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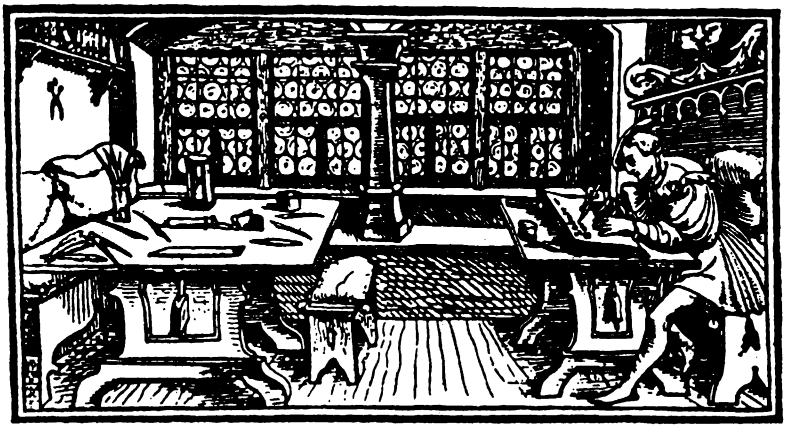

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STUDIA UNIV. "BABEŞ-BOLYAI", MATHEMATICA, Volume XLIII, Number 3, September 1998

PROFESSOR PAVEL ENGHIŞ AT HIS SEVENTIETH ANNIVERSARY

DORIN ANDRICA

The academic teaching and the mathematical research in Geometry at "Babeş-Bolyai" University of Cluj-Napoca in the last decades were influenced in a significant way by the life and scientifical work of Professor Pavel Enghiş, who was just celebrated his 70^{th} birthday.

The main purpose of this short presentation is to give some biographical dates concerning his life as well as to point out some important moments of his scientifical work.

Professor Pavel Enghiş is born on September 28, 1928, in Letca, a Transylvanian nice village situated on the river Someş. He attended the High School in Gherla and the Faculty of Mathematics and Physics at the "Victor Babeş" University of Cluj. In October 1951 he was promoted on a assistant position at the Chair of Geomtry, which was directed by Professor Tiberiu Mihăilescu. In 1957 he obtained a lecturer professor position in the same Faculty. The two universities of Cluj, the romanian one "Victor Babeş" and the hungarian one "Janos Bolyai", were unified in 1959 in the present "Babeş-Bolyai" University. Professor Pavel Enghiş' activity is connected more than 45 years on this University. He was appointed as associated professor at the Chair of Geometry in 1990 and as full professor at the same Chair in 1991. He was the Chief of Chair of Geometry until 1995 when he was retired.

During his activity at "Babeş-Bolyai" University, Professor Pavel Enghiş has teached many mathematical courses. Let us mention here: Special Mathematics, Geometry, Linear Algebra, Differential Geometry, Basic Geometry, Arithmetic and Foundations of Geometry, General Mathematics (for students in Phylosophy, Chemistry and Geology). His lectures were always very rigurous and straightful, being very appreciated by the students. Several of his lectures were published for didactical goals. We mention here the courses in Differential Geometry (1969) (with E. Frăţilă), (1985) (with M. Ţarină) as well as the Lectures on General Mathematics for Geologists I, II (1982) (with M. Balász, G. Goldner, Gr. Sălăgean).

The scientifical work of Professor Pavel Enghiş is related to the field of Geometry, mainly to Differential Geometry of Riemannian and affinely connected manifolds. Since now, he published more than 89 notes and papers in different volumes or mathematical journals, the most of them concerne the following topics:

1. The class of Riemannian spaces
2. The group of motion of Riemannian or Presudo-Riemannian spaces
3. The holomony groups of the affinely connected manifolds
4. The equivalence of the affinely connected manifolds
5. Affine connected spaces with recurrent tensors
6. Special classes of affine connections. Enghiş connections
7. The Geometry of Finsler Spaces

Now, when Professor Pavel Enghiş is 70 years old, he is an active presence in the teaching staff of our Faculty. All the colleagues and students take this opportunity to bring him a profound homage and to wish him full health and successes.

The table of publications of Professor Pavel Enghiş

I. Lectures

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2. Culegere de probleme de matematică pentru studenţii din anul pregătitor, Cluj-Napoca, 1977 (în colaborare)
3. Matematică I, II, Cluj-Napoca, 1982 (cu M. Balasz, G. Goldner, Gr. Sălăgean)
4. Curs de geometrie diferenţială, Cluj-Napoca, 1985 (cu M. Ţarină)

II. Scientifical papers

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HIGHER ORDER EINSTEIN-SCHRÖDINGER SPACES

ATANASIU GHEORGHE

Dedicated to Professor Pavel Enghiş at his 70th anniversary

Abstract. In 1945 A.Einstein [6] and E.Schrödinger [10] started form a generalized Riemann space, thas is, a space M associated with a nonsymmetric tensor $G_{ij}(x)$ and desired to find the set of all linear connections $\Gamma^i_{jk}(x)$ compatible with such a metric : $G_{ij/k} = 0$ (see also [1] and [2]).The geometry of this space $(M.G_{ij})$ is called **the Einsten - Schrödinger's geometry** [3], [4].

The purpose of this paper is to discuss a nonsymmetric tensor field $G_{ij}(x, y^{(1)}, ..., y^{(k)})$,where $(x, y^{(1)}, ..., y^{(k)})$ is a point of the k-osculator bundle $(Osc^k M, \pi, M)$ and to obtain the results for the Einstein - Schrödinger's geometry of the higher order in a natural case.

The fundamental notions and notations concerning the osculator bundle of the higher order are given in the papers [8] [9] and in the recent Miron's book [7] and we suppose them to be known.

For a nonsymmetric tensor field $G_{ij}(x, y^{(1)}, ..., y^{(k)})$ on $Osc^k M$,we have a symmetric tensor field $g_{ij}(x, y^{(1)}, ..., y^{(k)})$ and a skew-symmetric one $a_{ij}(x, y^{(1)}, ..., y^{(k)})$ from the spliting

$$G_{ij} = g_{ij} + a_{ij}, \tag{1}$$

where we suppose that

$$det \| g_{ij}(x, y^{(1)}, ..., y^{(k)})\| \cdot \|a_{ij}(x, y^{(1)}, ..., y^{(k)}) \| \neq 0 \tag{2}$$

and dim M = n = 2n'.

1991 *Mathematics Subject Classification.* 53C60, 53B50, 53C80.

Key words and phrases. osculator bundle, Einstein-Schrödinger geometry, higher order Lagrange spaces.

We denote

$$||g_{ij}(x, y^{(1)}, ..., y^{(k)})||^{-1} = ||g^{ij}(x, y^{(1)}, ..., y^{(k)})||,$$

$$||a_{ij}(x, y^{(1)}, ..., y^{(k)})||^{-1} = ||a^{ij}(x, y^{(1)}, ..., y^{(k)})||$$

We have from $G_{ij|k} = 0$, $G_{ij} \overset{(\alpha)}{|}_k = 0 \quad (\alpha = 1, ..., k)$ the following equations :

$$(3) \qquad g_{ij|k} = 0 \ , \ g_{ij} \overset{(\alpha)}{|}_k = 0 \ , \ a_{ij|k} = 0 \ , \ a_{ij} \overset{(\alpha)}{|}_k = 0 \quad (\alpha = 1, ..., k) \ ,$$

which is equivalent to

$$(4) \qquad g^{ij}_{\ \ |k} = 0 \ , \ g^{ij} \overset{(\alpha)}{|}_k = 0 \ , \ a^{ij}_{\ \ |k} = 0 \ , \ a^{ij} \overset{(\alpha)}{|}_k = 0 \quad (\alpha = 1, ..., k).$$

We investigate the set of all N-linear connections $D\Gamma(N) = (L^i{}_{jk}, \underset{(\alpha)}{C^i{}_{jk}})$ $(\alpha = 1, ..., k)$ for which we have (3) in the form

$$L^i{}_{jk} = \overset{\circ}{L^i}{}_{jk} + A^i_{jk} , \quad \underset{(\alpha)}{C^i{}_{jk}} = \underset{(\alpha)}{\overset{\circ}{C^i{}_{jk}}} + B^i_{jk} \quad (\alpha = 1, ..., k) \ ,$$

where $D \overset{\circ}{\Gamma} (N) = (\overset{\circ}{L^i}{}_{jk}, \underset{(\alpha)}{\overset{\circ}{C^i{}_{jk}}}) \ (\alpha = 1, ..., k)$ is a fixed N-linear connection on $Osc^k M$ and A^i_{jk} , $\underset{(\alpha)}{B^i_{jk}}$ are arbitrary tensor fields of type (1,2).

We obtain for A and for $\underset{(\alpha)}{B}$ the equations

$$(5) \qquad A^r_{ik} g_{rj} + A^r_{jk} g_{ir} = g_{ij \overset{\circ}{|} k} \ , \ A^r_{ik} a_{rj} + A^r_{jk} a_{ir} = a_{ij \overset{\circ}{|} k},$$

$$(6)\quad \begin{cases} \underset{(\alpha)}{B^r_{ik}}\, g_{rj} + \underset{(\alpha)}{B^r_{jk}}\, g_{ir} = g_{ij} \overset{(\alpha)}{\underset{o}{|}}_k , \\ \underset{(\alpha)}{B^r_{ik}}\, a_{rj} + \underset{(\alpha)}{B^r_{jk}}\, a_{ir} = a_{ij} \overset{(\alpha)}{\underset{o}{|}}_k \ (\alpha = 1, ..., k) \end{cases}$$

We do not know the general solution of the equation system (5) and (6) We give a solution for these equations in the following special case.

Definition 1. *An assymmetric metric (1) is called* **natural** *if we have*

$$(7)\qquad \Lambda^{rk}_{is}\Phi^{hs}_{rj} = \Phi^{rk}_{is}\Lambda^{hs}_{rj}$$

where

$$(8)\qquad \Lambda^{kh}_{ij} = \frac{1}{2}(\delta^k_i\delta^h_j - g_{ij}g^{kh}) , \ \Phi^{kh}_{ij} = \frac{1}{2}(\delta^k_i\delta^h_j - a_{ij}a^{kh}).$$

Theorem 1. *An assymmetric metric $G_{ij}(x, y^{(1)}, ..., y^{(k)})$ on $Osc^k M$ is natural if and only if there exist a function $\mu(x, y^{(1)}, ..., y^{(k)})$ on $Osc^k M$ such that*

$$(9)\qquad g_{ir}g_{js}a^{rs} = \mu g_{ij}.$$

Examples.

1. Let $f^i_j(x, y^{(1)}, ..., y^{(k)})$ be a tensor field of type (1,1) which gives an almost complex d-structure on $Osc^k M : f^2 = -\delta$. If we put:

$$(10)\qquad a_{ij} = f^r_i g_{rj},$$

then $a_{ij}(x, y^{(1)}, ..., y^{(k)})$ is alternating and $G_{ij} = g_{ij} + a_{ij}$ is an asymmetric metric on $Osc^k M$. In this case $\mu = -1$.

2. Let $q^i_j(x, y^{(1)}, ..., y^{(k)})$ be a tensor field of type (1,1) which gives an almost product d-structure on $Osc^k M : q^2 = +\delta$. If we put:

$$(11)\qquad a_{ij} = q^r_i g_{rj}$$

then $a_{ij}(x, y^{(1)}, ..., y^{(k)})$ is alternate and $G_{ij} = g_{ij} + a_{ij}$ is an asymmetric metric on $Osc^k M$. In this case $\mu = +1$.

Theorem 2. *If there exist a N-linear connection on $Osc^k M$ compatible with a natural asymmetric metric $G_{ij}(x, y^{(1)}, ..., y^{(k)})$, then the function μ is constant.*

Definition 2. *A natural asymmetric metric (1) is called* **elliptic** *if $\mu = -c^2$ and* **hyperbolic** *if $\mu = c^2$, where c is a positive constant.*

The converse of Theorem 2 holds as follows:

Theorem 3. *If a natural asymmetric metric (1) is elliptic or hyperbolic, then there exist N-linear connections $D\tilde{\Gamma}(N) = (\tilde{L}^i{}_{jk}, \underset{(\alpha)}{\tilde{C}^i{}_{jk}})$ compatible with $G_{ij}(x, y^{(1)}, ..., y^{(k)})$.*

Let $D\overset{\circ}{\Gamma}(N) = (\overset{\circ}{L^i}{}_{jk}, \underset{(\alpha)}{\overset{\circ}{C^i}{}_{jk}})$ be a given N-linear connection, then in the elliptic case we have

$$
(12)\quad
\begin{cases}
\tilde{L}^i{}_{jk} = \overset{\circ}{L^i}{}_{jk} + \frac{1}{4}\{g^{ir} g_{rj \overset{\circ}{|} k} + a^{ir} a_{rj \overset{\circ}{|} k} + f^r_j f^i_{r \overset{\circ}{|} k}\} \\[2ex]
\underset{(\alpha)}{\tilde{C}^i{}_{jk}} = \underset{(\alpha)}{\overset{\circ}{C^i}{}_{jk}} + \frac{1}{4}\{g^{ir} g_{rj} \overset{(\alpha)}{\overset{\circ}{|}}_k + a^{ir} a_{rj} \overset{(\alpha)}{\overset{\circ}{|}}_k + f^r_j f^i_r \overset{(\alpha)}{\overset{\circ}{|}}_k\}
\end{cases}
$$

$(\alpha = 1, ..., k)$, *and in the hyperbolic case we have*

$$
(13)\quad
\begin{cases}
\tilde{L}^i{}_{jk} = \overset{\circ}{L^i}{}_{jk} + \frac{1}{4}\{g^{ir} g_{rj \overset{\circ}{|} k} + a^{ir} a_{rj \overset{\circ}{|} k} - q^r_j q^i_{r \overset{\circ}{|} k}\} \\[2ex]
\underset{(\alpha)}{\tilde{C}^i{}_{jk}} = \underset{(\alpha)}{\overset{\circ}{C^i}{}_{jk}} + \frac{1}{4}\{g^{ir} g_{rj} \overset{(\alpha)}{\overset{\circ}{|}}_k + a^{ir} a_{rj} \overset{(\alpha)}{\overset{\circ}{|}}_k - q^r_j q^i_r \overset{(\alpha)}{\overset{\circ}{|}}_k\}
\end{cases}
$$

$(\alpha = 1, ..., k)$

Theorem 4. *The set of all N-linear connections $D\overset{*}{\Gamma}(N) = (\overset{*}{L}{}^i_{jk}, \overset{*}{\underset{(\alpha)}{C}}{}^i_{jk})$ compatible with a natural asymmetric metric (1) on $Osc^k M$ is given by*

$$\overset{*}{L}{}^i_{jk} = \tilde{L}^i{}_{jk} + \Lambda^{pi}_{jq}\Phi^{rq}_{ps}Y^s_{rk}, \quad \overset{*}{\underset{(\alpha)}{C}}{}^i_{jk} = \underset{(\alpha)}{\tilde{C}}{}^i_{jk} + \Lambda^{pi}_{jq}\Phi^{rq}_{ps}\underset{(\alpha)}{Z}{}^s_{rk} \tag{14}$$

where $D\tilde{\Gamma}(N)$ is the N-linear connection in Theorem 3 and Y^i_{jk}, $\underset{(\alpha)}{Z}{}^i_{jk}$ $(\alpha = 1, ..., k)$ are arbitrary tensor fields on $Osc^k M$.

If we put $D\overset{o}{\Gamma}(N) = D\overset{c}{\Gamma}(N) = (\overset{c}{L}{}^i{}_{jk}, \overset{c}{\underset{(\alpha)}{C}}{}^i{}_{jk})$, $(\alpha = 1, ..., k)$ for $g_{ij}(x, y^{(1)}, ..., y^{(k)})$, that is:

$$\begin{cases} \overset{c}{L}{}^i{}_{jk} = \frac{1}{2}g^{is}\left(\frac{\delta g_{js}}{\delta x^k} + \frac{\delta g_{sk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^s}\right), \\ \overset{c}{\underset{(\alpha)}{C}}{}^i{}_{jk} = \frac{1}{2}g^{is}\left(\frac{\delta g_{js}}{\delta y^{(\alpha)k}} + \frac{\delta g_{sk}}{\delta y^{(\alpha)j}} - \frac{\delta g_{jk}}{\delta y^{(\alpha)s}}\right), \end{cases} \tag{15}$$

the generalized Christoffel symbols we have:

Theorem 5. *The canonical N-linear connection compatible with a natural asymmetric metric $G_{ij}(x, y^{(1)}, ..., y^{(k)})$ is given in the elliptic case by:*

$$\begin{cases} L^i{}_{jk} = \overset{c}{L}{}^i{}_{jk} + \frac{1}{4}\{a^{ir}a_{rj\overset{c}{|}k} + f^r_j f^i{}_{r\overset{c}{|}k}\} \\ \underset{(\alpha)}{C}{}^i{}_{jk} = \overset{c}{C}{}^i{}_{jk} + \frac{1}{4}\{a^{ir}a_{rj}\overset{(\alpha)}{\overset{c}{|}}{}_k + f^r_j f^i_r \overset{(\alpha)}{\overset{c}{|}}{}_k\} \ (\alpha = 1, ..., k) \end{cases} \tag{16}$$

and in the hyperbolic case by

$$(17)\quad \begin{cases} L^i{}_{jk} = \overset{c}{L^i}{}_{jk} + \frac{1}{4}\{a^{ir}a_{rj|k}^{\;c} - q^r_j q^i{}_{r|k}^{\;c}\} \\ \underset{(\alpha)}{C^i{}_{jk}} = \underset{(\alpha)}{\overset{c}{C^i}{}_{jk}} + \frac{1}{4}\{a^{ir}a_{rj} \overset{(\alpha)}{\underset{c}{|}}{}_k - q^r_j q^i_r \overset{(\alpha)}{\underset{c}{|}}{}_k\}, \quad (\alpha = 1, ..., k) \end{cases}$$

Now, the Einstein equations, electromagnetic tensors, Maxwell equations for the higher order Einstein-Schrödinger geometry can be studied using these canonical N-linear connections.

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ON BIRECURRENT WEYL SPACES

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Dedicated to Professor Pavel Enghiş at his 70th anniversary

Abstract. In this paper, birecurrent Weyl spaces are defined and it is proved that the birecurrence tensor of a birecurrent Weyl space is symmetric if and only if the space is Riemannian. Moreover, some results concerning birecurrent hypersurfaces of a birecurrent Weyl space are obtained.

1. Introduction.

An n-dimensional manifold W_n is said to be a Weyl space if it has a conformal metric tensor g_{ij} and a symmetric connection ∇ satisfying the compatibility condition given by the equation

$$\nabla_k g_{ij} - 2T_k g_{ij} = 0 \tag{1.1}$$

where T_k denotes a covariant vector field and $\nabla_k g_{ij}$ denotes the usual covariant derivative.

Under a renormalization of the fundamental tensor of the form

$$\breve{g}_{ij} = \lambda^2 g_{ij} \tag{1.2}$$

the complementary vector T_k is transformed by the law

$$\breve{T}_k = T_k + \partial_k \ln \lambda \tag{1.3}$$

where λ is a function defined on W_n.

A quantity A is called a satellite with weight $\{p\}$ of the tensor g_{ij}, if it admits a transformation of the form

$$\breve{A} = \lambda^p A \tag{1.4}$$

under the renormalization (1.2) of the metric tensor g_{ij} ([1], [2]).

The prolonged covariant derivative of a satellite A of the tensor g_{ij} with weight $\{p\}$ is defined by ([1], [2])

$$\dot{\nabla}_k A = \nabla_k A - pT_k A. \tag{1.5}$$

We note that the prolonged covariant derivative preserves the weight.

According to Norden [3], we have

$$x^a_{,ijk} - x^a_{,ikj} = R^h_{ijk} x^a_h \tag{1.6}$$

where R^h_{ijk} is the curvature tensor of the Weyl space defined by

$$R^h_{ijk} = \frac{\partial}{\partial x^j}\Gamma^h_{ik} - \frac{\partial}{\partial x^k}\Gamma^h_{ij} + \Gamma^h_{lj}\Gamma^l_{ik} - \Gamma^h_{lk}\Gamma^l_{ij}. \tag{1.7}$$

The first and the second Bianchi identities for Weyl spaces are, by [4],

$$R^h_{ijk} + R^h_{kij} + R^h_{jki} = 0 \tag{1.8}$$

$$\dot{\nabla}_r R^i_{jkh} + \dot{\nabla}_k R^i_{jhr} + \dot{\nabla}_h R^i_{jrk} = 0. \tag{1.9}$$

2. Birecurrent Weyl Spaces

A Weyl space $W_n(g_{ij}, T_k)$ is called **recurrent**,[4], if the curvature tensor satisfies the following condition for some non-zero covariant vector field ϕ_s $(\neq T_s)$:

$$\dot{\nabla}_s R^h_{ijl} = \phi_s R^h_{ijl}. \tag{2.1}$$

We call a non-flat Weyl space $W_n(g_{ij}, T_k)$ **birecurrent** if the curvature tensor satisfies the condition

$$\dot{\nabla}_r \dot{\nabla}_s R^h_{ijk} = \phi_{sr} R^h_{ijk} \tag{2.2}$$

for some non-zero covariant tensor field ϕ_{sr}. Transvecting (2.2) by g_{hl} and remembering that the prolonged covariant differentiation preserves the metric, we obtain the equivalent form of (2.2) as

$$\dot{\nabla}_r \dot{\nabla}_s R_{lijk} = \phi_{sr} R_{lijk} \ , \ R_{lijk} = g_{lh} R^h_{ijk}. \tag{2.2'}$$

It is easy to see that a recurrent Weyl space is birecurrent. In fact, by taking the prolonged covariant derivative of (2.1) with respect to u^r, we get

$$\dot{\nabla}_r \dot{\nabla}_s R^h_{ijk} = (\phi_{s,r} + \phi_s \phi_r) R^h_{ijk}. \tag{2.3}$$

with $\phi_{sr} = \phi_{s,r} + \phi_s \phi_r$.

We examine Weyl spaces which satisfy (2.2), but not (2.1).

We remark that the definition of a birecurrent Weyl space agrees with that of a birecurrent Riemannian space if we take the complementary vector field of $W_n(g_{ij}, T_k)$ as zero.

Theorem 2.1. *The birecurrency tensor of a birecurrent Weyl space with a non-vanishing scalar curvature is symmetric if and only if the space is locally Riemannian.*

Proof. Assume ϕ_{sr} is a symmetric tensor. Transvecting (2.2)′ by $g^{hj} g^{ik}$ and remembering that the Ricci tensor R_{ij} and the scalar curvature R of the Weyl space are respectively defined by $R_{ij} = R^h_{ihj}$, $R = R_{ij} g^{ij}$, we get

$$\dot{\nabla}_r \dot{\nabla}_s R = \phi_{sr} R. \tag{2.4}$$

Changing the order of the indices r and s in (2.4) and subtracting the expression so obtained from (2.4), we have

$$\dot{\nabla}_{[r} \dot{\nabla}_{s]} R = \phi_{[sr]} R.$$

where the bracket indicates antisymmetrization.

Since, by assumption, ϕ_{sr} is a symmetric tensor, we get

$$\dot{\nabla}_{[r} \dot{\nabla}_{s]} R = 0. \tag{2.4$'$}$$

Expanding $\dot{\nabla}_{[r} \dot{\nabla}_{s]} R$ and remembering that R is a satellite of g_{ij} with weight $\{-2\}$, we find that

$$\dot{\nabla}_{[r} \dot{\nabla}_{s]} R = \nabla_{[r} \nabla_{s]} R + 2\nabla_{[r} T_{s]} R = 0 \text{ with } \nabla_{[r} \nabla_{s]} \mathrm{R} = 0.$$

Since $R \neq 0$, we have

$$\nabla_{[s} T_{r]} = 0$$

which means that W_n is locally Riemannian.

The sufficiency of the condition is a well-known fact from the Riemannian Geometry [5]. □

Corollary 2.1. *If* $\dot{\nabla}_r \dot{\nabla}_s R_{ijkh} = 0$, *then the Weyl space is locally Riemannian.*

Corollary 2.2. *If* $\dot{\nabla}_{[r} \dot{\nabla}_{s]} R_{ijkh} = 0$, *then the Weyl space is locally Riemannian.*

Theorem 2.2. *The birecurency tensor* ϕ_{is} *of a birecurrent Weyl space is the solution of the equation*

$$\phi_{is}(R^{i}_{..h} - R^{i}_{.h} + \delta^i_h R) = 0$$

where $R^{i}_{..h} = R^i_{jkh} g^{jk}$, $R^{i}_{.h} = R_{kh} g^{ik}$ *and* $R = R_{jk} g^{jk}$.

Proof. By taking the prolonged covariant derivative of (1.9) with respect to u^s, we get

$$\dot{\nabla}_s \dot{\nabla}_r R^i_{jkh} + \dot{\nabla}_s \dot{\nabla}_k R^i_{jhr} + \dot{\nabla}_s \dot{\nabla}_h R^i_{jrk} = 0 \tag{2.5}$$

from which, by (2.2), it follows that

$$\phi_{rs} R^i_{jkh} + \phi_{ks} R^i_{jhr} + \phi_{hs} R^i_{jrk} = 0. \tag{2.6}$$

Contracting (2.6) with respect to i and r and remembering that

$$R_{ij} = R^h_{ihj} \text{ and } R^h_{ilh} = -R^h_{ihl}$$

we get

$$\phi_{is} R^i_{jkh} - \phi_{ks} R_{jh} + \phi_{hs} R_{jk} = 0. \tag{2.7}$$

or,

$$\phi_{is}(R^i_{jkh} - \delta^i_k R_{jh} + \delta^i_h R_{jk}) = 0. \tag{2.8}$$

Transvection of (2.8) by g^{jk} yields

$$\phi_{is}(R^{i}_{..h} - R^{i}_{.h} + \delta^i_h R) = 0$$

where $R^{i}_{..h} = R^i_{jkh} g^{jk}$, $R^{i}_{.h} = R_{kh} g^{ik}$ and $R = R_{jk} g^{jk}$. □

Corollary 2.3. *If* $\det A^i_h \neq 0$, *then* W_n *is Riemannian, where* $A^i_h = R^{i}_{..h} - R^{i}_{.h} + \delta^i_h R$.

3. Hypersurfaces of Birecurrent Weyl Spaces.

Let $W_n(g_{ij}, T_k)$ be a hypersurface, with coordinates $u^i (i = 1, 2, \cdots, n)$ of a Weyl space $W_{n+1}(g_{ab}, T_c)$ with coordinates $x^a (a = 1, 2, \cdots, n+1)$. The metrics of W_n and W_{n+1} are connected by the relations

$$g_{ij} = g_{ab} x_i^a x_j^b \ (i, j = 1, 2, \cdots, n \ ; \ a, b = 1, 2, \cdots, n+1) \tag{3.1}$$

where x_i^a denotes the covariant derivative of x^a with respect to u^i.

It is easy to see that the prolonged covariant derivative of a satellite A, relative to W_n, and W_{n+1}, are related by

$$\dot{\nabla}_k A = x_k^c \dot{\nabla}_c A \ (k = 1, 2, \cdots, n \ ; \ c = 1, 2, \cdots, n+1) \tag{3.2}$$

Let n^a be the contravariant components of the vector field of W_{n+1} normal to W_n which is normalized by the condition

$$g_{ab} n^a n^b = 1. \tag{3.3}$$

The moving frame $\{x_a^i, n_a\}$ in W_n, reciprocal to the moving frame $\{x_i^a, n^a\}$ is defined by the relations [3]

$$n_a x_i^a = 0 \ n^a x_a^i = 0 \ x_i^a x_a^j = \delta_i^j. \tag{3.4}$$

Remembering that the weight of x_i^a is $\{0\}$, the prolonged covariant derivative of x_i^a with respect to u^k is found as

$$\dot{\nabla}_k x_i^a = \nabla_k x_i^a = \omega_{ik} n^a \tag{3.5}$$

where ω_{ik} is the second fundamental form. It can be shown that ω_{ik} is a satellite of g_{ij} with weight $\{1\}$.

The generalized Gauss and Mainardi-Codazzi equations are obtained in [4], respectively as

$$R_{pijk} = \Omega_{pijk} + \overline{R}_{dbce} x_p^d x_i^b x_j^c x_k^e \tag{3.6}$$

$$\dot{\nabla}_k \omega_{ij} - \dot{\nabla}_j \omega_{ik} + \overline{R}_{dbce} x_i^b x_j^c x_k^e n^d = 0, \tag{3.7}$$

where $\overline{R}_{dbce}$ is the covariant curvature tensor of W_{n+1} and Ω_{pijk} is the Sylvesterian of ω_{ij} defined by $\Omega_{pijk} = \omega_{pj}\omega_{ik} - \omega_{pk}\omega_{ij}$.

In the following we will use the notation

$$B^{ab\cdots cd}_{ij\cdots kl} = x^a_i x^b_j \cdots x^c_k x^d_l \tag{3.8}$$

the same as in [6].

Theorem 3.1. *For a hypersurface of a birecurrent Weyl space W_{n+1} with birecurence tensor ϕ_{ef} we have the identity*

$$\dot{\nabla}_r\dot{\nabla}_s R_{ijkl} - \phi_{rs}R_{ijkl} = \dot{\nabla}_r\dot{\nabla}_s\Omega_{ijkl} - \phi_{rs}\Omega_{ijkl} + S_{ijkl(sr)} + D_{ijkl}\omega_{sr} + \overline{R}_{abcd}\dot{\nabla}_r\dot{\nabla}_s B^{abcd}_{ijkl} \tag{3.9}$$

where

$$S_{ijklsr} = \dot{\nabla}_e\overline{R}_{abcd}x^e_s\dot{\nabla}_r B^{abcd}_{ijkl}\ ,\ D_{ijkl} = B^{abcd}_{ijkl}n^e\dot{\nabla}_e\overline{R}_{abcd} \text{ and } \phi_{rs} = \phi_{ef}B^{ef}_{rs}$$

and the paranthesis ( ) denotes symmetrization.

Proof. By taking the prolonged covariant derivative of Gauss equation with respect to u^s and u^r successively, we have

$$\begin{aligned}\dot{\nabla}_r\dot{\nabla}_s R_{ijkl} &= \dot{\nabla}_r\dot{\nabla}_s\Omega_{ijkl} + (\dot{\nabla}_f\dot{\nabla}_e\overline{R}_{abcd})B^{abcdef}_{ijklsr} + (\dot{\nabla}_e\overline{R}_{abcd})(\dot{\nabla}_r B^{abcde}_{ijkls})\\ &+ (\dot{\nabla}_f\overline{R}_{abcd})x^f_r\dot{\nabla}_r\dot{\nabla}_s B^{abcd}_{ijkl} + \overline{R}_{abcd}\dot{\nabla}_s B^{abcd}_{ijkl}.\end{aligned}$$

If W_{n+1} is birecurrent Weyl, then by the definition $\dot{\nabla}_f\dot{\nabla}_e\overline{R}_{abcd} = \phi_{ef}\overline{R}_{abcd}$ so that we have

$$\begin{aligned}\dot{\nabla}_r\dot{\nabla}_s R_{ijkl} &= \dot{\nabla}_r\dot{\nabla}_s\Omega_{ijkl} + \phi_{ef}\overline{R}_{abcd}B^{abcdef}_{ijklsr} + (\dot{\nabla}_e\overline{R}_{abcd})(\dot{\nabla}_r B^{abcde}_{ijkls})\\ &+ (\dot{\nabla}_f\overline{R}_{abcd})x^f_r\dot{\nabla}_r\dot{\nabla}_s B^{abcd}_{ijkl} + \overline{R}_{abcd}\dot{\nabla}_s B^{abcd}_{ijkl}.\end{aligned}$$

By using the Gauss equation (3.6), the above equation can be brought into the form

$$\begin{aligned}\dot{\nabla}_r\dot{\nabla}_s R_{ijkl} &= \phi_{sr}R_{ijkl} + \dot{\nabla}_s\dot{\nabla}_r\Omega_{ijkl} - \phi_{sr}\Omega_{ijkl} + \dot{\nabla}_e\overline{R}_{abcd}\dot{\nabla}_r B^{abcde}_{ijkls}\\ &+x^f_r\dot{\nabla}_f\overline{R}_{abcd}\dot{\nabla}_s B^{abcd}_{ijkl} + \overline{R}_{abcd}\dot{\nabla}_r\dot{\nabla}_s B^{abcd}_{ijkl}.\end{aligned}$$

where $\phi_{rs} = \phi_{ef}B^{ef}_{rs}$. Hence, by (3.5), the result follows. □

Theorem 3.2. *If a hypersurface of a Birecurrent Weyl space is birecurrrent then*

$$\dot{\nabla}_r\dot{\nabla}_s\Omega_{ijkl} - \phi_{rs}\Omega_{ijkl} + S_{ijkl(sr)} + \omega_{sr}D_{ijkl} + \overline{R}_{abcd}\dot{\nabla}_r\dot{\nabla}_s B^{abcd}_{ijkl} = 0. \tag{3.10}$$

or, equivalently,

$$\dot{\nabla}_{[r}\dot{\nabla}_{s]}\Omega_{ijkl} - \phi_{[rs]}\Omega_{ijkl} + \overline{R}_{abcd}\dot{\nabla}_{[r}\dot{\nabla}_{s]}B^{abcd}_{ijkl} = 0. \tag{3.11}$$

Proof. It is clear from (2.2) and Theorem 3.1. □

A hypersurface of a Weyl space is called totally geodesic if $\omega_{ij} = 0$.

Theorem 3.3. *Every totally geodesic hypersurface of a birecurrent Weyl space is birecurrent.*

Proof. Since the hypersurface is totally geodesic, by putting $\omega_{ij} = 0$ in (3.9) we get the result. □

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ON SEMI-DECOMPOSABLE PSEUDO-SYMMETRIC WEYL SPACES

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Dedicated to Professor Pavel Enghiş at his 70th anniversary

Abstract. In this paper, we first prove that, if a semi-decomposable Weyl space W_n can be written as the product of two Weyl spaces $\bar{W}_q$ and W^*_{n-q} , then W_n has homothetic metrics. Next, after having given the definitions of symmetric and pseudo-symmetric Weyl spaces, we have shown that the symmetric Weyl space W_n can be written as the product of the symmetric subspaces $\bar{W}_q$ and W^*_{n-q}, if and only if the complementary vector field of $\bar{W}_q$ is the gradient of $\ln\sqrt{\sigma}$. Finally, we prove two theorems concerning semi-decomposable pseudo-symmetric Weyl spaces.

1. Introduction

An n-dimensional manifold W_n is said to be a Weyl space if it has a conformal metric tensor g_{ij} and a symmetric connection ∇_k satisfying the compatibility condition given by the equation

$$\nabla_k g_{ij} - 2T_k g_{ij} = 0 \ , \tag{1.1}$$

where T_k denotes a covariant vector field [1].

Under a renormalization of the fundamental tensor of the form

$$\widetilde{g_{ij}} = \lambda^2 g_{ij} \tag{1.2}$$

the complementary vector field T_k is transformed by the law

$$\widetilde{T_i} = T_i + \partial_i \ln\lambda \ , \tag{1.3}$$

where λ is a scalar function defined on W_n.

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The coefficients Γ^i_{kl} of the Weyl connection ∇_k are given by

$$\Gamma^i_{kl} = \left\{ \begin{matrix} i \\ kl \end{matrix} \right\} - g^{im}\left(g_{mk}T_l + g_{ml}T_k - g_{kl}T_m\right) \ . \tag{1.4}$$

A quantity A is called a satellite with weight $\{p\}$ of the tensor g_{ij}, if it admits a transformation of the form

$$\widetilde{A} = \lambda^p A$$

under the renormalization (1.2) of the metric tensor g_{ij}[2].

The prolonged covariant derivative of a satellite A of the tensor g_{ij} with weight $\{p\}$ is defined by [2]

$$\dot{\nabla}_k A = \nabla_k A - p T_k A \ . \tag{1.5}$$

2. SEmi-decomposable Weyl spaces

As in the Riemannian case [3], we will say that an n-dimensional Weyl space W_n $(n > 2)$ is a semi-decomposable space if its metric can be given in some coordinate system by

$$ds^2 = g_{ij}\, dx^i dx^j = \overline{g}_{ab}\, dx^a dx^b + \sigma\, g^*_{\alpha\beta}\, dx^\alpha dx^\beta \tag{2.1}$$

$$(\ i, j, k, ... = 1, 2, ..., n \ ; \ a, b, c, ... = 1, 2, ..., q \ ; \ \alpha, \beta, \gamma, ... = q+1, q+2, ..., n \ )$$

where

$$g_{ab} = \overline{g}_{ab}\left(x^c\right) \ , \ g_{\alpha\beta} = \sigma g^*_{\alpha\beta}\left(x^\gamma\right) \tag{2.1$'$}$$

and σ is a function of $x^1, x^2, ..., x^q$ with weight $\{0\}$. The two parts of (2.1) are the metrics of the two Weyl spaces $\bar{W}_q$ and W^*_{n-q} which are called the complementary spaces of W_n.

Throughout this paper, objects denoted by a bar or a star will respectively assumed to be formed by $\overline{g}_{ab}$ and $g^*_{\alpha\beta}$ while $\dot{\nabla}$, $\dot{\overline{\nabla}}$, $\dot{\nabla}^*$ indicate prolonged covariant differentiation in W_n , $\bar{W}_q$ and W^*_{n-q} respectively. If, in particular $\sigma = 1$, then W_n reduces to a decomposable space.

Suppose that $\bar{\Gamma}^a_{bc}$, $\bar{R}_{abcd}$, $\bar{T}_a$ denote, respectively the connection coefficients, the curvature tensor and the complementary vector field of $\bar{W}_q$ and let $\Gamma^{*\alpha}_{\beta\gamma}$, $R^*_{\alpha\beta\gamma\delta}$, T^*_α

refer to the subspace W^*_{n-q} of a semi-decomposable Weyl space with non-constant function σ. We then have

$$g_{ab} = \overline{g}_{ab} \ , \ g_{\alpha\beta} = \sigma g^*_{\alpha\beta} \ , \ g^{ab} = \overline{g}^{ab} \ , \ g^{\alpha\beta} = \frac{1}{\sigma} g^{*\alpha\beta} \ , \ g_{a\alpha} = 0 \ , \ g^{a\alpha} = 0. \qquad (2.2)$$

From the compatibility condition (1.1) we get

$$T_a = \bar{T}_a \ , \ T_\alpha = T^*_\alpha \qquad (2.3)$$

and consequently the connection coefficients are related by

$$\Gamma^a_{bc} = \bar{\Gamma}^a_{bc} \ , \ \Gamma^\alpha_{\beta\gamma} = \Gamma^{*\alpha}_{\beta\gamma} \qquad (2.4)$$

$$\Gamma^a_{\beta\gamma} = -\sigma \overline{g}^{ab} \overline{u}_b \, g^*_{\beta\gamma} \ , \ \Gamma^\alpha_{a\beta} = \overline{u}_a \delta^\alpha_\beta \ , \ \Gamma^a_{b\alpha} = -\delta^a_b T^*_\alpha \ , \ \Gamma^\alpha_{ab} = \frac{1}{\sigma} \overline{g}_{ab} \, g^{*\alpha\beta} T^*_\beta \qquad (2.5)$$

where

$$\sigma_{,a} = \frac{\partial \sigma}{\partial x^a} \ , \ \overline{u}_a = \frac{1}{2\sigma} \sigma_{,a} - \overline{T}_a \ .$$

On the other hand, using the expression [4]

$$R_{ijkl} = g_{ih} R^h_{jkl} \ , \ R^h_{jkl} = \frac{\partial}{\partial x^k} \Gamma^h_{jl} - \frac{\partial}{\partial x^l} \Gamma^h_{jk} + \Gamma^h_{ki} \Gamma^i_{jl} - \Gamma^h_{li} \Gamma^i_{jk}$$

for the covariant curvature tensor R_{ijkl}, we show that the curvature tensors of W_n , $\bar{W}_q$ and W^*_{n-q} are related by

$$\begin{aligned} R_{abcd} &= \bar{R}_{abcd} + \frac{1}{\sigma} T^*_\alpha T^*_\beta \, g^{*\alpha\beta} \, \overline{A}_{abcd} \\ R_{\alpha\beta\gamma\delta} &= \sigma \overset{*}{R}_{\alpha\beta\gamma\delta} + \sigma^2 \, \overline{u}_a \overline{u}_b \, \overline{g}^{ab} \, \overset{*}{A}_{\alpha\beta\gamma\delta} \\ R_{a\alpha b\beta} &= -R_{a\alpha\beta b} = -R_{\alpha a b\beta} = -\sigma \, g^*_{\alpha\beta} \, \bar{A}_{ab} - \overline{g}_{ab} \, B^*_{\alpha\beta} \ , \end{aligned} \qquad (2.6)$$

where we have put

$$\overline{A}_{abcd} = \overline{g}_{ad} \overline{g}_{bc} - \overline{g}_{ac} \overline{g}_{bd} \ , \ \bar{A}_{ab} = \dot{\overline{\nabla}}_b \overline{u}_a + \overline{u}_a \overline{u}_b \ , \ B^*_{\alpha\beta} = -\dot{\nabla}^*_\beta T^*_\alpha + T^*_\alpha T^*_\beta \ . \qquad (2.7)$$

These relations are the Weyl versions of the relations obtained in [5] for a Riemannian semi-decomposable space. After some calculations and simplifications we find that

$$R_{a\beta\gamma\delta} = \sigma \overline{u}_a \left(g^*_{\beta\gamma} T^*_\delta - g^*_{\beta\delta} T^*_\gamma \right) , \; R_{\alpha a\gamma\delta} = \sigma \overline{u}_a \left(g^*_{\alpha\delta} T^*_\gamma - g^*_{\alpha\gamma} T^*_\delta \right)$$

$$R_{\alpha\beta a\delta} = \sigma \overline{u}_a \left(g^*_{\alpha\delta} T^*_\beta - g^*_{\beta\delta} T^*_\alpha \right) , \; R_{\alpha\beta\gamma a} = \sigma \overline{u}_a \left(g^*_{\beta\gamma} T^*_\alpha - g^*_{\alpha\gamma} T^*_\beta \right)$$

$$R_{\alpha bcd} = T^*_\alpha \left(\overline{g}_{bc} \overline{u}_d - \overline{g}_{bd} \overline{u}_c \right) , \; R_{a\alpha cd} = T^*_\alpha \left(\overline{g}_{ad} \overline{u}_c - \overline{g}_{ac} \overline{u}_d \right)$$

$$R_{ab\alpha d} = T^*_\alpha \left(\overline{g}_{ad} \overline{u}_b - \overline{g}_{bd} \overline{u}_a \right) , \; R_{abc\alpha} = T^*_\alpha \left(\overline{g}_{bc} \overline{u}_a - \overline{g}_{ac} \overline{u}_b \right)$$

$$R_{ab\alpha\beta} = \overline{g}_{ab} \left(\frac{\partial T^*_\alpha}{\partial x^\beta} - \frac{\partial T^*_\beta}{\partial x^\alpha} \right) , \; R_{\alpha\beta ab} = \sigma g^*_{\alpha\beta} \left(\frac{\partial \bar{T}_a}{\partial x^b} - \frac{\partial \bar{T}_b}{\partial x^a} \right) . \qquad (2.8)$$

If the Weyl space W_n is Riemannian, then all the quantities in (2.8) become zero which explain a well-known result for a semi-decomposable Riemannian space [5].

We first prove the following theorem concerning semi-decomposable Weyl spaces.

Theorem 2.1. *A semi-decomposable Weyl space which can be written as the product of two Weyl spaces has homothetic metrics.*

Proof. For the conformal change of the metric tensors g_{ab} , $\overline{g}_{ab}$ and $g^*_{\alpha\beta}$ we have

$$\tilde{g}_{ij} = \lambda^2 g_{ij} \; , \; \tilde{\overline{g}}_{ab} = \overline{\lambda}^2 \overline{g}_{ab} \; , \; \tilde{g}^*_{\alpha\beta} = \lambda^{*2} g^*_{\alpha\beta}$$

where

$$\lambda = \lambda(x^1, x^2, ..., x^n) \; , \; \overline{\lambda} = \overline{\lambda}(x^1, x^2, ..., x^q) \; , \; \lambda^* = \lambda^*(x^{q+1}, x^{q+2}, ..., x^n). \qquad (2.9)$$

Then, using (2.2) and (2.9) we obtain

$$\lambda = \overline{\lambda} = \lambda^*$$

which states that λ , $\overline{\lambda}$, λ^* are equal to the same constant c. But this means that W_n has a homothetic metric.

For a Weyl space with a homothetic metric tensor, the complementary vector field T_k is invariant under the transformation (1.2). So, such a Weyl space will be Riemannian if and only if the complementary vector field T_k is identically zero. □

Remark 1. It can be easily seen that a Weyl space W_n can not be written as the product of a Weyl space and a Riemannian space, unless W_n is Riemannian.

3. Pseudo-symmetric Weyl spaces

The Weyl space W_n whose curvature tensor R_{hijk} satisfies the condition

$$\dot{\nabla}_l R_{hijk} = 2\lambda_l\, R_{hijk} + \lambda_h\, R_{lijk} + \lambda_i\, R_{hljk} + \lambda_j\, R_{hilk} + \lambda_k\, R_{hijl} \tag{3.1}$$

will be called a pseudo-symmetric space and will be denote by PSW_n, λ_i being a covariant vector field with weight $\{0\}$.

Since the weight of R_{hijk} is $\{2\}$, by (1.5) we get

$$\dot{\nabla}_l R_{hijk} = \nabla_l\, R_{hijk} - 2T_l\, R_{hijk} \tag{3.2}$$

so that (3.1) becomes

$$\nabla_l R_{hijk} = 2(T_l + \lambda_l)\, R_{hijk} + \lambda_h\, R_{lijk} + \lambda_i\, R_{hljk} + \lambda_j\, R_{hilk} + \lambda_k\, R_{hijl}\ . \tag{3.3}$$

If $T_l = 0$, W_n becomes a Riemannian space and (3.3) reduces to

$$\nabla_l R_{hijk} = 2\lambda_l\, R_{hijk} + \lambda_h\, R_{lijk} + \lambda_i\, R_{hljk} + \lambda_j\, R_{hilk} + \lambda_k\, R_{hijl} \tag{3.4}$$

which is the definition of a pseudo-symmetric Riemannian space [6].

We will say that a Weyl space is symmetric if the condition

$$\dot{\nabla}_l R_{hijk} = 0 \tag{3.5}$$

is satisfied. This definition reduces to the definition of a symmetric Riemannian space if we take $T_k = 0$ in (3.5).

It can be shown that a symmetric Weyl space with $\lambda \neq$ *const.* is Riemannian since, in this case, the complementary vector field becomes locally a gradient [7].

Theorem 3.1. *A semi-decomposable, symmetric elliptic Weyl space W_n $(n > 2)$ with $\sigma \neq$ const. can be written as the product of two symmetric Weyl spaces $\bar{W}_q$ and W^*_{n-q}, if and only if* $\overline{T}_a = \left(\frac{\partial \ln \sqrt{\sigma}}{\partial x^a}\right)$.

Proof. Remembering that

$$\nabla_l R_{hijk} = \partial_l R_{hijk} - \Gamma^m_{hl} R_{mijk} - \Gamma^m_{il} R_{hmjk} - \Gamma^m_{jl} R_{himk} - \Gamma^m_{kl} R_{hijm}$$

and using (1.5) , (2.4) , (2.5) , (2.6) , (2.7) and (2.8), after some calculations and simplifications we obtain

$$\dot{\nabla}_e R_{abcd} = \dot{\overline{\nabla}}_e \overline{R}_{abcd} - \frac{1}{\sigma} g^{*\alpha\beta} T^*_\alpha T^*_\beta [\overline{A}_{ebcd} \overline{u}_a + \overline{A}_{aecd} \overline{u}_b + \overline{A}_{abed} \overline{u}_c + \overline{A}_{abce} \overline{u}_d + 2\overline{A}_{abcd} \overline{u}_e] \tag{3.6}$$

$$\dot{\nabla}_\eta R_{\alpha\beta\gamma\delta} = \sigma \dot{\nabla}^*_\eta \overset{*}{R}_{\alpha\beta\gamma\delta} + \sigma^2 \overline{g}^{ab} \overline{u}_a \overline{u}_b [\overset{*}{A}_{\eta\beta\gamma\delta} T^*_\alpha + \overset{*}{A}_{\alpha\eta\gamma\delta} T^*_\beta + \overset{*}{A}_{\alpha\beta\eta\delta} T^*_\gamma + \overset{*}{A}_{\alpha\beta\gamma\eta} T^*_\delta + 2\overset{*}{A}_{\alpha\beta\gamma\delta} T^*_\eta] \tag{3.7}$$

First, suppose that $\overline{W}_q$ and W^*_{n-q} are symmetric. By the definition we get

$$\dot{\overline{\nabla}}_e \overline{R}_{abcd} = 0 \ , \ \dot{\nabla}^*_\eta \overset{*}{R}_{\alpha\beta\gamma\delta} = 0 .$$

On the other hand, since W_n is symmetric we have

$$\dot{\nabla}_e R_{abcd} = 0 \ , \ \dot{\nabla}_\eta R_{\alpha\beta\gamma\delta} = 0 .$$

Under these symmetry conditions, (3.6) and (3.7) reduce, respectively to

$$g^{*\alpha\beta} T^*_\alpha T^*_\beta [\overline{A}_{ebcd} \overline{u}_a + \overline{A}_{aecd} \overline{u}_b + \overline{A}_{abed} \overline{u}_c + \overline{A}_{abce} \overline{u}_d + 2\overline{A}_{abcd} \overline{u}_e] = 0 \tag{3.8}$$

$$\overline{g}^{ab} \overline{u}_a \overline{u}_b [\overset{*}{A}_{\eta\beta\gamma\delta} T^*_\alpha + \overset{*}{A}_{\alpha\eta\gamma\delta} T^*_\beta + \overset{*}{A}_{\alpha\beta\eta\delta} T^*_\gamma + \overset{*}{A}_{\alpha\beta\gamma\eta} T^*_\delta + 2\overset{*}{A}_{\alpha\beta\gamma\delta} T^*_\eta] = 0 . \tag{3.9}$$

Since the space W_n is assumed to be elliptic, i.e. the metric is positive definite and W^*_{n-q} is not Riemannian the factor $g^{*\alpha\beta} T^*_\alpha T^*_\beta$ in (3.8) can not be zero. On the other hand, if in (3.9) $\overline{g}^{ab} \overline{u}_a \overline{u}_b = 0$, it follows that $\overline{u}_a = 0$, i.e. $\overline{T}_a = \left(\frac{\partial \ln \sqrt{\sigma}}{\partial x^a}\right)$ and consequently (3.8) and (3.9) are automatically satisfied.

Suppose now that

$$g^{*\alpha\beta} T^*_\alpha T^*_\beta \neq 0 \ , \ \overline{g}^{ab} \overline{u}_a \overline{u}_b \neq 0 .$$

In this case (3.8) and (3.9) are reduced to

$$[\overline{A}_{ebcd} \overline{u}_a + \overline{A}_{aecd} \overline{u}_b + \overline{A}_{abed} \overline{u}_c + \overline{A}_{abce} \overline{u}_d + 2\overline{A}_{abcd} \overline{u}_e] = 0 \tag{3.8$'$}$$

$$[\overset{*}{A}_{\eta\beta\gamma\delta} T^*_\alpha + \overset{*}{A}_{\alpha\eta\gamma\delta} T^*_\beta + \overset{*}{A}_{\alpha\beta\eta\delta} T^*_\gamma + \overset{*}{A}_{\alpha\beta\gamma\eta} T^*_\delta + 2\overset{*}{A}_{\alpha\beta\gamma\delta} T^*_\eta] = 0 . \tag{3.9$'$}$$

Now, transvecting (3.8)′ by $\overline{g}^{ad}$ and $\overline{g}^{bc}$ and (3.9)′ by $g^{*\beta\gamma}$ and $g^{*\alpha\delta}$, we get respectively

$$(q-1)\,\overline{u}_e\,(q+2)=0\ . \tag{3.10}$$

$$(n-q+2)(n-q-1)\,T^*_\eta=0 \tag{3.11}$$

from which it follows that, since $n>2$ and $T^*_\eta\neq 0$, the latter case can not happen. This proves the necessity of the condition.

Conversely, suppose that $\overline{T}_e=\dfrac{\partial(\ln\sqrt{\sigma})}{\partial x^e}$, i.e. $\overline{u}_e=0$. From (3.6) and (3.7) we conclude that

$$\dot{\overline{\nabla}}_e\overline{R}_{abcd}=0\ ,\ \dot{\nabla}^*_\eta\,\overset{\bullet}{R}_{\alpha\beta\gamma\delta}=0\ .$$

showing that the two subspaces $\bar{W}_q$ and W^*_{n-q} are symmetric . □

Theorem 3.2. *For a semi-decomposable PSW_n $(n>2)$, we have*

$$\lambda_a=-\overline{u}_a\ ,\ \lambda_\alpha=T^*_\alpha$$

unless $\overline{T}_a$ and T^*_α are gradients.

Proof. For a PSW_n we have from (3.1) that

$$\dot{\nabla}_a R_{\alpha\beta\gamma\delta}+\dot{\nabla}_a R_{\beta\alpha\gamma\delta}=2\lambda_a\,(R_{\alpha\beta\gamma\delta}+R_{\beta\alpha\gamma\delta}) \tag{3.12}$$

$$\dot{\nabla}_\alpha R_{abcd}+\dot{\nabla}_\alpha R_{bacd}=2\lambda_\alpha\,(R_{abcd}+R_{bacd})\,. \tag{3.13}$$

By using the relations (1.5) , (2.5) , (2.6) , (2.8), the left hand sides of (3.12) and (3.13) may be put into the form

$$\dot{\nabla}_a R_{\alpha\beta\gamma\delta} + \dot{\nabla}_a R_{\beta\alpha\gamma\delta} = -2\sigma\,\overline{u}_a\,(\overset{\bullet}{R}_{\alpha\beta\gamma\delta} + \overset{\bullet}{R}_{\beta\alpha\gamma\delta}) \tag{3.14}$$

$$\dot{\nabla}_\alpha R_{abcd} + \dot{\nabla}_\alpha R_{bacd} = 2T^*_\alpha\,(\overline{R}_{abcd} + \overline{R}_{bacd})\ . \tag{3.15}$$

If the relation [7]

$$R_{ijkl} + R_{jikl} = 2g_{ij}\,(T_{k,l} - T_{l,k}) \tag{3.16}$$

is taken into account, from (3.12),(3.13),(3.14),(3.15), we finally get

$$g^*_{\alpha\beta}\,(T^*_{\gamma,\delta} - T^*_{\delta,\gamma})\,(\overline{u}_a + \lambda_a) = 0 \tag{3.17}$$

$$\overline{g}_{ab}\,(\overline{T}_{c,d} - \overline{T}_{d,c})(T^*_\alpha - \lambda_\alpha) = 0. \tag{3.18}$$

Since the complementary vector fields $\overline{T}_a$ and T^*_α are not gradients, from (3.17) and (3.18) it follows that

$$\overline{u}_a + \lambda_a = 0\ ,\ T^*_\alpha - \lambda_\alpha = 0 \tag{3.19}$$

which completes the proof. □

Theorem 3.3. *For a semi-decomposable PSW_n the subspaces $\bar{W}_q$ and W^*_{n-q} are also pseudo-symmetric unless $\overline{T}_a$ and T^*_α are gradients.*

Proof. Using (2.6) , (3.1) , (3.6) , (3.7) and (3.19), after some calculations we obtain

$$\dot{\overline{\nabla}}_e \overline{R}_{abcd} = -2\overline{u}_e\,\overline{R}_{abcd} - \overline{u}_a\,\overline{R}_{ebcd} - \overline{u}_b\,\overline{R}_{aecd} - \overline{u}_c\,\overline{R}_{abed} - \overline{u}_d\,\overline{R}_{abce} \tag{3.20}$$

$$\dot{\overset{\bullet}{\nabla}}_\eta \overset{\bullet}{R}_{\alpha\beta\gamma\delta} = 2T^*_\eta\,\overset{\bullet}{R}_{\alpha\beta\gamma\delta} + T^*_\alpha\,\overset{\bullet}{R}_{\eta\beta\gamma\delta} + T^*_\beta\,\overset{\bullet}{R}_{\alpha\eta\gamma\delta} + T^*_\gamma\,\overset{\bullet}{R}_{\alpha\beta\eta\delta} + T^*_\delta\,\overset{\bullet}{R}_{\alpha\beta\gamma\eta} \tag{3.21}$$

stating that $\bar{W}_q$ and W^*_{n-q} are pseudo-symmetric. □

Corollary 3.4. *For a semi-decomposable PSW_n with $\sigma \neq$ const., the condition $\overline{T}_a = \dfrac{\partial}{\partial x^a}(\ln\sqrt{\sigma})$ implies that $\bar{W}_q$ is symmetric and that W^*_{n-q} is pseudo-symmetric provided that T^*_α is not a gradient.*

Proof. The truth of this assertion is clear from (2.6) , (3.7) , (3.17) and (3.20) if we take $\overline{u}_a = 0$. □

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A NOTE ON MINIMAX RESULTS FOR CONTINUOUS FUNCTIONALS

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Dedicated to Professor Pavel Enghiş at his 70^{th} anniversary

Abstract. In this paper we extend the Willem deformation lemma for continuous functionals and we traite also the equivariant case. With the aid of this results we extend the min-max results of Ghoussoub [21]. As application we give an another proof of some multiplicity results of Corvellec [5] and we give some multiplicity results for continuous functionals which contains a large class of multiplicity results for differentiable and locally Lipschitz functionals.

1. Introduction.

In many papers is studied the critical point theory for continuous functionals, see [3], [4], [5], [2], [?] and [8]. In this paper using some results from the paper of J.-N. Corvellec, M. Degiovanni and M. Marzocchi [4] we prove the Willem deformation lemma for continuous functionals. We traite also the equivariant case. With the aid of this results we give a simplified proof and generalize some min-max results of Ghoussoub [21], Fang [6], and Ribarska-Tsachev-Krastanov [9]. Using this result we give some multiplicity results of Ghoussoub [21], which represent an another proof of some multiplicity results of Corvellec [5]. As applications for different topological index we give some minmax and multiplicity results for continuous functionals, which represent generalizations for well known results, see Fadell [19], Santos [33], Chang [16], Marzocchi [28], Goeleven-Motreanu-Panagiotoulos [23], Mironescu-Radulescu [32] and another results.

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First we recall some definitions and result from the paper of M. Degiovanni and M. Marzocchi, see [3].

Definition 1.1. Let (X, d) be a complete metric space and let $f : X \to \mathbb{R}$ be a continuous function and $u \in X$ a fixed element. We denote by $|df|(u)$ the supremum of the $\sigma \in [0, \infty[$ such that there exist $\delta > 0$ and a continuous map

$$\mathcal{H} : B(u, \delta) \times [0, \delta] \to \mathbb{R}$$

such that $\forall v \in B(u, \delta)$ *for all* $t \in [0, \delta]$ we have

a) $d(\mathcal{H}(v,t), v) \leq t$

b) $f(\mathcal{H}(v,t)) \leq f(v) - \sigma t$

The extended real number $|df|(u)$ is called *the weak slope* of f at u.

Definition 1.2. Let $f : X \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous function. We define the function

$$\mathcal{G}_f : epi(f) \to \mathbb{R}$$

putting

$$epi(f) = \{(u, \xi) \in X \times \mathbb{R} : f(u) \leq \xi\} \quad and \quad \mathcal{G}_f(u, \xi) = \xi.$$

In the following $epi(f)$ will be endowed with the metric

$$d_{ep}((u, \xi), (v, \mu)) = (d(u, v)^2 + (\xi - \mu)^2)^{\frac{1}{2}}.$$

Of course $epi(f)$ is closed in $X \times \mathbb{R}$ and $\mathcal{G}_f$ is Lipschitz continuous of constant 1. Consequently $|d\mathcal{G}_f|(u, \xi) \leq 1$ for every $(u, \xi) \in epi(f)$.

Proposition 1.3. *Let $f : X \to \mathbb{R}$ be a continuous function and let $(u, \xi) \in epi(f)$. Then*

$$|\mathcal{G}_f|(u, \xi) = \begin{cases} \dfrac{|df|(u)}{\sqrt{1 + |df|(u)^2}}, & if \;\; f(u) = \xi and \;\; |df|(u) < \infty, \\ 1 & if \;\; f(u) < \xi or \;\; |df|(u) = \infty. \end{cases}$$

We recall a basic result from [4].

Theorem 1.4. *(Theorem 2.11, [4]) Let (X, d) be a complete metric space and let $f : X \to \mathbb{R}$ be a continuous function, C a closed subset of X and $\delta, \sigma > 0$ such that*

$$d(u, C) \leq \delta \Longrightarrow |df|(u) > \sigma.$$

Then there exists a continuous map $\eta : X \times [0, \delta] \to X$ such that

1. $d(\eta(u, t), u)) \leq t,$
2. $f(\eta(u, t)) \leq f(u),$
3. $d(u, C) \geq \delta \Longrightarrow \eta(u, t) = u,$
4. $u \in C \Longrightarrow f(\eta(u, t)) \leq f(u) - \sigma t.$

In the following, for every $c \in \mathbb{R}$ we use the next notations:

$$K_c(f) = \{x \in X : |df|(x) = 0 \text{ and } f(x) = c\};$$

$$f^c = \{x \in X : f(x) \leq c\} :$$

$$f_c = \{x \in X : f(x) \geq c\}.$$

2. Willem deformation lemma

In this section we extend the Willem deformation lemma for continuous functionals.

Theorem 2.1. *Let (X, d) be a complete metric space, $f : X \to \mathbb{R}$ a continuous function, C a closed subset of X and $c \in \mathbb{R}$ a real number. Let ε and $\delta > 0$ two number such that we have:*

$$\forall \ u \in f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap C_{2\delta} : \text{ we have } |df|(u) > \varepsilon. \tag{2.1}$$

Then there exists two real numbers $\varepsilon' \in (0, \varepsilon)$ and $\lambda > 0$ and a continuous map $\eta : X \times [0, 1] \to X$ such that:

a) $d(\eta(u, t), u) \leq \lambda t,$

b) $f(\eta(u, t)) \leq f(u),$

c) if $u \notin f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap C_{2\delta} : \eta(u, t) = u, \quad \forall \ t \in [0, 1]$

d) $\eta(f^{c+\varepsilon'} \cap C, 1) \subset f^{c-\varepsilon'},$

e) $\forall \ t \in]0, 1]$ *and* $\forall \ u \in f^c \cap C$ *we have* $f(\eta(t, u)) < c.$

Proof. First, we suppose that the function $f : X \to \mathbb{R}$ is Lipschitz continuous with constant 1. We consider the set:

$$C^* := \{u \in X | c - t_1 \leq f(u) \leq c + t_1, d(u, C) \leq 2\delta - \delta_1\}, \tag{2.2}$$

where $\delta_1 + t_1 < 2\varepsilon, \delta_1 < 2\delta$ and $\delta_1, t_1 > 0$, for example $\delta_1 := \min\{\varepsilon, \delta\}$ and $t_1 = \frac{\varepsilon}{2}$. Obvious the set C^* is a closed subset of X. We observe that from the relation $d(u, C^*) \leq \delta_1$ we get:

$$u \in f^{-1}([c - 2\varepsilon, c + 2\varepsilon) \cap C_{2\delta}. \tag{2.3}$$

Indeed, because $|f(v) - f(u)| \leq 1 \cdot d(u, v)$ for $\forall u, v \in X$ we obtain

$$-d(u, v) \leq f(u) - f(v) \leq d(u, v), \quad \forall v \in C^*.$$

Using this relation and the fact $d(u, C^*) \leq \delta_1$ we get

$$c - (t_1 + \delta_1 \leq f(u) \leq c + (t_1 + \delta_1).$$

Because $\delta_1 + t_1 < 2\varepsilon$ we obtain $u \in f^{-1}([c - 2\varepsilon, c + 2\varepsilon])$. It is easy to verify that $d(u, C^*) \leq \delta_1$ implies $u \in C_{2\delta}$.

Because $\varepsilon > \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}$ from the relation (2.3) we obtain $|df|(u) > \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}$. Now we can apply Proposition 2.4, for C^*, δ_1 $\frac{\varepsilon}{\sqrt{1+\varepsilon^2}}$ and we get a continuous function $\eta' : X \times [0, \delta] \to \mathbb{R}$ which satisfied the conditions 1)-4) from Theorem 2.4. Without loss of generality, we assume that $\lambda = \delta_1$, and define the function $\eta : X \times [0, 1] \to \mathbb{R}$ by $\eta(u, t) = \eta'(u, \lambda t)$. The properties a) and b) are obvious. Let $u \notin f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap C_{2\delta}$ since f is a Lipschitz function with constant 1, we have $d(u, C^*) > \delta_1$ and using Proposition 2.4, a) we get $\eta(u, t) = u$.

For the proof of d) let $\varepsilon' = \min\{t_1, \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}\}$ and we distinguish two cases:

2.4) If $u \in f^{c+\varepsilon'} \cap C$ and $f(u) \geq c - \varepsilon'$ it follows that $u \in C^*$, hence we have

$$f(\eta(u, 1)) \leq f(u) - \frac{\varepsilon}{\sqrt{1+\varepsilon^2}} \leq c + \varepsilon - \frac{\varepsilon}{\sqrt{1+\varepsilon^2}} \leq c - \varepsilon'.$$

2.5) If $u \in f^{c+\varepsilon'} \cap C$ and $f(u) < c - \varepsilon'$, then from b) we get

$$f(\eta(u, t)) \leq f(u) < c - \varepsilon'.$$

The part e) of the theorem is proved in same way as d).

Now we consider the general case. For this let $C^{**} = \{(u,\xi) \in epi(f) |\ u \in C\}$. The set $epi(f)$ is closed in $X \times \mathbb{R}$ and it follow that $epi(f)$ is a complete metric space. In the next we prove that for every $(u,\xi) \in epi(f)$ with $(u,\xi) \in \mathcal{G}_f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\delta}^{**}$, we have $|d\mathcal{G}_f|(u,\xi) > \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}$.

We distinguish two cases:

I) Let $f(u) = \xi$. In this case we have two subcases.

a) $|df|(u) < \infty$. If $(u, f(u)) \in \mathcal{G}_f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\delta}^{**}$, then we get $u \in f^{-1}([c-2\varepsilon, c+2\varepsilon])$ and $d_{ep}((u,f(u)), C^{**}) \leq 2\delta$. Because $d(u,C) \leq d_{ep}((u,f(u)), C^{**}) \leq 2\delta$ we get $u \in f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\delta}$ and using the hypothesis of theorem follow $|df|(u) > \varepsilon$.

Because $|df|(u) < \infty$ from Proposition 1.3 we have $|\mathcal{G}_f|(u,f(u)) = \frac{|df|(u)}{\sqrt{1+|df|^2(u)}}$ and using the fact that the function $x \mapsto \frac{x}{\sqrt{1+x^2}}$ is increasing we have $|d\mathcal{G}_f|(u,f(u)) > \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}$.

b) If $|df|(u) = \infty$ using Proposition 1.3 we get $|d\mathcal{G}_f|(u,f(u)) = 1 > \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}$.

II) If $f(u) < \xi$, then from Proposition 1.3 we have $|d\mathcal{G}_f|(u,f(u)) = 1 > \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}$.

From these we get that for every $(u,\xi) \in \mathcal{G}_f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\delta}^{**}$ implies $|d\mathcal{G}_f|(u,\xi) > \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}$.

The set $A := \mathcal{G}_f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\delta}^{**} \cap epi(f) \neq \emptyset$, because if $u \in C$, then $(u, f(u)) \in A$. We apply the previous step for $X := epi(f)$, $f := \mathcal{G}_f$ and $C := C^{**}$. Then there exists two positive numbers ε', $\lambda > 0$ and a continuous mapping $\overline{\eta} := (\overline{\eta_1}, \overline{\eta_2}) : epi(f) \times [0,1] \to epi(f)$ such that the following holds:

2.6) $d_{ep}((\overline{\eta}(u,\xi), t), (u,\xi)) \leq \lambda t, \quad \forall\ (u,\xi) \in epi(f), \forall\ t \in [0,1]$;

2.7) $\mathcal{G}_f(\overline{\eta}(u,\xi), t) = \overline{\eta}_2((u,\xi), t) \leq \xi = \mathcal{G}_f(u,\xi)$, for all $(u,\xi) \in epi(f)$, and $\forall\ t \in [0,1]$;

2.8) $\overline{\eta}((u,\xi), t) = (u,\xi)$ for every $(u,\xi) \in epi(f)$ with $(u,\xi) \notin \mathcal{G}_f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\delta}^{**}$;

2.9) $\overline{\eta}(\mathcal{G}_f^{c+\varepsilon'} \cap C^{**}, 1) \subset \mathcal{G}_f^{c-\varepsilon'}$;

2.10) $f(\overline{\eta}((u,\xi), t) < c$ for every $t \in]0,1]$ and $\forall\ (u,\xi) \in \mathcal{G}_f^c \cap C^{**}$.

We define the function $\eta : X \times [0,1] \to X$ by

2.11) $\eta(u,t) = \overline{\eta}_1((u, f(u)), t)$.

Because $\overline{\eta}$ takes its values in $epi(f)$, we have

2.12) $f(\overline{\eta}_1((u, f(u)), t) \leq \overline{\eta}_2((u, f(u)), t)$

From 2.6) we have:

$$d(\eta(u,t), u) = d((\overline{\eta}_1(u, f(u), t), u) \leq$$

$$\leq [d^2((\overline{\eta}_1(u, f(u)), t), u) + (\overline{\eta}_2((u, f(u)), t) - f(u))^2]^{\frac{1}{2}} =$$

$$= d_{ep}((\overline{\eta}(u, f(u)), t), (u, f(u)) \leq \lambda t.$$

From the relations 2.7) and 2.12) we get

$$f(\eta(u,t)) = f(\overline{\eta}_1(u, f(u)), t) \leq \overline{\eta}_2((u, f(u)), t) \leq f(u).$$

If $u \notin f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap C_{2\delta}$ then

2.13) $(u, f(u)) \notin \mathcal{G}_f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap C^{**}_{2\delta}$.

Now we assume that $(u, f(u)) \in \mathcal{G}_f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap C^{**}_{2\delta}$. From this follow that

2.14) $f(u) \in [c - 2\varepsilon, c + 2\varepsilon]$

and $(u, f(u)) \in C^{**}_{2\delta}$, which is equivalent with $d_{ep}((u, f(u)), C^{**}) \leq 2\delta$. But we have $d(u, C) = \inf\{d(u, v) | v \in C\} \leq \inf\{d_{ep}(u, f(u)), (v, \xi)) \mid (v, \xi) \in C^{**}\} = d_{ep}((u, f(u)), C^{**})$. From this and from 2.14) we get $u \in f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap C_{2\delta}$ which is a contradiction with assumption. If $u \notin f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap C_{2\delta}$, then from 2.8) we get $\eta(u,t) = \overline{\eta}_1((u, f(u)), t) = u$.

If $f(u) \leq c + \varepsilon'$ then from 2.9) and 2.12) we get

$$f(\eta(u, 1)) = f(\overline{\eta}_1(u, f(u)), 1) \leq \overline{\eta}_2((u, f(u)), 1) \leq c - \varepsilon'.$$

From 2.10) and 2.12) we get the relation e).

In the following we use the next form of Willem deformation theorem.

Corollary 2.2. *Let (X,d) be a complet metric space, $f : X \to \mathbb{R}$ a continuous function, C a closed subset of X and $c \in \mathbb{R}$ a real number. Let $\varepsilon > 0$ be a number such:*
$\forall \ x \in f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon}$: *we have* $|df|(x) > \varepsilon$.
Then there exists two real numbers $\varepsilon' \in (0,\varepsilon)$ and $\lambda > 0$ and a continuous map $\eta : X \times [0,1] \to X$ such that:

a') $d(\eta(u,t),u) \leq \lambda t$, for every $t \in [0,1]$.
b') $f(\eta(u,t)) \leq f(u)$, for every $t \in [0,1]$ and $x \in X$.
c') if $x \notin f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon} : \eta(x,t) = x, \ \forall \ t \in [0,1]$.
d') $\eta(f^{c+\varepsilon'} \cap C, 1) \subset f^{c-\varepsilon'}$ with $\varepsilon' = \dfrac{\varepsilon}{2\sqrt{1+\varepsilon^2}}$.
e') $\forall \ t \in]0,1]$ and $\forall \ x \in f^c \cap C$ we have $f(\eta(t,x)) < c$.

Proof. In the proof of Willem deformation lemma we take $\delta := \varepsilon, \ \ t_1 = \frac{\varepsilon}{2}$ and $\varepsilon' = \dfrac{\varepsilon}{2\sqrt{1+\varepsilon^2}}$.

3. A minmax result

Definition 3.1. Let B a closed subset of M. We shall say that the class $\mathcal{F}$ of subsets of M is homotopy stable with boundary B if:

(a) Every set in $\mathcal{F}$ contains B;

(b) For any set $A \in \mathcal{F}$ and any continuous function $\eta \in \mathcal{C}([0,1] \times M, M)$ verifying $\eta(t,x) = x$ for all $(t,x) \in (\{0\} \times M) \cup ([0,1] \times B)$ we have $\eta(1,A) \in \mathcal{F}$.

Definition 3.2. We say that a set F is dual $\mathcal{F}$ if F verifies the following conditions:

1°) $\operatorname{dist}(F,B) > 0$;

2°) $F \cap A \neq \emptyset$ for all $A \in \mathcal{F}$.

Denote by $\mathcal{F}^\star$ a family of subsets which are dual to $\mathcal{F}$ and we say that $\mathcal{F}^\star$ is dual family to $\mathcal{F}$. We have the following relation

$$c^\star := \sup_{F \in \mathcal{F}} \inf_{x \in F} f(x) \leq \inf_{A \in \mathcal{F}} \sup_{x \in A} f(x) =: c.$$

Examples:

3.1) Let K be a compact metric space, $K_0 \subset K$ a closed set, X a Banach space, $\chi \in \mathcal{C}(K,X)$. Then the set $\mathcal{F} = \{\, A = g(K) \,|\, g \in \mathcal{C}(K,X) \text{ with } g(K_0) = \chi(K_0) \,\}$ is a homotopy stable family wits boundary $B = \chi(K_0)$.

3.2) For each $n \in \mathbf{N}$ the families

$$\mathcal{F}_n = \{\, A \mid A \subset X \text{with} \operatorname{cat}_X(A) \geq n \,\}$$

is a homotopy stable family, where $\operatorname{cat}_X(A)$ denote the Ljusternik-Schnirelmann category.

3.3) For each $n \in \mathbf{N}$ the families

$$\mathcal{F}_n = \{\, A \mid Y \subset A \,\text{and} \operatorname{cat}_{(X,Y)}(A) \geq n \,\}$$

is a homotopy stable family with boundary Y, where $\operatorname{cat}_{(X,Y)}(A)$ denote the relative category, see [25].

3.4) We recall the definition of the P-ideal valued cohomological index. Let E be a paracompact space and $(X, A) \in \mathcal{E}_E$ whwre $\mathcal{E}_E$ is the category of paracompact pair (X, A) ion E for a fixed closed subset A of E. Let $H^*(,)$ be the Alexander-Spanier cohomology theory with a field coefficient K, see [34]. The cup product defines a multiplication on $H^*(X, A)$ as flows:

$$H^*(X, A) \otimes H^*(E) \stackrel{1 \otimes i^*}{\rightarrow} H^*(X, A) \otimes H^*(X) \rightarrow H^*(X, A),$$

where 1 is the identity on $H^*(X, A)$ and i is the inclusion map $X \stackrel{i}{\rightarrow} E$. Therefore , $H^*(X, A)$ is an $H^*(E)$ module. In particular, $H^*(A)$ is also an $H^*(E)$-module. We introduce the following notation: $\Lambda = H^*(E)$. For an $H^*(E)$-submodule P of $H^*(A)$ the P-ideal value cohomological index of (X, A) over K is an ideal denoted by

$$P - Index_E(X, A) = \{\lambda \in \Lambda \mid u \cdot \lambda = 0, \forall\, u \in M^*(X, A)\,\},$$

where $M^q(X, A) = \delta^q(P)$ for $q \geq -1$, $M^\circ(X, A) = \mathcal{E}(K)$, δ^* is the coboundary operator for the pair (X, A) and $\mathcal{E}$ is the augmentation. In the next we consider A and B two disjoint closed subsets of X. We say that A is P-ideal linking to B if and only if

$$P - Index_E(E \setminus B, A) \supset P - Index_E(E, A).$$

Let E be a connected paracompact space. We suppose that the following conditions holds:

1') There are two disjoint sets A and B such that A is P-ideal linking to $B, P \subset H^*(A)$;

2') There exists a closed set $\tilde{X} \supset A$ in E such that $\tilde{X} \setminus A$ is precompact and

$$P - Index_E(\tilde{X}, A) = P - Index_E(E, A).$$

We denote by $\alpha = P - Index_E(E, A)$ and $\beta = P - Index_E(E \setminus B, A)$. Since A is P-Ideal linking to B, we have $\beta \supset \alpha$ and $\beta \neq \alpha$. We define the set

$$\Sigma_\alpha = \{ (X, A) \in \mathcal{E}_E : P - Index_E(X, A) = \alpha \},$$

where $\mathcal{E}_E$ is the class of all paracompact pair (X, A) in$E$. Note that $\Sigma_\alpha \neq \emptyset$ since $(\tilde{X}, A) \in \Sigma_\alpha$. We prove that Σ_α is homotopy stable with boundary A. Let $(X, A) \in \Sigma_\alpha$ be a paracompact pair and $\eta \in \mathcal{C}([0,1] \times E, E)$ a deformation such that $\eta(t, x) = x$ for all $(t, x) \in (\{0\} \times E) \cup ([0,1] \times A)$. From the invariance property of the P-ideal valued index we get $\eta(1, X) \in \Sigma_\alpha$. Since A is P-Ideal linking to B, then for every $X \in \Sigma_\alpha$ we have $X \cap B \neq \emptyset$.

In the next we prove the main result of this section which generalize the main results from [21], [6] and [9].

Theorem 3.3. *Let (X, d) be a complete metric space and let $f : X \to \mathbb{R}$ be a continuous function. Consider a homotopy stable family $\mathcal{F}$ of subsets of X with boundary B and a dual family $\mathcal{F}^\star$ of $\mathcal{F}$. Let $F \in \mathcal{F}^\star$ be a fixed, element which verifies the following condition*

$$\inf_{x \in F} f(x) \geq c, \tag{3.1}$$

where $c := \inf_{A \in \mathcal{F}} \sup_{x \in A} f(x)$.

Let $\varepsilon \in (0, \frac{dist(B,F)}{2})$ and $\delta > 0$ be arbitrarly fixed numbers. Then for any $A \in \mathcal{F}$

which verifies the relation

$$\sup_{x\in A\cap F_\delta} f(x) \le c + \frac{\varepsilon}{2\sqrt{1+\varepsilon^2}}, \tag{3.2}$$

then there exists $x_\varepsilon \in X$ such that the following holds:

(i) $c - 2\varepsilon \le f(x_\varepsilon) \le c + 2\varepsilon$;

(ii) $|df|(x_\varepsilon) \le \varepsilon$;

(iii) $dist(x_\varepsilon, F) \le 2\varepsilon$;

(iv) $dist(x_\varepsilon, A) \le 2\varepsilon$.

Proof. From the definition of the number c, there exists a subset $A \subset X$ such that $\sup_{x\in A} f(x) \le c + \frac{\varepsilon}{2\sqrt{1+\varepsilon^2}}$. From this we get that for every $\delta > 0$ we have the following relation:

$$\sup_{x\in A\cap F_\delta} f(x) \le c + \frac{\varepsilon}{2\sqrt{1+\varepsilon^2}}.$$

Using the fact that $dist(x, A) = dist(x, \overline{A})$, the assertions i)-iv) from theorem is equivalent with:

$$\text{exists an } \ x_\varepsilon \in f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap \overline{F}_{2\varepsilon} \cap \overline{A}_{2\varepsilon} \ \text{ such that } \ |df|(x_\varepsilon) \le \varepsilon.$$

We suppose the contrary, i.e.

$$\forall \ \ x \in f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap \overline{F}_{2\varepsilon} \cap \overline{A}_{2\varepsilon} \quad \text{we have } \ |df|(x) > \varepsilon. \tag{3.3}$$

We consider the set $C := \overline{A \cap F}$ then we have $C_{2\varepsilon} = (\overline{F \cap A})_{2\varepsilon} \subset (\overline{F} \cap \overline{A})_{2\varepsilon} \subset \overline{F}_{2\varepsilon} \cap \overline{A}_{2\varepsilon}$. From the relation (3.3) we have the following implication;

$$\text{if } \ x \in f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_\varepsilon \ \text{ then } \ |df|(x) > \varepsilon.$$

From Corollary 2.2 we have a continuous function $\eta : X \times [0,1] \to X$ and $\lambda > 0$ which satisfies the assertions a')-e') with $\lambda \le \min\{\varepsilon, \delta\}$, see the proof of Theorem 2.1. Let $A_1 = \eta(A, 1)$. If $x \in C_X C_{2\varepsilon}$, where $C_X C_{2\varepsilon}$ denote the complementary of the set $C_{2\varepsilon}$ in X, then from the propertie c') we have $\eta(x,t) = x$ for every $t \in [0,1]$. Using the fact that $dist(F, B) > 2\varepsilon$, we get $B \subset C_X C_{2\varepsilon}$, thus we have $B = \eta(B, 1)$. Because $\mathcal{F}$ is a homotopy stable family with boundary B, result that $B = \eta(B,1) \subset \eta(A,1) = A_1$, thus $A_1 \in \mathcal{F}$. We have the following relation $\eta(A,1) \cap F \subset \eta(A \cap F_\lambda, 1)$.

Indeed, if $x \in \eta(A,1) \cap F$, then there exists an $y \in A$ such that $x = \eta(y,1) \in F$. But $d(y, \eta(y,1)) \leq \lambda$, thus $y \in F_\lambda$. From the relation $y \in A \cap F_\lambda$ it follows that $x = \eta(y,1) \in \eta(A \cap F_\lambda, 1)$. From (3.2) we have

$$A \cap F_\lambda \subset f^{c+\varepsilon'}, \quad \text{with } \varepsilon' = \frac{\varepsilon}{2\sqrt{1+\varepsilon^2}}.$$

Indeed, since $\lambda \leq \delta$, we have $A \cap F_\lambda \subset A \cap F_\delta$. But $\sup_{x \in A \cap F_\delta} f(x) \leq c + \varepsilon'$ we get $A \cap F_\lambda \subset f^{c+\varepsilon'}$. From the properties d') we have

$$\eta(A \cap F_\lambda, 1) \subset \eta(f^{c+\frac{\varepsilon}{2\sqrt{1+\varepsilon^2}}}, 1) \subset f^{c-\frac{\varepsilon}{2\sqrt{1+\varepsilon^2}}}. \tag{3.5}$$

From the relations (3.4) and (3.5) we get $A_1 \cap F \subset f^{c-\frac{\varepsilon}{2\sqrt{1+\varepsilon^2}}}$, which is equivalent to $f(x) \leq c - \frac{\varepsilon}{2\sqrt{1+\varepsilon^2}}$ for every $x \in A_1 \cap F$. From this relation we get

$$\inf_{x \in F} f(x) \leq \inf_{x \in F \cap A_1} f(x) \leq c - \frac{\varepsilon}{2\sqrt{1+\varepsilon^2}}.$$

From the relation (3.1) we have $c \leq \inf_{x \in F} f(x) \leq c - \frac{\varepsilon}{2\sqrt{1+\varepsilon^2}}$, which is a contradiction.

In the following we give a simplified proof for Theorem 1.5 from [6] without using the Ekeland's variational principle.

Corollary 3.4. *(Theorem 1.10,*[6]*) Let $f : X \to \mathbb{R}$ be a continuous functional on a complete metric space (X, d). We consider a homotopy stable family $\mathcal{F}$ of compact subsets of X with closed boundary B and a dual family $\mathcal{F}^*$ of $\mathcal{F}$. Assume that*

$$\sup_{F \in \mathcal{F}^*} \inf_{x \in F} f(x) = \inf_{A \in \mathcal{F}} \max_{x \in A} f(x) = c$$

and suppose that the number c is finite.

Let $\varepsilon > 0$ and F a subset of X dual to the family $\mathcal{F}$ and satisfying the relation

$$\inf_{x \in F} f(x) \geq c - \frac{\varepsilon}{3\sqrt{1+\varepsilon^2}}.$$

Suppose that $0 < \varepsilon < \frac{dist(B,F)}{2}$, the for any set $A \in \mathcal{F}$ satisfying

$$\sup_{x \in A} f(x) \leq c + \frac{\varepsilon}{2\sqrt{1+\varepsilon^2}},$$

then there exists an $x_\varepsilon \in X$ such that:

i) $c - 2\varepsilon \leq f(x_\varepsilon) \leq c + 2\varepsilon$;

ii) $|df|(x_\varepsilon) \leq \varepsilon$;

iii) $dist(x_\varepsilon, F) \leq 2\varepsilon$;

iv) $dist(x_\varepsilon, A) \leq 2\varepsilon$.

Proof. From the assumption of theorem we see that

$$\sup_{F\in\mathcal{F}^*} \inf_{x\in F} f(x) = \inf_{A\in\mathcal{F}} \max_{x\in A} f(x) = c,$$

and exists $A \in \mathcal{F}$ and $F \in \mathcal{F}^*$ such that:

$$\sup_{x\in A} f(x) \leq c + \frac{\varepsilon}{2\sqrt{1+\varepsilon^2}},$$

$$\inf_{x\in F} f(x) \geq c - \frac{\varepsilon}{3\sqrt{1+\varepsilon^2}}.$$

The proof is same to proof of Theorem 3.1, if we choose $C := A \cap F$. We have that C is closed, because A is compact set and $A_1 = \eta(A, 1)$ is compact, because the function $x \mapsto \eta(x, 1)$ is continuous. Therefore we have $A_1 \in \mathcal{F}$.

Remark 3.5 If we choose different homotopy stable family we give different min-max results. For example if we choose the homotopy stable family $\mathcal{F} = \{\, A = g(K) \,|\, g \in \mathcal{C}(K, X) \text{ with } g(K_0) = \chi(K_0)\,\}$, where K is a compact metric space and $K_0 \subset K$ we obtain a generalization for continuous functionals of the Theorem 4.3, see [26].

4. Equivariant version of min-max result

In this section we give a generalization of some min-max and multiplicity results of Ghoussoub [21] for continuous functionals which represent an another proof of some min-max and multiplicity results of some results of Corvellec [5]. First we recall some definition and results from [3] and [5].

In this section (X, d) will denote a metric space and G a group of isometries of X, i.e.

$$G = \{g : X \to X| \;\; d(g(x), g(y)) = d(x, y), \;\; \text{for all} \;\; x, y \in X \;\; \text{and} \;\; g \in G\}.$$

As usual, we say that

$$A \subset X \text{ is G-invariant if } g(A) = A \text{ for all} g \in G;$$

$$h : X \to \mathbb{R} \text{ is G-invariant if } h \circ g = h \text{ for all } g \in G;$$

$$h : X \to X \text{ is G-equivariant if } h \circ g = g \circ h \text{ for all} g \in G.$$

Definition 4.1. Let (X, d) be a complete G- metric space and let $f :\to \mathbb{R}$ be a continuous, G-ivariant function and $u \in X$ a fixed element. We denote by $|df|_G(u)$ the supremum of the $\sigma \in [0, \infty[$ such that there exist $\delta > 0$ and a continuous map

$$\mathcal{H} : B(Gu, \delta) \times [0, \delta] \to \mathbb{R}$$

such that $\forall v \in B(Gu, \delta)$ *for all* $t \in [0, \delta]$ we have

a) $\eta(., t)$ is G-invariant for each $t \in [0, \delta]$

b) $d(\mathcal{H}(v, t), v) \leq t$

c) $f(\mathcal{H}(v, t)) \leq f(v) - \sigma t$

The extended real number $|df|_G(u)$ is called *the G-weak slope* of f at u.

The epigraph function $\mathcal{G}_f$ is Lipschitz continuous of constant 1 and is G-invariant, because the function f is G-invariant.

Proposition 4.2. *Let $f : X \to \mathbb{R}$ be a continuous function, G-inariant and let $(u, \xi) \in epi(f)$. Then*

$$|\mathcal{G}_f|_G(u, \xi) = \begin{cases} \dfrac{|df|_G(u)}{\sqrt{1 + |df|_G(u)^2}}, & \text{if } f(u) = \xi \text{and } |df|_G(u) < \infty, \\ 1 & \text{if } f(u) < \xi \text{or } |df|_G(u) = \infty. \end{cases}$$

We recall a result from Corvellec [5].

Proposition 4.3. *Let (X, d) be a complete G- metric space and let $f : X \to \mathbb{R}$ be a continuous G-invariant function, C a closed G-invariant subset of X and $\delta, \sigma > 0$ such that*

$$d(u, C) \leq \delta \Longrightarrow |df|_G(u) > \sigma.$$

Then there exists a continuous G-equivariant map $\eta : X \times [0, \delta] \to X$ such that

1) $d(\eta(u,t),u)) \leq t$,

2) $f(\eta(u,t)) \leq f(u)$,

3) $d(u,C) \geq \delta \Longrightarrow \eta(u,t) = u$,

4) $u \in C \Longrightarrow f(\eta(u,t)) \leq f(u) - \sigma t$.

We have the following equivariant version of Willem deformation lemma.

Theorem 4.4. *Let* (X,d) *be a complet G-metric space,* $f : X \to \mathbb{R}$ *a continuous G-invariant function,* C *a closed G-invariant subset of* X *and* $c \in \mathbb{R}$ *a real number. Let* $\varepsilon > 0$ *number such:*

$\forall\ x \in f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon}$: *we have* $|df|_G(x) > \varepsilon$.

Then there exists two real numbers $\varepsilon' \in (0,\varepsilon)$ *and* $\lambda > 0$ *and a continuous G-equivariant map* $\eta : X \times [0,1] \to X$ *such that:*

a') $d(\eta(u,t),u) \leq \lambda t$, *for every* $t \in [0,1]$.

b') $f(\eta(u,t)) \leq f(u)$, *for every* $t \in [0,1]$ *and* $x \in X$.

c') *if* $x \notin f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\delta} : \eta(x,t) = x, \quad \forall\ t \in [0,1]$.

d') $\eta(f^{c+\varepsilon'} \cap C, 1) \subset f^{c-\varepsilon'}$ *with* $\varepsilon' = \dfrac{\varepsilon}{2\sqrt{1+\varepsilon^2}}$.

e') $\forall\ t \in]0,1]$ *and* $\forall\ x \in f^c \cap C$ *we have* $f(\eta(t,x)) < c$.

Definition 4.5. Let X be a paracompact space on wich act a compact Lie G. We denote by $\mathcal{P}_G(X) = \{\, A \subset X \mid A$ closed invariant subset of $X \,\}$. A **topological index** Ind_G associated to a compact Lie group G is a function $Ind_G : \mathcal{P}_G(X) \to \mathbf{N} \cup \{\infty\}$ verifying the following properties:

(I1) $Ind_G(A) = 0$ if and only if $A = \emptyset$;

(I2) If $f : A_1 \to A_2$ is a G-equivariant continuous map the $Ind_G(A_1) \leq Ind_G(A_2)$;

(I3) If K is a compact invariant, then there exists a closed invariant neighborhood U of K, such that $Ind_G(U) = Ind_G(K)$.

(I4) $Ind_G(A_1 \cup A_2) \leq Ind_G(A_1) + Ind_G(A_2)$

(I5) If K is compact invariant set with $K \cap I(G) = \emptyset$, then K contains at least n orbits provided $ind_G(K) \geq n$, where $I(G) = \{\, x \in X \mid \exists g \in G \backslash \{e\}\ \text{with}\, gx = x \}$.

(I6) If K is a compact invariant set with $K \cap I(G) = \emptyset$, then $Ind_G(K) < +\infty$.

Definition 4.6. Let X be a G-paracompact space. We introduce the following notation:

$$\widehat{\mathcal{P}}_G(X) = \{ (A, B) \mid B \subset A \subset X \text{ and } A, B \text{are closed and invariant} \}.$$

A **relative index** is a function $Ind_G(,) : \widehat{\mathcal{P}} \to \mathbf{N} \cup \{+\infty\}$ such that we have:

(R1) $Ind_G(, \emptyset)$ verifies the properties (I1)-(I6) of the index and will be denoted $Ind_G()$.

(R2) If $f : (A_1, B) \to (A_2, B)$ is equivariant and $f_{|B}$ is a homeomrphism, then $Ind_G(A_1, B) \leq Ind_G(A_2, B)$.

(R3) $Ind_G(A_1 \cup A_2) \leq Ind_G(A_1, B) + Ind_G(A_2)$.

Examples:

4.1) If we consider the cat$_G$-category introduced by [19], [14], [27] or $\mathcal{A}$- category or relative $\mathcal{A}$-category or the $\mathcal{A}$-genus, see [14], [15],[12] we get another class of G-index and relative index.

4.2) The relative cohomological index introduced by Fadell and Husseini see [20], the eqivariant cup-length see [13] and the ideal valued index see [33], is another relative index.

Definition 4.7. Let B a closed subset of M. We shall say that the class $\mathcal{F}$ of subsets of M is **G-homotopy stable with boundary** B if:

(a) Every set in $\mathcal{F}$ is G-invariant;

(b) Every set in $\mathcal{F}$ contains B;

(c) For any set $A \in \mathcal{F}$ and any G-equivariant $\eta \in \mathcal{C}([0,1] \times M, M)$ verifying $\eta(t, x) = x$ for all $(t, x) \in (\{0\} \times M) \cup ([0,1] \times B)$ we have $\eta(1, A) \in \mathcal{F}$.

Examples:

4.3) If we consider the Cat$_G$-category or $\mathcal{A}$- category or relative $\mathcal{A}$-category or the $\mathcal{A}$-genus, see [14], [15],[12] we get another class of G-homotpy stable family.

4.4) In general, if we consider an index or a relative index we get different G-homotopy stable families.

In the next we generalize the result of Ghoussoub in the equivariant case see [21], for continuous and G-ivariant functionals.

Theorem 4.8. *Let (X, d) be a complete G-metric space and let $f : X \to \mathbb{R}$ be a continuous G-invariant function. Consider a G-homotopy stable family $\mathcal{F}$ of subsets of X with boundary B and a dual family $\mathcal{F}^\star$ of $\mathcal{F}$. Let $F \in \mathcal{F}^\star$ be a fixed which element which verifies the following condition*

$$\inf_{x \in F} f(x) \geq c, \tag{4.1}$$

where $c := \inf_{A \in \mathcal{F}} \sup_{x \in A} f(x)$.
Let $\varepsilon \in (0, \frac{dist(B,F)}{2})$ and $\delta > 0$ be arbitrarly fixed numbers. Then for any $A \in \mathcal{F}$ which verifies the relation

$$\sup_{x \in A \cap F_\delta} f(x) \leq c + \frac{\varepsilon}{2\sqrt{1+\varepsilon^2}}, \tag{4.2}$$

then there exists $x_\varepsilon \in X$ such that the following holds:

(i) $c - 2\varepsilon \leq f(x_\varepsilon) \leq c + 2\varepsilon$;

(ii) $|df|_G(x_\varepsilon) \leq \varepsilon$;

(iii) $dist(x_\varepsilon, F) \leq 2\varepsilon$;

(iv) $dist(x_\varepsilon, A) \leq 2\varepsilon$.

With the aid of Theorem 4.8 it is easy to prove a result which is very useful in state differant multiplicity results. For this we need the following definition.

Definition 4.9. We say that the continuous and G-invariant function $f : M \to \mathbf{R}$ verifies the G- Palais-Smale condition at the level c and around the set F (shortly $G - (PS)_{F,c}$) along the sequence $(A_n)_n \subset \mathcal{F}$ if any sequence $(x_n)_n \subset M$ verifying $f(x_n) \to c$, $||df|_G(x_n) \to 0$, $\text{dist}(x_n, F) \to 0$ and $\text{dist}(x_n, A_n) \to 0$ has a convergent subsequence.

We shall denote by $A_\infty = \{x \in M \mid \lim_{n\to\infty} \text{dist}(x, A_n) = 0\}$. Under the hypothesis of Theorem 4.8 and assuming that $f : M \to \mathbf{R}$ verifies $(PS)_{F,c}$ along a minimaxing sequence $(A_n)_n$, the set $F \cap K_c \cap A_\infty$ is non empty.

We have the following two general multiplicity results of Ghoussoub [21] for continuous G-invariant functions. Because the proof it is same way we omit.

Theorem 4.10. *(Ghoussoub-Corvellec) Let G ,M and c as in Theorem 4.8 and $f : M \to \mathbb{R}$ a G-invariant continuous function veryfying the condition G-(PS). Let $(\mathcal{F}_j)_{j=1}^N$ be an decreasing sequence of G-homotopy stable family with boundaries $(B_j)_{j=1}^N$ and verifying the following excision property with respect to an index Ind_G:*

(E) For every $1 \le j \le j+p \le N$ any $A \in \mathcal{F}_{j+p}$ and any U open and invariant such that $\overline{U} \cap B_j = \emptyset$ and $Ind_G(\overline{U}) \ge p$ we have $A \setminus U \in \mathcal{F}_j$.

Let F be a closed invariant set such that for each $1 \le j \le N$, F verifies (F1) and $\sup f(B) \le \inf f(F)$ with respect to $\mathcal{F}_j$. Set $c_j = \inf_{A\in\mathcal{F}_j} \sup_{x\in A} f(x)$, $d = \inf f(F)$ and let $M = \sup\{k : c_k = d\} \vee 0$. Then we have:

(a) $Ind_G(K_{c_M} \cap F \cap A_\infty) \ge M$ for every minimaxing sequence $(A_n)_n$ in $\mathcal{F}_M$.

(b) For every $M < i \le j+p \le N$ such that $c_j = c_{j+p}$ we have $Ind_G(K_{c_j} \cap A_\infty \ge p+1$ for every minimaxing sequence $(A_n)_n$ in $\mathcal{F}_{j+p}$. In particular if $I(G) \subset (M \setminus F) \cap (f \le d)$ then:

(c) f has at least N distinct critical orbits.

(d) If $N \to \infty$ then f has an unbounded critical value.

If in the Theorem 4.8 we take $Ind_G = cat_G$ and $F = M$ we get the following multiplicity result.

Corollary 4.11. *Let $f : M \to \mathbf{R}$ a G-invariant continuous function, which satisfied the G-(PS) condition and is bounded below, then f has at least $cat_G(M)$ distinct critical orbit.*

This corollary is a generalization for continuous G-invariant function of the Fadell multiplicity result for cat_G, see [19].

As a consequence of Corollary 4.11 is the main result from [32] and the main result from section 1 of [23] and Theorem 4.12 see [26].

Corollary 4.12. *Let G be a discrete subgroup of the Banach space X and $f : X \to \mathbb{R}$ a G-invariant continuous function wich satisfied the G-(PS) condition and is bounded below. If the dimension n of the space generated by G is finit, then f has at least $n+1$ distinct critical orbit.*

Proof. Using Corollary 4.11 we get f has at least $\text{cat}_G(X)$ distinct critical orbit. But $cat_G(X) \geq cat(X/G) = cat_{T^n}(T^n) = n+1$, where T^n is the n-dimensional torus.

Now we consider the group Lie $G = (S^1)^k \text{or} G = (\mathbf{Z}_p)^k$, $k \geq 1$ and let X be an infinite dimensional ortogonal reprsentation of the group G. Using Corollary 3 from [12] we get $\text{cat}_G(SX) = \infty$ if $X^G = 0$ and $\text{cat}_G(SX) = 2$ if $X^G \neq 0$, where SX denote the unit sphere in X and X^G the fixed point set of the group action G on X.

Corollary 4.13. *Let G, X be as above, and $f : X \to \mathbf{R}$ a G-invariant continuous function. We suppose that the function f is bounded from belov on SX and f satisfies the G-(PS) condition on SX. If $X^G = 0$ then f has an infinitely many distinct critical orbits on SX.*

The Corollary 4.13 is a generalization of the main result from section 3 of [23], where the authors are considered $\mathbf{Z}/\mathbf{p}$-action.

Corollary 4.14. *(Li-Santos) Let E be a complete metric space and $f : X \to \mathbb{R}$ a continuous functional. We suppose that the following conditions holds:*

1. *f satisfied the (PS) conditions;*
2. *There are teo disjoint sets A and B such that A is P-ideal linking to B;*
3. $\sup_{x \in A} f(x) \leq \inf_{x \in B} f(x)$;
4. *There exists a closed subset $\tilde{X} \supset A$ in E such that $\tilde{X} \setminus A$ is precompact and*

$$P - Index_E(\tilde{X}, A) = P - Index_E(E, A).$$

Then f possesses at least one critical value $c \geq \inf_{x \in B} f(x)$.

Proof. We denote by $\alpha = P - Index_E(E, A)$ and $\beta = P - Index_E(E \setminus B, A)$. Since A is P-ideal linking to B, we have $\beta \supset \alpha$ and $\beta \neq \alpha$. We define the set

$$\Sigma_\alpha = \{(X, A) \in \mathcal{E}_E \quad : \quad P - Index_E(X, A) = \alpha \ \},$$

where $\mathcal{E}_E$ is the class of all paracompact pair (X, A) in E. Note that $\Sigma_\alpha \neq \emptyset$ since $(\tilde{X}, A) \in \Sigma_\alpha$ and is homotopy stable with boundary A. Thus the conditions of Theorem 4.10 are satisfied and the conclusion of this corollary is true.

If we consider the relative index we have the following result of Ghoussoub for G-invariant continuous functions.

Theorem 4.15. *(Ghoussoub) Let G and M as in Theorem 4.8 and let* $f : M \to \mathbb{R}$ *be a continuous G-invariant function satifying the G-(PS) condition. Let B and F be two disjoint closed and invariant subset of M such that:*

(1) $k = Ind_G(M \setminus F) < Ind_G(X, B) = n.$

(2) $\sup f(B) \leq \inf f(F)$

(3) $I(G) \subset B.$

Then f has at least $n - k$ *distinct critical orbits. Moreover, if* $Ind_G(X, B) = \infty$*, then f has an unbounded sequence of critical values.*

Remark 4.16 If in the Theorem 4.15 we take for relative index the relativ cohomological index we get a generalization of Theorem 5.6 see [19]. If we take different relative index we obtain different multiplicity results for continuous G-invariant function.

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MAXWELL EQUATIONS FOR A GENERALIZED LAGRANGE SPACE OF ORDER 2 IN INVARIANT FRAMES

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Dedicated to Professor Pavel Enghiş at his 70th anniversary

Abstract. The study of higher order Lagrange spaces founded on the notion of bundle of velocities of order k has been given by Radu Miron and Gheorghe Atanasiu in [2]. The bundle of accelerations correspond in this study to k=2. The notion of invariant geometry of order 2 was introduced by the author in [4]. In this paper we shall give the Maxwell equations of a generalized Lagrange space of order 2 in invariant frames.

1. General Invariant Frames

Let us consider the bundle $E = Osc^2M$, a nonlinear connection N with the coefficients $\left(\underset{(1)}{N^i{}_j} \ , \ \underset{(2)}{N^i{}_j} \right)$ and the duals $\left(\underset{(1)}{M^i{}_j} \ , \ \underset{(2)}{M^i{}_j} \right)$.

The invariant frames adapted to the direct decomposition

$$T_u(\ Osc^2M\ ) \ = \ N_0(u) \oplus N_1(u) \oplus V_2(u) \quad \forall u \in E \tag{1}$$

will be $\Re \ = \ (\ e^{(0)i}_{\alpha}\ , e^{(1)i}_{\alpha}\ , e^{(2)i}_{\alpha}\ )$ and the dual $\Re^* \ = \ (\ f^{(0)\alpha}_{i}\ , f^{(1)\alpha}_{i}\ , f^{(2)\alpha}_{i}\ )$.

The duality conditions are

$$< \ e^{(A)i}_{\alpha}\ , f^{(B)\alpha}_{j} \ > = \ \delta^i_j\, \delta^A_B \quad (\,A, B = 0, 1, 2\,) \tag{2}$$

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In this frame the adapted basis has the representation

$$\frac{\delta}{\delta x^i} = f_i^{(0)\alpha} \frac{\delta}{\delta s^{(0)\alpha}} \quad \frac{\delta}{\delta y^{(1)i}} = f_i^{(1)\alpha} \frac{\delta}{\delta s^{(1)\alpha}} \quad \frac{\delta}{\delta y^{(2)i}} = f_i^{(2)\alpha} \frac{\delta}{\delta s^{(2)\alpha}} \tag{3}$$

and the cobasis

$$\delta x^i = e_\alpha^{(0)i} \delta s^{(0)\alpha} \quad ; \delta y^{(1)i} = e_\alpha^{(1)i} \delta s^{(1)\alpha} \quad ; \delta y^{(2)i} = e_\alpha^{(2)i} \delta s^{(2)\alpha} \tag{4}$$

and we have the relations

$$\left\langle \frac{\delta}{\delta s^{(A)\alpha}}, \delta s^{(B)\beta} \right\rangle = \delta_\alpha^\beta \, \delta_A^B \qquad (A, B = 0, 1, 2) \tag{5}$$

This representation lead us to an invariant frames transformation group with the analitycal expressions

$$\bar{e}_\alpha^{(A)i} = \overset{A}{C}{}_\alpha^\beta \, (x, y^{(1)}, y^{(2)}), e_\beta^{(A)i} \quad ; \quad f_j^{(B)\alpha} = \overset{B}{\bar{C}}{}_\beta^\alpha \, \bar{f}_j^{(B)\beta} \tag{6}$$

isomorphic with the multiplicative nonsingular matrix group

$$\begin{pmatrix} \overset{0}{C}{}_\beta^\alpha & 0 & 0 \\ 0 & \overset{1}{C}{}_\beta^\alpha & 0 \\ 0 & 0 & \overset{2}{C}{}_\beta^\alpha \end{pmatrix}$$

A N-linear connection D has in the frame $\mathfrak{R}$ the coefficients

$$\overset{0A}{L}{}_{\beta\alpha}^\gamma = f_m^{(A)\gamma} \left(\frac{\delta e_\beta^{(A)m}}{\delta s^{(0)\alpha}} + e_\alpha^{(0)i} e_\beta^{(A)j} L_{ij}^m \right) \quad (A = 0, 1, 2) \tag{7}$$

$$\overset{BA}{C}{}_{\beta\alpha}^\gamma = f_m^{(A)\gamma} \left(\frac{\delta e_\beta^{(A)m}}{\delta s^{(B)\alpha}} + e_\alpha^{(B)i} e_\beta^{(A)j} \underset{(B)}{C_{ij}^m} \right) \quad (A = 0, 1, 2 \, ; \, B = 1, 2) \tag{8}$$

Definition 1.1. *If the vector field $X \in \chi(E)$ has the invariant components $X^{(A)\alpha}$ (A=0,1,2) and we denote by '$)$', $\overset{B}{)}$ the $h-$ and v_B, B=1,2 the covariant invariant derivative operators then*

$$X^{(A)\alpha}{}_{)\beta} = \frac{\delta X^{(A)\alpha}}{\delta s^{(0)\beta}} + \overset{0A}{L^{\alpha}_{\varphi\beta}} X^{(A)\varphi} \tag{9}$$
$$X^{(A)\alpha}\overset{(B)}{)_{\beta}} = \frac{\delta X^{(A)\alpha}}{\delta s^{(B)\beta}} + \overset{BA}{C^{\alpha}_{\varphi\beta}} X^{(A)\varphi}$$

The definition of the Lie bracket conduces us to the introduction of the non-holonomy coefficients of Vranceanu

$$\left[\frac{\delta}{\delta s^{(A)\alpha}}, \frac{\delta}{\delta s^{((B)\beta}}\right] = \underset{(AB)}{\overset{0}{W^{\gamma}_{\alpha\beta}}} \frac{\delta}{\delta s^{(0)\gamma}} + \underset{(AB)}{\overset{1}{W^{\gamma}_{\alpha\beta}}} \frac{\delta}{\delta s^{(1)\gamma}} + \underset{(AB)}{\overset{2}{W^{\gamma}_{\alpha\beta}}} \frac{\delta}{\delta s^{(2)\gamma}} \tag{10}$$

$(A, B = 0, 1, 2; \quad A \leq B)$.

2. Torsion and Curvature d-tensor Fields

The torsion tensor of the N-linear connection D on E

$$\mathcal{T}(X,Y) = D_X Y - D_Y X - [X,Y] \qquad \forall\, X,Y \in \chi(E) \tag{11}$$

in the invariant frame $\Re$, has a number of horizontal and vertical components corresponding to D^h , D^{v_1} , D^{v_2}

Theorem 2.1. *The torsion tensor of a N-linear connection D in the invariant frame $\Re$ is characterized by the d-tensor fields with local components*

$$\begin{cases} \underset{(0)}{T^{\gamma}{}_{\beta\alpha}} = \overset{(00)}{L^{\gamma}{}_{\beta\alpha}} - \overset{(00)}{L^{\gamma}{}_{\alpha\beta}} - \underset{(00)}{\overset{(0)}{W^{\gamma}{}_{\beta\alpha}}} \\ \underset{(0A)}{R^{\gamma}{}_{\beta\alpha}} = \underset{(00)}{\overset{(A)}{W^{\gamma}{}_{\beta\alpha}}} \end{cases} \tag{12}$$

$$\begin{cases} \underset{(1)}{K^{\gamma}{}_{\beta\alpha}} = -\underset{(1)}{\overset{(10)}{C^{\gamma}{}_{\beta\alpha}}} - \underset{(01)}{\overset{(0)}{W^{\gamma}{}_{\beta\alpha}}} \\ \underset{(11)}{P^{\gamma}{}_{\beta\alpha}} = \overset{(01)}{L^{\gamma}{}_{\beta\alpha}} + \underset{(01)}{\overset{(1)}{W^{\gamma}{}_{\beta\alpha}}} \\ \underset{(12)}{P^{\gamma}{}_{\beta\alpha}} = \underset{(01)}{\overset{(2)}{W^{\gamma}{}_{\beta\alpha}}} \end{cases} \tag{13}$$

$$\begin{cases} \underset{(2)}{K^{\gamma}{}_{\beta\alpha}} = -\underset{(2)}{\overset{(20)}{C^{\gamma}{}_{\beta\alpha}}} - \underset{(02)}{\overset{(0)}{W^{\gamma}{}_{\beta\alpha}}} \\ \underset{(21)}{P^{\gamma}{}_{\beta\alpha}} = \underset{(02)}{\overset{(1)}{W^{\gamma}{}_{\alpha\beta}}} - \underset{(01)}{\overset{(1)}{W^{\gamma}{}_{\beta\alpha}}} \\ \underset{(22)}{P^{\gamma}{}_{\beta\alpha}} = \overset{(02)}{L^{\gamma}{}_{\beta\alpha}} - \underset{(02)}{\overset{(2)}{W^{\gamma}{}_{\beta\alpha}}} \end{cases} \tag{14}$$

$$\begin{cases} \underset{(11)}{Q^{\gamma}{}_{\beta\alpha}} = \underset{(1)}{\overset{(11)}{C^{\gamma}{}_{\beta\alpha}}} - \underset{(1)}{\overset{(11)}{C^{\gamma}{}_{\alpha\beta}}} - \underset{(11)}{\overset{(1)}{W^{\gamma}{}_{\beta\alpha}}} \\ \underset{(21)}{\overset{(2)}{Q^{\gamma}{}_{\beta\alpha}}} = \underset{(11)}{\overset{(2)}{W^{\gamma}{}_{\beta\alpha}}} \end{cases} \tag{15}$$

$$\left\{\begin{array}{l} \underset{(12)}{Q^{\gamma}{}_{\beta\alpha}} = \underset{(2)}{\overset{(21)}{C^{\gamma}{}_{\beta\alpha}}} - \underset{(12)}{\overset{(1)}{W^{\gamma}{}_{\beta\alpha}}} \\ \underset{(22)}{Q^{\gamma}{}_{\beta\alpha}} = -\underset{(12)}{\overset{(1)}{C^{\gamma}{}_{\alpha\beta}}} - \underset{(12)}{\overset{(2)}{W^{\gamma}{}_{\beta\alpha}}} \end{array}\right. \tag{16}$$

$$\left\{ \underset{(2)}{S^{\gamma}{}_{\beta\alpha}} = \underset{(2)}{\overset{(22)}{C^{\gamma}{}_{\beta\alpha}}} - \underset{(2)}{\overset{(22)}{C^{\gamma}{}_{\alpha\beta}}} - \underset{(22)}{\overset{(2)}{W^{\gamma}{}_{\beta\alpha}}} \right. \tag{17}$$

Theorem 2.2. *The components given by Theorem 2.1 are the invariant components of the d-tensor fields of torsion of the N-linear connection D*

The curvature tensor field $\mathcal{R}$ of the N-linear connection D on $Osc^2(M)$ has the expression

$$\mathcal{R}(X,Y) = [\, D_X \,,\, D_Y \,]\, Z - D_{[X,Y]}Z \tag{18}$$

Theorem 2.3. *The curvature tensor field $\mathcal{R}$ of a N-linear connection D in the invariant frame $\Re$ is characterized by the following d-tensor fields on $Osc^2(M)$:*

$$\begin{aligned} R_{\gamma}{}^{\varphi}{}_{\beta\alpha} = {} & \frac{\delta \overset{(00)}{L^{\varphi}_{\gamma\beta}}}{\delta s^{(0)\alpha}} - \frac{\delta \overset{(00)}{L^{\varphi}_{\gamma\alpha}}}{\delta s^{(0)\beta}} + \overset{(00)}{L^{\eta}_{\gamma\beta}}\overset{(00)}{L^{\varphi}_{\eta\alpha}} - \overset{(00)}{L^{\eta}_{\gamma\alpha}}\overset{(00)}{L^{\varphi}_{\eta\beta}} - \\ & - \underset{(00)}{\overset{(0)}{W^{\psi}_{\beta\alpha}}}\ \overset{(00)}{L^{\varphi}_{\gamma\psi}} + \underset{(00)}{\overset{(1)}{W^{\psi}_{\beta\alpha}}}\ \underset{(1)}{\overset{(10)}{C^{\varphi}_{\gamma\psi}}} + \underset{(00)}{\overset{(2)}{W^{\psi}_{\beta\alpha}}}\ \underset{(2)}{\overset{(20)}{C^{\varphi}_{\gamma\psi}}} \end{aligned} \tag{19}$$

$$\underset{(1)}{P_{\gamma}{}^{\varphi}{}_{\beta\alpha}} = \frac{\delta \underset{(1)}{\overset{(10)}{C^{\varphi}_{\gamma\beta}}}}{\delta s^{(0)\alpha}} - \frac{\delta \overset{(00)}{L^{\varphi}_{\gamma\alpha}}}{\delta s^{(1)\beta}} + \underset{(1)}{\overset{(10)}{C^{\eta}_{\gamma\beta}}} \overset{(00)}{L^{\varphi}_{\eta\alpha}} - \overset{(00)}{L^{\eta}_{\gamma\alpha}} \underset{(1)}{\overset{(10)}{C^{\varphi}_{\eta\beta}}} - \underset{(01)}{\overset{(0)}{W^{\psi}_{\beta\alpha}}} \overset{(00)}{L^{\varphi}_{\gamma\psi}} + \underset{(01)}{\overset{(1)}{W^{\psi}_{\beta\alpha}}} \underset{(1)}{\overset{(10)}{C^{\varphi}_{\gamma\psi}}} + \underset{(01)}{\overset{(2)}{W^{\psi}_{\beta\alpha}}} \underset{(2)}{\overset{(20)}{C^{\varphi}_{\gamma\psi}}} \tag{20}$$

$$\underset{(2)}{P_{\gamma}{}^{\varphi}{}_{\beta\alpha}} = \frac{\delta \underset{(2)}{\overset{(20)}{C^{\varphi}_{\gamma\beta}}}}{\delta s^{(0)\alpha}} - \frac{\delta \overset{(00)}{L^{\varphi}_{\gamma\alpha}}}{\delta s^{(2)\beta}} + \underset{(2)}{\overset{(20)}{C^{\eta}_{\gamma\beta}}} \overset{(00)}{L^{\varphi}_{\eta\alpha}} - \overset{(00)}{L^{\eta}_{\gamma\alpha}} \underset{(2)}{\overset{(20)}{C^{\varphi}_{\eta\beta}}} - \underset{(02)}{\overset{(0)}{W^{\psi}_{\beta\alpha}}} \overset{(00)}{L^{\varphi}_{\gamma\psi}} + \underset{(02)}{\overset{(1)}{W^{\psi}_{\beta\alpha}}} \underset{(1)}{\overset{(10)}{C^{\varphi}_{\gamma\psi}}} + \underset{(02)}{\overset{(2)}{W^{\psi}_{\beta\alpha}}} \underset{(2)}{\overset{(20)}{C^{\varphi}_{\gamma\psi}}} \tag{21}$$

$$\underset{(11)}{S_{\gamma}{}^{\varphi}{}_{\beta\alpha}} = \frac{\delta \underset{(1)}{\overset{(10)}{C^{\varphi}_{\gamma\beta}}}}{\delta s^{(1)\alpha}} - \frac{\delta \underset{(1)}{\overset{(01)}{C^{\varphi}_{\gamma\alpha}}}}{\delta s^{(1)\beta}} + \underset{(1)}{\overset{(10)}{C^{\eta}_{\gamma\beta}}} \underset{(1)}{\overset{(10)}{C^{\varphi}_{\eta\alpha}}} - \underset{(1)}{\overset{(10)}{C^{\eta}_{\gamma\alpha}}} \underset{(1)}{\overset{(10)}{C^{\varphi}_{\eta\beta}}} - \underset{(11)}{\overset{(1)}{W^{\psi}_{\beta\alpha}}} \underset{(1)}{\overset{(10)}{C^{\varphi}_{\gamma\psi}}} - \underset{(11)}{\overset{(2)}{W^{\psi}_{\beta\alpha}}} \underset{(2)}{\overset{(20)}{C^{\varphi}_{\gamma\psi}}} \tag{22}$$

$$\underset{(21)}{S}{}_{\gamma\ \beta\alpha}^{\ \varphi} = \frac{\delta \overset{(20)}{\underset{(2)}{C}}{}^{\varphi}_{\gamma\beta}}{\delta s^{(1)\alpha}} - \frac{\delta \overset{(10)}{\underset{(1)}{C}}{}^{\varphi}_{\gamma\alpha}}{\delta s^{(2)\beta}} + \overset{(20)}{\underset{(2)}{C}}{}^{\eta}_{\gamma\beta} \overset{(10)}{\underset{(1)}{C}}{}^{\varphi}_{\eta\alpha} - \overset{(10)}{\underset{(1)}{C}}{}^{\eta}_{\gamma\alpha} \overset{(20)}{\underset{(2)}{C}}{}^{\varphi}_{\eta\beta} - \overset{(1)}{\underset{(12)}{W}}{}^{\psi}_{\beta\alpha} \overset{(10)}{\underset{(1)}{C}}{}^{\varphi}_{\gamma\psi} - \overset{(2)}{\underset{(12)}{W}}{}^{\psi}_{\beta\alpha} \overset{(20)}{\underset{(2)}{C}}{}^{\varphi}_{\gamma\psi} \tag{23}$$

$$\underset{(22)}{S}{}_{\gamma\ \beta\alpha}^{\ \varphi} = \frac{\delta \overset{(20)}{\underset{(2)}{C}}{}^{\varphi}_{\gamma\beta}}{\delta s^{(2)\alpha}} - \frac{\delta \overset{(20)}{\underset{(2)}{C}}{}^{\varphi}_{\gamma\alpha}}{\delta s^{(2)\beta}} + \overset{(20)}{\underset{(2)}{C}}{}^{\eta}_{\gamma\beta} \overset{(20)}{\underset{(2)}{C}}{}^{\varphi}_{\eta\alpha} - \overset{(20)}{\underset{(2)}{C}}{}^{\eta}_{\gamma\alpha} \overset{(20)}{\underset{(2)}{C}}{}^{\varphi}_{\eta\beta} - \overset{(2)}{\underset{(22)}{W}}{}^{\psi}_{\beta\alpha} \overset{(20)}{\underset{(2)}{C}}{}^{\varphi}_{\gamma\psi} \tag{24}$$

Theorem 2.4. *The components given by Theorem 2.3 are the invariant components of the d-tensor fields of curvature of the N-linear connection D*

Theorem 2.5. *In the frame $\mathfrak{R}$ the essential components of the curvature tensor field $\mathcal{R}$ are those given by Theorem 2.3.*

3. Fundamental Identities, Maxwell Equations

Begining from Jacoby identities we obtain

Theorem 3.1. *The non-holonomy coefficients W given by satisfy the following fundamental identities called Vranceanu identities:*

$$\sum_{\substack{cicl\\(\alpha\ \beta\ \gamma)}} \left\{ \underset{(00)}{\overset{(I)}{W}}{}^{\sigma}{}_{\beta\gamma}\ \underset{(0I)}{\overset{(J)}{W}}{}^{\eta}{}_{\alpha\sigma} + \frac{\underset{(00)}{\overset{(J)}{W}}{}^{\eta}{}_{\beta\gamma}}{\delta s^{(I)\alpha}} \right\} = 0 \qquad (25)$$

(I,J=0,1,2; summation also by I)

$$\underset{(\alpha,\beta)}{\succ\!\prec} \left\{ \underset{(0K)}{\overset{(I)}{W}}{}^{\sigma}{}_{\beta\gamma}\ \underset{(0I)}{\overset{(J)}{W}}{}^{\eta}{}_{\alpha\sigma} + \frac{\underset{(0K)}{\overset{(J)}{W}}{}^{\eta}{}_{\beta\gamma}}{\delta s^{(0)\alpha}} + \frac{1}{2}\ \underset{(00)}{\overset{(I)}{W}}{}^{\sigma}{}_{\alpha\beta}\ \underset{(IK)}{\overset{(J)}{W}}{}^{\eta}{}_{\sigma\gamma} + \frac{1}{2} \frac{\underset{(00)}{\overset{(J)}{W}}{}^{\eta}{}_{\alpha\beta}}{\delta s^{(K)\gamma}} \right\} = 0 \qquad (26)$$

$$\underset{(\beta,\gamma)}{\succ\!\prec} \left\{ \underset{(0K)}{\overset{(I)}{W}}{}^{\sigma}{}_{\gamma\alpha}\ \underset{(0I)}{\overset{(J)}{W}}{}^{\eta}{}_{\beta\alpha} + \frac{\underset{(0K)}{\overset{(J)}{W}}{}^{\eta}{}_{\gamma\alpha}}{\delta s^{(1)\beta}} + \frac{1}{2}\ \underset{(KK)}{\overset{(I)}{W}}{}^{\sigma}{}_{\beta\gamma}\ \underset{(KI)}{\overset{(J)}{W}}{}^{\eta}{}_{\alpha\sigma} + \frac{1}{2} \frac{\underset{(KK)}{\overset{(J)}{W}}{}^{\eta}{}_{\beta\gamma}}{\delta s^{(0)\alpha}} \right\} = 0 \qquad (27)$$

(I,J=0,1,2; K=1,2; ≻≺ meaning permutation of indexes and subtraction of results)

$$\underset{(12)}{\overset{(1)}{W}}{}^{\sigma}{}_{\beta\gamma}\ \underset{(01)}{\overset{(0)}{W}}{}^{\eta}{}_{\alpha\sigma} + \underset{(12)}{\overset{(2)}{W}}{}^{\sigma}{}_{\beta\gamma}\ \underset{(02)}{\overset{(0)}{W}}{}^{\eta}{}_{\alpha\sigma} + \underset{(02)}{\overset{(0)}{W}}{}^{\sigma}{}_{\gamma\alpha}\ \underset{(01)}{\overset{(0)}{W}}{}^{\eta}{}_{\sigma\beta} -$$

$$-\underset{(01)}{\overset{(0)}{W^{\sigma}{}_{\alpha\beta}}}\ \underset{(02)}{\overset{(0)}{W^{\eta}{}_{\sigma\gamma}}} + \Sigma\frac{\delta\, \underset{(12)}{\overset{(0)}{W^{\eta}{}_{\beta\gamma}}}}{\delta s^{(0)\alpha}} = 0 \tag{28}$$

$$\underset{(12)}{\overset{(I)}{W^{\sigma}{}_{\beta\gamma}}}\ \underset{(0I)}{\overset{(1)}{W^{\eta}{}_{\alpha\sigma}}} + \underset{(02)}{\overset{(I)}{W^{\sigma}{}_{\gamma\alpha}}}\ \underset{(I1)}{\overset{(1)}{W^{\eta}{}_{\sigma\beta}}} - \underset{(01)}{\overset{(I)}{W^{\sigma}{}_{\alpha\beta}}}\ \underset{(I2)}{\overset{(1)}{W^{\eta}{}_{\sigma\gamma}}} + \Sigma\frac{\delta\, \underset{(12)}{\overset{(1)}{W^{\eta}{}_{\beta\gamma}}}}{\delta s^{(0)\alpha}} = 0 \tag{29}$$

$$\underset{(12)}{\overset{(I)}{W^{\sigma}{}_{\beta\gamma}}}\ \underset{(0I)}{\overset{(2)}{W^{\eta}{}_{\alpha\sigma}}} + \underset{(02)}{\overset{(I)}{W^{\sigma}{}_{\gamma\alpha}}}\ \underset{(I1)}{\overset{(2)}{W^{\eta}{}_{\sigma\beta}}} - \underset{(01)}{\overset{(I)}{W^{\sigma}{}_{\alpha\beta}}}\ \underset{(I2)}{\overset{(2)}{W^{\eta}{}_{\sigma\gamma}}} + \Sigma\frac{\delta\, \underset{(12)}{\overset{(2)}{W^{\eta}{}_{\beta\gamma}}}}{\delta s^{(0)\alpha}} = 0 \tag{30}$$

(I=0,1,2; summation by I; Σ meaning sumation on simultaneous cycle on pairs $(0,\alpha); (1,\beta); (2,\gamma)$

Denoting by

$$q^{((1)\alpha} = s^{(1)\alpha}\ ; \quad q^{(2)\alpha} = s^{(2)\alpha} + \frac{1}{2}\underset{(1)}{M^{\alpha}{}_{\beta}}\, s^{(1)\beta} \tag{31}$$

then in the considerated invariant frame the Liouville vector fields are:

$$\overset{1}{\Gamma} = q^{(1)\alpha} e^{(1)i}_{\alpha} f^{(2)\beta}_{i} \frac{\delta}{\delta s^{(2)\beta}}$$

$$\overset{2}{\Gamma} = q^{(1)\alpha}\frac{\delta}{\delta s^{(1)\alpha}} + 2\, q^{(2)\alpha}\frac{\delta}{\delta s^{(2)\alpha}} \tag{32}$$

Let us consider the generalized Lagrange space $GL^{(2n)} = (M, g_{ij}(x, y^{(1)}, y^{(2)}))$ with g_{ij} symmetric and nondegenerated, the canonical metrical linear N-connection $L\Gamma(N)$ and the case when the three frames adapted to the three distributions are the same. Then

$$e^{(0)i}_{\alpha} = e^{(1)i}_{\alpha} = e^{(2)i}_{\alpha} = e^{i}_{\alpha} \tag{33}$$

and simillar for the duals.

In this frames the canonical metrical linear N-connection has the coefficients:

$$\overset{(00)}{L}{}^{\alpha}_{\beta\gamma} = \frac{1}{2}\underset{(00)}{\overset{(0)}{W}}{}^{\alpha}_{\beta\gamma} + f_i^{\alpha} e^j_{\beta} e^k_{\gamma} L^i_{jk}$$

$$\underset{(1)}{\overset{(11)}{C}}{}^{\alpha}_{\beta\gamma} = \frac{1}{2}\underset{(11)}{\overset{(1)}{W}}{}^{\alpha}_{\beta\gamma} + f_i^{\alpha} e^j_{\beta} e^k_{\gamma} \underset{(1)}{C}{}^{\alpha}_{\beta\gamma}$$

$$\underset{(2)}{\overset{(22)}{C}}{}^{\alpha}_{\beta\gamma} = \frac{1}{2}\underset{(22)}{\overset{(2)}{W}}{}^{\alpha}_{\beta\gamma} + f_i^{\alpha} e^j_{\beta} e^k_{\gamma} \underset{(2)}{C}{}^{\alpha}_{\beta\gamma} \tag{34}$$

We shall consider now the tensor fields

$$D^{(A)\alpha}_{\beta} = q^{(A)\alpha}_{\ \ )\beta} \qquad d^{(AB)\alpha}_{\beta} = q^{(A)\alpha}\overset{(B)}{)} \tag{35}$$

(A,B=1,2)

Theorem 3.2.. *The tensor fields defined above represent the invariant conponents of the deflection tensor of the canonical metrical N-linear connection.*

We define the invariant electromagnetic tensor field by:

$$F^{(A)}_{\alpha\beta} = \frac{1}{2}\left\{\frac{\delta q^{(A)}_{\alpha}}{\delta s^{(0)\beta}} - \frac{\delta q^{(A)}_{\beta}}{\delta s^{(0)\alpha}}\right\}$$

$$f^{(AB)}_{\alpha\beta} = \frac{1}{2}\left\{\frac{\delta q^{(A)}_{\alpha}}{\delta s^{(B)\beta}} - \frac{\delta q^{(A)}_{\beta}}{\delta s^{(B)\alpha}}\right\} \tag{36}$$

(A,B=1,2)

Theorem 3.3.. *The electromagnetic tensor fields have the expressions*

$$F^{(A)}_{\alpha\beta} = \frac{1}{2}\left(D^{(A)}_{\alpha\beta} - D^{(A)}_{\beta\alpha}\right)$$

$$f^{(AB)}_{\alpha\beta} = \frac{1}{2}\left(d^{(AB)}_{\alpha\beta} - d^{(AB)}_{\beta\alpha}\right) \tag{37}$$

and represent the invariant components of the electromagnetic tensor fields of the cononical metrical N-linear connection.

Using Ricci identities with respect to $C\Gamma(N)$ we prove

Theorem 3.4. *The electromagnetic tensor fields $F^{(A)}_{\alpha\beta}$ and $f^{(AB)}_{\alpha\beta}$ of the generalized Lagrange space $GL^{(2n)}$ satisfy the following Maxwell generalized equations:*

$$\Sigma F^{(A)}_{\alpha\beta|\gamma} = \Sigma \left\{ q^{(A)\eta} R_{\eta\beta\alpha\gamma} - \sum_{B=1}^{2} d^{(AB)}_{\beta\eta} \underset{(0B)}{R^{\eta}{}_{\alpha\gamma}} \right\}$$

$$\Sigma F^{(A)}_{\alpha\beta} \overset{(B)}{|}_{\gamma} + \Sigma f^{(AB)}_{\alpha\beta|\gamma} = \Sigma \left\{ q^{(A)\eta} \left(\underset{(B)}{P_{\eta\beta\alpha\gamma}} - \underset{(B)}{P_{\eta\beta\gamma\alpha}} \right) - \sum_{B=1}^{2} d^{(AB)}_{\beta\eta} \left(\underset{(B)}{P_{\eta\beta\alpha\gamma}} - \underset{(B)}{P_{\eta\beta\gamma\alpha}} \right) \right\}$$

$$\Sigma f^{(AB)}_{\alpha\beta} \overset{(B)}{|}_{\gamma} = \Sigma \left\{ q^{(A)\eta} \underset{(A)}{S_{\eta\beta\alpha\gamma}} - \sum_{C=1}^{2} d^{(AB)}_{\beta\eta} \left(\underset{(BC)}{R^{\eta}{}_{\alpha\gamma}} \right\}\right.$$

$$\Sigma f^{(AB)}_{\alpha\beta} \overset{(C)}{|}_{\gamma} = \Sigma \left\{ q^{(A)\eta} \underset{(BC)}{P_{\eta\beta\alpha\gamma}} - \sum_{D=1}^{2} d^{(AB)}_{\beta\eta} \left(\underset{(BC)}{\overset{(D)}{P^{\eta}{}_{\alpha\gamma}}} - d^{(AB)}_{\beta\eta} \left(\underset{(B)}{C^{\eta}{}_{\alpha\gamma}} - \underset{(B)}{C^{\eta}{}_{\gamma\alpha}} \right) \right\} \quad B \neq C \right. \tag{38}$$

Theorem 3.5. *If the canonical metrical N-linear connection is torsionless then* $F^{(A)}_{\alpha\beta}$ *and* $f^{(AB)}_{\alpha\beta}$ *satisfy the folowing generalized Maxwell equations:*

$$\begin{aligned} \Sigma F^{(A)}_{\alpha\beta|\gamma} &= 0 \\ \Sigma F^{(A)}_{\alpha\beta} \overset{(B)}{|}_{\gamma} + \Sigma f^{(AB)}_{\alpha\beta|\gamma} &= 0 \\ \Sigma f^{(AB)}_{\alpha\beta} \overset{(B)}{|}_{\gamma} &= 0 \\ \Sigma f^{(AB)}_{\alpha\beta} \overset{(C)}{|}_{\gamma} &= 0 \end{aligned} \tag{39}$$

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THE ASSOCIATED LOCUS OF SOME HYPERSURFACES IN $\mathbf{R}^{n+1}$

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Dedicated to Professor Pavel Enghiş at his 70th anniversary

Abstract. For a smooth hypersurface of the space $\mathbf{R}^{n+1}$ project orthogonally the origin of $\mathbf{R}^{n+1}$ on its tangent hyperplanes and call the set of all projections *the associated locus* of the given hypersurface. In this paper we are going to find the equation of the associated locus for some given hypersurfaces and to show that it is a smooth hypersurface diffeomorphic with the initial one. We will also show that in one particular case both of them, the hypersurface and its associated locus, are diffeomorphic with the n-dimensional sphere.

1. Introduction

In this section we recall a simple fact concerning homogeneous functions which will be very useful for all over this paper.

Definition 1.1. A function $f : \mathbf{R}^{n+1}\backslash\{0\} \to \mathbf{R}$ is called *homogeneous of order* $\alpha \in \mathbf{R}$ if $f(tx) = t^\alpha f(x)$ for all $t > 0$ and all $x \in \mathbf{R}^{n+1}\backslash\{0\}$.

Lemma 1.2. *If* $f : \mathbf{R}^{n+1}\backslash\{0\} \to \mathbf{R}$ *is a smooth homogeneous function of order* $\alpha \in \mathbf{R}^*$ *and* $c \in \mathbf{R}^*$, *then* $f^{-1}(c)$ *is either the empty set, or* $f^{-1}(c)$ *is a smooth hypersurface of* $\mathbf{R}^{n+1}$.

Example 1.3. Let α be a natural number, $\beta \in \{1, \dots, n+1\}$ and $a = (a_1, \dots, a_{n+1}) \in \mathbf{R}^{n+1}$ be such that $a_i \neq 0 \ \forall i \in \{1, \dots, n+1\}$. Then the set

$$H^\alpha_\beta = \left\{x = (x_1, \dots, x_{n+1}) \in \mathbf{R}^{n+1}\backslash\{0\} \,\middle|\, \frac{x_1^{2\alpha}}{a_1^{2\alpha}} + \dots + \frac{x_\beta^{2\alpha}}{a_\beta^{2\alpha}} - \frac{x_{\beta+1}^{2\alpha}}{a_{\beta+1}^{2\alpha}} - \dots - \frac{x_{n+1}^{2\alpha}}{a_{n+1}^{2\alpha}} = 1\right\}$$

is a hypersurface of $\mathbf{R}^{n+1}$.

Observe that H_β^α can be also represented as $H_\beta^\alpha = f^{-1}(1)$, where

$$f : \mathbf{R}^{n+1}\setminus\{0\} \to \mathbf{R},\ f(x_1, \dots, x_{n+1}) = \frac{x_1^{2\alpha}}{a_1^{2\alpha}} + \cdots + \frac{x_\beta^{2\alpha}}{a_\beta^{2\alpha}} - \frac{x_{\beta+1}^{2\alpha}}{a_{\beta+1}^{2\alpha}} - \cdots - \frac{x_{n+1}^{2\alpha}}{a_{n+1}^{2\alpha}}.$$

2. The associated locus of the hypersurface H_β^α

Let $x = (x_1, \dots, x_{n+1}) \in \mathbf{R}^{n+1}$ and $\alpha \in \mathbf{R}$ be such that x_i^α exist for all $i \in \{1, \dots, n+1\}$. Denote by x^α the vector $(x_1^\alpha, \dots, x_{n+1}^\alpha)$ and observe that $x^2 = ||x||^2$ for all $x \in \mathbf{R}^{n+1}$ and that $(tx)^\alpha = t^\alpha x^\alpha$ for all $t > 0$. Also, if there exist the vectors $x^{\alpha\beta}$ and $(x^\alpha)^\beta$, for the real numbers α, β, then $x^{\alpha\beta} = (x^\alpha)^\beta$. Using this notation the equation of H_β^α can be rewritten as follows:

$$H_\beta^\alpha : \varphi(a^{-2\alpha}, x^{2\alpha}) = 1 \tag{1}$$

where $\varphi : \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}$ is the nondegenerate biliniar symmetric form given by

$$\varphi(x, y) = x_1 y_1 + \cdots + x_\beta y_\beta - x_{\beta+1} y_{\beta+1} - \cdots - x_{n+1} y_{n+1}$$

for all $x = (x_1, \dots, x_{n+1}), y = (y_1, \dots, y_{n+1}) \in \mathbf{R}^{n+1}$.

Theorem 2.1. *The associated locus $\mathcal{L}_\beta^\alpha$ of H_β^α is the set*

$$\left\{x \in \mathbf{R}^{n+1}\setminus\{0\} \;\middle|\; ||x||^{\frac{4\alpha}{2\alpha-1}} = \varphi(a^{\frac{2\alpha}{2\alpha-1}}, x^{\frac{2\alpha}{2\alpha-1}})\right\}.$$

Proof. Denote by A_β^α the set

$$\left\{x \in \mathbf{R}^{n+1}\setminus\{0\} \;\middle|\; ||x||^{\frac{4\alpha}{2\alpha-1}} = \varphi(a^{\frac{2\alpha}{2\alpha-1}}, x^{\frac{2\alpha}{2\alpha-1}})\right\}$$

and consider $p^0 = (p_1^0, \dots, p_{n+1}^0) \in H_\beta^\alpha$. The tangent hyperplane $T_{p^0}(H_\beta^\alpha)$ of H_β^α at p^0 has the following equation:

$$T_{p^0}(H_\beta^\alpha) : \sum_{i+1}^{n+1} \frac{\partial f}{\partial x_i}(p^0)(x_i - p_i^0) = 0, \text{ or, equivalently}$$

$$T_{p^0}(H_\beta^\alpha) : \sum_{i+1}^{\beta} \frac{(p_i^0)^{2\alpha-1}}{a_i^{2\alpha}} x_i - \sum_{i=\beta+1}^{n+1} \frac{(p_i^0)^{2\alpha-1}}{a_i^{2\alpha}} x_i = 1. \tag{2}$$

The parametric equations of the straight line passing through $0 \in \mathbf{R}^{n+1}$ which is orthogonal on the tangent hyperplane $T_{p^0}(H^\alpha_\beta)$ are:

$$\begin{cases} x_i = t\frac{(p_i^0)^{2\alpha-1}}{a_i^{2\alpha}}, \; i \in \{1,\dots,\beta\} \\ \\ x_i = -t\frac{(p_i^0)^{2\alpha-1}}{a_i^{2\alpha}}, \; i \in \{\beta+1,\dots,n+1\}. \end{cases} \tag{3}$$

To find the orthogonal projections of $0 \in \mathbf{R}^{n+1}$ on the tangent hyperplane $T_{p^0}(H^\alpha_\beta)$, replace the x_i, $i \in \{1,\dots,n+1\}$ from equations (3) in the equation (2) and we get:

$$\sum_{i+1}^{n+1} t\frac{(p_i^0)^{4\alpha-2}}{a_i^{4\alpha}} = 1, \text{ that is } , t = \frac{1}{\langle a^{-4\alpha}, (p^0)^{4\alpha-2}\rangle},$$

where $\langle \cdot,\cdot \rangle$ is the usual scalar product on $\mathbf{R}^{n+1}$. Hence, the orthogonal projection $q^0 \in \mathcal{L}^\alpha_\beta$ of $0 \in \mathbf{R}^{n+1}$ on the tangent hyperplane $T_{p^0}(H^\alpha_\beta)$ has the following coordinates

$$\begin{cases} x_i = \frac{1}{\langle a^{-4\alpha},(p^0)^{4\alpha-2}\rangle}\frac{(p_i^0)^{2\alpha-1}}{a_i^{2\alpha}}, \; i \in \{1,\dots,\beta\} \\ \\ x_i = -\frac{1}{\langle a^{-4\alpha},(p^0)^{4\alpha-2}\rangle}\frac{(p_i^0)^{2\alpha-1}}{a_i^{2\alpha}}, \; i \in \{\beta+1,\dots,n+1\}. \end{cases} \tag{4}$$

Therefore, on the one hand, we have

$$||q^0||^2 = \sum_{i=1}^{n+1} x_i^2 = \frac{1}{\langle a^{-4\alpha}, (p^0)^{4\alpha-2}\rangle^2}\sum_{i=1}^{n+1}\frac{(p_i^0)^{4\alpha-2}}{a_i^{4\alpha}} = \frac{1}{\langle a^{-4\alpha}, (p^0)^{4\alpha-2}\rangle},$$

and on the other hand

$$\begin{cases} a_i x_i = \frac{1}{\langle a^{-4\alpha},(p^0)^{4\alpha-2}\rangle}\left(\frac{p_i^0}{a_i}\right)^{2\alpha-1}, \; i \in \{1,\dots,\beta\} \\ \\ a_i x_i = -\frac{1}{\langle a^{-4\alpha},(p^0)^{4\alpha-2}\rangle}\left(\frac{p_i^0}{a_i}\right)^{2\alpha-1}, \; i \in \{\beta+1,\dots,n+1\}. \end{cases} \tag{5}$$

From relations (5) it follows

$$\varphi(a^{\frac{2\alpha}{2\alpha-1}}, (q^0)^{\frac{2\alpha}{2\alpha-1}}) = \sum_{i=1}^{\beta}(a_i x_i)^{\frac{2\alpha}{2\alpha-1}} - \sum_{i=\beta+1}^{n+1}(a_i x_i)^{\frac{2\alpha}{2\alpha-1}} =$$

$$= \left(\frac{1}{\langle a^{-4\alpha}, (p^0)^{4\alpha-2}\rangle}\right)^{\frac{2\alpha}{2\alpha-1}}\varphi(a^{-2\alpha}, (p^0)^{2\alpha}) = \left(\frac{1}{\langle a^{-4\alpha}, (p^0)^{4\alpha-2}\rangle}\right)^{\frac{2\alpha}{2\alpha-1}},$$

namely

$$||q^0||^{\frac{4\alpha}{2\alpha-1}} = \varphi(a^{\frac{2\alpha}{2\alpha-1}}, (q^0)^{\frac{2\alpha}{2\alpha-1}}),$$

that is $q^0 \in A^\alpha_\beta$ and we just showed that $\mathcal{L}^\alpha_\beta \subseteq A^\alpha_\beta$. To prove the other inclusion, consider $x^0 = (x^0_1, \ldots, x^0_{n+1}) \in A^\alpha_\beta$, that is,

$$||x^0||^{\frac{4\alpha}{2\alpha-1}} = \varphi(a^{\frac{2\alpha}{2\alpha-1}}, (x^0)^{\frac{2\alpha}{2\alpha-1}}).$$

It easy to verify that x^0 is the orthogonal projection of the origin $0 \in \mathbf{R}^{n+1}$ on the tangent hyperplane $T_{p^0}(H^\alpha_\beta)$, where $p^0 = (p^0_1, \ldots, p^0_{n+1})$ and its components are given by:

$$\begin{cases} p^0_i = (a_i)^{\frac{2\alpha}{2\alpha-1}} \left(\frac{x^0_i}{||x^0||^2}\right)^{\frac{1}{2\alpha-1}}, & i \in \{1, \ldots, \beta\} \\ \\ p^0_i = -(a_i)^{\frac{2\alpha}{2\alpha-1}} \left(\frac{x^0_i}{||x^0||^2}\right)^{\frac{1}{2\alpha-1}}, & i \in \{\beta+1, \ldots, n+1\}, \end{cases} \tag{6}$$

and the theorem is completely proved.□

Let us mention that the associated locus of an ellipsoid appears in [Ca, pp. 90,91] as an exercise.

Theorem 2.2. *The associated locus $\mathcal{L}^\alpha_\beta$ of H^α_β is a smooth hypersurface of* $\mathbf{R}^{n+1}$

Proof. According to theorem 2.1 we have succesively

$$\mathcal{L}^\alpha_\beta = \left\{x \in \mathbf{R}^{n+1}\backslash\{0\} \,\Big|\, ||x||^{\frac{4\alpha}{2\alpha-1}} = \varphi\big(a^{\frac{2\alpha}{2\alpha-1}}, x^{\frac{2\alpha}{2\alpha-1}}\big)\right\} =$$

$$= \left\{x \in \mathbf{R}^{n+1}\backslash\{0\} \,\Big|\, \frac{\varphi\big(a^{\frac{2\alpha}{2\alpha-1}}, x^{\frac{2\alpha}{2\alpha-1}}\big)}{||x||^{\frac{4\alpha}{2\alpha-1}}} = 1\right\} = g^{-1}(1),$$

where

$$g : \mathbf{R}^{n+1}\backslash\{0\} \to \mathbf{R},\ g(x) = \frac{\varphi\big(a^{\frac{2\alpha}{2\alpha-1}}, x^{\frac{2\alpha}{2\alpha-1}}\big)}{||x||^{\frac{4\alpha}{2\alpha-1}}}.$$

For $t > 0$ and $x \in \mathbf{R}^{n+1}\backslash\{0\}$ we have:

$$g(tx) = \frac{\varphi\big(a^{\frac{2\alpha}{2\alpha-1}}, (tx)^{\frac{2\alpha}{2\alpha 1}}\big)}{||tx||^{\frac{4\alpha}{2\alpha-1}}} = \frac{\varphi\big(a^{\frac{2\alpha}{2\alpha-1}}, t^{\frac{2\alpha}{2\alpha-1}} \cdot x^{\frac{2\alpha}{2\alpha-1}}\big)}{t^{\frac{4\alpha}{2\alpha-1}}||x||^{\frac{4\alpha}{2\alpha-1}}} =$$

$$= \frac{t^{\frac{2\alpha}{2\alpha-1}}\varphi\big(a^{\frac{2\alpha}{2\alpha-1}}, x^{\frac{2\alpha}{2\alpha-1}}\big)}{t^{\frac{4\alpha}{2\alpha-1}}||x||^{\frac{4\alpha}{2\alpha-1}}} = t^{\frac{2\alpha}{1-2\alpha}} \cdot \frac{\varphi\big(a^{\frac{4\alpha}{2\alpha-1}}, x^{\frac{4\alpha}{2\alpha-1}}\big)}{||x||^{\frac{4\alpha}{2\alpha-1}}} = t^{\frac{2\alpha}{1-2\alpha}} \cdot g(x).$$

Therefore g is a smooth homogeneous function of order $\frac{2\alpha}{1-2\alpha}$. Because $(a_1, 0, \ldots, 0) \in g^{-1}(1)$, it follows that $g^{-1}(1) \neq \emptyset$, that is, according to lemma 1.2, $\mathcal{L}^\alpha_\beta = g^{-1}(1)$ is a smooth hypersurface of $\mathbf{R}^{n+1}$.□

The hypersurface H^{α}_{n+1} and its associated locus $\mathcal{L}^{\alpha}_{n+1}$ will be simply denoted by H^{α} and $\mathcal{L}^{\alpha}$ respectively. The equation of H^{α} is:

$$H^{\alpha} : \langle a^{-\alpha}, x^{2\alpha} \rangle = 1.$$

Corollary 2.3. *The associated locus of H^{α} is given by*

$$\mathcal{L}^{\alpha} = \left\{ x \in \mathbf{R}^{n+1} \backslash \{0\} \,\Big|\, ||x||^{\frac{4\alpha}{2\alpha-1}} = \langle a^{\frac{2\alpha}{2\alpha-1}}, x^{\frac{2\alpha}{2\alpha-1}} \rangle \right\}$$

and it is a smooth hypersurface of $\mathbf{R}^{n+1}$.

3. The diffeomorphism between H^{α}_{β} and $\mathcal{L}^{\alpha}_{\beta}$

Theorem 3.1. *The mappings $\chi : H^{\alpha}_{\beta} \to \mathcal{L}^{\alpha}_{\beta}$, $\chi_1 : \mathcal{L}^{\alpha}_{\beta} \to H^{\alpha}_{\beta}$ given by*

$$\chi(x) = \frac{1}{\langle a^{-4\alpha}, x^{4\alpha-2} \rangle} \left(\frac{x_1^{2\alpha-1}}{a_1^{2\alpha}}, \dots, \frac{x_{\beta}^{2\alpha-1}}{a_{\beta}^{2\alpha}}, -\frac{x_{\beta+1}^{2\alpha-1}}{a_{\beta+1}^{2\alpha}}, \dots, -\frac{x_{n+1}^{2\alpha-1}}{a_{n+1}^{2\alpha}} \right), \ x = (x_1, \dots, x_{n+1}) \in H^{\alpha}_{\beta}$$

$$\chi_1(x) = \frac{1}{||x||^{\frac{2}{2\alpha-1}}} \left(a_1^{\frac{2\alpha}{2\alpha-1}} \cdot x_1^{\frac{1}{2\alpha-1}}, \dots, a_{\beta}^{\frac{2\alpha}{2\alpha-1}} \cdot x_{\beta}^{\frac{1}{2\alpha-1}}, -a_{\beta+1}^{\frac{2\alpha}{2\alpha-1}} \cdot x_{\beta+1}^{\frac{1}{2\alpha-1}}, \dots, -a_{n+1}^{\frac{2\alpha}{2\alpha-1}} \cdot x_{n+1}^{\frac{1}{2\alpha-1}} \right),$$

for all $x = (x_1, \dots, x_{n+1}) \in \mathcal{L}^{\alpha}_{\beta}$, are well defined and they are inverse to each other.

Proof. Indeed on the one hand, for $x = (x_1, \dots, x_{n+1}) \in H^{\alpha}_{\beta}$, we have:

$$||\chi(x)||^{\frac{4\alpha}{2\alpha-1}} = \left(||\chi(x)||^2\right)^{\frac{2\alpha}{2\alpha-1}} = \frac{1}{\langle a^{-4\alpha}, x^{4\alpha-2} \rangle^{\frac{4\alpha}{2\alpha-1}}} \left(\sum_{i=1}^{n+1} \frac{x_i^{4\alpha-2}}{a_i^{4\alpha}} \right)^{\frac{2\alpha}{2\alpha-1}} =$$

$$= \frac{1}{\langle a^{-4\alpha}, x^{4\alpha-2} \rangle^{\frac{4\alpha}{2\alpha-1}}} \langle a^{-4\alpha}, x^{4\alpha-2} \rangle^{\frac{2\alpha}{2\alpha-1}} = \frac{1}{\langle a^{-4\alpha}, x^{4\alpha-2} \rangle^{\frac{2\alpha}{2\alpha-1}}},$$

and on the other hand,

$$\varphi\left(a^{\frac{2\alpha}{2\alpha-1}}, \chi(x)^{\frac{2\alpha}{2\alpha-1}}\right) = \frac{1}{\langle a^{-4\alpha}, x^{4\alpha-2} \rangle^{\frac{2\alpha}{2\alpha-1}}} \varphi\left(a^{\frac{2\alpha}{2\alpha-1} - \frac{4\alpha^2}{2\alpha-1}}, x^{2\alpha}\right) =$$

$$= \frac{1}{\langle a^{-4\alpha}, x^{4\alpha-2} \rangle^{\frac{2\alpha}{2\alpha-1}}} \varphi(a^{-2\alpha}, x^{2\alpha}) = \frac{1}{\langle a^{-4\alpha}, x^{4\alpha-2} \rangle^{\frac{2\alpha}{2\alpha-1}}}.$$

Therefore

$$||\chi(x)||^{\frac{4\alpha}{2\alpha-1}} = \varphi\left(a^{\frac{2\alpha}{2\alpha-1}}, \chi(x)^{\frac{2\alpha}{2\alpha-1}}\right) \text{ for all } x \in H^{\alpha}_{\beta},$$

that is $\chi(x) \in \mathcal{L}^{\alpha}_{\beta}$ for all $x \in H^{\alpha}_{\beta}$, which means that the mapping χ is well defined.

Analogously, for $x = (x_1, \dots, x_{n+1}) \in \mathcal{L}^\alpha_\beta$, we get:

$$\varphi(a^{-2\alpha}, \chi_1(x)^{2\alpha}) = \frac{1}{||x||^{\frac{4\alpha}{2\alpha-1}}}\varphi\big(a^{-2\alpha+\frac{4\alpha^2}{2\alpha-1}}, x^{\frac{2\alpha}{2\alpha-1}}\big) = \frac{1}{||x||^{\frac{4\alpha}{2\alpha-1}}}\varphi\big(a^{\frac{2\alpha}{2\alpha-1}}, x^{\frac{2\alpha}{2\alpha-1}}\big) = 1,$$

that is $\chi_1(x) \in H^\alpha_\beta$ for all $x = (x_1, \dots, x_{n+1}) \in \mathcal{L}^\alpha_\beta$, and the mapping χ_1 is well defined.

For $x = (x_1, \dots, x_{n+1}) \in \mathcal{L}^\alpha_\beta$ we also have:

$$(\chi \circ \chi_1)(x) = \chi(\chi_1(x)) =$$

$$= \chi\Big(\frac{1}{||x||^{\frac{2}{2\alpha-1}}}\Big(a_1^{\frac{2\alpha}{2\alpha-1}} \cdot x_1^{\frac{1}{2\alpha-1}}, \dots, a_\beta^{\frac{2\alpha}{2\alpha-1}} \cdot x_\beta^{\frac{1}{2\alpha-1}}, -a_{\beta+1}^{\frac{2\alpha}{2\alpha-1}} \cdot x_{\beta+1}^{\frac{1}{2\alpha-1}}, \dots, -a_{n+1}^{\frac{2\alpha}{2\alpha-1}} \cdot x_{n+1}^{\frac{1}{2\alpha-1}}\Big)\Big) =$$

$$= \frac{1}{\frac{1}{||x||^{\frac{8\alpha-4}{2\alpha-1}}}\langle a^{-4\alpha+4\alpha}, x^2\rangle} \cdot \frac{1}{||x||^2}(x_1, \dots, x_{n+1}) = x = id_{\mathcal{L}^\alpha_\beta}(x).$$

On the other hand, for $x = (x_1, \dots, x_{n+1}) \in H^\alpha_\beta$, we have

$$(\chi_1 \circ \chi)(x) = \chi_1(\chi(x)) = \chi\Big(\frac{1}{\langle a^{-4\alpha}, x^{4\alpha-2}\rangle}\Big(\frac{x_1^{2\alpha-1}}{a_1^{2\alpha}}, \dots, \frac{x_\beta^{2\alpha-1}}{a_\beta^{2\alpha}}, -\frac{x_{\beta+1}^{2\alpha-1}}{a_{\beta+1}^{2\alpha}}, \dots, -\frac{x_{n+1}^{2\alpha-1}}{a_{n+1}^{2\alpha}}\Big)\Big) =$$

$$= \frac{1}{\frac{1}{\langle a^{-4\alpha}, x^{4\alpha-2}\rangle^{\frac{2}{2\alpha-1}}}\Big(\sum_{i=1}^{n+1}\frac{x_i^{4\alpha-2}}{a_i^{4\alpha}}\Big)^{\frac{1}{2\alpha-1}}} \cdot \frac{1}{\langle a^{-4\alpha}, x^{4\alpha-2}\rangle^{\frac{1}{2\alpha-1}}}(x_1, \dots, x_{n+1}) = x = id_{H^\alpha_\beta}(x). \square$$

Corollary 3.2. *The mappings χ and χ_1 are diffeomorphisms between H^α_β and $\mathcal{L}^\alpha_\beta$.*

The next theorem can be proved in a completely analogous way.

Theorem 3.3. *The mapping $h : H^\alpha \to S^n$, $h(x) = \frac{x}{||x||}$ is a diffeomorphism and $h^{-1} : S^n \to H^\alpha$ is given by $h^{-1}(x) = \frac{x}{\langle a^{-2\alpha}, x^{2\alpha}\rangle^{1/2\alpha}}$.*

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V-COHOMOLOGY OF COMPLEX FINSLER MANIFOLDS

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Dedicated to Professor Pavel Enghiş at his 70th anniversary

Abstract. Starting from a natural decomposition of the exterior differential of a complex Finsler manifold, we define new cohomology groups and a Dolbeault type theorem is also proved.

1. The holomorphic tangent bundle.

Let us consider a complex manifold $M, \dim_C M = n$, $(U,(z^i))$ the complex coordinates in a local chart. The complexification $T_C M$ of the tangent bundle TM is decomposed in each point z after the $(1,0)$ vector fields and their conjugates of $(0,1)$ type, $T_C M = T'M \oplus T''M$. As it is well-known ([1],[2],[8]..), $T'M$ is also a complex manifold of dimension $\dim_C T'M = 2n$ and the natural projection $\pi_T : T'M \to M$ defines on $V(T'M) = \{\xi \in T'(T'M) \ / \ \pi_{T*}(\xi) = 0\}$ a structure of holomorphic vector bundle of rank n over $T'M$. We denote by $\mathcal{V}(T'M)$ the module of its sections, called *vector fields of* $v-$type.

A given supplementary subbundle $H(T'M)$ of $V(T'M)$ in $T'M$, i.e.

$$T'(T'M) = H(T'M) \oplus V(T'M) \tag{1}$$

defines a *nonlinear complex connection,* and we denote by $\mathcal{H}(T'M)$ the module of its sections, called *vector fields of* $h-$type.

Considering also their conjugates $\overline{V(T'M)}$ and $\overline{H(T'M)}$, we obtain the following decomposition of the complexification $T_C(T'M)$ of the real tangent bundle

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$T(T'M)$,

$$T_C(T'M) = H(T'M) \oplus V(T'M) \oplus \overline{H(T'M)} \oplus \overline{V(T'M)} \tag{2}$$

The elements of the conjugates are called *vector fields of $\overline{v}$−type, and $\overline{h}$−type,* respectively.

If $(U, (z^i, \eta^i))$ are the complex local coordinates on $T'M$ and if $N_i^j(z, \eta)$ are the coefficients of the complex nonlinear connection, which are changed at local change of the local chartafter the rule,

$$N_k'^i \frac{\partial z'^k}{\partial z^j} = \frac{\partial z'^i}{\partial z^k} N_j^k - \frac{\partial^2 z'^i}{\partial z^j \partial z^k} \eta^k \tag{3}$$

then the following set of complex vector fields

$$\{\frac{\delta}{\delta z^i} = \frac{\partial}{\partial z^i} - N_i^j \frac{\partial}{\partial \eta^j}\},\ \{\frac{\partial}{\partial \eta^i}\},\ \{\frac{\delta}{\delta \overline{z}^i} = \frac{\partial}{\partial \overline{z}^i} - \overline{N_i^j} \frac{\partial}{\partial \overline{\eta}^j}\},\ \{\frac{\partial}{\partial \overline{\eta}^i}\} \tag{4}$$

are called the local *adapted bases* of $\mathcal{H}(T'M), \mathcal{V}(T'M), \overline{\mathcal{H}(T'M)}$ and $\overline{\mathcal{V}(T'M)}$, respectively. The dual adapted bases are denoted by

$$\{dz^i\},\ \{\delta\eta^i = d\eta^i + N_j^i dz^j\},\ \{d\overline{z}^j\},\ \{\delta\overline{\eta}^i = d\overline{\eta}^i + \overline{N}_j^i d\overline{z}^j\} \tag{5}$$

2. Complex valued forms

Let us consider the set $F(T'M)$ of the complex valued differential forms on $T'M$ given by the direct sum,

$$F(T'M) = \bigoplus_{p,q,r,s=\overline{0.n}} F^{p,q,r,s}(T'M) \tag{6}$$

where $F^{p,q,r,s}(T'M)$ [or $F^{p,q,r,s}(U)$ for the open set U of $T'M$, or simply $F^{p,q,r,s}$ when there is no confusion danger] is the set of $(p+q+r+s)$−forms which can be non zero only when these act on p vector fields of h−type, on q vector fields of v−type, on r vector fields of $\overline{h}$−type, and on $\overline{s}$ vector fields of $\overline{v}$−type. The elements of $F^{p,q,r,s}(U)$ are called (p,q,r,s)*−forms* on U.

In the adapted dual bases we have the following local expression of (p,q,r,s) −forms ω,

$$\omega = \sum \omega_{i_1 \ldots i_p j_1 \ldots j_q h_1 \ldots h_r k_1 \ldots k_s} \cdot dz^{i_1} \wedge \ldots\ldots \wedge dz^{i_p} \wedge$$

$$\delta\eta^{j_1} \wedge \ldots.. \wedge \delta\eta^{j_q} \wedge d\overline{z}^{h_1} \wedge \ldots..d\overline{z}^{h_r} \wedge \delta\overline{\eta}^{k_1} \wedge \ldots.\delta\overline{\eta}^{k_s} \tag{7}$$

where the sum is after the indices $i_1 \prec \ldots \prec i_p$; $j_1 \prec \ldots \prec j_q$; $h_1 \prec \ldots \prec h_r$; $k_1 \prec \ldots \prec k_s$.

Now, let us consider f a complex valued differentiable function defined on $T'M$. In [2] the following operators are considered:

$$\begin{aligned} d'^h f &= \frac{\delta f}{\delta z^i} dz^i = \left(\frac{\partial f}{\partial z^i} - N_i^j \frac{\partial f}{\partial \eta^j}\right) dz^i \ ; \ d'^v f = \frac{\partial f}{\partial \eta^i} \delta\eta^i \\ d''^h f &= \frac{\delta f}{\delta \overline{z}^i} d\overline{z}^i = \left(\frac{\partial f}{\partial \overline{z}^i} - \overline{N}_i^j \frac{\partial f}{\partial \overline{\eta}^j}\right) d\overline{z}^i \ ; \ d''^v f = \frac{\partial f}{\partial \overline{\eta}^i} \delta\overline{\eta}^i \end{aligned} \tag{8}$$

and they give a natural decomposition of the exterior differential df of f.

We shall generalize these operators for any differential form. For this purpose we compute the exterior differential $d\omega$ of a (p,q,r,s)–form given by (7). A straightforward calculus, taking into account (5) and the properties of d, gives

$$d\omega \in F^{p+1,q,r,s} \oplus F^{p,q+1,r,s} \oplus F^{p,q,r+1,s} \oplus F^{p,q,r,s+1} \oplus F^{p+2,q-1,r,s} \oplus$$

$$F^{p+1,q-1,r+1,s} \oplus F^{p+1,q-1,r,s+1} \oplus F^{p+1,q,r+1,s-1} \oplus F^{p,q+1,r+1,s-1} \oplus F^{p,q,r+2,s-1} \tag{9}$$

Particularly, by using (8) we have

$$dF^{0,0,0,0} \subset F^{1,0,0,0} \oplus F^{0,1,0,0} \oplus F^{0,0,1,0} \oplus F^{0,0,0,1} \tag{10}$$

where $F^{0,0,0,0}$ denotes the set of complex valued differentiable functions on $T'M$. We also obtain

$$dF^{1,0,0,0} \subset F^{2,0,0,0} \oplus F^{1,1,0,0} \oplus F^{1,0,1,0} \oplus F^{1,0,0,1} \tag{11}$$

$$dF^{0,0,1,0} \subset F^{1,0,1,0} \oplus F^{0,1,1,0} \oplus F^{0,0,2,0} \oplus F^{0,0,1,1} \tag{12}$$

Now, we assume that M is a complex Finsler manifold with Finsler metric F ([2], definition 3.1) and we consider that N is the complex Rund connection on M (idem, definition 3.3). Then it is well-known that, [11]

$$\frac{\delta N_k^i}{\delta z^j} = \frac{\delta N_j^i}{\delta z^k}$$

and taking account the local expression (7) of the (p,q,r,s)–forms, the formulas (5) and the properties of the exterior differential, we give

$$dF^{0,1,0,0} \subset F^{1,1,0,0} \oplus F^{0,2,0,0} \oplus F^{0,1,1,0} \oplus F^{0,1,0,1} \oplus F^{1,0,1,0} \oplus F^{1,0,0,1} \tag{13}$$

$$dF^{0,0,0,1} \subset F^{1,0,0,1} \oplus F^{0,1,0,1} \oplus F^{0,0,1,1} \oplus F^{0,0,0,2} \oplus F^{1,0,1,0} \oplus F^{0,1,1,0} \tag{14}$$

From (9)-(14) it result the following

Proposition 01 *If M is a complex Finsler manifold endowed with the complex Rund connection then we have*

$$dF^{p,q,r,s} \subset F^{p+1,q,r,s} \oplus F^{p,q+1,r,s} \oplus F^{p,q,r+1,s} \oplus F^{p,q,r,s+1} \oplus$$

$$F^{p+1,q-1,r+1,s} \oplus F^{p+1,q-1,r,s+1} \oplus F^{p+1,q,r+1,s-1} \oplus F^{p,q+1,r+1,s-1} \tag{15}$$

From the above decomposition (15) it follows that we can define eight morphisms of complex vector spaces if we consider the different components, namely

$$d'^h : F^{p,q,r,s} \to F^{p+1,q,r,s} \ ; \ d'^v : F^{p,q,r,s} \to F^{p,q+1,r,s}$$
$$d''^h : F^{p,q,r,s} \to F^{p,q,r+1,s} \ ; \ d''^v : F^{p,q,r,s} \to F^{p,q,r,s+1}$$
$$\partial_1 : F^{p,q,r,s} \to F^{p+1,q-1,r+1,s} \ ; \ \partial_2 : F^{p,q,r,s} \to F^{p+1,q-1,r,s+1}$$
$$\partial_3 : F^{p,q,r,s} \to F^{p+1,q,r+1,s-1} \ ; \ \partial_4 : F^{p,q,r,s} \to F^{p,q+1,r+1,s-1}$$

We remark that these operators and the classical operators d' , d'' that appear in the decomposition $d = d' + d''$ of the differential on a complex manifold, are related by the following relations

$$d' = d'^h + d'^v + \partial_3 + \partial_4 \ ; \ d'' = d''^h + d''^v + \partial_1 + \partial_2 \tag{16}$$

Moreover, by equalizing the terms of the same type in the relation

$$d^2 = (d'^h + d'^v + \partial_3 + \partial_4 + d''^h + d''^v + \partial_1 + \partial_2)^2 = 0$$

we obtain:

$$(d'^h)^2 = 0, \quad (d'^v)^2 = 0, \quad (d''^h)^2 = 0, \quad (d''^v)^2 = 0$$
$$(\partial_1)^2 = 0, \quad (\partial_2)^2 = 0, \quad (\partial_3)^2 = 0, \quad (\partial_4)^2 = 0$$
$$d'^h d'^v + d'^v d'^h = 0, \quad d''^v d'^v + d'^v d''^v = 0, \quad d''^v d''^h + d''^h d''^v = 0$$

$$d'^h\partial_2 + \partial_2 d'^h = 0, \quad d'^v\partial_4 + \partial_4 d'^v = 0, \quad d''^h\partial_1 + \partial_1 d''^h = 0$$

$$d'^h\partial_3 + \partial_3 d'^h = 0, \quad d''^v\partial_2 + \partial_2 d''^v = 0, \quad d''^h\partial_4 + \partial_4 d''^h = 0$$

$$\partial_1\partial_2 + \partial_2\partial_1 = 0, \quad \partial_1\partial_3 + \partial_3\partial_1 = 0, \quad \partial_3\partial_4 + \partial_4\partial_3 = 0$$

$$d'^h d''^v + d''^v d'^h + d'^v\partial_2 + \partial_2 d'^v = 0, \; d'^v d''^h + d''^h d'^v + d''^v\partial_4 + \partial_4 d''^v = 0$$

$$d'^v\partial_3 + \partial_3 d'^v + d'^h\partial_4 + \partial_4 d'^h = 0, \quad d''^v\partial_1 + \partial_1 d''^v + d''^h\partial_2 + \partial_2 d''^h = 0$$

$$d''^h\partial_3 + \partial_3 d''^h + \partial_1\partial_4 + \partial_4\partial_1 = 0, \quad d'^h\partial_1 + \partial_1 d'^h + \partial_3\partial_2 + \partial_2\partial_3 = 0$$

$$d'^h d''^h + d''^h d'^h + \partial_1 d'^v + d'^v\partial_1 + \partial_2\partial_4 + \partial_4\partial_2 + \partial_3 d''^v + d''^v\partial_3 = 0 \quad .$$

By the same argument we have

$$d''^v(\omega \wedge \theta) = d''^v\omega \wedge \theta + (-1)^{\deg\omega}\omega \wedge d''^v\theta \tag{17}$$

for any $\omega \in F^{p,q,r,s}$, $\theta \in F^{p',q',r',s'}$ and similar equalities for the other operators defined above.

From (17) and from the linearity of d''^v we deduce that if $\omega \in F^{p,q,r,s}$ is locally given by (7) then

$$d''^v\omega = \sum \frac{\partial\omega_{i_1\ldots i_p j_1\ldots j_q h_1\ldots h_r k_1\ldots k_s}}{\partial\overline{\eta}^i}\delta\overline{\eta}^i \wedge dz^{i_1} \wedge \ldots\ldots \wedge dz^{i_p}$$

$$\wedge\delta\eta^{j_1} \wedge \ldots\ldots \wedge \delta\eta^{j_q} \wedge d\overline{z}^{h_1} \wedge \ldots\ldots d\overline{z}^{h_r} \wedge \delta\overline{\eta}^{k_1} \wedge \ldots\ldots\delta\overline{\eta}^{k_s} \tag{18}$$

where the sum is after the indices $i_1 \prec \ldots \prec i_p$; $j_1 \prec \ldots \prec j_q$; $h_1 \prec \ldots \prec h_r$; $k_1 \prec \ldots \prec k_s$.

We know ([8],proposition 1.1) that the local bases $\{\frac{\partial}{\partial\overline{\eta}^i}\}$, $\{\frac{\partial}{\partial\overline{\eta}'^i}\}$ of $\overline{V(T'M)}$, corresponding to a change of complex coordinates $\{z^i, \eta^i\} \to \{z'^i, \eta'^i\}$ on $T'M$, are related by

$$\frac{\partial}{\partial\overline{\eta}^i} = \frac{\partial\overline{z}'^j}{\partial\overline{z}^i}\frac{\partial}{\partial\overline{\eta}'^i} \tag{19}$$

The formulas (19) prove that if f is a complex valued differentiable function defined on $T'M$ then the condition

$$\frac{\partial f}{\partial\overline{\eta}^i} = 0 \quad ; \quad i = 1, 2\ldots n \tag{20}$$

is independent with respect to this change. Moreover, the of form $\omega \in F^{p,q,r,0}$ is d''^v−closed (i.e. $d''^v\omega = 0$) if and only if its local components satisfy the conditions (20). We denote by $\Phi^{p.q.r}$ the sheaf of germs of these forms.

Another property of the operator d''^v is a Grothendieck-Dolbeault type lemma, namely

Theorem 1 *Let ω be a d''^v-closed (p,q,r,s)−form defined on a neighborhood U on $T'M$ and $s \geq 1$. Then there exists a $(p,q,r,s-1)$−form θ defined on some neighborhood $U' \subset U$ and such that $d''^v\theta = \omega$ on U'.*

Proof. We use an argument inspired by the paper [12]. Let be h the index defined by the condition that the form ω, given by (7), does not contains $d\overline{z}^{h+1}, \ldots.., d\overline{z}^n$. We shall prove the assertion by using the induction on h.

For $h = 0$ we have

$\omega = \omega_{i_1\ldots i_p j_1\ldots j_q k_1\ldots k_s} dz^{i_1} \wedge \ldots.. \wedge dz^{i_p} \wedge \delta\eta^{j_1} \wedge \ldots.. \wedge \delta\eta^{j_q} \wedge \delta\overline{\eta}^{k_1} \wedge \ldots.\delta\overline{\eta}^{k_s}$

and then

$d''^v\omega = d''\omega$ (modulo the therms containing $d\overline{z}^1, \ldots., d\overline{z}^n$).

But $d''^v\omega = 0$ and if we consider $\overline{z}^1, \ldots., \overline{z}^n$ like parameters, then the Grothendieck-Dolbeault lemma can be applied and therefore there exists a $(p,q,0,s-1)$−form θ with the property

$\omega = d''\theta$ $(modulo\ \ d\overline{z}^1, \ldots., d\overline{z}^n)$

on some neighborhood $U' \subset U$. Hence we have

$$\omega = d''\theta + \sum_{i=1,n} \lambda_i \wedge d\overline{z}^i \tag{21}$$

where λ_i are $(p,q,0,s-1)$−forms. Now because $\omega \in F^{p,q,0,s}$, from (21) we obtain $\omega = d''^v\theta$ and the assertion is proved for $h = 0$.

We assume the result valid for the indices $h_0 \leq h-1$ and we prove it for h, i.e. for (p,q,r,s)−forms which do not contain $d\overline{z}^{h+1}, \ldots, d\overline{z}^n$. Such a form is expressed as follows

$$\omega = d\overline{z}^h \wedge \alpha + \beta \tag{22}$$

where $\alpha \in F^{p,q,r-1,s}$, $\beta \in F^{p,q,r,s}$ and do not contain $d\overline{z}^h, d\overline{z}^{h+1}, \ldots, d\overline{z}^n$. Therefore by using (17) we have

$$d''^v \omega = -d\overline{z}^h \wedge d''^v \alpha + d''^v \beta = 0$$

hence

$$d''^v \alpha = 0 \;\; ; \;\; d''^v \beta = 0$$

and by applying the induction hypothesis it follows that on some neighborhood $U' \subset U$ there are two forms $\alpha^* \in F^{p,q,r-1,s-1}$, $\beta^* \in F^{p,q,r,s-1}$ such that

$$\alpha = d''^v \alpha^* \quad ; \quad \beta = d''^v \beta^* \tag{23}$$

Now, from (22),(23) and taking into account (17) we obtain

$$\omega = d''^v(-d\overline{z}^h \wedge \alpha^* + \beta^*)$$

Q.E.D.

Let $\mathcal{F}^{p,q,r,s}$ be the sheaf of germs of $(p,q,r,s)-$forms and we denote by $i : \Phi^{p,q,r} \to \mathcal{F}^{p,q,r,0}$ the natural inclusion. The sheaves $\mathcal{F}^{p,q,r,s}$ are fine and taking into account the Theorem 1. it follows that the sequence of sheaves

$$0 \to \Phi^{p,q,r} \xrightarrow{i} \mathcal{F}^{p,q,r,0} \xrightarrow{d''^v} \mathcal{F}^{p,q,r,1} \ldots\ldots \xrightarrow{d''^v} \mathcal{F}^{p,q,r,s} \xrightarrow{d''^v} \mathcal{F}^{p,q,r,s+1} \ldots\ldots$$

is a fine resolution of $\Phi^{p,q,r}$ and we denote by $H^s(M, \Phi^{p,q,r})$ the cohomology groups of M with coefficients in the sheaf $\Phi^{p,q,r}$, called *$v-$cohomology groups of M*. then we obtain a de Rham type theorem, namely

Theorem 2. *The $v-$cohomology groups of the complex Finsler manifold M are given by*

$$H^s(M, \Phi^{p,q,r}) \approx Z^{p,q,r,s} \;/\; d''^v F^{p,q,r,s-1}(T'M)$$

where $Z^{p,q,r,s}$ is the space of d''^v- closed $(p,q,r,s)-$forms globally defined on $T'M$.

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CONFORMAL STRUCTURES IN THE LAGRANGE GEOMEETRY OF SECOND ORDER

MONICA PURCARU

Dedicated to Professor Pavel Enghiş at his 70^{th} anniversary

Abstract. In the present paper we introduce two d-structures on $E = Osc^2M$: the conformal metrical d-structure and the almost symplectic d-structure and we study the properties of this two d-structures.

1. Introduction

The literature on the higher order Lagrange spaces geometry highlights the theoretical and practical importance of these spaces.

Motivated by concrete problems in variational calculation, higher order Lagrange geometry has witnessed a wide acknowledgment due to the papers $[7-11]$ published by Acad.dr.R.Miron and Prof.dr.Gh.Atanasiu.

The study of higher order Lagrange spaces is grounded on the k-osculator bundle notion. The bundle space of accelerations (or 2-osculator bundle) corresponds in this study to $k = 2, [1], [7]$.

Very little research has been carried out with respect to the study of the important structures in the 2-osculator bundle.

In the present paper we define the conformal metrical d-structure notion, $\hat{g}$, in the Lagrange geometry of second order and we study the properties of this structure (§2). We also introduce the conformal almost symplectic d-structure notion, $\hat{a}$, in the Lagrange geometry of second order and we study the properties of this structure (§3).

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As to the terminology as notations we use those from [12], which are essentially based on M.Matsumoto's book [6].

2. The conformal metrical d-structure in the Lagrange geometry of second order.

Let M be a real n-dimensional C^∞-manifold, $(Osc^2 M, \pi, M)$ its 2-osculator bundle, or the bundle of accelerations. The local coordinates on $E = Osc^2 M$ are denoted by $(x^i, y^{(1)i}, y^{(2)i})$.

If N is a nonlinear connection on E, with the coefficients $N_{(1)}{}^i{}_j(x^i, y^{(1)i}, y^{(2)i})$, $N_{(2)}{}^i{}_j(x^i, y^{(1)i}, y^{(2)i})$, then let $D\Gamma(N) = (L^i_{jk}, C_{(1)}{}^i_{jk}, C_{(2)}{}^i_{jk})$ be an N- linear connection, D on E.

Let $L^{(2)n} = (M, L)$ be the second order Lagrange space, where $L : E \to R$ is a C^∞ differentiable regular Lagrangian of second order, whose fundamental metric d-tensor field, g_{ij}, has a constant signature on $\tilde{E} = \{(x, y^{(1)}, y^{(2)}) \in Osc^2 M$, rank $||y^{(1)i}|| = 1\}$:

$$g_{ij}(x, y^{(1)}, y^{(2)}) = \frac{1}{2} \frac{\partial^2 L}{\partial y^{(2)i} \partial y^{(2)j}}, \tag{2.1}$$

g_{ij} is a differentiable d-tensor field on $\tilde{E}$, symmetric, covariant of order two. Let (g^{ij}) be the inverse matrix of (g_{ij}):

$$g_{ik}(x, y^{(1)}, y^{(2)}) g^{kj}(x, y^{(1)}, y^{(2)}) = \delta^j_i. \tag{2.2}$$

Observation 2.1. *We can consider on E as g_{ij} any d-tensor field of type $(0,2)$ on E symmetric and nondegenerate.*

We associate to this d-structure Obata's operators:

$$\Omega^{ir}_{sj} = \frac{1}{2}(\delta^i_s \delta^r_j - g_{sj} g^{ir}),\ \Omega^{*ir}_{sj} = \frac{1}{2}(\delta^i_s \delta^r_j + g_{sj} g^{ir}), \tag{2.3}$$

Obata's operators have the same properties as the ones associated with a Finsler space [12].

Let $\mathcal{S}_2(E)$ be the set of all symmetric d-tensor fields of the type $(0,2)$ on E. As is easily shown, the relation for $a_{ij},\ b_{ij} \in \mathcal{S}_2(E)$ defined by:

$$a_{ij} \sim b_{ij} \Leftrightarrow \exists \rho(x, y^{(1)}, y^{(2)}) \in \mathcal{F}(E) \mid a_{ij} = e^{2\rho} b_{ij}, \tag{2.4}$$

is an equivalent relation on $\mathcal{S}_2(E)$.

Definition 2.1. *The equivalence class $\hat{g}$ of $\mathcal{S}_2(E)/_\sim$, to which the metric d-structure g_{ij} belongs, is called: conformal metrical d-structure on E.*

Every $g'_{ij} \in \hat{g}$ is a positive definite, symmetric d-tensor field, expressed by

$$g'_{ij} = e^{2\rho} g_{ij}. \tag{2.5}$$

We shall find the condition that in a differentiable manifold E, a given $g'_{ij} \in \mathcal{S}_2(E)$ belongs to a conformal metrical d-structure.

Lemma 2.1. *A given positive definite $g_{ij} \in \mathcal{S}_2(E)$ is a fundamental tensor field if and only if it holds:*

$$\frac{\partial g_{ij}}{\partial y^{(2)k}} y^{(2)j} = 0. \tag{2.6}$$

Theorem 2.1. *A given positive definite $g'_{ij} \in \mathcal{S}_2(E)$ belongs to a conformal metrical d-structure if and only if there exists a function $\rho(x, y^{(1)}, y^{(2)}) \in \mathcal{F}(E)$ satisfying:*

$$\frac{\partial g'_{ij}}{\partial y^{(2)k}} y^{(2)j} = 2 \frac{\partial \rho}{\partial y^{(2)k}} y_i^{(2)\prime}, \tag{2.7}$$

where $y_i^{(2)\prime} = g'_{ij} y^{(2)j}$.

Proof. Let g'_{ij} belongs to a conformal metrical d-structure. Since g'_{ij} satisfies (2.5), we obtain (2.7) from Lemma 2.1. Conversly, if there exists a function ρ satisfying (2.7), then $g_{ij} = e^{-2\rho} g'_{ij}$ satisfies (2.6). □

Obata's operators are defined for $g'_{ij} \in \hat{g}$ by putting $(g'^{ij}) = (g'_{ij})^{-1}$. Since equation (2.5) is equivalent to

$$g'^{ij} = e^{-2\rho} g^{ij}, \tag{2.8}$$

We have:

Proposition 2.1. *Obata's operators depend on the conformal metrical d-structure $\hat{g}$, and do not depend on its representative $g'_{ij} \in \hat{g}$.*

3. The conformal almost symplectic d-structure in the Lagrange geometry of second order.

Let M be a real n-dimensional C^∞-manifold, (Osc^2M, π, M) its 2-osculator bundle, or the bundle of accelerations. The local coordinates on the total space $E = Osc^2M$ are denoted by $(x^i, y^{(1)i}, y^{(2)i})$.

If N is a nonlinear connection on E with the coefficients $\underset{(1)}{N}{}^i{}_j(x^i, y^{(1)i}, y^{(2)i})$, $\underset{(2)}{N}{}^i{}_j(x^i, y^{(1)i}, y^{(2)i})$, then let $D\Gamma(N) = (L^i_{jk}, \underset{(1)}{C}{}^i_{jk}, \underset{(2)}{C}{}^i_{jk})$ be an N- linear connection, D, on E.

We consider on E an almost symplectic d-structure, defined by a d-tensor field of the type $(0,2)$, let us say $a_{ij}(x, y^{(1)}, y^{(2)})$, skewsymmetric

$$a_{ij}(x, y^{(1)}, y^{(2)}) = -a_{ji}(x, y^{(1)}, y^{(2)}), \tag{3.1}$$

and nondegenerate:

$$det\|a_{ij}(x, y^{(1)}, y^{(2)})\| \neq 0,\ \forall y^{(1)} \neq 0,\ \forall y^{(2)} \neq 0, \tag{3.2}$$

We asociate to this d-structure Obata's operators:

$$\Phi^{ir}_{sj} = \frac{1}{2}(\delta^i_s \delta^r_j - a_{sj}a^{ir}), \Phi^{*ir}_{sj} = \frac{1}{2}(\delta^i_s \delta^r_j + a_{sj}a^{ir}), \tag{3.3}$$

where (a^{ij}) is the inverse matrix of (a_{ij}):

$$a_{ij}a^{jk} = \delta^k_i. \tag{3.4}$$

Obata's operators have the same properties as ones associated with a Finsler space [14].

Let $L^{(2)n} = (M, L)$ be the second order Lagrange space, where $L : E \to R$ is a C^∞-differentiable, regular Lagrangian of second order.

Let $\mathcal{A}_2(E)$ be the set of all skewsymmetric d-tensor fields of the type $(0,2)$ on E. As is easily shown, the relation for $a_{ij},\ b_{ij} \in \mathcal{A}_2(E)$ defined by

$$a_{ij} \sim b_{ij} \Leftrightarrow \exists \rho(x, y^{(1)}, y^{(2)}) \in \mathcal{F}(E) | a_{ij} = e^{2\rho} b_{ij} \tag{3.5}$$

is an equivalent relation on $\mathcal{A}_2(E)$.

Definition 3.1. *The equivalence class, $\hat{a}$, of $\mathcal{A}_2(E)/_\sim$, to which the d-tensor field a_{ij} belongs, is called conformal almost symplectic d-structure on E.*

Every $a'_{ij} \in \hat{a}$ is a skewsymmetric and nondegenerate d-tensor field expressed by:

$$a'_{ij} = e^{2\rho} a_{ij}. \tag{3.6}$$

Obata's operators are defined for $a'_{ij} \in \hat{a}$ by putting $(a'^{ij}) = (a'_{ij})^{-1}$. Since equation (2.6) is equivalent to

$$(a'^{ij}) = e^{-2\rho} a^{ij}. \tag{3.7}$$

We have:

Proposition 3.1. *Obata's operators depend on the conformal almost symplectic d-structure $\hat{a}$, and do not depend on its representative $a'_{ij} \in \hat{a}$.*

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ON THE STABILITY OF THE QUADRATIC FUNCTIONAL EQUATION AND ITS APPLICATIONS

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Dedicated to Professor Pavel Enghiş at his 70th anniversary

Key words and phrases: quadratic functional equation, Abelian group, semigroup, Banach space, A-orthogonal pairs, stability of mappings

1. Introduction

To quote S.M. Ulam [22, p.63] for very general functional equations, one can ask the following question. When it is true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation? Similarly, if we replace a given functional equation by a functional inequality, when can one assert that the solutions of the inequality lie near the solutions of the strict equation?

The present paper will provide a solution of Ulam's problem for the case of the quadratic functional equation.

The quadratic functional equation

$$f(x+y)+f(x-y)-2f(x)-2f(y)=0 \tag{1}$$

clearly has $f(x)=cx^2$ as a solution with c an arbitrary constant when f is a real function of a real variable. We shall be interested in functions $f:E_1\to E_2$ where both E_1 and E_2 are real vector spaces, and we need a few facts concerning the relation between a quadratic function and a biadditive function sometimes called its polar. This relation is explained in Proposition 1, page 166, of the book by J. Aczél and J. Dhombres [1] for the case where $E_2=R$, but the same proof holds for functions $f:E_1\to E_2$. It follows then that $f:E_1\to E_2$ is quadratic if and only if there exists a unique symmetric function $B : E_1 \times E_1 \to E_2$, additive in x for fixed y, such that $f(x) = B(x,x)$. The biadditive function B, the polar of f, is given by

$$B(x,y) = \left(\frac{1}{4}\right)(f(x+y) - f(x-y)).$$

A stability theorem for the quadratic functional equation (1) was proved by F. Skof [18] for functions $f : X \to E$ where X is a normed space and E a Banach space. Her proof also works if X is replaced by an Abelian group G. In this form, the theorem was demonstrated by P.W. Cholewa [2]. A function $f : G \to E$ is called δ-quadratic if for a given $\delta > 0$ it satisfies the inequality

$$||f(x+y) + f(x-y) - 2f(x) - 2f(y)|| \leq \delta. \tag{2}$$

The statement of the Skof-Cholewa theorem follows:

Theorem 1. *If $f : G \to E$ is δ-quadratic for all x and y in G, then there exists a function $q : G \to E$ which is quadratic, i.e. q satisfies (1) for all x and y in G and also is the unique quadratic function such that $||f(x) - q(x)|| \leq \frac{\delta}{2}$ for all x in G. The function is given by*

$$q(x) = \lim_{n\to\infty} 4^{-n} f(2^n x) \tag{3}$$

for all x in G.

The proof is ommited, as it is a special case of that of the next theorem 2 due to S. Czerwik [4]. Czerwik's main result may be stated as follows.

Theorem 2. *Let E_1 be a normed vector space, E_2 a Banach space and $\varepsilon > 0$, $p \neq 2$ be real numbers. Suppose that the function $f : E_1 \to E_2$ satisfies*

$$||f(x+y) + f(x-y) - 2f(x) - 2f(y)|| \leq \varepsilon(||x||^p + ||y||^p). \tag{4}$$

Then there exists exactly one quadratic function $g : E_1 \to E_2$ such that

$$||g(x) - f(x)|| \leq c + k\varepsilon||x||^p \tag{5}$$

for all x in E_1, where: when $p < 2$, $c = \dfrac{||f(0)||}{3}$, $k = \dfrac{2}{4-2^p}$ and g is given by (3) with g instead of q. When $p > 2$, $c = 0$, $k = \dfrac{2}{2^p - 4}$ and $g(x) = \lim_{n\to\infty} 4^n f(2^{-n}x)$ for *all x un E_1. Also, if the mapping $t \to f(tx)$ from R to E_2 is continuous for each fixed x in E_1, then $g(tx) = t^2 g(x)$ for all t in R.*

Proof. Case 1. $p < 2$. In (4) set $y = x \neq 0$ and divide by 4. Then use the triangle inequality to obtain

$$\|4^{-1}f(2x) - f(x)\| \leq 4^{-1}\|f(0)\| + 4^{-1}(2\varepsilon\|x\|^p). \tag{6}$$

Make the induction hypothesis

$$\|4^{-n}f(2^n x) - f(x)\| \leq \|f(0)\|\sum_{k=1}^{n} 4^{-k} + 2\varepsilon\|x\|^p \sum_{k=1}^{n} 2^{(k-1)p}4^{-k} \tag{7}$$

which is true for $n = 1$ by (6). Assuming (7) true, replace x by $2x$ in it and divide by 4. Now combine the result with (6) to see that (7) remains true with n replaced by $n+1$, which establishes (7) for all positive integers n and all $x(\neq 0)$ in E_1. By summing the series on the right side of (7), we obtain

$$\|4^{-n}f(2^n x) - f(x)\| \leq c + \frac{2\varepsilon\|x\|^p}{4 - 2^p} = c + k\varepsilon\|x\|^p, \tag{8}$$

with $k = \dfrac{2}{4-2^p}$. In order to prove convergence of the sequence $g_n(x) = 4^{-n}f(2^n x)$, we divide inequality (8) by 4^m and also replace x by $2^m x$ to find that

$$\|g_{m+n}(x) - g_m(x)\| = \|4^{-(m+n)}f(2^{(m+n)}x) - 4^{-m}f(2^m x)\| \leq 4^{-m}c + 2^{-(2-p)m}k\varepsilon\|x\|^p, \tag{9}$$

which shows that the limit $g(x) = \lim_{n\to\infty} 4^{-4n}f(2^n x)$ exists for each non-zero x in E_1, since E_2 is a Banach space. By letting $n \to \infty$ in (8), we arrive at the formula (5) with $c = \dfrac{\|f(0)\|}{3}$ and $k = \dfrac{2}{4-2^p}$. To show that g is quadratic, replace x and y by $2^n x$ and $2^n y$, respectively, in (4) and divide by 4^n to get

$$\|g_n(x+y) + g_n(x-y) - 2g_n(x) - 2g_n(y)\| \leq 2^{-(2-p)n}\varepsilon(\|x\|^p + \|y\|^p).$$

Taking the limit as $n \to \infty$, we find that g satisfies (1) when x and y are different from zero. We now define $g(x)$ as $\lim_{n\to\infty} 4^{-n}f(2^n x)$ for all x in E_1; it follows that $g(0) = 0$. Thus, (1) holds for $x = y = 0$, when f is replaced by g in (1). When $y = 0$ and $x \neq 0$, we have $g(x+0) + g(x-0) - 2g(x) - 2g(0) = 0$. For

$$x \neq 0 \text{ and } y \neq 0, \quad g(x+y) + g(x-y) - 2g(x) - 2g(y) = 0,$$

and setting $y = x$, gives ys $g(2x) = 4g(x)$ for $x \neq 0$, but this last equation obviously also hods for $x = 0$. With $y = -x \neq 0$, we get $g(0) + g(2x) - 2g(x) - 2g(-x) = 0$ which reduces to $g(-x) = g(x)$, which again is clearly true for all x in E_1. Finally, for $x = 0$ and $y \neq 0$, we have $g(0+y) + g(0-y) - 2g(0) - 2g(y) = 0$. Therefore, $g : E_1 \to E_2$ is quadratic on E_1.

Case 2. $p > 2$. In (4), set $x = y = 0$ to see that $f(0) = 0$. Then replace both x and y by $\frac{x}{2}$ to obtain

$$\|f(x) - 4f(2^{-1}x)\| \leq \left(\frac{\varepsilon}{2}\right) \|x\|^p \cdot 2^{-(p-2)}. \tag{10}$$

Apply the induction hypothesis:

$$\|f(x) - 4^n f(2^{-n}x)\| \leq \left(\frac{\varepsilon}{2}\right) \|x\|^p \sum_{k=1}^{n} 2^{-k(p-2)} \tag{11}$$

for all x in E_1 and all positive integers n. In (11), replace x by $2^{-1}x$ and multiply by 4 to get

$$\|4f(2^{-1}x) - 4^{n+1} f(2^{-(n+1)}x)\| \leq \left(\frac{\varepsilon}{2}\right) \|x\|^p \sum_{k=2}^{n+1} 2^{-k(p-2)}.$$

Combine the last inequality with (10) to show that (11) remains true with n replaced by $n+1$, which completes the induction proof. Summing the series on the right side of (11), we get

$$\|f(x) - 4^n f(2^{-n}x)\| \leq k\varepsilon \|x\|^p, \tag{12}$$

where now $k = \frac{2}{2^p - 4}$. Putting $h_n(x) = 4^n f(2^{-n}x)$, multiplying (12) by 4^n and replacing x by $2^{-m}x$, we have

$$\|h_m(x) - h_{m+n}(x)\| = \|4^m f(2^{-m}x) - 4^{m+n} f(2^{-(m+n)}x)\| \leq 2^{-m(p-2)} k\varepsilon \|x\|^p. \tag{13}$$

This shows that $\{h_m(x)\}$ is a Cauchy sequence and this there exists $g : E_1 \to E_2$ with $g(x) = \lim_{n\to\infty} h_n(x)$ for all x in E_1. The proof that g is quadratic is similar to that in Case 1, except that here we replace x and y in (4) by $2^{-n}x$ and $2^{-n}y$ and multiply the result by 4^n.

To prove the uniqueness of the quadratic function g subject to (5), let us assume on the contrary that there is another quadratic function $h : E_1 \to E_2$ satisfying

(5), and a point y in E_1 with $a = ||g(y) - h(y)|| > 0$. Every quadratic function has a unique representation in terms of a symmetric, biadditive function. Thus, $g(x) = B(x,x)$, where $B : E_1 \times E_1 \to E_2$ is symmetric and biadditive. It follows that $g(rx) = r^2 g(x)$ for all rational numbers r. Similarly, $h(rx) = r^2 h(x)$ for rational r. Since both g and h satisfy (5),

$||g(x)-h(x)|| = ||g(x)-f(x)+f(x)-h(x)|| \leq 2c+2k\varepsilon|| \leq 2c+2k\varepsilon||x||^p$ for all x in E_1.

In particular, we have for $r > 0$ that

$$r^2 a = ||g(ry) - h(ry)|| \leq 2c + 2k\varepsilon r^p ||y||^p. \tag{14}$$

In case 1, where $p < 2$, we have by (14) that

$$a \leq \frac{2c}{r^2} + \frac{2k\varepsilon||y||^p}{r^{2-p}} \text{ for rational } r > 0, \text{ so } a = 0.$$

In case 2, where $p > 2$, $c = 0$. We set $r = \dfrac{1}{s}$ in (14), so that

$$\frac{a}{s^2} = \left\| g\left(\frac{y}{s}\right) - h\left(\frac{y}{s}\right) \right\| \leq \frac{2k\varepsilon||y||^p}{s^p}.$$

Hence, $a \leq \frac{2k\varepsilon||y||^p}{s^{p-2}}$ for all $s > 0$, and again we see that $a = 0$. The proof that $g(tx) = t^2 g(x)$ for all real t will be deferred until after the:

Corollary 3. *If in theorem 2 the function f is continuous everywhere in E_1, then g is also continuous for all $x \neq 0$ in E_1. When $p > 0$, thos restriction is unecessary.*

Proof of Corollary. In case $p < 0$, we must treat $x = 0$ as a special situation since the right members of inequalities (4), (5) and (9) may become infinite as $x \to 0$. Suppose that f is continuous for all x in E_1 and that x_0 in E_1 is not zero. Set $s = \dfrac{||x_0||}{2}$. For x in the open ball $B(x_0, s) = \{x \in E_1 : ||x - x_0|| < s\}$, we have $s < ||x|| < 3s$. In inequality (9), let $n \to \infty$, so that $||f(x) - g_m(x)|| < 4^{-m}c+2^{(2-p)m}k\varepsilon||x||^p$. For x in $B(x_0, s)$, we have $s^p > ||x||^p < (3s)^p$ when $p < 0$, while the inequalities are reversed when $p > 0$. Consequently, $g_m(x)$ converges uniformly to $g(x)$ in $B(x_0, s)$ as $m \to \infty$, and, since each function g_m is continuous in $B(x_0, s)$, it follows that the limit g is also continuous in $B(x_0, s)$. Thus, the quadratic function g

is continuous at each point $x_0 \neq 0$ in E_1. Clearly, the restriction $x_0 \neq 0$ is not needed when $p > 0$.

Proof of Theorem 2 (concluded). We have seen that $g)rx) = r^2 g(x)$ when r is a rational number. To prove that g is homogeneous of degree two for all real numbers as well, it is sufficient to prove that the map $t \to g(tx)$ is continuous in t for fixed x in E_1. By hypothesis, the map $t \to f(tx)$ is continuous in t for fixed x in E_1. Apply corollary 3 to the case where $E_1 = R$ to show that in case $x \neq 0$ and $t_0 \geq 0$ then $t \to g(tx)$ is continuous at $t = t_0$. Thus, $g(t_0 x) = t_0^2 g(x)$ for all $t_0 \neq 0$, $x \neq 0$. But this equality is obvious both $x = 0$ and $t_0 = 0$. Therefore, $g(tx) = t^2 g(x)$ for all real t and x in E_1.

Remark. S. Czerwik [4] proved that $g(tx) = t^2 g(x)$ under the weaker assumption that $tf(tx)$ was Borel measurable in t for each fixed x.

The exclusion of the case $p = 2$ in Theorem 2 is necessary as shown by the counterexample to be cited below, due to Czerwik [4; pp.63-64]. It is a modification of the example of Gajda [8].

Let $\phi : R \to R$ be defined by

$$\phi(x) = \begin{cases} \mu x^2 & \text{for } |x| < 1 \\ \mu & \text{for } |x| ge 1, \end{cases}$$

where $\mu > 0$, and put, for all x in R,

$$f(x) = \sum_{n=0}^{\infty} 4^{-n} \phi(2^n x).$$

Then f is bounded by $\frac{4\mu}{3}$ on R and satisfies the condition

$$|f(x+y) + f(x-y) - 2f(x) - 2f(y)| \leq 32\mu(x^2 + y^2) \tag{A}$$

for all x and y in R, as it will now be shown. If $x^2 + y^2$ is either 0 or $\geq \frac{1}{4}$, then the left side of (A) is less than 8μ, so (A) is true. Now suppose that $0 < x^2 + y^2 < \frac{1}{4}$. Then there exists a positive integer k such that

$$4^{-k-1} \leq x^2 + y^2 < 4^{-k}, \tag{B}$$

so that $4^{k-1}x^2 < 4^{-1}$, $4^{k-1}y^2 < 4^{-1}$ and $2^{k-1}x, 2^{k-1}y, 2^{k-1}(x \pm y)$ all belong to the interval $(-1,1)$. Hence, for $n = 0, 1, \ldots, k-1$, $2^n x, 2^n y, 2^n(x \pm y)$ all belong to this same interval and

$$\phi(2^n(x+y)) + \phi(2^n(x-y)) - 2\phi(2^n x) - 2\phi(2^n y) = 0 \text{ for } n = 0, 1, \ldots, k-1.$$

From the definition of f and from the inequality (B), we have

$$|f(x+y) + f(x-y) - 2f(x) - 2f(y)|$$
$$\leq \sum_{n=0}^{\infty} 4^{-n} |\phi(2^n(x+y)) + \phi(2^n(x-y)) - 2\phi(2^n x) - 2\phi(2^n y)|$$
$$\leq \sum_{n=k}^{\infty} 6\mu 4^{-n} = (2\mu)4^{1-k} < 32(x^2+y^2)\mu,$$

thus f satisfies (A).

Suppose now that there exists a quadratic function $g: R \to R$ and a constant $\beta > 0$ such that $|f(x) - g(x)| < \beta x^2$ for all x in R. Since f is bounded for all x, it follows that g is bounded on any open interval containing the origin, so that g has the form $g(x) = \eta x^2$ for x in R, where η os a constant (see, e.g. S. Kurepa [11]). Thus, we have

$$|f(x)| \leq (\beta + |\eta|)x^2, \text{ for } x \text{ in } R. \tag{C}$$

Let k be a positive integer with $k\mu > \beta + |\eta|$. If $x \in (0, 2^{1-k})$, then $2^n x \in (0,1)$ for $n \leq k-1$, and, for $x \in R$, we have

$$f(x) = \sum_{n=0}^{\infty} 4^{-n}\phi(2^n x) \geq \sum_{n=0}^{k-1} \mu 4^{-n}(2^n x)^2 = k\mu x^2 > (\beta + |\eta|)x^2,$$

which contradicts (C). □

Remark. Theorem 2 can be generalized without difficulty to the situation where the right side of inequality (4) is replaced by $\varepsilon H(||x||, ||y||)$, in which $H: R_+ \times R_+ \to R_+$ is positive homogeneous of degree $p \neq 2$, i.e. $H(cs, ct) = c^p H(s,t)$ when c, s and t are all positive. This statement is an analog for approximately quadratic functions of a theorem of Rassias and Šemrl [13] for approximately additive functions.

The relationship between a quadratic function and its biadditive polar is basic in the methods used by F. Skof and S. Terracini [19] in their study of the stability of the quadratic functional equation for functions defined on restricted domains in R with values in a Banach space E. They proved the following stability theorem for symmetric biadditive functions based on results from Skof [17].

Theorem 3. *Denote the set* $[0,r) \times [0,r)$ *in* R^2 *by* S, *where* $r > 0$, *and let* E *be a Banach space. Suppose that* $\phi : S \to E$ *is symmetric and* δ*-biadditive, i.e.* $\phi(x,y) = \phi(y,x)$ *for* (x,y) *in* E *and* $\|\phi(x,t+u) - \phi(x,t) - \phi(x,u)\| < \delta$ *for all* x *in* $[0,r)$ *and* t,u, *and* $t+u$ *in* $[0,r)$ *and some* $\delta > 0$. *Then there exists at least one function* $F : S \to E$ *which is symmetric, biadditive and such that* $\|\phi(x,y) - F(x,y)\| < 9\delta$ *for* (x,y) *in* S.

Proof. We will refer to the proof of the theorem of Skof [17]. By hypothesis, for each y in $[0,r)$, the function of $x : \phi_y(x) = \phi(x,y)$ is δ-additive on the set $T(r) = \{(x',x'') \in R^2 : \ x',x'' \in [0,r), \ x'+x'' \in [0,r)\}$. Following the proof of extension II above, we define the function $\phi_y^* : R_+ \to E$ for fixed y by $\phi_y^*(x) = n\phi_y\left(\frac{r}{2}\right) + \phi_y(\mu)$ for $x = \frac{nr}{2} + \mu$, $n = 1,2,\ldots$ and $0 \le \mu < \frac{r}{2}$. Thus, we have for x in $[0,r)$ that

$$\|\phi_y(x) - \phi_y^*(x)\| < \delta. \tag{15}$$

This function is extended to R by putting $\phi_y^*(x) = -\phi_y^*(-x)$ when $x < 0$. It follows that, for $y \in [0,r)$, $\phi_y^*(x)$ is 2 δ-additive on R^2 and hence, by theorem 1.1 above, there is a unique additive function

$$G_y^*(x) = \lim_{n\to\infty} 2^{-n}\phi_y^*(2^n x) \tag{16}$$

for all x in R and y in $[0,r)$ such that

$$\|\phi_y^*(x) - G_y^*(x)\| \le 2\delta, \quad x \in R, \ y \in [0,r). \tag{17}$$

Now define $G(x,y) = G_y^*(x)$ when $(x,y) \in S$. $G(x,y)$ is additive in the first variable. We will show that it is 2 δ-additive in its second variable on the set $T(r) = \{(y,z) \in R^2 : \ y,z \in [0,r), \ y+z \in [0,r)\}$. Indeed, fix $x \in [0,r)$ and $y,z \in [0,r)$ with $y+z \in [0,r)$. Put $2^n x = k_n\frac{r}{2} + \mu_n$, where $\mu_n \in \left[0,\frac{r}{2}\right)$, and k_n is a positive integer, so that $k_n = \left(\frac{2}{r}\right)(2^n x - \mu_n)$. Then we have

$$G(x,y+z) - G(x,y) - G(x,z) = \lim_{n\to\infty} 2^{-n}(\phi_{y+z}^*(2^n x) - \phi_y^*(2^n x) - \phi_z^*(2^n x)) =$$

$$= \lim_{n\to\infty} 2^{-n}(\phi_{y+z}(\mu_n)-\phi_y(\mu_n)-\phi_z(\mu_n))+\lim_{n\to\infty} 2^{-n}k_n\left(\phi_{y+z}\left(\frac{r}{2}\right)-\phi_y\left(\frac{r}{2}\right)-\phi_z\left(\frac{r}{2}\right)\right)=$$
$$=2\left(\frac{x}{r}\right)\left(\phi_{y+z}\left(\frac{r}{2}\right)-\phi_y\left(\frac{r}{2}\right)-\phi_z\left(\frac{r}{2}\right)\right).$$

Hence,

$$\|G(x,y+z)-G(x,y)-G(x,z)\|<2\delta \text{ when } x\in[0,r) \text{ and } y,z\in T(r). \tag{18}$$

Next we extend G to a function $H:[0,r)\times R\to E$. With x in $[0,r)$ and $y\in R_+$, let $y=\frac{nr}{2}+\mu$, where $\mu\in\left[0,\frac{r}{2}\right)$ and $n=1,2,\ldots$, and define H by

$$H(x,y)=nG\left(x,\frac{r}{2}\right)+G(x,\mu) \tag{19}$$

and for $y<0$ put

$$H(x,y)=-H(x,-y). \tag{20}$$

By extension II, we find that H is 4 δ-additive in the second variable and

$$\|H(x,y)-G(x,y)\|<2\delta \text{ for } (x,y) \text{ in } S. \tag{21}$$

Also, for each R,H is additive in the first variable on the set

$$T(r)=\{x',x'')\in R^2:\ x',x''\in[0,r),\ x'+x''\in[0,r)\},$$

since G has this property.

For each fixed x in $[0,r)$, it follows by Hyers's stability theorem [9] (see also [12]-[16]) that the function

$$F(x,y)=\lim_{n\to\infty}2^{-n}H(x,2^ny) \tag{22}$$

is additive in y and satisfies

$$\|F(x,y)-H(x,y)\|\le 4\delta,\quad (x,y)\in[0,r)\times R. \tag{23}$$

F is also additive in x on $T(r)$ by (22) since H is. By (15), (17), (21) and (23), we obtain the required inequality $\|\phi(x,y)-F(x,y)\|<9\delta$ when $(x,y)\in S$.

In order to prove the symmetry of F, observe that since φ is symmetric, $\|F(x,y)-F(y,x)\|<18\delta$. For $y=0$, we have $F(x,0)=0=(F(0,x)$ for $x\in[0,r)$. For a given $y\in(0,r)$ set, $h_y(x)=F(y,x)-F(x,y)$, $x\in[0,r)$. Noe $h_y(x)$ is additive on $T(r)$ and bounded, so it is the restriction to $[0,r)$ of a function of x of the form $h_y(x) = a(y)x$. But $h_y(y) = 0$ for all $y \in [0,r)$, so $a(y) \equiv 0$. □

In proving the next theorems of Skof and Terracini [19], we will make use of the following sets: $K(r) = \{(x,y) \in R^2 : \ 0 \le y \le x, \ x+y < r\}$ and $D(r) = \{(x,y) \in R^2 : \ |x+y| < r, \ |x-y| < r\}$.

Theorem 4. *Let $f : [0,r) \to E$ (a Banach space) be δ-quadratic on $K(r)$ for some $\delta > 0$. Then there exists a quadratic function $q : R \to E$ such that*

$$\|f(x) - q(x)\| < \frac{79\delta}{2} \text{ for } x \in [0,r) \tag{24}$$

Proof. Since f is δ-quadratic on $K(r)$,

$$\|f(0)\| < \frac{\delta}{2}, \quad \|f(2x) + f(0) - 4f(x)\| < \delta,$$

and thus

$$\|f(2x) - 4f(x)\| < \frac{3\delta}{2} \text{ for } x \in \left[0, \frac{r}{2}\right). \tag{25}$$

Extend the function f to the interval $(-r,0)$ by defining the extension φ as $\phi(x) = f(x)$ for $x \in [0,r)$ and $\phi(x) = f(-x)$ for $x \in (-r,0)$. We will show that φ is δ-quadratic on $D(r)$. For brevity, put $\mu(x,y) = \|\phi(x+y) + \phi(x-y) - 2\phi(x) - 2\phi(y)\|$, so that $\mu(x,y) < \delta$ on $K(r)$ by hypothesis. On the set $K_1(r) = \{(x,y) \in R^2 : \ 0 \le x \le y, \ y+x < r\}$, we have $\mu(x,y) = \|f(x+y) + f(-x+y) - 2f(x) - 2f(y)\| < \delta$ since $(y,x) \in K(r)$. On $K_2(r) = \{(x,y) \in R^2 : \ x < 0, \ y \ge 0, \ y - x < r\}$, we have $\mu(x,y) = \mu(-x,y) < \delta$ since $(-x,y) \in K_1(r) \cup K(r)$ and φ is even. Finally, if $(x,y) \in D(r)$ with $y < 0$, then $(x,-y) \in K(r) \cup K_1(r) \cup K_2(r)$, so again $\mu(x,y) < \delta$, as was asserted.

Next, define the auxiliary function $h : D(r) \to E$ by putting $4h(x,y) = \phi(x+y) - \phi(x-y)$. Clearly, $h(x,y) = h(y,x)$ for all $(x,y) \in D(r)$. When $y \in \left[0, \frac{r}{2}\right)$, h is δ-additive with respect to x on $T\left(\frac{r}{2}\right) = \left\{(u,v) \in R^2 : \ u,v \in \left[0, \frac{r}{2}\right), \ u+v \in \left[0, \frac{r}{2}\right)\right\}$, and also in y by the interchange of x and y. From the definition of h, it follows that

$$4(h(u+v,y) - h(u,y) - h(v,y)) =$$

$$= [\phi(u+v+y)+\phi(u-v-y)-2\phi(u)-2\phi(v+y)]-[\phi(u-y-v)+\phi(u-y+v)-2\phi(u-y)-2\phi(v)]-$$

$$-[\phi(u+y)+\phi(u-y)-2\phi(u)-2\phi(y)]+[\phi(v+y)+\phi(v-y)-2\phi(v)-2\phi(y)]$$

when $(u, v+y), (u-y, v), (u, y)$ and (v, y) are points of the set $D(r)$ where φ is δ-quadratic. Hence, $\|h(u+v, y) - h(u, y) - h(v, y)\| < \delta$.

Thus, h satisfies the hypothesis of Theorem 3, with $\frac{r}{2}$ in place of r, therefore exists a function $F : \left[0, \frac{r}{2}\right) \times \left[, \frac{r}{2}\right) \to E$ which is symmetric and biadditive such that

$$\|h(x,y) - F(x,y)\| < 9\delta \text{ when } (x,y) \in \left[0, \frac{r}{2}\right) \times \left[, \frac{r}{2}\right). \tag{26}$$

For $x \in \left[0, \frac{r}{2}\right)$, we have

$$\|h(x,x)-f(x)\| = 4^{-1}\|f(2x)-f(0)-4f(x)\| \le 4^{-1}\|f(2x)+f(0)-4f(x)\|+4^{-1}\|2f(0)\| < \frac{\delta}{2}$$

so that

$$\|h(x,x) - f(x)\| < \frac{delta}{2} \text{ for } x \in \left[0, \frac{r}{2}\right). \tag{27}$$

The function $F(x, x)$ is quadratic on $K\left(\frac{r}{2}\right)$. According to a theorem of Skof [18], it may be extended to a function $q : R \to E$ which is quadratic and such that $q(x) = F(x, x)$ when $x \in \left[0, \frac{r}{2}\right)$. Thus, when $x \in \left[0, \frac{r}{2}\right)$, we have $\|f(x) - F(x,x)\| \le \|f(x) - h(x,x)\| + \|h(x,x) - F(x,x)\|$, so, by (26) and (27),

$$\|f(x) - q(x)\| < \left(\frac{19}{2}\right)\delta, \quad x \in \left[0, \frac{r}{2}\right). \tag{28}$$

Now for $x \in \left[\frac{r}{2}, r\right]$, i.e. $\frac{x}{2} \in \left[\frac{r}{4}, \frac{r}{2}\right]$; taking account of (25) and (28), we have

$$\|f(x) - q(x)\| \le \left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| + \left\|4f\left(\frac{x}{2}\right) - 4q\left(\frac{x}{2}\right)\right\| < \frac{79\delta}{2},$$

and the theorem is proved. □

With the help of theorem 4, Skof and Terracini obtained the following

Theorem 5. *Denote the set* $\{(x,y) \in R^2 : \ |x+y| < r, \ |x-y| < r\}$ *by* $D(r)$. *Let* E *be a Banach space and suppose that* $f : (-r, r) \to E$ *is* δ*-quadratic on* $D(r)$ *for some* $\delta > 0$. *Then there exists a function* $q : R \to E$ *which is quadratic and satisfies the inequality* $\|f(x) - q(x)\| < \frac{81\delta}{2}$ *when* $-r < x < r$.

Proof. Observe that for a given $y \in (-r, r)$, with $(x, y) \in D(r)$ and $(x, -y) \in D(r)$ we have $\|f(y) - f(-y)\| = 2^{-1}\|(f(x+y) + f(x-y) - 2f(x) - 2f(y)) - (f(x-y) + f(x+y) - 2f(x) - 2f(-y))\| < \delta$. Denote by f_0 the restriction of f to $[0, r)$ and apply theorem 4 to obtain $\|f_0(x) - q(x)\| < k\delta$, where $k = \frac{79}{2}$. For $x \in (-r, 0)$, if f is an even function, we have

$$\|f(x) - q(x)\| \leq \|f(x) - f(-x)\| + \|f(-x) - q(-x)\| < (1+k)\delta = \frac{81\delta}{2}. \quad \square$$

The next topic on the stability of a certain type of conditional Cauchy equation is intriguing because it turns out to be q srot of hybrid between approximately additive and approximately quadratic mappings.

Functionals which are approximately additive on A-orthogonal vectors

Given a complex Hilbert space X, let $A : X \to X$ be a bounded selfadjoint linear operator whose range AX has dimension > 2. A functional $\phi : X \to C$ is said to be **additive on A-orthogonal pairs** if x, y in X with $(Ax, y) = 0$ implies that $\phi(x+y) = \phi(x) + \phi(y)$. Such functionals were studied by F. Vajzovic [23]. On page 80 of this reference, he proved the following:

Theorem 6. *If $\phi : X \to C$ is continuous and additive on A-orthogonal pairs, then there exists a unique scalar β and unique vectors u and v in X such that*

$$\phi(x) = (x, u) + (v, x) + \beta(Ax, x).$$

Here and in the remainder of this section, the inner product of the Hilbert space X will be denoted by means of parentheses.

H. Drljevic and Z. Mavar [6] considered a stability problem for such functionals as follows. Using the concept of approximate additivity of Th.M. Rassias [12], these authors defined a functional $\phi : X \to C$ to be **approximately additive on A0orthogonal pairs** if there exist constants $\theta > 0$ and p in the interval $[0, 1)$ such that

$$|\phi(x+y) - \phi(x) - \phi(y)| \leq \theta \left[|(Ax, x)|^{\frac{p}{2}} + |(Ay, y)|^{\frac{p}{2}} \right] \tag{29}$$

for all x, y in X for which $(Ax, y) = 0$. Their main theorem is:

Theorem 7. *Let X be a complex Hilbert space and $A : X \to X$ be a bounded linear selfadjoint operator whose range AX has dimension > 2. Suppose that $\phi : X \to C$ is approximately additive on orthogonal pairs, so that ϕ satisfies (29) for some $\theta > 0$ and some p in $[0, 1)$, and also that $\phi(tx)$ is continuous in the scalar t for each fixed x and all t in C.*

Then there exists a unique continuous functional $\phi : X \to C$ which is additive on A-orthogonal pairs and satisfies the inequality

$$|\phi(x) - \psi(x)| \leq \varepsilon(p, \theta)|(Ax, x|^{\frac{p}{2}}$$

for all x in X, where $\varepsilon(p, \theta)$ is a constant.

Moreover, by Theorem 6, ψ is of the form

$$\psi(x) = (x, u) + (v, x) + \beta(Ax, x),$$

where the vectors u, v and the scalar β are constants.

Proof. We decompose ϕ into its odd and even parts, putting:

$$G(x) = \frac{\phi(x) - \phi(-x)}{2} \text{ and } H(x) = \frac{\phi(x) + \phi(-x)}{2}.$$

It is easy to see by use of (29) that both G and H are approximately additive on A-orthogonal pairs with the same constants θ and p as those that appear in (29).

Properties of the odd functional G

As just stated, we have

$$|G(x + y) - G(x) - G(y)| \leq \theta \left[|(Ax, x)|^{\frac{p}{2}} + |(Ay, y)|^{\frac{p}{2}}\right] \tag{30}$$

for all x, y in which satisfy $(Ax, y) = 0$.

In the trivial case in which $(Ax, x) = 0$ for some x in X, we have $|G(2x) - 2G(x)| = 0$, so that $\dfrac{G(2x)}{2} = G(x)$ and it follows that, for all $n \in N$,

$$\frac{G(2^n x)}{2^n} = G(x) \text{ when } (Ax, x) = 0. \tag{31}$$

Lemma 8. *Let $(Ax, x) \neq 0$ for some fixed x. Then there exists a y in X such that $(Ay, y) \neq 0$ and $(Ax, y) = 0$.*

Proof. Suppose that, contrary to the Lemma, $(Ay, y) = 0$ for each y in the hyperplane $Y = \{y \in X : (Ax, y) = 0\}$. Then, for each pair y_1, y_2 in Y, we have $(A(y_1 + y_2), y_1 + y_2) = 0$, so that $(Ay_1, y_2) + (Ay_2, y_1) = 0$. Now replace y_2 by iy_2 in the last equality to get $-i(Ay_1, y_2)+i(Ay_2, y_1) = 0$, that is $-(Ay_1, y_2)+(Ay_2, y_1) = 0$, to see that $(Ay_2, y_1) = 0$ for each pair y_1, y_2 in Y. Since $(Ax, x) \neq 0$, it follows that x does not belong to Y. Thus, every z in X may be written in the form $z = \gamma x + y$ for some complex number γ and some y in Y. So for all z in X, $Az = \gamma Ax + Ay$ and $(Az, y) = (Ax, y) + (Ay, y) = 0$. Therefore, $Az = \gamma' Ax$, which is contrary to the hypothesis that $\dim(Ax) > 2$. □

Remark. Since A is selfadjoint it follows that (Ax, x) and (Ay, y) are always real. If x and y are those of Lemma 8, then, by multiplying y by an appropriate positive real number if necessary, we may assume that $(Ay, y) \pm (Ax, x)$, with $(Ax, y) = 0$.

Lemma 9. *Assume that* $(Ax, x) \neq 0$, $(Ax, y) = 0$ *and* $(Ay, y) = (Ax, x)$. *Then the limit* $\hat{G}(x) = \lim_{n\to\infty} \frac{G(2^n x)}{2^n}$ *exists.*

Proof. From the assumptions of the lemma, we see that $(A(x+y), x-y) = 0$, i.e. $x + y$ and $x - y$ form an A-orthogonal pair. Hence, by (30),

$$|G(x+y+x-y)-G(x+y)-G(x-y)| \leq \theta \left[|(A(x+y), x+y)|^{\frac{p}{2}} + |(A(x-y), x-y)|^{\frac{p}{2}}\right],$$

that is

$$|G(2x) - G(x+y) - G(x-y)| \leq 2\theta \cdot 2^{\frac{p}{2}} |(Ax, x)|^{\frac{p}{2}}.$$

From this inequality and (30), using the oddness of G, we obtain

$$|G(2x) - 2G(x)| \leq |G(2x) - G(x+y) - G(x-y)| + |G(x+y) - G(x) - G(y)| +$$

$$+ |G(x-y) - G(x) - G(-y)| \leq$$

$$\leq 2\theta \left(2^{\frac{p}{2}} |(Ax, x)|^{\frac{p}{2}} + |(Ax, x)|^{\frac{p}{2}} + |(Ax, x)|^{\frac{p}{2}}\right) = 2\theta \left(2 + 2^{\frac{p}{2}}\right) |(Ax, x)|^{\frac{p}{2}},$$

or

$$\left|\frac{G(2x)}{2} - G(x)\right| \leq \theta \left(2 + 2^{\frac{p}{2}}\right) |(Ax, x)|^{\frac{p}{2}}. \tag{32}$$

By mathematical induction, as in the proof of the theorem of Th.M. Rassias [12], we find that

$$|2^{-n}G(2^n x) - G(x)| \le \theta\left(2 + 2^{\frac{p}{2}}\right)|(Ax,x)|^{\frac{p}{2}} \sum_{k=0}^{n-1} 2^{k(p-1)},$$

or by summing the series indicated,

$$|2^{-n}G(2^n x) - G(x)| \le \theta|(Ax,x)|^{\frac{p}{2}} \frac{2\left(2+2^{\frac{p}{2}}\right)}{2-2^p}. \tag{33}$$

In order to show that $\left\{\dfrac{G(2^n x)}{2^n}\right\}$ is a Cauchy sequence, let m and n be integer with $m > n > 0$. Then

$$\left|\frac{G(2^m x)}{2^m} - \frac{G(2^n x)}{2^n}\right| = 2^{-n}\left|\frac{G(2^m x)}{2^{m-n}} - G(2^n x)\right|.$$

Use inequality (33) with x replaced by $2^n x$ and n by $m-n$ to find that

$$\left|\frac{G(2^m x)}{2^m} - \frac{G(2^n x)}{2^n}\right| \le \theta 2^{-n}|(A(2^n x), 2^n x)|^{\frac{p}{2}} \frac{2\left(2+2^{\frac{p}{2}}\right)}{2-2^p} =$$

$$= 2^{n(p-1)}\theta|(Ax,x)|^{\frac{p}{2}} \frac{2\left(2+2^{\frac{p}{2}}\right)}{2-2^p}.$$

Since $p-1<0$, the above sequence is a Cauchy sequence and so converges for each x in X. We put $\hat{G}(x) = \lim\limits_{n\to\infty} \dfrac{G(2^n x)}{2^n}$. □

Lemma 10. *If $(Ax,x) \ne 0$, $(Ax,y) = 0$ and $(Ay,y) = -(Ax,x)$, then the limit $\lim\limits_{n\to\infty} 2^{-n}G(2^n x)$ exists.*

Proof. From the hypothesis it follows that $A(x \pm y, x \pm y) = 0$ when the $\pm$ signs are consistent. Hence, by (31),

$$\frac{G[2^n(x \pm y)]}{2^n} = G(x \pm y), \quad n \in N.$$

In the inequality (30), replace x by $2^n x$, y by $2^n y$ and divide the result by 2^n to get

$$\left|\frac{G[2^n(x+y)]}{2^n} - \frac{G(2^n x)}{2^n} - \frac{G(2^n y)}{2^n}\right| \le 2\theta \cdot 2^{n(p-1)}|(Ax,x)|^{\frac{p}{2}}.$$

In this last inequality, replace y by $-y$ to obtain

$$\left|\frac{G[2^n(x-y)]}{2^n} - \frac{G(2^n x)}{2^n} + \frac{G(2^n y)}{2^n}\right| \le 2\theta \cdot 2^{n(p-1)}|(Ax,x)|^{\frac{p}{2}},$$

$$\left|G(x+y)-\frac{G(2^n x)}{2^n}-\frac{G(2^n y)}{2^n}\right| \leq 2\theta \cdot 2^{n(p-1)}|(Ax,x)|^{\frac{p}{2}},$$

and

$$\left|G(x-y)-\frac{G(2^n x)}{2^n}+\frac{G(2^n y)}{2^n}\right| \leq 2 \cdot 2^{n(p-1)}\theta|(Ax,x)|^{\frac{p}{2}}.$$

From the last two inequalities, we find that

$$\left|\frac{[G(x+y)+G(x-y)]}{2}-\frac{G(2^n x)}{2^n}\right| \leq 2 \cdot 2^{n(p-1)}\theta|(Ax,x)|^{\frac{p}{2}}.$$

Hence, the limit

$$\lim_{n\to\infty}\frac{G(2^n x)}{2^n}=\frac{G(x+y)+G(x-y)}{2} \text{ exists. } \square \tag{34}$$

From (31), and Lemmas 9 and 10, it follows that the limit $\hat{G}(x)=\lim_{n\to\infty}\frac{G(2^n x)}{2^n}$ exists for each x in X.

Lemma 11. *$\hat{G}$ is additive on A-orthogonal pairs.*

Proof. Assume that x, y in X satisfy $(Ax,y)=0$. Using (30), we find that

$$\left|\frac{G[2^n(x+y)]}{2^n}-\frac{G(2^n x)}{2^n}-\frac{G(2^n y)}{2^n}\right| \leq 2^{n(p-1)}\theta\left[|(Ax,x)|^{\frac{p}{2}}+|(Ay,y)|^{\frac{p}{2}}\right].$$

Since $p<1$, the right member of this inequality approaches 0 as $n\to\infty$, so that

$$\hat{G}(x+y)-\hat{G}(x)-\hat{G}(y)=0. \quad \square$$

Lemma 12. *The functional $\hat{G}: X\to C$ satisfies $\hat{G}(ax)=a\hat{G}(x)$ for each real a and each x in X.*

Proof. We first demonstrate that

$$\hat{G}(ax+bx)=\hat{G}(ax)+\hat{G}(bx) \text{ for all } a,b \text{ in } R \text{ and all } x \text{ in } X. \tag{35}$$

Case 0. Note that, if, for some x, we have $(Ax,x)=0$, then x,x is an orthogonal pair, so by Lemma 11 we have $G(ax+bx)=G(ax)+G(bx)$.

Assume that $(Ax,x)\neq 0$. Then we know by previous results that there exists y in X such that $(Ay,y)\neq 0$ and $(Ax,y)=0$ and that we may assume that $(Ay,y)=\pm(Ax,x)$.

Case 1. Let $(Ay,y)\neq 0$, $(Ax,y)=0$ and $(Ay,y)=(Ax,x)$.

Then $(A(x+y), x-y) = 0$, so that $x+y, x-y$ as well as x, y are orthogonal pairs. Hence, for real numbers a and b, we have

$$\hat{G}[a(x+y)+b(x-y)] = \hat{G}[a(x+y)] + \hat{G}[b(x-y)] = \hat{G}(ax) + \hat{G}(ay) + \hat{G}(bx) - \hat{G}(by).$$

Moreover,

$$\hat{G}[a(x+y)+b(x-y)] = \hat{G}[(a+b)x + (a-b)y] = \hat{G}[(a+b)x] + \hat{G}[(a-b)y].$$

It follows that

$$\hat{G}[(a+b)x] + \hat{G}[(a-b)y] = \hat{G}(ax) + \hat{G}(bx) + \hat{G}(ay) - \hat{G}(by). \tag{36}$$

Now interchange a and b to obtain

$$G[(a+b)x] - G[(a-b)y] = G(ax) + G(bx) + G(by) - G(ay). \tag{37}$$

By adding (36) and (37) and then dividing the result by 2, we have $G(ax+bx) = G(ax) + G(bx)$ in Case 1.

Case 2. Assume that $(Ay, y) \neq 0$, $(Ax, y) = 0$ and $(Ay, y) = -(Ax, x)$.

Then, as we have seen before, $(A(x \pm y), x \pm y) = 0$, so that $x+y, x+y, x-y, x-y$ as well as x, y are orthogonal pairs. Thus, we have

$$\hat{G}[a(x+y)+b(x+y)] = \hat{G}[a(x+y)] + \hat{G}[b(x+y)] = \hat{G}(ax) + \hat{G}(bx) + \hat{G}(ay) + \hat{G}(by),$$

$$\hat{G}[a(x+y)+b(x+y)] = \hat{G}[(a+b)x + (a+b)y] = \hat{G}[(a+b)x] + \hat{G}[(a+b)y],$$

so that

$$\hat{G}[(a+b)x] + \hat{G}[(a+b)y] = \hat{G}(ax) + \hat{G}(bx) + \hat{G}(ay) + \hat{G}(by). \tag{38}$$

In (38), replace y by $-y$ to get

$$\hat{G}[(a+b)x] - \hat{G}[(a+b)y] = \hat{G}(ax) + \hat{G}(bx) - \hat{G}(ay) - \hat{G}(by), \tag{39}$$

where we have used the fact that G is odd. From (38) and (39), it follows that $\hat{G}(ax+bx) = \hat{G}(ax) + \hat{G}(bx)$ in Case 2. Thus, this equality holds for all $x \in X$ when a and b are real numbers.

In order to complete the proof of Lemma 12, let a mapping $\Phi_x(t) = \hat{G}(tx)$, t in R, be defined from R into C. By what has just been proved, we have

$$\Phi_x(a+b) = \Phi_x(a) + \Phi_x(b)$$

for all real a and b. Put $\Phi_{xn}(t) = \dfrac{G(2^n tx)}{2^n}$, so that

$$\Phi_x(t) = \lim_{n\to\infty} \Phi_{xn}(t).$$

Each of the functions $\Phi_{xn}(t)$ is continuous in t for all t by a hypothesis of Theorem 7. Hence, $\Phi_x(t) : R \to C$ is measurable in t since it is a limit of continuous functions. Since $\Phi_x(t)$ is both additive and measurable in t, it follows that $\Phi_x(a) = a\Phi_x(1)$ for a in R and each x in X. That is, $\hat{G}(ax) = a\hat{G}(x)$ for each a in R and each x in X. □

As to the estimate of the difference $\hat{G}-G$, we have shown that, when $(Ax,x) = 0$, this difference is zero according to (31). In Case 1, where $(Ax,x) \neq 0$, $(Ax,y) = 0$ and $(Ax,x) = (Ay,y)$, it follows from (33) that

$$|\hat{G}(x) - G(x)| \leq \varepsilon_1(p,\theta) \text{ for all } x \text{ in } X \tag{40}$$

where $\varepsilon_1(p,\theta) = \dfrac{2\theta\left(2+2^{\frac{p}{2}}\right)}{2-2^p}$.

We now turn to Case 2, where $(Ax,x) \neq 0$, $(Ax,y) = 0$ and $(ay,y) = -(Ax,x)$. In this Case, $\hat{G}(x) = \frac{(x+y)-G(x-y)}{2}$, and, by (34) and (30), we obtain

$$\left|\lim_{n\to\infty} \frac{G(2^n x)}{2^n} - G(x)\right| = \left|\frac{G(x+y)+G(x-y)}{2} - G(x)\right| =$$

$$= \left|\frac{G(x+y)-G(x)-G(y)}{2} + \frac{G(x-y)-G(x)-G(-y)}{2}\right| \leq$$

$$\leq 2^{-1}|G(x+y)-G(x)-G(y)| + 2^{-1}|G(x-y)-G(x)-G(-y)| \leq$$

$$\leq \theta|(Ax,x)|^{\frac{p}{2}} + \theta|(Ax,x)|^{\frac{p}{2}} = 2\theta|(Ax,x)|^{\frac{p}{2}}.$$

Therefore, in all three cases, and hence for all x in X, we have

$$|\hat{G}(x) - G(x)| \leq \varepsilon_1(p,\theta)(Ax,x)^{\frac{p}{2}} \tag{41}$$

where $\varepsilon_1(p,\theta) = 2\theta\dfrac{2+2^{\frac{p}{2}}}{2-2^p} > 2\theta$.

Since by Lemma 12 the mapping $\hat{G}$ is homogeneous of degree one with respect to real numbers, Drljevic and Mavar (1982) concluded that this property may be substituted for the continuity of the functional φ in Vajzovic's result cited above as Theorem 6. Thus, they found that the odd function $\hat{G}$, which was shown above to be additive on orthogonal pairs, is of the form

$$\hat{G}(x) = (x, u) + (v, x), \tag{42}$$

and hence is continuous and additive on X.

Properties of the even functional H

Consider the even functional $H(x) = \dfrac{\phi(x) + \phi(-x)}{2}$. By (29), it follows immediately that when $(Ax, y) = 0$, then

$$|H(x+y) - H(x) - H(y)| \leq \theta \left[|(Ax,x)|^{\frac{p}{2}} + |(Ay,y)|^{\frac{p}{2}} \right]. \tag{43}$$

Clearly $\phi(0) = 0$ implies that $H(0) = 0$.

Lemma 13. *For each x in X, the limit $\hat{H}(x) = \lim_{n\to\infty} 4^{-n} H(2^n x)$ exists.*

Proof. Case 0. Suppose that $(Ax, x) = 0$ for some x in X. Then, as before, in considering the functional G, we have $h(2x) = 2H(x)$, and $(Ax, -x) = 0$, so by (43) with y replaced by $-x$ it follows that $h(x) = 0$. Thus, $H(2x) = 0$ and $\dfrac{H(2^n x)}{4^n} = 0$ hold for all $n \in N$, and we have $\hat{H}(x) = 0$ in this case.

Suppose that $(Ax, x) \neq 0$ for some x. By Lemma 8, we know that there exists y in $Y = \{y : \ (Ax, y) = 0\}$ with $(Ay, y) \neq 0$.

Case 1. $(Ax, y) = 0$, $(Ax, x) \neq 0$ and (Ax, x) and (Ay, y) have the same sign. As we have seen previously, we may assume that $(Ax, x) = (Ay, y)$, so that $(A(x+y), x-y) = 0$. This in turn implies that

$$|H(2x) - H(x+y) - H(x-y)| \leq \theta \cdot 2^{1+\frac{p}{2}} |(Ax,x)|^{\frac{p}{2}}. \tag{44}$$

Notice that

$$H(y) - H(x) = \left| H\left(\frac{y+x}{2} + \frac{y-x}{2}\right) - H\left(\frac{y+x}{2}\right) - H\left(\frac{y-x}{2}\right) \right| +$$

$$+ \left| H\left(\frac{y+x}{2}\right) + H\left(\frac{y-x}{2}\right) - H\left(\frac{y+x}{2} + \frac{x-y}{2}\right) \right| \leq$$

$$\leq \left|H\left(\frac{y+x}{2}+\frac{y-x}{2}\right)-H\left(\frac{y+x}{2}\right)-H\left(\frac{y-x}{2}\right)\right|+$$
$$+\left|H\left(\frac{x+y}{2}\right)+H\left(\frac{x-y}{2}\right)-H\left(\frac{x+y}{2}+\frac{x-y}{2}\right)\right|\leq$$
$$\leq 2\theta\left[\left|\left(A\left(\frac{x+y}{2}\right),\frac{x+y}{2}\right)\right|^{\frac{p}{2}}+\left|\left(A\left(\frac{x-y}{2}\right),\frac{x-y}{2}\right)\right|^{\frac{p}{2}}\right],$$

or

$$|H(y)-H(x)|\leq 2^2\theta\cdot 2^{-\frac{p}{2}}|(Ax,x)|^{\frac{p}{2}}=\theta\cdot 2^{2-\frac{p}{2}}|(Ax,x)|^{\frac{p}{2}}. \tag{45}$$

By using (44) and (45), we have

$$|H(2x)-4H(x)|=|H(2x)-H(x+y)-H(x-y)+H(x+y)+H(x-y)-4H(x)|\leq$$
$$\leq |H(2x)-H(x+y)-H(x-y)|+$$
$$+|H(x+y)-H(x)-H(y)+H(x-y)-H(x)-H(-y)+2(H(y)-H(x))|\leq$$
$$\leq |H(2x)-H(x+y)-H(x-y)|+$$
$$+|H(x+y)-H(x)-H(y)+H(x-y)-H(x)-H(-y)+2(H(y)-H(x))|\leq$$
$$\leq \theta\cdot 2^{1+\frac{p}{2}}|(Ax,x)|^{\frac{p}{2}}+4\theta|(Ax,x)|^{\frac{p}{2}}+\theta\cdot 2^{3-\frac{p}{2}}|(Ax,x)|^{\frac{p}{2}}.$$

Therefore,

$$|H(2x)-4H(x)|\leq 2\theta\left(2+2^{\frac{p}{2}}+2^{2-\frac{p}{2}}\right)|(Ax,x)|^{\frac{p}{2}}. \tag{46}$$

Let us put

$$\mu(p,\theta)=\theta\left(1+2^{\frac{p}{2}-1}+2^{1-\frac{p}{2}}\right). \tag{47}$$

Divide inequality (46) by 4 and use the abbreviation (47) to obtain

$$\left|\frac{H(2x)}{4}-H(x)|\right|\leq \mu(p,\theta)|(Ax,x)|^{\frac{p}{2}}. \tag{48}$$

In (48), replace x by $2x$ and divide the result by 4 to get

$$\left|\frac{H(2^2x)}{4^2}-\frac{H(2x)}{4}\right|\leq 2^{p-2}\mu(p,\theta)|(Ax,x)|^{\frac{p}{2}}.$$

Combining the last two inequalities, we have

$$\left|\frac{H(2^2x)}{4^2}-H(x)\right|\leq \mu(p,\theta)|(Ax,x)|^{\frac{p}{2}}(1+2^{p-2}).$$

By mathematical induction, we find that

$$\left|\frac{H(2^n x)}{4^n} - H(x)\right| \le \mu(p,\theta)|(Ax,x)|^{\frac{p}{2}} \sum_{k=0}^{n} 2^{k(p-2)} \text{ for all } n \text{ in } N.$$

By summing the series indicated, we may write

$$\left|\frac{H(2^n x)}{4^n} - H(x)\right| \le \frac{4\mu(p,\theta)|(Ax,x)|^{\frac{p}{2}}}{4-2^p}. \tag{49}$$

Using a method similar to that used above in the case of the functional G, we find that the sequence $\left\{\frac{H(2^n x)}{4^n}\right\}$ converges in Case 1.

Case 2. $(Ax,y) = 0$, while (Ay,y) and (Ax,x) have opposite signs. Again we put $Y = \{y : (Ax,y)\} = 0$. Without loss of generality, we may assume that $(Ax,x) > 0$ and that $(Ay',y') \le 0$ for each y' in Y. We can also find a y in Y such that $(Ay,y) = -(Ax,x)$. Let P be a projection of the space X onto Y parallel to the vector Ax. Then $Ay = \alpha(y)Ax + PAy$, so that $(PAy,y) = (Ay,y)$. We note that the last equality holds if y is replaced by any y' in Y. Let $Z = \{z : z \in Y, (PAy,z) = 0\}$. Id $(PAz,z) = 0$ for all z in Z, then since $Z \subset Y$m it follows that $(Az,z) = (PAz,z) = 0$ for all z in Z. Thus, Az is perpendicular to z, so that $PAz = \beta(z)PAy$. Clearly, $x \notin Y$. Also, from $(Ay,y) = -(Ax,x) \ne 0$, it follows that $y \notin Z$. Moreover, x and y are linearly independent. For, iff $\alpha x + \beta y = 0$ for some scalars α and β, then $\alpha Ax + \beta Ay = 0$ and $\alpha PAx + \beta PAy = 0$, that is $\beta PAy = 0$. Hence, $\beta(PAy,y) = \beta(Ay,y) = 0$, so $\beta = 0$. Hence, $\alpha x = 0$ so $\alpha = 0$, since $x \ne 0$.

Thus, $y' \in Y$ can be written in the form $y' = \alpha x + \beta y + z$, with z in Z, and so $PAy' = \alpha PAx + \beta PAy + PAz = \beta PAy + PAz = (\beta + \beta(z))PAy$. It follows that $(PAz,z) = 0$ for all z in Z implies that

$$PAy' = (\beta + \beta(z))PAy \text{ for all } y' \in Y.$$

For each u in X, we write

$$Au = \alpha(u)Ax + PAu \tag{50}$$

Let $u = \alpha_1 x + \beta_1 y + z$. Then $PAu = \alpha_1 PAx + \beta_1 PAy + PAz$. As shown above, $PAz = \beta(z)PAy$, so that $PAu = (\beta_1 + \beta(z))PAy$ for all u in X. Thus, (50) becomes $A(u) = \alpha(u)Ax + (\beta_1 + \beta(z))PAy$ for all u in X, which is a contradiction to the hypothesis that the dimension of $A(X)$ is greater than two. Therefore, there exists a z' in Z such that $(PAz', z') \neq 0$. We may choose z in Z so that $(PAz, z) = (PAy, y)$, or $(Az, z) = (Ay, y)$. Also, $(Ay, z) = 0$, for we have $Ay = \alpha(y)Ax + PAy$ from the definition of P, so that $(Ay, z) = (\alpha(y)Ax, z) + (PAy, z)$. The first term of the right side of the last equality vanishes because $z \in Y$, and the second term vanishes by the definition of Z.

Thus, we have an element y in X with $(Ay, y) \neq 0$ and an element z in X satisfying $(Ay, z) = 0$ and $(Az, z) = (Ay, y)$. So we can use the results of Case 1, replacing x by y and y by z, to conclude that the sequence $\left\{\frac{H(2^n y)}{4^n}\right\}$ converges. On the other hand, since $(Ay, y) = -(Ax, x)$ and $(Ax, y) = 0$ implies that $(A(x \pm y), x \pm y) = 0$, it follows from Case 0 that $\frac{H(2^n(x \pm y))}{2^n} = 0$ for $n \in N$. For the A-orthogonal pair x, y, we have $|H(x+y) - H(x) - H(y)| \leq 2\theta|(Ax, x)|^{\frac{p}{2}}$. Since $2^n x, 2^n y$ is also A-orthogonal pair with $(A(2^n y), 2^n y) = -(A(2^n x, 2^n x))$, we have

$$\left|\frac{H(2^n(x+y))}{4^n} - \frac{H(2^n x)}{4^n} - \frac{H(2^n y)}{4^n}\right| \leq 2\theta \cdot 2^{n(p-2)}|(Ax, x)|^{\frac{p}{2}},$$

or

$$\left|\frac{H(2^n x)}{4^n} + \frac{H(2^n y)}{4^n}\right| \leq 2\theta \cdot 2^{n(p-2)}|(Ax, x)|^{\frac{p}{2}}.$$

Since the sequence $\left\{\frac{H(2^n y)}{4^n}\right\}$ converges, the same is true for the sequence $\left\{\frac{H(2^n x)}{4^n}\right\}$. From the results of Cases 0,1 and 2, we conclude that the sequence $\left\{\frac{H(2^n x)}{4^n}\right\}$ converges for each x in X. □

Lemma 14. *The functional $\hat{H}$ defined by $\hat{H} = \lim_{n\to\infty} \frac{H(2^n x)}{4^n}$ has the following properties:*

(1) $\hat{H}$ is additive on A-orthogonal pairs.

(2) $\hat{H}(x) = 0$ if $(Ax, x) = 0$.

(3) If $(Ax, x) = (Ay, y) \neq 0$ and $(Ax, y) = 0$, then $\hat{H}(x) = \hat{H}(y)$, while, if $(Ax, x) = -(Ay, y)$ and $(Ax, y) = 0$, then $\hat{H}(x) = -\hat{H}(y)$.

Proof. For (1), let x, y in X satisfy $(Ax, y) = 0$. Then, by (43), it follows that

$$\left|\frac{H(2^n(x+y))}{4^n} - \frac{H(2^n x)}{4^n} - \frac{H(2^n y)}{4^n}\right| \le 2^{n(p-2)}\theta\left[|(Ax,x)^{\frac{p}{2}} + |(Ay,y)|^{\frac{p}{2}}\right].$$

Taking the limit as $n \to \infty$, we get $\hat{H}(x+y) - \hat{H}(x) - \hat{H}(y) = 0$. For (2), from the definition of $\hat{H}$ in terms of H, it is clear that $\hat{H}(0) = 0$. Now let $(Ax, x) = 0$. Then, by the last inequality with $y = -x$, we have

$$\left|\frac{H(0)}{4^n} - \frac{H(2^n x)}{4^n} - \frac{H(-2^n x)}{4^n}\right| \le 0 \text{ for all } n \in N,$$

so $\hat{H}(0) - 2\hat{H}(x) = 0$ and $\hat{H}(x) = 0$. For (3), let $(Ax, x) = (Ay, y) \neq 0$ and $(Ax, y) = 0$. Then $(A(x+y), x-y) = 0$, so, using the A-orthogonal pairs $\frac{x+y}{2}$ and $\frac{x-y}{2}$, we get

$$\hat{H}(x) = \hat{H}\left(\frac{x+y}{2} + \frac{x-y}{2}\right) = \hat{H}\left(\frac{x+y}{2}\right) + \hat{H}\left(\frac{x-y}{2}\right),$$

while, using $\frac{x+y}{2}$ and $\frac{y-x}{2}$, we get $\hat{H}(y) = \hat{H}\left(\frac{x+y}{2} + \frac{y-x}{2}\right) = \hat{H}\left(\frac{x+y}{2}\right) + \hat{H}\left(-\frac{x-y}{2}\right) = \hat{H}(x)$, since $\hat{H}$ is even. On the other hand, suppose that $(Ax, x) = -(Ay, y)$ and $(Ax, y) = 0$. Then $(A(x+y), x+y) = 0$. Using the properties (2) and (1) above, we have $0 = \hat{H}(x+y) = \hat{H}(x) + \hat{H}(y)$, so that $\hat{H}(x) = -\hat{H}(y)$ in this case. □

Lemma 15. *For all x in X and all complex numbers a we have*

$$\hat{H}(ax) = |a|^2 \hat{H}(x).$$

Proof. From property (3) of Lemma 14, we conclude that the functional $\hat{H}$ is a function of (Ax, x). Let $\hat{H}(x) = \Gamma((Ax, x))$. Given x in X, we can find a y in X such that $(Ax, y) = 0$ and $(Ax, x) = \pm(Ay, y)$.

Case 1. $(Ax, x) = (Ay, y) > 0$.

Since $\hat{H}$ is additive in orthogonal pairs, it follows that, for real a and b, $\hat{H}(ax + by) = \hat{H}(ax) + H(by)$. By the definition of the function Γ, putting $u = a^2(Ax, x)$ and $v = b^2(Ax, x) = b^2(Ay, y)$, we have

$$\Gamma(u + v) = \Gamma(u) + \Gamma(v) \text{ for all } u, v \ge 0. \tag{51}$$

We may extend the function to all real numbers in a well known way, so that (51) will hold for all real u and v. Now define the mapping $\phi : R \to R$ by $\psi(a) = \hat{H}(ax) = \Gamma((a(ax), ax)) = \Gamma(a^2(Ax, x))$. Then, for real numbers a, b,

$$\phi(a+b) = \Gamma((a+b)^2(Ax,x)) = \Gamma(a^2(Ax,x) + 2ab(Ax,x) + b^2(Ax,x)),$$

$$\phi(a-b) = \Gamma((a-b)^2(Ax,x)) = \Gamma(a^2(Ax,x) - 2ab(Ax,x) + b^2(Ax,x)).$$

Using (51), we obtain

$$\psi(a+b) + \psi(a-b) = 2\phi(a) + 2\psi(b). \tag{52}$$

Since ψ is measurable, by a known theorem of S. Kurepa [11], it may be written as $\psi(a) = \alpha(x)a^2$, or $\hat{H}(ax) = \Gamma(a^2(Ax, x))$. Put $a = 1$ to get $\hat{H}(x) = \alpha(x)$; so that $\hat{H}(ax) = a^2\hat{H}(x)$ for x in X and a in R. Now suppose that $a = r\exp(i\omega)$ is a complex number ($r = |a|$). Then $(A(ax), ax) = (A(r\exp(i\omega)x, r\exp(i\omega)x) = r^2(Ax, x))$. Thus, $\hat{H}(ax) = \Gamma(A(ax), ax) = \Gamma(r^2(Ax, x))$ or $\hat{H}(ax) = |a|^2\hat{H}(x)$ for all x in X and all complex numbers a, in Case 1.

Case 2. $(Ax, x) = -(Ay, y)$ for all y in Y.

As before, we can find a z in Y such that $(Az, z) = (Ay, y)$ and $(Ay, z) = 0$. Hence, by the result of Case 1, we have

$$\hat{H}(ay) = |a|^2\hat{H}(y).$$

Since, by the condition of Case 2, we have $\hat{H}(x) = -\hat{H}(y)$ by Lemma 14, (3), it follows that

$$\hat{H}(ax) = -\hat{H}(ay) = -|a|^2\hat{H}(y) = |a|^2\hat{H}(x).$$

Therefore, $\hat{H}(ax) = |a|^2\hat{H}(x)$ holds for all x in X and all complex numbers a. □

From inequality (49), it follows that

$$|\hat{H}(x) - H(x)| \le \varepsilon_2(p,\theta)|(Ax,x)|^{\frac{p}{2}}, \text{ with } \varepsilon_2(p,\theta) = \frac{4\mu(p,\theta)}{4-2^p}. \tag{53}$$

On the basis of the property $\hat{H}(ax) = |a|^2\hat{H}(x)$ of $\hat{H}$, the authors conclude that the conclusion of Theorem 6 above holds, so that H is of the form

$$\hat{H}(x) = (x,c) + (d,x) + \beta(Ax,x),$$

where the constant vectors c, d in X and the complex constant β are uniquely determined by the functional $\hat{H}$. Thus, $\hat{H}$ is continuous. Also, since $\hat{H}$ is an even functional, it follows that $\hat{H}(x) = \beta(Ax,x)$.

To complete the proof of Theorem 7, we note that, by (41) and (53), we have

$$|\phi(x)-[\hat{G}(x)+\hat{H}(x)]| = |G(x)+H(x)-\hat{G}(x)-\hat{H}(x)| \leq |G(x)-\hat{G}(x)|+|H(x)-\hat{H}(x)| \leq$$

$$\leq \varepsilon(p,\theta)|(Ax,x)|^{\frac{p}{2}}, \text{ where } \varepsilon(p,\theta) = \varepsilon_1(p,\theta) + \varepsilon_2(p,\theta).$$

Therefore, the required functional of Theorem 7 is by (42):

$$\psi(x) = \hat{G}(x) + \hat{H}(x) = (x,u) + (v,x) + \beta(Ax,x).$$

To prove the uniqueness of ψ, suppose on the contrary that there is another functional $\psi_1 \neq \psi$ which is continuous, additive on A-orthogonal pairs and which satisfies

$$\psi_1(x) - \phi(x)| \leq \varepsilon'|(Ax,x)|^{\frac{p}{2}} \text{ for some constant } \varepsilon' > 0 \text{ and all } x \in X.$$

Since ψ_1 is continuous and additive on A-orthogonal pairs, it follows that it is of the form $\psi_1(x) = (x,c) + (d,x) + \gamma(Ax,x)$, where c, d are constant vectors in X and $\gamma \in C$. Then

$$|\psi(x) - \psi_1(x)| \leq |\psi(x) - \phi(x)| + |\phi(x) - \psi_1(x)| \leq (\varepsilon(p,\theta) + \varepsilon')|(Ax,x)|^{\frac{p}{2}},$$

that is, for all x in X, we have

$$|(x,u-c) + (v-d,x) + (\beta-\gamma)(Ax,x)| \leq (\varepsilon(p,\theta) + \varepsilon')(Ax,x)^{\frac{p}{2}}.$$

In this last inequality, replace x with nx to obtain

$$|(nx,u-c) + (v-d,nx) + (\beta-\gamma)n^2(Ax,x)| \leq (\varepsilon(p,\theta) + \varepsilon')n^p|(Ax,x)|^{\frac{p}{2}}. \quad (54)$$

Divide (54) by n^2 to get

$$|n^{-1}(x,u-c)+n^{-1}(v-d,x)+(\beta-\gamma)(Ax,x)| \le (\varepsilon(p,\theta)+\varepsilon')n^{p-2}|(Ax,x)|^{\frac{p}{2}},$$

and, letting $n \to \infty$, we obtain $\beta = \gamma$. Thus, (54) now becomes

$$|(nx,u-c)+(v-d,nx)| \le (\varepsilon(p,\theta)+\varepsilon')n^p|(Ax,x)|^{\frac{p}{2}}.$$

Divide this last inequality by n and then let $n \to \infty$ to get $(x,u-c)+(v-d,x)=0$. Now, if we first put $x=u-c$ and second put $x=i(u-c)$, to obtain $||u-c||^2+(v-d,u-c)=0$ and $i||u-c||^2-i(v-d,u-c)=0$, we find that $u=c$ and $v=d$. The uniqueness property of the functional ψ has been proved. □

Comments

In their paper, Drljevic and Mavar [6,p.171], stated without proof the following:

Theorem 16. *Let X be a Banach space and h a functional on X such that $h(tx)$ is continuous in the scalar t for each fixed x in X. Let $\theta \ge 0$ and $p \in [0,2)$ be real numbers such that*

$$|h(x+y)+h(x-y)-2h(x)-2h(y)| \le \theta(||x||^p+||y||^p) \text{ for each } x,y \text{ in } X.$$

Then there exists a unique quadratic functional h_1 on X such that

$$|h(x)-h_1(x)| \le \frac{4\theta}{4-2^p}||x||^p.$$

This anticipated, in the case of functionals, one of the results of Czerwik (see Case 1 of Theorem 2 above).

Later, Drljevic [7] proved the following:

Theorem 17. *Let X be a complex Hilbert space of dimension ≥ 3, $A: X \to X$ a bounded self-adjoint linear operator with $\dim AX \ge 2$, and let the real numbers $\theta \ge 0$ and $p \in [0,2)$ be given. Suppose that $h: X \to C$ is continuous and satisfies the inequality $|h(x+y)+h(x-y)-2h(x)-2h(y)-2h(y)| \le \theta\left[|(Ax,x)|^{\frac{p}{2}}+|(Ay,y)|^{\frac{p}{2}}\right]$ whenever $(Ax,y)=0$.*

Then the limit $h_1(x)=\lim_{n\to\infty}\dfrac{h(2^n x)}{4^n}$ exists for each $x \in X$ and the functional h_1 is continuous and satisfies $h_1(x+y)+h_1(x-y)=2h_1(x)+2h_1(y)$ whenever

$(Ax, y) = 0$. *Moreover, there exists a real number* $\varepsilon > 0$ *such that* $|h(x) - h_1(x)| \leq \varepsilon |(Ax, x)|^{\frac{p}{2}}$.

The methods of proof of this theorem are based in part on those explained above in the proof of Theorem 7.

Approximately homogeneous mappings

This topic has been studied by S. Czerwik [5] and by Jacek and Jozef Tabor [21]. We begin with a presentation of Czerwik's work. The following notations will be used. R *denotes the set of all real numbers,* R_+ *the set of non-negative reals and* R_0 *the set of non-zero reals. For each* α *in* R *and each* p *in* R_0*, we define* $U_p = \{\alpha \in R : \ \alpha^p \text{ exists in } R\}$.

Lemma 18. *Let* X *be a real vector space and* Y *a real normed space. Given* $f : X \to Y$, p *in* R_0 *and* $h : R \times X \to R_+$ *which satisfy the inequality*

$$||f(\alpha x) - \alpha^p f(x)|| \leq h(\alpha, x) \tag{55}$$

for all (α, x) *in* $U_p \times X$*, then the inequality*

$$||f(\alpha^n x) - \alpha^{np} f(x)|| \leq \sum_{s=0}^{n-1} |\alpha|^{sp} h(\alpha, \alpha^{n-s-1} x) \tag{56}$$

holds for all $n \in N$ *and* $(\alpha, x) \in U_p \times X$.

Proof. *We use (56) as an induction hypothesis. Note that it is true for* $n =$ *by (55). In (56), replace* x *by* αx *to get*

$$||f(\alpha^{n+1} x) - \alpha^{np} f(\alpha x)|| \leq \sum_{s=0}^{n-1} |\alpha|^{sp} h(\alpha, \alpha^{n-s} x).$$

Now multiply (55) by α^{np} *to obtain*

$$||\alpha^{np} f(\alpha x) - \alpha^{(n+1)p} f(x)|| \leq |\alpha|^{np} h(\alpha, x).$$

Combine the last two inequalities to find that

$$||f(\alpha^{n+1} x) - \alpha^{(n+1)p} f(x)|| \leq \sum_{s=0}^{n} |\alpha|^{sp} h(\alpha, \alpha^{n-s} x),$$

which completes the induction proof. □

Theorem 19. *Let the assumptions of Lemma 18 be satisfies, and let Y be a Banach space. Suppose that for some β in U_p with $\beta \neq 0$ the series*

$$\sum_{n=1}^{\infty} |\beta|^{-np} h(\beta, \beta^n x) \tag{57}$$

converges for each x in X, and that

$$\liminf_{n\to\infty} |\beta|^{-np} h(\alpha, \beta^n x) = 0 \tag{58}$$

for each (α, x) in $U_p \times X$. Then there exists a unique mapping $g : X \to Y$ such that $g(\alpha x) = \alpha^p g(x)$ for each (α, x) in $U_p \times X$ and which satisfies

$$\|g(x) - f(x)\| \leq \sum_{n=1}^{\infty} |\beta|^{-np} h(\beta, \beta^{n-1} x) \tag{59}$$

for each x in X.

Proof. For $n \in N$, set

$$g_n(x) = \beta^{-np} f(\beta_n x), \quad x \in X. \tag{60}$$

From (56), we get when $n \in N$ and $x \in X$:

$$\|g_n(x) - f(x)\| \leq \sum_{s=1}^{n} |\beta|^{-sp} h(\beta, \beta^{s-1} x). \tag{61}$$

In order to show that the $g_n(x)$ form a Cauchy sequence, we note that, if in (56) we replace x by $\beta^n x$ and n by $n-m$ where $n > m$, we have

$$\|g_n(x) - g_m(x)\| \leq |\beta|^{-np} \|f(\beta^n x) - \beta^{(n-m)p} f(\beta^m x)\| =$$

$$= \||\beta|^{-np} \|f(\beta^{n-m}(\beta^m x) - \beta^{(n-m)p} f(\beta^m x))\| \leq$$

$$\leq |\beta|^{-np} \sum_{s=0}^{n-m-1} |\beta|^{sp} h(\beta, \beta^{n-s-1}) = \sum_{s=0}^{n-m-1} |\beta|^{(s-n)p} h(\beta, \beta^{n-s-1} x).$$

This inequality may be written as $\|g_n(x) - g_m(x)\| \leq \sum_{k=m+1}^{n} |\beta|^{-kp} h(\beta, \beta^{k-1} x)$, and, by hypothesis, it follows that $\{g_n(x)\}$ is a Cauchy sequence for each x in X. From (60), (55) and (58), we obtain

$$\|g(\alpha x) - \alpha^p g(x)\| = \lim_{n\to\infty} \|\beta^{-np}[f(\alpha\beta^n x) - \alpha^p f(\beta^n x)] \leq$$

$$\leq \lim_{n\to\infty} |\beta|^{-np} h(\alpha, \beta^n x) = 0.$$

Thus, g is a p-homogeneous mapping when $\alpha \in U_p$. Also, from (61), we get (59).

It remains to prove that g is the unique p-homogeneous mapping that satisfies (59). Suppose that there are two such mappings, say g_1 and g_2. Then, for $m \in N$,

$$\|g_1(x) - g_2(x)\| = |\beta|^{-mp}\|g_1(\beta^m x) - g_2(\beta^m x)\| \leq$$

$$\leq |\beta|^{-mp}[\|g_1(\beta^m x) - f(\beta^m x)\| + \|g_2(\beta^m x) - f(\beta^m x)\|] \leq$$

$$\leq |\beta|^{-mp} \cdot 2\sum_{s=1}^{n} |\beta|^{-sp} h(\beta, \beta^{s+m-1}) \leq 2 \sum_{k=m+1}^{\infty} |\beta|^{-kp} h(\beta, \beta^{k-1}x).$$

Consequently, since the series (57) converges, it follows that $g_2 = g_1$. □

Corollary 20. *Let the assumptions of the Lemma 18 be satifies with* $h(\alpha, x) = \delta + |\alpha|^p \varepsilon$ *for given positive numbers* δ *and* ε, *and let* Y *be a Banach space. Then there is a unique p-homogeneous mapping* $g : X \to X$ *such thart*

$$\|g(x) - f(x)\| \leq \varepsilon \text{ for all } x \text{ in } X. \tag{62}$$

Proof. Assume that $p > 0$. By Theorem 19, for every $\beta = m \in N$, $\beta \geq 2$, there exists a p-homogeneous mapping

$$g_m(x) = \lim_{n\to\infty} m^{-np} f(m^n x), \quad x \in X,$$

such that

$$\|g_m(x) - f(x)\| \leq \sum_{n=1}^{\infty} m^{-np} h(m, m^{n-1}x) \leq \sum_{n=1}^{\infty} m^{-np}(\delta + m^p \varepsilon)$$

or

$$\|g_m(x) - f(x)\| \leq \frac{\delta + m^p \varepsilon}{m^p - 1}, \quad x \in X. \tag{63}$$

Now we shall show that, for each pair $m > 1$, $r > 1$ in N, we have $g_m = g_r$. By (63), for $n \in N$,

$$\|g_m(x) - g_r(x)\| = 2^{-np}\|g_m(2^n x) - g_r(2^n x)\| \leq 2^{-np}\left(\frac{\delta + m^p \varepsilon}{m^p - 1} + \frac{\delta + r^p \varepsilon}{r^p - 1}\right).$$

Thus, since $p > 0$, if we let $n \to \infty$, we get $g_m = g_r$. We pit $g(x) = g_2(x)$, $x \in X$. By (63) we have $\|g(x) - f(x)\| \leq \frac{\delta + m^p\varepsilon}{m^p - 1}$, and now, letting $m \to \infty$, we find that $\|g(x) - f(x)\| \leq \varepsilon$.

In the case where $p < 0$, we can take $\beta = \dfrac{1}{m}$ and say $q = -p$ and carry out a similar proof. □

Example. Take $f(x) = \sin x$ for x in R. Then $|\sin(\alpha x) - \alpha^p \sin x| < 1 + |\alpha|^p$ for (α, x) in $Up \times R$. This shows that not all cases under Corollary 20 are superstable.

Corollary 21. *Let the assumptions of Lemma 18 be satisfied with* $h(\alpha, x) = \delta + |\alpha|^p\varepsilon$, δ, ε *in* R_+, *and let* Y *be a Banach space. Then, if either* δ *or* ε *is zero,* $f(\alpha x) = \alpha^p f(x)$ *for all* (α, x) *in* $(U_p \setminus \{0\}) \times X$.

Proof. Suppose that $\delta = 0$. Then

$$\|f(\alpha x) - \alpha^p f(x)\| \leq |\alpha|^p \varepsilon \text{ for } (\alpha, x) \text{ in } U_p \times X. \tag{64}$$

Putting $x = \dfrac{y}{\alpha}$ with α in $U_p \setminus \{0\}$, we get $\left\| f(y) - \alpha^p f\left(\dfrac{y}{\alpha}\right) \right\| \leq |\alpha|^p \varepsilon$. Assume that $p > 0$. Then $f(y) = \lim_{\alpha \to 0} \alpha^p f\left(\dfrac{y}{\alpha}\right)$ for y in X. Therefore, for (β, x) in $(U_p \setminus \{0\}) \times X$, we have $f(\beta x) = \lim \alpha^p f\left(\dfrac{\beta x}{\alpha}\right) = \lim \beta^p \left(\dfrac{\alpha}{\beta}\right)^p f\left(\dfrac{\beta x}{\alpha}\right) = \beta^p f(x)$, so the corollary is verified for $\delta = 0$ and $p > 0$. If $p < 0$, then, from (64) we get $\lim\limits_{|\alpha| \to \infty} \alpha^p f\left(\dfrac{y}{\alpha}\right) = f(y)$, and as before we find that the corollary holds for $\delta = 0$ and $p < 0$.

On the other hand, suppose that $\varepsilon = 0$. Then $\|\alpha^{-p} f(\alpha x) - f(x)\| \leq |\alpha|^{-p}\delta$ for (α, x) in $(U_p \setminus \{0\}) \times X$. Hence, when $p > 0$, $f(x) = \lim\limits_{|\alpha| \to \infty} \alpha^{-p} f(\alpha x)$, and when $p < 0$, $f(x) = \lim\limits_{\alpha \to 0} \alpha^{-p} f(\alpha x)$. As before, it is easily shown that Corollary 21 holds in these cases as well. □

Czerwik [5] remarked that the problem remained open for $p = 0$ except when $X = Y = R$.

Jacek and Josef Tabor [21] have used a different definition of approximately homogeneous mappings from that of S. Czerwik. Jozef Tabor [20], in connection with his study of approximately linear mappings has already proved that every mapping from one real normed space X to anothe Y which for a given $\varepsilon > 0$ satisfies $\|f(\alpha x) -$

$\alpha f(x)\| \leq \varepsilon$ for all real α and x in X is homogeneous (see Corollary 1 of J. Tabor [20]).

In the seminar of R. Ger (Katowice, October 1992), K. Baron asked if the conclusion still holds if ε in the above inequality is replaced by $\varepsilon|\alpha|$. In this particular case, it turns out that these two conditions are equivalent. However, Baron's question led Jacek Tabor and Jozef Tabor to consider some generalizations of the inequality $\|f(\alpha x) - \alpha f(x)\| \leq \varepsilon|\alpha|$, which lead to the results given below. They began with a very general statement:

Lemma 22. *Consider a set X, a Hausdorff topological space Y and mappings $g_1 : X \to X$, $g_2 : Y \to Y$ and $f : X \to Y$. Suppose that g_2 is continuous on Y. Then the following two conditions are equivalent:*

(i) $g_2(f(x)) = f(g_1(x))$ for all x in X.

(ii) There exists a sequence of mappings $f_n : X \to Y$ such that

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ and } \lim_{n\to\infty} g_2(f_n(x)) = f(g_1(x)), \quad x \text{ in } X.$$

Proof. Observe that (i) implies (ii) because we may put $f_n = f$ for n in N. Suppose that (ii) holds. Since g_2 is continuous we get $f(g_1(x)) = \lim_{n\to\infty} g_2(f_n(x)) = g_2(f(x))$ for x in X. □

Definition. Given a set X and a semigroup G with unit I, we say that G acts on X if there is a mapping $\phi : G \times X \to X$ such that $\phi(\beta, \phi(\alpha, x)) = \phi(\beta\alpha, x)$ for α, β in G, x in X, where $\phi(1, x) = x$. In what follows, we shall write $\phi(\alpha, x)$ as a multiplication, e.g., $\alpha \circ x$ or $\alpha * x$.

Notation. R_+ denotes the non-negative real numbers, K denotes the field of either real of complex numbers and $0^0 = 1$.

Theorem 23. *Given a set X, a metric space (Y, d) and a semigroup G with identity acting on X (denoted $\alpha \circ x$) and also on Y (denoted $\alpha * y$). Assume that for each α in G the mapping $y \to \alpha * y$ is continuous in y for all y in Y. For a given* *mapping* $g : G \times X \to R_+$ *suppose that* $f : X \to Y$ *satisfies the inequality*

$$d(f(\alpha \circ x), \alpha * f(x)) \leq g(\alpha, x). \tag{65}$$

Assume also that there exists a sequence of invertible elements α_n *in* G *such that, for* α *in* G *and* x *in* X, *we have*

$$\lim_{n\to\infty} g(\alpha\alpha_n, (\alpha_n)^{-1} \circ x) = 0. \tag{66}$$

Then $f(\alpha \circ x) = \alpha * f(x)$ *for all* α *in* G *and* x *in* X.

Proof. In (65), replace α by $\alpha\alpha_n$ and x by $(\alpha_n)^{-1} \circ x$:

$$d(f(\alpha \circ x), \alpha\alpha_n * (\alpha_n^{-1} \circ x)) \leq g(\alpha\alpha_n, (\alpha_n)^{-1} \circ x).$$

Hence, by (66), we have

$$\lim_{n\to\infty} \alpha\alpha_n * f(\alpha_n^{-1} \circ x) = f(\alpha \circ x). \tag{67}$$

Taking $\alpha = 1$ in (67), we obtain

$$\lim_{n\to\infty} \alpha_n * f(\alpha_n^{-1} \circ x) = f(x) \text{ for } x \text{ in } X. \tag{68}$$

For an arbitrary α in G, put $g_1(x) = \alpha\circ$, $g_2(y) = \alpha * y$ and $f_n(x) = \alpha_n * f(\alpha_n^{-1} \circ x)$. By (68) and (67), f_n satisfies condition (ii) of Lemma 22. By this lemma, we get $f(g_1(x)) = g_2(f(x))$, that is $f(\alpha \circ x) = \alpha * f(x)$ for α in G and x in X. □

Corollary 24. *Let* X *be a normed space, where* $L(X)$ *denotes the semigroup of continuous linear operators on* X *with composition as the binary operation, and let* p_1, p_2 *be non-negative real numbers with* $p_1 \neq p_2$. *Let* $k : X \to R_+$ *be a mapping such that*

$$k(Ax) \leq ||A||^{p_2} k(x) \text{ for } A \text{ in } L(X), \quad x \text{ in } X. \tag{69}$$

Suppose that $f : X \to X$ *satisfies the inequality*

$$||f(Ax) - Af(x)|| \leq ||A||^{p_1} k(x) \text{ for } A \text{ in } L(X), \quad x \text{ in } X. \tag{70}$$

Then there exists an α *in* K *such that* $f(x) = \alpha x$.

Proof. Put $g(A,x) = ||A||^{p_1}k(x)$ for A in $L(X)$, x in X, and $A_n = \alpha_n I$, where $I =$ the identity map and

$$\alpha_n = \begin{cases} 1/n & \text{if } p_1 > p_2, \\ n & \text{if } p_1 < p_2. \end{cases}$$

By (70), the inequality (65) is satisfied. By (69) we have for x in X

$$g(AA_n, A_n^{-1}x) = ||AA_n||^{p_1}k(A_n^{-1}x) \le ||A||^{p_1}||A_n||^{p_1}||A_n^{-1}||^{p_2}k(x) \le ||A||^{p_1}|\alpha_n|^{p_1-p_2}.$$

Thus, $g(AA_n, A_n^{-1}x) \to 0$ as $n \to \infty$, so condition (66) holds. Hence, by Theorem 23, we have

$$f(Ax) = Af(x) \text{ for all } A \text{ in } L(X) \text{ and } x \text{ in } X. \tag{71}$$

It remains to prove that $f(x) = \alpha x$ for some α in K. In (71), put $x = 0$ and $A = 2I$ to see that $f(0) = 2f(0)$, so that $f(0) = 0 = \alpha 0$ for α in K. Suppose that, contrary to the statement in question, there exists an x in X, $x \neq 0$ such that $f(x) \neq \alpha x$ for each α in K. Then x and $f(x)$ ate linearly independent, so that there exists an A in $L(X)$ with $Af(x) = 0$ and $Ax = x$. Hence, by (71), $f(x) = 0$, a contradiction, and we conclude that for each x in X there exists an α such that $f(x) = \alpha x$. Now we must show that α does not depend on x. Let x_1, x_2 in X satisfy $x_1 \neq 0$, $x_2 \neq 0$ and $x_1 \neq x_2$, with $f(x_1) = \alpha_1 x_1$ and $f(x_2) = \alpha_2 x_2$. Take an A in $L(X)$ such that $Ax_1 = x_2$. Then, by (71), $\alpha_1 x_2 = \alpha_1 Ax_1 = A(\alpha_1 x_1) = A(f(x_1)) = f(Ax_1) = f(x_2) = \alpha_2 x_2$ and so $\alpha_1 = \alpha_2$. □

Corollary 25. *Let K be the real or complex field and let p, p_1 and p_2 be non-negative real numbers with $p_1 \neq p_2$. With X a vector space over K and Y a normed vector space over K, let $k : X \to R_+$ be a mapping such that*

$$k(\alpha x) \le |\alpha|^{\beta_2}k(x) \text{ for all } \alpha \text{ in } K \text{ and } x \text{ in } X. \tag{72}$$

If a mapping $f : X \to Y$ satisfies the inequality

$$||f(\alpha x) - |\alpha|^p f(x)|| \le |\alpha|^{p_1}k(x), \tag{73}$$

then

$$f(\alpha x) = |\alpha|^p f(x), \quad \alpha \text{ in } K, x \text{ in } X. \tag{74}$$

Proof. Here, K acting on X means the usual multiplication by scalars, but K acting on Y will be defined by $\alpha * y = |\alpha|^p y$ for α in K and y in Y. We put $g(\alpha, x) = |\alpha|^{p_1} k(x)$. Then (73) implies that (65) is satisfied. Again we take $\alpha = \frac{1}{n}$ if $p_1 > p_2$ and $\alpha_n = n$ if $p_1 < p_2$. Then, by (72), we have for α in K, x in X, $g(\alpha\alpha_n, \alpha_n^{-1}x) \leq |\alpha|^{p_1}|\alpha_n|^{p_1-p_2}k(x) \to 0$ as $n \to \infty$. Thus, condition (66) holds. By Theorem 23, we have

$$f(\alpha x) = \alpha * f(x) = |\alpha|^p f(x) \text{ for } \alpha \text{ in } K \text{ and } x \text{ in } X. \quad \square$$

In a similar way, the authors proved:

Corollary 26. *If* $k(\alpha x) < |\alpha|^{p_2} k(x)$ *and* $\|f(\alpha x) - \alpha f(x)\| \leq |\alpha|^{p_1} k(x)$, *then* $f(\alpha x) = \alpha f(x)$.

These authors also generalized these results to the case where Y is a topological vector space over K, and where the domain of f is a subset X_1 of X which is closed under multiplication by scalars, with a similar substitution for Y. Their generalization of Corollary 25 reads as follows:

Theorem 27. *Let X be a vector space over K, Y a topological vector space over K and let X_1 and Y_1 be subsets of X and Y, respectively, such that $KX_1 \subset X$ and $KY_1 \subset Y$. We are given a bounded set $V \subset Y$, a mapping $g : K \times Y_1 \to K$ and a sequence of non-zero elements α_n of K such that*

$$\lim_{n\to\infty} g(\alpha\alpha_n^{-1}x) = 0 \text{ for } \alpha \in K,\ x \in X_1.$$

Suppose that the mapping $f : X_1 \to Y_1$ satisfies the condition

$$f(\alpha x) - |\alpha|^p f(x) \in g(\alpha, x)V \text{ for all } \alpha \text{ in } K \text{ and } x \text{ in } X_1.$$

Then

$$f(\alpha x) = |\alpha|^p f(x) \text{ for all } \alpha \text{ in } K \text{ and } x \text{ in } X_1.$$

Comments

It is interesting to compare the results of S. Czerwik and of J. and J. Tabor on the subject of approximately homogeneous mappings, which were clearly arrived at independently. Consider the case where α is a real and non-negative and where X is a real vector space, Y a Banach space and let $f : X \to Y$ satisfy $\|f(\alpha x) - \alpha^p f(x)\| \leq h(\alpha, p, x)$. The Tabors looked at cases where h (their g) was constant ($h = \varepsilon$) or where h has a sub-homogeneity property. In both cases, superstability resulted. However, in Corollary 20, together with the Example which follows, Czerwik showed that, if h is the sum of a non-zero constant and a particular homogeneous function, superstability fails. On the other hand, the Tabors succeeded in generalizing their results to more general spaces.

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