| \sigma + (1 + \|D^{-1}P\| L)(\theta + \ldots + \theta^{p+1}) < 1$, *where* $\theta = \|H(I-P)\| L$,

*then* $T$ *is a contraction.*

*Proof.* Let $T(v_i, w_i) = (V_i, W_i), i = 1, 2$. We have

$$\|W_1 - W_2\| \leq \|H(I-P)\| L(\|v_1 - v_2\| + \|w_1^p - w_2^p\|)$$

But

$$\|w_1^p - w_2^p\| \leq (\|v_1 - v_2\| + \|w_1^{p-1} - w_2^{p-1}\|) \|H(I-P)\| L$$

thus

$$\|W_1 - W_2\| \leq \|v_1 - v_2\| (\theta + \ldots + \theta^{p+1}) + \theta^{p+1} \|w_1 - w_2\|$$

Consequently,

$$\|V_1 - V_2\| + \|W_1 - W_2\| \leq \left[\left\|D^{-1}\right\| \sigma + (1 + \left\|D^{-1}P\right\| L)(\theta + \ldots + \theta^{p+1})\right] \|v_1 - v_2\| +$$
$$+(1 + \left\|D^{-1}P\right\| L)\theta^{p+1} \|w_1 - w_2\| \leq \mu (\|v_1 - v_2\| + \|w_1 - w_2\|)$$

Hence $T$ has an unique fixed point $(v, w) = (V, W) \in C$ that may be obtained by the iteration procedure $(v_{k+1}, w_{k+1}) = T(v_k, w_k)$.

**Theorem 5.** *If the conditions of the theorems 4,5 hold, then* $u = V + W$ *is a solution of the equation* $Eu = Nu$.

*Proof.* We have $w^1 = W, \ldots, w^p = W$, that is $W = H(I - P)N(V + W)$. Then $V = V - D^{-1}P(EV - N(V + W))$ and consequently, $P(EV - N(V + W)) = 0$ and $Eu = Nu$ from theorem 3.

### 3. The optimization of the numerical computation of the solutions

We approximate the $2\pi$−periodic solutions of the equation

$$-u''(t) = f(t, u(t))$$

Let $X$ be the Banach space of $2\pi$−periodic, continuous functions $u : \mathbb{R} \to \mathbb{R}, \|u\| = \sup_{t\in[0,2\pi]} |u(t)|$, $f$ a continuous, $2\pi$− periodic function on $t$, differentiable in $u$, with locally bounded derivative. Let $X_E = H^2(0, 2\pi), X_N = X, Eu = -u'', Nu = f(\cdot, u)$.

If $u \in X$ let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt)$$

his Fourier series. We define, for $m \in \mathbb{N}$,

$$P_m u = \frac{a_0}{2} + \sum_{k=1}^{m}(a_k \cos kt + b_k \sin kt)$$

$$H(I - P_m)u = \sum_{k=m+1}^{\infty} (a_k \cos kt + b_k \sin kt)/k^2$$

¿From [3] we have $\|H(I - P_m)\| \to 0$ when $m \to \infty$. For an approximation $v_0 \in P_m X$ we define the sequence $w^s = H(I - P_m)N(v + w^{s-1})$, $w^0 = 0$, for $s = 1, 2, \ldots, p + 1$. If $m$ is sufficiently great then $(v, w^s) \in C$ if $\|v - v_0\| \leq a$, $v \in P_m X$. The second equation $P(EPu - Nu) = 0$ becomes an equation for the Fourier coefficients of $v$, $F(c) = 0$, where $c = (a_0/2, a_1, b_1, \ldots a_m, b_m)$.

If the Jacobian $J(c_0)$ of $F$ in $v_0$ is invertible, let $D = J(c_0)$ and we use a theorem of Urabe [4]:

**Theorem 6.** *Let us consider the system* $F(c) = 0, F = (F_1, \ldots, F_n)$, $c = (c_1, \ldots, c_n)$ *for* $n \in \mathbb{N}$. *We suppose that* $F \in C^1(\Omega)$ *and that there exists* $k \in [0, 1)$ *and* $\delta > 0$ *such that*

1. $\Omega_\delta = \{c \in P_m X \, |||c - c_0|| \leq \delta\} \subset \Omega$
2. $||J(c) - J(c_0)|| \leq k/M$
3. $Mr/(1-k) \leq \delta$

*where* $M \geq \left\|J^{-1}(c_0)\right\|, r \geq ||F(c_0)||$.

*Then the system* $F(c) = 0$ *has an unique solution* $\bar{c} \in \Omega_\delta$ *and* $||\bar{c} - c_0|| \leq Mr/(1-k)$.

For a sufficiently great $m$ the conditions of theorems 4,5 are consequences of the hypothesis of the Urabe's theorem. Hence the $2\pi$-periodic solution $u$ of the equation $-u'' = f(t, u)$ is $u = V + W$, where $W$ is obtained by a fixed point iteration procedure for $P_m + H(I - P_m)N$ and $V$ is obtained by the Newton's algorithm for the system $F(c) = 0$ (every step requires the iterations for $W$).

We consider the following error sources:

a) The computation of the Fourier coefficients (cf. [5]) of $w^s = H(I - P_m)N(v + w^{s-1})$.

**Theorem 7.** *If* $g(t)$ *is* $p$ *times continuously differentiable,* $2\pi$*-periodic and his Fourier series is*

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt)$$

*then the Fourier coefficients may be approximated by*

$$a_k \approx \frac{1}{N}\sum_{i=1}^{2N} g(t_i) \cos nt_i \qquad b_k \approx \frac{1}{N}\sum_{i=1}^{2N} g(t_i) \sin nt_i$$

*where* $t_i = (2i-1)\pi/(2N), i = 1, ..., 2N$ *and* $k = 1, ..., N-1$ *and the approximation error is*

$$2\sigma_p(N-1)\left[\frac{1}{2\pi}\int_0^{2\pi} g^{(p)}(t)^2 dt\right]^{1/2}$$

*where*

$$\sigma_p(N-1) = \sqrt{2}\left[\frac{1}{N^{2p}} + \frac{1}{(N+1)^{2p}} + ...\right]^{1/2} < \sqrt{\frac{2}{2p-1}}(N-1)^{-p+1/2}$$

b) The truncation of the Fourier series at rank $N-1$ (cf. [5]). We have

$$\left|g(t) - \frac{a_0}{2} - \sum_{k=1}^{N-1}(a_k \cos kt + b_k \sin kt)\right| \le \sigma_p(N-1)\left[\frac{1}{2\pi}\int_0^{2\pi} g^{(p)}(t)^2 dt\right]^{\frac{1}{2}}$$

Consequently, if $w^s$ is approximated by $\widetilde{w}^s$ we have

$$\|w^s - \widetilde{w}^s\| \le 2\sqrt{N-m}\sigma_p(N-1)\left[\tfrac{1}{2\pi}\int_0^{2\pi} N\left(v + w^{s-1}\right)^{(p)}(t)^2 dt\right]^{\frac{1}{2}}(\sigma(m) - \sigma(N)) +$$
$$+\sigma_p(N-1)\left[\tfrac{1}{2\pi}\int_0^{2\pi} N\left(v + w^{s-1}\right)^{(p)}(t)^2 dt\right]^{\frac{1}{2}}$$

where

$$\sigma(m) = \left(\sum_{i=m+1}^{\infty}\frac{1}{i^2}\right)^{\frac{1}{2}}$$

At every step we have an error $\varepsilon_s \le \mathcal{K}(N_s - 1)^{-p+1/2}$, where

$$\mathcal{K} = \sqrt{\frac{2}{2p-1}}\left(1 + 2\sqrt{N-m}\right)\sigma(m)\max_{s\le S}\left[\frac{1}{2\pi}\int_0^{2\pi} N\left(v + w^{s-1}\right)^{(p)}(t)^2 dt\right]^{\frac{1}{2}}$$

if $N_s \le N$ for $s = 1, 2, ..., S$.

The whole error for $S$ iterations is

$$\|W - \widetilde{w}^{s+1}\| \le \frac{\theta^{s+1}b}{1-\theta} + \sum_{i=0}^{S}\frac{\theta^S}{\theta^i}\frac{\mathcal{K}}{(N_i - 1)^{p-1/2}} \equiv \varepsilon_0$$

for a computational effort proportional to $2(N_0 + ... + N_S)$.

Our problem is now to minimize this effort for a given error $\varepsilon_0$. Let $S \in \mathbb{N}$. We have to minimize $N_0 + ... + N_S$ if

$$\sum_{i=0}^{S}\frac{1}{\theta^i(N_i - 1)^{p-1/2}} = \frac{\frac{\varepsilon_0}{\theta^S} - \frac{b\theta}{1-\theta}}{\mathcal{K}} \equiv A_S$$

By the Lagrange multipliers rule, let

$$L = N_0 + ... + N_S + \lambda \left( \sum_{i=0}^{S} \frac{1}{\theta^i (N_i - 1)^{p-1/2}} - A_S \right)$$

We have the system

$$1 - \frac{\lambda \left(p - \frac{1}{2}\right)}{\theta^i (N_i - 1)^{p+1/2}} = 0 \text{ for } i = 0, 1, ..., S$$

$$\sum_{i=0}^{S} \frac{1}{\theta^i (N_i - 1)^{p-1/2}} = A_S$$

from where

$$N_i = 1 + \frac{\left(\theta^{-\frac{2(S+1)}{2p+1}} - 1\right)^{\frac{2}{2p-1}}}{A_S^{\frac{2}{2p-1}} \theta^{i \frac{2}{2p+1}} \left(\theta^{-\frac{2}{2p+1}} - 1\right)^{\frac{2}{2p+1}}}$$

for $i = 0, 1, ..., S$. Now we can choose $S$ for which the computing effort is minimum.

As an example, let us consider the equation (cf. [5])

$$u'' = \sin t - u^3 (t)$$

For $m = 1, p = 2$, $\theta = 0.4$, $N_0 = 4$, $N_1 = 4$, $N_2 = 5$, $N_3 = 7$, $N_4 = 10$, $N_5 = 14$, $N_6 = 20$ and at every step the fixed point $W$ was obtained by 64 evaluations of $Nu \equiv \sin u - u^3$ (instead of 120 evaluations if at every step we choose $N = 20$, for the same precision).

## References


[1] Harder, A.M., Hicks, T.L., *Stability results for fixed point iteration procedures,* Math. Japonica, **33**, 5, 1988, pp. 693-706.

[2] Sanchez, D.A., *An iteration scheme for boundary value alternative problems,* Rep. 1412, Jan., 1984, Univ. Wisconsin.

[3] Cesari, L., *Functional analysis and Galerkin's method,* Mich. Math.J., **11**, 1984.

[4] Urabe, M., *Galerkin procedure for nonlinear periodic systems,* Arch. rat. mech. anal., **20**, 2, 1965, pp. 120-152.

[5] Urabe, M., Reiter, A., *Numerical computation of nonlinear forced oscillations by Galerkin's procedure,* J. Math. Anal. Appl., **14**, 1, 1966, pp. 107-140.

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