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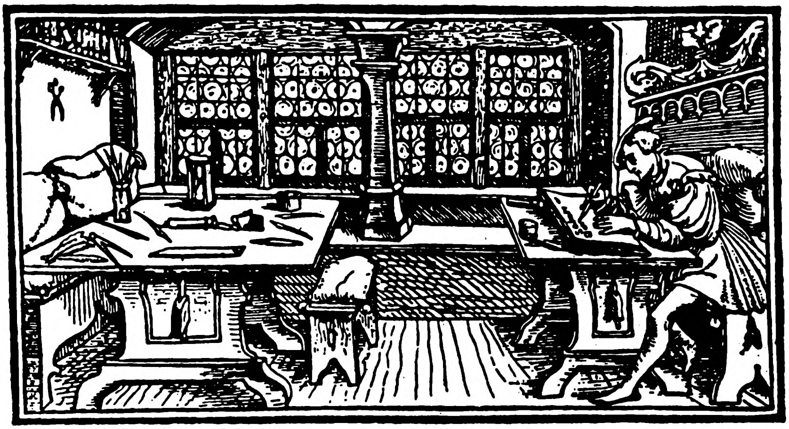

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UNIVERSITATIS "BABEŞ-BOLYAI"

MATHEMATICA

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ON THE MAXIMUM PRINCIPLE FOR BERNSTEIN-TYPE OPERATORS

JOSÉ A. ADELL, JESÚS DE LA CAL, AND IOAN RASA

Abstract. Let X be a separable metric space. Each particular family L of Bernstein-type (i.e. probabilistically defined) operators acting on $C(X)$ determines a topology on X, referred to as the L-topology, which is coarser than the metric topology. The concept of L-topology is used to establish a "maximum principle" which is applicable to a large class of examples and contains, as particular cases, some known results.

1. Introduction

Let $B_n f$ be the n-th Bernstein polynomial of a real function f defined on the standard m-simplex Δ_m, i.e.,

$$B_n f(\mathbf{x}) := \sum_{\mathbf{u}\in U_n} f\left(\frac{\mathbf{u}}{n}\right) \mu_n^{\mathbf{x}}\left\{\frac{\mathbf{u}}{n}\right\},$$

where U_n is the set of m-tuples of nonnegative integers $\mathbf{u} := (u_1, \ldots, u_m)$ such that $\sum_{i=1}^{m} u_i \leq n$, and, for $x := (x_1, \ldots, x_m) \in \Delta_m$,

$$\mu_n^{\mathbf{x}}\left\{\frac{\mathbf{u}}{n}\right\} := \frac{n!}{\prod_{i=1}^{m} u_i!\left(n - \sum_{i=1}^{m} u_i\right)!} \prod_{i=1}^{m} x_i^{u_i} \left(1 - \sum_{i=1}^{m} x_i\right)^{n - \sum_{i=1}^{m} u_i}$$

Chang and Davis [4] have shown that if f is convex, then $B_n f \geq f$, $n = 1, 2, \ldots$; see [1] for a probabilistic proof of the same result and [2,9,10] for related results in an infinite-dimensional setting.

Chang and Zang [5] have proved the following converse theorem of convexity:

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Theorem A. *Let $f \in C(\Delta_2)$. If*

$$B_n f \geq f, \quad n = 1, 2, \ldots,$$

then f cannot have a strict local maximum at an interior point of Δ_2.

In [6], it is reported that Chang and Zang have extended Theorem A higher dimensional simplices. A related version of Theorem A is the following "maximum principle" shown by Dahmen and Micchelli [6].

Theorem B. *Let $f \in C(\Delta_m)$. If*

$$B_n f \geq f, \quad n = 1, 2, \ldots, \tag{1}$$

then f achieves its maximum on the boundary of Δ_m.

In [11], Sauer gives a short and elementary proof of Theorem B, with the additional conclusion that f achieves its maximum at one of the vertices of Δ_m. From a result shown by Altomare and Rasa (cf. [2, Sec.6.1], [10]), it follows that Theorem B is valid in a more general setting (see Section 3 below).

The aim of this paper is to obtain a general maximum principle, which is applicable to many Bernstein-type operators and contains the above mentioned version of Theorem B. (In particular, we show that condition (1) can be weakened to $B_1 f \geq f$.)

Our approach is based upon two main ingredients: On the one hand, the probabilistic representation of the operators considered; on the other hand, the concept of L-topology associated with each particular family L of Bernstein-type operators that we introduce in the next section.

2. The L-topology and the maximum principle

Let X be a separable metric space and denote by $C(X)$ the set of all real continuous functions defined on X. Let $L := (L_i)_{i \in I}$ be a family of positive linear operators having the form

$$L_i f(x) = \int_X f d\mu_i^x, \quad i \in I, \ x \in X, \tag{2}$$

where, for $i \in I$ and $x \in X$, μ_i^x is a Borel probability measure on X and $f \in C(X)$ satisfies

$$\int_X |f| d\mu_i^x < \infty, \quad i \in I, \ x \in X. \tag{3}$$

Our basic assumption will be the following:

$$x \in S_x := \overline{\bigcup_{i \in I} S(\mu_i^x)}, \quad \text{for all } x \in X, \tag{4}$$

where $\overline{A}$ denotes the closure of A in the metric topology of X and $S(\mu_i^x)$ stands for the support of μ_i^x, which is defined by

$$S(\mu_i^x) := \bigcap \{C \subset X : \; C \text{ is closed and } \mu_i^x(C) = 1\}.$$

Since X is separable, it is clear that $S(\mu_i^x)$ is the least closed subset of X accumulating the whole mass of μ_i^x. This property may not hold if the separability assumption on X is dropped (see, for instance [12]).

With these notations and assumptions, we give the following:

Lemma 1. *The subsets C of X such that*

$$C = \overline{\bigcup_{x \in C} \bigcup_{i \in I} S(\mu_i^x)}$$

are the closed sets of a topology on X which will be called the L-topology. A closed set in this topology will be called an L-closed set.

Proof. It follows easily from elementary properties of the closure operation, as well as from assumption (4). □

Remark 1. Condition (4) is fulfilled if L is a net of operators having the form (2) and approximating every real continuous bounded function defined on X, that is,

$$\lim_i L_i f(x) = f(x), \quad x \in X, \; f \in CB(X),$$

or, equivalently (cf. [12]),

$$w - \lim_i \mu_i^x = \delta_x, \quad x \in X,$$

where "$w - \lim$" stands for weak limit and δ_x is the unit mass at x. Actually, since S_x is closed in the metric topology of X, we have (cf. [12])

$$1 = \limsup_i \mu_i^x(S_x) \leq \delta_x(S_x), \quad x \in X,$$

which implies (4). On the other hand, in the context of this paper, no generality is lost if it is assumed that the identity operator is an element of L, and this also guarantees that (4) is satisfied.

In what follows, M_f will denote the set of all points of X on which $f \in C(X)$ attains i maximum value. Note that M_f is a (possibly empty) closed set in the metric topolo of X.

Theorem 1. *Let $f \in C(X)$ satisfying (3). If*

$$M_f \subset \{y \in X : \; L_i f(y) \geq f(y), \; i \in I\},$$

then M_f is an L-closed set.

Proof. By hypothesis, we have

$$\int_X |f - f(x)| d\mu_i^x = 0, \quad i \in I, \; x \hat{\imath} M_f,$$

which implies

$$\mu_i^x(M_f) = 1, \quad i \in I, \; x \in M_f$$

and, therefore,

$$\overline{\bigcup_{x \in M_f} \bigcup_{i \in I} S(\mu_i^x)} \subset M_f.$$

Finally, the converse inclusion follows immediately from (4).

Corollary 1. *Let X be a compact metric space. If $f \in C(X)$ satisfies*

$$L_i f \geq f, \quad i \in I,$$

then M_f is (nonempty) L-closed set.

Remark 2. Recall that, from the Riesz representation theorem, if X is a compact me space, then every positive linear operator acting on $C(X)$ and preserving the const has the form (2).

3. Bernstein-Schnabl operators

In this section, we describe the L-topology generated by the Bernstein-Sch operators associated to an Altomare projection.

Let X be a metrizable compact convex subset of a locally convex Hausdorff space. Let $T : C(X) \to H \subset C(X)$ be an Altomare projection (cf. [9,10]). This m that:

(i) T is a linear positive projection on H.

(ii) H contains all the affine continuous functions on X.

(iii) For all $h \in X$, $z \in X$, $a \in [0,1]$, the functions $x \mapsto h(ax + (1-a)z)$ belongs to H.

For $x \in X$, let ν_x be the Borel probability measure on X such that

$$Tf(x) = \int_X f d\nu_x, \quad f \in C(X).$$

Let $P_n : X^n \to X$ be given by

$$P_n(x_1, \dots, x_n) := (x_1 + \cdots + x_n)/n.$$

The associated Bernstein-Schnabl operators B_n^* are defined (see [2,9,10]) by

$$B_n^* f(x) := \int_{X^n} f \circ P_n d\nu_x^n = \int_X f d\nu_n^x, \quad x \in X,\ n = 1, 2, \dots,$$

where ν_x^n denotes the product measure $\nu_x \otimes \cdots \otimes \nu_x$, with n factors, and $\mu_n^x := \nu_x^n \circ P_n^{-1}$.

For $A \subset X$, denote by $co(A)$ the convex hull of A, and set

$$m(A) := \bigcup_{n \geq 1} P_n(A^n).$$

Lemma 2. *We have:*

(a) For each $x \in X$

$$\overline{m(S(\nu_x))} = \overline{co(S(\nu_x))}.$$

(b) For each $x \in X$

$$m(S(\nu_x)) = \bigcup_{n \geq 1} S(\mu_n^x).$$

(c) For each $C \subset X$

$$C \subset \overline{\bigcup_{x \in C} m(S(\nu_x))}.$$

Proof. Part (a) has been shown in [2, Lemma 6.1.16] and [10, Lemma 2.1]. Part (b) is a consequence of the following equality

$$S(\mu_n^x) = P_n((S(\nu_n))^n), \quad n \geq 1.$$

Finally, (c) follows from (a) since x is the barycenter of ν_x. □

The following theorem gives a characterization of the B^*-topology associated to the sequence $B^* := (B_n^*)_{n \geq 1}$.

Theorem 2. *A set $C \subset X$ is B^*-closed if and only if the two following conditions* [cut off]
satisfied:

(a) C is closed in the metric topology.

(b) $\bigcup_{x \in C} \overline{co(S(\nu_x))} \subset C$.

Proof. By Lemma 2(b), $C \subset X$ is B^*-closed if and only if

$$C = \overline{\bigcup_{x \in C} m(S(\nu_x))}.$$

Thus, the conclusion follows from Lemma 2(a,c).

Theorem 3 below characterizes the Choquet boundary of X with respect t[cut off]
denoted by $Ch(H)$, in terms of the B^*-topology. Firstly, from a result shown by Altom[cut off]
[cf. [2; (3.3.4)], we have

$$x \in Ch(H) \Leftrightarrow Tf(x) = f(x) \text{ for all } f \in C(X) \Leftrightarrow S(\nu_x) = \{x\}.$$

We also have, by [2; (6.1.8)],

$$\bigcup_{x \in X} S(\nu_x) \subset Ch(H).$$

These facts, together with Theorem 2, immediately yield the following

Theorem 3. *(a) $Ch(H)$ is B^*-closed.*

(b) The set $\{x\}$ is B^*-closed if and only if $x \in Ch(H)$.

(c) $Ch(H)$ is the smallest subset of X which has nonempty intersection w[cut off] every nonempty B^*-closed set.

Combining Corollary 1, Theorem 2 and Theorem 3, we obtain the follow[cut off]
known result.

Corollary 2. *([2; Sec.6.1], [10]) If $f \in C(X)$ satisfies*

$$B_n^* f \geq f, \quad n \geq 1,$$

then:

(a) f is constant on $\bigcup_{x \in M_f} \overline{co(S(\nu_x))}$.

(b) f achieves its maximum on $Ch(H)$.

Remark 3. Let X be a metrizable Bauer simplex and let T be the canonical Altomare projection associated with X (cf. [9,10]). For each $x \in X$, let F_x be the closed face of X generated by x. Then $\overline{co(S(\nu_x))} = F_x$. We conclude that $C \subset X$ is B^*-closed if and only if it is closed in the metric topology and $F_x \subset C$ for all $x \in C$.

In particular, if X is the standard m-simplex Δ_m, B^* is just the sequence $B := (B_n)_{n \geq 1}$ of the Bernstein operators mentioned in the introduction. Then, the B-closed sets are the unions of k-dimensional faces of Δ_m ($k = 0, 1, \ldots, m$) and Corollary 2(b) is an extension of Theorem B.

The last result in this section is an improvement of Corollary 2(b).

Corollary 3. *If $f \in C(X)$ satisfies $B_1^* f \geq f$, then f achieves its maximum on $Ch(H)$.*

Proof. Let $M := \max\{f(x) : x \in Ch(H)\}$, and set $g := f - M$. Then $g \geq 0$ on $Ch(H)$. Using (5), we conclude that $Tg \leq 0$ on X. On the other hand,

$$Tg = Tf - M = B_1^* f - M \geq f - m = g.$$

Thus, $g \leq Tg \leq 0$ on X, which implies $f \leq M$. □

4. Other classical Bernstein-type operators

The Baskakov operator K_t, $(t > 0)$, has the form (2), with $X = [0, \infty)$ and μ_t^x given by

$$\mu_t^x \left\{ \frac{k}{t} \right\} = \binom{t+k-1}{k} \frac{x^k}{(1+x)^{t+k}}, \quad k = 0, 1, 2, \ldots$$

Hence, if $K = (K_t)_{t \in I}$, where I is any unbounded set of positive real numbers, the only K-closed sets are $\emptyset$, X and $\{0\}$.

As an example of bivariate (Baskakov-type) operator, distinct from a tensor product, we shall mention the following (cf. [1]): Let $X = [0, \infty) \times [0, \infty)$ and let L_t be defined by

$$L_t f(x, y) = \sum_{k,h=0}^{\infty} f\left(\frac{k}{t}, \frac{h}{t}\right) \binom{t+k+h-1}{k} \binom{t+h-1}{h} \frac{x^k y^h}{(1+x+y)^{t+k+h}},$$

where $t > 0$, $(x, y) \in X$ and $f \in C(X)$ is bounded. If $L = (L_t)_{t \in I}$, where I is any unbounded set of positive real numbers, then any proper L-closed set is a union of the following sets: the two semi-axes and the origin.

In the same way, Theorem 1 applies to many other univariate (such as Szász, Cheney and Sharma, Bleimann-Butzer-Hahn, etc.) and multivariate approximation operators. Details are omitted.

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References

[1] J.A. Adell, J. de la Cal and M. San Miguel, *On the property of monotonic convergence fo multivariate Bernstein-type operators*, J. Approx. Theory, to appear.
[2] F. Altomare and M. Campiti, *Korovkin-type Approximation Theory and Applications*, W de Gruyter, Berlin, 1994.
[3] R.B. Ash, *Real Analysis and Probability*, Academic Press, New York, 1972.
[4] G.Z. Chang and P.J. Davis, *The convexity of Bernstein polynomials over triangles*, J. Ap prox. Theory, 40(1984), 11-28.
[5] G.Z. Chang and J. Zhang, *Converse theorems of convexity for Bernstein polynomials ove triangles*, J. Approx. Theory, 61(1990), 265-278.
[6] W. Dahmen and C.A. Micchelli, *Convexity and Bernstein polynomials on k-simploids*, Act Math. Appl. Sinica, 6(1990), 50-66.
[7] J. de la Cal and F. Luquin, *Probabilistic methods in approximation theory: a general settin* Atti Sem. Mat. Fis. Univ. Modena, 40(1992), 137-147.
[8] G.G. Lorentz, *Bernstein Polynomials*, 2nd ed., Chelsea, New York, 1986.
[9] I. Rasa, *Altomare projections and Lototsky-Schnabl operators*, Rend. Circ. Mat. Palermo (2) Suppl., 33(1993), 439-451.
[10] I. Rasa, *On some properties of Altomare projections*, Conf. Sem. Mat. Univ. Bari, 253(1993) 1-17.
[11] T. Sauer, *On the maximum principle of Bernstein polynomials on a simplex*, J. Approx Theory, 71(1992), 121-122.
[12] N.N. Vakhania, V.I. Tarieladze and S.A. Chobanian, *Probability distributions on Banac spaces*, Kluwer, Dordrecht, 1987.

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APPROXIMATION OF FUNCTIONS OF SEVERAL VARIABLES BY OPERATORS OF PROBABILISTIC TYPE

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Abstract. In this paper we define a new type of linear positive operator associated to Pearson's χ^2-distribution, used in the approximation of multivariate functions and we study its approximation properties.

1. Introduction

In our papers [2], [4] we defined and investigated a linear positive operator which was associated with Pearson's-χ^2 distribution:

$$(C_nf)(x) = e\left[f\left(\frac{1}{n}\sum_{k=1}^{n}X_k^2\right)\right] = \frac{1}{(2x)^{n/2}\Gamma(n/2)}\int_0^{\infty} t^{\frac{n}{2}-1}e^{-\frac{t}{2x}}f\left(\frac{t}{n}\right)dt \tag{1.1}$$

where the sequence of independent random variables $(X_k)_{k\in N^*}$ having the same normal distribution $N(0,\sqrt{x})$, $x>0$, $E(X_k)=0$, $D^2(X_k)=x$, $(\forall)\ k\in N^*$ and f is a real function bounded on $(0,+\infty)$ such that the mean value of the random variable $f\left(\frac{1}{n}\sum_{k=1}^{n}X_k^2\right)$ exists, for any $n\in N^*$. This operator was called by F. Altomare, M. Campiti [1] "the n-th Cismasiu operator". This linear positive operator was extension in the case of two variables [5], when f is a given function defined and bounded over $\Omega_2=\{(x,y)\in R^2|\ x>0,\ y>0\}$:

$$(C_nf)(x,y) = \frac{1}{(4xy)^{n/2}(\Gamma(n/2))^2}\int_0^{\infty}\int_0^{\infty}(uv)^{\frac{n}{2}-1}e^{-\frac{1}{2}\left(\frac{u}{x}+\frac{v}{y}\right)}f\left(\frac{u}{n},\frac{v}{n}\right)du\,dv. \tag{1.2}$$

Now, in according with [6], we consider the following extension of the operators (1.1) and (1.2) to the case of several variables:

$$(C_nf)(x_1,x_2,\dots,x_s) =$$

$$= \frac{1}{(2^sx_1x_2\dots x_s)^{n/2}\left(\Gamma\left(\frac{n}{2}\right)\right)^s}\int_0^{\infty}\cdots\int_0^{\infty}(t_1t_2\dots t_s)^{\frac{n}{2}-1}e^{-\frac{1}{2}\left(\frac{t_1}{x_1}+\dots+\frac{t_s}{x_s}\right)}.$$

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$$\cdot f\left(\frac{t_1}{x_2},\dots,\frac{t_s}{x_s}\right) dt_1\dots dt_s \quad (1.$$

where f is a given function defined and bounded over $\Omega_s = \{x_1, x_2, \dots, x_s) \in R^s | x_1$ $0,\ x_2 > 0, \dots, x_s > 0\}$. Indeed, let be a sequence of s-dimensional random vecto $\{X_k = (X_{k1}, X_{k2}, \dots, X_{ks})\}_{k\in N^*}$, where X_{kj}, $j = \overline{1,s}$ are independent random variabl having the same normal distribution $N\left(0, \sqrt{x_j}\right)$, $x_j > 0$, $E(X_{kj}) = 0$, $D^2(X_{kj}) =$ $k \in N^*$, $j = \overline{1,s}$.

We assume that the components Y_{nv} of the random vector Y_n represent t arithmetic means of the first n components X_{kv}^2, $k = \overline{1,n}$, $v = \overline{1,s}$: $Y_{nv} = \frac{1}{n}\sum_{k=1}^{n} X_{}$ $Y_n = (Y_{n1}, Y_{n2}, \dots, Y_{ns})$.

These components Y_{nv} have a Pearson's-χ^2 distribution with n degrees of fr dom and parameters $x_v > 0$, $v = \overline{1,s}$. If f is a real function bounded on $\Omega_s = (0,+\infty)$ $\cdots\times(0,+\infty)$ such that the mean value of the random variables $f(Y_{n1}, Y_{n2}, \dots, Y_{ns})$ exi for any $n \in N^*$, then (1.1) become (1.3).

2. Approximation property of operators

In this section we investigate the approximation properties of the operators (1.

Theorem 2.1. *If f is a bounded uniform continuous function on $(0,a)\times\cdots\times(0,a)$, a 0, then the sequence $\{(C_nf)(x_1, x_2, \dots, x_s)\}_{n\in N^*}$ converge uniformly to $f(x_1, x_2, \dots,$ on $(0,a)\times\cdots\times(0,a)$, $a > 0$.*

Proof. In accordance with the limit theorem of [5] is sufficient that $\lim_{n\to\infty} \sigma_{n,v}^2 = 0$, $(\forall)\ v$ $\overline{1,s}$ where

$$\sigma_{n,v}^2 = D^2\left(\frac{1}{n}\sum_{k=1}^{n} X_{k,v}^2\right) =$$

$$= \frac{1}{(2^s x_1 x_2 \dots x_s)^{n/s}\left(\Gamma\left(\frac{n}{2}\right)\right)^s}\int_0^\infty \cdots \int_0^\infty \left(\frac{t_v}{n} - x_v\right)^2 \cdot$$

$$\cdot (t_1 t_2 \dots t_s)^{\frac{n}{2}-1} e^{-\frac{1}{2}\left(\frac{t_1}{x_1}+\frac{t_2}{x_2}+\dots+\frac{t_s}{x_s}\right)} dt_1 \dots dt_s$$

is the variances of the Pearson's-χ^2 distribution.

But $\sigma_{n,v}^2 = (2x_v^2/n)$, $v = \overline{1,s}$ and $\lim_{n\to\infty} \sigma_{n,v}^2 = 0$, $(\forall)\ v = \overline{1,s}$. We conclude th $\lim_{n\to\infty}(C_nf)(x_1, \dots, x_s) = f(x_1, \dots, x_s)$ uniformly on $(0,a)\times\cdots\times(0,a)$, $a > 0$ for a function f uniform continuous.

3. Estimate of the order of approximation

We shall now proceed to estimate the order of approximation of the function f by the operator (1.3). It is convenient to make use of the modulus of continuity, defined as follows:

$$\omega(f;\delta_1,\delta_2,\ldots,\delta_s) =$$

$$= \sup\{|f(x_1'',x_2'',\ldots,x_s'') - f(x_1',x_2',\ldots,x_s')|;\ |x_1''-x_1'|<\delta_1,\ldots,|x_s''-x_s'|<\delta_s\}$$

where $(x_1'',x_2'',\ldots,x_s'')$ and $(x_1',x_2',\ldots,x_s')$ are point of $(0,a)\times\cdots\times(0,a)$, $a>0$.

Theorem 3.1. *If f is a bounded and uniform continuous function on $(0,a)\times\cdots\times(0,a)$, $a>0$, then*

$$|f(x_1,x_2,\ldots,x_s) - (C_nf)(x_1,x_2,\ldots,x_s)| < \left(1+sa\sqrt{2}\right)\omega\left(f;\frac{1}{\sqrt{n}},\ldots,\frac{1}{\sqrt{n}}\right).$$

Proof. Using the following properties to the modulus of continuity:

$$|f(x_1'',x_2'',\ldots,x_s'') - f(x_1',x_2',\ldots,x_s')| < \omega(f;|x_1''-x_1'|,\ldots,|x_s''-x_s'|)$$

and

$$\omega(f,\lambda_1\delta_1,\ldots,\lambda_s\delta_s) < (1+\lambda_1+\cdots+\lambda_s)\omega(f,\delta_1,\delta_2,\ldots,\delta_s)$$

where

$$\lambda_1>0,\ldots,\lambda_s>0,$$

we have

$$|f(x_1'',x_2'',\ldots,x_s'') - f(x_1',x_2',\ldots,x_s')| < \omega\left(f;\frac{1}{\delta_1}|x_1''-x_1'|\delta_1,\ldots,\frac{1}{\delta_s}|x_s''-x_s'|\delta_s\right).$$

Now,

$$|f(x_1,x_2,\ldots,x_s) - (C_nf)(x_1,x_2,\ldots,x_s|) \le$$

$$\le \int_0^\infty\cdots\int_0^\infty\left|f(x_1,x_2,\ldots,x_s) - f\left(\frac{t_1}{x_1},\ldots,\frac{t_s}{x_s}\right)\right|\rho(t_1\ldots t_s;x_1\ldots x_s)dt_1\ldots td_s$$

where

$$\rho_n(t_1,t_2,\ldots,t_s;x_1,x_2,\ldots,x_s) =$$

$$= \begin{cases} \frac{1}{(2^s x_1x_2\ldots x_s)^{\frac{n}{2}}}(t_1t_2\ldots t_s)^{\frac{n}{2}-1}e^{-\frac{1}{2}\left(\frac{t_1}{x_1}+\cdots+\frac{t_s}{x_s}\right)}, & t_i>0,\ x_i>0,\ i=\overline{1,s} \\ 0, & t_i\le 0. \end{cases}$$

We may therefore write:

$$|f(x_1,x_2,\ldots,x_s) - (C_nf)(x_1,x_2,\ldots,x_s)| <$$

$$< \left[1 + \sum_{v=1}^{s} \left(\frac{1}{\delta_v} C_n \left(\left|x_v - \frac{t_v}{n}\right|; x_1, \ldots, x_s\right)\right)\right] \omega(f; \delta_1, \ldots, \delta_s).$$

In accordance with the Cauchy-Schwarz inequality, we have:

$$C_n\left(\left|x_v - \frac{t_v}{n}\right|; x_1, \ldots, x_s\right) \le$$

$$\le \left(\int_0^{\infty} \cdots \int_0^{\infty} \left(x_v - \frac{t_v}{n}\right)^2 \rho_n(t_1, \ldots, t_s; x_1, \ldots, x_s) dt_1 \ldots dt_s\right)^{1/2} =$$

$$= \sigma_{n,v} = \left(\frac{2x_v^2}{n}\right)^{1/2}.$$

So:

$$|f(x_1, x_2, \ldots, x_s) - (C_n f)(x_1, x_2, \ldots, x_s)| < \left(1 + \sum_{v=1}^{s} \frac{x_v \sqrt{2}}{\delta_v \sqrt{n}}\right) \omega(f; \delta_1, \ldots, \delta_s).$$

For $\delta_v = 1/\sqrt{n}$, $v = \overline{1,s}$ and $\sup\{x_v\sqrt{2} | \; x_v \in (0,a)\} = a\sqrt{2}$, we obtain:

$$|f(x_1, x_2, \ldots, x_s) - (C_n f)(x_1, x_2, \ldots, x_s)| < \left(1 + sa\sqrt{2}\right) \omega\left(f; 1/\sqrt{n}, \ldots, 1/\sqrt{n}\right).$$

□

4. Asymptotic estimate of the remainder

We next turn to the task of establishing an asymptotic estimate of the remainder

$$R_n(f; x_1, x_2, \ldots, x_s) = f(x_1, x_2, \ldots, x_s) - (C_n f)(x_1, x_2, \ldots, x_s)$$

which corresponds to a result of Voronovskaja about Bernstein polynomials.

Theorem 4.1. *If f is a function defined and bounded on $(0, +\infty) \times \cdots \times (0, +\infty)$ and at an interior point $(x_1, x_2, \ldots, x_s)$ of Ω_s the second differential $d^2 f(x_1, x_2, \ldots, x_s)$ exists, then we have the asymptotic formula:*

$$\lim_{n \to \infty} n[|f(x_1, x_2, \ldots, x_s) - (C_n f)(x_1, x_2, \ldots, x_s)|] = -\sum_{v=1}^{s} x_v^2 f''_{x_v^2}(x_1, \ldots, x_s).$$

Proof. Let $(t_1, t_2, \ldots, t_s) \in (0, +\infty) \times \cdots \times (0, +\infty)$ be. Under the hypothesis of the theorem, exists a function $g(t_1, t_2, \ldots, t_s)$ defined on $(0, +\infty) \times \cdots \times (0, +\infty)$ such that when $(t_1, t_2, \ldots, t_s) \to (x_1, x_2, \ldots, x_s)$ we have $g(t_1, t_2, \ldots, t_s) \to 0$ and

$$f\left(\frac{t_1}{n}, \ldots, \frac{t_s}{n}\right) = f(x_1, x_2, \ldots, x_s) + \sum_{v=1}^{s} \left(\frac{t_v}{n} - x_v\right) f'_{x_v}(x_1, \ldots, x_s) +$$

$$+\frac{1}{2}\sum_{v,j=1}^{s}\left(\frac{t_v}{n}-x_v\right)\left(\frac{t_j}{n}-x_j\right)f''_{x_vx_j}(x_1,\ldots,x_s)+\left(\sum_{v=1}^{s}\left(\frac{t_v}{n}-x_v\right)^2\right)g\left(\frac{t_1}{n},\ldots,\frac{t_s}{n}\right).$$

Multiply by $\rho_n(t_1,\ldots,t_s;x_1,\ldots,x_s)$ and then integrate into $t_1,t_2,\ldots,t_s$ with $t_1>0,\ldots,t_s>0$, we have:

$$R_n(f;x_1,x_2,\ldots,x_s)=-\frac{1}{2}\sum_{v=1}^{s}\frac{2x_v^2}{n}f''_{x_v^2}(x_1,x_2,\ldots,x_s)+\alpha_n(x_1,x_2,\ldots,x_s)$$

where

$$\alpha_n(x_1,x_2,\ldots,x_s)=$$

$$=\int_0^\infty\cdots\int_0^\infty\left(\sum_{v=1}^{s}\left(\frac{t_v}{n}-x_v\right)^2\right)g\left(\frac{t_1}{n},\ldots,\frac{t_s}{n}\right)\rho_n(t_1,\ldots,t_s;x_1,\ldots,x_s)dt_1\ldots dt_s.$$

Since $g\left(\frac{t_1}{n},\ldots,\frac{t_s}{n}\right)\to 0$ as $\frac{t_v}{n}\to x_v$, $v=\overline{1,s}$, it follows that for any positive $\varepsilon>0$ there are the positive numbers $\delta_1,\ldots,\delta_s$ such that $\left|g\left(\frac{t_1}{n},\ldots,\frac{t_s}{n}\right)\right|<\varepsilon$ whenever $\frac{t_v}{n}\to x_v$, $v=\overline{1,s}$. In view of the fact that

$$\alpha_n(x_1,x_2,\ldots,x_s)\le$$

$$\le\int_0^\infty\cdots\int_0^\infty\left(\sum_{v=1}^{s}\left(\frac{t_v}{n}-x_v\right)^2\right)\left|g\left(\frac{t_1}{n},\ldots,\frac{t_s}{n}\right)\right|\rho_n(t_1,\ldots,t_s;x_1,\ldots,x_s)dt_1\ldots dt_s$$

we may proceed further in the same way as in the case of one variable [3] and reach the conclusion that

$$\alpha_n(x_1,x_2,\ldots,x_s)=\frac{\varepsilon_n(x_1,x_2,\ldots,x_s)}{n},$$

where $\varepsilon_n(x_1,x_2,\ldots,x_s)\to 0$, $n\to\infty$. □

References

[1] Altomare, F., Campiti, M., *Korovkin-type approximation theory and its applications*, Walter de Gruyter, Berlin New York, 1994.

[2] Cismasiu, C.S., *About an Infinitely divisible distribution*, Proceedings of the Colloquium on Approximation and Optimization, Cluj-Napoca, 1984, 53-58.

[3] Cismasiu, C.S., *Probabilistic interpretation of Voronovskaja's theorem*, Bul. Univ. Brasov, Seria C, vol.XXVII, 1985, 7-12.

[4] Cismasiu, C.S., *A linear positive operator associated with the Pearson's-χ^2 distribution*, Studia Univ. Babes-Bolyai, Mathematica, XXXII, 4, 1987, 21-23.

[5] Cismasiu, C.S., *A new linear positive operator in two variables associated with the Pearson's-χ^2 distribution* (to appear).

[6] Stancu, D.D., *Probabilistic methods in the theory of approximation of functions of several variables by linear positive operators*, Approximation Theory, Intern. Symp. Univ. Lancaster, July 1969, London 1970, 329-342.

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ALGORITHM FOR THE CALCULUS OF THE CONVEX FUNCTION OF BEST UNIFORM APPROXIMATION

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Abstract. The present article deals with an algorithm for the calculus of the convex function of best uniform approximation using divided difference.

1. Introduction

Let $I = [a, b]$ be a compact real interval and $C(I)$ be the Banach space of all continuous real functions f on I equiped with the uniform norm $\|f\| = \sup\{|f(x)| : x \in I\}$. We denote by $Conv(I)$ the set of all continuous and convex functions on I.

Given an f in $C(I)$, we define its greatest convex minorant $\bar{f}$ to be the largest convex function which does not exceed f at any point in I:

$$\bar{f}(x) = \sup \{ g(x) : g \in Conv(I),\ g(x) \leq f(x) \text{ for all } \ x \in I \}, \quad x \in I.$$

The problem of convex approximation on I (see [1]) implies that the practically determination of the convex function of best uniform approximation is equivalent with an algorithm for the calculus of $\bar{f}$.

2. Main results

By the help of the following lemma we can formulate the desired algorithm. Furthemore, we shall prove the convergence of the algorithm for every $f \in C(I)$.

Lemma. *Let* $a = z_0 < z_1 < \cdots < z_{i-1} < z < z_i < \cdots < z_{m-1} < z_m = b$ *and* $g \in Conv(I)$ *a linear function on every interval* $[z_{i-1}, z_i]$, $i = \overline{1, m}$.

If $M < g(z)$ *then there exist* $g_z \in Conv(I)$ *and* $J(z, M) \subseteq \{0, 1, 2, \ldots, m\}$ *such that* $g_z \leq g$ *on* I *(i.e.* $g_z(x) \leq g(x)$ *for all* $x \in I$ *),* $g_z(z) = M$, g_z *is linear on every interval* $[z_{j-1}, z_j]$ $(j = \overline{1, i-1})$, $[z_{i-1}, z]$, $[z, z_i]$, $[z_j, z_{j+1}]$ $(j = \overline{i, m-1})$ *and* $g_z(z_i) = g(z_i)$ *for every* $i \in J(z, M)$, *where* $J(z, M)$ *is maximal with this property.*

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Under the notations of Lemma we have the following algorithm:

(i) Let $\varphi_0(x) = f(a) + [a,b;f]\cdot(x-a)$ for $x \in I$ and $A_0 = \{a,b\}$ ($[a,b;f]$ represents the divided difference of the function f on the indicated nodes);

(ii) Set $n = 0$;

(iii) Given the points $a = x_0 < x_1 \cdots < x_{2^n-1} < x_{2^n} = b$, $x_i = a + \frac{i}{2^n}(b-a)$, $i = \overline{0, 2^n}$, we choose $y_1 = \frac{1}{2}(x_0 + x_1)$. If $f(y_1) < \varphi_n(y_1)$, then using Lemma for $m = 2^n$, $g = \varphi_n$, $M = f(y_1)$ and $z = y_1$ we obtain the function g_{y_1} such that $g_{y_1} \leq \varphi_n$. If $f(y_1) \geq \varphi_n(y_1)$ then let $g_{y_1} = \varphi_n$.

Let $y_2 = \frac{1}{2}(x_1 + x_2)$. If $f(y_2) < g_{y_1}(y_2)$ then we use the Lemma for the nodes $a = x_0 < y_1 < x_1 < x_2 < \cdots < x_{2^{n-1}-1} < x_{2^n} = b$, and for $g = g_{y_1}$, $M = f(y_2)$ and $z = y_2$. So we obtain the function g_{y_2} such that $g_{y_2} \leq g_{y_1}$. If $f(y_2) \geq g_{y_1}(y_2)$ then let $g_{y_2} = g_{y_1}$.

Finally, let $y_{2^n} = \frac{1}{2}(x_{2^n-1} + x_{2^n})$. If $f(y_{2^n}) < g_{y_{2^n-1}}(y_{2^n})$ then there exists the function $g_{y_{2^n}}$ such that $g_{y_{2^n}} \leq g_{y_{2^n-1}}$, by Lemma. In this case we apply the Lemma for the nodes $a = x_0 < y_1 < x_1 < y_2 < x_2 < \cdots < x_{2^n-1} < x_{2^n} = b$, and for $g = g_{y_{2^n-1}}$, $M = f(y_{2^n})$ and $z = y_{2^n}$. If $f(y_{2^n}) \geq g_{y_{2^n-1}}(y_{2^n})$ then let $g_{y_{2^n}} = g_{y_{2^n-1}}$;

(iv) Let $\varphi_{n+1} = g_{y_{2^n}}$ and we define the following set: $A_{n+1} = A_n \cup \{y_1, y_2, \ldots, y_{2^n}\}$. Let $x_0, x_1, \cdots, x_{2^{n+1}}$ denote the elements of A_{n+1}, where $a = x_0 < x_1 < x_2 < \cdots < x_{2^{n+1}-1} < x_{2^{n+1}} = b$;

(v) Set $n = n + 1$;

(vi) Go to (iii).

The execution of the algorithm stops when the function obtained at the nth iteration satisfies some demands.

The method (i) - (vi) generates a sequence of function $\{\varphi_n\}_{n\geq 0}$ for which we have the following result:

Theorem. *We have the following statements:*

a) the sequence $\{\varphi_n\}_{n\geq 0}$ *converges uniformly to* $\bar{f}$ *on* I;

b) $\|\varphi_n - \bar{f}\| \leq \|g_n\|$, *for all* $n \geq 0$, *where* $g_n : I \to R$,

$$g_n(x) = \begin{cases} \varphi_n(x) - f(x), & \text{if } \varphi_n(x) > f(x) \\ 0, & \text{if } \varphi_n(x) \leq f(x). \end{cases}$$

3. Proofs

Proof of the lemma. Define the function $h : I \to R$ as follows:

$$h(x) = \begin{cases} g(z_{i-1}) + \frac{M-g(z_{i-1})}{z-z_{i-1}} \cdot (x - z_{i-1}), & \text{if } x \in [z_{i-1}, z) \\ M, & \text{if } x = z \\ M + \frac{g(z_i)-M}{z_i-z} \cdot (x - z), & \text{if } x \in (z, z_i]. \end{cases}$$

Let us suppose that $2 \le i \le m-1$ and $[z_{i-1}, z; h] \ge [z_{i-2}, z_{i-1}; h]$, $[z, z_i; h] \le [z_i, z_{i+1}; h]$. Then $g_z(x) = h(x)$ for all $x \in I$.

If $i = 1$ and $[z, z_1; h] \le [z_1, z_2; h]$ then $g_z(x) = h(x)$ for all $x \in I$.

If $i = m$ and $[z_{m-1}, z; h] \ge [z_{m-2}, z_{m-1}; h]$ then $g_z(x) = h(x)$ for all $x \in I$.

In the opposite case we may assume that $[z_{i-1}, z; h] < [z_{i-2}, z_{i-1}; h]$. Let $k = \min\{ j : 1 \le j \le i-1, \ [z_j, z; h] < [z_{j-1}, z_j; h] \}$. Then

$$g_z(x) = \begin{cases} h(x), & \text{if } x \in [a, z_{k-1}) \\ h(z_{k-1}) + \frac{M-h(z_{k-1})}{z-z_{k-1}}(x - z_{k-1}), & \text{if } x \in [z_{k-1}, z]. \end{cases}$$

For $[z, z_i; h] > [z_i, z_{i+1}; h]$ let $l = \max\{ j : i \le j \le n-1, \ [z, z_j; h] > [z_j, z_{j+1}; h] \}$. We define the function g_z as follows:

$$g_z(x) = \begin{cases} M + \frac{h(z_{l+1})-M}{z_{l+1}-z} \cdot (x - z), & \text{if } x \in [z, z_{l+1}) \\ h(x), & \text{if } x \in [z_{l+1}, b]. \end{cases}$$

If $z \in (a, z_1)$ then $g_z(x) = g(a) + \frac{M-g(a)}{z_1-a}(x-a)$ for $x \in [a, z]$, and we apply the above construction on $[z, b]$.

If $z \in (z_{m-1}, b)$ then $g_z(x) = M + \frac{g(b)-M}{b-z}(x-z)$ for $x \in [z, b]$, and we apply the above construction on $[a, z]$. □

Proof of the theorem. a). The reader can readily verify that $\varphi_n \in Conv(I)$ and $\varphi_n \ge \varphi_{n+1}$ on I for all $n \ge 0$. On the other hand, $\varphi_n \ge \bar{f}$ on I for all $n \ge 0$.

Indeed, it is clear that $\varphi_0 \ge \bar{f}$ on I. If $\varphi_n \ge \bar{f}$ on I and $\varphi_n \ne \varphi_{n+1}$ then there exists $y \in I$ such that $\varphi_n(y) > \varphi_{n+1}(y) = f(y) \ge \bar{f}(y)$. Because $\bar{f} \in Conv(I)$ and $\varphi_n \ge \bar{f}$ on I, we have $\varphi_{n+1} \ge \bar{f}$ on I by the construction of φ_{n+1}.

So, there exists $\varphi(x) = \lim_{n\to\infty} \varphi_n(x)$ for all $x \in I$. Because $\varphi_n \in Conv(I)$ we have $\varphi \in Conv(I)$. But $\varphi_n \ge \bar{f}$ for all $n \ge 0$, therfore $\varphi \ge \bar{f}$. If we prove that $f \ge \varphi$ on I, then $\varphi = \bar{f}$.

Indeed, if there exists $x_0 \in I - \{a,b\}$ such that $f(x_0) < \varphi(x_0)$ then there is a neighbourhood V of x_0 such that $f(x) < \varphi(x)$ for all $x \in I \cap V$, because f and φ are continuous functions. But $\overline{\cup_{n\geq 0} A_n} = I$, therefore there exists $n_0 \in N$ and $i \in \{0,1,\ldots,2^{n_0}\}$ such that $x_i \in (A_{n_0} - A_{n_0-1}) \cap V$ and $f(x_i) < \varphi(x_i)$. The inequality $\varphi_n \geq \varphi$ on I $(n \geq 0)$ implies that $\varphi_{n_0-1}(x_i) \geq \varphi(x_i)$. So $\varphi_{n_0-1}(x_i) > f(x_i)$. Using the lemma for the points of $A_{n_0-1}, g = \varphi_{n_0-1}, M = f(x_i)$ and $z = x_i$, we have $\varphi_{n_0} \leq \varphi_{n_0-1}$ and $\varphi_{n_0}(x_i) = f(x_i)$. But $\varphi_n \geq \varphi$ on I for all $n \geq 0$, so in particular $\varphi_{n_0} \geq \varphi$ on I. Then $f(x_i) = \varphi_{n_0}(x_i) \geq \varphi(x_i) > f(x_i)$, contradiction.

Using Dini's theorem ([3], $p.136$), it follows that the sequence $\{\varphi\}_{n\geq 0}$ converges uniformly to $\bar{f}$ on I.

b). Let $J \subseteq I$ be a closed interval and $g \in C(I)$. Then we employ the usual norm notation: $\|g\|_J = \sup\{|g(x)| : x \in J\}$. It is clear that

$$\|\varphi_n - \bar{f}\| = \max_{1\leq i\leq 2^n} \|\varphi_n - \bar{f}\|_{J_i} = \|\varphi_n - \bar{f}\|_{J_{i_0}},$$

where $J_i = [x_{i-1}, x_i]$, $i = \overline{1,2^n}$ and $i_0 \in M = \{\, k \in \{1,2,\ldots,2^n\} : \|\varphi_n - \bar{f}\| = \|\varphi_n - \bar{f}\|_{J_k} \}$.

If $f(x) \geq f(a) + [a,b;f]\cdot(x-a)$, $x \in I$, then $g_n \equiv 0$ and $\varphi_n = \bar{f}$ for all $n \geq 0$.

If there exists $x_0 \in I - \{a,b\}$ such that $f(x_0) < f(a) + [a,b;f]\cdot(x_0 - a)$ then $[x_{i_0-1}, x_{i_0}; \varphi_n] = [x_{i_0-1}, x_{i_0}; \bar{f}]$ or there exists $x \in J_{i_0}$ such that $f(x) = \bar{f}(x)$.

Indeed, in the opposite case we have $f(x) > \bar{f}(x)$ for all $x \in J_{i_0}$, therefore $\bar{f}$ is linear on J_{i_0}. Hence

$$\|\varphi_n - \bar{f}\|_{J_{i_0}} = \max\{\varphi_n(x_{i_0-1}) - \bar{f}(x_{i_0-1})\ ,\ \varphi_n(x_{i_0}) - \bar{f}(x_{i_0})\}.$$

We may assume that $\|\varphi_n - \bar{f}\|_{J_{i_0}} = \varphi_n(x_{i_0}) - \bar{f}(x_{i_0})$, because $[x_{i_0-1}, x_{i_0}; \varphi_n] \neq [x_{i_0-1}, x_{i_0}; \bar{f}]$. On the other hand, there exists $\varepsilon > 0$ such that $\bar{f}$ is linear on $[x_{i_0}, x_{i_0} + \varepsilon]$ and $[x_{i_0}, x_{i_0} + \varepsilon; f] = [x_{i_0-1}, x_{i_0}; \bar{f}]$. Because $\varphi_n \in Conv(I)$, we have $[x_{i_0}, x_{i_0} + \varepsilon; \varphi_n] \geq [x_{i_0-1}, x_{i_0}; \varphi_n]$. Hence $\varphi_n(x_{i_0} + \varepsilon) - \bar{f}(x_{i_0} + \varepsilon) > \varphi_n(x_{i_0}) - \bar{f}(x_{i_0})$, cotradiction with the choice of the interval J_{i_0}.

If $[x_{i_0-1}, x_{i_0}; \varphi_n] = [x_{i_0-1}, x_{i_0}; \bar{f}]$ then, by $f(a) = \bar{f}(a)$ and $f(b) = \bar{f}(b)$, there exist $k \in M - \{i_0\}$ and $x \in J_k$ such that $\|\varphi_n - \bar{f}\| = \|\varphi_n - \bar{f}\|_{J_k}$ and $f(x) = \bar{f}(x)$, respectively.

Assume that $\|\varphi_n - \bar{f}\| = \|\varphi_n - \bar{f}\|_{J_{i_0}}$ and there exists $x \in J_{i_0}$ such that $f(x) = \bar{f}(x)$. The functions φ_n and $\bar{f}$ are continuous on J_{i_0}, so there exists $x_1 \in J_{i_0}$ such that

$\|\varphi_n - \bar{f}\|_{J_{i_0}} = \varphi_n(x_1) - \bar{f}(x_1)$. If $\bar{f}(x_1) \neq f(x_1)$ then there exists $\varepsilon > 0$ such that $\bar{f}$ is linear on $J_{i_0} \cap [x_1 - \varepsilon, x_1 + \varepsilon]$ and $[x_1 - \varepsilon, x_1 + \varepsilon; f] = [x_{i_0-1}, x_{i_0}; \varphi_n]$.

Let $M_1 = \{y \in J_{i_0} : \bar{f}(y) < f(y)\}$. Because $f(x) = \bar{f}(x)$ it follows that $f(x_2) = \bar{f}(x_2)$ or $f(x_3) = \bar{f}(x_3)$, where $x_2 = \sup M_1$ and $x_3 = \inf M_1$ (if they there exist).

At the same time $\bar{f}$ remains linear on $[x_3, x_2] \cap J_{i_0}$, therefore either $\bar{f}(x_2) = \bar{f}(x_1)$ or $\bar{f}(x_3) = \bar{f}(x_1)$. Hence $\|\varphi_n - \bar{f}\|_{J_{i_0}} = \varphi_n(x_2) - \bar{f}(x_2) = \varphi_n(x_2) - f(x_2)$ or $\|\varphi_n - \bar{f}\|_{J_{i_0}} = \varphi_n(x_3) - \bar{f}(x_3) = \varphi_n(x_3) - f(x_3)$.

Let $M_2 = \{y \in J_{i_0} : \varphi_n(y) \geq f(y)\}$. Then $\varphi_n(y) \geq f(y) \geq \bar{f}(y)$ for all $y \in M_2$ and $x_1 \in M_2$, $x_2 \in M_2$ or $x_3 \in M_2$. Therefore $\|\varphi_n - \bar{f}\|_{J_{i_0}} \geq \|g_n\|_{J_{i_0}} \geq \varphi_n(x_2) - f(x_2)$(resp.$\varphi_n(x_3) - f(x_3)$). Hence $\|\varphi_n - \bar{f}\|_{J_{i_0}} = \|g_n\|_{J_{i_0}} \leq \|g_n\|$. □

Remark. a). We can choose another points instead of the points x_i $(i = \overline{1, 2^n})$ in our algorithm, but we must have the condition $\overline{\cup_{n\geq 0} A_n} = I$.

Indeed, without this condition, we can choose the sets A_n $(n \geq 0)$ such that φ_n does not converge uniformly to $\bar{f}$ on I. Our example is the following:
let $f : [0, 5] \to R$,

$$f(x) = \begin{cases} x + 1, & \text{if } x \in [0, 1) \\ -x + 3, & \text{if } x \in [1, 3) \\ x - 3, & \text{if } x \in [3, 5]. \end{cases}$$

Then $\bar{f} : [0, 5] \to R$,

$$\bar{f}(x) = \begin{cases} -\frac{x}{3} + 1, & \text{if } x \in [0, 3) \\ x - 3, & \text{if } x \in [3, 5]. \end{cases}$$

Define the function φ by $\varphi : [0, 5] \to R$,

$$\varphi(x) = \begin{cases} 1, & \text{if } x \in [0, 4) \\ x - 3, & \text{if } x \in [3, 5]. \end{cases}$$

The sequence $\{A_n\}_{n\geq 0}$ is the following:

$$A_0 = \{0, 5\}, \quad A_1 = \{0, 2, 5\}$$

$$A_2 = \left\{0, x_1, x_2, x_3, 2, 4 + \frac{1}{3}, 4 + \frac{1}{2}, x_7, 5\right\}$$

and generally

$$A_n = \{0,\ x_1,\ x_2, \ldots,\ x_{2^{n-1}-1},\ 2,\ 4+\frac{1}{n},\ 4+\frac{1}{n-1},\ x_{2^{n-1}+3},$$
$$4+\frac{1}{n-2},\ x_{2^{n-1}+4},\ x_{2^{n-1}+5},\ x_{2^{n-1}+6},\ 4+\frac{1}{n-3}, \ldots,\ 4+\frac{1}{2},$$
$$x_{2^n-2^{n-2}+1},\ x_{2^n-2^{n-2}+2}, \ldots,\ x_{2^n-1},\ 5\}.$$

With the aid of the set A_n it is possible to define the function φ_n as follows: $\varphi_n : [0,5] \to R$,

$$\varphi_n(x) = \begin{cases} 1, & \text{if } x \in [0,2) \\ \frac{x}{2n+1} + \frac{2n-1}{2n+1}, & \text{if } x \in [2, 4+\frac{1}{n}) \\ x-3, & \text{if } x \in [4+\frac{1}{n}, 5]. \end{cases}$$

It is clear that $\|\varphi_n - \varphi\| = \varphi_n(4) - \varphi(4) = \frac{2}{2n+1} \to 0 \quad (n \to \infty)$, so φ_n converges uniformly to φ on $[0,5]$ $(n \to \infty)$, but $\varphi \neq \bar{f}$ on $[0,5]$.

b). Let $A = \{x \in I : f(x) = \bar{f}(x)$ and there exists $r_x > 0$ such that $f(y) > \bar{f}(y)$ for all $y \in (x - r_x, x) \cap I$ or $f(y) > \bar{f}(y)$ for all $y \in (x, x + r_x) \cap I\}$. Then the set A is closed and the cardinality of A is arbitrary.

Because f and $\bar{f}$ are continuous functions on I, we obtain that A is closed set. By some examples we show the cardinality of the set A can be arbitrary. Let us distinguish the following cases:

1. if $f \in Conv(I)$ then the cardinality of A is zero;
2. if $n \in N^*$ and $f : [0, 2n] \to R$, $f(x+2) = f(x)$, for all $x \in [0, 2n-2]$ and

$$f(x) = \begin{cases} -x+1, & \text{if } x \in [0,1) \\ x-1, & \text{if } x \in [1,2], \end{cases}$$

then $A = \{1, 3, \ldots, 2n-1\}$. So the cardinality of A is n;

3. let

$$f(x) = \begin{cases} 0, & \text{if } x = \frac{1}{n},\ n \in N^*, & n \text{ is odd} \\ \frac{1}{n}, & \text{if } x = \frac{1}{n},\ n \in N^*, & n \text{ is even} \end{cases};$$

let $f(0) = 0$ and let f be linear in each interval $\left[\frac{1}{n+1}, \frac{1}{n}\right]$, $n \in N^*$. Then $A = \left\{\frac{1}{2n-1} : n \in N^*\right\}$. Therefore the cardinality of A is the the cardinality of N;

4. let us define a continuous mapping of $[0,1]$ onto $[0,1]$ that assumes every value an uncountable number of times. So, let $g : [0,\infty) \to R$, $g(x+2) = g(x)$, for all $x \geq 0$ and

$$g(x) = \begin{cases} 0, & \text{if } x \in \left[0, \frac{1}{3}\right) \\ 3x-1, & \text{if } x \in \left[\frac{1}{3}, \frac{2}{3}\right) \\ 1, & \text{if } x \in \left[\frac{2}{3}, \frac{4}{3}\right) \\ -3x+5, & \text{if } x \in \left[\frac{4}{3}, \frac{5}{3}\right) \\ 0, & \text{if } x \in \left[\frac{5}{3}, 2\right]. \end{cases}$$

Then the function $f : [0,1] \to [0,1]$,

$$f(x) = \frac{1}{2} \cdot g(x) + \frac{1}{2^2} \cdot g(3^2 x) + \frac{1}{2^3} \cdot g(3^4 x) + \dots$$

has the wanted properties (see [2], p.134, 9.).

In the same time, $\{x \in [0,1] : f(x) = 0\} \subseteq A$, because there not exists an interval $[\alpha, \beta]$ such that $[\alpha,\beta] \cap [0,1] \subseteq \{x \in [0,1] : f(x) = 0\}$.

Indeed, there exist $m, n \in N^*$ such that $\frac{m}{3^n} \in (\alpha, \beta)$ for all m, $1 \leq m \leq 3^n - 1$. Let $l \in N^*$ such that $3^l \leq m < 3^{l+1}$, where $0 \leq l < n-1$. Then we can choose m in such a way that let $k = n - 1 - l$ be even and for that k we have: $3^{n-1} \leq m \cdot 3^k < 5 \cdot 3^{n-1}$. Therefore $\frac{1}{3} \leq 3^k \cdot \frac{m}{3^n} < \frac{5}{3}$. Hence $g\left(3^k \cdot \frac{m}{3^n}\right) \neq 0$, so $f\left(\frac{m}{3^n}\right) \neq 0$. This implies the cardinality of A is equal with the cardinality of R.

References

[1] Z. Finta *Best piecewise convex uniform approximation* Studia Univ.Babeş - Bolyai (to appear)
[2] B.R.Gelbaum and J.M.H.Olmsted *Counterexamples in Analisys* Holden - Day, Inc. San Francisco, London, Amsterdam , 1963
[3] W.Rudin *Principles of mathematical analysis*, McGraw - Hill Book Company, New York, San Francisco, Toronto, London, 1964

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COMMON FACTORS AND DISJOINTNESS OF EXTENSIONS OF MINIMAL TOPOLOGICAL TRANSFORMATIONS SEMIGROUPS

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Abstract. There are given some results concerning the common factors and disjointness of finite families of extensions of minimal topological transformation semigroups.

1. Introduction

We shall develop the theory of disjointness of minimal sets and their extensions. The notion of disjointness of sets (extensions) was introduced and first studied by Furstenberg [1] (Shapiro [2]). The books by Bronstein I.U. [3] and by van der Woude, J.C.S.P. [4] contains many results about disjointness theory in case of the topological transformation groups.

Here we give some results concerning the common factors and disjointness of finite families of extensions of minimal topological transformation semigroups.

In our research we will use some algebraic technique and the τ-topology [5,6]. Developing them [6] we will follow the idea of the article [5]. But the way of realization of this idea will not be the same. The constructions in [5] are based on Stone-Cech compactification of phase (discrete) group. We are starting from the Ellis enveloping semigroup of universal minimal topological transformation semigroup of the class of minimal topological transformation semigroups with the compact Hausdorff phase space and fixed phase semigroup.

We use terminology and denotations generally accepted at present in theory of topological transformations groups. We give only necessary, in our view, definitions of concepts and facts; for more detailed discussions the reader is referred to [1-8].

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2. Basic definitions and notes

A topological transformation semigroup (abbreviation: TTS) is a triple (X, S, π), where X is a nonempty compact Hausdorff topological space with unique uniformity $\mathcal{U}[X]$ (phase space), S is a topological semigroup with unit element e (phase semigroup) and $\pi : X \times S \to X$ is a continuous mapping satisfying the following conditions:

1) $\forall\, x \in X\ (x, e)\pi = x$;

2) $\forall\, x \in X\ \forall\, s, t \in S\ ((x, s)\pi, t)\pi = (x, st)\pi$.

We shall refer to the $TTS(X, S)$ rather than (X, S, π).

Let (X, S, π) be a TTS, $s \in S$, $A \subset X$. Usually we shall write π^s for the map $X \to X$ defined by $x\pi^s = (x, s)\pi$ $(x \in X)$; $xs = x\pi^s$ and $xS = \{xs|\ s \in S\}$ $(x \in X)$. A is called minimal if $A \neq \emptyset$ and $\overline{xS} = A$ for every $x \in A$. A $TTS(X, S)$ is called minimal if the set X is minimal. If for $x \in X$ we have $\overline{xS}$ is minimal than x is called an almost periodic point. The class of minimal $TTSs$ with fixed phase semigroup S will be denoted by $K(S)$.

An extension (a homomorphism) $\varphi : (X, S, \pi) \to (Y, S, \rho)$ of $TTSs$ is a continuous surjection $\varphi : X \to Y$ such that for $\forall\, x \in X\ \forall\, s \in S\ x\pi^s\varphi = x\varphi\rho^s$.

Let $n \in N$ be a natural number, $(X_i, S), (Y, S) \in K(S)$, $\varphi_i : (X_i, S) \to (Y, S)$ be an extension $(i = 1, \ldots, n)$. The family $\{\varphi_i\}_1^n$ of extensions φ_i is called the n-fan of (Y, S) (or n-fan). The class of all n-fans of (Y, S) will be denoted by $K^n(Y, S)$. Let $\{\varphi_i\}_i^n \in K^n(Y, S)$. We denote:

$$R_{\varphi_1 \ldots \varphi_n} = \{(x_1, \ldots, x_n)|\ x_i \in X_i\ (i = 1, \ldots, n) \wedge x_1\varphi_1 = \cdots = x_n\varphi_n\};$$

$R_{\varphi_1 \ldots \varphi_n} J$ the set all almost periodic points from $R_{\varphi_1 \ldots \varphi_n}$.

Let $\varphi : (X, S) \to (Y, S) \in K^1(Y, S)$. Then define:

$$\Delta(X) = \{(x, x)|\ x \in X\}; \quad R_\varphi = R_{\varphi\varphi};$$

$$P(R_\varphi) = \bigcap_{\alpha \in \mathcal{U}[X]} \bigcup_{s \in S} \{(x, y)|\ (x, y) \in R_\varphi \wedge (xs, ys) \in \alpha\};$$

$$Q(R_\varphi) = \bigcap_{\alpha \in \mathcal{U}[X]} \overline{\bigcup_{s \in S} \{(x, y)|\ (x, y) \in R_\varphi \wedge (xs, ys) \in \alpha\}};$$

$Q^*(R_\varphi)$ the smallest closed invariant equivalence relation containing $Q(R_\varphi)$.

An extension φ is called distal (regionally distal), if $P(R_\varphi) = \Delta(X)$ $(Q(R_\varphi) = \Delta(X))$.

An extension φ is called B-extension, if $R_\varphi = \overline{R_\varphi J}$.

An extension φ is called RIC-extension, if $R_{\varphi\psi}$ is a minimal set for each proximal extension $\psi \in K^1(Y,S)$.

An extension φ is called distal in the point $x \in X$, if $xP(R_\varphi) = \{x\}$.

An extension φ is called stable in the fiber $\varphi^{-1}(y)$ $(y \in Y)$, if

$$\forall\ \alpha \in \mathcal{U}[X]\ \exists\ \beta \in \mathcal{U}[X]\ ((\varphi^{-1}(y) \times \varphi^{-1}(y)) \cap \beta)S \subset \alpha.$$

An extension φ is called homogeneous, if for the each point $(x_1, x_2) \in R_\varphi$ there exists an automorphism ψ of the $TTS(X,S)$ such that $x_1\psi = x_2$.

There exists a universal minimal $TTS(U,S,\sigma)$ for $K(S)$. Let E be the Ellis enveloping semigroup of (U,S,σ), I be a fixed minimal right ideal of E, $u \in I$ be a fixed idempotent. It is known, that $(I,S) \in K(S)$. For $(X,S) \in K(S)$ there exists a commutative diagram

$$\begin{array}{ccc}
(U,S) & \xleftarrow{\ \Gamma\ } & (E,S) \\
{\scriptstyle\Phi}\big\downarrow & & \big\downarrow{\scriptstyle\Theta} \\
(X,S) & \xleftarrow[\ \rho_{x_0}\ ]{} & (E(X,S),S),
\end{array}$$

where $E(X,S)$ is the Ellis enveloping semigroup of (X,S), phi is a homomorphism from definition of universality of (U,S), Θ is a homomorphism induced by Φ, $p\rho_{x_0} = x_0p$ $(p \in E(X,S))$, $x_0 \in X$ is a fixed point, Γ is a map defined analogously with ρ_{x_0}. E acts naturally on X : $xp = x(p\Theta)$ $(x \in X,\ p \in E)$.

Let $(X,S,\pi) \in K(S)$ and 2^X be the collection of nonempty closed subsets of X endowed with the Vietoris topology. Then $(2^X, S, \pi^*)$, defined by $(A,s)\pi^* = A\pi^s$, also TTS $(A \in 2^X,\ s \in S)$ and E acts on 2^X too. If $p \in E$ and $\lim\limits_i \sigma^{s_i} = p$ for any net $\{s_i\} \subset S$, $A \in 2^X$ then we define $A \odot p = \lim\limits_i A\sigma^{s_i} = \lim\limits_i \{a\sigma^{s_i} \mid a \in A\}$, where the limit it is understood in the Vietoris topology. If $A \subset X$ is not necessarily closed nonempty subset of X, we define $A \odot p = \overline{A} \odot p$. For $A = \emptyset$ we define $A \odot p = \emptyset$.

The operation $c : c(A) = A \odot u \cap Xu$ $(A \subset Xu)$ defines a closure operator on Xu. We call the topology associated with closure operator c the τ-topology.

Let $\mathcal{E} = Iu$. Then $(\mathcal{E}, \tau)$ is a T_1 compact semitopological group (with identity element u).

Let T be a τ-closed subset of $\mathcal{E}$, $u \in T$; $N(T)$ be the neighbourhoods filter for the τ-topology on T at u, $H(T) = \bigcap_{V \in N(T)} cl_{S_\tau} V$. We now define inductively for all transfinite numbers α the set $H^\alpha(T)$: 1) $H^0(T) = T$; 2) let $H^\alpha(T)$ be defined for every ordinal α, $\alpha < \beta$; if $\beta = \alpha + 1$, then we consider $H^\beta(T) = H(H^\alpha(T))$; if β is a limit ordinal, then we consider $H^\beta(T) = \bigcap_{\alpha<\beta} H^\alpha(T)$.

If T is a τ-closed subgroup of $\mathcal{E}$, then $H^\alpha(T)$ is a τ-closed normal subgroup of $\mathcal{E}$ for every transfinite number α.

For $(X, S) \in K(S)$ and $x_0 \in Xu$ we define the Ellis group of (X, S) : $\mathcal{G}(X, x_0) \equiv \mathcal{G}(X) = \{p |\, p \in \mathcal{E} \wedge x_0 p = x_0\}$. $\mathcal{G}(X)$ is a τ-closed subgroup of $\mathcal{E}$.

Let $\{\varphi_i : (X_i, S) \to (Y, S)\}_1^n \in K^n(Y, S)$, $x_i^0 \in X_i u$, $y^0 = x_i^0 \varphi_i$; $A_i = \mathcal{G}(X_i, x_i^0)$ and $T = \mathcal{G}(Y, y^0)$ are the Ellis groups of $TTS(X_i, S)$ and $TTS(Y, S)$ respectively ($i = 1, \ldots, n$), $[A_1, \ldots, A_n]$ the smallest τ-closed subgroup of T containing A_i for every $i = 1, \ldots, n$. For $(X, S) \in K(S)$ and $x \in X$ let J_x be the set all idempotents v from I with $xv = x$.

An extension $\varphi : (X, S) \to (Y, S)$ is called common factor of $\{\varphi_i\}_1^n$, if it is a factor for every extension φ_i ($i = 1, 2, \ldots$), i.e., for all $i = 1, 2 \ldots, n$ exists a homomorphism ψ_i of (X_i, S) onto (X, S) such that $\varphi_i = \psi_i \circ \varphi$. A common factor φ of $\{\varphi_i\}_1^n$ is called a D-factor (RD-factor, P-factor, B-factor), if it is a distal (regionally distal, proximal, B-) extension. A n-fan is called a Dn-fan (RDn-fan, Pn-fan, Bn-fan), if every its common factor is a D-factor (RD-factor, P-factor, B-factor). A n-fan is called prime (D-prime, RD-prime, P-prime, B-prime), if every common factor (D-factor, RD-factor, P-factor, B-factor) of its is trivial, i.e., isomorphism. A n-fan $\{\varphi_i\}_1^n$ is called disjoint, if $R_{\varphi_1 \ldots \varphi_n}$ is minimal.

Let θ be some ordinal and $\{\varphi_\alpha |\, \alpha < \theta\} \equiv \{\varphi_\alpha : (X_\alpha, S) \to (Y, S) |\, \alpha < 0\}$ be a transfinite sequence of extensions of $TTSs$, $x_\alpha^0 \in X_\alpha u$, $y^0 = x_\alpha^0 \varphi_\alpha$ ($\alpha < \theta$), $\{\varphi_\alpha^\beta : (X_\beta, S) \to (X_\alpha, S) |\, \alpha \leq \beta < \theta\}$ be a family of extensions of $TTSs$ with

$$x_\beta^0 \varphi_\alpha^\beta = x_\alpha^0, \quad \varphi_\alpha^\beta \circ \varphi_\alpha = \varphi_\beta, \quad \varphi_\alpha^\beta \circ \varphi_\gamma^\alpha = \varphi_\gamma^\beta,$$

$$\varphi_\alpha^\alpha \text{ is the identity map} \quad (\alpha \leq \beta \leq \gamma < \theta). \tag{1}$$

Now let $x_\theta^0 \in \prod_{\alpha<\theta} X_\alpha$ with $Pr_{X_\alpha} = x_\alpha^0$ ($\alpha < \theta$). There are the sub$TTS(\overline{x_\theta^0 S}, S)$ of direct product $\left(\prod_{\alpha<\theta} X_\alpha, S\right)$ of $TTSs(X_\alpha, S)$ ($\alpha < \theta$) and the extension $\varphi_\theta : (\overline{x_\theta^0 S}, S) \to$

(Y, S) defined by $x_\theta^0 p\varphi_\theta = y^0 p$ $(p \in I)$. We denote:

$$(\overline{x_\theta^0 S}, S) = \varprojlim((X_\alpha, S), \varphi_\alpha^\beta)_0^\theta, \quad \varphi_\theta = \varprojlim(\varphi_\alpha, \varphi_\alpha^\beta)_0^\theta.$$

Let μ be an ordinal and $\{\varphi_\alpha : (X_\alpha, S) \to (Y, S) |\ \alpha \leq \mu\} \subset K(Y, S)$ be a family of extensions of $TTSs$ and $\{\varphi_\alpha^\beta |\ \alpha \leq \beta \leq \mu\}$ be a family of their morphisms satisfying condition (1). Assume also that for each limit ordinal θ, $\theta \leq \mu$, we have

$$\varphi_\theta = \varprojlim(\varphi_\alpha, \varphi_\alpha^\beta)_0^\theta, \quad (X_\theta, S) = \varprojlim((X_\alpha, S), \varphi_\alpha^\beta)_0^\theta.$$

Then the system $\{\varphi_\alpha, \varphi_\alpha^\beta\}_0^\mu$ is called a projective system of extensions.

An extension φ is called a PRD-extension, if there exists a projective system $\{\varphi_\alpha : (X_\alpha, S) \to (Y, S),\ \varphi_\alpha^\beta : (X_\beta, S) \to (X_\alpha, S)\}_0^\mu$, such that:

1) $X_0 = Y$;

2) $\varphi_\mu = p \circ \varphi$ for some proximal homomorphism $p : (X_\mu, S) \to (X, S) \in K^1(X, S)$;

3) $\varphi_\alpha^{\alpha+1}$ is a composition proximal extension and regionally distal extension $(\alpha < \mu)$.

Let α be a transfinite number. A n-fan $\{\varphi\}_1^n$ is called α-pseudostable if $H^\alpha(T) \subset [A_1, \ldots, A_n]$. A n-fan is called pseudostable, if it is 1-pseudostable.

Remark 1. 1) A n-fan containing some extension with a stable fibre is pseudostable.

2) PRD-extension is a α-pseudostable for some ordinal α.

3) A n-fan containing some PRD-extension is α-pseudostable for some ordinal α.

4) An α-pseudostable for some transfinite number α and a RD-prime Bn-fan is prime.

Proof. 1) The proof is similar that of the proposition:

For an extension $\varphi : (X, S) \to (Y, S) \in K^1(Y, S)$ with a stable fibre there exists the commutative diagram

$$\begin{array}{ccc} (X, S) & \xleftarrow{\ p\ } & (X^*, S) \\ \varphi \downarrow & & \downarrow \varphi^* \\ (Y, S) & \xleftarrow{\ q\ } & (Y*, S), \end{array}$$

where φ^* is a regionally distal minimal extension (hence $H(\mathcal{G}(Y^*)) \subset \mathcal{G}(X^*)$) and the extensions p and q are proximal.

2) This follows from definition of the PRD-extension by the principle of transfinite induction.

3) and 4) are obvious. □

3. Results

a) n-fans of extensions

Theorem 1. *If for some $k = 1, 2, \dots, n$ we have*

$$\underbrace{T \times \cdots \times T}_{n-1} = \left(\prod_{\substack{i=1,2\dots,n \\ i \neq k}} A_i \right) A_k, \tag{2}$$

then $R_{\varphi_1 \dots \varphi_n}$ contains a unique minimal subset. If $R_{\varphi_1 \dots \varphi_n}$ contains a unique minimal subset, then for every $k = 1, 2, \dots, n$ we have (2).

Proof. The proof is similar to that of the proposition

$$R_{\varphi_1 \dots \varphi_n} J = (\{x_1^0 T \times \cdots \times x_n^0 T) I = (x_1^0 T \times \{x_2^0\} \times x_3^0 T \times \cdots \times x_n^0 T) I = \cdots =$$

$$(x_1^0 T \times x_2^0 T \times \cdots \times x_{n-2}^0 T \times \{x_{n-1}^0\} \times x_n^0 T_I = (x_1^0 T \times x_2^0 T \times \cdots \times x_{n-1}^0 T \times \{x_n^0\}) I. \tag{3}$$

□

Theorem 2. *1) If for some $k = 1, 2, \dots, n$ and for every $x \in X_k$ we have*

$$\prod_{\substack{i=1,2,\dots,n \\ i \neq k}} \varphi_i^{-1}(x\varphi_k) = \left(\prod_{\substack{i=1,2,\dots,n \\ i \neq k}} \varphi_i^{-1}(x\varphi_k) \right) J_x, \tag{4}$$

then $R_{\varphi_1 \dots \varphi_n} J = R_{\varphi_1 \dots \varphi_n}$. If $R_{\varphi_1 \dots \varphi_n} J = R_{\varphi_1 \dots \varphi_n}$, then for all $k = 1, 2, \dots, n$ and for $\forall\ x \in X_k$ we have (4).

2) If for some $k = 1, 2, \dots, n$ and for every $x \in X_k$ there exists a point $v \in J_x$, such that

$$\prod_{\substack{i=1,2,\dots,n \\ i \neq k}} \varphi_i^{-1}(x\varphi_k) = \left(\prod_{\substack{i=1,2,\dots,n \\ i \neq k}} \varphi_i^{-1}(x\varphi_k) v \right) \odot J_x, \tag{5}$$

then $\overline{R_{\varphi_1 \dots \varphi_n} J} = R_{\varphi_1 \dots \varphi_n}$. If $\overline{R_{\varphi_1 \dots \varphi_n} J} = R_{\varphi_1 \dots \varphi_n}$, then for every $k = 1, 2, \dots, n$, for every $x \in X_k$ and for every $v \in J_x$ we have (5).

Proof. This follows from (3). □

Corollary 1. *If* $\{\varphi_i\}_1^n$ *contains* $n-1$ *distal extensions (RIC-extensions), then* $R_{\varphi_1\ldots\varphi_n}J = R_{\varphi_1\ldots\varphi_n}(\overline{R_{\varphi_1\ldots\varphi_n}J} = R_{\varphi_1\ldots\varphi_n}$.

Corollary 2. *1) If* $R_{\varphi_1\ldots\varphi_n}J = R_{\varphi_1\ldots\varphi_n}$, *then* $\{\varphi_i\}_1^n$ *is Dn-fan.*

2) If $\overline{R_{\varphi_1\ldots\varphi_n}J} = R_{\varphi_1\ldots\varphi_n}$, *then* $\{\varphi_i\}_1^n$ *is Bn-fan.*

3) If $T = [A_1, \ldots, A_n]$ *(specifically* $R_{\varphi_1\ldots\varphi_n}$ *contains a unique minimal subset), then* $\{\varphi_i\}_1^n$ *is Pn-fan.*

4) If $\{\varphi_i\}_1^n$ *satisfies the conditions of item 2) and 3) of our corollary, then it is prime.*

5) If $\{\varphi_i\}_1^n$ *is disjoint, then* $\{\varphi_i\}_1^n$ *is prime.*

Let $(V, S) \to (Y, S) \in K^1(Y, S)$ be a universal distal (regionally distal) extension and D_1 (D_2) is the Ellis group of (V, S). It is known, that $D_2 = D_1 H(T)$.

Theorem 3. *1) For every n-fan* $\{\varphi_i\}_1^n$ *there exist its maximal D-factor (RD-factor)* $\varphi : (X, S) \to (Y, S)$ *and* $\mathcal{G}(X) = [A_1, \ldots, A_n]D_1(\mathcal{G}(X) = [A_1, \ldots, A_n]D_2)$*. Consequently* $\{\varphi_i\}_1^n$ *is D-prime (RD-prime) iff* $T = [A_1, \ldots, A_n]D_1$ $(T = [A_1, \ldots, A_n]D_2)$.

2) Let $\varphi : (X, S) \to (Y, S)$ *be a maximal RD-factor of* $\{\varphi_i\}_1^n$. *If* $\overline{R_{\varphi_1\ldots\varphi_n}J} = R_{\varphi_1\ldots\varphi_n}$, *then* $\mathcal{G}(X) = [A_1, \ldots, A_n]H(T)$. *Consequently* $\{\varphi_i\}_1^n$ *is a RD-prime iff* $T = [A_1, \ldots, A_n]H(T)$.

Proof. The assertion 1) is obvious. Let $\overline{R_{\varphi_1\ldots\varphi_n}J} = R_{\varphi_1\ldots\varphi_n}$ and let $M = (x_1^0, \ldots, x_n^0)I$ be the minimal subset of $R_{\varphi_1\ldots\varphi_n}$. We define a relation ρ in M as follows: $\forall\ p, q \in I$

$$(x_1^0, \ldots, x_n^0)p\rho(x_1^0, \ldots, x_n^0)q \ \Leftrightarrow$$

$$\Leftrightarrow\ q(pu)^{-1} \in [A_1, \ldots, A_n]H(T) \wedge (x_1^0 p, \ldots, x_{n-1}^0 p, z_n^0 q) \in R_{\varphi_1\ldots\varphi_n}.$$

Since ρ is a closed invariant equivalence relation, then there are defined a $TTS(M/\rho, S)$ and an extension $\eta : (M/\rho, S) \to (Y, S)$ where $((x_1^0, \ldots, x_n^0)p)\rho\eta = y^0 p$ ($p \in I$). η is a maximal RD-factor of $\{\varphi_i\}_1^n$ and $\mathcal{G}(M/\rho, (x_1^0, \ldots, x_n^0)\rho) = [A_1, \ldots, A_n]H(T)$. □

Theorem 4. *1) For a pseudostable n-fan* $\{\varphi_i\}_1^n$ *there exist a maximal RD-factor* φ *of* $\{\varphi\}_1^n$ *and a B-prime n-fan* $\{\psi\}_1^n$, *such that* $\varphi_i = \psi_i \circ \varphi$ *for every* $i = 1, 2, \ldots, n$.

3) If $\{\varphi\}_1^n$ is pseudostable and $R_{\varphi_1\ldots\varphi_n}J = R_{\varphi_1\ldots\varphi_n}$, then there exist maximal RD-factor φ of $\{\varphi_i\}_1^n$ and prime n-fan $\{\psi\}_1^n$, such that $\varphi_i = \psi_i \circ \varphi$ for every $i = 1, 2, \ldots, n$.

Proof. This follows from Theorem 3 and Corollary 2. □

Corollary 3. *A n-fan $\{\varphi\}_1^n$ is disjoint, iff $\overline{R_{\varphi_1\ldots\varphi_n}J} = R_{\varphi_1\ldots\varphi_n}$ and for some $k = 1, 2, \ldots, n$*

$$\underbrace{T \times \cdots \times T}_{n-1} = \left(\prod_{\substack{i=1,2,\ldots,n \\ i \neq k}} A_i \right) A_k.$$

Theorem 5. *Let $\varphi_i\}_1^n$ contains $n-1$ pseudostable RIC-extensions and φ_i^0 be a maximal RD-factor of φ_i $(i = 1, 2, \ldots, n)$. If $\{\varphi_i^0\}_1^n$ is disjoint, then $\{\varphi_i\}_1^n$ is disjoint.*

Proof. Let $\varphi_1, \ldots, \varphi_{n-1}$ be pseudostable RIC-extensions and $\varphi_i^0 : (X_i^0, S) \to (Y, S)$ be a maximal RD-factor of φ $(i = 1, \ldots, n)$. Then $\overline{R_{\varphi_1\ldots\varphi_n}J} = R_{\varphi_1\ldots\varphi_n}$ and $D_2 \subset A_i = \mathcal{G}(X_i^0)$ for $i = 1, \ldots, n-1$ and $\mathcal{G}(X_n^0) = A_n D_2$. If $\{\varphi_i^0\}_1^n$ is disjoint, then

$$\underbrace{T \times \cdots \times T}_{n-1} = (A_1 \times \cdots \times A_{n-1})A_n D_2 \subset (A_1 D_2 \times \cdots \times A_{n-1} D_2)A_n =$$

$$= (A_1 \times \cdots \times A_{n-1})A_n \subset \underbrace{T \times \cdots \times T}_{n-1},$$

hence $\underbrace{T \times \cdots \times}_{n-1} = (A_1 \times \cdots \times A_{n-1})A_n$. Therefore $\{\varphi_i\}_1^n$ is disjoint by Corollary 3. □

b) Pairs of extensions

We shall study hereafter of the pairs (φ, ψ) of extensions $\varphi : (X, S) \to (Y, S)$, $\psi : (Z, S) \to (Y, S) \in K^1(Y, S)$. Let $x_0 \in Xu$, $z_0 \in Zu$, $y_0 = x_0\varphi = z_0\psi$; $A = \mathcal{G}(X, x_0)$, $B = \mathcal{G}(Z, z_0)$ and $T = \mathcal{G}(Y, y_0)$ are the Ellis groups of TTS $(X, S), (Z, S)$ and (Y, S) respectively; $[A, B]$ the smallest τ-closed subgroup of T containing $A \cup B$; $\varphi_0 : (X_0, S) \to (Y, S)$ $(\varphi_0' : (X_0', S) \to (Y, S))$ and $\psi_0 : (Z_0, S) \to (Y, S)$ $((\psi_0' : (Z_0', S) \to (Y, S))$ be a maximal RD-factor (D-factor) of φ and ψ respectively. If $(\varphi_1, \varphi_2) \in K^2(Y, S)$ is disjoint, then we denote $\varphi_1 \perp \varphi_2$. If (Y, S) is trivial, then we shall refer to the $(X, S) \perp (Z, S)$ rather than $\varphi \perp \psi$.

Corollary 4. *$\varphi \perp \psi$ if $T = AB$ and $R_{\varphi\psi} = \overline{R_{\varphi\psi}J}$.*

Corollary 5. *Of the semigroup S is commutative, then $(X, S) \perp (Z, S)$ iff $\mathcal{E} = AB$.*

Corollary 6. *Any of the following conditions implies $\varphi \perp \psi$:*

1) φ or ψ is RIC-extension and $T = AB$;

2) φ is stable in some fibre and ψ is RD-prime RIC-extension;

3) φ is distal (regionally distal) and ψ is D-prime (RD-prime).

Theorem 6. *If AB is group and $R_{\varphi\psi} = R_{\varphi\psi}J$ (in particular, φ or ψ is homogeneous (group extension), then (φ, ψ) is D-prime iff $\varphi \perp \psi$.*

Proof. Let (φ, ψ) be a D-prime. Since AB is group, then by Theorem 10 from [7] there exists a D-factor $\eta : (W, S) \to (Y, S)$ of (φ, ψ) with $\mathcal{G} = AB$. Because η is trivial, then $F = AB$. By Corollary 4 we have $\varphi \perp \psi$. □

Corollary 7. *Let the semigroup S be a σ-compact and $Ss \subset sS$ $(s \in S)$ or S be a group. If AB is group and $R_{\varphi\psi} = R_{\varphi\psi}J$, then $\varphi_0 \perp \psi_0 \Leftrightarrow (\varphi, \psi)$ is RD-prime $\Leftrightarrow (\varphi, \psi)$ is D-prime $\Leftrightarrow (\varphi, \psi)$ is prime $\Leftrightarrow \varphi \perp \psi$.*

Corollary 8. *Let AB be a group and φ or ψ be a pseudostable RIC-extension. Then (φ, ψ) is D-prime iff $\varphi \perp \psi$.*

Theorem 7. *Let AB be a group, $R_{\varphi\psi} = R_{\varphi\psi}J$ and for any D-prime (RD-prime) extension $\delta \in K^1(Y, S)$ the set $R_{\varphi\delta}$ or $R_{\psi\delta}$ contains a unique minimal subset. Then the following statements are mutually equivalent:*

1) (φ, ψ) is D-prime (RD-prime);

2) $\varphi_0' \perp \psi_0'$ $(\varphi_0 \perp \psi_0)$;

3) $\varphi \perp \psi$;

4) (φ, ψ) is prime.

Proof. We prove the theorem in the distal case. Let 1) be true. Then $T = ABD_1$ and $T = (AD_1)(BD_1)$, hence by Corollary 4 $\varphi_0' \perp \psi_0'$, i.e., 2) is true. Let 2) be true. Then $T = (AD_1)(BD_1) = ABD_1$. Since AB is τ-closed subgroup of T, then there are the minimal extension $\theta : (A * B, S) \to (Y, S)$, where $A * B = \{(AB) \odot p |\ p \in I\}$, $((AB) \odot p)\theta = y_0 p$ $(p \in I)$ and if $\omega = (AB) \odot u$, then $\mathcal{G}A * B, \omega) = AB$ (see Proposition 3 from [8]). Therefore from $T = ABD_1$ implies, that θ prime. Because $R_{\varphi\theta}$ or $R_{\psi\theta}$ contains a unique minimal subset, then $T = A(AB) = AB$ or $T = B(AB) = AB$, i.e., $T = AB$. By Corollary 4 $\varphi \perp \psi$, i.e., 3) is true. The implications 3) $\Rightarrow$ 4) $\Rightarrow$ 1) are obvious. □

Theorem 8. *Let $ABH(T)$ be a group, $R_{\varphi\psi} = \overline{R_{\varphi\psi}J}$ and $H^{\alpha}(T) \subset AB$ for some transfinite number α. Then the following statements are mutually equivalent:*

1) (φ, ψ) is RD-prime;

2) $\varphi_0 \perp \psi_0$;

3) $\varphi \perp \psi$;

4) (φ, ψ) is prime.

Proof. Let 1) be true. Then $T = ABH(T)$ and $T = ABH(T) = ABD_2 = (AD_2)(BD_2) = \mathcal{G}(X_0)\mathcal{G}(Z_0)$, i.e., $T = \mathcal{G}(X_0)\mathcal{G}(Z_0)$. By Corollary 4 $\varphi_0 \perp \psi_0$, i.e., 2) is true. Let 2) be true. Then $T = \mathcal{G}(X_0)\mathcal{G}(Z_0) = (AD_2)(BD_2) = ABD_2 = [A, B]D_2$, i.e., $T = [A, B]D_2$. At this point the pair (φ, ψ) is RD-prime. Than $T = [A, B]H(T) = [A, B, H(T)] = ABH(T) = ABH^{\alpha}(T) = AB$, i.e., $T = AB$ and by Corollary 4 we get $\varphi \perp \psi$. Thus 2) implies 3). The implications 3) $\Rightarrow$ 4) $\Rightarrow$ 1) are obvious. □

Theorem 9. *Let:*

1) φ or ψ be a composition proximal extension and B-extension;

2) $H^{\alpha}(T) \subset AB$ for some transfinite number α;

3) $\varphi_0 \perp \psi_0$.

Then $R_{\varphi\psi}$ contains a unique minimal subset. If, in addition, $R_{\varphi\psi} = \overline{R_{\varphi\psi}J}$, then $\varphi \perp \psi$.

Proof. Since $\mathcal{G}(X_0) = AH(T)$ and |mcG$(Z_0) = BD_2$ or $\mathcal{G}(X_0) = AD_2$ and $\mathcal{G}(Z_0) = BH(T)$, then $\varphi_0 \perp \psi_0$ implies $T = AH(T)BD_2$ or $T = AD_2BH(T)$. Because $D_2 \subset AH(T)$ and $D_2 \subset BH(T)$, then $T = ABH(T)$. Therefore $T = ABH^{\alpha}(T) = AB$, i.e. $T = AB$, hence $R_{\varphi\psi}$ contains a unique minimal subset. □

Corollary 9. *If φ or ψ is RIC-extension and $H^{\alpha}(T) \subset AB$ for some transfinite number α, then $\varphi_0 \perp \psi_0$ implies $\varphi \perp \psi$.*

Corollary 10. *Let the semigroup S be a commutative, $(X_0, S)((Z_0, S))$ be a maximal RD-factor of $(X, S)((Z, S))$. If $H^{\alpha}(\mathcal{E}) \subset AB$ for some transfinite number α, then $(X_0, S) \perp (Z_0, S)$ implies $(X, S) \perp (Z, S)$.*

Lemma 1. *Let φ or ψ be a composition proximal extension and B-extension, ABD_2 be a group. If $(W, S) \to (Y, S)$ is a maximal RD-factor of the pair (φ, ψ), then $\mathcal{G}(W) = ABH(T)$.*

Proof. Since $\mathcal{G}(X_0) = AD_2$ and $\mathcal{G}(Z_0) = BD_2$, then $[\mathcal{G}(X_0), \mathcal{G}(Z_0)] = ABD_2$. Because $(W,S) \to (Y,S)$ is maximal RD-factor of the pair (φ_0, ψ_0), then $\mathcal{G}(W) = [\mathcal{G}(X_0), \mathcal{G}(Z_0)]H(T) = ABD_2H(T)$, i.e., $\mathcal{G}(W) = ABD_2H(T)$. Since φ or ψ is a composition proximal extension and B-extension, then $\mathcal{G}(X_0) = AH(T)$ or $\mathcal{G}(Z_0) = BH(T)$. Because φ_0 and ψ_0 are regionally distal, then $D_2 \subset \mathcal{G}(X_0)$ and $D_2 \subset \mathcal{G}(Z_0)$, hence $D_2 \subset ABH(T)$. Therefore $\mathcal{G}(W) = ABD_2H(T)$ implies $\mathcal{G}(W) = ABH(T)$. □

Theorem 10. *Let ABD_2 be a group, $H^\alpha(T) \subset AB$ for some transfinite number α and φ or ψ be a composition proximal extension and B-extension. Then the following statements are mutually equivalent:*

1) (φ, ψ) is RD-prime;

2) $\varphi_0 \perp \psi_0$;

3) $R_{\varphi\psi}$ contains a unique minimal subset;

4) (φ, ψ) is P-pair;

5) (φ, ψ) is prime.

Proof. Let 1) be true. Then $T = ABH(T)$ by Lemma 1. Since $D_2 \subset \mathcal{G}(X_0) = AH(T)$ or $D_2 \subset \mathcal{G}(Z_0) = BH(T)$, then $T = ABH(T)$ implies $T = AH(T)BD_2$ or $T = AD_2BH(T)$. By Corollary 4 $\varphi_0 \perp \psi_0$, i.e., 2) is true. The implications 2) $\Rightarrow$ 3) $\Rightarrow$ 4) $\Rightarrow$ 5) $\Rightarrow$ 1) are obvious. □

Lemma 2. *If the semigroup S is commutative and the $TTS(X,S)$ is pseudostable, then A is invariant subgroup of $\mathcal{E}$.*

Proof. There exists a proximal extension $(X,S) \to (W,S)$ with a regionally distal $TTS(W,S)$. Then $\mathcal{G}(W) = A$. Since (W,S) is homogeneous, then $\mathcal{G}(W)$ is invariant subgroup of $\mathcal{E}$. □

Lemma 3. *Let the semigroup S be a commutative. If (W,S) is maximal RD-factor of the pair $((X,S),(Z,S))$, then $\mathcal{G}(W) = ABH(\mathcal{E})$. Consequently the pair $((X,S),(Z,S))$ is RD-prime if $ABH(\mathcal{E}) = \mathcal{E}$.*

Proof. The proof is obvious. □

Theorem 11. *Let the semigroup S be a commutative and the pair $((X,S),(Z,S))$ be a α-pseudostable for some transfinite number α; $(X_0,S)((Z_0,S))$ be a maximal RD-factor of $(X,S)((Z,S))$. Then $((X,S),(Z,S))$ is RD-prime $\Leftrightarrow$ $(X_0,S) \perp (Z_0,S)$ $\Leftrightarrow$ $((X,S),(Z,S))$ is prime.*

Proof. The proof is obvious. [

Theorem 12. *Let the semigroup S be a commutative and $H^{\alpha}(\mathcal{E} \subset AB$ for some tran finite number α, $(X_0, S)((Z_0, S))$ be a maximal RD-factor of $(X, S)((Z, S))$. The $((X, S), (Z, S))$ is RD-prime $\Leftrightarrow$ $(X_0, S) \perp (Z_0, S)$ $\Leftrightarrow$ $(X, S) \perp (Z, S)$ $\Leftrightarrow$ $((X, S), (Z,$ is prime.*

Proof. The proof is obvious.

Consider the commutative diagram

$$\begin{array}{ccccc} (X,S) & \xrightarrow{\delta} & (W,S) & & (Z,S) \\ & \searrow\varphi & \downarrow\eta & \psi\swarrow & \\ & & (Y,S) & & \end{array} \qquad ($$

And let $\omega_0 = x_0\delta$, $D = \mathcal{G}(W, \omega_0)$ the Ellis group of $TTS(W, S)$.

Theorem 13. *Let in (6):*

1) δ be a proximal;

2) $R_{\eta\psi}$ contains a unique minimal subset;

3) $R_{\varphi\psi} = \overline{R_{\varphi\psi}J}$.

Then $\varphi \perp \psi$.

Proof. Since 1) $\Rightarrow$ $A = D$ and 2) $\Rightarrow$ $T = BD$, then $T = AB$. By Corollar $\varphi \perp \psi$.

Corollary 11. *Let in (6):*

1) δ be a proximal and η be a distal (regionally distal);

2) (η, ψ) be a D-prime (RD-prime);

3) AB be a group and $R_{\varphi\psi} = \overline{R_{\varphi\psi}J}$.

Then $\varphi \perp \psi$.

Proof. It follows from Theorem 6 and 13.

Corollary 12. *Let the semigroup S be a commutative, the $TTS(W, S)$ be a distal and extension $\delta : (X, S) \to (W, S)$ be a proximal. If AB is group and the pair $((X, S), (Z,$ is D-prime, then $(X, S) \perp (Z, S)$.*

Corollary 13. *If the semigroup S is commutative, the extension $\delta : (X,S) \to (W,S)$ is proximal and $(W,S) \perp (Z,S)$, then $(X,S) \perp (Z,S)$.*

Theorem 14. *Let in (6):*

1) δ be a proximal;

2) ψ be a RD-prime RIC-extension;

3) η be a pseudostable (stable in the some fibre).

Then $\varphi \perp \psi$.

Proof. Since 1) and 3) $\Rightarrow$ $H(T) \subset A$ and 2) $\Rightarrow$ $T = BH(T)$, then $T = AB$. Also 2) $\Rightarrow$ $R_{\varphi\psi} = \overline{R_{\varphi\psi}J}$. By Corollary 4 $\varphi \perp \psi$. □

Corollary 14. *Let the semigroup S be a commutative, the extension $\delta : (X,S) \to (W,S)$ be a proximal and $TTS(W,S)$ be a pseudostable. If $TTS(Z,S)$ is RD-prime, then $(X,S) \perp (Z,S)$.*

Corollary 15. *Let in (6):*

1) φ or ψ be a pseudostable;

2) φ or ψ be a RIC-extension;

3) η be a maximal RD-factor of φ;

4) $\eta \perp \psi$.

Then $\varphi \perp \psi$.

Theorem 15. *Let in (6):*

1) ϖ or ψ be a pseudostable;

2) φ or ψ be a RIC-extension;

3) $AH(D) = D$;

4) $R_{\eta\psi}$ contains a unique minimal subset.

Then $\varphi \perp \psi$.

Proof. Since 1) $\Rightarrow$ $H(T) \subset AB$ and 4) $\Rightarrow$ $T = DB$, then $T = DB = AH(T)B \subset AB \subset T$, hence $T = AB$. At this point $\varphi \perp \psi$ by Corollary 4. □

Corollary 16. *Let the semigroup S be a commutative, (X,S) or (Z,S) be a pseudostable and the extension $\delta : (X,S) \to (W,S)$ such that $AH(D) = D$. If $(W,S) \perp (Z,S)$, then $(X,S) \perp (Z,S)$.*

Theorem 16. *Let in (6):*

1) ψ be a distal (regionally distal);

2) η be a maximal D-factor (RD-factor) of φ;

3) $\eta \perp \psi$.

Then $\varphi \perp \psi$.

Proof. Since conditions 2) – 3) $\Rightarrow$ $T = AD_i$ and $T = DB$, then $T = AD_iB$ $(i = 1, 2)$. Because 1) $\Rightarrow$ $D_i \subset B$ $(i = 1, 2)$, then at this point $T = AB$ and $\varphi \perp \psi$ by Corollary 4. □

Theorem 17. *Let in (6):*

1) $R_{\eta\psi}$ contains a unique minimal subset;

2) φ or ψ be a B-extension or $R_{\varphi\psi} = \overline{R_{\varphi\psi}J}$;

3) $H^{\alpha}(D) \subset [A, B]$ for some transfinite number α;

4) $R_{\delta} \subset Q^{*}(R_{\varphi})$.

Then pair (φ, ψ) is prime.

Proof. Since condition 1) $\Rightarrow$ $T = BD$, then by Proposition 7 [6] we have

$$H(T) \subset BH(D). \tag{7}$$

Conditions 1), 2) and 4) implies

$$T = [A, B]H(T). \tag{8}$$

Conditions (7) and (8) implies

$$T = [A, B]H(D). \tag{9}$$

By transfinite induction from (8) and (9) we prove $T = [A, B]H^{\alpha}(D)$, hence $T = [A, B]$ by condition 3). At this point (φ, ψ) is prime. □

Corollary 17. *Let in (6):*

1) φ or ψ be a composition proximal extension and B-extension;

2) $R_{\delta} \subset Q^{*}(R_{\delta})$;

3) $R_{\eta\psi}$ contains a unique minimal subset;

4) $H^{\alpha}(D) \subset AB$ for some transfinite number α;

5) $R_{\varphi\psi} = \overline{R_{\varphi\psi}J}$.

Then $\varphi \perp \psi$ iff $\eta \perp \psi$.

Definition 1. An extension $\varphi : (X,S) \to (Y,S)$ is called almost distal (almost automorphic), if the set of points, where extension φ is distal (one to one), is dense in X.

Remark 2. If the extension φ is distal (one to one) in the point $x \in X$ and $\varphi^{-1}(x\varphi s) = \varphi^{-1}(x\varphi)s$ $(s \in S)$, then it is almost distal (almost automorphic).

Proof. If the extension φ is distal (one to one) in the point x, then extension φ is distal (one to one) in each point of xS. Since $\overline{xS} = X$, then φ is almost distal (almost automorphic). □

Theorem 18. *Let in (6):*

1) δ be a almost distal (almost automorphic);

2) $R_{\eta\psi} = R_{\eta\psi}J$;

3) ψ be open or ψ be semiopen and φ be open.

Then $R_{\varphi\psi} = \overline{R_{\varphi\psi}J}$.

Proof. Let, for clarity, the extension δ be almost distal and V be any open nonempty subset of $R_{\varphi\psi}$. There exist open subsets V_1 and V_2 of X and Z respectively such that for $\forall\ x \in V_1\ \exists\ z \in V_2$ with $(x,z) \in V$. Let $x \in V_1$, δ is distal in x and $z \in V_2$ with $(x,z) \in V$. Because $(x\delta, z) \in R_{\eta\psi}$, then by 2) $(x\delta, z)v = (x\delta, z)$ for some idempotent $v \in I$. Therefore $x\delta v = x\delta$ and $zv = z$. Since $xv\delta = x\delta v = x\delta$, then $xv \in \delta^{-1}(x)$. Because δ is distal in x, then $xv = x$. Therefore $(x,z)v = (x,z)$ and $(x,z) \in R_{\varphi\psi}J \cap V$, hence $R_{\varphi\psi} = \overline{R_{\varphi\psi}J}$. □

Corollary 18. *Let in (6):*

1) φ or ψ be a composition proximal extension and B-extension;

2) ψ be open or ψ be a semiopen and φ be open;

3) δ be a almost distal;

4) $H^\alpha(D) \subset AB$ for some transfinite number α;

5) $R_\delta \subset Q^*(R_\varphi)$.

Then $\varphi \perp \psi$ iff $\varphi_0 \perp \psi_0$.

Corollary 19. *Let in (6):*

1) δ be a almost distal (almost automorphic);

2) $R_{\eta\psi} = R_{\eta\psi}J$;

3) ψ be open or ψ be semiopen and φ be open;

4) $R_{\varphi\psi}$ contains a unique minimal subset.

Then $\varphi \perp \psi$.

Corollary 20. *If φ is almost automorphic and ψ is open, then $\varphi \perp \psi$.*

Corollary 21. *If the extension $\delta : (X,S) \to (W,S)$ is almost automorphic and $(W,S) \perp (Z,S)$, then $(X,S) \perp (Z,S)$.*

References

[1] Furstenberg H., *Disjointness in Ergodic Theory, Minimal Sets, and a Problem in Diophantine Approximation*, Math. Systems Theory, 1, 1967, 1-50.

[2] Shapiro L., *Distal and Proximal Extensions of Minimal Flows*, Math. Systems Theory, 5, 1971, 76-88.

[3] Bronštein I.U., *Extensions of minimal transformation groups*, Sijthoff & Noordhoff International Publishers, 1979, 319 p.

[4] Woude, J.C.S.P., van der, *Topological dynamics*, Mathematisch Centrum, Amsterdam, 1982, 300 p.

[5] Ellis R., Glasner Sh., Shapiro L., *Proximal-Isometric ($\mathcal{PI}$)-Flows*, Advances in Math., 17, 1975, 213-260.

[6] Gerko A.I., *$\tau(A)$-topology and some properties of groups connected with topological transformation semigroups*, Izv. Akad. Nauk Resp. Moldova, N2, 1991, 26-31.

[7] Gerko A.I., *Some generalization of the Galois theory of distal extensions transformation semigroups*, Mat. Issled. (118), AN Mod. SSR, 1990, 28-35.

[8] Gerko A.I., *RIC-extensions and structure of minimal transformation semigroups*, Mat. Issled. (77), AN Mold. SSR, 1984, 47-60.

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NECESSARY CONDITIONS FOR EXISTENCE OF SOME STOKES FLOWS

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Abstract. In this paper there are obtained a couple of necessary conditions for the existence of two-dimensional Stokes flows for circular obstacles. There are examined in details some examples.

1. Mathematical formulation

It is well known that the motion equations of an incompressible viscous flow, which has very small Reynolds number, can be reduced to the following Stokes equations:

$$\begin{aligned} \Delta\vec{u} - \nabla p = 0 \\ \nabla \cdot \vec{u} = 0 \end{aligned} \tag{1}$$

where by *vecu* and p we denote the fluid velocity and the fluid pressure, respectively.

From the continuity equation (1.b) we deduce that there exists a stream function $\psi = \psi(r, \theta)$ such that

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \quad u_\theta = -\frac{\partial\psi}{\partial r} \tag{2}$$

where u_r, u_θ are the velocity components with respect to the polar coordinates (r, θ). From (2), it results that the motion equations (1.a) and (1.b) can be written as follows:

$$\Delta^2\psi(r, \theta) = 0. \tag{3}$$

Let $\psi_0 = \psi_0(r, \theta)$ be the stream function of a given unbounded Stokes flow. If in this flow we introduce a circular cylinder $r = a$, then the stream function ψ of a resulting flow satisfies the biharmonic equation (3) and also the following boundary conditions:

$$\begin{cases} \psi(a, \theta) = 0 \\ \frac{1}{r}\frac{\partial\psi}{\partial\theta}(a, \theta) = 0, \quad \frac{\partial\psi}{\partial r}(a, \theta) = 0 \end{cases} \tag{4}$$

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On the other hand, at infinity the function ψ has the same form as ψ_0, so:

$$\psi \sim \psi_0, \text{ as } r \to \infty. \quad (5)$$

If we consider the following representation of the function ψ:

$$\psi = \psi_0 + \tilde{\psi} \quad (6)$$

where $\tilde{\psi}(r,\theta)$ denotes the perturbed stream function corresponding to the presence o the circular cylinder in the given Stokes flow, then the function $\tilde{\psi}$ is a solution of th next system:

$$\begin{cases} \Delta^2\tilde{\psi}(r,\theta) = 0 \\ \tilde{\psi}(a,\theta) = -\psi_0(a,\theta) \\ \frac{1}{a}\frac{\partial\tilde{\psi}}{\partial\theta}(a,\theta) = -\frac{1}{a}\frac{\partial\psi_0}{\partial\theta}(a,\theta) \\ \frac{\partial\tilde{\psi}}{\partial r}(a,\theta) = -\frac{\partial\psi_0}{\partial r}(a,\theta) \end{cases} \quad (7$$

In the following, we suppose that at infinity $\tilde{\psi}$ can be written as $b + c\ln r +$ $d\cos 2\theta + e\sin 2\theta$, where the unknowns constants b, c, d, e, will be determined using th boundary conditions $(7)_2 - (7)_4$.

2. Necessary result for the existence of solution

Now, we consider the following problem, corresponding to a biharmonic functio ϕ:

$$\Delta^2\phi(r,\theta) = 0 \text{ in } \theta, \quad (8$$

with the boundary conditions:

$$\begin{cases} \phi(a,\theta) = f(\theta) \\ \frac{\partial\phi}{\partial r}(a,\theta) = g(\theta) \end{cases} \quad (9$$

and with the asymptotic form at infinity $b + c\ln r + d\cos 2\theta + e\sin 2\theta$, as $r \to \infty$.

Here Ω is the outer domain of circle $r = a$.

From the above asymptotic condition, we deduce that ϕ may be written unde the form [1]:

$$\phi(r,\theta) = b + c\ln r + d\cos 2\theta + e\sin 2\theta + \quad (10$$

$$+\sum_{n=1}^{\infty}\frac{A_n\cos n\theta + B_n\sin n\theta}{r^n} + \sum_{n=3}^{\infty}\frac{C_n\cos n\theta + D_n\sin n\theta}{r^{n-2}}$$

where A_n, B_n, C_n, D_n are some constants which will be determined using the boundary conditions (9). From (9) we obtain:

$$b + c\ln a + d\cos 2\theta + e\sin 2\theta + \tag{11}$$

$$+\sum_{n=1}^{\infty}\frac{A_n\cos n\theta + B_n\sin n\theta}{a^n} + \sum_{n=3}^{\infty}\frac{C_n\cos n\theta + D_n\sin n\theta}{a^{n-2}} = f(\theta)$$

$$\frac{c}{a} - \sum_{n=1}^{\infty}\frac{n(A_n\cos n\theta + B_n\sin n\theta}{a^{n+1}} - \sum_{n=3}^{\infty}\frac{(n-2)(C_n\cos n\theta + D_n\sin n\theta}{a^{n-1}} = g(\theta). \tag{12}$$

We suppose that the functions f and g can be written as:

$$f(\theta) = \frac{f_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta) \tag{13}$$

$$g(\theta) = \frac{g_0}{2} + \sum_{n=1}^{\infty}(c_n\cos n\theta + d_n\sin n\theta). \tag{14}$$

If we use the relations (11)-(14), then we deduce:

$$\begin{cases} b + c\ln a = f_0/2, \quad c = ag_0/2 \\ A_1/a = a_1, \quad B_1/a = b_1, \quad -A_1/a^2 = c_1, \quad -B_1/a^2 = d_1 \\ d + A_1/a^2 = a_2, \quad a^2e + B_2 = a^2b_2, \quad -2A_2 = a^3c_2, \quad -2B_2 = a^3d_2 \end{cases} \tag{15}$$

$$\begin{cases} A_n + a^2C_n = a^na_n, \quad B_n + a^2D_n = b_n \\ -nA_n - (n-2)a^2C_n = a^{n+1}c_n \quad -nB_n - (n-2)a^2D_n = a^{n+1}d_n \end{cases} \quad \text{for } n \geq 3. \tag{16}$$

As a consequence of (15) it results that:

$$a_1 = -ac_1, \quad b_1 = -ad_1 \tag{17}$$

Hence, we obtain the necessary condition for existence of the function ϕ:

$$\int_0^{2\pi} f(\theta)e^{i\theta}\,d\theta = -a\int_0^{2\pi} g(\theta)e^{i\theta}\,d\theta. \tag{18}$$

Using the above arguments, we can formulate the following result:

Theorem 1. *If the function $\phi(r,\theta)$ is a solution of the problem (8)-(10), then the functions f and g must satisfy the necessary condition (18).*

Also, using the relations (15)-(16), we deduce:

Corollary 1. *If $\phi(r,\theta)$ is a solution of the next problem:*

$$\begin{cases} \Delta^2\phi(r,\theta) = 0 \text{ in } \Omega \\ \phi(a,\theta) = f(\theta) \\ \frac{\partial\phi}{\partial r}(a,\theta) = g(\theta) \\ \phi \to 0, \text{ as } r \to \infty \end{cases} \tag{19}$$

then the functions f and g must satisfy the following conditions:

$$\begin{cases} \int_0^{2\pi} f(\theta)d\theta = \int_0^{2\pi} g(\theta)d\theta = 0 \\ \int_0^{2\pi} f(\theta)e^{i\theta}d\theta = -a\int_0^{2\pi} g(\theta)e^{i\theta}d\theta \\ \int_0^{2\pi} f(\theta)e^{2i\theta}d\theta = -\frac{a}{a}\int_0^{2\pi} g(\theta)e^{2i\theta}d\theta \end{cases} \tag{20}$$

The proof is immediately, since in this case we have $b = c = d = e = 0$, s $f_0 = g_0 = 0$ and $a_2 = -\frac{a}{2}c_2$, $b_2 = -\frac{a}{2}d_2$.

Remark. The problem formulated in the Corollary 1 is a directly consequence of the cas when the perturbations at infinity are zero.

From Theorem 1 we deduce that the stream function ψ_0 of a given Stokes flov in the presence of circular cylinder $r = a$, must satisfy the necessary condition:

$$\int_0^{2\pi} \psi_0(a,\theta)e^{i\theta}d\theta = -a\int_0^{2\pi} \frac{\partial\psi_0}{\partial r}(a,\theta)e^{i\theta}d\theta. \tag{21}$$

The following result gives the perturbation produced by a circular cylinder in mersed in a given Stokes flow. The proof of this result may be found in [6].

Theorem 2. *Let $\psi_0(r,\theta)$ the stream function of a given Stokes flow be defined in tl plane (r,θ), with except perhaps some singularities located in a bounded domain outsi of circle $r = a$. We suppose that the function ψ_0 satisfies the condition (21). If a circul cylinder $r = a$ is immersed in this flow, then the stream function ψ of resulting flo which has same singularities as $\psi_0(r,\theta)$, has the following form:*

$$\psi(r,\theta) = \psi_0(r,\theta) + \psi_0^*(r,\theta),$$

where ψ_0^* is given by:

$$\psi_0^*(r,\theta) = \psi_0(r,\theta) + \frac{r^4 - 2r^2a^2}{a^4}\psi_0\left(\frac{a^2}{r},\theta\right) - \frac{r^3}{a^4}(a^2 - r^2)\frac{\partial}{\partial r}\psi_0\left(\frac{a^2}{r},\theta\right) + \tag{2}$$

$$+\frac{r^2}{4a^4}(a^2-r^2)\Delta\left[r^2\psi_0\left(\frac{a^2}{r},\theta\right)\right]+\frac{a^2+r^2}{4a^2r}\int^r\Delta\left(\rho^2\psi_0\left(\frac{a^2}{\rho},\theta\right)\right)d\rho-$$

$$-\frac{1}{2a}\int_0^a\Delta\left(\rho^2\psi_0\left(\frac{a^2}{\rho},\theta\right)\right)d\rho.$$

Remark. From the above hypothesis it results that the added function ψ_0^* has not singularities in the prescribed motion domain $r > a$.

3. Two simple examples

a) Let consider a Stokes flow with the uniform velocity $\vec{U} = U\vec{i}$, past a cylinder $r = a$. Then, the stream function ψ_0 is given by:

$$\psi_0(r,\theta) = Ur\sin\theta.$$

On the other hand, since the next relations are satisfied:

$$\int_0^{2\pi}\psi_0(a,\theta)e^{i\theta}d\theta = Ua\pi i, \quad -a\int_0^{2\pi}\frac{\partial\psi_0}{\partial r}(a,\theta)e^{i\theta} = -Uai\pi,$$

it is clear that the necessary condition (21) is not satisfied. Hence it is impossible to obtain a solution for the above considered problem. In fact this is the well known Stokes paradox [1].

b) Let the stream function ψ_0 be defined by:

$$\psi_0(r,\theta) = r^2\sin 2\theta.$$

Then, we obtain:

$$\int_0^{2\pi}\psi_0(a,\theta)e^{i\theta}d\theta = -a\int_0^{2\pi}\frac{\partial\psi_0}{\partial r}(a,\theta)e^{i\theta}d\theta = 0,$$

hence, the necessary condition (21) is satisfied. It is easily to prove that ψ_0 is a biharmonic function, and applying the result of Theorem 2, we deduce that the stream function ψ has the form:

$$\psi(r,\theta) = r^2\sin 2\theta + 2a^2\sin 2\theta + \frac{a^4}{r^2}\sin 2\theta.$$

4. A necessary result for existence of biharmonic functions in an outside domain of a sphere

Let $\psi_0 = \psi_0(r,\theta,\varphi)$ be a biharmonic function given in an unbounded three dimensional domain. We denote with ψ a biharmonic function be defined in the outside domain Ω of sphere $r = a$. This function satisfies the following equation and conditions:

$$\Delta^2\psi(r,\theta,\varphi) = 0 \text{ in } \Omega, \tag{23}$$

$$\begin{cases} \psi(a,\theta,\varphi) = 0 \\ \frac{\partial\psi}{\partial r}(a,\theta,\varphi) = 0 \end{cases}, \quad 0 \le \theta \le \pi, \quad 0 \le \varphi \le 2\pi. \tag{24}$$

At infinity ψ has the same form as ψ_0, hence:

$$\psi \sim \psi_0, \text{ for } r \to \infty. \tag{25}$$

If we consider the perturbed stream function $\tilde{\psi}$, given by $\psi = \psi_0 + \tilde{\psi}$, then this function is a solution of the next problem:

$$\begin{cases} \Delta^2\tilde{\psi}(r,\theta,\varphi) = 0, \text{ in } \Omega \\ \tilde{\psi}(a,\theta,\varphi) = -\psi_0(a,\theta,\varphi) \\ \frac{\partial\tilde{\psi}}{\partial r}(a,\theta,\varphi) = -\frac{\partial\psi_0}{\partial r}(a,\theta,\varphi) \end{cases}, \quad 0 \le \theta \le \pi, \quad 0 \le \varphi \le 2\pi. \tag{26}$$

Let $\phi(r,\theta,\varphi)$ be a solution of the following problem:

$$\begin{cases} \Delta^2\phi(r,\theta,\varphi) = 0 \text{ in } \Omega \\ \phi(a,\theta,\varphi) = f(\theta,\varphi) \\ \frac{\partial\phi}{\partial r}(a,\theta,\varphi) = g(\theta,\varphi) \end{cases} \tag{27}$$

then, from [1] and since any solution of the biharmonic equation can be represented in the spherical coordinates as:

$$\phi(r,\theta,\varphi) = h(r,\theta,\varphi) + r^2 s(r,\theta,\varphi),$$

where $\Delta h(r,\theta,\varphi) = \Delta s(r,\theta,\varphi) = 0$, it results that the function ϕ may be written under the form:

$$\phi(r,\theta,\varphi) = \sum_{n=0}^{\infty}\left(a_n r^n + b_n r^{n+2}\right) S_n(\theta,\varphi) + \tag{28}$$

$$+\sum_{n=2}^{\infty}\left(c_n r^{-(n+1)} + d_n r^{-(n-1)}\right) T_n(\theta,\varphi) + \frac{c_1}{r^2}T_1(\theta,\varphi) + \frac{c_0}{r},$$

where

$$S_n(\theta,\varphi) = \sum_{m=0}^{n} P_n^m(\cos\theta)(A_{mn}\cos m\varphi + B_{mn}\sin m\varphi),$$

$$T_n(\theta,\varphi) = \sum_{m=0}^{n} P_n^m(\cos\theta)(C_{mn}\cos m\varphi + D_{mn}\sin m\varphi),$$

for all $n \geq 1$.

Also, in the above relations P_n^m mean the Legendre's functions, given by

$$P_n^0(x) = P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right],$$

$$P_n^m(x) = C\frac{d^{m+n}}{dx^{n+m}}\left[(x^2-1)^n\right], \quad -1 < x < 1,$$

for all $n \in \mathbb{N}$ and $m \in \{0,\dots,n\}$.

Using (27) we deduce that:

$$\begin{cases} \sum\limits_{n=0}^{\infty}\left(a_n a^n + b_n a^{n+2}\right)S_n(\theta,\varphi) + \sum\limits_{n=2}^{\infty}\left(c_n a-(n+1) + d_n a^{-(n-1)}\right)T_n(\theta,\varphi)+ \\ +\frac{c_1}{a^2}T_1(\theta,\varphi) + \frac{c_0}{a} = f(\theta,\varphi), \\ \sum\limits_{n=0}^{\infty}\left(na_n a^{n-1} + (n+2)b_n a^{n+1}\right)S_n(\theta,\varphi)- \\ -\sum\limits_{n=2}^{\infty}\left((n+1)c_n a^{-(n+2)} + (n-1)d_n a^{-n}\right)T_n(\theta,\varphi) - \frac{2c_1}{a^3}T_1(\theta,\varphi)+ \\ -\frac{c_0}{a^2} = g(\theta,\varphi). \end{cases} \tag{29}$$

On the other hand, we suppose that the functions f and g may be written as:

$$f(\theta,\varphi) = \sum_{n=1}^{\infty} f_n F_n(\theta,\varphi) + f_0, \quad g(\theta,\varphi) = \sum_{n=1}^{\infty} g_n G_n(\theta,\varphi) + g_0 \tag{30}$$

where

$$\begin{cases} F_n(\theta,\varphi) = \sum\limits_{m=0}^{n}(f_{mn}^1\cos m\varphi + f_{mn}^2\sin m\varphi)P_n^m(\cos\theta) \\ G_n(\theta,\varphi) = \sum\limits_{m=0}^{n}(g_{mn}^1\cos m\varphi + g_{mn}^2\sin m\varphi)P_n^m(\cos\theta) \end{cases} \tag{31}$$

for all $n \geq 1$.

From (29)-(31), we obtain the following equalities:

$$\frac{c_0}{a} + a_0 + b_0 a^2 = f_0, \quad -\frac{c_0}{a^2} + 2b_0 a = g_0, \quad \frac{c_1}{a^2}C_{10} + \left(a_1 a + b_1 a^3\right)A_{10} = f_1 f_{10}^1$$

$$-\frac{2c_1}{a^3}C_{10} + \left(a_1 + 3b_1 a^2\right)A_{10} = g_1 g_{10}^1, \quad \frac{c_1}{a^2}C_{11} + \left(a_1 + 3b_1 a^2\right)A_{11} = f_1 f_{11}^1$$

$$-\frac{2c_1}{a^3}C_{11} + \left(a_1 + 3b_1 a^2\right)A_{11} = g_1 g_{11}^1$$

$$\begin{cases} \frac{c_1}{a^2}D_{11} + (a_1 a + b_1 a^2)B_{11} = f_1 f_{11}^2 \\ -\frac{2c_1}{a^3}D_{11} + (a_1 + 3b_1 a^2)B_{11} = g_1 g_{11}^2 \end{cases}$$

$$\left(a_n a^n + b_n a^{n+2}\right) A_{mn} + \left(c_n a^{-(n+1)} + d_n a^{-(n-1)}\right) C_{mn} = f_n f_{mn}^1$$

$$\left(a_n a^n + b_n a^{n+2}\right) B_{mn} + \left(c_n a^{-(n+1)} + d_n a^{-(n-1)}\right) D_{mn} = f_n f_{mn}^2$$

$$\left(n a_n a^{n-1} + (n+2) b_n a^{n+1}\right) A_{mn} - \left((n+1) c_n a^{-(n+2)} + (n-1) d_n a^{-n}\right) C_{mn} = g_n g_n^1$$

$$\left(n a_n a^{n-1} + (n+2) b_n a^{n+1}\right) B_{mn} - \left((n+1) c_n a^{-(n+2)} + (n-1) d_n a^{-n}\right) D_{mn} = g_n g_n^2$$

If we suppose that $\phi \to 0$, as $r \to \infty$, then we obtain the next relations:

$$f_0 = -a g_0, \quad f_1 f_{10}^1 = -\frac{a}{2} g_{10}^1 g_1, \quad f_1 f_{11}^1 = -\frac{a}{2} g_1 g_{11}^1, \quad f_1 f_{11}^2 = -\frac{a}{2} g_1 g_{11}^2$$

which assure the necessary conditions:

$$\int_0^\pi \int_0^{2\pi} f(\theta, \varphi) \sin\theta d\theta\, d\varphi = -a \int_0^\pi \int_0^{2\pi} g(\theta, \varphi) \sin\theta d\theta\, d\varphi,$$

$$\int_0^\pi \int_0^{2\pi} f(\theta, \varphi) \sin 2\theta d\theta\, d\varphi = -\frac{a}{2} \int_0^\pi \int_0^{2\pi} g(\theta, \varphi) \sin 2\theta d\theta\, d\varphi.$$

Here, we used the orthogonality properties of Legendre's functions

$$\int_{-1}^1 P_n(x) P_m(x) = \begin{cases} 0, & \text{for } m \neq n \\ \frac{2}{2n+1}, & \text{for } m = n \geq 0 \end{cases}$$

and also,

$$P_0(x) = 1, \quad P_1(x) = x, \quad \text{for } -1 \leq x \leq 1.$$

Now, we can formulate the following result:

Theorem 3. *If $\phi(r, \theta, \varphi)$ is a solution of the problem (27), with the assumption ϕ as $r \to \infty$, then the function f and g must satisfy the conditions (32) and (33).*

From the above result we conclude that the biharmonic function ψ_0 from must satisfy the necessary conditions:

$$\int_0^\pi \int_0^{2\pi} \psi_0(a, \theta, \varphi) \sin\theta d\theta\, d\varphi = -a \int_0^\pi \int_0^{2\pi} \frac{\partial \psi_0}{\partial r}(a, \theta, \varphi) \sin\theta d\theta\, d\varphi$$

$$\int_0^\pi \int_0^{2\pi} \psi_0(a, \theta, \varphi) \sin 2\theta d\theta = -\frac{a}{2} \int_0^\pi \int_0^{2\pi} \frac{\partial \psi_0}{\partial r}(a, \theta, \varphi) \sin\theta d\theta\, d\varphi$$

References

[1] L. Dragoş, *Principles of Continuous Mechanics Media* (in Romanian), Ed. Tehnică, Bucureşti, 1983.

[2] J.J.L. Higdon, *Stokes flow in arbitrary two-dimensional domains: shear flow over ridges and cavities*, J. Fluid Mech. (1985), 159, 195-226.

[3] O.A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow* (New York: Gordon and Breach, 1969).

[4] R.C. MacCamy, *On a class of two-dimensional Stokes-flows*, Arch. Rat. Mech. Anal., 21, 246-258.

[5] J. Martinek, H.P. Thielman, *Circle and Sphere Theorems for the Biharmonic Equation (Interior and exterior problem)*, ZAMP, 16(1965), 494-501.

[6] R. Usha, K. Hemalatha, *A note on plane Stokes flow past a shear free impermeable cylinder*, ZAMP, 44(1993), 73-84.

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SUBCLASS OF MEROMORPHIC STARLIKE FUNCTIONS

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Abstract. In the present paper we have introduced a subclass $\sum_S^*(\alpha,\beta,\gamma)$ of meromorphic univalent functions, and we prove coefficient inequality and distorsion theorem. Lastly, we have obtained radii of convexity for function $f(z)$ belonging to $\sum_S^*(\alpha,\beta,\gamma)$. Various results obtained in the present paper are shown to be sharp.

1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \tag{1.1}$$

which are regular in $U^* = \{z : 0 < |z| < 1\}$ with simple pole at origin and residue 1 there. Further, Σ^* denote the subclass of Σ consisting of analytic and univalent functions $f(z)$ in U^*. A function $f(z)$ in Σ^* is said to be meromorphically starlike of order α if

$$\text{Re}\,\{-zf'(z)/f(z)\} < \alpha, \quad (z \in U^*)$$

for some α, $(0 \le \alpha < 1)$. We denote by $\Sigma^*(\alpha)$, the class of meromorphically starlike functions of order α. Let Σ_s denote the subclass of Σ whose members have the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z_n, \quad a_n \ge 0. \tag{1.2}$$

Definition. A function $f(z)$ in Σ_s is in the class $\Sigma_s^*(\alpha,\beta,\gamma)$ if it satisfies the condition

$$\left|\frac{zf'(z)/g(z)+1}{zf'(z)/g(z)-(1-2\beta)}\right| < \gamma, \tag{1.3}$$

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for $z \in U^*$, $0 \le \beta < 1$, $0 < \gamma \le 1$ and $g(z) \in \Sigma^*(\alpha)$, with $g(z)$ of the form

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n, \quad b_n \ge 0.$$

The systematic study of aforementioned class has been motivated by recent work of Aouf [1], Srivastava and Owa [3], and also by Gupta [2], Uralegaddi and Ganigi [4].

2. Coefficient Inequalities and Distortion Theorem

We state following Lemma due to Uralegaddi and Ganigi [4], which we are going to use in our further investigations.

Lemma 1. *A function* $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$, $(a_n \ge 0)$ *is in* $\Sigma^*(\alpha)$ *if and only if*

$$\sum_{n=1}^{\infty} (n+\alpha) a_n \le (1-\alpha). \tag{2.1}$$

The result is sharp.

Theorem 1. *Let the function* $f(z)$ *defined by (1.2) be in the class* $\Sigma_s^*(\alpha,\beta,\gamma)$. *Then*

$$\sum_{n=1}^{\infty} \left\{ (1+\gamma) n a_n + \frac{(1-\alpha)(1-\gamma+2\beta\gamma)}{(n+\alpha)} \right\} \le 2\gamma(1-\beta). \tag{2.2}$$

Proof. Since $f \in \Sigma_s^*(\alpha,\beta,\gamma)$, there exists a function $g(z)$ in the class $\Sigma^*(\alpha)$ such that

$$\left| \frac{zf'(z) + g(z)}{zf'(z) - (1-2\beta)g(z)} \right| < \gamma, \quad z \in U^*. \tag{2.3}$$

It follows from (2.3) that

$$\mathrm{Re} \left\{ \frac{\sum_{n=1}^{\infty} [na_n + b_n] z^{n+1}}{2(1-\beta) - \sum_{n=1}^{\infty} [na_n - (1-2\beta) b_n] z^{n+1}} \right\} < \gamma, \quad z \in U^*. \tag{2.4}$$

Choose the values of z on the real axis so that $zf'(z)/g(z)$ is real. Thus, upon clearing denominator in (2.4) and letting $z \to 1$ through real values we have

$$\sum_{n=1}^{\infty} [na_n + b_n] \le \gamma[2(1-\beta) - \sum_{n=1}^{\infty} (na_n - (1-2\beta) b_n)] \tag{2.5}$$

or equivalently

$$\sum_{n=1}^{\infty} \{(1+\gamma) n a_n + (1-\gamma+2\beta\gamma) b_n\} \le 2\gamma(1-\beta). \tag{2.6}$$

Using Lemma 1, we have

$$\sum_{n=1}^{\infty}\left\{(1+\gamma)na_n+\frac{(1-\alpha)(1-\gamma+2\beta\gamma)}{(n+\alpha)}\right\}\leq 2\gamma(1-\beta). \tag{2.7}$$

□

Corollary. *Let the function $f(z)$ defined by (1.2) be in the class $\Sigma_s^*(\alpha,\beta,\gamma)$. Then*

$$a_n\leq\frac{2\gamma(n+\alpha)(1-\beta)-(1-\alpha)(1-\gamma+2\gamma\beta)}{n(n+\alpha)(1+\gamma)},\quad n\geq 1. \tag{2.8}$$

The result (2.7) is sharp for the function

$$f(z)=z+\frac{2\gamma(n+\alpha)(1-\beta)-(1-\alpha)(1-\gamma+2\gamma\beta)}{n(n+\alpha)(1+\gamma)}z^n,\quad n\geq 1, \tag{2.9}$$

with respect to

$$g(z)=z+\frac{(1-\alpha)}{(n+\alpha)}z^n,\quad n\geq 1. \tag{2.10}$$

Theorem 2. *Let the function $f(z)$ defined by (1.2) be in the class $\Sigma_s^*(\alpha,\beta,\gamma)$. Then*

$$\frac{1}{|z|}-|z|\frac{(1-\alpha)(1+\gamma)+2\gamma(1-\beta)}{(1+\alpha)(1+\gamma)}\leq|f(z)|\leq$$

$$\leq\frac{1}{|z|}+|z|\frac{(1-\alpha)(1+\gamma)+2\gamma(1-\beta)}{(1+\alpha)(1+\gamma)},\quad z\in U^*.$$

The result is sharp for the function

$$f(z)=\frac{1}{z}+\frac{(1-\alpha)(1+\gamma)+2\gamma(1-\beta)}{(1+\alpha)(1+\gamma)}z$$

with respect to

$$g(z)=z+\frac{1-\alpha}{1+\alpha}z.$$

Proof. Since $f\in\Sigma_s^*(\alpha,\beta,\gamma)$, we have

$$(1+\gamma)\sum_{n=1}^{\infty}a_n+(1-\gamma+2\beta\gamma)\sum_{n=1}^{\infty}b_n\leq 2\gamma(1-\beta). \tag{2.11}$$

For $g\in\Sigma^*(\alpha)$, Lemma 1 yields

$$\sum_{n=1}^{\infty}b_n\leq\frac{1-\alpha}{1+\alpha},$$

so that (2.11) reduce to

$$\sum_{n=1}^{\infty}a_n\leq\frac{(1-\alpha)(1+\gamma)+2\gamma(1-\beta)}{(1+\alpha)(1+\gamma)}.$$

Hence

$$|f(z)| \geq \frac{1}{|z|} - \sum_{n=1}^{\infty} a_n |z|^n \geq \frac{1}{|z|} - |z| \sum_{n=1}^{\infty} a_n \geq$$

$$\geq \frac{1}{|z|} - |z| \frac{(1-\alpha)(1+\gamma)+2\gamma(1-\beta)}{(1+\alpha)(1+\gamma)},$$

and

$$|f(z)| \leq \frac{1}{|z|} + \sum_{n=1}^{\infty} a_n |z|^n \leq \frac{1}{|z|} + |z| \sum_{n=1}^{\infty} a_n \leq$$

$$\leq \frac{1}{|z|} + |z| \frac{(1-\alpha)(1+\gamma)+2\gamma(1-\beta)}{(1+\alpha)(1+\gamma)}.$$

Theorem 3. *Let the function $f(z)$ defined by (1.2) be in the class $\Sigma_s^*(\alpha,\beta,\gamma)$. The $f(z)$ is convex in the disk*

$$0 < |z| < r = r(\alpha,\beta,\gamma),$$

where

$$r(\alpha,\beta,\gamma) = \inf_n \left\{ \frac{(1+\alpha)(1+\gamma)}{n(1-\alpha)(1+\gamma)+2\gamma(1-\beta)} \right\}^{1/n+1}, \quad n \geq 1.$$

The result is sharp.

Proof. We try to show that

$$\left| \frac{zf''(z)/f'(z)+2}{zf''(z)/f'(z)} \right| < 1, \quad 0 < |z| < r(\alpha,\beta,\gamma).$$

Noting that

$$\left| \frac{zf''(z)/f'(z)+2}{zf''(z)/f'(z)} \right| = \left| \frac{\sum_{n=1}^{\infty} n(n+1)a_n z^{n+1}}{2 - \sum_{n=1}^{\infty} n(n-1)a_n z^{n+1}} \right| \leq$$

$$\leq \frac{\sum_{n=1}^{\infty} n(n+1)a_n |z|^{n+1}}{2 - \sum_{n=1}^{\infty} n(n-1)a_n |z|^{n+1}}. \tag{2.12}$$

The expression (2.12) is bounded above by 1 whenever

$$\sum_{n=1}^{\infty} n^2 a_n |z|^{n+1} \leq 1. \tag{2.13}$$

In view of Theorem 1, we have

$$\sum_{n=1}^{\infty} n a_n \leq \frac{(1+\alpha)(1+\gamma)}{(1-\alpha)(1+\gamma)+2\gamma(1-\beta)}. \tag{2.14}$$

With the aid of (2.14), (2.13) is true if

$$n|z|^{n+1} \le \frac{(1+\alpha)(1+\gamma)}{(1-\alpha)(1+\gamma)+2\gamma(1-\beta)}. \tag{2.15}$$

Hence we have the desired result as

$$r(\alpha,\beta,\gamma) = \inf_n \left\{ \frac{(1+\alpha)(1+\gamma)}{n(1-\alpha)(1+\gamma)+2\gamma(1-\beta)} \right\}^{1/n+1}, \quad n \ge 1.$$

The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{(1-\alpha)(1+\gamma)+2\gamma(1-\beta)}{(1+\alpha)(1+\gamma)} z^n, \quad n \ge 1.$$

□

References

[1] M.R. Aouf, *A certain subclass of meromorphically starlike functions with negative coefficients*, Rend. Mat., 9(1989), 225-235.

[2] V.P. Gupta, *Convex class of starlike functions*, Yokohama Math. J., (32), (1984), 55-59.

[3] H.M. Srivastava and S. Owa, *Certain subclasses of starlike functions* I, J. Math. Anal Appl., 2(1), (1991), 405-415.

[4] B.A. Uralegaddi and M.D. Ganigi, *A certain class of meromorphically starlike functions with positive coefficients*, Pure Appl. Math. Sci., XXVI, 1-2(1987), 75-81.

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LAGRANGE-JACOBI RELATION FOR PARTICLE SYSTEMS WITH QUASIHOMOGENEOUS POTENTIALS

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Abstract. One considers the n-body problem in the case of quasihomogeneous force fields. One obtains a relation between the moment of inertia, the force function and the energy constant, the analogous of the Lagrange-Jacobi relation in the Newtonian case. The relation obtained is particularized for diffrent force fields.

1. Basic formulae

Consider n interacting particles of masses $m_i > 0$ $(i = \overline{1,n})$ in the Euclidean space $\mathbb{R}^3$, having the coordinates $q_i = (x_i, y_i, z_i)$ in an absolute reference frame. Let $q = (q_1, q_2, \ldots, q_n) \in \mathbb{R}^{3n}$ be the configuration of the system of particles, and let the force field be defined by the quasihomogeneous potential function $W = U + V$, where (cf.[3])

$$U : \mathbb{R}^{3n} \setminus \Delta \to \mathbb{R}_+, \quad U(q) = \sum_{1 \le i < j \le n} A(m_i, m_j) q_{ij}^{-\alpha}, \tag{1}$$

$$V : \mathbb{R}^{3n} \setminus \Delta \to \mathbb{R}_+, \quad V(q) = \sum_{1 \le i < j \le n} B(m_i, m_j) q_{ij}^{-\beta}, \tag{2}$$

are homogeneous functions of degree $-\alpha$ and $-\beta$, respectively $(1 \le \alpha \le \beta)$, $q_{ij} = |q_i - q_j|$ is the Euclidean distance between particles i and j, Δ stands for the collision set

$$\Delta = \bigcup_{1 \le i < j \le n} \{q |\ q_i = q_j\},$$

while A and B are symmetric positive functions of masses, that is, $A(m_i, m_j) = A(m_j, m_i) > 0$ and likewise for B.

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The equations of motion read

$$m_i\ddot{q}_i = \partial_i W(q), \quad i = \overline{1,n},$$

(where $\partial_i W$ is the i-th gradient of W), or

$$m_i\ddot{x}_i = \partial W/\partial x_i, \quad m_i\ddot{y}_i = \partial W/\partial y_i, \quad m_i\ddot{z}_i = \partial W/\partial z_i, \quad i = \overline{1,n}. \qquad ($$

It is known that along a solution we have

$$T(\dot{q}) - W(q) = h, \qquad ($$

which represents the integral of energy, where $T : \mathbb{R}^{3n} \to [0,\infty)$,

$$T(\dot{q}) = \sum_{i=1}^{n} m_i|\dot{q}_i|^2/2 = \sum_{i=1}^{n} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)/2, \qquad ($$

is the kinetic energy of the system, while $h \in \mathbb{R}$ is the constant of energy.

We also define the moment of inertia: $J : \mathbb{R}^{3n} \to [0,\infty)$,

$$J(q) = \sum_{i=1}^{n} m_i|q_i|^2/2 = \sum_{i=1}^{n} m_i(x_i^2 + y_i^2 + z_i^2)/2. \qquad ($$

2. Lagrange-Jacobi relation

Differentiating twice (6) with respect to time, we get successively

$$\dot{J}(\mathbf{q}) = \sum_{i=1}^{n} m_i(x_i\dot{x}_i + y_i\dot{y}_i + z_i\dot{z}_i),$$

$$\ddot{J}(\mathbf{q}) = \sum_{i=1}^{n} m_i(\dot{x}_i^2 + x_i\ddot{x}_i + \dot{y}_i^2 + y_i\ddot{y}_i + \dot{z}_i^2 + z_i\ddot{z}_i). \qquad ($$

Now, using (1) and (2), it is easy to verify that

$$\sum_{i=1}^{n}\left(x_i\frac{\partial W}{\partial x_i} + y_i\frac{\partial W}{\partial y_i} + z_i\frac{\partial W}{\partial z_i}\right) = -\alpha U(\mathbf{q}) - \beta V(\mathbf{q}),$$

or, taking intto account (3),

$$\sum_{i=1}^{n} m_i(x_i\ddot{x}_i + y_i\ddot{y}_i + z_i\ddot{z}_i) = -\alpha U(\mathbf{q}) - \beta V(\mathbf{q}). \qquad ($$

By (4) and (5) it results

$$\sum_{i=1}^{n} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) = 2U(\mathbf{q}) + 2V(\mathbf{q}) + 2h. \qquad ($$

Finally, adding together (8) and (9), and replacing the resulting expression in (7), we get

$$\ddot{J}(\mathbf{q}) = (2-\alpha)U(\mathbf{q}) + (2-\beta)V(\mathbf{q}) + 2h. \tag{10}$$

This constitutes the analogous of the Lagrange-Jacobi relation for the case of quasihomogeneous potentials.

3. Particular cases

Suppose that

$$\alpha = \beta = 1, \quad A(m_i, m_j) = B(m_i, m_j) = Gm_im_j/2,$$

where G denotes the Newtonian gravitational constant. In this case W is just the Newtonian potential function, and (10) becomes the classical Lagrange-Jacobi relation

$$\ddot{J}(q) = W(q) + 2h.$$

Suppose now that

$$\alpha = \beta > 1, \quad A(m_i, m_j) = B(m_i, m_j) = Gm_im_j/2.$$

This is the generalized Newtonian attraction law (see [2]), for which (10) becomes

$$\ddot{J}(\mathbf{q}) = (2-\alpha)W(\mathbf{q}) + 2h.$$

(The interesting properties of the case $\alpha = 2$ were pointed out in [2].)

Lastly, consider that

$$\alpha = 1, \quad \beta = 1, \quad A(m_i, m_j) = Gm_im_j,$$

$$B(m_i, m_j) = 3G^2m_im_j(m_i + m + j)/(2c^2),$$

where c is the speed of light. This is Maneff's field (see e.g. [1,4]), for which (10) acquires the form

$$\ddot{J}(\mathbf{q}) = U(\mathbf{q}) + 2h.$$

Observe that in Maneff's field case the Lagrange-Jacobi relation has exactly the same form as in the classical Newtonian field. The expression of the energy constant h is however different (cf. e.g. [5]).

References

[1] Delgado, J., Diacu, F.N., Lacomba, E.A., Mingarelli, A., Mioc, V., Perez, E., Stoica, C., *The Global Flow of the Manev Problem*, J. Math. Phys., 37(1996) (to appear).

[2] Diacu, F.N., *Total Collapse Dynamics for Particle Systems*, Libertas Math., 10(1990), 161-170.

[3] Diacu, F.N., *Near-Collision Dynamics for Particle Systems with Quasihomogeneous Potentials*, J. Diff. Eq. (to appear).

[4] Diacu, F.N., Mingarelli, A., Mioc, V., Stoica, C., *The Manev Two-Body Problem: Quantitative and Qualitative Theory*, in R.P. Agarwal (ed.), *Dynamical Systems and Applications*, World Scientific Series in Applicable Analysis, Vol.4, World Scientific Publ. Co., Singapore, 1995, 213-227.

[5] Stoica, C., Mioc, V., *Radial Motion in Maneff's Field*, Studia Univ. Babeş-Bolyai, ser. Mathematica, 40(1995), No.4, 99-105.

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THE CONVERGENCE OF NUMERICAL DIFFERENTIATION FOR JACOBI ULTRASPHERICAL EVEN NODES

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Abstract. A theorem which states the convergence of numerical differentiation formulas (1) of interpolatory type, having the zeros of Jacobi ultraspherical polynomials $P_{2n}^{(\alpha)}$ as nodes, for all real continuously differentiable function defined on the interval [-1,1] and for each $\alpha > -1$, is established.

1. Preliminaries

Denote by $P_n^{(\alpha)}$, $n \in \mathbb{N}$, $\alpha > -1$, the Jacobi ultraspherical polynomials $P_n^{(\alpha,\alpha)}$, namely

$$P_n^{(\alpha)}(x) = \frac{(-1)^n}{2^n n!}(1-x^2)^{-\alpha}[(1-x^2)^{\alpha+n}]^{(n)}, \quad |x| < 1$$

and let $y_n^k = \cos\theta_n^k$, $1 \le k \le n$, $0 < \theta_n^1 < \theta_n^2 < \cdots < \theta_n^n < \pi$, be the zeros of the polynomial $P_n^{(\alpha)}$, $n \ge 1$.

Suppose that the strictly increasing sequence of natural numbers $(j_n)_{n\ge1}$, the "nodes" x_n^k, with $-1 \le x_n^1 < x_n^2 < \cdots < x_n^{j_n} \le 1$ and the real "coefficients" a_n^k, $n \ge 1$, $1 \le k \le j_n$, are given and let's consider the following numerical differentiation formulas:

$$f'(0) = D_n f + R_n f, \quad n \ge 1, \tag{1}$$

associated to the space C_1 of all real functions f which are continuous together with their first derivatives on the interval [-1,1], where

$$D_n f = \sum_{k=1}^{j_n} a_n^k f(x_n^k), \quad n \ge 1. \tag{2}$$

In what follows, we suppose that the formulas defined by (1) are of interpolatory type, that is the equality $f'(0) = D_n f$ holds for each polynomial whose degree doesn't

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exceed $j_n - 1$. In this case, the coefficients a_n^k can be computed by the formulas:

$$a_n^k = \begin{cases} -\frac{w_n'(0)x_n^k + w_n(0)}{(x_n^k)^2 w_n'(x_n^k)}, & \text{if } x_n^k \neq 0 \\ -\frac{w_n''(0)}{2w_n'(0)}, & \text{if } x_n^k = 0, \end{cases} \tag{3}$$

where $w_n(x)$ is a polynomial of degree j_n whose roots are x_n^k, $1 \leq k \leq j_n$.

For $j_n = 2n+1$ and $w_n(x) = x(1-x^2)P_{2n-2}^{(\alpha)}(x)$, it was shown, [3], that the numerical differentiation formulas (1) are convergent on the space C_1 for $\alpha \geq -\frac{1}{2}$, that is $D_n f \to f'(0)$ for all f in C_1, if $\alpha \geq -\frac{1}{2}$.

The aim of this paper is to prove the convergence of the numerical differentiation formulas (1) on the space C_1 for each $\alpha > -1$, if $j_n = 2n$ and $w_n(x) = P_{2n}^{(\alpha)}(x)$.

2. Expressing the functionals D_n for Jacobi nodes

Let $x_n^k = y_{2n}^{2n+1-k} = \cos\theta_{2n}^{2n+1-k}$, $1 \leq k \leq 2n$, $n \geq 1$. Since $P_{2n}^{(\alpha)}$ is an even polynomial, it is easy to see that

$$0 < \theta_{2n}^1 < \cdots < \theta_{2n}^n < \frac{\pi}{2} < \theta_{2n}^{n+1} < \cdots < \theta_{2n}^{2n} < \pi,$$

$$-1 < x_n^1 < \cdots < x_n^n < 0 < x_n^{n+1} < \cdots < x_n^{2n} < 1,$$

and

$$x_n^{n-k+1} = -x_n^{n+k}, \quad 1 \leq k \leq n. \tag{4}$$

Moreover, the same reasons imply the following formulas for a_n^k of (3), taking $w_n(x) = P_{2n}^{(\alpha)}(x)$:

$$\begin{cases} a_n^k = -\frac{P_{2n}^{(\alpha)}(0)}{(x_n^k)^2 (P_{2n}^{(\alpha)})'(x_n^k)}, & 1 \leq k \leq 2n \\ a_n^{n-k+1} = -a_n^{n+k}, & 1 \leq k \leq n. \end{cases} \tag{5}$$

Now the expression of $D_n f$ becomes, according to (2), (4) and (5):

$$D_n f = \sum_{k=1}^{2n} a_n^k f(x_n^k) = \sum_{k=1}^{n} a_n^k f(x_n^k) + \sum_{k=1}^{n} a_n^{n+k} f(x_n^{n+k})$$

or

$$D_n f = \sum_{k=1}^{n} a_n^{n+k} [f(x_n^{n+k}) - f(-x_n^{n+k})].$$

For each polynomial P of degree at most $(2n-1)$, we obtain:

$$D_n f - f'(0) = D_n(f - P) + P'(0) - f'(0)$$

or

$$|D_n f - f'(0)| \leq 2\left(\sum_{k=1}^{n} |a_n^{n+k}|\right) \|f - P\| + \|f' - P'\|, \tag{6}$$

where $\|g\|$ denotes the uniform norm of a continuous function $g : [-1,1] \to \mathbb{R}$.

3. Estimates for the coefficients a_n^{n+k}, $1 \leq k \leq n$

In what follows, given the sequences of real numbers (u_n) and (t_n), we shall write $u_n \sim t_n$ if two real numbers A and B which don't depend on n exist so that $t_n \neq 0$ and $0 < A \leq \left|\frac{u_n}{t_n}\right| \leq B$, for all $n \geq 1$.

By (3), with $w_n(x) = P_{2n}^{(\alpha)}(x)$, we obtain:

$$a_n^{n+k} = -\frac{P_{2n}^{(\alpha)}(0)}{(x_n^{n+k})^2 (P_{2n}^{(\alpha)})'(x_n^{n+k})}, \quad 1 \leq k \leq n. \tag{7}$$

Since

$$P_{2n}^{(\alpha)}(0) = (-1)^n \frac{\Gamma(2n+\alpha+1)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+\alpha+1)(2n)!\sqrt{\pi}}$$

and

$$|(P_n^{(\alpha,\beta)})'(\cos\theta_n^k)| \sim k^{-\alpha-3/2} n^{\alpha+2} \quad \text{for } 0 < \theta_n^k \leq \frac{\pi}{2},$$

see [4], [5], we deduce, using also the relation $\Gamma(n+\alpha+1) \sim n! n^{\alpha}$, $\alpha > -1$:

$$|P_{2n}^{(\alpha)}(0)| \sim \frac{1}{\sqrt{n}} \tag{8}$$

and

$$|(P_{2n}^{(\alpha)})'(x_n^{n+k})| \sim \frac{n^{\alpha+2}}{(n-k+1)^{\alpha+3/2}}, \quad 1 \leq k \leq n. \tag{9}$$

By (7), (8) and (9) we derive:

$$|a_n^{n+k}| \sim \frac{1}{(x_n^{n+k})^2} \left(\frac{n-k+1}{n}\right)^{\alpha+1} \frac{\sqrt{n(n-k+1)}}{n^2}, \quad 1 \leq k \leq n. \tag{10}$$

4. Estimates for the roots x_n^{n+k}, $1 \leq k \leq n$

Since $\theta_{2n}^k = \frac{k\pi}{2n} + \frac{O(1)}{2n}$, $1 \leq k \leq 2n$, see [4], [5], it follows that a real $\lambda > 0$ which doesn't depend on n exists so that $\theta_{2n}^k = \pi \frac{k+u_n^k}{2n}$, where $|u_n^k| \leq \lambda$, $1 \leq k \leq 2n$, $n \geq 1$. This gives:

$$x_n^{n+k} = \cos\frac{k-1-u_n^{n-k+1}}{2n}\pi = \sin\frac{k+t_n^k}{2n}\pi, \quad 1 \leq k \leq n, \tag{11}$$

where

$$|t_n^k| \le 1 + \lambda, \quad 1 \le k \le n \tag{12}$$

(since $t_n^k = -1 - u_n^{n-k+1}$).

From $0 < \theta_{2n}^{n-k+1} < \frac{\pi}{2}$, $1 \le k \le n$, we obtain:

$$k + t_n^k > 0 \quad \text{and} \quad n - k - t_n^k > 0, \quad 1 \le k \le n. \tag{13}$$

Now, the usual inequalities $\frac{2}{\pi}x \le \sin x \le x$ for $0 \le x \le \frac{\pi}{2}$ and the relations (11 (12) and (13) give:

$$x_n^{n+k} \sim \frac{k + t_n^k}{n}, \quad 1 \le k \le n, \quad n \ge 1. \tag{14}$$

5. Evaluating the sum $\sum_{k=1}^{n} |a_n^{n+k}|$

In what follows we denote by M_j, $j \ge 1$, some positive constants which don depend on n.

Let $[\lambda]$ be the integer satisfying the inequalities $[\lambda] \le \lambda < [\lambda] + 1$ and suppo $n \ge s$, where $s = [\lambda] + 2$. Since $t_n^k \ge -\lambda - 1$, we deduce the inequality $t_n^k s \ge k(-\lambda - 1)$ f all natural k so that $s \le k \le n$. It follows by here that $k + t_n^k \ge \frac{s-1-\lambda}{s} k$ for $s \le k \le$ which implies the inequality

$$\sum_{k=s}^{n} \frac{1}{(k + t_n^k)^2} \le \frac{s^2}{(s - 1 - \lambda)^2} \sum_{k=s}^{n} \frac{1}{k^2} \le M_1.$$

Now, this inequality and the relations (10), (13) and (14) give:

$$|a_n^{n+k}| \le M_2 \frac{n^2}{(k + t_n^k)^2} \frac{\sqrt{n(n-k+1)}}{n^2} \le M_2 n \frac{1}{(k + t_n^k)^2}$$

and

$$\sum_{k=1}^{n} |a_n^{n+k}| \le M_2 n \left[\sum_{k=1}^{s-1} \frac{1}{(k + t_n^k)^2} + \sum_{k=s}^{n} \frac{1}{(k + t_n^k)^2} \right] \le M_2 n \left(\frac{s-1}{\delta^2} + M_1 \right),$$

where $\delta = \min\{k + t_n^k : 1 \le k \le s - 1\} > 0$. So, we obtain:

$$\sum_{k=1}^{n} |a_n^{n+k}| \le M_3 n. \tag{15}$$

Remark. It is easy to see that $k + t_n^k \le (\lambda + 2)k$ for all $k \ge 1$, so that we get:

$$\sum_{k=1}^{n} |a_n^{n+k}| \ge M_4 \sum_{k=1}^{n} \frac{n^2}{(k + t_n^k)^2} \left(\frac{n-k+1}{n}\right)^{\alpha+3/2} \frac{1}{n} \ge M_5 n,$$

which leads, together with (15), to the estimate

$$\sum_{k=1}^{n} |a_n^{n+k}| \sim n.$$

6. The convergence of the numerical differentiation formulas (1)

Now, we can prove the following statement:

Theorem. *The numerical differentiation formulas (1) of interpolatory type, having the zeros of Jacobi ultraspherical polynomials $P_{2n}^{(\alpha)}$ as nodes, are convergent on the space C_1 for each $\alpha > -1$ fixed, that is the equality $\lim_{n\to\infty} D_n f = f'(0)$ holds for all f in C_1.*

Proof. Let $E_n(f)$ be the degree of approximation of a continuous function $f : [-1,1] \to \mathbb{R}$ by algebraic polynomials of degree at most n. The inequality (6) implies:

$$|D_n f - f'(0)| \le 2\left(\sum_{k=1}^{n} |a_n^{n+k}|\right) E_{2n-1}(f) + E_{2n-2}(f'), \quad f \in C_1.$$

Taking into account the estimate (15), the last inequality becomes:

$$|D_n f - f'(0)| \le M_6 n E_{2n-1}(f) + E_{2n-2}(f'), \quad f \in C_1. \tag{16}$$

Denote by $\omega(f;\cdot)$ the modulus of continuity of a real continuous function f defined on the interval [-1,1]. It is known that:

$$\begin{cases} E_n(f) \le M_7 n^{-1} \omega\left(f'; \frac{1}{n}\right) \quad \text{and} \\ E_n(f') \le M_8 \omega\left(f'; \frac{1}{n}\right), \quad f \in C_1, \end{cases} \tag{17}$$

see [2], [5].

Now, from (16) and (17) we obtain:

$$|D_n f - f'(0)| \le M_9 \omega\left(f'; \frac{1}{n}\right),$$

which completes the proof. □

Remark. An equivalent proof can be based on Theorem 1 of [3].

References

[1] Lorenz, G.G., *Approximation of Functions*, Chelsea Publ. Comp., New-York, 1986.
[2] Lorenz, R.A., *Convergence of Numerical Differentiation*, J. Approx. Th., **30**(1980), 59-70.
[3] Mitrea, A.I., *On the Convergence of a Class of Numerical Differentiation Formulas*, Ana Num. Th. Approx. (to print).
[4] Szabados, J. and Vértesi, P., *Interpolation of Functions*, World Sci. Publ. Sci., Singapore New Jersey - London - Hong Kong, 1990.
[5] Szegö, G., *Orthogonal Polynomials*, Amer. Math. Soc., New-York, 1939.

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REAL STAR-CONVEX FUNCTIONS

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Abstract. This paper contains a survey of the properties of a class of real functions, which is intermediate between the class of convex functions and the class of starshaped functions. We present some known as well as new results or new proofs and examples.

1. Introduction

Let $\mathbb{R}$ be the real axis and let $I \subseteq \mathbb{R}$ be an interval (closed or not, bounded or not). A function $f : I \to \mathbb{R}$ is said to be *convex* on I if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y), \tag{1}$$

for all $x, y \in I$ and all $\lambda \in [0,1]$.
The function f is called *starshaped* on I if

$$f(\lambda x) \leq \lambda f(x), \tag{2}$$

for all $x \in I$ and all $\lambda \in [0,1]$. For $\lambda = 0$ we get $f(0) \leq 0$, which also implies $0 \in I$.

The aim of this survey paper is to analyse an intermediate concept, which connects the property of convexity with that of starshapedness by means of a parameter $\alpha \in [0,1]$. This concept was introduced in [9] and it was inspired by the notion of α-convexity defined for complex functions in [3]. We shall present here some results obtained in [1], [4], [7], [9] and [10] as well as some new results or new proofs.

2. α-Star-convex functions

We begin with the definition and some general properties of α-star-convex functions.

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Definition 1. [9] Given $\alpha \in [0,1]$, the function $f : I \to \mathbb{R}$ is said to be *α-star-convex* on I if

$$f(\lambda x + (1-\lambda)\alpha y) \le \lambda f(x) + (1-\lambda)\alpha f(y), \tag{3}$$

for all $x, y \in I$ and all $\lambda \in [0,1]$.

Remark 1. If $\alpha = 1$, then (3) reduces to (1), i.e. an 1-star-convex function is convex. If $\alpha = 0$, then (3) reduces to (2), i.e. a 0-star-convex function is starshaped. As in this last case, in [10] it was shown that it is natural to put the conditions

$$0 \in I \text{ and } f(0) \le 0. \tag{4}$$

In fact, taking $x = y = 0$ from (3) we get the second part of (4) but only for $\alpha \neq 1$. Remark that $y \in I$ implies $\alpha y \in I$ and so for $\alpha \in (0,1)$ we have $(0,y] \subseteq I$. This gives

Lemma 1. *If f is α-star-convex on I, $0 \in I$, then f is starshaped on I.*

Proof. For any $x \in I$ and $\lambda \in [0,1]$ we have

$$f(\lambda x) = f(\lambda x + (1-\lambda)\alpha \cdot 0) \le \lambda f(x) + (1-\lambda)\alpha f(0) \le \lambda f(x). \square$$

Theorem 1. *If f is α-star-convex on I, $0 \in I$ and $0 \le \beta \le \alpha$, then f is also β-star-convex.*

Proof. If $x, y \in I$ and $\lambda \in [0,1]$, then by using Lemma 1 we deduce

$$f(\lambda x + (1-\lambda)\beta y) = f(\lambda x + (1-\lambda)\alpha \frac{\beta y}{\alpha}) \le$$

$$\le \lambda f(x) + (1-\lambda)\alpha f(\frac{\beta}{\alpha} y) \le \lambda f(x) + (1-\lambda)\beta f(y). \square$$

Remark 2. A.W. Roberts and D.E. Varberg [6] defined the class of functions $f : I \to \mathbb{R}$ that satisfy the condition

$$f(sx + ty) \le sf(x) + tf(y)$$

for all $x, y \in I$ and all (s,t) in a given set M. Note that for example Jensen convexity corresponds to $M = \{(1/2, 1/2)\}$, subadditivity corresponds to $M = \{(1,1)\}$ and α-star-convexity is also of this type, with M given by the segment joining the points $A(1,0)$ and $B(0,\alpha)$.

Remark 3. The concept of α-star-convexity has the following geometric interpretation. If $y \in I$ is fixed and if we consider the point $M = M(\alpha y, \alpha f(y))$, then for all $x \in I$ the graph Γ_f of the function f on the interval $[x, \alpha y]$ or $[\alpha y, x]$ lies under the segment

MP, where $P = P(x, f(x))$. This means that Γ_f is starshaped with respect to the point M (see Figure 1).

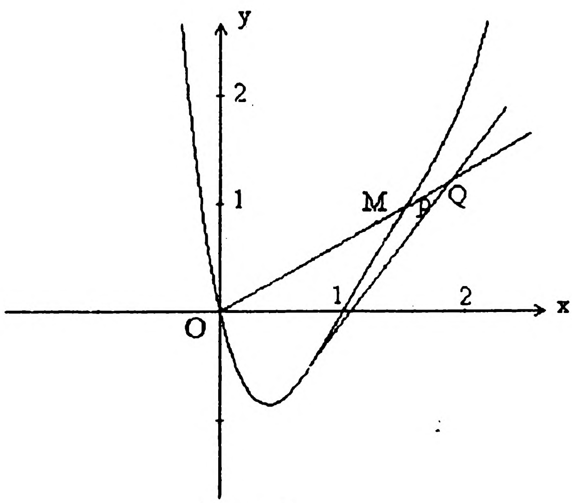

Figure 1

In view of Theorem 1, in [4] it was given the following definition.

Definition 2. Given a starshaped function $f : I \to \mathbf{R}$ we define the ***order of star-convexity*** of f by

$$\alpha = \alpha^*[f] = \sup\{\beta : f \text{ is } \beta\text{-star-convex on } I\}. \tag{5}$$

In this case we say that f is ***star-convex of order*** α.

Remark 4. The geometric interpretation mentioned in Remark 3 allows us to obtain the order of star-convexity of the function f given by (5) in the following way. Take a point $P \in \Gamma_f$ and starting from $O = O(0,0)$ let consider the point M on the segment OP at a longest distance from O with the property that the graph Γ_f is starshaped with respect to M. Then

$$\alpha = \alpha^*[f] = \inf\{\frac{OM}{OP} : P \in \Gamma_f\}. \tag{6}$$

Given $\alpha \in [0, 1]$ a natural problem is to find a function f such that $\alpha^*[f] = \alpha$. The answer to this problem is given by the following simple example [4].

Example 1. Let $\alpha \in (0,1]$ and let $f : \mathbb{R}_+ \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -x, & \text{if } 0 \le x \le 1; \\ x - 2, & \text{if } 1 \le x \le 2; \\ \frac{\alpha}{2-\alpha}(x-2), & \text{if } 2 \le x \le \frac{2+\alpha}{\alpha}. \\ 1 + a\left[x - \frac{2+\alpha}{\alpha}\right], & \text{if } \frac{2+\alpha}{\alpha} \le x,\ a > 1. \end{cases}$$

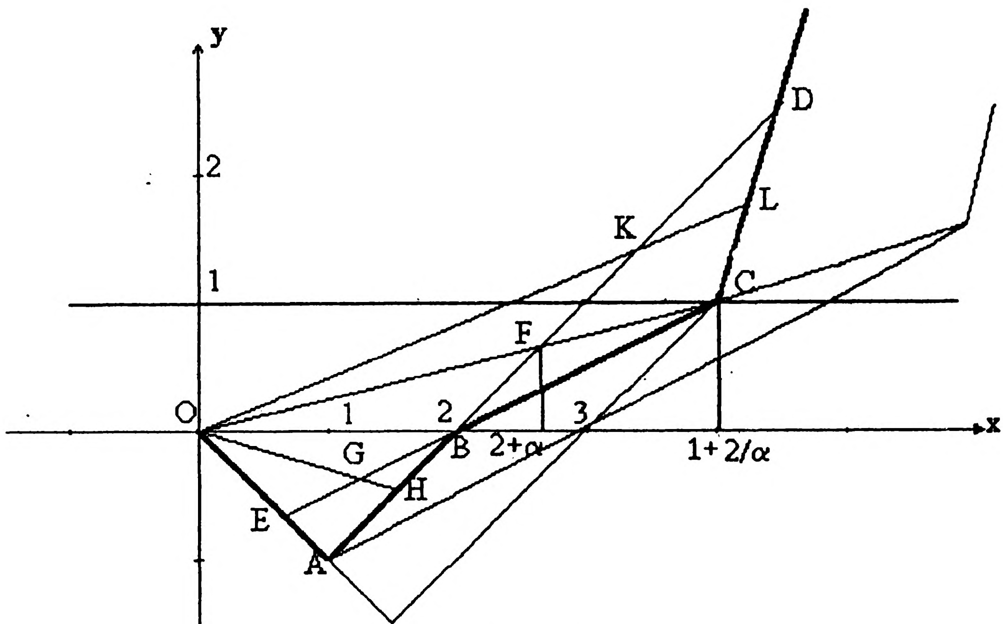

Figure 2

If $\alpha = 0$, then we take $f(x) = 0$, for $x \ge 2$. The graph Γ_f is given in Figure 2.

By using (5) and some elementary geometric considerations we easily find that $\alpha^*[f] = \alpha$. In Figure 2 we have $OE/OA = OF/OC = \alpha$ and $OK/OL > OF/OC$, OG/OH OE/OA.

Remark 5. If $\alpha > 1$ the only functions with $f(0) = 0$ which are α-star-convex are of the form $f_0(x) = ax$ and in this case $\alpha^*[f_0] = \infty$. Hence the significant range for α in Definition 1 is the interval [0, 1].

Remark 6. As in the case of convex functions, in [1] the following inequality of Jensen type is given: If $f : I \to \mathbb{R}$ is an α-star-convex function with condition (4) then for all

$p_i \geq 0$, with $\sum_{i=0}^{n} p_i = 1$ and all $x_i \in I$, $i = 0, 1, ..., n$, we have

$$f(p_0 x_0 + \alpha p_1 x_1 + ... + \alpha^n p_n x_n) \leq p_0 f(x_0) + \alpha p_1 f(x_1) + ... + \alpha^n p_n f(x_n).$$

3. The boundedness of star-convex functions

It is known that a convex function is bounded on every compact interval but a starshaped function is not. Let us study the boundedness of α-star-convex functions.

Lemma 2. *If the function f is starshaped on $[0, b]$, then it is bounded from above by* $M = \max\{0, f(b)\}$.

Proof. For every $x \in [0, b]$, there is a $t \in [0, 1]$ such that $x = tb$. So we have

$$f(x) \leq t f(b) \leq M. \quad \square$$

Analogously we can prove the boundedness from above on $[a, 0]$ and thus to deduce

Theorem 2. *If the function f is α-star-convex on I,, with $\alpha \in [0, 1]$, then it is bounded from above on every closed interval of I.*

It is easy to find examples of starshaped functions which are not bounded from below, but for α strictly positive we have the following result.

Theorem 3. *If the function $f : I \to \mathbb{R}$ is α-star-convex, with $\alpha \in (0, 1]$, then it is also bounded from below on every closed interval $[a, b] \subseteq I$.*

Proof. We have

$$f\left(\frac{a + \alpha b}{2}\right) = f\left(\frac{1}{2}(a + \alpha t) + \frac{1}{2}\alpha(b - t)\right) \leq$$

$$\leq \frac{1}{2} f(a + \alpha t) + \frac{1}{2}\alpha f(b - t).$$

If $t \in [0, b - a]$, we have $a + \alpha t \in [a, a + \alpha(b - a)] \subseteq [a, b]$, so that if we denote by M the upper bound of f on $[a, b]$, we get

$$f(b - t) \geq \frac{2}{\alpha}\left[f\left(\frac{a + \alpha b}{2}\right) - \frac{1}{2} f(a + \alpha t)\right] \geq$$

$$\geq \frac{2}{\alpha}\left[f\left(\frac{a + \alpha b}{2}\right) - \frac{M}{2}\right] = m,$$

hence m is a lower bound of f on $[a, b]$. $\square$

4. The Lipschitz continuity of α-star-convex functions

It is easy to observe that a function $f:[a,b]\to\mathbb{R}$, with $0\in[a,b]$ is starshaped on $[a,b]$ if and only if f can be written in the form

$$f(x)=\begin{cases} xg_+(x), & \text{if } x\in(0,b];\\ f(0), & \text{if } x=0;\\ xg_-(x), & \text{if } x\in[a,0),\end{cases}$$

where $f(0)\le 0$, $g_+:(0,b]\to\mathbb{R}$ and $g_-:[a,0)\to\mathbb{R}$ are increasing functions on $(0,b]$ and $[a,0)$ respectively. From this representation it immediately follows that f has at most a countable number of discontinuity of the first kind. Moreover, the point $x=0$ can be a discontinuity point of the second kind. We shall show that if α is strictly positive an α-star-convex function is Lipschitz on certain interval.

Theorem 4. *Let $\alpha\in(0,1]$ and let $a<b$ with $0\in[a,b]$. If the function $f:[a,b]\to\mathbb{R}$, is α-star-convex on $[a,b]$, then f is Lipschitz continuous on each compact interval $K=[a_1,a_2]\subseteq(\alpha a,\alpha b)$, where $a_1<a_2$.*

Proof. Since $K\subseteq(\alpha a,\alpha b)$, there exists $h>0$ such that $K_h=[a_1-\alpha h,a_2+\alpha h]\subseteq(\alpha a,\alpha b)$, and hence $K_h^1=[a_1/\alpha-h,a_2/\alpha+h]\subseteq(a,b)$. Let m_h be the greatest lower bound of f on K_h and let M_h be the least upper bound of f on K_h^1. From the definition of the least upper bound there is a sequence $(\varepsilon_n)_{n\ge1}$, with $\varepsilon_n\searrow 0$ and a corresponding sequence $(x_n)_{n\ge1}$, $x_n\in K_h^1$, such that $M_h-\varepsilon_n=f(x_n)$. Since $\alpha x_n\in K_h$ we have

$$M_h-\varepsilon_n=f(x_n)=f(\frac{1}{\alpha}\alpha x_n)\ge\frac{1}{\alpha}f(\alpha x_n)\ge\frac{1}{\alpha}m_h,$$

hence $\alpha M_h\ge m_h$.

Let denote by $\overline{f}'(x_0+0)$, $\overline{f}'(x_0-0)$, $\underline{f}'(x_0+0)$, and $\underline{f}'(x_0-0)$ the upper-right, upper-left, lower-right and lower-left Dini derivatives at $x_0\in K$ respectively. If in (3) we let $x=x_0$, $y=x_0/\alpha+h$ and devide by $(1-\lambda)\alpha h$, we deduce

$$\lambda\frac{f(x_0+(1-\lambda)\alpha h)-f(x_0)}{(1-\lambda)\alpha h}\le$$

$$\le\frac{\alpha f(x_0/\alpha+h)-f(x_0+(1-\lambda)\alpha h)}{\alpha h}\le\frac{\alpha M_h-m_h}{\alpha h},$$

and by letting $\lambda\nearrow 1$ we obtain

$$\overline{f}'(x_0+0)\le\frac{\alpha M_h-m_h}{\alpha h},\ \forall x_0\in K.$$

Analogously, if in (3) we let $x = x_0$ and $y = x_0/\alpha - h$ we deduce

$$\underline{f}'(x_0 - 0) \geq \frac{m_h - \alpha M_h}{\alpha h}, \ \forall x_0 \in K.$$

If in (3) we let $x = x_0 - (1-\lambda)\alpha h$, $y = x_0/\alpha + \lambda h$ and devide by $(1-\lambda)\alpha h$, then we get

$$\lambda \frac{f(x_0) - f(x_0 - (1-\lambda)\alpha h)}{(1-\lambda)\alpha h} \leq$$

$$\leq \frac{\alpha f(x_0/\alpha + \lambda h) - f(x_0)}{\alpha h} \leq \frac{\alpha M_h - m_h}{\alpha h},$$

and by letting $\lambda \nearrow 1$, we deduce

$$\overline{f}'(x_0 - 0) \leq \frac{\alpha M_h - m_h}{\alpha h}, \ \forall x_0 \in K.$$

Analogously, if in (3) we let $x = x_0 + (1-\lambda)\alpha h$ and $y = x_0/\alpha - \lambda h$, we obtain

$$\underline{f}'(x_0 + 0) \geq \frac{m_h - \alpha M_h}{\alpha h}, \ \forall x_0 \in K.$$

Therefore we deduce that f satisfies the Lipschitz condition with the constant $(\alpha M_h - m_h)/(\alpha h)$ on $K \subseteq (\alpha a, \alpha b)$. □

Corollary 1 [7]. *If $f : [a,b] \to \mathbb{R}$, with $a < b$, $0 \in [a,b]$, is α-star-convex, where $\alpha \in (0,1]$, then f is continuous on $(\alpha a, \alpha b)$. In particular, if $f : \mathbb{R} \to \mathbb{R}$ is α-star-convex then f is continuous on $\mathbb{R}$, and Lipschitz continuous on each compact interval of $\mathbb{R}$.*

5. Other characterizations of α-star-convex functions

Suppose now that the function f has a right-hand derivative $f'(x+0)$ and a left-hand derivative $f'(x-0)$ at each point $x \in I$. If we let $u = \lambda x + (1-\lambda)\alpha y$, then from (1) we obtain

$$f(u) - f(x) \leq (1-\lambda)[\alpha f(y) - f(x)].$$

If $u > x$, i.e. $\alpha y > x$, then we have

$$\frac{f(u) - f(x)}{u - x} \leq \frac{\alpha f(y) - f(x)}{\alpha y - x},$$

and if we let $\lambda \nearrow 1$ we deduce

$$f'(x+0) \leq \frac{\alpha f(y) - f(x)}{\alpha y - x},$$

hence

$$f(y) \geq \frac{f(x)}{\alpha} + f'(x+0)(y - \frac{x}{\alpha}), \ \forall y > \frac{x}{\alpha}.$$

In a similar way we obtain

$$f(y) \geq \frac{f(x)}{\alpha} + f'(x-0)(y - \frac{x}{\alpha}),\ \forall y < \frac{x}{\alpha}.$$

The above results have the following geometric interpretation [4]: Take a point P $P(x, f(x)) \in \Gamma_f$ and consider the point Q on the ray OP such that $OP/OQ = 1$ (see Figure 1). Then the graph Γ_f lies above the reunion of the half lines

$$Y = \frac{f(x)}{\alpha} + f'(x+0)(X - \frac{x}{\alpha}),\ X > \frac{x}{\alpha},$$

and

$$Y = \frac{f(x)}{\alpha} + f'(x-0)(X - \frac{x}{\alpha}),\ X < \frac{x}{\alpha}.$$

(see Figure 2.)

From the geometric interpretation mentioned in Remark 3 we deduce the foll ing characterization of an α-star-convex function [9].

Theorem 5. *The function $f : I \to \mathbb{R}$, with condition (4) is α-star-convex on I if only if for all $y \in I$ the function $\varphi_y : I \setminus \{\alpha y\} \to \mathbb{R}$ defined by*

$$\varphi_y(x) = \frac{f(x) - \alpha f(y)}{x - \alpha y}$$

is increasing on each interval $\{x \in I : x < \alpha y\}$ and $\{x \in I : x > \alpha y\}$.

If we suppose that the function f is differentiable on I, then from Theore and (7) we deduce that f is α-star-convex on I if and only if for each $x, y \in I$ following inequality holds

$$f'(x)(x - \alpha y) - [f(x) - \alpha f(y)] \geq 0$$

or

$$xf'(x) - f(x) - \alpha[yf'(x) - f(y)] \geq 0.$$

Since an α-star-convex function is necessarily starshaped we have $xf'(x$ $f(x) \geq 0$. If $yf'(x) - f(y) \leq 0$ then (8) holds for all positive α. If we suppose $yf'(x) - f(y) > 0$, then from (8) we deduce

$$\alpha \leq \frac{xf'(x) - f(x)}{yf'(x) - f(y)} \equiv \Phi(x, y).$$

From this inequality we obtain in (5) of Definition 2 the following formula [4]:

$$\alpha^*[f] = \inf\{\frac{xf'(x) - f(x)}{yf'(x) - f(y)} : yf'(x) - f(y) > 0,\ x, y \in I\}.$$

If there exist $x_0, y_0 \in I$ such that $x_0 f'(x_0) = f(x_0)$ and $y_0 f'(x_0) - f(y_0) > 0$, then $\alpha^*[f] = 0$.

Suppose now that $xf'(x) - f(x) > 0$ for all $x \in I \setminus \{0\}$ (i.e. f is *strictly starshaped* on I) and that f is twice differentiable on I. Then the system

$$\frac{\partial \Phi}{\partial x} = 0, \ \frac{\partial \Phi}{\partial y} = 0$$

is equivalent to

$$f''(x) = 0, \ \ f'(x) = f'(y). \tag{10}$$

Hence in certain cases $\alpha^*[f]$ given by (9) can be obtained by solving the system (10).

Example 2. [4]. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$f(x) = x^4 - 5x^3 + 9x^2 - 5x.$$

If we let $g(x) = f(x)/x$, then $g'(x) = 3x^2 - 10x + 9 > 0$, hence f is strictly starshaped on $\mathbb{R}$. We also have

$$f'(x) = 4x^3 - 15x^2 + 18x - 5$$

and

$$f''(x) = 6(2x^2 - 5x + 3).$$

The equation $f''(x) = 0$ has the roots $x_1 = 1$ and $x_2 = 3/2$. For $x_1 = 1$ equation $f'(y) = f'(x_1)$ has the root $y_1 = 7/4$ and we have $\Phi(x_1, y_1) = 512/539 \approx 0.949\ldots$. For $x_2 = 3/2$ equation $f'(y) = f'(x_2)$ has the root $y_2 = 3/4$ and we have $\Phi(x_2, y_2) = 16/17 = 0.941\ldots$. Hence from (9) we deduce $\alpha^*[f] = 16/17$.

The graph of the function f is given in Figure 1.

6. Hermite - Hadamard inequalities

It is known that if f is convex on $[a, b]$ then the following Hermite -Hadamard inequalities

$$f(\frac{a+b}{2}) \le \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x \le \frac{f(a) + f(b)}{2} \tag{11}$$

hold. A variant of (11) for α-star-convex functions was given in [1]. We give here another one.

Theorem 6. *If the function f is α-star-convex on $[a,b]$, $a<b$ with $\alpha\in(0,1]$, then*

$$\frac{1}{\alpha b-a}\int_a^{\alpha b} f(x)\,\mathrm{d}x \le \frac{f(a)+\alpha f(b)}{2}. \tag{12}$$

Proof. Integrating

$$f(ta+\alpha(1-t)b)\le tf(a)+\alpha(1-t)f(b)$$

for $t\in[0,1]$ we get (12). □

Theorem 7. *If the function f is α-star-convex on $[a,b]$, $a<b$ with $\alpha\in(0,1]$ then*

$$f\left(\frac{a+\alpha b}{2}\right)\le\frac{1+\alpha}{2\alpha(b-a)}\int_a^{\frac{a+\alpha b}{1+\alpha}} f(x)\,\mathrm{d}x+\alpha\frac{1+\alpha}{2(b-a)}\int_{\frac{a+\alpha b}{1+\alpha}}^{b} f(x)\,\mathrm{d}x. \tag{13}$$

Proof. We have

$$f\left(\frac{a+\alpha b}{2}\right)=f\left[\frac{1}{2}(a+\alpha t)+\frac{1}{2}\alpha(b-t)\right]\le$$

$$\le\frac{1}{2}f(a+\alpha t)+\frac{1}{2}\alpha f(b-t)$$

and integrating for $t\in[0,(b-a)/(1+\alpha)]$ we get (13). □

Note that if we take $\alpha=1$ in (12) and (13) then we obtain (11).

7. Weighted arithmetic means

In [10] it was studied the problem of the conservation of α-star-convexity by a weighted arithmetic mean of the form

$$A_g[f](x)=\frac{1}{g(x)}\int_0^x g'(t)f(t)\,\mathrm{d}t. \tag{14}$$

Let us denote by $K_\alpha(b)$ the set of α-star-convex functions on $[0,b]$, such that $f(0)=0$. In [10] the following results were obtained.

Theorem 8. *If $A_g[f]\in K_\alpha(b)$ for all $f\in K_\alpha(b)$ then*

$$g(x)=kx^\gamma,$$

for some $k\ne 0$ and $\gamma>0$. In this case

$$A_g[f](x)=A_\gamma[f](x)=\frac{\gamma}{x^\gamma}\int_0^x t^{\gamma-1}f(t)\,\mathrm{d}t=\int_0^1 f(xs^{1/\gamma})\,\mathrm{d}s.$$

If we denote by $M^\gamma K_\alpha(b)$ the set of functions f with the property that $A_\gamma[f]\in K_\alpha(b)$, then we have

Theorem 9. *If $0 < \alpha < \beta < 1$ and $\gamma > 0$ then the following inclusions*

$$\begin{array}{ccccccc} K_1(b) & \subseteq & K_\beta(b) & \subseteq & K_\alpha(b) & \subseteq & K_0(b) \\ \cap & & \cap & & \cap & & \cap \\ M^\gamma K_1(b) & \subseteq & M^\gamma K_\beta(b) & \subseteq & M^\gamma K_\alpha(b) & \subseteq & M^\gamma K_0(b) \end{array}$$

hold.

In fact an α-star-convex function can be mapped onto a β-star-convex function with $\beta > \alpha$, as was shown in [4] by the following example, for $\gamma = 1$.

Example 3. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$f(x) = 5x^4 - 20x^3 + 27x^2 - 10x$$

and let

$$F(x) = \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t = x^4 - 5x^3 + 9x^2 - 5x,$$

which is the function given in Example 2. By using the system (10) we obtain $\alpha^*[f] = 0.302\ldots$, while $\alpha^*[F] = 16/17 = 0.941\ldots$.

8. Star-convexity and Bernstein polynomials

For a function $f : [0,1] \to \mathbb{R}$ let us denote by $B_n(f)$ the Bernstein polynomial of order n of f defined by

$$B_n(f)(x) = \sum_{k=0}^{n} C_n^k\, x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0,1].$$

A well known result of classical analysis (see D.D. Stancu [8] p. 264) asserts that if f is convex in $[0,1]$ then:

$$B_n(f)(x) \geq f(x), \quad \forall\, x \in [0,1]. \tag{15}$$

First we consider an example of a starshaped function not verifying the inequality (15).

Example 4. Let $f : [0,1] \to \mathbb{R}$, be given by

$$f(x) = \begin{cases} -x, & \text{if } x \in [0, 1/3] \\ 2x, & \text{if } x \in (1/3, 2/3] \\ 4x, & \text{if } x \in (2/3, 1]. \end{cases}$$

We have

$$B_2(f)(x) = 2x + 2x^2 < 4x = f(x), \ \forall\, x \in (2/3, 1).$$

But the function $x \to f(x)/x$ being increasing on $(0,1]$, f will be starshaped on $[0,1]$.

In the following lemma of independent interest we give a generalization of equality (15) for α-star-convex functions, $\alpha \in [0,1]$. Particularly, for $\alpha = 1$ one obtai again (15).

Lemma 3. *Given $\alpha \in [0,1]$, let us denote by S_n^α the real function defined on $[0,1]$ b*

$$S_n^\alpha(x) = \alpha^{nx}, \ \forall\, x \in [0,1].$$

If f is α-star-convex on $[0,1]$ then:

$$B_n(S_n^\alpha \cdot f)(x) \geq f(B_n(S_n^\alpha \cdot J)(x)), \ \forall\, x \in [0,1],$$

where J is the identity mapping on $[0,1]$.

Proof. If we let in the Jensen type inequality (mentioned in Remark 6)

$$p_k = C_n^k x^k (1-x)^{n-k}, \ x_k = k/n, \ k = 0,1,...,n,$$

x being fixed in $[0,1]$ one obtains

$$f\left(\sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} \alpha^{n\cdot k/n} \frac{k}{n}\right) \leq \sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} \alpha^{n\cdot k/n} f\left(\frac{k}{n}\right),$$

and this yields the conclusion. □

In [2] it was proved that if the starshaped function $f : [0,1] \to \mathbb{R}$, verifies properties $f(0) = 0$, $f(x) \geq 0$, $\forall x \in [0,1]$ and $f \in C[0,1]$, then $B_n(f)$ is starshap $B_n(f)(0) = 0$ and $B_n(f)(x) \geq 0$, $\forall x \in [0,1], n = 1,2,...$. The proof in [2] can extended with some minor changes to a little more general setting. So, if f is arbitrary starshaped function on [0, 1] then $B_n(f)$ is also starshaped on $[0,1]$, for $n \geq 1$.

Now, a natural problem is: Given an $\alpha \in [0,1]$ and a starshaped function f $[0,1]$ with $\alpha^*(f) = \alpha$, does it follows that $\alpha^*[B_n(f)] = \alpha$, $n = 1,2,...$? The answe negative.

Example 5. Let f be defined on $[0,1]$ with $f(0) = 0$ a function such that $\alpha^*[f] =$ $[0,1)$. Then: $B_1(f)(x) = f(1) \cdot x$, and so $B_1(f)$ is convex. On the other hand

$$B_2(f)(x) = 2f(\frac{1}{2})x + \left[\frac{f(1)}{1} - \frac{f(1/2)}{1/2}\right] x^2$$

and $B_2(f)$ is also a convex function on $[0,1]$. However, f is not a convex function.

Example 6. In this example the function $f : [0,1] \to \mathbb{R}$ is starshaped and $B_3(f)$ is convex. Letting

$$f(x) = 3x^4 - 10x^3 + 11x^2, \ \forall\, x \in [0,1],$$

we have that f is starshaped and because $f''(0) = 22 > 0$, $f''(1) = -2 < 0$ f is not convex on $[0,1]$. The third Bernstein polynomial

$$B_3(f)(x) = \frac{8}{3}x + \frac{20}{9}x^2 - \frac{8}{9}x^3$$

is starshaped but from $B_3(f)''(0) = 40/9 > 0$ and $B_3(f)''(1) = -8/9 < 0$, it follows the non-convexity of $B_3(f)$.

Let f be a continuous starshaped function on $[0,1]$. We will be interested to obtain informations on the order of star-convexity of $B_n(f)$, $n = 1, 2, \dots$, when we know the order of star-convexity of f. For a particular case one obtains effectively $\alpha^*[B_n(f)]$. A comparison of this order to $\alpha^*[f] \in [0,1]$ will be made. The study of the asymptotic behaviour of the sequence $(\alpha^*[B_n(f)])_{n\geq 1}$ is our main purpose in the sequel.

Lemma 4 *Suppose that for the continuous function f on $[0,1]$, $\alpha^*[f] \leq 1$. Then*

$$\overline{\lim_{n\to\infty}} \alpha^*[B_n(f)] \leq \alpha^*[f], \quad n = 1, 2, \dots .$$

Proof. If $\alpha^*[f] = 1$, then f is convex on $[0,1]$ and from a well known result [5], $B_n(f)$ is convex on $[0,1]$, for all $n \leq 1$. Since $\alpha^*[f] = 1$, it follows that there exists $n_0 \in \mathbb{N}$ such that degree$(B_n(f)) \geq 2$, $\forall n \geq n_0$. Then $\alpha^*[B_n(f)] = 1$, $\forall n \geq n_0$ and in this case Lemma is proved. Let now suppose that $\alpha^*[f] < 1$ and $\varepsilon > 0$ be given. Because f isn't $(\alpha^*[f] + \varepsilon)$-star-convex this means that there exist $\lambda_0, x_0, y_0 \in [0,1]$ such that

$$f(\lambda_0 x_0 + (1-\lambda_0)(\alpha^*[f]+\varepsilon)y_0) - \lambda_0 f(x_0) - (1-\lambda_0)(\alpha^*[f]+\varepsilon)f(y_0) = d > 0.$$

From the uniform convergence of $(B_n(f))_{n\geq 1}$ to f it follows that

$$B_n(f)(\lambda_0 x_0 + (1-\lambda_0)(\alpha^*[f]+\varepsilon)y_0) -$$

$$-\lambda_0 B_n(f)(x_0) - (1-\lambda_0)(\alpha^*[f]+\varepsilon)B_n(f)(y_0) \geq \frac{d}{2} > 0,$$

for all $n \geq n_0 \in \mathbb{N}$. This implies that $B_n(f)$ is not $(\alpha^*[f]+\varepsilon)$-star-convex, for $n \geq n_0$ and

$$\overline{\lim_{n\to\infty}} \alpha^*[B_n(f)] \leq \alpha^*[f] + \varepsilon, \ \forall \varepsilon > 0. \square$$

Remark 7. In particular it follows that if $\alpha^*[f] = 0$ then $\lim_{n\to\infty} \alpha^*[B_n(f)] = 0$.

Example 7. Let $f : [0,1] \to \mathbb{R}$, be given by $f(x) = -2x^3 + 5x^2 + 6x$. After some simple computations one obtains that $\alpha^*[f] = 27/28$. Moreover the infimum in formula (9) giving $\alpha^*[f]$ is attained for $x = 1$ and $y = 2/3$.

The Bernstein polynomials $B_n(f)$ are

$$B_n(f)(x) = \frac{6n^2+5n-2}{n^2}x + \frac{(n-1)(5n-6)}{n^2}x^2 - \frac{2(n-1)(n-2)}{n^2}x^3,$$

$n = 1, 2, \dots$. It follows that $B_n(f)$, $n = 1, 2, \dots$, is starshaped and

$$\alpha^*[B_n(f)] = \frac{27}{4} \cdot \frac{(n+2)(n-2)^2}{n^2(7n-18)}, \ \forall n \geq 6.$$

Moreover the sequence $(\alpha^*[B_n(f)])_{n\geq 6}$ is decreasing and $\lim_{n\to\infty} \alpha^*[B_n(f)] = 27/28 = \alpha^*[f]$. Also $\alpha^*[B_n(f)] > \alpha^*[f], \forall n \geq 1$ and the infimum in formula (9) giving $\alpha^*[B_n(f)]$ is attained for $x = 1$ and $y = 2n/(3n-6)$, $n \geq 6$. In this example we have that $\alpha^*[B_n(f)] \geq \alpha^*[f], \forall n \geq 1$. We expect that generally

$$\lim_{n\to\infty} \alpha^*[B_n(f)] \geq \alpha^*[f].$$

Proposition 1. *Let $\alpha \in [0,1]$ be fixed and let $(f_n)_{n\geq 1}$ be a sequence of real functions on $[0,1]$. Suppose that $\alpha^*[f_n] \geq \alpha, \forall n \geq 1$ and that $f_n(x) \to f(x)$, for any $x \in [0,1]$. Then $\alpha^*[f] \geq \alpha$.*

Proof. Indeed, for a given pair $(x,y) \in [0,1]^2$ making $n \to \infty$ in the inequality

$$f_n(\lambda x + (1-\lambda)\alpha y) \leq \lambda f_n(x) + (1-\lambda)\alpha f_n(y),$$

one obtains that $\alpha^*[f] \geq \alpha$. □

Example 8. Let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = (4/n)x(x-1), \forall x \in [0,1], n = 1, 2, \dots$. Then $\alpha^*[f_n] = 1$, $n = 1, 2, \dots$ and the sequence $(f_n)_{n\geq 1}$ converges uniformly to the null function g_0 on $[0,1]$. But $\alpha^*[g_0] = \infty$.

Lemma 5. a) *Let $f \in C^1[0,1]$ be a strictly starshaped function. If $f(0) < 0$ then:*

$$\lim_{n\to\infty} \alpha^*[B_n(f)] \geq \alpha^*[f].$$

b) *Let $f \in C^2[0,1]$ be a strictly starshaped function. If $f(0) = 0$ and $f''(0) \neq 0$, then:*

$$\lim_{n\to\infty} \alpha^*[B_n(f)] \geq \alpha^*[f].$$

Proof. a) Suppose that

$$\lim_{n\to\infty} \alpha^*[B_n(f)] = a < \alpha^*[f].$$

Let $\varepsilon > 0$ be small enough such that $a + 2\varepsilon < \alpha^*[f]$. Then there exists a sequence of indices $(n_k)_{k\geq 1}$ such that

$$\lim_{k\to\infty} \alpha^*[B_{n_k}(f)] = a,$$

and for $k \geq k_0 \in \mathbb{N}$ we have

$$\alpha^*[B_{n_k}(f)] \in (a-\varepsilon, a+\varepsilon). \tag{16}$$

This means that for $k \geq k_0$, $B_{n_k}(f)$ is not $(a+2\varepsilon)$-star-convex. There exist the sequences $(x_{n_k})_{k\geq 1}$, $(y_{n_k})_{k\geq 1}$ of reals in $[0,1]$ with the property

$$x_{n_k} B_{n_k}(f)'(x_{n_k}) - B_{n_k}(f)(x_{n_k}) - \tag{17}$$

$$-(a+2\varepsilon)(y_{n_k} B_{n_k}(f)'(x_{n_k}) - B_{n_k}(f)(y_{n_k})) < 0,$$

for all $k \geq k_0$. We can suppose that the sequences $(x_{n_k})_{k\geq 1}$, $(y_{n_k})_{k\geq 1}$ are convergent. Let $x = \lim_{k\to\infty} x_{n_k}$, $y = \lim_{k\to\infty} y_{n_k}$. Because $B_{n_k}(f) \rightrightarrows f$, $B_{n_k}(f)' \rightrightarrows f'$, from (16) we obtain

$$xf'(x) - f(x) - (a+2\varepsilon)(yf'(x) - f(y)) \leq 0. \tag{18}$$

On the other hand from (16) and (17) it follows

$$y_{n_k} B_{n_k}(f)'(x_{n_k}) - B_{n_k}(f)(y_{n_k}) \geq 0, \ \forall k \geq k_0.$$

Then $yf'(x) - f(y) \geq 0$. But, f being $\alpha^*[f]$-star-convex, from (18) we have

$$xf'(x) - f(x) - (a+2\varepsilon)(yf'(x) - f(y)) = 0 \tag{19}$$

and

$$xf'(x) - f(x) - \alpha^*[f](yf'(x) - f(y)) \geq 0.$$

From this and (19) we have:

$$(-\alpha^*[f] + a + 2\varepsilon)(yf'(x) - f(y)) \geq 0,$$

so

$$yf'(x) - f(y) = 0 \text{ and } xf'(x) - f(x) = 0, \tag{20}$$

which contradicts $f(0) < 0$ or the strict starshapedness of f.

b) Suppose now that $f \in C^2[0,1]$, $f(0)=0$, $f''(0) \neq 0$. One observe that $f''(0) > 0$. Indeed

$$f''(0) = \lim_{x \searrow 0} \frac{f(2x) - 2f(x) + f(0)}{x^2} = \lim_{x \searrow 0} \frac{2}{x}\left[\frac{f(2x)}{2x} - \frac{f(x)}{x}\right] \geq 0.$$

Suppose that $f''(0) = d > 0$. Using the same arguments as in the case a) and supposing that $\underline{\lim}_{n\to\infty} \alpha^*[B_n(f)] = a < \alpha^*[f]$, we have again (20) with $x = \lim_{k\to\infty} x_{n_k}$ and $y = \lim_{k\to\infty} y_{n_k}$.

Now, from strictly starshapedness of f it follows that (20) yields $x = 0$. But $f'(0) = \lim_{x\searrow 0} f(x)/x < f(z)/z$, $\forall z \in (0,1]$. This means that $y = 0$ and $x = y = 0$. From $f''(0) = d > 0$ and from the continuity of f'' it follows that f is strictly convex on a neighbourhood of 0. More precisely $f''(x) > d/2$, $\forall x \in [0,\delta]$ with $\delta > 0$ sufficiently small. From $B_{n_k}(f)'' \rightrightarrows f''$ it follows that for $k \geq k_1 \in \mathbb{N}$, $B_{n_k}(f)''(x) \geq d/4$, $\forall x \in [0,\delta]$. Then $B_{n_k}(f)$, is convex on $[0,\delta]$ for $k \geq k_1$. But for $k \geq k_2 \in \mathbb{N}$, $x_{n_k}, y_{n_k} \in [0,\delta]$ and (17) will be contradicted for all $k > k_3 = \max\{k_1, k_2\}$. □

Theorem 10. *If f verifies the conditions* a) *or* b) *in Lemma* 5 *then*

$$\lim_{n\to\infty} \alpha^*[B_n(f)] = \alpha^*[f].$$

References

[1] Dragomir, S.S., Toader, Gh., *Some inequalities for m-convex functions,* Studia Univ. Babeş-Bolyai, Mathematica, 38 (1993), 1, 21-28.

[2] Lupaş, L., *A property of the S.N. Bernstein operator,* Mathematica, 9 (32) (1967), 2, 299-301.

[3] Mocanu, P.T., *Une propriété de convexité generalisée dans la théorie de la représentation conforme,* Mathematica 11 (34) (1969), 127-133.

[4] Mocanu, P.T., *Star-convex functions of one real variable,* Lucrările Seminarului Didactica Matematicii, 11, (1995), 97-104.

[5] Popoviciu, T., *Sur l'approximation des fonctions convexes d'ordre supérieur,* Mathematica 10 (1935), 49-54.

[6] Roberts, A.W., Varberg, D.E., *Convex functions,* Academic Press, New York, 1973.

[7] Şerb, I., *The continuity of α-star-convex functions of a real variable,* Lucrările Seminarului Didactica Matematicii, 11, (1995), 105-108.

[8] Stancu, D.D., *Curs şi culegere de probleme de analiză numerică,* vol I., Cluj-Napoca 1977.

[9] Toader, Gh., *Some generalizations of the convexity,* Proceedings of the Colloquium on Approximation and Optimization, Cluj-Napoca, (1984), 329-338.

[10] Toader, Gh., *On a generalization of the convexity,* Mathematica 30 (53),(1988), 1, 83-87.

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CONNECTIONS IN VECTOR BUNDLES OF FINSLER TYPE

GHEORGHE MUNTEANU AND VASILE LAZĂR

Abstract. In paper [9] D. Opriş introduces the notion of Finsler vector bundle as being the associate bundle of a vector bundle $\pi : E \to M$. This notion generalizes the Finsler bundle notion studied by Akbar-Zadeh (and others) [1] and systematically by M. Matsumoto in his monography [7]. The present paper we shall consider also the bundle structure (when there exist) on base M of a Finsler bundle. In this case are studied the nonlinear connections. An interesting application is that of second order tangent bundle.

1. Vector bundles of Finsler type

Let $\pi : E \to M$ be a vector bundle by R^m fibre and the base manifold M, $\dim M = n$. Consider $p : TM \to M$ the tangent bundle of M and π^*TM the pull-back bundle TM in E,

$$\pi^*TM = \{(y, z) \in TM \times E |\ p(y) = \pi(z)\}. \tag{1.1}$$

We have the vector bundles morphism $\pi! : TE \to \pi^*TM$, $\pi!(X_u) = (u, \pi_u^T(X_u))$. A nonlinear connection on E is a left splitting in the exact sequence:

$$0 \longrightarrow Ve \xrightarrow{i} TE \xrightarrow{\pi^*} \pi^*TM \longrightarrow 0. \tag{1.2}$$

If $E = L(M) = (L, M, \pi_L, G(u))$ is the bundle of linear frames of a manifold M, then π^*TM is called the Finsler bundle and is denoted by $F(M)$ ([7]).

D. Opriş in [9] names the vector bundle by Finsler type, associated to one vector bundle $\pi : E \to M$, the submanifold F of $E \times E$ given by:

$$F = \{(y, z) \in E \times E |\ \pi(y) = \pi(z)\}. \tag{1.1'}$$

F has an induced vector bundle structure $\pi_1 : F = \pi^*E \to E$ with fibre R^m. Locally, if (U, φ) is a map in $x = (x_i) \in M$ and (U, ψ) is a vector chart in $u = (x^i, y^\alpha)$ from

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E_u, then $v = (x^i, y^\alpha, z^\alpha)$ will be the local coordinates in E_v. At the change $\tilde{x}^i = \tilde{x}^i(x^j)$ on M, there result the change of coordinates on F:

$$\tilde{x}^i = \tilde{x}^i(x^j), \quad \tilde{y}^\alpha = M^\alpha_\beta y^\beta, \quad \tilde{z}^\alpha = M^\alpha_\beta z^\beta, \tag{1.3}$$

where $(M^\alpha_\beta) : U \cap U' \to GL(m, R)$, $i, j = \overline{1, n}$, $\alpha, \beta = \overline{1, m}$.

Particularly, if $E = TM$ then $F = \pi^* TM$ and the matrix (M^α_β) from (1.3) is $\left(\frac{\partial \tilde{x}^i}{\partial x^k}\right)$. This is called the Finsler tangent bundle $T_F M$ ([2]). If E is changed by his dual bundle E^*, then $u = (x^i, p_\alpha)$ and the matrix (M^α_β) from (1.3) is replaced with its inverse ([6]).

The geometry of F bundle is nothing but that one of the vertical bundle VE of TE. Indeed, if $\chi_u = X^i \frac{\partial}{\partial x^i} + Y^\alpha \frac{\partial}{\partial y^\alpha}$ is a tangent vector at $T_u F$, then $V_u E$ is spanned by $\left\{Y^\alpha \frac{\partial}{\partial y^\alpha}\right\}$ so that VE has the local cooedinates $(x^i, y^\alpha, 0, Y^\alpha)$ that are locally isomorphically with $(x^i, y^\alpha, Y^\alpha)$ and the law of change is (1.3). On the other hand, it is known the canonic isomorphism $r : VE \to E$ and hence result the study of Finsler geometry based on the fibre bundle of Finsler type.

There exist also on F a bundle structure with base M and fibre R^{2m}. The mapping $\pi_2 = \pi \circ \pi_1 : F \to M$ is a surjection. If:

$$\psi_{U,x} : \pi^{-1}(U) \to U \times R^m \quad \text{and} \quad \phi_{\pi^{-1}(U),x} : \pi_2^{-1}(U) \to U \times R^m$$

then the mapping $\theta_{U,x} = \phi_{\pi^{-1}(U),x} \circ \psi_{U,x} \circ r_{U,x}$ is a bijection, but $\theta_{V,x} \circ \theta^{-1}_{U,x}$ isn not always compulsory linear. Therefore, generally this structure is not of the vector bundle. When that structure is one of the vector bundle $\pi_2 : F \to M$ we shall denoted it by F_M. Typical example of F_M structure are $E = TM$ or T^*M.

Taking into account that $\phi^{-1}\tilde{\phi}$ and $\psi^{-1}\tilde{\psi}$ are isomorphisms and $(\{x\} \times R^m) \times (\{x\} \times R^m) \to \{x\} \times R^{2m}$ is a local isomorphism there results the local ispmorphism $\mathcal{S} : F_M \to E \oplus E$.

For the vector bundle of Finsler type F we shall consider the distributions:

1. The *vertical distribution*: $\mathcal{V}_V = \{X_V \in T_V F / \pi_1^\perp(X_V) = 0\}$ spanned locally by $\left\{\frac{\partial}{\partial z^\alpha}\right\}$. The reunion of this distribution defined the vertical subbundle VF.

2. Let $p_2 : F \to E$ be the induced mapping, $p_2(y, z) = z$. The distribution $\mathcal{V}_V E = \{X_V \in T_V F / p_2^\perp(X_V) = 0\}$ is called the induced vertical distribution by E. A local base in $\mathcal{V}_V E$ is $\left\{\frac{\partial}{\partial y^\alpha}\right\}$. The reunion of this distributions define the induced vertical subbundle VFE.

3. The *quasivertical distribution*, $V_V QF = \{X_V \in T_V F / \ \pi^\perp \circ \pi_1^\perp X = 0\}$, locally spaned by $\left\{\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial z^\alpha}\right\}$. The reunion of these distributions defines the quasivertical subbundle VQE.

Obviously, $VQE = VF \oplus VFE$ and the following sequence is exact:

$$0 \longrightarrow VF \xrightarrow{i} VQE \xrightarrow{\pi_1} VFE \longrightarrow 0. \tag{1.4}$$

2. Connections in vector bundles of Finsler type

Let's $\pi_1 : F \to E$ be a vector bundle by Finsler type and TF its tangent bundle. A *nonlinear connection* $\mathcal{D}$ in F ([5], [8]) is a splitting on left in the exact sequence:

$$0 \longrightarrow VF \xrightarrow{i} TF \xrightarrow{\pi_1} \pi^* E \longrightarrow 0. \tag{2.1}$$

The connection mapping $D^F : TF \to F$ associated to the nonlinear connection $\mathcal{D}$ has the local form for $\chi_V = X^i \frac{\partial}{\partial x^i} + Y^\alpha \frac{\partial}{\partial y^\alpha} + Z^\alpha \frac{\partial}{\partial z^\alpha}$:

$$D^F(x, y, X, Y, Z) = (x, y, Z + \omega^1(x, y, z) + \omega^2(x, y, z)X), \tag{2.2}$$

where ω^1, ω^2 are locally characterized by the connection coefficients: $\omega^1(x, y, z)(e_i) = \Gamma_i^\alpha(x, y, z)e_\alpha$; $\omega^2(x, y, z)(e_\beta) = C_\beta^\alpha(x, y, z)e_\alpha$; $\alpha, \beta = \overline{1, m}$, $\{e_i\}, \{e_\alpha\}$ the canonic bases in R^n respectively R^m.

At the change (1.3) the coefficients of nonlinear connection are changed after the rules ([9]):

$$\tilde{\Gamma}_j^\alpha \frac{\partial \tilde{x}^j}{\partial x^i} = M_\beta^\alpha \Gamma^\beta - C_\beta^\alpha \frac{\partial M_\gamma^\beta}{\partial x^i} y^\gamma - \frac{\partial M_\beta^\alpha}{\partial x^i} z^\beta \tag{2.3}$$

$$\tilde{C}_\beta^\alpha M_\delta^\beta = C_\delta^\gamma M_\gamma^\alpha.$$

If $X = X^i \frac{\partial}{\partial x^i} + Y^\alpha \frac{\partial}{\partial y^\alpha} \in \chi(E)$ and $A = A^\alpha s_\alpha \in Sect(F)$, then the covariant derivative D_X^F has the local expression:

$$D_X^F A = \left\{ X^i \left(\frac{\partial A^\gamma}{\partial x^i} + \Gamma_i^\gamma \right) + Y^\alpha \left(\frac{\partial A^\gamma}{\partial y^\alpha} + C_\alpha^\gamma \right) \right\} s_\gamma. \tag{2.4}$$

Let suppose that D^F is a linear connection on F,

$$\Gamma_i^\alpha = \Gamma_{i\beta}^\alpha(x, y)z^\beta, \quad C_\alpha^\beta = C_{\alpha\gamma}^\beta(x, y)z^\gamma.$$

Through the isomorphism r to the linear connection D^F corresponding a linear connection $\overline{D}$ on VF, that admit a prolongation to TF which preserve the vertical distribution VF and one supplementary distribution HF determined by a nonlinear connection N on E. Denoting by D this prolongation. If $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^\alpha \frac{\partial}{\partial y^\alpha} = \delta_i$ is local base in HF, then we have: $D_{\delta_i}\partial_\alpha = L^\beta_{\alpha i}\partial_\beta$, where $L^\beta_{\alpha i} = \Gamma^\beta_{\alpha i} - C^\beta_{\alpha j}N^j_i$; $\partial_\beta = \frac{\partial}{\partial y^\beta}$ $D_{\partial_\alpha}\partial_\beta = C_\beta\gamma_\alpha\partial_\gamma$ and $D_{\delta_i}\delta_j = L^k_{ji}\delta_k$; $D_{\partial_\alpha}\delta_i = C^k_{i\alpha}\delta_k$ ($L^k_{ji}, C^k_{i\alpha}$ are notations). It obtained the notion of linear d-connection with the well-known rules of change ([8]).

From (2.3) results that every nonlinear connection $\Gamma^\alpha_i(x,y)$ on E induces a non linear connection on F with coefficients: $\overline{\Gamma}^\alpha_i = \Gamma^\alpha_i$ and $C^\alpha_\beta = 0$. If $\Gamma^\alpha_i(x,y)$ are the coefficients of a nonlinear connection on E, then the connection $\overset{\beta}{\Gamma}{}^\alpha_i = \frac{\partial\Gamma^\alpha_i}{\partial y^\beta}z^\beta$; $C^\alpha_\beta = 0$ linear on F and is called the Berwald connection.

Now, let's consider the F_M structure, when F is a vector bundle over M an R^{2m} fibre.

The following sequence of bundles are exact:

$$0 \longrightarrow VQF \xrightarrow{\ i\ } TF \xrightarrow{\pi_2'} \pi^*TM \longrightarrow 0. \tag{2.5}$$

A left splitting φ in the exact sequence (2.5) defines a *nonlinear connection* i bundle F_M. Having in mind that $VQF = VF \oplus VFE$, results that $\varphi = (\varphi_1, \varphi_2)$ where

$$\varphi : (x,y,z,X,Y,Z) \to (x,y,z,0,\varphi_1(x,y,z,X,Y,Z),\varphi_2(x,y,z,X,Y,Z)).$$

The condition $\varphi \circ i = Id_{VQF}$ implies $\varphi_1(x,y,z,0,Y,Z) = Y$, $\varphi_2(x,y,z,0,Y,Z) = Z$ and from linearity of φ_1 and φ_2 it results that $\varphi_1'(x,y,z,X,Y,Z) - Y$ an $\varphi_2'(x,y,z,X,Y,Z) - Z$ don't depends of Y and respectively Z and hence:

$$\varphi_1(x,y,z,X,Y,Z) = Y + \mathcal{M}(x,y,z)X; \quad \varphi_2(x,y,z,X,Y,Z) = Z + \mathcal{N}(x,y,z)X.$$

Then the connection mapping K^F associated to φ is locally written:

$$K^F(x^i,y^\alpha,z^\alpha,X^i,Y^\alpha,Z^\alpha) = (x^i; Y^\alpha + \mathcal{M}^\alpha_i(x,y,z)X^i; Z^\alpha + \mathcal{N}^\alpha_i(x,y,z)X^i). \tag{2.6}$$

At the change (1.3) the coefficients of nonlinear connection $\varphi = (\mathcal{M},\mathcal{N})$ ar changed after the rules:

$$\tilde{\mathcal{M}}^\alpha_j \frac{\partial\tilde{x}^j}{\partial x^i} = M^\alpha_\beta\mathcal{M}^\beta_i - \frac{\partial M^\alpha_\beta}{\partial x^i}y^\beta \tag{2.7}$$

$$\tilde{\mathcal{N}}^\alpha_j \frac{\partial\tilde{x}^j}{\partial x^i} = M^\alpha_\beta\mathcal{N}^\beta_i - \frac{\partial M^\alpha_\beta}{\partial x^i}z^\beta. \tag{2.8}$$

From (2.7) results that $\mathcal{M}(x,y,z)$ has the same rule as that of nonlinear connection on E.

We shall say that a nonlinear connection $\varphi = (\mathcal{M}, \mathcal{N})$ on F_M is a *nonlinear connection of Finsler type* if that provides from a nonlinear connection on E, e.i. $\mathcal{M}_i^\alpha(x, y, z) = \mathcal{M}_i^\alpha(x, y)$.

Proposition 2.1. *The projection $\pi_1^\perp(\varphi)$ of a nonlinear connection on F_M is a nonlinear connection on E.*

Conversely, let suppose that on E is given a nonlinear connection $\mathcal{M}_i^\alpha(x, y)$. Then the pair $(\mathcal{M}_i^\alpha, \mathcal{N}_i^\alpha)$, where:

$$\mathcal{N}_i^\alpha = \frac{\partial \mathcal{M}_i^\alpha}{\partial y^\beta} z^\beta \tag{2.9}$$

determines a nonlinear connection on F_M of Finsler type. There results that:

Theorem 2.2. *There exists a correspondent between the nonlinear connection on F and Finsler type connection on F_M.*

The nonlinear connection on F_M defines the following decomposition:

$$TF = \mathcal{N}F \oplus VWF = \mathcal{N}F \oplus VF \oplus VFE.$$

The subbundle $\mathcal{N}F$ will called the *normal subbundle* of F.

The normal lift of a vector field X on M is a vector field X^n on F_M so that $\pi_2^\perp(X^n) = X$. From $K^F(X^n) = 0$ we have that

$$\chi = X^i \frac{\partial}{\partial x^i} + Y^\alpha \frac{\partial}{\partial y^\alpha} + Z^\alpha \frac{\partial}{\partial z^\alpha}$$

is normal if $Y^\alpha = -\mathcal{M}_i^\alpha$ and $Z^\alpha = -\mathcal{N}_i^\alpha$.

The normal lift of the field $\frac{\partial}{\partial x^i}$ is:

$$\frac{aa}{aax^i} = \frac{\partial}{\partial x^i} - \mathcal{M}_i^\alpha \frac{\partial}{\partial y^\alpha} - \mathcal{N} \frac{\partial}{\partial z^\alpha}, \tag{2.10}$$

with:

$$-\frac{aa}{aa\tilde{x}^i} = \frac{\partial x^j}{\partial \tilde{x}^i} \frac{aa}{aax^j}.$$

We get the following local base: $\left\{\frac{aa}{aax^i}\right\}$, $\left\{\frac{\partial}{\partial y^\alpha}\right\}$, $\left\{\frac{\partial}{\partial z^\alpha}\right\}$ respectively in $\mathcal{N}_V F$, $\mathcal{V}_V F$, $\mathcal{V}_V FE$.

Let's remark that $\mathcal{M}_i^\alpha = -\frac{aa}{aax^i}$ and $\mathcal{N}_i^\alpha = -\frac{aaz^\alpha}{aax^i}$ and the relations (2.7), (2.8) are written equivalently:

$$\frac{aa\tilde{y}^\alpha}{aa\tilde{x}^j} \frac{\partial \tilde{x}^j}{\partial x^i} = M_\beta^\alpha \frac{aay^\beta}{aax^i} + \frac{\partial M_\beta^\alpha}{\partial x^i} y^\beta \tag{2.7'}$$

$$\frac{aa\tilde{z}^\alpha}{aa\tilde{x}^j}\frac{\partial \tilde{x}^j}{\partial x^i} = M^\alpha_\beta \frac{aaz^\beta}{aax^i} + \frac{\partial M^\alpha_\beta}{\partial x^i} y^\beta. \qquad (2$$

A *vertical connection* in F_M is a left splitting in the exact sequence:

$$0 \longrightarrow VF \xrightarrow{i} VQF \xrightarrow{\pi_1!} \pi_1^* VE \longrightarrow 0. \qquad (2.$$

A vertical connection is characterized by a system of functions $\mathcal{H}^\alpha_\beta$ with the

of transformation:

$$\tilde{H}^\alpha_\beta M^\beta_\delta = \mathcal{H}^\gamma_\delta M^\alpha_\gamma.$$

Proposition 2.3. *Every nonlinear connection on F induces a vertical connection vector bundle of Finsler type F_M.*

Proof. Let $\mathcal{M}^\alpha_i(x, y)$ be a nonlinear connection on F and $t^i(x, y)$ an arbitrary vector fi on F. Then $\mathcal{H}^\alpha_\beta = \frac{\partial^2(\mathcal{M}^\alpha_j)t^j}{\partial y^\beta \partial y^\gamma} z^\gamma$ are the coefficients of a vertical connection on F_M.

Of course if $t^j = 0$ then $\mathcal{H}$ is the zero vertical connection.

The vertical connection $\mathcal{H}$ determines the decomposition $VQF = HQF \oplus V$ The vector fields $\left\{\frac{\delta}{\delta y^\beta} = \frac{\partial}{\partial y^\beta} - \mathcal{H}^\alpha_\beta \frac{\partial}{\partial z^\alpha}\right\}$ is a local base in HQF called the *horizo subbundle.*

The covariant derivative $D^{F_M} : \chi(M) \times Sect(F_M) \to Sect(F_M)$ is given $(X, L) \to D^{F_M}_X L = K^F \circ L^T \circ X$, where K^F is the connection mapping associated to nonlinear connection $(\mathcal{M}, N)$. Let $\{s_\alpha\}$, $\alpha = \overline{1, m}$ be a base of local section on F. ' isomorphism $\mathcal{S} : F_M \to E \oplus E$ determines a base of local sections on F_M: $s^1_\alpha = (s_\alpha$ and $s^2_\alpha = (0, s_\alpha)$. If $L = A^\alpha s^1_\alpha + B^\alpha s^2_\alpha \in Sect(F_M)$ and $X = X^i \frac{\partial}{\partial x^i} \in \chi(M)$, then L the mapping:

$$L^T : (x^i, Y^i) \to \left(x^i, A^\alpha, B^\alpha, X^i, \frac{\partial A^\alpha}{\partial x^i} X^i, \frac{\partial B^\alpha}{\partial x^i} X^i\right).$$

It results that:

$$D^{F_M}_X L = X^i \left(\frac{\partial A^\alpha}{\partial x^i} + \mathcal{M}^\alpha_i\right) s^1_\alpha + X^i \left(\frac{\partial B^\alpha}{\partial x^i} + \mathcal{N}^\alpha_i\right) s^2_\alpha. \qquad (2$$

D^{F_M} satisfies the known properties of a derivative law.

Let's remark that if (D^1, D^2) is a pair of derivative laws on F, by the isomorph $\mathcal{S}$, $D^{F_M} : \chi(M) \times Sect(F) \equiv \chi(M) \times Sect(F) \times Sect(F) \to Sect(F_M)$ is given by:

$$D^{F_M} : (X, L) \to D^F_X L = \mathcal{S}^{-1}(D^1_X \mathcal{S}^1(L) \oplus D^2_X \mathcal{S}^2(L)), \qquad (2.$$

where $\mathcal{S} = (\mathcal{S}^1, \mathcal{S}^2)$. If $\mathcal{S}^1 D^F_X L = 0$ ($\mathcal{S}^2 D^F_X L = 0$), then we say that L is paralle respect to X by first order (second order).

Proposition 2.4. *If $(\mathcal{M}, N)$ is a nonlinear connection of Finsler type then $\pi_1^{\perp}(D_X^{F_M} L)$ is a derivative law on F.*

The nonlinear connection $(\mathcal{M}, N)$ is called linear of Finsler type if $\mathcal{M}_i^{\alpha}(x,y)$ is linear in y and $\mathcal{N}_i^{\alpha}(x,y,z)$ is bilinear in (y,z), i.e. $\mathcal{M}_i^{\alpha} = \Gamma_{i\beta}^{\alpha} y^{\beta}$ and $\mathcal{N}_i^{\alpha}(x,y,z) = \theta_{i\beta}^{\alpha}(x)(x,z)$.

Theorem 2.5. *Every linear connection on E induces a linear connection of Finsler type on F_M.*

Proof. Let $\Gamma_{i\beta}^{\alpha}(x)$ be the coefficients of linear connection on E. Then the connection:

$$\mathcal{M}_i^{\alpha} = \Gamma_{i\beta}^{\alpha}(x) y^{\beta}; \quad \mathcal{N}_i^{\alpha} = \Gamma_{i\beta}^{\alpha}(x) z^{\beta} \tag{2.14}$$

is linear by Finsler type on F_M. Moreover, $\mathcal{N}_i^{\alpha}(x,y,z) = \mathcal{N}_i^{\alpha}(x,0,z)$. □

3. The second order tangent bundle

The second order tangent bundle T^2M ([3], [4], [10],...) is the bundle locally characterized in $x \in M$ by a $3n$ system of coordinates (x^i, y^i, z^i) with the law of transformation:

$$\tilde{x}^i = \tilde{x}^i(x^j) \tag{3.1}$$

$$\tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j} y^j$$

$$2\tilde{z}^i = \frac{\partial^2 \tilde{x}^i}{\partial x^k \partial x^1} y^k y^1 + 2\frac{\partial \tilde{x}^i}{\partial x^k} z^k.$$

In [3] is emphasized the subbundle structure of T^2M in bundle $\mathcal{P} : TTM \to TM$, $T^2M = \{v \in TTM / \mathcal{P}_V = p^* v\}$.

Here the local coordinates are (x^i, y^i, y^i, z^i) induced by $u = (x^i, y^i)$ on TM, where $\chi = y^i \frac{\partial}{\partial x^i} + z^i \frac{\partial}{\partial y^i}$ is a tangent vector to TM. T^2M has no vector bundle structure over M.

Let assume that on M is given a nonlinear connection $N_i^j(x,y)$ and $\left\{\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}; \frac{\partial}{\partial y^i}\right\}$ is an adapted local base on TM.

Then

$$\chi_u = y^i \frac{\partial}{\partial x^i} + z^i \frac{\partial}{\partial y^i} = y^i \frac{\delta}{\delta x^i} + (z^i + y^j N_j^i) \frac{\partial}{\partial y^i}$$

Because $\frac{\delta}{\delta \tilde{x}^i} = \frac{\partial x^j}{\partial \tilde{x}^i} \cdot \frac{\delta}{\delta x^j}$ it results that a given nonlinear connection on M determines on T^2M a F_M vector bundle structure of Finsler type relative to the following local coordinates:

$$\bar{x}^i = x^i; \quad \bar{y}^i = y^i; \quad \bar{z}^i = z^i + N^i_j y^j \tag{3.2}$$

with the transformation rules deduced from (1.3):

$$\tilde{x}^i = \tilde{x}^i(x^j); \quad \tilde{\bar{y}}^i = \frac{\partial \tilde{x}^i}{\partial x^j} y^j; \quad \tilde{\bar{z}}^i = \frac{\partial \tilde{x}^i}{\partial x^j} \bar{z}^j. \tag{3.3}$$

Let's consider $(\mathcal{M}, \mathcal{N})$ a nonlinear connection in F_M vector bundle T^2M. From (2.6) results that $\mathcal{M}^j_i$ and $\mathcal{N}^j_i$ is changed after the rules:

$$\tilde{\mathcal{M}}^k_j \frac{\partial \tilde{x}^j}{\partial x^i} = \frac{\partial \tilde{x}^k}{\partial x^j} \mathcal{M}^j_i - \frac{\partial^2 \tilde{x}^k}{\partial x^i \partial x^j} y^j, \tag{3.4}$$

$$\tilde{\mathcal{N}}^k_j \frac{\partial \tilde{x}^j}{\partial x^i} = \frac{\partial \tilde{x}^k}{\partial x^j} \mathcal{N}^j_i - \frac{\partial^2 \tilde{x}^k}{\partial x^i \partial x^j} \bar{z}^j - \frac{\partial \tilde{x}^k}{\partial x^j} \cdot \frac{\partial N^j_1 y^1}{\partial x^i} - \frac{\partial \tilde{x}^k}{\partial x^j} \cdot \frac{\partial (N^j_1 y^1)}{\partial y^m} \mathcal{M}^m_i \tag{3.5}$$

Obviously, if $(\mathcal{M}, N)$ is a nonlinear connection of Finsler type then we can consider $\mathcal{M} = N(x, y)$. The normal lift of $\frac{\partial}{\partial x^i}$ is:

$$\frac{aa}{aax^i} = \frac{\partial}{\partial x^i} - \mathcal{M}^j_i \frac{\partial}{\partial y^j} - \mathcal{N}^j_i \frac{\partial}{\partial \bar{z}^j}.$$

From $\frac{aa}{aa\tilde{x}^i} = \frac{\partial x^j}{\partial \tilde{x}^i} \cdot \frac{aa}{aax^j}$ we have:

$$\frac{aa\tilde{\bar{z}}^k}{\partial \tilde{x}^j} \cdot \frac{\partial \tilde{x}^j}{\partial x^i} = \frac{\partial^2 \tilde{x}^k}{\partial x^i \partial x^j} \bar{z}^j + \frac{aa\bar{z}^j}{aax^i} \cdot \frac{\partial \tilde{x}^k}{\partial x^j}. \tag{3.6}$$

According to (2.9), results that giving a nonlinear connection $\mathcal{M}^j_i(x, y)$ on M, then the pairs $(\mathcal{M}^j_i, \mathcal{N}^j_i)$:

$$\mathcal{N}^k_i = \frac{\partial \mathcal{M}^k_i}{\partial y^j} \bar{z}^j + \frac{\partial \mathcal{M}^k_j}{\partial x^i} y^j - \mathcal{M}^1_i \frac{\partial (\mathcal{M}^k_j y^j)}{\partial y^1} \tag{3.7}$$

determine a nonlinear connection of Finsler type on F_M vector bundle T^2M.

The covariant derivative of $L = A^i \frac{\delta}{\delta x^i} + B^i \frac{\partial}{\partial y^i}$ in respect to $X = X^i \frac{\partial}{\partial x^i}$ is given by (2.12):

$$D_X L = X^i \left(\frac{\partial A^j}{\partial x^i} + \mathcal{M}^j_i \right) \frac{\delta}{\delta x^j} + X^i \left(\frac{\partial B^j}{\partial x^i} + \mathcal{N}^j_i \right) \frac{\partial}{\partial y^j}. \tag{3.8}$$

The vertical connection in T^2M is defined as above.

The coefficients of a vertical connection $\mathcal{H}^j_i$ are changed after the rule:

$$\tilde{\mathcal{H}}^k_j \frac{\partial \tilde{x}^j}{\partial x^i} = \frac{\partial \tilde{x}^k}{\partial x^j} \mathcal{H}^j_i - \frac{\partial \tilde{x}^k}{\partial x^j} \cdot \frac{\partial (N^j_1 y^1)}{\partial y^i}. \tag{3.9}$$

The vector fields $\frac{\delta}{\delta y^i} = \frac{\partial}{\partial y^i} - \mathcal{H}_i^j \frac{\partial}{\partial \bar{z}^j}$ is changed after the rule: $\frac{\delta}{\delta y^i} = \frac{\partial x^j}{\partial \bar{x}^i} \cdot \frac{\delta}{\delta y^j}$. If $\mathcal{M}_i^j(x,y)$ is a given nonlinear connection on M then:

$$\mathcal{H}_i^j = \frac{\partial^2(\mathcal{M}_k^j)t^k}{\partial y^i \partial y^1}\bar{z}^1 - \frac{\partial(\mathcal{M}_k^j y^k)}{\partial y^i} \tag{3.10}$$

os a vertical connection on T^2M, where $t^i(x)$ are the components of an arbitrary vector field on M.

Definition. A nonlinear connection in F_M vector bundle T^2M is a triad $(\mathcal{M}, H, N)$ where $(\mathcal{M}, N)$ is a nonlinear connection of Finsler type and $\mathcal{H}$ is a vertical connection in F_M vector bundle T^2M.

A nonlinear connection in F_M vector bundle T^2M determines decomposition $T(T^2M) = VT^2M \oplus \mathcal{H}T^2M \oplus \mathcal{N}T^2M$, where the distribution vertical, horizontal and normal has respectively the local bases: $\left\{\frac{\partial}{\partial \bar{z}^i}\right\}$, $\left\{\frac{\delta}{\delta y^i}\right\}$, $\left\{\frac{aa}{aax^i}\right\}$.

References

[1] Akbar-Zadeh H., *Les spaces de Finsler et certain de leurs generalisations*, Ann. Sci. Ecole Norm. Sup. (3)80(1963), 1-79.

[2] Albu I.D., Opris D., *On the differential geometry of Finsler tangent bundle*, The Proc. of Nat. Sem. on Finsler Spaces, Vol.II, Braşov, 1983, 17-24.

[3] Bowman R., *Second order connections*, J. Diff. Geom. 7(1972), 549-561.

[4] Dodson C.T.J., Radivoiovici M., *Tangent and frame bundles of order two*, Analele St. Univ. "Al.I.Cuza" Iaşi, tome XXVIII, I-a, 1982, f 1., 63-71.

[5] Duc T.V., *Sur la geometrie differentialle des fibres vectoriels*, Koday Math. Sem. Rep. 36(1975), 439-408.

[6] Ianus S., *On differential geometry of the dual of a vector bundle*, The Proc. of 5th Nat. Sem. of Finsler and Lagrange spaces, Braşov, 1988, 173-180.

[7] Matsumoto M., *Foundations of Finsler geometry and special Finsler spaces*, Kyoto 1982.

[8] Miron R., Anastasiei M., *Fibrate vectoriale. Spaţii Lagrange. Aplicaţii în teoria relativităţii*, Ed. Academiei R.S.R. Bucureşti 1987.

[9] Opris D., *Fibres vectoriels de Finsler et connexions associes*, The Proc. of Nat. Sem. on Finsler Spaces, Braşov, 1980, 185-193.

[10] Yano K., Ishihara S., *Tangent and cotangent bundles*, Marcel Dekker Inc. New York, 1973.

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INEQUALITIES RELATED TO THE ZEROS OF SOLUTIONS OF CERTAIN SECOND ORDER DIFFERENCE EQUATION

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Abstract. The aim of the present paper is to establish two new inequalities related to the zeros of the solutions of certain second order difference equations by using elementary analysis. An application to prove the boundedness of oscillatory solutions of the associated difference equations is given.

1. Introduction

Consider the following second order nonlinear difference equations:

$$\Delta(r(n)|\Delta y(n)|^{\alpha-1}\Delta y(n)) + c(n)|y(n)|^{\beta-1}y(n) = 0, \qquad (A)$$

$$\Delta(r(n)(|y(n+1)|^p + |y(n)|^p)|\Delta y(n)|^{q-2}\Delta y(n)+$$
$$+(n)|y(n)|^{p+q-2}y(n) = 0, \qquad (B)$$

where $n \in I_\infty$, $I_\infty = \{a, a+1, a+2, \dots\}$. a an integer, $\alpha \geq 1$, $\beta \geq 1$, $p \geq 1$, $q \geq 2$ are real constants and $q > p$, the operator Δ is defined by $\Delta y(n) = y(n+1) - y(n)$ for $n \in I_\infty$, $r(n)$, $c(n)$, $n \in I_\infty$ are real-valued functions and $r(n) > 0$. We shall define the subset I of I_∞ by $I = \{a, a+1, a+2, \dots, b\}$, $a, b = a+m$ $(m \geq 2)$ are integers, and denote by I^0 the interior of I and assume that I^0 is nonempty.

The continuous analogues of the equations like (A) and (B) have been recently dealt with in [1-7, 9-12] with various view points. The object of this paper is to derive two new finite difference inequalities related to the zeros of the solutions of equations (A) and (B) which can be used as handy tools in the study of qualitative behavior of solutions of the corresponding finite difference equations. We also present some immediate applications to convey the importance of our results to the literature. The reader is refered to papers [8, 12] for similar results.

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2. Main results

Our main results are established in the following theorems.

Theorem 1. *Let $y(n)$, $n \in I$ be a solution of equation (A) such that $y(z) = y(b) = 0$, $y(n) \neq 0$ for $n \in I^0$. If k be a point in I^0 where $|y(n)|$ is maximized, then*

$$1 \leq M^{\beta-\alpha} \left(\sum_{s=a}^{b-1} r^{-(1/\alpha)}(s) \right)^{\alpha} \left(\sum_{s=a}^{b-1} |c(s)| \right), \tag{1}$$

where $M = \max |y(n)| = |y(k)|$, $k \in I^0$.

Proof. Let $M = |y(k)|$, $k \in I^0$. Since $y(a) = y(b) = 0$, it is easy to observe that

$$M^2 = y^2(k) = \sum_{s=a}^{k-1} \Delta y^2(s) = -\sum_{s=k}^{b-1} \Delta y^2(k), \quad k \in I. \tag{2}$$

From (2) we observe that

$$2M^2 \leq \sum_{s=a}^{b-1} |\Delta y^2(s)| \leq \sum_{s=a}^{b-1} (|y(s+1)| + |y(s)|)|\Delta y(s)|) =$$

$$= \sum_{s=a}^{b-1} [r^{-(1/(\alpha+1))}(s)(|y(s+1)|) + |y(s)|)] \times [r^{(1/(\alpha+1))}(s)|\Delta y(s)|]. \tag{3}$$

Now by applying the Hölder's inequality on the right side of (3) with indices $(\alpha+1)/\alpha$, $\alpha+1$, the following formula of summation by parts

$$\sum_{s=a}^{b-1} u(s)\Delta v(s) = (u(b)v(b) - u(a)v(a)) - \sum_{s=a}^{b-1} v(s+1)\Delta u(s), \tag{4}$$

the facts that $y(a) = y(b) = 0$ and $y(n)$ is a solution of equation (A) we observe that

$$2M^2 \leq \left(\sum_{s=a}^{b-1} r^{-(1/\alpha)}(s)(|y(s+1)| + |y(s)|)^{(\alpha+1)/\alpha} \right)^{\alpha/(\alpha+1)} \times \left(\sum_{s=a}^{b-1} r(s)|\Delta y(s)|^{\alpha+1} \right)^{1/(\alpha+1)}$$

$$= \left(\sum_{s=a}^{b-1} r^{-(1/\alpha)}(s)(|y(s+1)| + |y(s)|)^{(\alpha+1)/\alpha} \right)^{\alpha/(\alpha+1)} \times$$

$$\times \left(\sum_{s=a}^{b-1} (r(s)|\Delta y(s)|^{\alpha-1} \Delta y(s)) \Delta y(s) \right)^{1/(\alpha+1)} =$$

$$= \left(\sum_{s=a}^{b-1} r^{-(1/\alpha)}(s)(|y(s+1)| + |y(s)|)^{(\alpha+1)/\alpha} \right)^{\alpha/(\alpha+1)} \times$$

$$\times \left(-\sum_{s=a}^{b-1} y(s+1) \Delta(r(s)|\Delta y(s)|^{\alpha-1} \Delta y(s)) \right)^{1/(\alpha+1)} =$$

$$= \left(\sum_{s=a}^{b-1} r^{-(1/\alpha)}(s)(|y(s+1)|+|y(s)|)^{(\alpha+1)/\alpha}\right)^{\alpha/(\alpha+1)} \times$$

$$\times \left(\sum_{s=a}^{b-1} u(s+1)c(s)|y(s)|^{\beta-1}y(s)\right)^{1/(\alpha+1)} \leq$$

$$\leq (2M)(M)^{(\beta+1)/(\alpha+1)} \left(\sum_{s=a}^{b-1} r^{-(1/\alpha)}(s)\right)^{\alpha/(\alpha+1)} \times \left(\sum_{s=a}^{b-1} |c(s)|\right)^{1/(\alpha+1)} \tag{5}$$

Dividing both sides of (5) by $2M^2$ and then raising the power $(\alpha+1)$ to both sides of the resulting inequality, we get the desired inequality in (1). The proof is complete. □

Theorem 2. *Let $y(n)$, $n \in I$ be a solution of equation (B) such that $y(a) = y(b) = 0$, $y(n) \neq 0$ for $n \in I^0$. If k be a point in I^0 where $|y(n)|$ is maximized, then*

$$1 \leq (1/2) \left(\sum_{s=a}^{b-1} r^{-(1/(q-1))}(s)\right)^{(q-1)} \left(\sum_{s=a}^{b-1} |c(s)|\right). \tag{6}$$

Proof. By following the proof of Theorem 1, we have the following inequality

$$2M^2 \leq \sum_{s=a}^{b-1} (|y(s+1)|+|y(s)|)|\Delta y(s)| =$$

$$= \sum_{s=a}^{b-1} [r^{-(1/q)}(s)(|y(s+1)|+|y(s)|)^{1-(p/q)}] \times$$

$$\times [r^{(1/q)}(s)(|y(s+1)|+|y(s)|)^{p/q}|\Delta y(s)|]. \tag{7}$$

Now by aplying the Hölder's inequality on the right side of (7) with indices $q/(q-1)$, q, the elementary inequality $(c+d)^p \leq 2^{p-1}(c^p+d^p)$, $c \geq 0$, $d \geq 0$, $p \geq 1$ reals, the formula (4), the facts that $y(a) = y(b) = 0$ and $y(n)$ is a solution of equation (B) we observe that

$$2M^2 \leq \left(\sum_{s=a}^{b-1} r^{-(1/(q-1))}(s)(|y(s+1)|+|y(s)|)^{(q-p)(q-1)}\right)^{(q-1)/q} \times$$

$$\times \left(\sum_{s=a}^{b-1} r(s)(|y(s+1)|+|y(s)|)^p|\Delta y(s)|^q\right)^{(1/q)} \leq$$

$$\leq \left(\sum_{s=a}^{b-1} r^{-(1/(q-1))}(s)(|y(s+1)|+|y(s)|)^{(q-p)(q-1)}\right)^{(q-1)/q} \times$$

$$\times 2^{(p-1)/q} \left(\sum_{s=a}^{b-1} (r(s)(|y(s+1)|^p+|y(s)|^p)|\Delta y(s)|^{q-2}\Delta y(s)) \times \Delta y(s)\right)^{1/q} =$$

$$= 2^{(p-1)q} \left(\sum_{s=a}^{b-1} r^{-(1/(q-1))}(s)(|y(s+1)| + |y(s)|)^{(q-1)/(q-1)} \right)^{(q-1)/q} \times$$

$$\times \left(- \sum_{s=a}^{b-1} y(s+1)\Delta(r(s)(|y(s+1)|^p + |y(s)|^p)|\Delta y(s)|^{q-2}\delta y(s)) \right)^{1/q} =$$

$$= 2^{(p-1)/q} \left(\sum_{s=a}^{b-1} r^{-(1/(q-1))}(s)(|y(s+1)| + |y(s)|)^{(q-p)/(q-1)} \right)^{(q-1)/q} \times$$

$$\times \left(\sum_{s=a}^{b-1} y(s+1)c(s)|y(s)|^{p+q-2}y(s) \right)^{1/q} \leq$$

$$\leq 2^{(p-1)/q}(2M)^{(q-p)/q}(M)^{(p+q)/q} \times$$

$$\times \left(\sum_{s=a}^{b-1} r^{-(1/(q-1))}(s) \right)^{(q-1)/q} \left(\sum_{s=a}^{b-1} |c(s)| \right)^{1/q} =$$

$$= 2^{(q-1)/q} M^2 \left(\sum_{s=a}^{b-1} r^{-(1/(q-1))}(s) \right)^{(q-1)/q} \left(\sum_{s=a}^{b-1} |c(s)| \right)^{1/q} \tag{8}$$

Dividing both sides of (8) by $2M^2$ and then raising the power q to both sides of the resulting inequality, we get the required inequality in (6). □

3. Some applications

In the following theorem we apply our inequalities given in Theorem 1 and 2 to study the boundedness of oscillatory solutions of equations (A) and (B).

Theorem 3. *(i) If*

$$\sum_{s=a}^{\infty} r^{-(1/\alpha)}(s) < \infty, \quad \sum_{s=a}^{\infty} |c(s)| < \infty, \tag{9}$$

then every oscillatory solution of equation (A) is bounded on I_∞.

(ii) If

$$\sum_{s=a}^{\infty} r^{-(1/(q-1))}(s) < \infty, \quad \sum_{s=a}^{\infty} |c(s)| < \infty, \tag{10}$$

then every oscillatory solution of equation (B) is bounded on I_∞.

Proof. Here we will prove (ii) only. The proof of (i) can be completed similarly. Suppose $y(n)$ is an oscillatory solution of equation (B) on I_∞. Because of (1), we can choose $T \geq a$ large enough so that for every $t \geq T$,

$$\sum_{s=t}^{\infty} r^{-(1/(q-1))}(s) < 2^{(1/(q-1))}, \quad \sum_{s=t}^{\infty} |c(s)| < 1. \tag{11}$$

Suppose to the contrary that $\limsup |y(n)| = \infty$. Indeed since $y(n)$ is oscillatory, there exists an interval (n_1, n_2) such that $n_1 > T$, $y(n_1) = y(n_2) = 0$, $|y(n)| > 0$ on (n_1, n_2) and $M = \max\{|y(n)| : n_1 \leq n \leq n_2\}$. Choose k is (n_1, n_2) such that $|y(k)| = M$. Clearly, the inequality (6) in Theorem 2 is true on the interval (n_1, n_2) and we have

$$1 \leq (1/2) \left(\sum_{s=n_1}^{n_2-1} r^{-(1/(q-1))}(s) \right)^{(q-1)} \left(\sum_{s=n_1}^{n_2-1} |c(s)| \right). \tag{12}$$

From (12) and (11) we have

$$1 \leq (1/2) \left(\sum_{s=n_1}^{\infty} r^{-(1/(q-1))}(s) \right)^{(q-1)} \left(\sum_{s=n_1}^{\infty} |c(s)| \right) < 1.$$

This contradiction shows that the oscillatory solution $y(n)$ of (B) is bounded on I_∞. The proof is complete. □

Finally, we note that our results in Theorem 1-2 can be very easily extended to the following more general equations of the forms:

$$\Delta(r(n)|\Delta y(n)|^{\alpha-1}\Delta y(n)) + c(n)|y(n)|^{\beta-1}y(n)f(n, y(n)) = 0, \tag{13}$$

$$\Delta(r(n)(|y(n+1)|^p + |y(n)|^p)|\Delta y(n)|^{q-2}\Delta y(n)) +$$

$$+ c(n)|y(n)|^{p+q-2}y(n)f(n, y(n)) = 0, \tag{14}$$

where $\alpha, \beta, p, q, r(n), c(n)$ are as defined in equations (A) and (B) and the function $f : I_\infty \times \mathbb{R} \to \mathbb{R}$ satisfies the condition $|f(n, y)| \leq w(n, |y|)$, where the function $w : I_\infty \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $w(n, u) \leq w(n, v)$ for $0 \leq u \leq v$. By following the proofs of Theorems 1 and 2, corresponding to the equations (13) and (14) the inequalities obtained in (1) and (6) takes the forms

$$1 \leq M^{\beta-\alpha} \left(\sum_{s=a}^{b-1} r^{-(1/\alpha)}(s) \right)^{\alpha} \left(\sum_{s=a}^{b-1} |c(s)| w(s, M) \right), \tag{15}$$

and

$$1 \leq (1/2) \left(\sum_{s=a}^{b-1} r^{-(1/(q-1))}(s) \right)^{(q-1)} \left(\sum_{s=a}^{b-1} |c(s)| w(s, M) \right), \quad (16$$

where $M = |y(k)|$, $k \in I^0$. For a similar result in continuous case of equation like (A) see [11].

References

[1] L.E. Bobisud, *Steady-state turbulent flow with reaction*, Rocky Mountain J. Math., 21(1991) 993-1007.

[2] L.E. Bobisud, *Existence of solutions of some nonlinear diffusion problems*, J. Math. Ana Appl., 168(1992), 413-424.

[3] M. Del Pino and R. Manasevich, *Oscillation and non-oscillation for* $(|u'|^{p-2}u')' \cdot a(t)|u|^{p-2}u = 0$, $p > 1$, Houstion J. Math., 14(1988), 173-177.

[4] M. del Pino, M. Elgueta and R. Manasevich, *A homotopic deformation along p of a lear Schauder degree result and existence for* $(|u'|^{p-2}u')' + f(t, u) = 0$, $u(0) = u(T) = 0$, $p >$ 1 J. Diff. Eq., 80(1989), 1-13.

[5] P. Hartman, *Ordinary Differential Equations*, Wiley, New York, 1964.

[6] T. Kusano and N. Yoshida, *Nonoscillation theorems for a class of quasilinear differenti equations of second order*, J. Math. Anal. Appl., 189*1995), 115-127.

[7] K. Nishihara, *Asymptotic behavior of solutions of second order differential equations*, Math. Anal. Appl., 189(1995), 424-441.

[8] B.G. Pachpatte, *On Lyapunov type finite difference inequality*, Tamkang J. Math., 21(1990 337-339.

[9] B.G. Pachpatte, *On the zeros of solutions of certain differential equations*, Demonstratu Mathematica, 25(1992), 825-833.

[10] B.G. Pachpatte, *A Lyapunov type inequality for a certain second order differential equatio* Proc. Nat. Acad. Sci. India, 64(A)(1994), 69-73.

[11] B.G. Pachpatte, *An inequality suggested by Lyapunov's inequality*, Centre de Rech. Matl Pures Neuchatel Chambery, Fasc. 26, ser.I(1995), 1-4.

[12] W.T. Patula, *On the distance between zeros*, Proc. Amer. Math. Soc., 52(1975), 247-251.

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CLASS OF MEROMORPHIC CLOSE-TO-CONVEX FUNCTIONS

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Abstract. Some sufficient conditions for meromorphically close-to-convex functions are obtained.

1. Introduction.

The writing of this paper has been motivated by a recent paper of Cho and Kim [1]. Let Σ denote the class of functions of the form $f(z) = \frac{1}{z} + \sum_{m=0}^{\infty} a_m z^m$ that are analytic in the punctured disk $E = \{z : 0 < |z| < 1\}$. For any real number α, let the operator I^α operating on $f \in \Sigma$ be defined by

$$I^\alpha f(z) = \frac{1}{z} + \sum_{m=0}^{\infty} (m+2)^{-\alpha} a_m z^m.$$

Clearly we have

$$I^\alpha(I^\beta f(z)) = I^{\alpha+\beta} f(z)$$

for all real numbers α and β. For any non positive integer α, the operators I^α are the differential operators studied by Uralegaddi and Somanatha [7]. Also the operators I^α are closely related to the multiplier transformations introduced by Flett [2].

For any real number α and $1 < \lambda \leq 3/2$ let $C_\alpha(\lambda)$ denote the class of functions $f \in \Sigma$ satisfying the condition

$$\operatorname{Re}\left\{\frac{(I^{\alpha-1} f(z))'}{(I^\alpha f(z))'} - 2\right\} > -\lambda \quad z \in U = \{z : |z| < 1\}.$$

A function

$$g(z) = e^{i\beta} z^{-1} + \sum_{n=0}^{\infty} b_n z^n \quad (\beta, \text{ real number}) \tag{1}$$

which is analytic in E with a simple pole at $z = 0$ is said to starlike if Re $zg'(z)/g(z) < 0$, $|z| < 1$. A function f in Σ is said to be close-to-convex of order α $(0 \leq \alpha < 1)$ in

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E, denoted $f \in \Sigma_c(\alpha)$ if there exists a meromorphic starlike univalent function $g(z$ given by (1) such that Re $zf'(z)/g(z) \geq \alpha$ for $|z| < 1$. Observe that $\Sigma_c(0) = \Sigma_c$ th class of meromorphic close-to-convex functions. The concept of close to convexity t the meromorphic case was introduced by R.J. Libera and M.S. Robertson [3] and wa extended by R.J. Libera [4]. It is known that meromorphic starlike functions are univalen but close-to-convex functions need not be univalent.

In this paper we shall show that all functions in $C_0(\lambda)$ are meromorphicall close-to-convex of order $1/(2\lambda - 1)$. Further for the class $C_\alpha(\lambda)$ of functions in Σ prov that $C_\alpha(\lambda) \subset C_{\alpha+1}(\lambda)$. Hence for α a non positive real number, all members in $C_\alpha(\lambda$ are meromorphically close-to-convex.

2. We need the following lemma which is due to Miller and Mocanu [5].

Lemma. *Let $\phi(u,v)$ be a complex valued function, $\phi : D \to C$, $D \subset C^2$ (C is th complex plane) and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function $\phi(u,v$ satisfies the following conditions*

i) $\phi(u,v)$ is continuous in D;

ii) $(1,0) \in D$ and Re $\{\phi(1,0)\} > 0$

iii) Re $\{\phi(iu_2, v_1)\} \leq 0$ *for all $(iu_2, v_1) \in D$ such that*

$$v_1 < -\frac{1+u_2^2}{2}.$$

Let $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ be analytic in U such that $(p(z), zp'(z)) \in D$ fo all $z \in U$. If Re $\{\phi(p(z), zp'(z)\} > 0$ $(z \in U)$, *then* Re $\{p(z)\} > 0$ $(z \in U)$.

Theorem 1. *If $f(z) \in C_0(\lambda)$ then $f(z)$ is meromorphically close-to-convex of orde $1/(2\lambda - 1)$.*

Proof. It suffices to show that Re $\{-z^2 f'(z)) > 1/(2\lambda - 1)$. Define the function $p(z)$ by

$$-z^2 f'(z) = \gamma + (1-\gamma)p(z) \tag{2}$$

where $\gamma = 1/(2\lambda - 1)$, $1/2 \leq \gamma > 1$.

We see that $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ is analytic in U.

Differentiating (2) and simplifying we get

$$z\frac{f''(z)}{f'(z)} + 1 + \lambda = \lambda - 1 + \frac{(1-\gamma)zp'(z)}{\gamma + (1-\gamma)p(z)}.$$

Hence

$$\text{Re}\left\{z\frac{f''(z)}{f'(z)}+1+\lambda\right\}=\text{Re}\left\{\lambda-1+\frac{(1-\gamma)zp'(z)}{\gamma+(1-\gamma)p(z)}\right\}>0.$$

Let

$$\phi(u,v)=\lambda-1+\frac{(1-\gamma)v}{\gamma+(1-\gamma)u}.$$

Then $\phi(u,v)$ satisfies

i) $\phi(u,v)$ is continuous in $D=(C-(\gamma/\gamma-1))\times C$;

ii) $(1,0)\in D$ and Re $\{\phi(1,0)\}=\lambda-1>0$;

iii) for all $(iu_2,v_1)\in D$ such that $v_1\leq-\frac{1+u_2^2}{2}$,

$$\text{Re}\ \{\phi(iu_2,v_1)\}=\lambda-1+\frac{\gamma(1-\gamma)v_1}{\gamma^2+(1-\gamma)^2u_2^2}\leq\lambda-1-\frac{\gamma(1-\gamma)(1+u_2^2)}{2[\gamma^2+(1-\gamma)^2u_2^2]}=$$

$$=\lambda-1-\frac{\gamma(1-\gamma)(1+u_2^2)}{2\gamma^2\left[1+\frac{(1-\gamma)^2}{\gamma^2}u_2^2\right]}<\lambda-1-\frac{\gamma(1-\gamma)}{2\gamma^2}\leq 0.$$

Thus the function $\phi(u,v)$ satisfies the conditions of the above stated lemma. Hence Re $\{p(z)\}>0$ $(z\in U)$. Thus

$$\text{Re}\ \{-z^2f'(z)\}>\gamma$$

i.e.

$$\text{Re}\ \{-z^2f'(z)\}>\frac{1}{2\lambda-1}.$$

For $\lambda=3/2$ we get the earlier result [Corollary 3, 6] □

Theorem 2. *If $f\in C_\alpha(\lambda)$ then $f\in C_{\alpha+1}(\mu)$ where*

$$\mu=\frac{5+2\lambda-\sqrt{(3-2\lambda)^2+8}}{4}.$$

Proof. Define the function $p(z)$ by

$$\frac{(I^\alpha f(z))'}{(I^{\alpha+1}f(z))'}=\gamma+(1-\gamma)p(z) \tag{3}$$

where

$$\gamma=\frac{3-2\lambda-\sqrt{(3-2\lambda)^2+8}}{4}\quad\left(\frac{1}{\sqrt{2}}\leq\gamma<1\right).$$

We see that $p(z)=1+p_1z+p_2z^2+\ldots$ is analytic in U. Logarithmic differentiation of (3) gives

$$\frac{(I^\alpha f(z))''}{(I^\alpha f(z))'}-\frac{(I^{\alpha+1}f(z))''}{(I^{\alpha+1}f(z))'}=\frac{(1-\gamma)p'(z)}{\gamma+(1-\gamma)p(z)}. \tag{4}$$

From the following identity

$$z(I^{\alpha}f(z))' = I^{\alpha-1}f(z) - 2I^{\alpha}f(z)$$

we get

$$z(I^{\alpha}f(z))'' = (I^{\alpha-1}f(z))' - 3(I^{\alpha}f(z))'. \quad (5$$

Using (5) the equation (4) reduces to

$$\frac{(I^{\alpha-1}f(z))'}{(I^{\alpha}f(z))'} = \gamma + (1-\gamma)p(z) + \frac{(1-\gamma)zp'(z)}{\gamma+(1-\gamma)p(z)}$$

$$\text{Re}\left\{\frac{(I^{\alpha-1}f(z))'}{(I^{\alpha}f(z))'} - 2 + \gamma\right\} = \text{Re}\left\{\lambda + \gamma - 2 + (1-\gamma)p(z) + \frac{(1-\gamma)zp'(z)}{\gamma+(1-\gamma)p(z)}\right\} > 0.$$

Let $\phi(u, v)$ be the function defined by

$$\phi(u, v) = \lambda + \gamma - 2 + (1-\gamma)u + \frac{(1-\gamma)v}{\gamma+(1-\gamma)u}.$$

Then $\phi(u, v)$ satisfies

i) $\phi(u, v)$ is continuous in $D = (C - (\gamma/\gamma - 1)) \times C$;

ii) $(1, 0) \in D$ and Re $\{\phi(1, 0)\} = \lambda - 1 > 0$;

iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\frac{1+u_2^2}{2}$,

$$\text{Re}\,\{\phi(iu_2, v_1)\} = \text{Re}\left\{\lambda + \gamma - 2 + \frac{(1-\gamma)v_1}{\gamma+(1-\gamma)iu_2}\right\} = \lambda+\gamma-2-\frac{\gamma(1-\gamma)(1+u_2^2)}{2[\gamma^2+(1-\gamma)^2u_2^2]} \leq$$

Thus the function $\phi(u, v)$ satisfies the conditions of above stated Lemma. Hence Re $\{p(z)\} > 0$ $(z \in U)$.

Therefore

$$\text{Re}\left\{\frac{(I^{\alpha}f(z))'}{(I^{\alpha+1}f(z))'}\right\} > \gamma \quad (z \in U)$$

i.e.

$$\text{Re}\left\{\frac{(I^{\alpha}f(z))'}{(I^{\alpha+1}f(z))'} - 2\right\} > -\mu.$$

Hence $C_{\alpha}(\lambda) \subset C_{\alpha+1}(\mu)$.

Since $\mu - \lambda < 0$ we have

Corollary 1. $C_{\alpha+1}(\mu) \subset C_{\alpha+1}(\lambda)$.

Corollary 2. *If* $f(z) = \frac{1}{z} + \sum_{m=0}^{\infty} a_m z^m$ *is meromorphically close-to-convex then so is*

$$F(z) = \frac{1}{z^2}\int_0^z tf(t)dt.$$

References

[1] N.A. Cho and J.A. Kim, *On certain classes of meromorphically starlike functions*, Internat. J. Mat. and Math. Sci., 18, No.3(1995), 463-468.

[2] T.M. Flett, *The dual of an inequality of Hardy and Littlewood and some related inequalities*, J. Math. Anal. Appl., 38*1972), 746-765.

[3] R.J. Libera and M.S. Robertson, *Meromorphic close-to-convex functions*, Michigan Math. J., 8(1961), 167-175.

[4] R.J. Libera, *Meromorphic close-to-convex functions*, Duke Math. J., 32(1965), 121-128.

[5] S.S. Miller and P.T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl., 65(1978), 289-305.

[6] S.M. Sarangi and Suguna B. Uralegaddi, *Certain sufficient conditions for close-to-convexity and starlikeness of meromorphic functions* (to appear in Maths student).

[7] B.A. Uralegaddi and C. Somanatha, *Certain differential operators for meromorphic functions*, Houstion J. Math., 17(1991), 279-284.

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RADIAL MOTION WITH ZERO INITIAL VELOCITY IN MANEFF-TYPE FIELDS

CRISTINA STOICA AND VASILE MIOC

Abstract. The paper investigates the radial motion of the particle in the Maneff field, by using the energy integral, for all the possible values of the parameters of the field and of the initial radius vector. There are emphasized collision or escape trajectories, libration motions between forbidden domains, as well as equilibrium points.

1. Introduction

Consider a force field, whose nature is not necessarily gravitational and may remain unspecified, characterized by a quasihomogeneous potential function of the form $A/r + b/r^2$, where r = distance between two particles, and A, B = constants. We shall call it Maneff-type field (see [2-9]).

The two-body problem in such a field can be reduced to a central force problem for a unit mass particle. The field being central, the motion will be planar and described by the equation

$$\ddot{\mathbf{r}} = -(A/r^3 + 2B/r^4)\mathbf{r}, \tag{1}$$

where dots mark time-differentiation.

Different expressions assigned to the parameters A and B can model various situations. The motion in certain post-Newtonian nonrelativistic fields (even in Maneff's one; see e.g. [3, 5, 12-14]) or in certain relativistic fields (e.g. in Fock's one [10] or in that featured by the Reissner Nordström metric, truncating the negligible terms) are such situations. The motion in the photogravitational field (see e.g. [11, 15, 17]) generated by a source of constant luminosity, and the two-body problem with constant equivalent gravitational parameter [16] also join this model. Actually, a force field of the above form

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has implications not only in (celestial) mechanics, but also in astrodynamics, cosmogon astrophysics, even in atomic physics [1, 3], and so forth.

In the present paper we shall investigate the behaviour of the unit mass partic in a Maneff-type field for zero initial velocity (in other words, at the initial instant th particle is set free in the field). It is clear that the motion will be radial, because the forc field is central. The analysis will be performed for all possible values of the paramete A and B and of the initial radius vector.

2. Equations of motion and first integrals

Using polar coordinates (r, u), equation (1) transform into

$$\ddot{r} - r\dot{u}^2 = -(A + 2B/r)/r^2, \quad (2$$

$$r\ddot{u} + 2\dot{r}\dot{u} = 0, \quad (3$$

system to which we attach the initial conditions

$$(r, u, \dot{r}, \dot{u})(t_0) = (r_0, u_0, V_0 \cos\alpha, V_0 \sin\alpha/r_0), \quad (4$$

where $V = |\dot{\mathbf{r}}|$ is the velocity, $V_0 = V(t_0)$, α is the angle between the initial radius vecto and the initial velocity.

Since the force field is central, the angular momentum is conserved, and (3 provides the first integral

$$r^2\dot{u} = C, \quad (5$$

where $C = r_0 V_0 \sin\alpha$ is the constant angular momentum. The first integral of energ can also be easily obtained, by the usual technique, as

$$V^2 = \dot{r}^2 + (r\dot{u})^2 = 2(A + B/r)/r + h, \quad (6$$

where the constant of energy h results to have the expression

$$h = V_0^2 - 2(A + B/r_0)r_0. \quad (7$$

The hypothesis of zero initial velocity ($V_0 = 0$) leads to $C = 0$ and implies radia motion. The same hypothesis makes (7) become

$$h = -2(A + B/r_0)/r_0, \quad (8$$

so (6) reads

$$V^2 = \dot{r}^2 = 2(1/r - 1/r_0)[A + B(1/r + 1/r_0)]. \tag{9}$$

In what follows, on the basis of these formulae, we shall analyze the behaviour of the particle for all possible values of A and B, and for different domains (with respect to A and B) in which r_0 can lie. The domains in which the motion is possible, featured by the condition $V^2 \geq 0$, as well as the characteristics of the motion, will be pointed out.

3. Behaviour of the particle

Let us first introduce some abridging notations. If $A < 0$ and $B > 0$, or if $A > 0$ and $B < 0$, denote

$$r_c = -2B/A. \tag{10}$$

Also denote

$$r_1 = -B(A + B/r_0), \tag{11}$$

when the interplay among A, B and r_0 makes positive the expression in the right-hand side of (11). Now, recalling that all motions are rectilinear, we can start the analysis.

3.1. Case $B > 0$. We distinguish three subcases:

(i) Subcase $A < 0$. First observe that

$$r_0 < r_c/2 \;\Rightarrow\; h < 0,\; r_0 = r_c/2 \;\Rightarrow\; h = 0,\; r_0 > r_c/2 \;\Rightarrow\; h > 0.$$

If $r_0 < r_c$, the particle will move inwards on a collision trajectory: r decreases, while V increases tending to an infinite value at collision ($r \to 0$).

If $r_0 = r_c$, the particle remains in r_0, but the equilibrium in unstable. Any radial perturbing force, no matter how small, makes the particle follow either a collision path (as in the previous situation), or an escape trajectory (unbound motion outwards) on which the velocity increases (with decreasing acceleration), tending to $V = \sqrt{h}$ (one sees that $h > 0$) for $r \to \infty$.

If $r_0 > r_c$, we also have $h > 0$, while the particle moves outwards on an escape trajectory, and $V \to \sqrt{h}$ for $r \to \infty$.

(ii) Subcase $A = 0$. We have $h < 0$. For any value of r_0, the particle moves inwards on a collision path. The velocity tends to an infinite value when $r \to 0$.

(iii) Subcase $A > 0$. The particle will behave according to the same scenario ($h < 0$, collision trajectory with $V \to \infty$ for $r \to 0$).

3.2. Case $B = 0$. We also distinguish three subcases:

(i) Subcase $A < 0$. The constant of energy is positive and has the expression $h = -2A/r_0$. Whatever r_0 is, the particle follows an escape path, moving outwards. The velocity increases (with decreasing acceleration) and tends to $\sqrt{h}$ for $r \to \infty$.

(ii) Subcase $A = 0$. In this subcase $h = 0$. In addition, the force field vanishes (that is, the resultant of all radial forces composing the field is zero). Under these conditions, whatever r_0 is, the particle remain at rest in its initial position.

(iii) Subcase $A > 0$. We have $h < 0$. The particle moves inwards on a collision trajectory, for any value r_0. The velocity increases (and the acceleration as well), tending to infinity for $r \to 0$.

3.3. Case $B < 0$. The three subcases are the same:

(i) Subcase $A < 0$. The constant of energy is positive. Whatever r_0 is, the particle will move outwards, following an escape trajectory. The velocity increases (with decreasing acceleration), tending to the value $\sqrt{h}$ when $r \to \infty$.

(ii) Subcase $A = 0$. The constant of energy is also positive and has the expression $h = -2B/r_0^2$. The particle will behave exactly as in the previous subcase (escape path with $V \to \sqrt{h}$ for $r \to \infty$).

(iii) Subcase $A > 0$. First observe that

$$r_0 < r_c/2 \Rightarrow h > 0, \; r_0 = r_c/2 \Rightarrow h = 0, \; r_0 > r_c/2 \Rightarrow h < 0.$$

If $r_0 < r_c/2$, the particle will follow an escape trajectory: r increases, V also increases. When $r = r_c$, the velocity reaches a maximum value given by

$$V_{\max}^2 = -(A + 2B/r_0)^2/(2B), \tag{12}$$

then decreases, tending asymptotically to $\sqrt{h}$ when $r \to \infty$.

If $r_0 = r_c/2$, the particle will also move outwards, on an escape path. The radius vector increases, V increases, too. When $r = r_c$, the velocity reaches a maximum value given by

$$V_{\max}^2 = -A^2/(2B), \tag{13}$$

then decreases, tending asymptotically to zero when $r \to \infty$.

In the next situations, for which $h < 0$, the particle behaviour will be essentially different.

If $r_c/2 < r_0 < r_c$, the motion starts outwards (r increases). The velocity increases up to a maximum value given by (12), which is reached for $r = r_c$, then decreases, and for $r = r_1$ we have $V = 0$. From r_1 the particle starts inwards, with exactly the same evolution of the velocity $V = V(r)$ (but in the opposite sense) up to r_0, for which $V = 0$. From here the scenario is repeated infinitely many times. In other words, in this situation the particle librates in the domain $[r_0, r_1]$. It can neither go outside the sphere of radius r_1, nor penetrate inside the sphere of radius r_0 (both spheres centered in origin).

If $r_0 = r_c$, the particle remains in r_0 (as in Case 3.1, Subcase (i)), but this time the equilibrium is stable.

Lastly suppose $r_0 > r_c$. In this situation one sees easily that $r_1 < r_c$. The particle will librate exactly as for $r_c/2 < r_0 < r_c$, but this time the motion starts inwards and takes place in the interval $[r_1, r_0]$. The forbidden domain for the particle is now consisting of the exterior of the sphere of radius r_0 and the interior of the sphere of radius r_1 (both spheres centered in origin).

Assume for this last situation that the particle comes from such a great distance that we may put $1/r_0 = 0$. The motion is performed inwards, the velocity increases up to a maximum value given by (13), which is reached for $r = r_c$, then the motion is decelerated and for $r = r_c/2$ we have $V = 0$. From here the motion is performed outwards, with the same evolution of velocity $V = V(r)$, but in the opposite sense. For $r \to \infty$, the velocity tends again to zero.

4. Concluding remarks

Reviewing the results exposed in Section 3, we see that a particle with zero initial velocity may behave in a Maneff-type field according to one of the following scenarios:

(a) Escape trajectory (unbound curve in the upper halfplane of the phase plane $(r, \dot{r})$).

(b) Collision trajectory (unbound curve in the lower halfplane of the phase plane).

(c) Libration (starting outwards or inwards) between two forbidden domains (closed curve in the phase plane).

(d) Rest in the initial position (point in the phase plane on the axis $\dot{r} = 0$), in stable or unstable equilibrium. This situation includes the particular case when the resultant of the forces featuring the field is zero, making the field vanish.

References

[1] Belenkii, I.M., *A Method of Regularizing the Equations of Motion in the Central Force-Field*, Celest. Mech., 23(1981), 9-32.

[2] Delgado, J., Diacu, F.N., Lacomba, E.A., Mingarelli, A., Mioc, V., Perez, E., Stoica, C., *The Global Flow of the Manev Problem*, J. Math. Phys., 37(1996), No.5.

[3] Diacu, F.N., *The Planar Isosceles Problem for Maneff's Gravitational Law*, J. Math. Phys., 34(1993), 5671-5690.

[4] Diacu, F.N., *Near-Collision Dynamics for Particle Systems with Quasihomogeneous Potentials*, J. Diff. Eq. (1996) (to appear).

[5] Diacu, F.N., Mingarelli, A., Mioc, V., Stoica, C., *The Manev Two-Body Problem: Quantitative and Qualitative Theory*, in R.P. Agarwal (ed.), Dynamical Systems and Applications, World. Sci. Ser. Appl. Anal., Vol.4, World Scientific Publ. Co., Singapore, 1995, pp.213-227.

[6] Maneff, G., *La gravitation et le principe de l'égalité de l'action et de réaction*, C.R. Acad. Sci. Paris, 178(1924), 2159-2161.

[7] Maneff, G., *Die Gravitation und das Prinzip von Wirkung und Gegenwirkung*, Z. Ohys., 31(1925), 786-802.

[8] Maneff, G., *Le principe de la moindre action et la gravitation*, C.R. Acad. Sci. Paris, 190(1930), 963-965.

[9] Maneff, G., *La gravitation et l'énergie au zéro*, C.R. Acad. Sci. Paris, 190(1930), 1374-1377.

[10] Mioc, V., *Elliptic-Type Motion in Fock's Gravitational Field*, Astron. Nachr., 315(1994), 175-180.

[11] Mioc, V., Radu, E., *Orbits in an Anisotropic Radiation Field*, Astron. Nachr., 313(1992), 353-357.

[12] Mioc, V., Stoica, C., *Discussion et résolution complète du problème des deux corps dans le champ gravitationnel de Maneff*, C.R. Acad. Sci. Paris, 320(1995), sér.I, 645-648.

[13] Mioc, V., Stoica, C., *Discussion et résolution complète du problème des deux corps dans le champ gravitationnel de Maneff* (II), C.R. Acad. Sci. Paris, 321(1995), sér.I, 961-964.

[14] Mioc, V., Stoica, C., *Unperturbed Trajectories in Maneff's Gravitational Field are Ellipses in Velocity Plane*, Bull. Astron. Belgrade, No.152(1995), 43-47.

[15] Saslaw, W.C., *Motion around a Source whose Luminosity Changes*, Astrophys. J., 226(1978), 240-252.

[16] Şelaru, D., Cucu-Dumitrescu, C., Mioc, V., *On a Two-Body Problem with Periodically Changing Equivalent Gravitational Parameter*, Astron. Nachr., 313(1992), 257-263.

[17] Şelaru, D., Cucu-Dumitrescu, C., Mioc, V., *Periodic Motions around Pulsating Stars*, Astrphys. Space Sci., 202(1993), 11-19.

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