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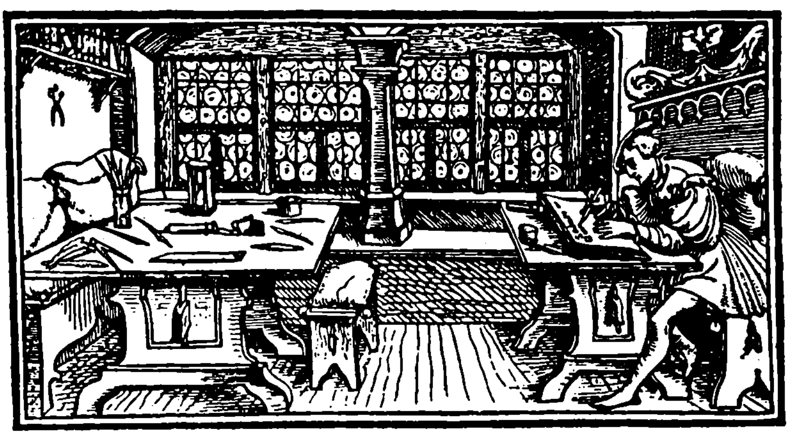

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SUMAR - CONTENTS - SOMMAIRE

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A BIVARIATE EXTENSION OF THE BERNSTEIN POLYNOMIALS

OCTAVIAN AGRATINI

Abstract. In this paper it is given an extension to two variables of the Bernstein $B_{n,r}f$ operators and there are investigated their approximation properties.

1. In the paper [2] is introduced and studied a new sequence of Bernstein type polynomials. Namely, for any function $f \in C^r[0,1]$ is defined the associated polynomial:

$$(B_{n,r}f)(x) = \sum_{k=0}^{n}\sum_{i=0}^{r} \frac{f^{(i)}\left(\frac{k}{n}\right)}{i!}\left(x - \frac{k}{n}\right)^i \binom{n}{k} x^k(1-x)^{n-k}. \tag{1}$$

It has been proved the estimation:

$$\|f - (B_{n,r}f)\| = O\left(n^{-\frac{r}{2}}\omega\left(f^{(r)}; n^{-1/2}\right)\right),$$

where $\|g\| = \sup\{|g(x)| : x \in [0,1]\}$ for $g \in C[0,1]$ and ω is the modulus of continuity of the function f, defined as usually:

$$\omega(f;s) = \sup\{|f(x) - f(y)| : x, y \in [0,1],\ |x-y| \le s\}.$$

The aim of this paper is to give an extension to two variables of the $B_{n,r}f$ and to investigate their approximation properties. It should be mentioned that there are many extensions to two variables of the linear operators of approximation (see, e.g. [3], [4], [5]).

2. Let $E = [0,1] \times [0,1]$ and $f : E \to \mathbf{R}$ differentiable of order r on E. The Taylor polynomial of degree r associated to the function f, in a point $(a,b) \in E$, is defined by:

$$(T_rf)(a,b;x,y) = f(a,b) + \frac{1}{1!}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right) f(a,b) +$$

$$+\frac{1}{2!}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right)^2 f(a,b) + \cdots + \frac{1}{r!}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right)^r f(a,b).$$

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The corresponding Taylor approximation formula is

$$f(x,y)=(T_rf)(a,b;x,y)+(R_rf)(a,b;x,y). \tag{2}$$

We introduce the notations:

$$(D^kf)(a,b;x,y)=\left((x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right)^k f(a,b)=\sum_{i=0}^{k}\binom{k}{i}\frac{\partial^k f(a,b)}{\partial x^{k-i}\partial y^i}(x-a)^{k-i}(y-b)^i.$$

Then the formula (2) becomes:

$$f(x,y)=f(a,b)+\sum_{i=1}^{r}\frac{1}{1!}(D^if)(a,b;x,y)+(R_rf)(a,b;x,y).$$

Definition. A generalized Bernstein polynomial of two variables of order $(n,m;r)$ for a differentiable function f of order r on E is a polynomial having the form:

$$(B_{n,m}^{(r)}f)(x,y)=\sum_{k=0}^{n}\sum_{l=0}^{m}(T_rf)\left(\frac{k}{n},\frac{l}{m};x,y\right)\binom{n}{k}\binom{m}{l}(1-x)^{n-k}x^k(1-y)^{m-l}y^l,$$

where

$$(T_rf)\left(\frac{k}{n},\frac{l}{m};x,y\right)=\sum_{i=0}^{r}\frac{1}{i!}(D^if)\left(\frac{k}{n},\frac{l}{m};x,y\right).$$

It is evident that for $r=0$

$$(T_0f)\left(\frac{k}{n},\frac{l}{m};x,y\right)=f\left(\frac{k}{n},\frac{l}{m}\right)$$

and consequently $B_{n,m}^{(0,)}f$ becomes the classical polynomial of Bernstein of two variables.

3. We intend to evaluate the quantity:

$$(E_{n,m}^{(r)}f)(x,y)=|f(x,y)-(B_{n,m}^{(r)}f)(x,y)|.$$

We need to use the identity:

$$\sum_{k=0}^{n}\sum_{l=0}^{m}\binom{n}{k}\binom{m}{l}x^k(1-x)^{n-k}y^l(1-y)^{m-l}=1.$$

By multiplying it by $f(x,y)$ we can write successively:

$$(E_{n,m}^{(r)}f)(x,y)\le\sum_{k=0}^{n}\sum_{l=0}^{m}\binom{n}{k}\binom{m}{l}\left|f(x,y)-(T_rf)\left(\frac{k}{n},\frac{l}{m};x,y\right)\right|x^k(1-x)^{n-k}y^l(1-y)^{m-l}=$$

$$=\sum_{k=0}^{n}\sum_{l=0}^{m}\binom{n}{k}\binom{m}{l}\left|(R_rf)\left(\frac{k}{n},\frac{l}{m};x,y\right)\right|x^k(1-x)^{n-k}y^l(1-y)^{m-l}. \tag{3}$$

It is known that if f has continuous partial derivatives of order r in a neighbourhood of $\left(\frac{k}{n}, \frac{l}{m}\right)$, then the remainder can be expressed under the form:

$$(R_r f)\left(\frac{k}{n}, \frac{l}{m}; x, y\right) = \frac{1}{r!} s_{\left(\frac{k}{n}, \frac{l}{m}\right)}(x, y) \rho^r_{\left(\frac{k}{n}, \frac{l}{m}\right)}(x, y), \tag{4}$$

where $s_{\left(\frac{k}{n}, \frac{l}{m}\right)}$ is a continuous in $\left(\frac{k}{n}, \frac{l}{m}\right)$ and vanishes in this point. Also, we notice that:

$$\rho_{\left(\frac{k}{n}, \frac{l}{m}\right)}(x, y) = \left(\left(x - \frac{k}{n}\right)^2 + \left(y - \frac{l}{m}\right)^2\right)^{1/2}$$

In the next stage we shall consider that the remainder of Taylor's formula expressed by (4) fulfills the following condition: exist the real constants A, B and the numbers $p \geq 2,\ q \geq 2$ so as

$$|s_{\left(\frac{k}{n}, \frac{l}{m}\right)}(x, y)| \leq A\left|x - \frac{k}{n}\right|^p + B\left|y - \frac{l}{m}\right|^q, \quad (x, y) \in E. \tag{5}$$

Because (x, y) and $\left(\frac{k}{n}, \frac{l}{m}\right)$ belong to E, we can deduce:

$$\rho_{\left(\frac{k}{n}, \frac{l}{m}\right)}(x, y) \leq 2^{\frac{r}{2}}. \tag{6}$$

From (4), (5) and (6) we get:

$$\left|(R_r f)\left(\frac{k}{n}, \frac{l}{m}; x, y\right)\right| \leq \frac{2^{\frac{r}{2}}}{r!}\left(A\left|x - \frac{k}{n}\right|^p + B\left|y - \frac{l}{m}\right|^q\right) \leq \frac{2^{\frac{r}{2}}}{r!}\left(A\left(x - \frac{k}{n}\right)^2 + \left(y - \frac{l}{m}\right)^2\right).$$

The following inequalities are well-known:

$$\sum_{k=0}^{n}\sum_{l=0}^{m}(k - nx)^2\binom{n}{k}\binom{m}{l}x^k(1-x)^{n-k}y^l(1-y)^{m-l} \leq \frac{n}{4}$$

and

$$\sum_{k=0}^{n}\sum_{l=0}^{m}(l - my)^2\binom{n}{k}\binom{m}{l}x^k(1-x)^{n-k}y^l(1-y)^{m-l} \leq \frac{n}{4}.$$

They lead to the next result:

Theorem. *Let $f : E \to \mathbf{R}$ with all partial derivatives of order r continuous on E. If the remainder of Taylor's formula fulfills the condition (5) then we obtain the inequality:*

$$|f(x, y) - (B^{(r)}_{m,n} f)(x, y)| \leq \frac{2^{\frac{r}{2}}}{4r!}\left(\frac{A}{n} + \frac{B}{m}\right).$$

Corollary. *Under the hypothesis of this theorem we can further write:*

$$\lim_{m,n \to \infty} ||f - B^{(r)}_{m,n} f|| = 0,$$

where $\|\cdot\| = \max\limits_{E} |\cdot|$.

4. We mention that in [1] we have introduced a class of linear polynomial approximating operators $(L_{nr})_{n\geq 1}$, $r = 0, 1, 2, \ldots$ for the functions $f \in C^r[0,1]$. In order to construct them we used the Taylor polynomial of degree r and a class of linear positive operators generated by a probabillistic method. Also, we studied the order of approximation using the moduli of continuity of first and second order, $(L_{nr}f)_{n\geq 1}$ including as a special case the generalized Bernstein polynomials defined in [2].

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A GLOBAL BIFURCATION THEOREM FOR PROPER FREDHOLM MAPS OF INDEX ZERO

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Abstract. This paper presents a global bifurcation theorem for the nonlinear eigenvalue problem $f_\lambda(x) = (I-\lambda L)x - G(\lambda, x) = 0$ under the assumption that for all $\lambda \in R$, the map f_λ is a proper C^r Fredholm map of index zero, $r \geq 2$. This is a generalization of the global bifurcation theorem of Rabinowitz (1971), which has been proved under the assumption that L and G are compact.

0.Introduction

In many parts of mathematical physics, in particular in fluid dynamics and in the elasticity theoiy, there are problems which solving leads to a nonlinear eigenvalue problem of the form:

$$f(\lambda, x) = (I - \lambda L)x - G(\lambda, x) = 0. \qquad (*)$$

Thus the nature of the structure of the set of its solutions is an important question. Under the assumption that the operators L and G are compact, Rabinowitz (1971) has proved a global bifurcation theorem for (*), being a global extension of Krasnoselski's theorem. The phenomenon of interest is bifurcation which is studied by means of the Leray-Schauder theory. A detailed description of the Leray-Schauder degree and index and there properties can be found e.g. in [Ni].

This paper is structured in three sections and references containing 14 titles, as follows: in Section 1 we present the bifurcation theorems of Krasnoselski ([Kr]) and Rabinowitz ([Ra]). Section 2 consists of a review of the definition of the Brouwer degree for proper C^r Fredholm maps of index zero and its properties; for more details, see [El,Tr1] and [El,Tr2]. The main result is presented in Section 3 and consists in a generalization of Rabinowitz's theorem. The idea of the generalization of Rabinowitz's theorem derives from both the facts that because L and G are compact, f is a Fredholm map of index

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zero and that Elworthy and Tromba (1970) have generalized the Leray-Schauder degr (which will be denoted in the sequel by deg, defining the Brouwer degree of proper (Fredholm maps of index zero, $r \geq 2$ (denoted by dg. The present paper tries to ope a direction of research such as to obtain the result even if the hypothesis (6) from th Section 3 appear in a weaker form.

1. The bifurcation theorems of Krasnoselski and Rabinowitz

Suppose X is a real Banach space with the norm $\|\cdot\|$ and $R \times X$ has the produc topology. By a *nonlinear eigenvalue problem* we mean an equation of the form

$$x = F(\lambda, x) \tag{1.1}$$

where $\lambda \in R$, $x \in X$ and $F : R \times X \to X$. *A solution* of (1.1) is a pair $(\lambda, x) \in R \times X$ and a *trivial solution* of (1.1) is a pair $(\lambda, 0) \in R \times X$. Of course this equation is too general to study without imposing more conditions on F. Rabinowitz ([Ra]) proved the global bifurcation theorem for nonlinear eigenvalue problem under the hypotheses:

(1) F is compact and

$$F(\lambda, x) = \lambda Lx + G(\lambda, x), \tag{1.2}$$

(2) $L : X \to X$ is a compact linear map and

(3) $G : R \times X \to X$ is a compact nonlinear map and $G(\lambda, x) = 0(\|x\|)$ near $x = 0$ (this means $G(\lambda, x) \to 0$ as $\|x\| \to 0$) uniformly on bounded λ intervals.

Next we suppose (1)-(3) satisfied. With the above assumptions, the equation (1.1) possesses *the line of trivial solutions*

$$\{(\lambda, 0); \lambda \in R\}.$$

Definition 1.1. (i) We call $(\mu, 0)$ a *bifurcation point* of (1.1) with respect to the line of trivial solutions if every neighbourhood of $(\mu, 0)$ contains nontrivial solutions of (1.1).

(ii) Let X be a topological space. A *subcontinuum* of X is a closed connected subset of X and a *component* of X is maximal (with respect to the inclusion) subcontinuum.

Let $r(L)$ denotes the set of a real characteristic values of L. It is well known that a necessary condition for $(\mu, 0)$ to be a bifurcation point for (1.1) with respect to the line of trivial solutions is that $\mu \in r(L)$. This follows since $\mu \notin r(L)$, then $I - \lambda L$ is invertible for all λ near μ. Hence for λ near μ, (1.1) is equivalent to

$$x = (I - \lambda L)^{-1} G(\lambda, x). \tag{1.3}$$

Since the right hand side of (1.3) is $0(||x||)$ for x near 0 while the left hand side not, $(\lambda, 0)$ is an isolated solution of (1.1) ir $\{\lambda\} \times X$ uniformly in λ, for λ neat μ. Consequently $(\mu, 0)$ cannot be a bifurcation point.

The above necessary condition is not sufficient as simple examples show. E.G. let $X = R^2$, $x := (x, y)$ and

$$\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -y^3 \\ x^3 \end{pmatrix}. \tag{1.4}$$

Multiplying the first equation by y, the second by x and substracting, shows that (1,(0,0)) is not a bifurcation point.

In Chapter 4 of his book [Kr], Krasnoselski has given a general sufficient condition for a point to be a bifurcation point of (1.1) with respect to the line of trivial solution - within the category of compact operators. Though we shall present a more general result later, we first present this result. The proof can be also find in [Ni].

Theorem 1.2. *(Krasnoselski 1964). If $\mu \in r(L)$ is of odd multiplicity, $(\mu, 0)$ is a bifurcation point of (1.1). (The multiplicity of a characteristic value μ is: $n_\mu = \dim \bigcup_{j \in N} \ker(I - \mu L)^j$).*

Rabinowitz proved that bifurcation in this situation is a *global*, rather than a *local* phenomenon.

Theorem 1.3. *(Rabinowitz 1971). If $\mu \in r(L)$ is of odd multiplicity n_μ, S possesses a component C containing $(\mu, 0)$. Moreover either*

(a) C is unbounded or

(b) C contains $(\hat{\mu}, 0)$ where $\hat{\mu} \in r(L)$ and $\hat{\mu} \neq \mu$.

The original proof of this theorem can be found in ([Ra]), but there is also another proof of this theorem in ([Ni]), using a very nice lemma of Ize ([Iz] or [Ni]).

2. Degree theory for proper Fredholm maps of index zero

We shall be discussing an extension of the Leray-Schauder degree theory using some simple techniques of differential topology. This part follows the ideas of Elworthy and Tromba ([El,Tr1], [El,Tr2]) and it is presented on infinite dimensional real Banac spaces. Unless the contrary is explicitly stated, X and Y are real Banach spaces.

Definition 2.1. The linear operator $A : X \to Y$ is a *Fredholm operator* iff:

(a) A is a continuous operator,

(b) the kernel of A, ker$A$, has finite dimension and

(c) the range of A, im$A$, has finite codimension. It is well known (see e.g. [Go that im$A$ is a closed subspace of Y and then the codimension of im$A$ is defined,

$$\text{codim im}A = \dim(Y/\text{im}A).$$

With each Fredholm operator, we can associate an integer - its *Fredholm index*:

$$\text{ind}A = \dim\ker A - \text{codim im}A.$$

A linear operator is bounded iff it is continuous ([Ze]). The set $L(X,Y)$ o bounded (continuous) linear operators from X to Y together with the norm

$$||A|| = \sup\{||Ax||;\ ||x|| \leq 1\}$$

forms an infinite dimensional real Banach space.

The basic results about Fredholm operators can be found in [Pa].

In his paper [Sm], Smale introduced the definition of the Fredholm map between Banach manifolds.

Definition 2.2. If $D \subseteq X$ is a domain of X (i.e. an open connected subset of X), then a C^1-map $f : \overline{D} \to Y$ is a *Fredholm map* iff for each $x \in D$, the differential of f, $Df_x : X \to Y$ is a Fredholm operator. The index of f, ind$f$, is defined to be indDf_x for some x. In [Gk,Ke] is proved that the set $\phi(X,Y)$ of Fredholm operators from X to Y is open in the space $L(X,Y)$ of all bounded (continuous) operators in the norm topology and the index is continuous in $\phi(X,Y)$. So, since f is C^1 and D is connected, the definition does not depend on x.

Finally, in this section we shall present some notations and definitions which will be useful in the sequel.

Spaces of the form $L(X,X)$ will be denoted by $L(X)$. Let $\phi_n(X,Y)$ be the set of Fredholm operators of index n; $L_c(X,Y)$ the set of all $T \in L(X,Y)$ which are compact operators; $L_f(X,Y)$ the set of all $T \in L(X,Y)$ with finite dimensional rank

(rank$T$ = dim im$T$). Then $L_f(X,Y) \subseteq L_c(X,Y)$ and those two sets are linear subspaces of $L(X,Y)$. For any $T \in L(X,Y)$ we define

$$T - L_f(X,Y) = \{T - C \in L(X,Y);\ C \in L_f(X,Y)\}$$

and

$$T - L_c(X,Y) = \{T - C \in L(X,Y);\ C \in L_c(X,Y)\}.$$

In [La] it is shown that $L_c(X,Y)$ is closed in $L(X,Y)$ and if $T \in \phi_n(X,Y)$, then $T - L_c(X,Y) \subseteq \phi_n(X,Y)$ and, of course, $T - L_f(X,Y) \subseteq \phi_n(X,Y)$. Because the identity map $I: Y \to Y$ belongs to $\phi_0(Y)$, we obtain a very important class of Fredholm operators of index zero which are the maps of the set $I - L_c(Y)$. Let $GL(X,Y)$ be the subgroup of $L(X,Y)$ of all invertible maps and

$$GL_c(X) = (I - L_c(X)) \cap GL(X),$$

$$GL_f(X) = (I - L_f(X)) \cap GL(X).$$

It is well known (see e.g. [Šv]) that $GL_f(X)$ and $GL_c(X)$ have two path-components each, which are $GL_f^+(X)$ and $GL_f^-(X)$ and respectively $GL_c^+(X)$ and $GL_c^-(X)$.

Definition 2.3. If $D \subseteq X$ is a domain of X and $f: \overline{D} \to Y$ is a C^k-map ($k \leq \infty$) such that $Df_x : X \to Y$ is in some subset σ of $L(X,Y)$ for all $x \in X$, then f is called a $C^k - \sigma$-map. The $C^k - \phi_m(X,Y)$-maps are also called $C^k - \phi_m$-maps.

Definition 2.4. A $\phi(I)$-map is a map $f: X \to X$ with property that $Df_x \in I - L_c(X)$ for all $x \in X$.

Definition 2.5. Let $f: X \to Y$ be a continuous map between Banach spaces X and Y; f is proper iff for each compact subset $K \subseteq Y$, $f^{-1}(K)$ is compact in X.

Theorem 2.6. *If $f: X \to Y$ is continuous and proper, then it is closed.*

Proof. Let $A \subseteq X$ be a closed subset of X. We have to show that if any sequence $f(a_n)$ converges in Y to y with $a_n \in A$, then $y \in f(A)$. The set consisting of y and of the $f(a_n)$ is compact, so its preimage by f is a compact set which contains (a_n). The (a_n) have a subsequence converging to some x which must be in A, because A is closed. By continuity, $f(x) = y$, hence $y \in f(A)$. □

Definition 2.7. A Banach space X is said to admit a C-structure modeled on a Banach space Y iff there is a collection of charts $\{(U_i, \gamma_i)\}$ covering X, $\gamma_i : U_i \to \gamma_i(U_i) \subseteq Y$, such that

$$D(\gamma_i \circ \gamma_j^{-1})_{\gamma_j(x)} \in GL_c(Y) \quad \forall\, x \in X.$$

The following theorem shows that $C^r - \phi_0$-maps are often $C^r - \phi(I)$-maps.

Theorem 2.8. *(Pull back theorem) Let $f : X \to Y$ be a $C^r - \phi_0$-map. Then X admits an unique C-structure as a manifold modeled on Y with respect to which f become a $C^r - \phi(I)$-map.*

Proof. Let $a \in X$ be fixed. Then $Df_a : X \to Y$ is a linear Fredholm operator of index zero and we can consider the following splits:

$$X = \ker Df_x \oplus X_2$$

and

$$Y = Y_1 \oplus \mathrm{im} Df_a$$

where X_2 and Y_1 are corresponding complements of $\ker Df_a$ and $\mathrm{im} Df_a$ in X and Y respectively. Then every $x \in X$ is a pair of two elements (x_1, x_2) with $x_1 \in \ker Df_a$ and $x_2 \in X_2$ and we have the same form for every $y \in Y$, $y = (y_1, y_2)$ with $y_1 \in Y_1$ and $y_2 \in \mathrm{im} Df_a$. Then

$$\dim \ker Df_a = \dim Y_1 < \infty$$

and we can find $\theta : \ker Df_a \to Y_1$ like a fixed isomorphism between $\ker Df_a$ and Y_1. Let $\pi_1 : Y \to Y_1$ and $\pi_2 : Y \to \mathrm{im} Df_a$ be the canonical projections. We define

$$\Theta : X \to Y \qquad \Theta(x_1, x_2) = (\theta(x_1), \pi \circ f(x_1, x_2)).$$

Thus $D\Theta_a$ is a linear isomorphism. Then it follows from the Local Inverse Mapping Theorem that there is $U \subseteq X$ an open neighbourhood of a such that

$$\gamma = \Theta|_U : U \to V - \Theta(U) \subseteq Y$$

has an inverse $\gamma^{-1} : V \to U$. Hence, for every $y = (y_1, y_2) \in V$ we can compute:

$$f \circ \gamma^{-1}(y_1, y_2) = (\pi_1 \circ f \circ \gamma^{-1}(y_1, y_2), \pi_2 \circ f \circ \gamma^{-1}(y_1, y_2)) =$$

$$(\pi_1 \circ f \circ \gamma^{-1}(y_1, y_2), \pi_2 \circ \gamma \circ \gamma^{-1}(y_1, y_2)) = (\pi_1 \circ f \circ \gamma^{-1}(y_1, y_2), y_2) =$$

$$= (y_1, y_2) - (y_1 - \pi_1 \circ f \circ \gamma^{-1}(y_1, y_2), 0).$$

Thus we can write

$$f \circ \gamma^{-1} = I - h \in I - L_f(V, Y)$$

where

$$h : V \to Y \qquad h(y_1, y_2) = (y_1 - \pi_1 \circ f \circ \gamma^{-1}(y_1, y_2), 0)$$

has a finite dimensional range. Clearly, $D(f \circ \gamma^{-1})_{\gamma(x)} \in I - L_c(Y)$ for all $x \in U$.

For each $a \in X$ we obtain such a chart (U, γ). Thus this collection will define a C-structure with respect to which f become a $C^r - \phi(I)$-map if we can show that for every U_1, U_2 such that $U_1 \cap U_2 \neq \emptyset$, we have

$$D(\gamma_1 \circ \gamma_2^{-1})_{\gamma_2(x)} \in GL_c(Y) \quad \forall x \in U_1 \cap U_2.$$

Let $(U_1, \gamma_1), (U_2, \gamma_2)$ be such charts. Hence

$$D(f \circ \gamma_2^{-1})_{\gamma_2(x)} = D(f \circ \gamma_1^{-1})_{\gamma_1(x)} \circ D(\gamma_1 \circ \gamma_2^{-1})_{\gamma_2(x)}$$

and since

$$D(f \circ \gamma_1^{-1})_{\gamma_1(x)} \in I - L_c(Y)$$

and

$$D(f \circ \gamma_2^{-1})_{\gamma_2(x)} \in I - L_c(Y)$$

we have

$$D(\gamma_1 \circ \gamma_2^{-1})_{\gamma_2(x)} \in I - L_c(Y).$$

Because γ_1 and γ_2 are invertible maps on $U_1 \cap U_2$, we obtain

$$D(\gamma_1 \circ \gamma_2^{-1})_{\gamma_2(x)} \in GL_c(Y).$$

This also proved the uniqueness and completes the proof. □

Definition 2.9. A Banach manifold with C-structure modeled on the Banach space Y is orientable with respect to the C-structure iff there is a collection R of charts contained in the C-structure such that if $(U_i, \gamma_i), (U_j, \gamma_j) \in R$ and $U_i \cap U_j \neq \emptyset$, then

$$D(\gamma_i \circ \gamma_j^{-1})_{\gamma_j(x)} \in GL_c^+(Y) \quad \forall\, x \in U_i \cap U_j.$$

We shall take R to be maximal and we shall call it an orientation. We say that the Banach space X is completely orientable like a Banach manifold on the Banach space Y, if it is orientable with respect to every C-structure it admits.

Definition 2.10. Let $D \subset X$ be an open subset of X and $f : \overline{D} \to Y$ be a C^1-map. A point $x \in D$ is called a regular point of f iff the linear map $Df_x : X \to Y$ is bijective, and it is called critical or singular iff is not regular. The images of the critical points under f are called the critical values and their complement on Y, the regular values. Note that if $y \in Y$ is a regular value, then $f^{-1}(y)$ is either empty or consists of regular points only.

We recall here, without proof (see [Sm]), an extension Sard-Smale's theorem which is an extension of the well-known Sard's theorem.

Theorem 2.11. *(Sard-Smale 1964). Suppose $f \in \phi_n(X,Y)$ is C^r with $r > \max(n,0)$. Then the set of critical values has measure zero on Y. Moreover, if $y \in Y$ is a regular value, then $f^{-1}(y)$ is a subspace of X whose dimension is equal to n, the Fredholm index of f, or is empty.*

Justified by the motivation presented in [El,Tr1] and [Sm], we shall define the Brouwer degree for proper $C^r - \phi_0$-maps with $r \geq 2$. It follows from Sard-Smale's theorem that for $y \in Y$ a regular value of f, $f^{-1}(y)$ is either empty or a closed subspace of X of dimension equal to zero. Since f is proper, it follows that $f^{-1}(y) \subseteq D$ is either empty or consists of a finite number of regular points. Thus we have the following two definitions:

Definition 2.12. Let X be completely orientable like a Banach manifold modeled on Y, $f : X \to Y$ be a $C^r - \phi_0$-map and $x \in X$ be a regular point. We consider X with the C-structure modeled on Y defined on Theorem 2.8. Then we define $signDf_x = 1$ iff for some chart (U,γ) of x, $D(f \circ \gamma^{-1})_{\gamma(x)} \in GL_c^+(X)$ and $signDf_x = -1$ iff $D(f \circ \gamma^{-1})_{\gamma(x)} \in GL_c^-(X)$. Clearly, $signDf_x$ is independent of the chart selected.

Further, let $D \subseteq X$ be a bounded domain of X and $f : \overline{D} \to Y$ a proper $C^r - \phi_0$-map, $r \geq 2$. We may define an integer $dg(f,D,y)$ by looking at the inverse image of the regular point y of f lying in the component of y in $Y \setminus f(\partial D)$. This is possible since $f(\partial D)$ will be closed because a proper map is a closed map (see Theorem 2.6).

Definition 2.13. (Definition of the Brouwer degree for regular values) Let $y \in Y \setminus f(\partial D)$ be a regular value of f. The degree of f at y relative to D, $dg(f,D,y)$, is defined to be zero if $f^{-1}(y) \cap D = \emptyset$ or

$$dg(f,D,y) = \sum_{x \in f^{-1}(y) \cap D} signDf_x.$$

Remark. (i) This Brouwer degree is constant for all regular values of f which are in the same component of $Y \setminus f(\partial D)$ (see [El,Tr1]).

(ii) In their paper [El,Tr1], Elworthy and Tromba show that if $f : \overline{D} \to X$ belongs to the $I - L_c(X)$ and is smooth of class C^2, the Leray-Schauder degree and the Brouwer degree defined in this section are equals,

$$deg(f, D, y) = dg(f, D, y).$$

In fact, we consider X with the C-structure $\{(U_i, \gamma_i)\}$ modeled on Y. Thus we can define locally the Brouwer degree $dg(f, U_i, y)$ to be the Leray-Schauder degree of $f \circ \gamma_i^{-1}$, that is

$$dg(f, U - i, y) = deg(f \circ \gamma_i^{-1}, \gamma_i(U_i), y)$$

when $y \in Y \setminus \{f(\partial U_i) \cup f \circ \gamma_i^{-1}(\partial \gamma_i(U_i))\}$.

Definition 2.14. (Definition of the Brouwer degree for critical values) Let $z \in Y \setminus f(\partial D)$ be a critical value of f. By Sard-Smale's theorem, in any neighbourhood of z we can find $y \in Y \setminus f(\partial D)$ a regular value of f. By Remark (i), we may define:

$$dg(f, D, z) = dg(f, D, y).$$

This "oriented" degree has all the natural properties of a degree, so $dg(I, D, y) = 1$ if $y \in D$; if $dg(f, D, y) \neq 0$ than $\exists x \in D$ such that $f(x) = y$; decomposition of domain property; excision property; (see [El,Tr1], [El,Tr2]). But the homotopy invariance property appear in a weaker form. Hence, it is not an invariant of proper homotopy through ϕ_0-maps, a sign change may appear. In [El,Tr2] is given the following example. Suppose E is an infinite dimensional Hilbert space, $T \in GL_c^-(E)$, $D \subseteq E$ a bounded domain of E and $y \in E \setminus T(\partial D)$. Then $dg(T, D, y) = -1$, although $GL(E)$ is connected and so T is homotopic in $GL(E)$ to the identity map which has degree $+1$. To get a proper homotopy invariant through ϕ_0-maps one has to take the absolute value of the degree. It is homotopic invariant only through $\phi(I)$-maps, so only if we consider properly admissible homotopy, as follows:

Definition 2.15. If f and g are proper ϕ_0-maps, f is properly ϕ_0 homotopic to g iff there is a C^r proper map $F : [0,1] \times \overline{D} \to Y$ such that $F_t \in \phi_0(\overline{D}, Y)$ for all $t \in [0,1]$ and $F_0 = f$, $F_1 = g$. A proper ϕ_0-homotopy $F : [0,1] \times \overline{D} \to Y$ between f and g is said to be admissible with respect to a C-structure on X iff for each $t \in [0,1]$, F_t becomes a $\phi(I)$-map with respect to the same C-structure on X. In [El,Tr1] it is shown the way we may obtain from X both the statement that $[0,1] \times \overline{D}$ is a manifold with C-structure and it is orientable.

This definition of course implies that if f and g are properly admissible homotopic, they are $\phi(I)$-maps with respect to the same C-structure on X.

Combining the homotopy invariance property with decomposition of domain property and the excision property, we obtain also two more general forms of the homotopy invariance. Because they are useful in the sequel, we recall them here.

Theorem 2.16. *Suppose D_* is a bounded domain of $[0,1] \times X$ and $f : D_* \to X$ is a proper $C^r - \phi_0$-map, $r \geq 2$. Let f_t denote the map $x \mapsto f(t,x)$ and let*

$$D_t = \{x \in X;\ (t,x) \in D_*\} \subseteq X.$$

Suppose that $y \notin f_t(\partial D_t)$ for all $t \in [0,1]$.

(i) Then $|dg(f_t, D_t, y)|$ is independent of $t \in [0,1]$.

(ii) Suppose that $\forall t \in [0,1]$, f_t becomes $\phi(I)$-map with respect to the same C-structure on X. Then $dg(f_t, D_t, y)$ is independent of $t \in [0,1]$.

3. A generalization of Rabinowitz's theorem

Let X be a real Banach space with the norm $\|\cdot\|$ and $R \times X$ has the product topology. Suppose that the map $f : R \times X \to X$ defined by $f(\lambda, x) = (I - \lambda L)x - G(\lambda, x)$ satisfies the following assumptions:

(1) $L : X \to X$ is a linear map,

(2) $G : R \times X \to X$ is a nonlinear map and $G(\lambda, x) = 0(\|x\|)$ near $x = 0$ (this means $G(\lambda, x) \to 0$ as $\|x\| \to 0$) uniformly on bounded λ intervals.

(3) $\forall \lambda \in R$, the map $f_\lambda : X \to X$, $x \mapsto f(\lambda, x)$ is a proper $C^r - \phi_0$-map, $r \geq 2$.

Under the assumptions (1)-(3) we want to study the nonlinear eigenvalue problem

$$f(\lambda, x) = (I - \lambda L)x - G(\lambda, x) = 0 \tag{3.1}$$

which possesses the line of the trivial solutions

$$\{(\lambda, 0);\ \lambda \in R\}.$$

In order to present the generalization of Rabinowitz's theorem (Theorem 3.3), some terminology and two technical lemmas are needed first. Let S denote the closure of the set of nontrivial solution of (3.1). Thus the only trivial solutions in S are bifurcation point. By a δ-neighbourhood of a set A we mean the set of points within a distance δ of A. B_ρ denotes the closed ball in X of radius ρ centered at the origin.

Lemma 3.1. *If A and B are disjoint closed subsets of a compact set K such that no component of K intersects both A and B, there exists a separation $K = K_a \cup K_b$ where K_a and K_b are disjoint compact sets containing A and B respectively.*

For proof, see [Wh].

Lemma 3.2. *Let $\mu \in r(L)$ be isolated and let C denote the component of $S \cup \{(\mu, 0)\}$ to which $(\mu, 0)$ belongs. Suppose that*

(a) C is compact and

(b) does not contain $(\hat{\mu}, 0)$ for any $\hat{\mu} \in r(L)$, $\hat{\mu} \neq \mu$.

Then there exists a bounded open set $M \subseteq R \times X$ such that:

- *$C \subseteq M$,*
- *$S \cap \partial M = \emptyset$ and*
- *the only trivial solutions contained in M consist of the segment*

$$\{(\lambda, 0);\ |\lambda - \mu| < \varepsilon\}$$

for some $\varepsilon < \varepsilon_0$, where ε_0 is the distance from μ to $r(L) \setminus \{\mu\}$.

Proof. Let U_δ be a δ-neighbourhood of C, where $\delta < \varepsilon_0$. Therefore $\partial U_\delta \cap C = \emptyset$. Because of the only trivial solutions in S are bifurcation point and $\lambda \notin r(L)$, the trivial solution $(\lambda, 0)$ cannot be a bifurcation point and so it will be contained in a neighbourhood disjoint from S and a fortiori C. With the aid of this observation together with (b) we can assume that the only trivial solutions U_δ contains are

$$\{(\lambda, 0);\ |\lambda - \mu| < \delta\}.$$

Let $K = \overline{U}_\delta \cap S$. Then K is a compact metric space under the induced topology from $R \times X$. Since $\partial U_\delta \cap C = \emptyset$, be Lemma 3.1 there exist disjoint compact subsets K_1 and K_2 of K such that:

$$K_1 \supseteq C, \quad K_2 \supseteq (\partial U_\delta) \cap S \quad \text{and} \quad K = K_1 \cup K_2.$$

Thus if M is an ε-neighbourhood (in $R \times X$) of K_1, where $\varepsilon < \delta$ and less then the distance from K_1 to K_2 (which is a positive nonzero number), then M satisfies all requirements. □

Let

$$M_\lambda = \{x \in X;\ (\lambda, x) \in M\}$$

and

$$(\partial M)_\lambda = \{x \in X;\ (\lambda, x) \in \partial M\}.$$

We make the following hypotheses:

(4) $\mu \in r(L)$ is isolated such that $(\mu, 0)$ is a bifurcation point of (3.1) and after we change f_μ with respect to the C-structure specified in (6), μ becomes a characteristic value of odd multiplicity n_μ for the differential of h (by h we mean the operator with finite rank appeared in the new form of f_μ).

(5) Each M_λ is completely orientable like a Banach manifold on X.

(6) For ε sufficiently small, X has a C-structure (modeled on X) such that for $\lambda \in (\mu - \varepsilon, \mu + \varepsilon)$, every f_λ becomes a $\phi(I)$-map with respect to this C-structure.

Theorem 3.3. *Under the assumptions (1)-(6), S possesses a component C containing $(\mu, 0)$. Moreover either*

(a) C is not compact or

(b) C contains $(\hat{\mu}, 0)$ where $\mu \neq \hat{\mu} \in r(L)$.

Proof. If $(\mu, 0)$ is a bifurcation point of (3.1), then $(\mu, 0) \in S$ and S possesses a component C containing $(\mu, 0)$.

We suppose no alternatives of the theorem can appear. Then there exists a bounded open set M and $\delta > \varepsilon > 0$ as in Lemma 3.2. Foe $0 < |\lambda - \mu| \leq \delta$, $(\lambda, 0)$ is an isolated solution of (3.4.1) in $\{\lambda \times X$. Therefore there exists $\rho(\lambda) > 0$ such that $(\lambda, 0)$ is the only solution of (3.1) in $\{\lambda\} \times B_{\rho(\lambda)}$. Define

$$\rho(\lambda) = \rho(\mu + \delta) \quad \text{for} \quad \lambda > \mu + \delta$$

and

$$\rho(\lambda) = \rho(\mu - \delta) \quad \text{for} \quad \lambda < \mu - \delta.$$

By choosing $\rho(\mu \pm \delta)$ small enough, it follows from the properties of M that

$$B_{\rho(\lambda)} \cap (\partial M)_\lambda = \emptyset \quad \text{if} \quad |\lambda - \mu| \geq \delta.$$

Since for $\lambda \neq \mu$ there are no solution of (3.1) on $\{\lambda\} \times \partial(M_\lambda \setminus B_{\rho(\lambda)})$,

$$dg(f_\lambda, M_\lambda \setminus B_{\rho(\lambda)}, 0)$$

is defined. We shall prove that

$$dg(f_\lambda, M_\lambda \setminus B_{\rho(\lambda)}, 0) = 0 \quad \text{for} \quad \lambda \neq \mu \tag{3.2}$$

and then show that (3.2) is incompatible with the odd multiplicity assumption for μ.

Let $\lambda > \mu$ and $\lambda^* > \lambda$ be chosen so that $\lambda^* - \mu$ is greater than the diameter of M. Then $M_{\lambda^*} = \emptyset$. Defining

$$\rho = \inf\{\rho(\theta);\ \theta \in [\lambda, \lambda^*]\},$$

it follows from remarks made earlier that $\rho > 0$. Let

$$U = M \setminus \{[\lambda, \lambda^*] \times B_\rho\}.$$

Then U is a bounded open set in $[\lambda, \lambda^*] \times X$ and by construction $f(\zeta, x) \neq 0$ for all $(\zeta, x) \in \partial U$, where ∂U refers to the boundary of U in $[\lambda, \lambda^*] \times X$. Since for all $\zeta \in [\lambda, \lambda^*]$, $dg(f_\zeta, M_\zeta \setminus B_\rho, 0)$ is defined and $M_{\lambda^*} \setminus B_\rho = \emptyset$, we have

$$dg(f_{\lambda^*}, M_{\lambda^*} \setminus B_\rho, 0) = 0.$$

Since f_{λ^*} and f_λ are properly ϕ_0-homotopic, it follows from Theorem 2.16 that

$$|dg(f_\zeta, M_\zeta \setminus B_\rho, 0)| \equiv \text{constant} \quad \forall\, \zeta \in [\lambda, \lambda^*]$$

and hence

$$dg(f_\lambda, M_\lambda \setminus B_\rho, 0) = 0. \tag{3.3}$$

Let $intB_{\rho(\lambda)}$ denote the interior of the closed ball $B_{\rho(\lambda)}$. Since f_λ has no zeros in $\{\lambda\} \times (intB_{\rho(\lambda)}) \setminus B_\rho$,

$$dg(f_\lambda, (intB_{\rho(\lambda)}) \setminus B_\rho, 0) = 0. \tag{3.4}$$

By decomposition of domain property we have

$$dg(f_\lambda, M_\lambda \setminus B_\rho, 0) = dg(f_\lambda, M_\lambda \setminus B_{\rho(\lambda)}, 0) + dg(f_\lambda, (intB_{\rho(\lambda)}) \setminus B_\rho, 0) \tag{3.5}$$

and then relation (3.2) follows from relations (3.3)-(3.5) for $\lambda > \mu$. The argument for $\lambda < \mu$ is the same.

Let $\varepsilon > 0$ be sufficiently small such that we have the hypothesis (6) satisfied. For $|\lambda - \mu| < \varepsilon$, $dg(f_\lambda, M_\lambda, 0)$ is defined. Select $\underline{\lambda}, \overline{\lambda}$ so that

$$\mu - \varepsilon < \underline{\lambda} < \mu < \overline{\lambda} < \mu + \varepsilon.$$

By (6), f_λ and $f_{\overline{\lambda}}$ are properly admissible homotopic and it follows from Theorem 2.16 that

$$dg(f_{\underline{\lambda}}, M_{\underline{\lambda}}, 0) = dg(f_{\overline{\lambda}}, M_{\overline{\lambda}}, 0). \tag{3.6}$$

By decomposition of domain property and the fact that for $\lambda \notin r(L)$, $(\lambda, 0)$ is an isolated solution of (3.1) in $\{\lambda\} \times X$, we have

$$dg(f_{\underline{\lambda}}, M_{\underline{\lambda}}, 0) = dg(f_{\underline{\lambda}}, M_{\underline{\lambda}} \setminus B_{\rho(\underline{\lambda})}, 0) + dg(f_{\underline{\lambda}}, B_{\rho(\underline{\lambda})}, 0) \tag{3.7}$$

and

$$dg(f_{\overline{\lambda}}, M_{\overline{\lambda}}, 0) = dg(f_{\overline{\lambda}}, M_{\overline{\lambda}} \setminus B_{\rho(\overline{\lambda})}, 0) + dg(f_{\overline{\lambda}}, B_{\rho(\overline{\lambda})}, 0). \tag{3.8}$$

Combining (3.2), (3.6)-(3.8) we obtain

$$dg(f_{\underline{\lambda}}, B_{\rho(\underline{\lambda})}, 0) = dg(f_{\overline{\lambda}}, B_{\rho(\overline{\lambda})}, 0). \tag{3.9}$$

For $\varepsilon > 0$ sufficiently small $\underline{\lambda}$ and $\overline{\lambda}$ sufficiently close to μ), we can choose the same chart (V, γ) from the C-structure of X to compute the degrees of $f_{\underline{\lambda}}$ and $f_{\overline{\lambda}}$. Thus, in relation (3.9) we can use the Leray-Schauder degree:

$$deg(f_{\underline{\lambda}} \circ \gamma^{-1}, \gamma(B_{\rho(\underline{\lambda})}, 0) = deg(f_{\overline{\lambda}} \circ \gamma^{-1}, \gamma(B_{\rho(\overline{\lambda})}, 0)$$

and so the Leray-Schauder index

$$ind(f_{\underline{\lambda}} \circ \gamma^{-1}, 0) = ind(f_{\overline{\lambda}} \circ \gamma^{-1}, 0). \tag{3.10}$$

Since μ is a characteristic value of odd multiplicity n_μ for the differential of h ($f_\mu \circ \gamma^{-1} = I - h$), by index jump property of the Leray-Schauder degree, we obtain

$$ind(f_{\underline{\lambda}} \circ \gamma^{-1}, 0) = (-1)^{n_\mu} ind(f_{\overline{\lambda}} \circ \gamma^{-1}, 0). \tag{3.11}$$

Thus (3.10) and (3.11) are contradictory and the result follows. □

Conclusion. At first sight, the hypothesis (6) is a little bit too restrictive, but I think it can occur in a weaker form under some natural conditions. A further work may be devoted to prove this. Some examples anu applications in mathematical physics will be interesting to be found.

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THE STABILITY OF THE RITZ APPROXIMATE SOLUTION FOR A HYDRODYNAMICAL PROBLEM

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Abstract. In this paper we have performed a functional study for an operatorial differential equation obtained in [1] by using the Kantorovich variational method to a hydrodinamical problem (the flow of an potential incompressible fluid between two solid walls). By analyzing the properties of trigonometric functions (1.10) used in [1] as trial functions for the Ritz approximate solution, we prove the stability and boundedness of the conditioning number for the Ritz algorithm and we find the constants of the method which justify the numerical results obtained in [1].

1. Introduction

Problem formulation. An incompressible fluid in a two-dimensional, potential and stationary motion between two solid walls, $\overline{DA}$ and $\overline{CB}$ ($y = 0,\ u = y_p(x)$), fig.1 is considered. In the paper [1] it is shown that the determination of the stream function ψ, given by $\psi = \dfrac{qy}{y_p(x)} - v(x,y)$, where q (=const.) is the flow-rate in the Ox direction, is reduced to solving the following Dirichlet problem ($'' = d^2/dx^2$)

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = qy(y_p^{-1}(x))'' \text{ on } \Omega \tag{1.1}$$

$$v(x,0) = v(x, y_p(x)) = v(\pm a_0, y) = 0 \tag{1.2}$$

where

$$\overline{\Omega} = \{(x,y) |\ -a_0 \le x \le a_0,\ 0 \le y \le y_o(x)\}$$

$$(\psi_{3,4} \equiv \psi(\mp a_0, y) = qy_p^{-1}(\mp a_0)y). \tag{1.2'}$$

By using the Kantorovich variational method with the first approximate solution $v_1 = a_1(x)[y - y_p(x)]y$ and considering that $\overline{CB}$ is plane ($y_p \equiv p(t) = \alpha_0 a_0 t + \beta_0$) the

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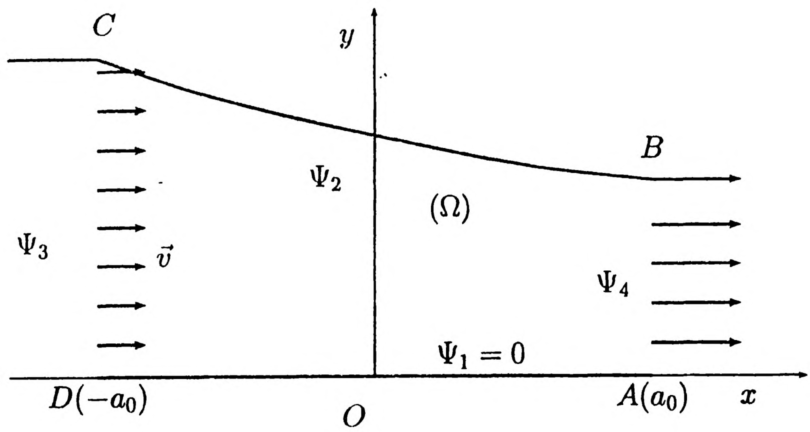

Figure 1

unknown function $a_1(x) \equiv u(t)$ verifies the self-adjoint operatorial equation [4]

$$Au = f(t) \text{ on } H = L_2(-1,1) \tag{1.3}$$

where $f = 5qp'^2p$. The differential operator $A : D(A) \subset H \to H$ is given in [1] by

$$Au = -\frac{d}{dt}\left(p^5(t)\frac{du}{dt}\right) + 10a_0^2p^3u \tag{1.4}$$

and

$$D(A) = \left\{u \in L_2(\Omega) |\ u \in C^2(\Omega) \cap C(\overline{\Omega});\ Au \in L_2(\Omega),\ \Omega = (-1,1),\ u(-1) = u(1) = 0\right\}.$$

On the energetic space H_A associated to $D(A)$,

$$H_A = \left\{u \in H_0^1(-1,1) |\ \|u\|_{H_0^1} = \|u\|_A\right\} \tag{1.5}$$

where $H_0^1(-1,1)$ is Sobolev space, the energetic norm

$$\|u\|_A^2 = \int_{-1}^1 p^3(p^2u'^2 + 10a_0^2u^2)dt \tag{1.6}$$

and the energetic scalar product

$$(u,v)_A = \int_{-1}^1 p^3(p^2u'v' + 10a_0^2uv)dt$$

introduced are defined.

The variational problem (equivalent to (1.3))

$$F(u) = \|u\|_A^2 - 10q\alpha_0^2a_0^2\int_{-1}^1 pu\,dt \to \min, \quad u \in H_A \tag{1.7}$$

has the Ritz solution (the n-order approximate solution)

$$u_n(x,y) = \sum_1^n c_k \varphi_k(x,y), \quad c_k \in \mathbb{R}^1 \tag{1.8}$$

which has been determined by solving the Ritz system

$$\sum_{k=1}^n K_{jk} c_k = b_j, \quad j = \overline{1,n}, \tag{1.9}$$

where $K_{jk} = (\varphi_j, \varphi_k)_A, \quad b_j = 5q\alpha_0^2 a_0^2 (p, \varphi_j)_{L_2(-1,1)}$.

With trigonometric trial functions $\varphi_k \in H_A$ (here $\varphi_k \in D(A)$)

$$\varphi_k(t) = \frac{1}{k\pi} \sin k\pi t, \quad t \in [-1,1] \tag{1.10}$$

the following results have been obtained in the $n = 3$ case ($a_0 = 1$, $\alpha_0 = -1/4$, $\beta_0 = 3/4$, $q = 1$, $u \equiv a_1$) [1]

$$a_1(x) = \sum_1^n c_k \varphi_k(x), \quad \psi(x,y) = \{y_p^{-1}(x) - a_1(x)[y - y_p(x)]\} y$$

$$c_1 = -0,01899506749, \quad c_2 = 0,00546525160, \quad c_3 = -0,00271135686$$

and in the $n = 4$ case

$$c_1 = -0,01899477457, \quad c_2 = 0,00543857854,$$

$$c_3 = -0,00242113158, \quad c_4 = 0,00189391796.$$

A good concordance has been obtained between the approximate stream lines $\psi(x,y) = k$ (const.), $0 < k < 1, \ldots$ and the exact ones $y = \frac{k}{4}(3-x)$. For example, in the $n = 3$ case, on the stream line $\psi = 3/4$ for $x = 1/2$ the corresponding values of y coordinate are $y = 0,46903$ (approximation value) and $y_e = 0,46875$ (exact value) with the error $\varepsilon = |y - y_e| = 0,00028$ ($< 3 \cdot 10^{-4}$).

The aim of this paper is to complete and justify results in order to conclude that the Ritz procedure may continue for higher approximation order by using trigonometric trial functions of the form (1.10). For this purpose the differentiability, completeness, basic functions are studied on the energetic space of the operator A.

2. The functional study of the operatorial equation (1.3). The properties of the operator A and the test functions φ_k on the energetic space H_A, (1.5)

Let us consider the auxiliary operator B defined by the value Bu and its domain $D(B)$:

$$Bu = -u''(t), \quad t \in (-1,1), \quad D(B) = D(A) \tag{2.1}$$

having the energetic space H_B and energetic norm $\|\cdot\|_B$ given by

$$H_B = H_0^1(-1,1), \quad \|u\|_B^2 = \int_{-1}^{1} u'^2 dx, \quad u \in H_B \text{ (norm Dirichlet)}. \tag{2.2}$$

a) Properties of operators A and B. It is known that B is a linear, symmetrical and positive definite operator, [4], on H_B and also, with a discrete spectrum (result of the Rellich theorem, A-positive definite).

remark 2.1. In this case the Rellich theorem which prove that H_B is compact imbedded in $L_2(-1,1)$, can be easily justified. Indeed, let M be a set of functions from H_B, bounded in H_B i.e. $\|u\|_B < C$ (const), $\forall\, u \in M$. Taking into account that

$$\|u\|_B = \|u'\|_{L_2(-1,1)} < C, \quad \forall\, u \in M \tag{$\tilde{\alpha}$}$$

and since all functions from $H_0^1(-1,1)$ are absolutely continuous, there exists the integral representation

$$u(t) = \int_{-1}^{t} u'(s)ds \quad \text{or} \quad u(t) = \int_{-1}^{1} k(s,t)u'(s)ds \tag{$\tilde{\beta}$}$$

if

$$k(s,t) = \begin{cases} 1, & -1 \le s \le t \\ 0, & t < s \le 1 \end{cases}$$

It is known that Fredholm's integral operator $Kv = u(t) = \int_{-1}^{1} k(s,t)v(s)ds$, $v \in L_1(-1,1)$ is compact on $L_2(-1,1)$ (it transforms any bounded set from $L_2(-1,1)$ into a compact set in $L_2(-1,1)$). So, if we put $v(t) = u'(t)$ (which is possible since $u \in H_0^1(-1,1)$ implies $u' \in L_2(-1,1)$) it follows from ($\tilde{\beta}$) and ($\tilde{\alpha}$) that the functions set $u \in M$ bounded in the norm $\|\cdot\|_B$ will be compact in the norm $\|\cdot\|_{L_2(-1,1)}$ (the imbeding of H_B onto $L_2(-1,1)$ is compact).

It follows that there exists an eigenvalue $\lambda_1 > 0$ (strictly positive, the lowest one) for the positive definite operator B which, according to the lowest eigenvalue theorem, is given by the equality:

$$\lambda_1 = \inf_{u\in H_B} \frac{(u,u)_B}{(u,u)_{L_2}} \quad \left(\leq \frac{\|u\|_B^2}{\|u\|_{L_2}^2}\right) \Rightarrow \|u\|_B^2 \geq \lambda_1 \|u\|^2_{L_2(-1,1)}, \quad u \in H_B. \tag{2.3}$$

On the other hand, the eigenvalues can be determined by solving the problem $-u'' = \lambda u$, $u \in D(B)$ with a solution $u(x) \not\equiv 0$; here we find $\lambda_1 = \pi^2/4$.

H_B and H_A can be identified with $H_0^1(-1,1)$ (since these spaces are formed with the same functions).

The operator A is linear, symmetrical and positive definite operator and semi-analogous one with B, A having a discrete spectrum. First, for the quadratic functional (Au, u) we have

$$(Au,u)_{L_2} \geq \gamma_0^2 \|u\|_B^2, \quad \gamma_0^2 = \min_{t\in(-1,1)} p^5(t) = p^5(1). \tag{2.4}$$

Hence, with (2.3), we obtain the inequality:

$$(Au,u)_{L_2} \geq \alpha^2 (u,u)_{L_2}, \quad u \in D(A) \quad (\subset L_2(-1,1)), \quad \alpha = \frac{\gamma_0 \pi}{2} \tag{2.5}$$

which proves that A is positive definite operator on $D(A)$ with the constant $\alpha^2 > 0$.

Since $D(A)$ is everywhere dense in H_A, the inequalities (2.5)-(2.4) can be extended to H_A in the form:

$$\|u\|_A \geq \gamma_0 \|u\|_B, \quad \|u\|_A \geq \alpha \|u\|_{L_2(-1,1)}; \quad u \in H_A. \tag{2.6}$$

Second, since the function u from $D(A) \subset H_A = H_0^1(-1,1)$ is absolutely continuous (the functions from $H_0^1(-1,1)$ are absolutely continuous) we are able to write that:

$$u \in H_0^1(-1,1) \Rightarrow u(t) = \int_{-1}^1 u'(s)ds \leq \sqrt{t+1}\left(\int_{-1}^1 u'^2 dt\right)^{1/2} \Rightarrow$$

$$\Rightarrow \int_{-1}^1 u^2 dt \leq 2\int_{-1}^1 u'^2 dt = 2\|u\|_B^2. \tag{2.6'}$$

Then, by dominating procedure, we obtain, for $\|\cdot\|_A$, the least upper bound:

$$\|u\|_A^2 \leq k_2^2 \|u\|_B^2, \quad k_2^2 = \max_{t\in(-1,1)} [p^5(t) + 20a_0^2 p^3(t)]. \tag{2.7}$$

Plus, the following energetic norm equivalence holds:

$$k_1 \|u\|_B \leq \|u\|_A \leq k_2 \|u\|_B, \quad u \in H_A \quad (= H_0^1(-1,1)) \tag{2.8}$$

where

$$k_1 = \gamma_0 = p^2(1)\sqrt{p(1)}, \quad k_2 = p(-1)\sqrt{p(-1)[p^2(-1) + 20a_0^2]}. \tag{2.9}$$

Having $H_A = H_B = H_0^1(-1,1)$ and the norm equivalence (2.8) we conclude that A and B are semianalogous operators.

At last, A has a discrete spectrum. To justify this, the same procedure can be used as for the operator B, replacing $(\tilde{\alpha})$ by

$$\|u\|_A^2 \geq p^5(1)\|u'\|_{L_2(-1,1)}^2.$$

b) Choice of test functions φ_k from H_A and their properties (standard basis). As test functions of the Ritz method the eigenfunctions of the operator A are usually chosen. These functions can not be easily found (and they are not with φ_k from (1.10)). But, for φ_k one can use the eigenfunctions of the operator B (with a discrete spectrum): for B there exist eigenvalues: $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \ldots$, $\lambda_n \to \infty$ to which corresponds the system of eigenfunctions $\{\psi_i\}_1^\infty$, linear independent, orthogonal and complete in the energetic space H_B (orthonormal and complete in $H = L_2(-1,1)$). The eigenvalues λ_k and the eigenfunctions ψ_k can be determined by solving the two-point boundary value problem $-\psi''(t) = \lambda\psi(t)$, $\psi(-1) = \psi(1) = 0$ (the case $\lambda \leq 0$ is excluded according to the boundary conditions).

We obtain:

$$\lambda_k = (k\pi)^2, \quad \psi_k(t) = \sin k\pi t; \quad k = 1, 2, \ldots; \tag{2.10}$$

$$\|\psi_k\|_H^2 = 1 \text{ in } H = L_2(-1,1)$$

i.e., $\{\psi_k\}$ is orthonormal in the space $H = L_2(-1,1)$. Then, from $B\psi_k = \lambda\psi_k$ we can write that $(B\psi_k, h)_{L_2} = \lambda_k(\psi_k, h)_{L_2}$, for any h from H_B and then, taking into account the definition (energetic scalar product) $(\psi_k, h)_B = (B\psi_k, h)$, $\forall\, h \in H_B$, we conclude that ψ_k and λ_k verify the identity:

$$(\psi_k, h)_B = \lambda_k(\psi_k, h), \quad \forall\, h \in H_B;$$

i.e. ψ_k and λ_k are generalized eigenfunctions and generalized eigenvalues respectively. With $h = \psi_k$ (element from H_B) we have $(\psi_k, \psi_k)_B = \|\psi_k\|_B^2 = \lambda_k$; so $\{\psi_k\}$ it is not orthonormal in H_B, [6].

Proposition 2.1. *The system of functions $\psi_k(t)$:*

$$\varphi_k(t) = \frac{\psi_k(t)}{\|\psi_k\|_B} = \frac{1}{k\pi}\sin k\pi t, \quad t \in [-1,1], \tag{2.11}$$

linear independent, complete and orthonormal in H_B (and orthogonal in H_A), is uniform linearly independent basis in the energetic space H_A i.e. for any linear combination (including the Ritz combination $\sum_1^n c_k\varphi_k$)

$$v_n(t) = \sum_1^n a_k\varphi_k(t), \quad \forall\ a = (a_1, \dots, a_n) \in \mathbb{R}^n \tag{2.12}$$

with

$$\|a\|_2^2 = \sum_1^2 a_i^2 \neq 0$$

the following inequalities hold

$$k_1^2\|a\|_2^2 \le \|v_n\|_A^2 \le k_2^2\|a\|_2^2 \tag{2.13}$$

with the constants k_1 and k_2, independent of n, given by (1.9).

Proof. Let us calculate, according to the norm definition in the space $L_2(-1,1)$, the following norms:

$$\left\|\frac{dv_n}{dt}\right\|_{L_2}^2 = \int_{-1}^{2} \sum_{i,j=1}^{n} a_i a_j \cos i\pi t \cos j\pi t\, dt = \sum_{i=1}^{n} a_i^2 = \|a\|_2^2;$$

$$\|v_n\|_{L_2}^2 \le 2\|a\|_2^2 \quad \text{(calculus like in (2.6'); } v_n \in H_0^2(-1,1)).$$

Now, let evaluate the energetic norm $\|v_n\|_A$ generated by the operator A given in (0.4), in which $p(1) \le p(t) \le p(-1)$, $t \in [-1,1]$

$$\|v_n\|_A^2 = \int_{-1}^{1} v_n A_n v_n\, dt \ge -p^5(1)\int_{-1}^{1} v_n\, dv_n' = p^5(1)\|a\|_2^2$$

$$\|v_n\|_A^2 = -\int_{-1}^{1} v_n\, d(p^5 v_n') + 10a_0^2\int_{-1}^{1} p^3 v_n^2\, dt \le p^3(-1)[p^2(-1) + 20a_0^2]\|a\|_2^2.$$

Hence, the basis $\{\varphi_k\}_1^n$, given in (2.11), has the property (2.13) (stability). □

remark 2.2. The standard basis $\{\varphi_k\}_1^n$ (orthonormal and complete in H_B and orthogonal in H_A) considered in (2.11), which satisfies (2.13) is called uniform linearly independent basis, [3].

3. The stability of the Ritz approximate solution $u_n(t)$ in the energetic space H_A

We will show that the $\{\varphi_k\}$ basis, (2.11), verifies a stability criterion of the Ritz solution u_n in the space H_A.

Let assume that in the Ritz system (1.9) written in a matrix form:

$$K_n AA^{(n)} = BB_n, \quad K_n = [K_{ij}]_{n \times n}, \quad AA^{(n)} = \left(c_1^{(n)}, \dots, c_n^{(n)}\right)^T, \quad BB_n = (b_1, \dots, b_n)^T$$

the matrices K_n, BB_n have been numerically calculated; their elements, K_{ij} and b_i have been approximated by decimal numbers (with calculus and quadrature errors). Hence, in reality, we have to solve, effectively, an inaccurate Ritz system (perturbated)

$$(K_n + \Delta K_n) AA^{(n)} = BB_n + \Delta BB_n \tag{3.1}$$

such that if $AA^{(n)}$ and $AA^{*(n)}$ are exact solutions for the exact Ritz system and perturbated respectively, we have the equalities:

$$(\bar{1}) \qquad K_n AA^{(n)} = BB_n;$$

$$(\bar{2}) \qquad (K_n + \Delta K_n) AA^{*(n)} = BB_n + \Delta BB_n; \quad AA^{*(n)} = AA^{(n)} + \Delta AA^{(n)} \tag{3.2}$$

to which correspond the exact Ritz solutions u_n and u_n^*:

$$u_n = \sum_{k=1}^{n} c_k^{(n)} \varphi_k, \quad u_n^* = \sum_{k=1}^{n} c_k^{*(n)} \varphi_k \tag{3.3}$$

with

$$\Delta u_n = u_n^* - u_n = \sum_{k=1}^{n} \Delta c_k^{(n)} \varphi_k$$

where δ indicates small perturbations and Δu_n is the error (called "hereditary") owed to the approximate construction of the Ritz system (3.1), and the modification with ΔK_n and ΔBB_n respectively.

We analyze the stability of u_n with respect to ΔK_n and ΔBB_n. If we use the property of linearly independence of $\{\varphi\}_1^n$, i.e. (2.13), we obtain the following results (properties):

P1. The perturbated matrix ΔK_n verifies the condition:

$$\|\Delta K_n\|_2 \equiv \left(\sum_{i,j=1}^{n} (\Delta K_{ij})^2 \right)^{1/2} \leq c_0, \quad c_0 \text{ (const) } > 0, \ \forall\, n \in \mathbf{N} \tag{3.4}$$

P2. The perturbated system ($\overline{2}$) has an unique solution ($\forall\ \Delta BB_n$). First, we notice that K_n is a positive definite matrix since we have:

$$\left(\forall\ a = (a_1, \ldots, a_n) \in \mathbb{R}^n, \quad \|a\|_2 = \left(\sum_1^n a_i^2\right)^{1/2} \neq 0, \quad a\ -\ \text{vector of (2.12)}\right)$$

$$a^T \cdot K_n a \equiv (K_n a, a)_2 = \sum_{i,j=1}^n a_i a_j (\varphi_i, \varphi_j)_A = \left(\sum_{i=1}^n a_i \varphi_i, \sum_{j=1}^n a_j \varphi_j\right)_A = (v_n, v_n)_A > 0 \tag{3.5}$$

remark 3.1. According to the extremal properties of the eigenvalues of symmetrical matrix M, we have: $(Ma, a)_2 \leq \rho\|a\|_2^2$, $\forall\ a \in \mathbb{R}^n$ (with $\rho = \max\{\lambda_1, \ldots, \lambda_n\}$ where λ_i are eigenvalues of the symmetrical matrix M) and the norm (Schmidt) $\|\Delta K_n\|_2 = |\rho(\Delta K_n)|$ $(= \max\{|\lambda_i|,\ 1 \leq i \leq n\})$.

Hence

$$(\Delta K_n a, a)_2 \geq -\|\Delta K_n\|_2 \|a\|_2^2 \tag{3.5'}$$

The matrix $K_n + \Delta K_n$ of the perturbated system ($\overline{2}$) is also positive definite since we have (taking into account the left member of (2.13) and (3.5')):

$$a^T \cdot (K_n + \delta K_n) a = a^T \cdot K_n a + a^T \cdot (\Delta K_n) a = (v_n, v_n)_A + a^T \cdot (\Delta K_n) a \geq$$

$$\geq k_1 \|a\|_2^2 - \|\Delta K_n\|_2 \|a\|_2^2 \geq (p^5(1) - \|\Delta K_n\|_2)\|a\|_2^2 \geq \beta\|a\|_2^2 > 0 \tag{3.6}$$

if we choose $\|\Delta K_n\|_2 < c_0$ so that $c_0 < p^5(1)$ and we put

$$\beta = p^5(1) - c_0; \quad (\beta > 0) \tag{3.6'}$$

The matrix $K_n + \Delta K_n$ being positive definite, the system (2.2) – ($\overline{2}$) has an unique solution.

Property 3.. *The following estimation holds: there exist the positive constants* $c)1$ *and* c_2*, independent of* n*, so that:*

$$\|\Delta u_n\|_A \leq c_1 \|\Delta K_n\|_2 + c_2 \|\Delta BB_n\|_2; \tag{3.7}$$

and furthermore, the constants c_1 *and* c_2 *have the values:*

$$c_1 = \frac{k_2}{k_1} \frac{1}{\alpha\beta} \|f\|_{L_2(-1,1)}; \quad c_2 = \frac{1}{k_1}\left(1 + \frac{c_0}{\beta}\right); \tag{3.8}$$

where k_1 and k_2 are given in (2.9), $\alpha = \gamma_0(\pi/2) = \sqrt{p^5(1)}(\pi/2)$, β is given in (2.6') and c_0 is defined by means of the property P1.

Proof. Let us return to (2.2). Replacing c^* in $(\overline{2})$ by its expression and taking into account $(\overline{1})$ we obtain (after the scalar multiplication by $\Delta AA^{(n)T}$; T denoting the transpose matrix):

$$\Delta AA^{(n)T} \cdot K_n \Delta AA^{(n)} = \Delta AA^{(n)T} \cdot (-\Delta K_n AA^{*(n)} + \Delta BB_n); \quad (P_s \equiv P_d) \tag{3.9}$$

Let us estimate both members (denoted by P_s and P_d) of (3.9) by using (3.2). If we choose

$$a = \Delta c^{(n)} = c^{*(n)} - c^{(n)}, \quad v_n = \sum_1^n \Delta c_k^{(n)} \varphi_k = \Delta u_n, \tag{3.9'}$$

with (3.5) we obtain

$$P_s = (\Delta u_n, \Delta u_n)_A = ||\Delta u_n||_A^2 \geq \alpha^2 ||\Delta u_n||_H^2; \quad \text{(owing to (1.6))}.$$

Now, let us consider the right member P_d. Using (3.2) we can write:

$$||P_d||_2 \leq \left(||\delta K_n||_2 ||AA^{*(n)}||_2 + ||\Delta BB_n||_2\right) ||\delta AA^{(n)}||_2 \equiv$$

$$\equiv N\left(||AA^{*(n)}||_2\right) ||\Delta A^{(n)}||_2 \leq N\left(||c^{*(n)}||_2\right) \frac{||\Delta u_n||_A}{\sqrt{p^5(1)}} \tag{3.9''}$$

$$\left((1.13) \;\Rightarrow\; p^5(1)||\Delta A^{(n)}||_2^2 \leq ||\delta u_n||_A^2\right).$$

Using (3.9') and (3.9"), from (3.9) it follows that

$$\alpha ||\delta u_n||_H \leq ||\delta u_n||_A \leq \frac{1}{\sqrt{p^5(1)}} N\left(||AA^{*(n)}||_2\right). \tag{3.10}$$

We calculate the Euclidean norm of the vector $AA*(n)$ which verifies the perturbated $(\overline{2})$. According to (2.6) ($K_n + \Delta K_n$ is a positive definite matrix) we obtain:

$$\beta ||AA^{*(n)}||_2^2 \leq A^{*(n)T} \cdot (K_n + \Delta K_n) AA^{*(n)} = AA^{*(n)} \cdot (BB_n + \Delta BB_n) =$$

$$= c^{*(n)} \cdot BB_n + c^{*(n)} \cdot \Delta b_n = (f, u_n^*)_H + AA^{*(n)} \Delta BB_n \leq ||f||_H ||u_n^*||_H + ||c^{*(n)}||_2 ||\Delta BB_n||_2$$

$$\left(AA^{*(n)} BB_n = \sum_{i=1}^n c_i^{*(n)} b_i = \sum_{i=1}^n (f, \varphi_i)_H c_i^{*(n)} = (f, u_n^*)_H\right).$$

We have, after (2.13), the inequalities

$$||u_n^*||_H \leq \frac{1}{\alpha} ||u_n^*||_A \leq (k_2/\alpha) ||AA^{*(n)}||_2,$$

hence

$$\|AA^{*(n)}\|_2 \leq \frac{1}{\beta}\left(\frac{k_2}{\alpha}\|f\|_H + \|\Delta BB_n\|_2\right). \tag{3.11}$$

By using the estimation (3.11) from (3.10) we obtain (3.7) with the constants c_1 and c_2 given by (3.8). □

We admit the following definition: if the properties P1-P3 hold, we say that the Ritz solution is stable in the energetic space H_A.

So, the uniform linearly independent system $\{\varphi_k\}$, (2.11), assures the stability of the Ritz solution u_n in H_A (in the sense of the definition) and because of that it can be used in the Ritz method for high order approximations.

4. The solving of the Ritz system. The calculus error and the Ritz matrix conditioning number

First let us mention that the Ritz matrix K_n, being a positive definite matrix (non-singular), the Ritz system has the grade n (algebraic Cramer system). Let $AA^{(n)}$ the calculated solution, (manual or by computer) by a direct method (Gauss - we exclude the iterative methods which are always approximate methods and which yield different kind of errors). The solutions $AA^{(n)}$ and $\overline{AA}^{(n)}$ verify the equations:

$$K_n AA^{(n)} = BB_n, \quad K_n\overline{AA}^{(n)} = BB_n + \delta BB_n; \quad (K\overline{AA}^{(n)} \neq BB_n) \tag{4.1}$$

where δBB_n are perturbations (residue) owed to the rounding error admitted in the arithmetical operations performed in the solving method of the algebraic (linear) system. We deduce at once the relative error in the form ($\|\cdot\|$ - any norm; A, B - matrix $\|AB\| \leq \|A\| \cdot \|B\|$):

$$\frac{\|\overline{AA}^{(n)} - \mathring{A}^{(n)}\|}{\|AA^{(n)}\|} \leq \mu(K_n)\frac{\|\delta BB_n\|}{\|b_n\|} \quad \text{with} \quad \mu(K_n) = \|K_n\| \cdot \|K_n^{-1}\| \tag{4.2}$$

where $\mu(K_n)$ is the conditioning number of non-singular Gram matrix K_n; this number has to be bounded (preferably as little as possible).

The Ritz matrix K_n (symmetric and positive definite) has the spectral norms:

$$\|K_n\|_s = \lambda_M \quad \text{and} \quad \|K_n^{-1}\|_s = \frac{1}{\lambda_m}.$$

It follows that

$$\mu(K_n) = \frac{\lambda_M(K_n)}{\lambda_m(K_n)}.$$

The eigenvalues $\lambda_i(K_n)$ of the matrix K_n are n (positive) real numbers of Therefore, we denote $\lambda_i(K_n) = \lambda_i^{(n)}$ (> 0) and consider that $\lambda_m^{(n)} = \lambda_1^{(n)} \leq \cdots$ $\lambda_n^{(n)} = \lambda_M^{(n)}$. It is known that for $n \to \infty$, $\lambda_1^{(n)}$ does not increase and $\lambda_n^{(n)}$ does n decrease; so $\mu(K_n)$ does not decrease and if $\lambda_1^{(n)} \to 0$ and $\lambda_n^{(n)} \to \infty$ (with $n \to \infty$ then $\mu(K_n) \to \infty$. For $\mu(K_n)$ to rest bounded it is necessary and sufficient that, all n, to have $\lambda_0 \leq \lambda_k^{(n)}(K_n) \leq \Lambda_0$ where the bounds λ_0 and Λ_0 are positive consta independent of n.

remark 4.1. In this case we can say that the system of trial functions $\{\varphi_k\}_1^n$, wh defines the Gram matrix K_n, is almost orthonormal in the energetic space H_A (a particularly strong minimal in H_A).

Hence, we must have $\mu(K_n) \leq \Lambda_0/\lambda_0$. Let us evaluate the limits λ_0 and Λ_0. do this, we consider $\{\varphi_k\}_1^n$ given in (2.11) and Gram matrix (in H_A and H_B) K_n $[(\varphi_i, \varphi_j)_A]$ and $K_n^{(B)} = [(\varphi_i, \varphi_j)_B] = [\delta_{ij}]$ having $n \times n$ dimension and the correspondi eigenvalues $\lambda_1^{(n)} \leq \cdots \leq \lambda_n^{(n)}$ and $\mu_k = 1$ ($k = \overline{1,n}$) with $\lambda_0^B = \Lambda_0^B = 1$ (since $\{\varphi_k\}_1^n$ orthonormal in H_B). Let $a = (a_1, \ldots, a_n)$ be an arbitrary vector in $\mathbb{R}^n$ with the Euclid norm $||a||_2^2 = \sum_1^n a_k^2 \neq 0$ and let $\sum_1^n a_k\varphi_k$ ($\equiv v_n \in H_A$) be an arbitrary element fr H_A. From (3.6) and (3.2) we obtain:

$$\lambda_1^{(n)} = \min_{a\in\mathbb{R}^n} \frac{(K_n a, a)_2}{(a,a)_2} = \min_{a\in\mathbb{R}^n} \frac{\left\|\sum_1^n a_k\varphi_k\right\|_A^2}{||a||^2} = \min_{a\in\mathbb{R}^n} \frac{||v_n||_A^2}{||a||_2^2} \geq k_1^2$$

$$\lambda_n^{(n)} = \max_{a\in\mathbb{R}^n} \frac{(K_n a, a)_2}{(a,a)_2} \leq k_2^2.$$

Hence it results that $\{\varphi_k\}_1^n$, (2.11), is a strong minima; system in H_A (it assu the stability) and that for the conditioning number $\mu(K_n)$ we obtain the estimation (le upper bound; $\mu(K_n) \geq 1$):

$$\forall\, n \in \mathrm{N}, \quad \mu(K_n) \leq \frac{k_2^2}{k_1^2} = \frac{p^3(-1)[p^2(-1) + 20a_0^2]}{p^5(1)}; \qquad (4$$

if we use (2.9).

So, the solution of the Ritz system is stable: in the studied problem and appr imatively solved with the trial functions $\{\varphi_k\}$ given by (2.11), the algebraic (exact) R n-order system can be replaced by the approached algebraic system.

Conclusions

It is known that in the practical application of the Ritz method there appear two main difficulties: the choice of trial functions and the assurance of the stability of the method. Both these aspects have been theoretically studied in this paper in order to justify some good numerical results obtained in [1] by applying the Kantorovich-Ritz variational method to a hydrodinamical problem. Our paper shows that the system of trial functions (1.10) assures the convergence and stability of the Ritz method:

1. - a stability criterion of the Ritz solution is tested

2. - the least upper bound of the conditioning number of the Ritz matrix is evaluated

3. - the constants which appear in the inequalities for stability and for boundedness of the conditioning number are numerically (effectively) determined.

In conclusion, the Ritz variational method, used in [1], can be applied with trial functions (1.10) to a more general problem, as the flow of an ideal fluid between curvilinear walls.

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A NEW LINEAR POSITIVE OPERATOR IN TWO VARIABLES ASSOCIATED WITH THE PEARSON'S χ DISTRIBUTION

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Abstract. In this paper we defined a new linear positive operator, associated to a Pearson's χ^2 distribution, by using a probabilistic method, and we studied its approximation properties.

1. Introduction

In our papers [1], [3], we defined and investigated a linear positive operator, which was associated, using a probabilistic method, with Pearson's χ^2 distribution:

$$(L_n f)(x) = e\left[f\left(\frac{1}{n}\sum_{k=1}^{n} X_k^2\right)\right] = \frac{1}{(2x)^{n/2}\Gamma\left(\frac{n}{2}\right)}\int_0^{\infty} t^{\frac{n}{2}-1}e^{-t/(2x)} f\left(\frac{t}{n}\right) dt,$$

where the sequence of independent random variables $(X_k)_{k\in N^*}$ having the same normal distribution $N(0,\sqrt{x})$, $x>0$, $E(X_k)=0$, $D^2(X_k)=x$, $(\forall)k\in N^*$ and f is a real function bounded on $(0,+\infty)$ such that the mean value of the random variable $f\left(\frac{1}{n}\sum\limits_{k=1}^{n} x_k^2\right)$ exists, for any $n\in N^*$.

Now, we consider the following extension of this operator, to the case of two variables:

$$L_n(f;x,y) = \frac{1}{(4xy)^{n/2}\left(\Gamma\left(\frac{n}{2}\right)\right)^2}\int_0^{\infty}\int_0^{\infty}(uv)^{\frac{n}{2}-1}e^{-\frac{1}{2}\left(\frac{u}{x}+\frac{v}{y}\right)} f\left(\frac{u}{n},\frac{v}{n}\right) du\,dv \tag{1.1}$$

where f is a given function on defined and bounded over

$$\Omega_2 = \{(x,y)\in R^2 |\ x>0,\ y>0\}.$$

This new operator can be obtained by a probabilistic method, which was presented in [4].

Indeed, we consider a sequence of 2-dimensional random vectors $Z_n = (X_n, Y_n)$, $n\in N^*$, where X_n for any $n\in N^*$ are independent random variables having the same

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normal distribution $N(0,\sqrt{x})$, $x > 0$, $E(X_n) = 0$, $D^2(X_n) = x$, $n \in N^*$ and Y_n any $n \in N^*$ are independent random variables having the same normal distribu $N(0,\sqrt{y})$, $y > 0$, $E(Y_n) = 0$, $D^2(Y_n) = x$, $n \in N^*$. Then, the arithmetic mean the first n components X_k^2, $\frac{1}{n}\sum_{k=1}^{n} X_k^2$, have a Pearson chi^2 distribution with n degree freedom and also $\frac{1}{n}\sum_{k=1}^{n} Y_k^2$ have a Pearson chi^2 distribution with n degrees of freed Also we consider that, for any $n \in N^*$ the components X_n and Y_n are independent.

If f is a real function bounded on $(0,+\infty)\times(0,+\infty)$, such that the mean vali the random variable $f\left(\frac{1}{n}\sum_{k=1}^{n} X_k^2, \frac{1}{n}\sum_{k=1}^{n} Y_k^2\right)$ exists, for any $n \in N^*$, then (1.1) becoi

$$L_n(f;x,y) = E\left(f\left(\frac{1}{n}\sum_{k=1}^{n} X_k^2, \frac{1}{n}\sum_{k=1}^{n} Y_k^2\right)\right).$$

2. Approximation properties of the operator

In this section we investigate the approximation properties of the operator (

Theorem 2.1. *If f is a bounded uniform continuous function on $(0,a)\times(0,a)$, a then the sequence $\{L_n(f;x,y)\}_{n\in N^*}$ convergence to $f(x,y)$ uniformly on $(0,a)\times($*

Proof. In accordance with the Bohmann-Korovkin-Volkov theorem of two variable [6], it is sufficient that

$$L_n(1;x,y) = 1; \quad L_n(u;x,y) \xrightarrow[n\to\infty]{} x; \quad L_n(v;x,y)n \xrightarrow[n\to\infty]{} y$$

$$L_n(u^2;x,y) \xrightarrow[n\to\infty]{} x^2; \quad L_n(uv;x,y) \xrightarrow[n\to\infty]{} xy; \quad L_n(v^2;x,y) \xrightarrow[n\to\infty]{} y^2$$

or, in accordance with Stancu [4] to have for the variances of the components:

$$\lim_{n\to\infty} \sigma_{n,1}^2 = \lim_{n\to\infty} D^2\left(\frac{1}{n}\sum_{k=1}^{n} X^2\right) = 0, \quad \lim_{n\to\infty}\left(\sigma_{n,2}^2 = D^2\left(\frac{1}{n}\sum_{k=1}^{n} Y_k^2\right)\right) = 0.$$

But,

$$L_n(1;x,y) = \frac{1}{(4xy)^{n/2}\left(\Gamma\left(\frac{n}{2}\right)\right)^2}\int_0^\infty u^{\frac{n}{2}-1}e^{-u/(2x)}du\int_0^\infty v^{\frac{n}{2}-1}e^{-v(2y)}dv =$$

$$= \frac{(2x)^{n/2}(2y)^{n/2}}{(4xy)^{n/2}\left(\Gamma\left(\frac{n}{2}\right)\right)^2}\underbrace{\int_0^\infty z^{\frac{n}{2}-1}e^{-z}dz}_{\Gamma\left(\frac{n}{2}\right)}\underbrace{\int_0^\infty t^{\frac{n}{2}-1}e^{-t}dt}_{\Gamma\left(\frac{n}{2}\right)} = 1,$$

$$L_n(u;x,y) = \frac{1}{(4xy)^{n/2}\left(\Gamma\left(\frac{n}{2}\right)\right)^2}\underbrace{\int_0^\infty u^{\frac{n}{2}}e^{-u/(2x)}\frac{1}{n}du}_{(2x)^{\frac{n}{2}+1}\frac{1}{n}\Gamma\left(\frac{n}{2}+1\right)}\underbrace{\int_0^\infty v^{\frac{n}{2}-1}e^{-v/(2y)}dv\,dv}_{(2y)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} =$$

$$= \frac{2x}{n}\frac{n}{2} = x,$$

$$L_n(v;x,y) = \frac{1}{(4xy)^{n/2}\left(\Gamma\left(\frac{n}{2}\right)\right)^2} \underbrace{\int_0^\infty u^{\frac{n}{2}-1}e^{-u/(2x)}du}_{(2x)^{n/2}\Gamma(\frac{1}{2})} \underbrace{\int_0^\infty v^{n/2}e^{-v/(2y)}\frac{1}{n}dv\,dv}_{(2y)^{\frac{n}{2}+1}\frac{1}{n}\Gamma(\frac{n}{2}+1)} = \frac{2y}{n}\frac{n}{2} = y,$$

$$L_n(u^2;x,y) = \frac{1}{(4xy)^{n/2}\left(\Gamma\left(\frac{n}{2}\right)\right)^2} \underbrace{\int_0^\infty \frac{1}{n^2}u^{\frac{n}{2}+1}e^{-u/(2x)}du}_{\frac{1}{n^2}(2x)^{\frac{n}{2}+2}\Gamma(\frac{n}{2}+2)} \underbrace{\int_0^\infty v^{\frac{n}{2}-1}e^{-v/(2y)}\,dv}_{(2y)^{\frac{n}{2}}\Gamma(\frac{n}{2})} =$$

$$= \frac{(2x)^2\left(\frac{n}{2}+1\right)\Gamma\left(\frac{n}{2}\right)}{n^2} = x^2 + \frac{2x^2}{n} \to x^2 \quad (n\to\infty),$$

$$L_n(v^2;x,y) = y^2 + \frac{2y^2}{n} \to y^2 \quad (n\to\infty),$$

$$L_n(uv;x,y) = \frac{1}{(4xy)^{n/2}\left(\Gamma\left(\frac{n}{2}\right)\right)^2}\frac{1}{n^2}\int_0^\infty u^{\frac{n}{2}}e^{-u/(2x)}du\int_0^\infty v^{\frac{n}{2}}e^{-v/(2y)}dv =$$

$$= \frac{(2x)^{\frac{n}{2}+1}\Gamma\left(\frac{n}{2}+1\right)(2y)^{\frac{n}{2}+1}\Gamma\left(\frac{n}{2}+1\right)}{(4xy)^{n/2}\left(\Gamma\left(\frac{n}{2}\right)\right)^2 n^2} = \frac{4xy\left(\frac{n}{2}\right)^2}{n^2} = xy.$$

So that, $\lim\limits_{n\to\infty} L_n(f(u,v);x,y) = f(x,y)$ uniformly on $(0,a)\times(0,a)$, $a>0$, whenever $f(u,v)$ is one of the functions $1, u, v, u^2, uv, v^2$. In probabilistic form, the conditions (2.1) means (2.2), where

$$\sigma_{n,1}^2 = D^2\left(\frac{1}{n}\sum_{k=1}^n X_k^2\right) =$$

$$= \frac{1}{(4xy)^{n/2}\left(\Gamma\left(\frac{n}{2}\right)\right)^2}\int_0^\infty \left(\frac{u}{n}-x\right)^2 u^{\frac{n}{2}-1}e^{-u/(2x)}du\int_0^\infty v^{\frac{n}{2}-1}e^{-v/(2y)}dv =$$

$$= \frac{1}{(2x)^{n/2}\Gamma\left(\frac{n}{2}\right)}\int_0^\infty\left(\frac{u^2}{n^2} - 2\frac{u}{n}x + x^2\right)u^{\frac{n}{2}-1}e^{-u/(2x)}du =$$

$$= \frac{1}{(2x)^{n/2}\Gamma\left(\frac{n}{2}\right)}\left[\int_0^\infty u^{\frac{n}{2}+1}e^{-u/(2x)}du - 2x\int_0^\infty u^{\frac{n}{2}}e^{-u/(2x)}du + \right.$$

$$\left. +x^2\int_0^\infty u^{\frac{n}{2}-1}e^{-u/(2x)}du\right] =$$

$$= \frac{(2x)^2}{n^2}\left(\frac{n}{2}+1\right)\left(\frac{n}{2}\right) - \frac{(2x)^2}{n}\left(\frac{n}{2}\right) + x^2 = \frac{2x^2}{n} \to \infty \quad (n\to\infty)$$

and similarly

$$\sigma_{n,2}^2 = D^2\left(\frac{1}{n}\sum_{k=1}^n Y_k^2\right) = \frac{1}{(4xy)^{n/2}\left(\Gamma\left(\frac{n}{2}\right)^2\right)}\int_0^\infty u^{\frac{n}{2}-1}e^{-u/(2x)}du\cdot$$

$$\cdot\int_0^\infty\left(\frac{v}{n}-y\right)^2 v^{\frac{n}{2}-1}e^{-v/(2y)}dv = \frac{2y^2}{n} \to \infty, \quad n\to\infty.$$

We conclude that $\lim_{n\to\infty} L_n(f;x,y) = f(x,y)$ uniformly on $(0,a)\times(0,a)$, $a >$ for any function f uniform continuous on $(0,a)\times(0,a)$, $a > 0$. [

3. Estimate of the order of approximation

We shall now proceed to estimate the order of approximation of the function by the operator (1.1).

It is convenient to make use of the modulus of continuity, defined as follows:

$$\omega(f;\delta_1,\delta_2) = \sup\{|f(x'',y'') - f(x',y')|;\ |x''-x'| < \delta_1,\ |y''-y'| < \delta_2\}$$

where (x',y') and (x'',y'') are points from $(0,a)\times(0,a)$, $a > 0$ such that $|x''-x'| \leq \delta$ and $|y''-y'| \leq \delta_2$, δ_1,δ_2 being positive numbers.

Theorem 3.1. *If f is a bounded and uniform continuous function on $(0,a)\times(0,a$ $a > 0$ then:*

$$|f(x,y) - L_n(f;x,y)| \leq \left(1+2a\sqrt{2}\right)\omega\left(f;\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}}\right).$$

Proof. Using the following properties to the modulus of continuity

$$|f(x'',y'') - f(x',y')| \leq \omega(f;|x''-x'|,|y''-y'|)$$

and

$$\omega(f;\lambda_1\delta_1,\lambda_2\delta_2) \leq (1+\lambda_1+\lambda_2)\omega(f;\delta_1,\delta_2),\quad \lambda_1 > 0,\ \lambda_2 > 0,$$

we have

$$|f(x'',y'') - f(x',y')| \leq \omega\left(f;\frac{1}{\delta_1}|x''-x'|,\frac{1}{\delta_2}|y''-y'|\right) \leq$$
$$\leq \left(1+\frac{1}{\delta_1}|x''-x'|+\frac{1}{\delta_2}|y''-y'|\right)\omega(f;\delta_1,\delta_2).$$

Now

$$|f(x,y) - L_n(f;x,y)| \leq \int_0^\infty\int_0^\infty \left|f(x,y) - f\left(\frac{u}{v},\frac{v}{n}\right)\right|\rho_n(u,v;x,y)\,du\,dv$$

where

$$\rho_n(u,v;x,y) = \begin{cases} 0, & u<0,\ v<0 \\ \frac{1}{(4xy)^{n/2}\Gamma\left(\frac{n}{2}\right)^2}(uv)^{\frac{n}{2}-1}e^{-\frac{1}{2}\left(\frac{u}{x}+\frac{v}{y}\right)}, & u\geq 0,\ v\geq 0,\ x>0,\ y>0 \end{cases}$$

We may therefore write:

$$|f(x,y) - L_n(f;x,y)| \leq \left(1+\frac{1}{\delta_1}L_n\left(\left|x-\frac{u}{n}\right|;x,y\right)+\frac{1}{\delta_2}L_n\left(\left|y-\frac{v}{n}\right|;x,y\right)\right).$$

In accordance with the Cauchy-Schwarz inequality we have:

$$L_n\left(\left|x-\frac{u}{n}\right|;x,y\right) \leq \left(\int_0^\infty\int_0^\infty \left(x-\frac{u}{n}\right)^2 \rho_n(u,v;x,y)du\,dv\right)^{1/2} = \sigma_{n,1} = \left(\frac{2x^2}{n}\right)^{1/2}$$

$$L_n\left(\left|y-\frac{v}{n}\right|;x,y\right) \leq \left(\int_0^\infty\int_0^\infty \left(y-\frac{v}{n}\right)^2 \rho_n(u,v;x,y)du\,dv\right)^{1/2} = \sigma_{n,2} = \left(\frac{2y^2}{n}\right)^{1/2}$$

So,

$$|f(x,y) - L_n(f;x,y)| \leq \left(1 + \frac{x\sqrt{2}}{\delta_1\sqrt{n}} + \frac{y\sqrt{2}}{\delta_2\sqrt{n}}\right)\omega(f;\delta_1,\delta_2).$$

For $\delta_1 = \delta_2 = \frac{1}{\sqrt{n}}$ and $\sup\limits_{x\in(0,a)} x\sqrt{2} = a\sqrt{2}$, $\sup\limits_{y\in(0,a)} y\sqrt{2} = a\sqrt{2}$, we obtain

$$|f(x,y) - L_n(f;x,y)| \leq \left(1 + 2a\sqrt{2}\right)\omega\left(f;\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}}\right).$$

□

4. Asymptotic estimate of the remainder

We next turn to the task of establishing an asymptotic of the remainder

$$f(x,y) - L_n(f;x,y) = R_n(f;x,y),$$

which corresponds to a result of Voronovskaya.

Theorem 4.1. *If f is a function defined and bounded on $(0,+\infty)\times(0,+\infty)$ and at an interior point (x,y) of $(0,+\infty)\times(0,+\infty)$ the second differential $d^2f(x,y)$ exists, then we have the asymptotic formula:*

$$\lim_{n\to\infty} n[f(x,y) - L_n(f;x,y)] = -x^2 f''_{xx}(x,y) - y^2 f''_{xx}(x,y).$$

Proof. Let $(u,v) \in (0,+\infty)\times(0,+\infty)$. Under the hypothesis of the theorem, exists a function $g(u,v)$ defined on $(0,+\infty)\times(0,+\infty)$, such that when $(u,v)\to(x,y)$, we have $g(u,v)\to 0$ and

$$f\left(\frac{u}{n},\frac{v}{n}\right) = f(x,y) + \left(\frac{u}{n}-x\right)f'_x(x,y) + \left(\frac{v}{n}-y\right)f'(x,y) + \frac{1}{2}\left(\frac{u}{n}-x\right)^2 f''_{xx}(x,y) +$$

$$+\left(\frac{u}{n}-x\right)\left(\frac{v}{n}-y\right)f''_{xy}(x,y) + \frac{1}{2}\left(\frac{v}{n}-y\right)^2 f''_{yy}(x,y) + \left[\left(\frac{u}{n}-x\right)^2 + \left(\frac{v}{n}-y\right)^2\right]g\left(\frac{u}{v},\frac{v}{n}\right).$$

Multiplying by $\rho_n(u,v;x,y)$ and then integrating with respect to u and v, with $u>0$, $v>0$, we have

$$-R_n(f;x,y) = \frac{1}{2}\frac{2x^2}{n}f''_{xx}(x,y) + \frac{1}{2}\frac{2y^2}{n}f''_{yy}(x,y) + \alpha_n(x,y),$$

where

$$\alpha_n(x,y) = \int_0^\infty \int_0^\infty \left[\left(\frac{u}{n} - x^2\right) + \left(\frac{v}{n} - y\right)^2 \right] g\left(\frac{u}{n}, \frac{v}{n}\right) \rho_n(u,v;x,y) du\, dv.$$

Since $g\left(\frac{u}{n}, \frac{v}{n}\right) \to 0$ as $\frac{u}{n} \to x$ and $\frac{v}{n} \to y$, it follows that for every posit $\varepsilon > 0$ there exist the positive numbers δ_1 and δ_2, such that $\left|g\left(\frac{u}{v}, \frac{v}{n}\right)\right| < \varepsilon$, whene $\left|\frac{u}{n} - x\right| \le \delta_1$ and $\left|\frac{v}{n} - y\right| \le \delta_2$. In view of the fact that

$$|\alpha_n(x,y)| = \int_0^\infty \int_0^\infty \left[\left(\frac{u}{n} - x\right)^2 + \left(\frac{y}{n} - y\right)^2 \right] g\left(\frac{u}{n}, \frac{v}{n}\right) \rho_n(u,v;x,y) du\, dv,$$

we may proceed further in the same way as in the case of one variable [2] and reach conclusion that

$$\alpha_n(x,y) = \varepsilon_n(x,y)/n,$$

where $\varepsilon_n(x,y) \to 0$, when $n \to \infty$.

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ON THE APPROXIMATION BY KANTOROVICH VARIANT OF A FAVARD-SZASZ TYPE OPERATOR

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Abstract. In this paper one studies the order of approximation of a function by means of Kantorovich variant of Jakimovski-Leviatan operators. By making use of k functional and Steklov's function one obtains estimates expressed with the second order modulus of continuity.

1. Introduction

In 1969, A. Jakimovski and D. Leviatan [4] have introduced a generalized Favard-Szasz operator, obtained by means of Appell polynomials. Let us remind it. One considers $g(z) \equiv \sum_{n=0}^{\infty} a_n z^n$ an analytic function in the disk $|z| < R$, $R > 1$ and suppose $g(1) \neq 0$. One defines the Appell polynomials $p_k(x)$, $k \geq 0$, by

$$g(u)e^{ux} \equiv \sum_{k=0}^{\infty} p_k(x)u^k. \tag{1}$$

One denotes by $\mathcal{E}$ the class of real functions of exponential order,

$$\mathcal{E} = \{f : [0,\infty) \to \mathbf{R} \text{ for which there are } A, B \in \mathbf{R} \text{ such that } |f(x)| \leq Be^{Ax},\ \forall\, x \geq 0\}.$$

A. Jakimovski and D. Leviatan have considered the operator $P_n : \mathcal{E} \to \mathbf{C}[0,\infty)$,

$$(P_n f)(x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \quad n > 0. \tag{2}$$

The authors have studied the approximation properties of this operator. They also have considered the Kantorovich variant of this operator. One considers

$$\mathcal{E}_k = \left\{ f : [0,\infty) \to \mathbf{R},\ f \in C[0,\infty) \text{ with the property that } \exists\, A \in \mathbf{R} \text{ such that } F(t) \equiv \int_0^t f(u)du = \mathcal{O}(e^{At}), \quad (t \to \infty) \right\}.$$

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The Kantorovich variant of the operator P_n is $K_n : \mathcal{E}_k \to C[0,\infty)$ defined by

$$(K_nf)(x) = n\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx)\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt. \qquad (3$$

This operator is positive if and only if $\frac{a_n}{g(1)} \geq 0, \ n = 0, 1, \ldots$ [6].

In our paper [2] we have studied some properties of the operator K_n. The imag of the test functions are given by

Lemma 1.1. *For all $x \geq 0$, we have*

$$(K_ne_0)(x) = 1$$

$$(K_ne_1)(x) = x + \frac{1}{n}\left(\frac{1}{2} + \frac{g'(1)}{g(1)}\right)$$

$$(K_ne_2)(x) = x^2 + 2\frac{x}{n}\left(1 + \frac{g'(1)}{g(1)}\right) + \frac{1}{n^2}\left(\frac{g''(1) + 2g'(1)}{g(1)} + \frac{1}{3}\right)$$

where $e_i(x) = x^i, \ i \in \{0, 1, 2\}$.

2.The aim of this paper is to study the order of approximation of the function f l means of the operator K_n. In order to establish the main results we need the followin

Definition 1. For $t \geq 0$, the second order modulus of continuity of $f \in C_B[0,\infty)$ is

$$\omega_2(f;t) = \sup_{h\leq t} ||f(\circ + 2h) - 2f(\circ + h) + f(\circ)||_{C_B},$$

where $C_B[0,\infty)$ is the class of real valued functions defined on $[0,\infty)$ which are bound and uniformly continuous with the norm $||f||_{C_B} = \sup_{x\in[0,\infty)} |f(x)|$.

Definition 2. [1] The K-functional of function $f \in C_B$ is defined by

$$K(f;t) = \inf_{g\in C_B^2}\left\{||f - g||_{C_B} + t||g||_{C_B^2}\right\},$$

where $C_B^2 = \{f \in C_B |\ f', f'' \in C_B\}$ with the norm

$$||f||_{C_B^2} = ||f||_{C_B} + ||f'||_{C_B} + ||f''||_{C_B}.$$

It is known that

$$K(f;t) \leq A\left\{\omega_2\left(f;\sqrt{t}\right) + \min(1,t)||f||_{C_B}\right\} \qquad ($$

for all $t \in [0,\infty)$. The constant A is independent of t and f.

Lemma 2.1. *[3] If $z \in C^2[0,\infty)$ and (P_n) is a sequence of positive linear operators with the property $P_n e_0 = e_0$, then*

$$|(P_n z)(x) - z(x)| \le \|z'\|\sqrt{(P_n(t-x)^2)(x)} + \frac{1}{2}\|z''\|(P_n(t-x)^2)(x).$$

Theorem 2.2. *If $f \in \mathcal{E}_k$, then for every $x \in [0,a]$, we have*

$$|(K_n f) - f(x)| \le \frac{2h}{a}\|f\| + \frac{3}{4}\omega_2(f;h)\left(3 + \frac{a}{h}\right),$$

where

$$h = \sqrt{\frac{x}{n} + \frac{1}{n^2}\left(\frac{g''(1) + 2g'(1)}{g(1)} + \frac{1}{3}\right)}.$$

Proof. Let f_h be the Steklov function attached to the function f. We will use a result of V.V. Juk [5]:

If $f \in C[0,a]$ and $h \in (0, \frac{b-a}{2})$, then $\|f - f_h\| \le \frac{3}{4}\omega_2(f;h)$ and $\|f_h''\| \le \frac{3}{2}\frac{1}{h^2}\omega_2(f;h)$. Since $(K_n e_0)(x) = e_0$, we can write

$$|(K_n f)(x) - f(x)| \le |(K_n(f - f_h))(x)| + |(K_n f_h)(x) - f_h(x)| +$$

$$+|f_h(x) - f(x)| \le 2\|f - f_h\| + |(K_n f_h)(x) - f_h(x)|.$$

By making use of Lemma 2.1 for function $f_h \in C^2[a,b]$, we obtain

$$|(K_n f_h)(x) - f_h(x)| \le \|f_h'\|\sqrt{(K_n(t-x)^2)(x)} + \frac{1}{2}\|f_h''\|(K_n(t-x)^2)(x).$$

According to Landau's inequality, which is generalized in [3] and [5], we have

$$\|f_h'\| \le \frac{2}{a}\|f_h\| + \frac{a}{2}\|f_h''\| \le \frac{2}{a}\|f\| + \frac{a}{2}\|f_h''\| \le \frac{2}{a}\|f\| + \frac{3a}{4}\frac{1}{h^2}\omega_2(f;h).$$

It resulte that

$$|(K_n f_h)(x) - f_h(x)| \le$$

$$\le \left(\frac{2}{a}\|f\| + \frac{3a}{4}\frac{1}{h^2}\omega_2(f;h)\right)\sqrt{(K_n(t-x)^2)(x)} + \frac{3}{4}\frac{1}{h^2}\omega_2(f;h)(K_n(t-x)^2)(x).$$

By inserting into it

$$h = \sqrt{(K_n(t-x)^2)(x)} = \sqrt{\frac{x}{n} + \frac{1}{n^2}\left(\frac{g''(1) + 2g'(1)}{g(1)} + \frac{1}{3}\right)},$$

we obtain

$$|(K_n f_h)(x) - f_h(x)| \le \frac{2}{a}\|f\|h + \frac{3a}{4}\frac{1}{h}\omega_2(f;h) + \frac{3}{4}\omega_2(f;h),$$

and therefore,

$$|(K_n f)(x) - f(x)| \le \frac{3}{2}\omega_2(f;h) + \frac{2h}{a}\|f\| + \frac{3a}{4}\frac{1}{h}\omega_2(f;h) + \frac{3}{4}\omega_2(f;h) =$$

$$= \frac{2h}{a}\|f\| + \frac{3}{4}\omega_2(f;h)\left(3 + \frac{a}{h}\right).$$

Theorem 2.3. *For every function* $f \in C_B^2[0,\infty)$, *we have*

$$|(K_n f)(x) - f(x)| \le \frac{1}{n}\left(x + \frac{g''(1) + 2g'(1)}{g(1)} + \frac{1}{3}\right)\|f\|_{C_B^2}, \quad x \in [0,\infty).$$

Proof. Applying the Taylor expansion for $f \in C_B^2[0,\infty)$, we can write:

$$(K_n f)(x) - f(x) = f'(x)(K_n(t-x))(x) + \frac{1}{2}f''(\xi)(K_n(t-x)^2)(x)$$

where $\xi \in (t,x)$. By making use of Lemma 2.1 we obtain:

$$|(K_n f)(x) - f(x)| \le \frac{1}{n}\left(\frac{1}{2} + \frac{g'(1)}{g(1)}\right)\|f'\|_{C_B} +$$

$$+\frac{1}{2}\left[\frac{x}{n} + \frac{1}{n^2}\left(\frac{g''(1) + 2g'(1)}{g(1)} + \frac{1}{3}\right)\right]\|f''\|_{C_B} \le$$

$$\le \frac{1}{n}\left(\frac{1}{2} + \frac{g'(1)}{g(1)}\right)\|f'\|_{C_B} + \frac{1}{n}\left(x + \frac{g''(1) + 2g'(1)}{g(1)} + \frac{1}{3}\right)\|f''\|_{C_B} \le$$

$$\le \frac{1}{n}\left(x + \frac{g''(1) + 2g'(1)}{g(1)} + \frac{1}{3}\right)(\|f'\|_{C_B} + \|f''\|_{C_B}) \le$$

$$\le \frac{1}{n}\left(x + \frac{g''(1) + 2g'(1)}{g(1)} + \frac{1}{3}\right)\|f\|_{C_B^2}.$$

Theorem 2.4. *For* $f \in C_B[0,\infty)$, *we have*

$$|(K_n f)(x) - f(x)| \le 2A\{\omega_2(f;h) + \lambda_n(x)\|f\|_{C_B}\}$$

where

$$h = \sqrt{\frac{1}{2n}\left(x + \frac{g''(1) + 2g'(1)}{g(1)} + \frac{1}{3}\right)},$$

A being a constant independent of f and h, and

$$\lambda_n(x) = \min\left(1; \frac{1}{2n}\left(x + \frac{g''(1) + 2g'(1)}{g(1)} + \frac{1}{3}\right)\right).$$

Proof. We will use the K-functional, the relation (4) and the previous theorem. $f \in C_B[0,\infty)$ and $z \in C_B^2[0,\infty)$, we have

$$|(K_n f)(x) - f(x)| \le |(K_n f)(x) - (K_n z)(x)| + |(K_n z)(x) - z(x)| + |z(x) - f(x)| \le$$

$$\le 2\|z - f\|_{C_B} + \frac{1}{n}\left(x + \frac{g''(1) + 2g'(1)}{g(1)} + \frac{1}{3}\right)\|z\|_{C_B^2} =$$

$$= 2\left[\|z-f\|_{C_B} + \frac{1}{2n}\left(x + \frac{g''(1)+2g'(1)}{g(1)} + \frac{1}{3}\right)\|z\|_{C_B^2}\right].$$

Because the left side of this inequality does not depends of the function $z \in C_B^2$, it results that

$$|(K_n f)(x) - f(x)| \le 2K\left(f; \frac{1}{2n}\left(x + \frac{g''(1)+2g'(1)}{g(1)} + \frac{1}{3}\right)\right) \le$$

$$\le 2A\left\{\omega_2\left(f; \sqrt{\frac{1}{2n}\left(x + \frac{g''(1)+2g'(1)}{g(1)} + \frac{1}{3}\right)}\right) + \right.$$

$$\left. + \min\left(1, \frac{1}{2n}\left(x + \frac{g''(1)+2g'(1)}{g(1)} + \frac{1}{3}\right)\right)\|f\|_{C_B}\right\}.$$

Thus we have obtained the desired result. □

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REMARKS ON THE CONVERGENCE OF SOME ITERATIVE METHODS OF THE TRAUB TYPE

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Abstract. One studies the convergence of an iterative method for finding a solution of a nolinear operational equation in a normed space.

Let us consider X and Y two normed linear spaces and $f : X \to Y$ a nonlinear mapping. In relation with the above elements the equality:

$$f(x) = \theta_Y \tag{1}$$

is called an operational nonlinear equation. In this equality θ_Y represents the null element of the space Y. Solving the operational equation (1) is resumed to the determination of an element $\overline{x} \in X$ so that for $x = \overline{x}$ the equality (1) is true; the element $\overline{x}$ is thus called solution of the equation (1).

In order to determine such a solution $\overline{x}$ we will consider the so-called iterative methods. An iterative method consists in building a sequence $(x_n)_{n\in N} \subseteq X$, starting from an initial element $x_0 \in X$. This is the well-known iterative method of the type Newton-Kantorovich where the sequence $(x_n)_{n\in N}$ verifies the relation:

$$f'(x_n)(x_{n+1} - x_n) + f(x_n) = \theta_Y, \tag{2}$$

for any $n \in N$. Here $f'(x) \in (X, Y)^*$ represents the Fréchet derivative of the mapping f in point x, derivative the existence of which is considered on a set $D \subseteq X$. We noted $(X, Y)^*$ the set of linear and continuous mappings defined on X with values in Y.

Supposing that for any $n \in N$ the mapping $[f'(x_n)]^{-1}$ from $(Y, X)^*$ exists, relation (2) is equivalent with:

$$x_{n+1} = x_n - [f'(x_n)]^{-1} f(x_n) \tag{3}$$

which is the well-known form of the Newton-Kantorovich method.

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We can consider a modification of method (2) in order to accelerate the conv gence by using an operator $Q : X \to X$ first applied upon the approximation x_n. In t way the following relation of recurrence for the sequence $(x_n)_{n\in N}$ is obtained:

$$f'(x_n)(x_{n+1} - Q(x_n)) + f(Q(x_n)) = \theta_Y,$$

with $n \in N$, or, considering that the mapping $[f'(x_n)]^{-1}$ exists for any $n \in N$, relation (4) will be written in the following way:

$$x_{n+1} = Q(x_n) - [f'(x_n)]^{-1} f(Q(x_n)).$$

Regardless of the form of the relation of recurrence, any of the above mentioi methods are resumed to the determination of the mapping $[f'(x_n)]^{-1}$ for every $n \in$ that is the solving an operational linear equation for every $n \in N$. In the same way in papers [1], [2], [3] we will eliminate this difficulty by using a supplementary seque $(A_n)_{n\in N} \subseteq (Y, X)^*$ in order to approximate simultaneously the solution $\overline{x}$ of the equat (1) and the mapping $[f'(\overline{x})]^{-1}$.

In this way, using as initial elements $x_0 \in X$ and the mapping $A_0 \in (Y,$ we will build the sequences $(x_n)_{n\in N} \subseteq X$ and $(A_n)_{n\in N} \subseteq (Y, X)^*$ using the relatioi recurrence:

$$\begin{cases} x_{n+1} = Q(x_n) - A_n \sum\limits_{k=0}^{r} (I_Y - f'(x_n)A_n)^k f(Q(x_n)) \\ A_{n+1} = A_n \sum\limits_{k=0}^{q} (I_Y - f'(x_{n+1}A_n)^k. \end{cases}$$

In the relation (6), r and $q \in N$ are given, and I_Y is the identical mappin the space Y.

We should mention that if we have a given mapping $A \in (X, Y)^*$ and there ex $A_0 \in (X, Y)^*$ so that

$$\|I_Y - AA_0\| < 1,$$

then $A_d^{-1} \in (Y, X)^*$ exists; this mapping will be called the inversion to the right of mapping A and it verifies the equality $A \cdot A_d^{-1} = I_Y$. The mapping A_d^{-1} will be obtai as a limit of the sequence $(A_n)_{n\in N}$ formed through the relation of recurrence:

$$A_{n+1} = A_n \sum_{k=0}^{p} (I_Y - AA_n)^k, \quad p \in N.$$

If the inequality

$$\|A_d^{-1} - A_n\| \leq \|A_d^{-1} \cdot \|I_Y - AA_0\|^{(p+1)^n}, \quad n \in N, \tag{8}$$

takes place, we observe that the order of approximation is $p+1$. For this reason we can define the mapping:

$$S_{p+1} : (X,Y)^* \times (Y,X)^* \to (Y,X)^*$$

which for $A \in (X,Y)^*$ and $A_0 \in (Y,X)^*$ is given through

$$S_{p+1}(A, A_0) = A \sum_{k=0}^{p} (I_Y - AA_0)^k.$$

According to the relation (8), we will have:

$$\|A_d^{-1} - S_{p+1}(A, A_n)\| \leq \|A_d^{-1} \cdot \|I_Y - AA_0\|^{(p+1)^n}, \quad n \in N. \tag{9}$$

Because of the above reasons the mapping $S_{p+1}(A, A_0)$ is called $p+1$ approximant of the mapping A_d^{-1}.

We mention the following result: If a mapping f admits a Fréchet derivative on the open and convex set $D \subseteq X$, derivative that verifies Lipschitz's condition for this set, that is the existence of $L > 0$ so that for any $x, y \in D$ we have the inequality:

$$\|f'(x) - f'(y)\| \leq L\|x - y\|, \tag{10}$$

then for any $x, y \in D$ we also have the relation:

$$\|f(x) - f(y) - f'(y)(x - y)\| \leq \frac{L}{2}; \qquad [2], [3]. \tag{11}$$

In what the existence of the solution $\overline{x}$ of the equation (1) and the convergence of the sequences $(x_n)_{n \in N}$ and $(A_n)_{n \in N}$ generated by method (6) are concerned we have the following theorem:

Theorem 1. *If X is a Banach space, the open and convex set $D \subset X$ exists so that:*

a) the mapping $f : X \to Y$ admits the Fréchet derivative in every point $x \in D$ and the mapping $f' : D \to (X,Y)^*$ verifies Lipschitz's condition, that is $L > 0$ exists so that for every $x, y \in D$ the inequality (10) is true,

b) $p \in N,\ p > 1,\ L, M > 0$ exists so that for every $x \in D$ we have:

$$\|f(Q(x))\| \leq K\|f(x)\|^p$$

and

$$\|Q(x)-x\| \le M\|f(x)\|,$$

c) the initial element $x_0 \in D$ *and the initial mapping* $A_0 \in (Y,X)^*$ *verify th*

conditions:

$$d = \max\left\{\frac{1}{C_1}\|f(x_0)\|, \frac{1}{C_2}\|I_Y - f'(x_0)A_0\|\right\} < \theta$$

where $\theta \in]0,1[$ *is the solution of the equation:*

$$x^s - x^{s-1} - (T+1)x + 1 = 0$$

where

$$T = \frac{\overline{R}}{\overline{R}-1}$$

with

$$\overline{R} = \frac{2}{\left(\sqrt{b^2+4a}+b\right)\|[f'(x_0)]^{-1}\|},$$

$$a = LKC_1^p\frac{1-C_2^{r+1}}{1-C_2}, \quad b = LMC_1, \quad s = \min(2p, r+p+1, p+1, q+1);$$

and C_1 *and* C_2 *are the solution in* $]0,1[$ *of the system:*

$$\begin{cases} \frac{K^2(1+C_2)^2(1-C_2^{r+1})^2}{2LM^2C_1^2(1-C_2)^2} + KC_1^p\left[C_2^{r+1} + (1+C_2)\frac{1-C_2^{r+1}}{1-C_2}\right] \le C_1 \\ \left[1+2C_2+\frac{K(1+C_2)^2C_1^{p-2}}{4M}\cdot\frac{1-C_2^{r+1}}{1-C_2}\right]^{q+1} \le C_2 \\ \sqrt{b^2+4a}+b < \frac{2}{\|[f'(x_0)]^{-1}\|}; \end{cases}$$

d) the inclusion $S(x_0,\delta) = \{x \in X \mid \|x-x_0\| \le \delta\} \subseteq D$ *where:*

$$\delta \ge \left[2MC_1 + KC_1^{p-1}\frac{1+C_2}{LM}\cdot\frac{1-C_2^{r+1}}{1-C_2}\right]\frac{d}{1-d^{s-1}}$$

takes place;

then:

j) the sequences $(x_n)_{n\in N}$ *and* $(A_n)_{n\in N} \subseteq (Y,X)^*$ *are convergent;*

jj) the equation (1) admits the solution $\overline{x} \in S(x_0,\delta)$ *where* $\overline{x} = \lim_{n\to\infty} x_n$,

jjj) there exists the mapping $\overline{A} = [f'(\overline{x})]_d^{-1} \in (Y,X)^*$, *which represents inverse to the right of the mapping* $f'(\overline{x})$,

jv) the following inequalities are true:

$$\|x_{n+1}-x_n\| \le \left[MC_1 + \frac{KC_1^{p-1}(1-C_2^{r+1})}{LM(1-C_2)}\right]d^{s^n},$$

$$\|\overline{x}-x_n\| \le \left[MC_1 + \frac{KC_1^{p-1}(1-C_2^{r+1})}{LM(1-C_2)}\right]\frac{d^{s^n}}{1-d^{s^n(s-1)}},$$

$$\|A_{n+1}-A_n\| \le \frac{1+C_2}{LMC_1}\cdot\frac{\alpha d^{s^n}-(\alpha d^{s^n})^{q+1}}{1-\alpha d^{s^n}},$$

$$\|\overline{A}-A_n\| \le \frac{1+C_2}{LMC_1}\sum_{i=1}^{q}\frac{(\alpha d^{s^n(s-1)})^i}{1-d^{is^n(s-1)}},$$

where:

$$\alpha = C_2+\frac{1+C_2}{LMC_1}\left[LMC_1+\frac{KC_1^{p-1}(1-C_2^{r+1})}{M(1-C_2)}\right].$$

Proof. We will prove using the method of mathematical induction that for any $n\in N$ the following properties are true:

i) $x_n \in S(x_0,\delta)$,

ii) $\rho_n = \|f(x_n)\| \le C_1 d^{s^n}$, $\delta_n = \|I_y - f'(x_n)A_n\| \le C_2 d^{s^n}$,

iii) $Q(x_n)\in S(x_0,\delta)$;

iv) the mapping $[f'(x_n)]^{-1}$ exists and

$$\|[f'(x_n)]^{-1}\| \le \|[f'(x_0)]^{-1}\|\exp\frac{d}{(1-d)(1-d^{s-1})}$$

v) $\|A_n\| \le \|f'(x_0)^{-1}\|(1+C_2)\exp\frac{d}{(1-d)(1-d^{s-1})}$.

The verification of these properties for $n=0$ is instantaneous. Indeed, it is evidently true that $x_0\in S(x_0,\delta)$, as centre of the ball.

As $d=\max\left\{\frac{1}{C_1}\|f(x_0)\|,\frac{1}{C_2}\|I_Y-f'(x_0)A_0\|\right\}<1$, we can deduce that $\rho_0 = \|f(x_0)\|\le C_1 d$ and $\delta_0 = \|I_Y - f'(x_0)A_0\|\le C_2 d$.

From $d<1$ we deduce that:

$$\|Q(x_0)-x_0\| \le M\|f(x_0)\| \le MC_1 d < MC_1 <$$

$$< 2MC_1 + BKC_1^p\frac{(1+C_2)(1-C_2^{r+1})}{1-C_2} < \delta,$$

so $Q(x_0)\in S(x_0,\delta)$.

The existence of the mapping $[f'(x_0)]^{-1}$ is made certain by the hypothesis of the theorem. As $\frac{d}{(1-d)(1-d^{s-1})}>0$ we deduce that $\exp\frac{d}{(1-d)(1-d^{s-1})}>1$, so for $n=0$ the inequality from property iv) is true.

Nevertheless:

$$\|A_0\| = \|[f'(x_0)]^{-1}+[f'(x_0)]^{-1}f'(x_0)A_0-[f'(x_0)]^{-1}\| \le$$

$$\le \|[f'(x_0)]^{-1}\|(1+\|I_Y-f'(x_0)A_0\|) \le$$

$$\le \|[f'(x_0)]^{-1}\|(1+C_2)\exp\frac{d}{(1-d)(1-d^{s-1})}.$$

Let us now suppose that these properties are true for $n \le k$ and demons that they are true for $n = k+1$.

i) Let $j \in \{0, 1, \ldots, k\}$. We have:

$$\|x_{j+1} - x_j\| \le \|Q(x_j) - x_j\| + \|S_{r+1}(f'(x_j), A_j)\| \cdot \|f(Q(x_j))\|.$$

As from $x_j \in S(x_0, \delta)$ we have:

$$\|Q(x_j) - x_j\| \le M\|f(x_j)\| \le MC_1 d^{s^j};$$

$$\|S_{r+1}(f'(x_j), A_j)\| \le \|A_j\| \sum_{i=0}^{r} \|I_Y - f'(x_j)A_j\|^i \le$$

$$\le B\sum_{i=0}^{r}(C_2 d^{s^j})^i < B\sum_{i=0}^{r} C_2^i = B\frac{1 - C_2^{r+1}}{1 - C_2},$$

where:

$$B = \|[f'(x_0)]^{-1}\|(1 + c_2)\exp\frac{d}{(1-d)(1-d^{s-1})},$$

we deduce that:

$$\|x_{j+1} - x_j\| \le \left(MC_1 + KC_1^p d^{(p-1)s^n} B\frac{1 - C_2^{r+1}}{1 - C_2}\right) s^{s^j}.$$

In this way:

$$\|x_{k+1} - x_0\| \le \sum_{j=0}^{k} \|x_{j+1} - x_j\| \le \left(2MC_1 + KBC_1^p\frac{1 - C_2^{r+1}}{1 - C_2}\right)\frac{d}{1 - d^{s-1}} < \delta,$$

from where it follows that $x_{k+1} \in S(x_0, \delta)$.

ii) Evidently we have the inequality:

$$\|f(x_{k+1})\| \le \|f(x_{k+1}) - f(Q(x_k)) - f'(Q(x_k))(x_{k+1} - Q(x_k))\| +$$

$$+\|f(Q(x_k)) + f'(Q(x_k))(x_{k+1} - Q(x_k))\|.$$

From the remark mentioned before, using Lipschitz's condition for the n $f' : X \to (X, Y)^*$ for $S(x_0, \delta) \subset D \subset X$ and the fact that x_{k+1}, $Q(x_k) \in S(x_0, \delta)$ have:

$$\|f(x_{k+1}) - f(Q(x_k)) - f'(Q(x_k))(x_{k+1} - Q(x_k))\| \le \frac{L}{2}\|x_{k+1} - Q(x_k)\|^2.$$

Because of the fact that:

$$\|x_{k+1} - Q(x_k)\| \le \|A_k\| \cdot \|f(Q(x_k))\| \sum_{j=0}^{r} \|I_Y - f'(x_k)A_k\|^j \le$$

$$\leq B(1+C_2)K\|f(x_k)\|^p \sum_{j=0}^{r} \|I_Y - f'(x_k)A_k\|^j,$$

from (14) we will deduce:

$$\|f(x_{k+1} - f(Q(x_k)) - f'(Q(x_k))(x_{k+1} - Q(x_k))\| \leq \frac{L}{2}B^2(1+C_2)^2K^2\rho_k^{2p}\left(\sum_{j=0}^{r}\delta_k^j\right)^2. \tag{15}$$

Nevertheless:

$$\|f(Q(x_k)) + f'(Q(x_k))(x_{k+1} - Q(x_k))\| \leq \tag{16}$$

$$\leq \|I_Y - f'(Q(x_k))(S_r(f'(x_k)A_k)\| \cdot \|f(Q(x_k))\| \leq$$

$$\|I_Y - f'(x_k)S_r(f'(x_k), A_k)\| + \|f'(Q(x_k)) - f'(x_k)\| \cdot \|S_r(f'(x_k), A_k)\| \cdot \|f(Q(x_k))\|.$$

As $x_k, Q(x_k) \in S(x_0, \delta)$, we have the following:

$$\|f'(Q(x_k)) - f'(x_k)\| \leq L\|Q(x_k) - x_k\| \leq LM\|f(x_k)\|;$$

then:

$$\|S_r(f'(x_k), A_k)\| \leq \|A_k\| \sum_{j=0}^{r} \|I_Y - f'(x_k)A_k\|^j \leq B(1+C_2)\sum_{j=0}^{r}\delta_k^j$$

and

$$\|I_Y - f'(x_k)S_r(f'(x_k), A_k)\| \leq \|I_Y - f'(x_k)A_k\|^{r+1} \leq \delta_k^{r+1}.$$

In this way, from (16) we will deduce:

$$\|f(Q(x_k)) + f'(x_k)(x_{k+1} - Q(x_k))\| \leq K\left[\delta_k^{r+1} + LMB(1+C_2)\rho_k\sum_{j=0}^{r}\delta_k^j\right]\rho_k^p \tag{17}$$

then from (13), (15) and (17) we can deduce:

$$\rho_{k+1} \leq \frac{1}{2}B^2(1+C_2)^2K^2\rho_k^{2p}\left(\sum_{j=0}^{r}\delta_k^j\right)^2 + K\rho_k^p\left[\delta_k^{r+1} + LMB(1+C_2)\rho_k\sum_{j=0}^{r}\delta_k^j\right]. \tag{18}$$

We also note that:

$$\|I_Y - f'(x_{k+1})A_{k+1}\| = \|I_Y - f'(x_{k+1})S_q(f'(x_{k+1}), A_k)\| \leq \|I - f'(x_{k+1})A_k\|^{q+1} \leq$$

$$\leq (\|I_Y - f'(x_k)A_k\| + \|A_k\| \cdot \|f'(x_{k+1}) - f'(x_k)\|)^{q+1} \leq$$

$$\leq (\|I_Y - f'(x_k)A_k\| + LB(1+C_2)\|x_{k+1} - x_k\|)^{q+1}.$$

But:

$$\|x_{k+1} - x_k\| \leq M\|f(x_k)\| + BK(1 + C_2)\|f(x_k)\|^p \sum_{j=0}^{r} \|I_Y - f'(x_k)A_k\|^j$$

and in this way we will have:

$$\delta_{k+1} \leq \left(\delta_k + LMB(1 + C_2)\rho_k + LB^2K(1 + C_2)^2\rho_k^p \sum_{j=0}^{r} \delta_k^j\right)^{q+1} \tag{19}$$

In relations (18) and (19) the number B depends on d and we will thus show that we will be able to use, instead of B, a number R independently of d.

Let

$$\overline{R} = \frac{\sqrt{b^2 + 4a} - b}{2a\|[f'(x_0)]^{-1}\|} = \frac{2}{\|[f'(x_0)]^{-1}\| \left(\sqrt{b^2 + 4a} + b\right)}$$

and as

$$\sqrt{b^2 + 4a} + b < \frac{2}{\|[f'(x_0)]^{-1}\|},$$

we deduce $\overline{R} > 1$.

From $d \in]0, \theta[$, where θ is the root from the interval $]0, 1[$ of the equation

$$d^s - d^{s-1} - (T + 1)d + 1 = 0,$$

we will deduce that:

$$d^s - d^{s-1} - (T + 1)d + 1 > 0,$$

from where one can easily deduce that $t = \frac{d}{(1-d)(1-d^{s-1})} < 1 - \frac{1}{\overline{R}}$ and so $\frac{1}{1-t} < \overline{R}$. But $t < 1 - \frac{1}{\overline{R}} < 1$ and thus $\frac{1}{1-t} < \overline{R}$ so $B = \|[f'(x_0)]^{-1}\| \exp t < \frac{2}{\sqrt{b^2+4a}+b} = R$. But R is the solution of the equation $\varphi(x) = 0$, where $\varphi(x) = ax^2 + bx - 1$. As $a > 0$, the roots of φ are separated from 0, $\varphi\left(\frac{1}{b}\right) = \frac{a}{b^2} > 0$, so $R < \frac{1}{b}$ and thus $\frac{1}{b} = \frac{1}{LMC_1}$.

Using the inequality $B < \frac{1}{LMC_1}$ and the inequalities (18) and (19) we will obtain the following system of inequalities:

$$\begin{cases} \rho_{k+1} \leq \frac{K^2}{2LM^2C_1^2}(1 + C_2)^2\rho_k^{2p} \left(\sum_{j=0}^{r} \delta_k^j\right)^2 + K\rho_k^p \left[\rho_k^{r+1} + \frac{1+C_2}{C_1}\rho_k \sum_{j=0}^{r} \delta_k^j\right] \\ \delta_{k+1} \leq \left[\delta_k + \frac{1+C_2}{C_1}\rho_k + \frac{K(1+C_2)^2}{LMC_1^2}\rho_k^p \sum_{j=0}^{r} \delta_k^j\right]^{q+1} \end{cases} \tag{20}$$

From the hvpothesis of the induction we have $\rho_k \le C_1 d^{s^k}$ and $\delta_k \le C_2 d^{s^k}$ and in this way from (20) we will deduce:

$$\begin{cases} \rho_{k+1} \le \frac{K^2}{2LM^2C_1^2}(1+C_2)^2 d^{2ps^k}\left(\sum_{j=0}^{r} C^j d^{js^k}\right)^2 + \\ \quad + KC_1^p d^{ps^k}\left[C_2^{r+1} d^{(r+1)s^k} + \frac{1+C_2}{C_1} C_1 d^{s^k} \sum_{j=0}^{r} C_2^j d^{js^k}\right], \\ \delta_{k+1} \le \left[C_2 d^{s^k} + \frac{1+C_2}{C_1} C_1 d^{s^k} + \frac{K(1+C_2)^2}{LMC_1^2} C_1^p d^{ps^k} \sum_{j=0}^{r} C_2^j d^{js^k}\right]^{q+1} \end{cases} \tag{21}$$

As $s = \min(2p, p+r+1, p+1, q+1)$ we deduce that $d^{2p} \le d^s$, $d^{p+r+1} \le d^s$, $d^{p+1} \le d^s$, $d^{q+1} \le d^s$, and keeping in mind that C_1 and C_2 are the solution of the system from the enunciation of the theorem, from (21) we deduce that:

$$\delta_{k+1} \le C_1 d^{s^{k+1}} \quad \text{and} \quad \delta_{k+1} \le C_2 d^{s^{k+1}},$$

which proves that the properties ii) are true for $n = k+1$.

iii) We evidently have:

$$\|Q(x_{k+1}) - x_0\| \le \|Q(x_{k+1}) - x_{k+1}\| + \|x_{k+1} - x_0\| \le M\|f(x_{k+1})\| + \|x_{k+1} - x_0\| \le$$

$$\le MC_1 d^{s^{k+1}} + \left[MC_1 + B(1+C_2)KC_1^p \frac{1-C_2^{r+1}}{1-C_2}\right] \frac{d}{1-d^{s-1}}.$$

As $d^{s^{k+1}} < d < \frac{d}{1-d^{s-1}}$ we deduce that:

$$\|Q(x_{k+1}) - x_0\| < \left(2MC_1 + B(1+C_2)KC_1^p \frac{1-C_2^{r+1}}{1-C_2}\right) \frac{d}{1-d^{s-1}} < \delta$$

so $Q(x_{k+1}) \in S(x_0, \delta)$.

iv) The hypothesis of the induction certifies the existence of the mapping $f'(x_k)]^{-1}$. From $x_k, x_{k+1} \in S(x_0, \delta)$ we have:

$$\|[f'(x_k)]^{-1}(f'(x_k) - f'(x_{k+1}))\| \le \|[f'(x_k)]^{-1}\| \cdot \|f'(x_k) - f'(x_{k+1})\| \le \tag{22}$$

$$\le L\|[f'(x_k)]^{-1}\| \cdot \|x_{k+1} - x_k\| \le BC\left(MC_1 + KC_1^p B \frac{1-C_2}{1-C_2}\right) d^{s^k} = (aB^2 + bB) d^{s^k}.$$

From $0 < B < R = \frac{\sqrt{b^2+4a}-b}{2a}$, as R is the positive root of the equation $ax^2 + bx - 1 = 0$ and $a > 0$ we deduce that $aB^2 + bB < 1$. As also $d < 1$ we have therefore:

$$\|[f'(x_k)]^{-1}(f'(x_k) - f'(x_{k+1}))\| < 1.$$

According to the remark that precedes the enunciation of the theorem we dedu that the mapping:

$$H_k = I_X - [f'(x_k)]^{-1}(f'(x_k) - f'(x_{k+1})) \in (X,X)^*$$

is inversable. So $H_k^{-1} \in (X,X)^*$ exists and:

$$\|H_k^{-1}\| \leq \frac{1}{1-(aB^2+bB)d^{s^k}} < \frac{1}{1-d^{s^k}}.$$

From the expression of H_k we deduce that:

$$H_k = [f'(x_k)]^{-1} f'(x_{k+1}) \quad \text{or} \quad f'(x_{k+1}) = f'(x_k)H_k.$$

In this way from the inversion of the mappings $f'(x_k)$ and H_k we deduce inversability of the mapping $f'(x_k)H_k$, so the inversability of the mapping $f'(x_{k+1})$ well and:

$$[f'(x_{k+1})]^{-1} = H_k[f'(x_k)]^{-1}$$

and

$$\|[f'(x_{k+1})]^{-1}\| \leq \|H_k^{-1}\| \cdot \|[f'(x_k)]^{-1}\| \leq \frac{\|[f'(x_k)]^{-1}\|}{1-d^{s^k}}. \qquad ($$

The relations similar to (23) are true if instead of the number k we use natural number smaller or equal to k, so step by step we have:

$$\|[f'(x_{k+1})]^{-1}\| \leq \frac{\|[f'(x_0)]^{-1}\|}{(1-d)(1-d^s)(1-d^{s^2})\dots(1-d^{s^k})}. \qquad ($$

Using the inequality of means, the fact that $d<1$, so $d^{s^i} \leq d$ for any $i \leq k$ a

$$\sum_{i=0}^{k} d^{s^i} < \frac{d}{1-d^{s-1}},$$

we deduce:

$$\prod_{i=0}^{k} \frac{1}{1-d^{s^i}} \leq \left[\frac{1}{k+1}\sum_{i=0}^{k}\frac{1}{1-d^{s^i}}\right]^{k+1} = \left[1+\frac{1}{k+1}\sum_{i=0}^{k}\frac{d^{s^i}}{1-d^{s^i}}\right]^{k+1} \leq$$

$$\leq \left[1+\frac{1}{k+1}\frac{\sum_{i=0}^{k} d^{s^i}}{1-d}\right]^{k+1} < \left[1+\frac{1}{k+1}\frac{d}{(1-d)(1-d^{s-1})}\right]^{k+1} < \exp\frac{d}{(1-d)(1-d^{s-}}$$

So from (24) we deduce:

$$\|[f'(x_{k+1})]^{-1}\| \leq \|[f'(x_0)]^{-1}\| \exp\frac{d}{(1-d)(1-d^{s-1})}. \qquad ($$

v) In the same way as for $A_0 \in (Y,X)^*$ we have:

$$\|A_{k+1}\| = \|[f'(x_{k+1})]^{-1} + [f'(x_{k+1})]^{-1} f'(x_{k+1})A_{k+1} - [f'(x_{k+1})]^{-1}\| \leq$$

$$\leq \|[f'(x_{k+1})]^{-1}(1 + \|I_Y - f'(x_{k+1})A_{k+1}\|) \leq \|[f'(x_{k+1})]^{-1}\|(1 + C_2 d^{s^k}) \leq$$

$$\leq \|[f'(x_0)]^{-1}\|(1 + C_2)\exp\frac{d}{(1-d)(1-d^{s-1})}.$$

As shown above all the properties i)-v) are true for $n = k+1$, so, according to the principle of mathematical induction these are true for any $n \in N$.

The space X is a Banach space; from here we deduce that $(Y,X)^*$ is also a Banach space, so in these spaces we will be able to demonstrate the convergence of the sequences using their characteristic of being Cauchy sequences.

For the sequence $(x_k)_{n\in N}$ from X we have:

$$\|x_{n+m} - x_n\| \leq \sum_{j=n}^{n+m-1} \|x_{j+1} - x_j\| \leq \sum_{j=n}^{n+m-1} \left(MC_1 + KC_1^p B \frac{1 - C_2^{r+1}}{1 - C_2}\right) d^{s^j} < \tag{26}$$

$$< \left[MC_1 + \frac{KC_1^{p-1}(1 - C_2^{r+1})}{LM(1 - C_2)}\right] \frac{d^{s^n}}{1 - d^{s^n(s-1)}},$$

and for the sequence $(A_h)_{n\in N}$ from $(Y,X)^*$ we have:

$$\|A_{j+1} - A_j\| \leq \|A_j\| \sum_{i=1}^{q} \|I_Y - f'(x_{j+1})A_j\|^i \leq$$

$$\leq B(1 + C_2)\sum_{i=1}^{q}[\|I_Y - f'(x_j)A_j\| + \|f'(x_j) - f'(x_{j+1})\| \cdot \|A_j\|]^i \leq$$

$$\leq B(1 + C_2)\sum_{i=1}^{q}\left[C_2 + \left(LMC_1 + \frac{KC_1^{p-1}(1 - C_2^{r+1})}{M(1 - C_2)}\right) B(1 + C_2)\right]^i d^{is^j} \leq$$

$$\leq \frac{1 + C_2}{LMC_1}\sum_{i=1}^{q}\left(\alpha d^{s^j}\right)^i < \frac{1 + C_2}{LMC_1}\,\frac{\alpha d^{s^j} - \left(\alpha d^{s^j}\right)^{q+1}}{1 - \alpha d^{s^j}},$$

then:

$$\|A_{n+m} - A_n\| \leq \sum_{j=n}^{n+m-1} \|A_{j+1} - A_j\| \leq \frac{1 + C_2}{LMC_1}\sum_{i=1}^{q}\sum_{j=n}^{n+m-1}\left(\alpha d^{s^j}\right)^i \leq \tag{27}$$

$$\leq \frac{1 + C_2}{LMC_1}\sum_{i=1}^{q}\frac{\left(\alpha d^{s^n(s-1)}\right)^i}{1 - d^{s^n(s-1)i}}.$$

As:

$$\lim_{n\to\infty}\frac{d^{s^n}}{1 - d^{s^n(s-1)}} = \lim_{n\to\infty}\sum_{i=1}^{q}\frac{\left(\alpha d^{s^n(s-1)}\right)^i}{1 - d^{s^n(s-1)i}} = 0$$

we deduce that $(x_n)_{n\in N}$ is a fundamental sequence in X, and $(A_n)_{n\in N}$ is a fundamen sequence in $(Y,X)^*$ and from their characteristic of being Banach spaces results t convergence of these sequences, so $\overline{x} \in X$; $\overline{x} = \lim\limits_{n\to\infty} x_n$ and $\overline{A} \in (Y,X)^*$, $\overline{A} = \lim\limits_{n\to\infty}$ exists.

If we make $m \to \infty$ in the inequalities (26) and (27) we obtain the inequalit of evaluation from the conclusions of the theorem.

From the inequality $\|f(x_n)\| \leq C_1 d^{s^n}$, true for any $n \in N$, using $d < 1$, deduce $\lim\limits_{n\to\infty} C_1 d^{s^n} = 0$ so $\lim\limits_{n\to\infty} \|f(x_n)\| = 0$ and using the continuity of f and the no we deduce $\|f(\overline{x})\| = 0$ so $f(\overline{x}) = \theta_Y$; in this way the existence of the solution of equation $f(x) = \theta_Y$ is proved.

From the inequality which limits $\|\overline{x} - x_n\|$, for $n = 0$ we deduce:

$$\|\overline{x} - x_0\| \leq \left[MC_1 + \frac{KC_1^{p-1}(1 - C_2^{r+1})}{LM(1 - C_2)}\right] \frac{d}{1 - d^{s-1}} \leq \delta$$

so $\overline{x} \in S(x_0, \delta)$.

From the inequality $\|I_Y - f'(x_n)A_n\| \leq C_2 d^{s^n}$ and $d < 1$, we deduce t $\lim\limits_{n\to\infty} \|I_Y - f'(x_n)A_n\| = 0$ or $\|I_Y - f'(\overline{x})\overline{A}\| = 0$ from where we deduce that $[f'(\overline{x})]_d^{-1}$ = exists.

The theorem is thus demonstrated.

Remark. Out of the numbers r, p, q, the number p has a fixed value and thus we obtain the maximum value of the number s, which represents the order of convergenc the method we have studied; choosing $r = 0$, $q = p$, this maximum value will be $s = p$

This affirmation is immediate from the relation:

$$s = \min(2p, r + p + 1, p + 1, q + 1).$$

In this case the relation of recurrence for the sequences $(x_n)_{n\in N}$ and (A_n), will be:

$$\begin{cases} x_{n+1} = Q(x_n) - A_n f(Q(x_n)) \\ A_{n+1} = A_n \sum\limits_{j=0}^{p} (I_Y - f'(x_{n+1})A_n); \quad n \in N. \end{cases}$$

From theorem 1, we deduce for the iterative method (28) the following coroll

Corollary 2. *If X is a Banach space; the open and convex set $D \subseteq X$ exists so tha* *a) the mapping $f : X \to Y$ admits a Frechet derivative in every point $x \in D$ and the mapping $f' : D \to (X, Y)^*$ verifies Lipschitz's condition, that is $L > 0$ exists so that for every $x, y \in D$ the inequality (10) is true,*

b) $p \in N,\ p > 1,\ L > 0,\ M > 0$ exists, so that for any $x \in D$ we have:

$$\|f(Q(x))\| \leq K\|f(x)\|^p$$

and

$$\|Q(x) - x\| \leq M\|f(x)\|^p,$$

c) the initial element $x_0 \in D$ and the initial mapping $A_0 \in (Y, X)^*$ verify the condition:

$$d = \max\left\{\frac{1}{C_1}\|f(x_0)\|, \frac{1}{C_2}\|I_Y - f'(x_0)A_0\|\right\} < \theta,$$

where $\theta \in]0, 1[$ is the root of the equation:

$$x^{p+1} - x^p - (T + 1)x + 1 = 0, \quad T = \frac{\overline{R}}{\overline{R} - 1},$$

$$\overline{R} = \frac{2}{(\sqrt{b^2 + 4a} + b)\,\|[f'(x_0)]^{-1}\|}, \quad a = LKC_1^p, \quad b = LMC_1$$

and (C_1, C_2) are the solution from $]0, 1[$ of the system:

$$\begin{cases} \frac{K^2(1+C_2)^2}{2LM^2C_1} + KC_1^p(1 + 2C_2) & \leq C_1 \\ \left[1 + 2C_2 + \frac{KC_1^{p-1}(1+C_2)}{4M}\right]^{q+1} & \leq C_2 \\ C_1\sqrt{L(LM^2 + 4KC_1^{p-2})} + LMC_1 & < \frac{2}{\|[f'(x_0)]^{-1}\|} \end{cases}$$

d) the inclusion $S(x_0, \delta) = \{x \in X |\ \|x - x_0\| < \delta\} \subseteq D$ where:

$$\delta \geq \left[2MC_1 + \frac{KC_1^{p-1}(1 + C_2)}{LM}\right] \frac{d}{1 - d^p}$$

then the conclusions j)-jjj) of theorem 1 and the following evaluations are true:

$$\|x_{n+1} - x_n\| \leq \left(MC_1 + \frac{KC_1^{p-1}}{LM}\right) d^{(p+1)^n},$$

$$\|\overline{x} - x_n\| \leq \left(MC_1 + \frac{KC_1^{p-1}}{LM}\right) \frac{d^{(p+1)^n}}{1 - d^{p(p+1)^n}},$$

$$\|A_{n+1} - A_n\| \leq \frac{1 + C_2}{LMC_1} \frac{\alpha d^{(p+1)^n} - \left(\alpha d^{(p+1)^n}\right)^{p+1}}{1 - \alpha d^{(p+1)^n}},$$

$$\|\overline{A} - A_n\| \leq \frac{1 + C_2}{LMC_1} \sum_{i=1}^{p} \frac{\left(\alpha d^{p(p+1)^n}\right)^i}{1 - d^{ip(p+1)^n}},$$

where:

$$\alpha = C_2 + \frac{1+C_2}{LMC_1}\left(LMC_1 + \frac{KC_1^{p-1}}{M}\right).$$

In this case the speed of convergence of the method is $p+1$.

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AN APPLICATION OF (P,Q)-ANALYTIC FUNCTIONS TO STUDY OF BORDA MODEL FOR AN AXIALLY-SYMMETRIC IDEAL JET

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Abstract. In this paper there is presented the mathematical model for the compressible fluids motion in the axially-symmetrical case or the planar case. By using the properties of the (P,Q)-analytical functions and those of the quasiconformal transformations for elliptical PDE of Beltrami-type, one presents a method for the solution of these spatial problems. The Borda model problem in the axially-symmetric case is solved.

We shall consider a stationary, irrationally motion of an ideal barotrop fluid and we shall suppose that the massic forces are null.

The isentropic law is given by the following relation:

$$p = p^1 \left(\frac{\rho}{\rho^1}\right) \tag{1}$$

where p is the pression, ρ is the fluid's density and p^1, ρ^1 are the characteristic values for p, ρ, respectively. Here p^1, ρ^1 may be p_0, ρ_0 in the points of null velocity, or p^0, rho^0 which are calculated on the free surface of the jet where the velocity is $V_0 = \text{const}$. Then the Bernoulli equation is:

$$\frac{1}{2}V^2 + \int_{p_0}^{p} \frac{dp}{\rho} = 0 \tag{2}$$

where V is the algebraic velocity.

The sound velocity for ρ, p, and C_0 which correspond for ρ_0, p_0 are defined by the following formulas:

$$C^2 = \frac{dp}{d\rho}, \quad C_0^2 = \left(\frac{dp}{d\rho}\right)_0. \tag{3}$$

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Using the Ciaplighin variable [1]

$$\tau = \frac{\gamma - 1}{2}\frac{V^2}{V_0^2} = \frac{V^2}{V_{\max}^2}$$

where γ is the adiabatic constant, then we obtain:

$$\rho = \rho_0(1-\tau)^\beta, \quad p = p_0(1-\tau)^{\beta+1}, \quad C^2 = C_0^2(1-\tau).$$

We shall consider the system of cylindrical coordinates (Ox, Or), $z = x +$ where Ox is the radial axis, which in the case of plane motions is the Oy axis. Then motion equation are given by the following equations:

$$\begin{cases} \frac{\partial}{\partial x}(\rho r^k u) + \frac{\partial}{\partial r}(\rho r^k v) = 0 \\ \frac{\partial u}{\partial r} - \frac{\partial v}{\partial x} = 0 \end{cases}$$

where the first equation is the continuity equation and the secondly equation is irotationality equation (rot $V = 0$). Here $V = (u, v)$.

For $k = 1$ the motion is axially symmetric and for $k = 0$ the motion is plane

By (5) we introduce the velocity potential $\varphi(x, r)$ and we obtain the next syst

$$u = \frac{\partial \varphi}{\partial x} = p\frac{\partial \psi}{\partial r}, \quad v = \frac{\partial \varphi}{\partial r} = -p\frac{\partial \psi}{\partial x}, \quad p = r^k\frac{\rho}{\rho^1}.$$

In this case the complex potential is given by $f(z) = \varphi(x, r) + i\psi(x, r)$ and complex velocity by $w = u + iv$, respectively.

Using the hodographic plane (V, θ), where $u = V\cos\theta$, $v = V\sin\theta$, with $V =$ and $\theta = \arg(u + iv)$, or the plane (τ, θ), where $V^2 = 2a\tau$, then by (3) and (6), we h the following system:

$$\begin{cases} \frac{\partial \tau}{\partial \psi} = \frac{2\tau}{p} \cdot \frac{\partial \phi}{\partial \varphi} \\ \frac{\partial \tau}{\partial \varphi} = -\frac{2\tau(1-\tau)p}{1-(2\beta+1)\tau} \cdot \frac{\partial \theta}{\partial \psi} - \frac{k}{r} \cdot \frac{1-\tau}{1-(2\beta+1)\tau} \cdot \frac{\sin\theta}{2}\sqrt{\frac{2\tau}{\alpha}} \end{cases}$$

Here $\beta = (\alpha - 1)^{-1}$ and $p = r^k\frac{\rho}{\rho^1}$.

This is the Ciaplighin fundamental system for the axially-symmetric case on domain $D_w(\tau, \theta)$.

When the motion has the free surface (ρ^0, p^0, τ_0), by considering $\rho^1 = \rho^0$, obtain the following relation:

$$p^{-1} = \frac{1}{r^k}\left(\frac{1-\tau_0}{1-\tau_0 V^2/V_0^2}\right)^\beta.$$

The fundamental system for the plane case corresponding to $k = 0$ is given by the following equations [1]:

$$\frac{\partial \tau}{\partial \theta} = 2\tau(1-\tau)^{-\beta}\frac{\partial \psi}{\partial \tau}, \quad \frac{\partial \varphi}{\partial \tau} = -\frac{1-(2\beta+1)\tau}{2\tau(1-\tau)^{\beta+1}} \cdot \frac{\partial \psi}{\partial \theta}. \tag{8}$$

If the fluid is incompressible ($C \to \infty$, $\beta \to 0$), then from (7), we obtain:

$$\frac{\partial \tau}{\partial \psi} = \frac{2\tau}{r^k} \cdot \frac{\partial \theta}{\partial \varphi}, \quad \frac{\partial \tau}{\partial \varphi} = -2\tau r^k \frac{\partial \theta}{\partial \psi} - \frac{k}{r}\sqrt{\frac{2\tau}{\alpha}} \sin\theta. \tag{9}$$

For the plane problems it is well known the hodographic method due to Ciaplighin-Iacob-Falkovici (C-J-F) [1].

Because the system (7) is very complicated it is applied the theory of generalized functions or (P,Q)-analytic functions.

For this aim we consider the canonic domain $D_\zeta = \zeta + i\eta$ and we transform conformally the domains D_z, D_f, D_w on D_ζ.

This canonic domain may be an semicircle or halfplane. We prefer the halfplane $\zeta = \xi + i\eta$, $\eta \geq 0$ and we introduce the Jukowski function ω given by the following relation:

$$\begin{cases} \omega = \ln \frac{V_0}{V} + i\theta = t + i\theta, \quad W = Ve^{i\theta} \\ \overline{W} = V_0 e^{-\omega(\zeta)} \end{cases} \tag{10}$$

Definition 1. [6] The complex function $f(z) = U(x,y) + iV(x,y)$, $z = x + iy$, is called (P,Q)-analytic in a domain D_z if it satisfies the Beltramy system:

$$\begin{cases} PU_x + QU_y - V_y = 0 \\ -QU_x + PU_y + V_x = 0 \end{cases} \tag{11}$$

where $P(x,y)$, $Q(x,y)$, $P > 0$ and U, V satisfy the regularity conditions.

Here

$$U_x = \frac{\partial U}{\partial x} \text{ etc.}$$

When $Q = 0$, f is called P-analytic function in D_z if [7]:

$$PU_x = V_y, \quad PU_y = -V_x. \tag{12}$$

The system (6) has the form (12). When the system (12) has the elliptical form, then it is equivalent with the following equation:

$$W_{\overline{z}} = AW_z + B\overline{W}_{\overline{z}} + C \tag{13}$$

where

$$W(z) = PV + i(V - QU), \quad W_z = \frac{\partial W}{\partial z}, \quad W_{\overline{z}} = \frac{\partial W}{\partial \overline{z}}.$$

In this case, the one-to-one maps between D_z and D_f, which are defined by t (P, Q)-analytic functions, are called the quasiconform transformations [6].

Let the quasiconformal system be defined by

$$\begin{cases} -\psi_y + a_{11}\varphi_x + a_{12}\varphi_y + a_0\varphi + b_0\psi = h \\ \psi_x + a_{21}\varphi_x + a_{22}\varphi_y + c_0\varphi + d_0\psi = i \end{cases} \quad (1$$

where $a_{ij}, a_0, b_0, c_0, d_0, h, i$ are measurable and bounded functions by (x, y, φ, ψ).

Then the function $f = \varphi + i\psi$, which is a solution of this system and al an homeomorphism between (x, y) and (φ, ψ), is a quasiconformal transformation q-quasiconformal if it verifies the following differential condition [7]:

$$\varphi_x^2 + \varphi_y^2 + \psi_x^2 + \psi_y^2 \leq \left(q + \frac{1}{q}\right)(\varphi_x\psi_y - \varphi_y\psi_x) \quad ($$

$$f_{\overline{z}} + \mu_1 f_z + \mu_2 \overline{f}_{\overline{z}} = F, \quad ($$

where the coefficients and F are measurable and analytic functions with respect $z, \overline{z}, f, \overline{f}$. Then the q-quasiconformality condition (15) becomes:

$$|f_{\overline{z}}| \leq \frac{q-1}{q+1}|f_z|.$$

Let the function $f(z) = \varphi + i\psi$, where $z = x + ir$, be defined on a domain L

Definition 2. [6] The P-derivation operator of f has the following form

$$\frac{d_P f(z)}{dz} = \frac{P\varphi_x + \psi_r}{2} + i\frac{\psi_x - P\psi_r}{2}. \quad ($$

The (P, Q)-derivation operator of f has the following form:

$$\frac{d_{(P,Q)} f(z)}{dz} = \frac{P\psi_x - \varphi_r + \psi_r}{2} + i\frac{\psi_x - Q\varphi_x - P\psi_r}{2}. \quad ($$

We are presenting four immediate results obtained by using the analytic ger alized functions.

Using (6) and (12), we observe that the complex potential f is a P-anal function.

Theorem 1. *[4] The velocity $W(z)$ is an analytic generalized function of z and W verifies the conditions (11) or (13), which are written on the following forms:*

$$\frac{d\overline{W}}{d\overline{z}} = -\frac{1}{2P}\cdot\frac{d\overline{W}}{dz} - \frac{1}{2P}\cdot\frac{dW}{dz}, \quad P = r^k\frac{\rho}{\rho^1}. \tag{19}$$

The solution of these equations realizes the quasi-conformal transformation between D_z and D_w.

Theorem 2. *[4] The function $\omega(f)$ defined by (10) is an analytic generalized on D_f and verifies the following equation which is the type (11) or (13):*

$$\omega_{\overline{f}} - q_1\omega_f - q_2\overline{\omega}_{\overline{f}} = F \tag{20}$$

where

$$q_1 = \frac{P^2+D-1}{N}, \quad q_2 = \frac{PD}{N}, \quad N = (P+1)(P-D+1)$$

$$F = \frac{k}{r}\cdot\frac{\partial r}{\partial\varphi}, \quad D = \frac{2\beta V^2}{V_{\max}^2 - V^2}.$$

Theorem 3. *[4] If there exists a conformally transformation between D_f and D_ζ, then the function which map the domain D_z on D_ζ ($z = z(\zeta)$) is a (P', Q')-analytic of ζ and satisfies the equation:*

$$\omega_\zeta = q_1^*\omega_{\overline{\zeta}} - q_2^*\overline{\omega}_\zeta = F^*, \tag{21}$$

where

$$q_1^* = \frac{(\varphi_\xi + i\varphi_\eta)(P^2+D-1)}{(\varphi_\xi - i\varphi_\eta)},$$

$$q_2^* = \frac{DP}{N^*}, \quad N^* = (P+1)(P+D^*),$$

$$D^* = \frac{1-M}{1-\frac{M^2}{2\beta+1}}$$

$$F^* = \frac{kr}{V^2N^*}\cdot\frac{\varphi_\xi^2+\varphi_\eta^2}{\varphi_\xi - i\varphi_\eta}$$

$$D = 1 - D^*,$$

and M is Mach's number.

Remark. For solving the equations (19), (20) and (21) are applied the numerical or functional methods. We will approximate the equation (21) by the plane case $\omega_{\overline{\zeta}} = 0$ and other formulas are exactly.

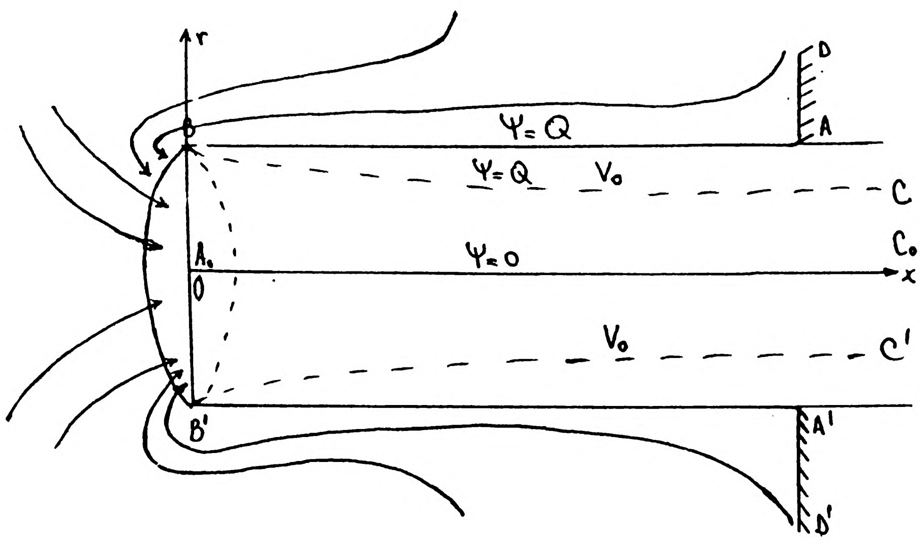
The Borda Model for the axially-symmetric fluid

We consider an ideal fluid jet which comes to an infinite cylindrical tube $ABB'A$ with the axially section xOr.

The motion has the irrotationally and axially-symmetric character. Then the O axis is a symmetric axis for the motion domain. The cylindrical tube is an expansion hence $V(A_0) = V(A) = 0$.

When the upstream fluid meets the vertically walls AD and $A'D'$, which ar situated at the great distances, then it must returns and comes at the cylinder. So wil appear the free lines BC and $B'C'$ respectively, where the velocity is constant and equa with V_0.

We suppose that on the Ox axis we have $\psi = 0$ and on AB and BC we hav $\psi = Q$, where Q is a constant such that $\rho^0 Q$ represents the flux of the flow.



This is the Borda case for the axially symmetric ideal jet.

The plane problem was studied by C. Iacob [1].

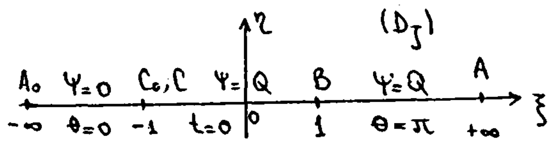
We shall determine an analytic function $f = \varphi + i\psi$ of the motion and th complex speed W.

We transform conformally the half-plane D_f from the motion domain

$$\{0 \leq \psi \leq Q,\ -\infty \leq \varphi \leq +\infty\}$$

in the half-plane D_ζ, where $\zeta = \xi + i\eta,\ \xi \in (-\infty, \infty),\ \eta \geq 0$.

Hence, it will be determined the analytical function $f = f(\zeta)$ on D_ζ, with the following correspondence:



$$f : D_\zeta \to \mathbf{C},$$

$$\psi|_{\eta=0} \equiv \begin{cases} 0, & \text{on } (-\infty,-1), \quad (A_0, C_0) \\ Q, & \text{on } (-1,1), \quad (C, B) \\ Q, & \text{on } (1,+\infty), \quad (B, A) \end{cases} \tag{22}$$

The solution of this Dirichlet's problem for the half-plane will be directly determined by following the Cissotti's formula [1]:

$$f(\zeta) = -\frac{Q}{\pi}\log(\zeta + 1) + iQ. \tag{23}$$

We remark that for $\eta \to 0$ we have

$$\frac{\partial \varphi}{\partial \xi} = -\frac{Q}{\pi} \cdot \frac{1}{\xi + 1}, \quad \frac{\partial \varphi}{\partial \eta} \to 0. \tag{24}$$

We map the motion domain from D_ω one-to-one onto the domain D_ζ and from Theorem 4 we obtain that $\omega = \omega(\zeta)$ is a generalized analytic function in D_ζ.

In this way with

$$\omega = \ln \frac{V_0}{V} + i\theta = t + i\theta$$

we will know the values of $\omega = \omega(\zeta)$ on the boundary, as follows:

$$\begin{cases} A_0C_0: & \theta = \operatorname{Im} \omega = 0 \\ BC: & t = \operatorname{Re} \omega = 0 \\ AB: & \operatorname{Im} \omega = \theta = \pi \end{cases} \tag{25}$$

By using the function

$$S(\zeta) = -\frac{i\omega}{\sqrt{\zeta^2 - 1}}$$

we have

$$\begin{cases} \text{Re } S = 0 \text{ on } (-\infty, -1) & (A_0, C_0), \quad \eta = 0 \\ \text{Re } S = 0 \text{ on } (-1, 1) & (B, C), \quad \eta = 0 \\ \text{Re } S = \frac{\pi}{\sqrt{\xi^2-1}} \text{ on } (1, +\infty) & (A, B), \quad \eta = 0 \end{cases} \quad (2$$

This is a Dirichlet's problem.

Hence, by following the Cissotti's formula we deduce

$$\omega(\zeta) = \sqrt{\zeta^2 - 1} \int_1^{\infty} \frac{ds}{\sqrt{s^2 - 1}(s - \zeta)}.$$

By solving this integral equation we deduce that

$$\omega(\zeta) = \log\left(\frac{-1}{\zeta + \sqrt{\zeta^2 - 1}}\right). \quad ($$

From the following transformation formula

$$W - Ve^{i\theta}, \quad \overline{W} = V_0 e^{-\omega(\xi)}$$

we obtain:

$$\overline{W}(\zeta) = \frac{V_0}{\sqrt{\zeta^2 - 1 - \zeta}}.$$

Hence, we have the correspondence between D_w and D_ζ.

So, using the formulas (21) and (24) we obtain the mapping $z = z(\zeta)$ be defi

in the next:

$$\begin{cases} \frac{\partial x}{\partial \xi} = \frac{u}{V^2} \cdot \frac{\rho^1(\xi)}{\rho r^k(\xi)} \cdot \varphi_\xi \\ \frac{\partial r}{\partial \xi} = \frac{v}{V^2} \cdot \frac{\rho^1(\xi)}{\rho r^k(\xi)} \cdot \varphi_\xi \end{cases} \quad ($$

In this way from (27) and (28) the conditions of Theorem 1 are satisfied, so get the correspondences between D_w and D_z, and also, between D_f and D_z, respectiv

The distribution for speeds, pressures and densities on the boundary of motion domain d_ζ, $\eta = 0$, are deduce using the formulas (4) and (27).

On A_0C_0 : $-\infty < \xi < -1$, $\eta = 0$, $u = V_0 f_1(\xi)$, $v = 0$

$$p = p_0 \left(1 - \tau_0 f_1^2\right)^{\beta+1}, \quad \rho = \rho_0 \left(1 - \tau_0 f_1^2\right)^{\beta},$$

where

$$f_1(\xi) = \frac{1}{\sqrt{\xi^2 - 1 - \xi}}.$$

On BC : $-1 < \xi < 1,\ \eta = 0,\ u = V_0 \cos f_2,\ v = V_0 \sin f_2,$

$$p = p^0 = p_0(1-\tau_0)^{\beta+1}, \quad \rho = \rho^0 = \rho_0(1-\tau_0)^{\beta},$$

where

$$f_2(\xi) = \pi - \arccos \xi.$$

On AB : $1 < \xi < +\infty,\ \eta = 0,\ u = V_0 f_3(\xi),$

$$p = p_0\left(1-\tau_0 f_3^2\right)^{\beta+1}, \quad \rho = \rho_0\left(1-\tau_0 f_3^2\right)^{\beta},$$

where

$$f_3(\xi) = -\frac{1}{\xi+\sqrt{\xi^2-1}}.$$

Also, it is easy to see that

$$V(B) = -V_0 = v; \quad V(A) = \lim_{\xi\to\infty}(V_0 f_3(\xi)) = 0$$

$$v(B) = 0, \quad u(B) = i(\xi = 1) = -V_0$$

$$V(A_0) = \lim_{\xi\to-\infty} V_0 f_1(\xi) = 0$$

$$V(C_0) = V(\xi = -1) = V_0 = v(C).$$

So, these formulas are satisfied and they are used to obtain the velocity on some important points of our configuration.

Using the above distribution for speed and density we can obtain the equations of the boundaries by integrating the equations (28).

On BC (which is the free surface) we obtain:

$$\frac{\partial r}{\partial \xi} = -\frac{Q}{\pi}\cdot\frac{\sin f_2}{V_0 r^k(\xi)}\cdot\frac{1}{\xi+1}\cdot\frac{\rho_0}{\rho^0}$$

where $\rho^0 = \rho_0(1-\tau_0)^{\beta}$ and $\tau_0 = \frac{V_0^2}{V_{max}^2}$, hence by integrating, we have the following relations:

$$\left.\frac{r^{k+1}}{k+1}\right|_1^{\xi} = -\frac{Q}{\pi V_0}\cdot\frac{\rho_0}{\rho^0}\int_1^{\xi}\sin f_2(\xi)\left[\frac{1}{1+\xi}\right]d\xi$$

$$r(\xi) = \left\{r_B^{k+1} - \frac{Q(k+1)}{\pi V_0}\cdot\frac{\rho_0}{\rho^0}\int_1^{\xi}\sin f_2(\xi)\left[\frac{1}{1+\xi}\right]d\xi\right\}^{\frac{1}{k+1}}$$

So, we deduce that:

$$\frac{\partial x(\xi)}{\partial \xi} = -\frac{\cos f_2}{V_0}\cdot\frac{\rho_0}{\rho^0}\cdot\frac{Q}{\pi}\left[\frac{1}{1+\xi}\right]\frac{1}{r^k(\xi)}.$$

Hence, we obtain the following relation:

$$x(\xi) = -\frac{Q}{\pi V_0} \cdot \frac{\rho_0}{\rho^0} \int_1^{\xi} \frac{\cos f_2(\xi)}{r^k(\xi)} \left[\frac{1}{1+\xi}\right] d\xi.$$

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INTEGRAL OPERATOR OF SINGH AND HARDY CLASSES

GHEORGHE MICLAUS

Abstract. In this paper one obtains the Hardy classes for integral operator (4) and in particular one obtains the Hardy classes for Libera and Bernardi operators determined in [2] and Hardy classes for the class of bounded Mocanu variation functions determined in [5].

1. Introduction

Let A denote the set of functions $f(z) = z + a_2z^2 + \dots$ which are analytic in the unit disk $U = \{z : |z| < 1\}$. In [2] sharp results concerning the boundary behaviour of $L(f)$ when f belongs to the Hardy space H^p, $0 < p \leq \infty$, were obtained, where $L(f)$ is the integral operator (Bernardi) defined by

$$L(f)(z) = \frac{1}{z^\gamma}\int_0^z f(t)t^{\gamma-1}dt, \quad \gamma \in \mathbb{C},\ Re\gamma \geq 0. \tag{1}$$

P.T. Mocanu, investigating in [4] a more general integral operator, defines the second order integral operator A by

$$A(f)(z) = \frac{1}{z^\gamma}\int_0^z \left(t^{\gamma-\beta-1}\int_0^t f(s)s^{\beta-1}ds\right)dt. \tag{2}$$

In fact, A is defined by $A = C \circ B$ where

$$B(f)(z) = \frac{1}{z^\beta}\int_0^z f(t)t^{\beta-1}dt \quad \text{and} \quad C(f)(z) = \frac{1}{z^\gamma}\int_0^z f(t)t^{\gamma-1}dt. \tag{3}$$

In 1973 R. Singh [5] showed that if

$$I(f)(z) = \left[\frac{\beta+\gamma}{z^\gamma}\int_0^z f^\beta(t)t^{\gamma-1}dt\right]^{\frac{1}{\beta}}, \ z \in U \tag{4}$$

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and if $\beta, \gamma = 1, 2, 3 \ldots$ then $I(S^*) \subset S^*$, where S^* denote subsets of starlike functio In this paper we obtain the Hardy classes for integral operator (4) and in particular obtain the Hardy classes for Libera and Bernardi operators determined in [2] and Har classes for the class of bounded Mocanu variation functions determined in [5]. We def the n-order integral operator for (4) and we determine the Hardy classes.

2. Preliminaries

For f analytic and $z = re^{i\theta}$, $z \in U$ we denote

$$M(r,f) = \begin{cases} \left(\frac{1}{2\pi} \int\limits_0^{2\pi} \left| f(re^{i\theta}) \right|^p d\theta \right)^{\frac{1}{p}}, & \text{for} \quad 0 < p < \infty \\ \sup\limits_{0 \le \theta \le 2\pi} \left| f(re^{i\theta}) \right|, & \text{for} \quad p = \infty. \end{cases}$$

A function is said to be of Hardy class $H^p (0 < p \le \infty)$ if $M(r,f) < \infty$ as $r \to 1^-$. is class of bounded analytic functions in the unit disk. We shall need the following lemn

Lemma 1. *If $f' \in H^p$, $0 < p < 1$, then $f \in H^{\frac{p}{1-p}}$. If $f' \in H^p$, $p \ge 1$, then $f \in H^\infty$*

Lemma 2. *If $f \in H^p$ and $g \in H^q$ then $f \cdot g \in H^{\frac{pq}{p+q}}$.*

Lemma 3. *If $f \in H^p$ and $F(z) = \int_0^z f(t)dt$ then $F \in H^{\frac{p}{1-p}}$, for $0 < p < 1$ and $F \in$ for $p \ge 1$.*

These lemmas are well known [1].

3. Results for the integral operator I

Theorem 1. *If I is an integral operator defined by (4) then $I = J_{\frac{1}{\beta}} \circ K \circ G \circ J_\beta$ wl*

$$J_\alpha(f)(z) = z \left[\frac{f(z)}{z} \right]^\alpha;$$

$$G(f)(z) = \int\limits_0^z f(t) t^{\beta+\gamma-2} dt;$$

$$K(f) = \frac{\beta+\gamma}{z^{\beta+\gamma-1}} f(z).$$

Proof. A straightforward computation shows that has this decomposition.

Theorem 2. *If $f \in H^p$ then $J_\alpha(f) \in H^{\frac{p}{\alpha}}$.*

Proof.

$$M_p(r,f) = \frac{1}{2\pi}\int_0^{2\pi}\left|z\left[\frac{f(z)}{z}\right]^\alpha\right|^p d\theta = \frac{1}{2\pi}\int_0^{2\pi}\left|re^{i\theta}\frac{f^\alpha(re^{i\theta})}{r^\alpha e^{i\theta\alpha}}\right|^p d\theta =$$
$$= \frac{1}{2\pi}\frac{1}{r^{\alpha-1}}\int_0^{2\pi}|f(re^{i\theta})|^{\alpha p}d\theta$$

and we have $\lim_{r\to 1^-} M(r, J_\alpha) < \infty$ if $J_\alpha(f) \in H^{\frac{p}{\alpha}}$. □

Theorem 3. *If $f \in A$ then*

(i) $f \in H^p$, $0 < p < 1$ implies $G(f) \in H^{\frac{p}{1-p}}$;

(ii) $f \in H^p$, $1 \le p \le \infty$ implies $G(f) \in H^\infty$.

Proof. If f is analytic in U then $G(f)$ exists and is an analytic function in U and $G'(f)(z) = f(z)z^{\beta+\gamma-2}$ and we obtain $G'(f) \in H^p$. From Lemma 1 we obtain $G(f) \in H^{\frac{p}{1-p}}$ for $0 < p < 1$ and $G(f) \in H^\infty$ for $p \ge 1$. □

Theorem 4. *If $f \in H^p$ then $K(f) \in H^p$.*

Proof. Immediate. □

Theorem 5. *If $f \in A$, $f \in H^p$, $\beta \in \mathbb{R}_+^*$ then*

(i) if $\beta > p$ then $I(f) \in H^{\frac{\beta p}{\beta-p}}$;

(ii) if $\beta \le p$ then $I(f) \in H^\infty$.

Proof. *(i)* If $f \in A$ and $f \in H^p$ then $J_\beta(f) \in H^{\frac{p}{\beta}}$, from Theorem 2. Applying Theorem 3 we have $G(J_\beta(f)) \in H^p$, $q = \frac{\frac{p}{\beta}}{1-\frac{p}{\beta}} = \frac{p}{\beta-p}$, $\beta > p$. From Theorem 4 $K(G(J_\beta(f))) \in H^{\frac{p}{\beta-p}}$. Applying Theorem 2 again, we have $J_{\frac{1}{\beta}}(K(G(J_\beta(f)))) \in H^{\frac{\beta p}{\beta-p}}$.

(ii) If $f \in H^p$ then $J_\beta(f) \in H^{\frac{p}{\beta}}$, and from Theorem 3 $G(J_\beta(f)) \in H^\infty$ and $I(f) \in H^\infty$.

□

Theorem 6. *If $f \in H^p$, $0 < \beta < 1$ then*

(i) $I'(f) \in H^{\frac{\beta p}{\beta p+\beta-p}}$ if $\beta < p$;

(ii) $I'(f) \in H^{\frac{p}{\beta}}$ if $\beta \le p$ or $\beta = 1$.

Proof.

$$I'(f) = \frac{-\gamma}{\beta z}I(f) + \frac{\beta+\gamma}{\beta z}[f(z)]^\beta[I(f)]^{1-\beta} = M_1(f) + M_2(f).$$

(i) If $\beta > p$ then $I(f) \in H^{\frac{\beta p}{\beta - p}}$ and

$$M_1(f) \in H^{\frac{\beta p}{\beta-p}} : [f(z)]^{\beta},\ [I(f)]^{1-\beta} \in H^{\frac{\beta p}{\beta-p}\cdot\frac{1}{1-\beta}} \quad \text{and} \quad M_2(f) \in H^{\lambda},$$

where $\lambda = \dfrac{\frac{p}{\beta}\cdot\frac{\beta p}{\beta-p}\cdot\frac{1}{1-\beta}}{\frac{p}{\beta}+\frac{\beta p}{\beta-p}\cdot\frac{1}{1-\beta}} = \dfrac{\beta p}{\beta p+\beta-p}$;

(ii) $\beta \leq p$ implies $I(f) \in H^{\infty}$ and $M_1(f) \in H^{\infty}$. $[I(f)]^{1-\beta} \in H^{\infty}$ and $M_2(f) \in H^{\frac{p}{\beta}}$ implies $I'(f) \in H^{\frac{p}{\beta}}$.

∎

4. The n-order integral operator of Singh

Let be $f \in A$. We define the n-order integral operator of Singh I^n:

$$I^n(f) = I_n \circ I_{n-1} \circ \cdots \circ I_1, \quad (5)$$

where

$$I_i(f) = \left(\frac{\beta_i+\gamma_i}{z^{\gamma_i}}\int_0^z f^{\beta_i}(t)t^{\gamma_i-1}dt\right)^{\frac{1}{\gamma_i}}, \quad \beta_i, \gamma_i \in \mathbb{C},\ i \in \{1,2,\ldots,n\}. \quad (6)$$

Theorem 7. *If $f \in H^p, \beta_i \in \mathbb{R}_+^*$ then*

(i) if $\beta_1 > p$ and

$$\beta_{i+1} > \frac{\beta_1\beta_2\ldots\beta_i p}{\beta_1\beta_2\ldots\beta_i - p(\beta_2\beta_3\ldots\beta_i+\beta_1\beta_3\ldots\beta_i+\cdots+\beta_1\beta_2\ldots\beta_{i-1})},$$

$i \in \{1,2,3,\ldots,n-1\}$, then $I^n(f) \in H^{\lambda}$, where

$$\lambda = \frac{\beta_1\beta_2\ldots\beta_n p}{\beta_1\beta_2\ldots\beta_n - p(\beta_2\beta_3\ldots\beta_i+\beta_1\beta_3\ldots\beta_i+\cdots+\beta_1\beta_2\ldots\beta_{n-1})}$$

(ii) if $\beta_1 \leq p$ or $(\exists i)$ $i \in \{1,2,\ldots,n-1\}$ such that

$$\beta_{i+1} \leq \frac{\beta_1\beta_2\ldots\beta_i p}{\beta_1\beta_2\ldots\beta_i - p(\beta_2\beta_3\ldots\beta_i+\beta_1\beta_3\ldots\beta_i+\cdots+\beta_1\beta_2\ldots\beta_{i-1})}$$

then $I^n(f) \in H^{\infty}$.

Proof. (i) From Theorem 5, we have that if $\beta_1 > p$ then $I_1(f) \in H^{\frac{\beta_1 p}{\beta_1 - p}}$. We suppose that $I^{n-1}(f) \in H^{\lambda}$ where

$$\lambda = \frac{\beta_1\beta_2\ldots\beta_{n-1} p}{\beta_1\beta_2\ldots\beta_{n-1} - p(\beta_2\beta_3\ldots\beta_{n-1}+\beta_1\beta_3\ldots\beta_i+\cdots+\beta_1\beta_2\ldots\beta_{n-2})}$$

then $I^n(f) = I_n(I^{n-1}(f)) \in H^\lambda$, where

$$\lambda = \frac{\beta_n \dfrac{\beta_1\beta_2\ldots\beta_{n-1}p}{\beta_1\beta_2\ldots\beta_{n-1} - p(\beta_2\beta_3\ldots\beta_{n-1} + \cdots + \beta_1\beta_2\ldots\beta_{n-2})}}{\beta_n - \dfrac{\beta_1\beta_2\ldots\beta_{n-1}p}{\beta_1\beta_2\ldots\beta_{n-1} - p(\beta_2\beta_3\ldots\beta_{n-1} + \cdots + \beta_1\beta_2\ldots\beta_{n-2})}}$$
$$= \frac{\beta_1\beta_2\ldots\beta_n p}{\beta_1\beta_2\ldots\beta_n - p(\beta_2\beta_3\ldots\beta_n + \beta_1\beta_3\ldots\beta_n + \cdots + \beta_1\beta_2\ldots\beta_{n-1})}.$$

(ii) If $\beta_1 \leq 1$ then, from Theorem 4, $I_1(f) \in H^\infty$ and $I_2(I_1(f)) \in H^\infty$ etc. If $(\exists i)\ i \in \{1, 2, \ldots, n-1\}$,

$$\beta_{i+1} = \frac{\beta_1\beta_2\ldots\beta_i p}{\beta_1\beta_2\ldots\beta_n - p(\beta_2\beta_3\ldots\beta_i + \beta_1\beta_3\ldots\beta_i + \cdots + \beta_1\beta_2\beta_{i-1}},$$

then $I^n(f) \in H^\infty$.

□

Remark. For $\beta_1 = \beta_2 = \cdots = \beta_n = \beta$ we obtain:

(i) if $\beta > np$ then $I^n(f) \in H^{\frac{\beta p}{\beta - np}}$;

(ii) if $\beta \leq np$ then $I^n(f) \in H^\infty$.

5. Some particular cases

1) For $\beta = 1$, $\gamma = 0$ the integral operator I becomes $I(f) = \int_0^z \frac{f(t)}{t}dt$, and we obtain Alexander's operator. If $p\frac{1}{n}$ then $I^n(f) \in H^{\frac{p}{1-np}}$ and if $p \geq \frac{1}{n}$ then $I^n(f) \in H^\infty$.
2) For $\beta = 1$, $\gamma = 1$ the integral operator I becomes $I(f) = \frac{2}{z}\int_0^z f(t)dt$, and we obtain Libera's operator. If $p < \frac{1}{n}$ then $I^n(f) \in H^{\frac{p}{1-np}}$ and if $p \geq \frac{1}{n}$ then $I^n(f) \in H^\infty$.
3) For $\beta = \alpha$, $\gamma = 0$ we obtain the operator $I(f) = \left[\alpha \int_0^z f^\alpha(t)t^{-1}dt\right]^{\frac{1}{\alpha}}$. In 1974 S.S. Miller, P.T. Mocanu, and M.O. Reade showed that $I(S^*) \subset S^*$.. We obtain Hardy classes for that operator. If $p < \frac{\alpha}{n}$ then $I^n(f) \in H^{\frac{\alpha p}{\alpha - np}}$ and if $p \geq \frac{\alpha}{n}$ then $I^n(f) \in H^\infty$.
4) If S denotes a subset of A consisting of univalent functions and if $S^*, K, M_\alpha, U_K, V_K$ and $MV[\alpha, k]$ denote subsets of A consisting of starlike, convex, α-convex, bounded argument rotation, bounded boundary rotation and bounded Mocanu variation functions respectively, then it's easy to determine Hardy classes for $I(S)$, $I(S^*)$, $I(K)$, $I(U_K)$, $I(V_K)$ and $I(MV[\alpha, k])$.
5) For $\beta = \frac{1}{\alpha}$, $\gamma = 0$ we obtain the operator $I(f) = \left(\frac{1}{\alpha}\int_0^z f^{\frac{1}{\alpha}}(t)t^{-1}dt\right)^\alpha$. This is the integral representation for functions with bounded Mocanu variation for $f \in U_k$. If

$\beta > p$ we have $\frac{1}{\alpha} > \frac{2}{k+2}$ then $\frac{2\alpha}{k+2} < 1$. . From Theorem 5

$$F(z) = I(f) \in H^{\lambda} \quad \lambda = \frac{\beta p}{\beta - p} = \frac{\frac{1}{\alpha} \cdot \frac{2}{k+2}}{\frac{1}{\alpha} - \frac{2}{k+2}} = \frac{2}{k+2-2\alpha}.$$

From Theorem 6 we have

$$F'(z) = I'(f) \in H^{\mu},\ \mu = \frac{\beta p}{\beta p + \beta - p} = \frac{\frac{1}{\alpha} \cdot \frac{2}{k+2}}{\frac{1}{\alpha} \cdot \frac{2}{k+2} + \frac{1}{\alpha} - \frac{2}{k+2}} = \frac{2}{k+4-2\alpha}.$$

If $\beta \leq p$ then $I(f) \in H^{\infty}$ implies $\frac{2\alpha}{k+2} \geq 1$ then $F \in H^{\infty}$. From Theorem 6 $I'(f)$ $H^{\frac{p}{\beta}}$. Hence $F' \in H^{\frac{2\alpha}{k+2}}$. . These results were obtained in [5].

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ORBITAL PERIOD VARIATONS IN THE GRAVITOMAGNETIC FIELD OF A ROTATING MASS

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Abstract. A test particle moving in the gravitomagnetic field generated by a rotating central body is subject to a relativistic (Lense-Thirring)acceleration. The influence of this acceleration on the nodal period of the test particle is determined is studied in the framework of perturbation theory. The first order variation of the nodal period is determined analitically up to the third order in excentricity.

1. Introduction

Consider a test particle orbiting a rotating central body. As it is known since the second decade of our century (J. Lense and H. Thirring have shown that in 1918), the rotation of the central mass produces a gravitomagnetic field, implying the "inertial frame dragging" effect. In other words (e.g. [5, 8]), the spacetime in the neighbourhood of the rotating central mass is influenced by this one as if it were immersed in a viscous fluid which transfers a part of its rotational energy to the surrounding medium by means of drag forces.

Of course, under the action of this inertial frame dragging, the test particle will undergo a relativistic acceleration. This effect, due to the influence of the gravitomagnetic field, is usually called Lense-Thirring acceleration. In terms of changes in Keplerian orbital parameters of the test particle, the Lense-Thirring effect entails periodic variations for the majority of these elements and secular changes only for longitude of ascending node (orbit precession) and argument of pericentre (apsidal motion).

These results of first order were obtained by applying a perturbative treatment to the motion of the test particle in the gravitomagnetic field generated by the rotating central ma·.. In other words, the Lense-Thirring acceleration was considered to be a

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perturbing acceleration altering the Keplerian motion in the Newtonian field of the bod (in the point mass approximation),

Since, as our knowledge goes, the variation of the orbital period due to th Lense-Thirring effect was not tackled yet (the existing analytic results being confined t Keplerian orbital elements), we shall do that in the sequel. The nodal period will b analyzed, because this one allows the study of very low eccentric (even circular) orbit too (for such orbits the anomalistic period is not well defined or even undefined). Th same above mentioned perturbative way will be followed. The analytic results (to fir order in the small parameter featuring the Lense-Thirring acceleration) will be give with a third order accuracy in the orbital eccentricity.

2. Basic equations of the method

Treating the problem in a perturbative manner, and recalling that it is th nodal period to be studied, we naturally start from Newton-Euler equations writte with respect to u (argument of latitude) in the form (e.g.[6])

$$\begin{aligned} dp/du &= 2(Z/\mu)r^3T, \\ dq/du &= (z/\mu)\{r^3kBCW/(pD) + r^2T[r(q+A)/p + A] + r^2BS\}, \\ dk/du &= (Z/\mu)\{-r^3qBCW/(pD) + r^2T[r(k+B)/p + B] - r^2AS\}, \\ d\Omega/du &= (Z/\mu)r^3BW/(pD), \\ di/du &= (Z/\mu)r^3AW/p, \\ dt/du &= Zr^2/\sqrt{\mu p}, \end{aligned} \qquad (1$$

where $Z = [1 - r^2C(d\Omega/dt)/\sqrt{\mu p}]^{-1/2}$, μ = gravitational parameter of the central mas r = radius vector of the test particle, p = semilatus rectum, $q = e\cos\omega$, $k = e\sin$ (e = eccentricity, ω = argument of pericentre), Ω = longitude of ascending node, i inclination, $A = \cos u$, $B = \sin u$, $C = \cos i$, $D = \sin i$, S, T, W = radial, transverse, an binormal components of the perturbing acceleration, respectively.

The changes of the orbital elements $y \in \{p, q, k, \Omega, i\}$ between the initial (u_0 and current (u) positions are given by

$$\Delta y = \int_{u_0}^{u} (dy/du)du, \qquad (2$$

with the integrands provided by the first five equations (1). The integrals (2) are estimated by successive approximations, with $Z \approx 1$, limiting the process to first order approximation.

The nodal period, defined as

$$T_\Omega = \int_0^{2\pi} (dt/du)du, \tag{3}$$

is determined starting from the last equation (1), by means of a method proposed by I.D. Zhongolovich [9], and extended in [3]. The principles of this method have already been exposed elsewhere (see e.g. [3]) and will not be repeated here. It provides the difference ΔT_Ω between the real (perturbed) nodal period T_Ω and the corresponding Keplerian period T_0. To first order in a small parameter σ which features the perturbing factor, this difference reads

$$\Delta T_\Omega = I_1 + I_2 + I_3 + I_4, \tag{4}$$

with

$$\begin{aligned} I_1 &= (3/2)\sqrt{p_0/\mu}\int_0^{2\pi}(1+q_0A+k_0B)^{-2}\Delta p\, du, \\ I_2 &= -2p_0\sqrt{p_0/\mu}\int_0^{2\pi}(1+q_0A+k_0B)^{-3}A\Delta q\, du, \\ I_3 &= -2p_0\sqrt{p_0/\mu}\int_0^{2\pi}(1+q_0A+k_0B)^{-3}B\Delta k\, du, \\ I_4 &= \int_0^{2\pi}\{\partial[r^4C(d\Omega/dt)/(\mu p]/\partial\sigma\}\sigma du, \end{aligned} \tag{5}$$

and Δy, $y \in \{p,q,k\}$, are the first order (in σ) perturbations provided by (2). Subscript "0" added to a quantity means the value of the respective quantity for $u = u_0$.

3. Equations of motion in the gravitomagnetic field

So far (Section 2) the nature of the perturbation remained unspecified, formulae (1)-(5) being valid for any perturbing factor. Now we shall write the equations of motion for our problem, namely for the motion in the gravitomagnetic field generated by the rotating central mass.

The Lense-Thirring acceleration components read (e.g. [8])

$$\begin{aligned} S &= KhC/r^4, \\ T &= -KhCe\sin\nu/(pr^3), \\ W &= KhD(2B + Are\sin\nu/p)/r^4, \end{aligned} \tag{6}$$

where ν is the true anomaly and $h = \sqrt{\mu p}$. The factor K has the expression (cf. [8])

$$K = 2(\gamma + 1)\mu w R^2/(5c^2), \tag{7}$$

with γ = space curvature parameter, w (constant) = rotational angular velocity of the field-generating body, R = radius of this body, c = speed of light.

Taking into consideration the relation $u = \omega + \nu$ and the definition of q and k, expressions (6) become

$$\begin{aligned} S &= KhC/r^4, \\ T &= -K(h/p)C(Bq - Ak)/r^3, \\ W &= KhD[2B + r(ABq - A^2k)/p]/r^4. \end{aligned} \tag{8}$$

Substituting expressions (8) in equations (1), then using the orbit equation in polar coordinates $r = p/(1 + e\cos\nu)$ under the form

$$r = p/(1 + Aq + Bk) \tag{9}$$

to replace r in the resulting expression, the equations of motion acquire the form

$$\begin{aligned} dp/du &= -2ZxbpC(Bq - Ak), \\ dq/du &= ZxbC[B + (2 + 2B^2)k - Bq^2 + (2A + 3AB^2)qk + 3B^3k^2], \\ dk/du &= -ZxbC[A + (2 + 2B^2)q + (A + 3AB^2)q^2 + (B + 2B^3)qk - Ak^2], \\ d\Omega/du &= Zxb[2B^2 + 3AB^2q + (2B - 3A^2B)k], \\ di/du &= ZxbD[2AB + 3A^2Bq + (2A - 3A^3)k], \\ dt/du &= (Z/b)(1 + Aq + Bk)^{-2}, \end{aligned} \tag{10}$$

where we introduced the notation

$$x = K/\mu = 2(\gamma + 1)wR^2/(5c^2), \tag{11}$$

$$b = h/p^2 = \sqrt{\mu/p}/p. \tag{12}$$

4. Results

As shown in Section 2, equations (10) are solved by successive approximations, with $Z \approx 1$, limiting the process to first order approximation. So, the first order variations of the elements p, q, k, Ω, i in the interval $[u_0, u]$, written in a compact form, are

$$\begin{gathered}\Delta p = -2xb_0p_0C_0(I_{01}q_0 - I_{10}k_0),\\ \Delta q = xb_0C_0[I_{01} + 2(I_{00} + I_{02})k_0 - I_{01}q_0^2 + (2I_{10} + 3I_{12})q_0k_0 + 3I_{03}k_0^2],\\ \Delta k = -xb_0C_0[I_{10} + 2(I_{00} + I_{02})q_0 + (I_{10} + 3I_{12})q_0^2 +\\ +(I_{01} + 3I_{03})q_0k_0 - I_{10}k_0^2],\\ \Delta\Omega = xb_0[2I_{02} + 3I_{12}q_0 + (3I_{03} - I_{01})k_0],\\ \Delta i = xb_0D_0[2I_{11} + 3(I_{01} - I_{03})q_0 + (3I_{12} - I_{10})k_0],\end{gathered} \tag{13}$$

where we did not explicit yet the quantities I_{mn}, functions of u

$$I_{mn} = \int_{u_0}^{u} A^m B^n du. \tag{14}$$

Let us now perform the first three integrals (5). For this purpose we expand the first factor of each integrand to third order in q_0, k_0 (that is, in eccentricity). Then we explicit the first three expressions (13) with respect to u, and introduce them in the respective integrands. Lastly, performing the resulting integrals and taking into account notation (12), we get to third order q_0, k_0

$$\begin{gathered}I_1 = -3\pi xC_0(2A_0q_0 + 2B_0k_0 + 2q_0^2 - 2q_0k_0 + 2k_0^2 +\\ +3A_0q_0^3 + 3B_0q_0^2k_0 + 3A_0q_0k_0^2 + 3B_0k_0^3),\\ I_2 = \pi xC_0\{2 + 6A_0q_0 + 7q_0^2 + 6[A_0B_0 + 3(\pi - u_0)]q_0k_0 - 3k_0^2 +\\ +9A_0q_0^3 - 6B_0(B_0^2 + 2)q_0^2k_0 - 3(2A_0^3 - 11A_0 - 16)q_0k_0^2\},\\ I_3 = \pi xC_0\{2 - 12q_0 + 6B_0k_0 + 17q_0^2 - 6[A_0B_0 + 3(\pi - u_0)]q_0k_0 + 7k_0^2 -\\ -24q_0^3 + 3B_0(2B_0^2 + 7)q_0^2k_0 + 6(A_0^3 - 3A_0 - 8)q_0k_0^2 + 9B_0k_0^3\}.\end{gathered} \tag{15}$$

As to the fourth integral (5), we consider x to be the small parameter σ. Using the fourth and the last equations (1), as well as the expansion of (9), and taking again into account notation (12), we get after integration

$$I_4 = 2\pi xC_0(1 + k_0^2). \tag{16}$$

Finally, introducing expressions (15)-(16) in (4), one obtains

$$\Delta T_\Omega = 6\pi xC_0(1 - 2q_0 + 3q_0^2 + q_0k_0 - 4q_0^3 + A_0q_0k_0^2). \tag{17}$$

5. Concluding remarks and comments

Formula (17) gives, with an accuracy of third order in eccentricity, the first orde (in x) perturbation of the nodal period (that is, the difference between the real noda period and the corresponding Keplerian one) due to the Lense-Thirring acceleratio undergone by the test particle moving in the gravitomagnetic field generated by th uniformly rotating central mass. Since the nodal period was chosen, the consideration o very low eccentric (even circular) orbits is allowed; for the anomalistic period such orbit must be avoided. It is true that in our case equatorial orbits may not be considered but this is a smaller shortcoming. Lastly, the third order accuracy in eccentricity make expression (17) be a good approximation for orbits eccentric enough, up to $e_0 = 0.$ roughly.

Expression (17) emphasizes some connections between ΔT_Ω and the initial orbi of the test particle. So, for initially polar orbits ($C_0 = 0$) the nodal period is not affecte by the Lense-Thirring effect. The closer to equator an orbit is (but not so close t make the nodes not well defined), the stronger the Lense-Thirring effect on the noda period will be. Also observe that up to $e_0 = 0.05$ roughly the sign of ΔT_Ω depends onl on the inclination (through C_0). More precisely, if the test particle has direct motio ($C_0 > 0$), then $T_\Omega > T_0$, hence the Lense-Thirring effects acts to decelerate the motion for retrograde orbits the motion will be accelerated. Finally observe that to second orde in eccentricity ΔT_Ω does not depend on the initial position of the test particle u_0 (throug A_0); such a dependence appears only for an accuracy of third order in e.

By (17), and observing from (11) that x does not depend on the initial orbit one can formulate a surprising conclusion. The first order (in x) perturbation of th nodal period depends on the shape and orientation of the initial orbit (through e_0, ω_0, i contained in q_0, k_0, C_0) and on the initial position of the test particle (through $A_0 =$ $\cos u_0$), but not on the orbit dimensions (through a linear element p_0, say). This mean that a gravitomagnetic field generating source will produce the same difference ΔT for all homothetic orbits in the field. We must emphasize that this independence o the linear element (on distance for circular orbits) vanishes at the second order (in x perturbation of the nodal period. As a matter of fact, V. Mioc and E. Radu [4] obtaine a similar result while studying the first order effect of the Lorentz force on the noda period of a charged artificial satellite moving in the dipolic geomagnetic field (cf. [1, 2])

Another surprising conclusion arises from formulae (17) and (11). The first order (in x) perturbation of the nodal period does not depend on the mass of the body which generates the gravitomagnetic field. Such a dependence does appear only at the second order approximation.

The fact that, in our first order approximation, the nodal period variation does not depend on either mass of the field source or initial orbit dimensions is more surprising if we observe that, in the same approximation, the variations (13) of the orbital elements depend on these factors (through b_0 given by formula (12)).

To push the analytic calculations to the second order (in x) approximation could present only a theoretical interest from the above standpoint; this is useless from a practical point of view. To perform some numerical estimates, we referred to the solar system (see also [7]). With the known value of c, and with $\gamma \approx 1$, we assigned to R and w values corresponding to Sun and major planets. We obtained that for circumsolar orbits ΔT_Ω is of order $10^{-4}s$; for circumplanetar orbits the order is $10^{-4}s$ (Jupiter, Saturn), $10^{-6}s$ (Earth), $10^{-9}s$ (Mercury, Venus), and so forth. This enables us to give up a second order perturbation [3] and to consider (17) a good approximation for the perturbation of the nodal period of a test particle moving in the gravitomagnetic field generated by a rotating central mass.

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ON THE NUMERICAL SOLUTION OF A SYSTEM OF THIRD ORDER DIFFERENTIAL EQUATIONS BY SPLINE FUNCTIONS

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Abstract. The purpose of this paper is to construct Spline function approximations for solving the system of differential equations:

$$y''' = f_1(x, y, y', z, z'), \quad z''' = f_2(x, y, y', z, z') \quad \text{with} \quad y^{(i)}(x_0) = y_0^{(i)}$$

and $z^{(i)}(x_0) = z_0^{(i)}$ where $i = 0(1)2$. The approximating functions used in the method are polynomial splines. It is shown that the method is a one-step method $O(h^{\alpha+r})$ in $y^{(i)}(x)$, $z^{(i)}(x)$, $i = 0(1)2$ and $O(h^{\alpha+r+3-q})$ in $y^{(q)}(x)$, $z^{(q)}(x)$ where $q = 3(1)r+3$, assuming $f_1, f_2 \in C^r([0,1]) \times \mathbf{R}^4)$, $r \in I^+$, $0 < \alpha \leq 1$. It is also shown that the method is stable.

1. Assumptions and procedures

Consider the system of differential equations:

$$y''' = f_1(x, y, y', z, z'), \quad y^{(i)}(x_0) = y_0^{(i)} \tag{1.1}$$

$$z''' = f_2(x, y, y', z, z'), \quad z^{(i)}(x_0) = z_0^{(i)} \tag{1.2}$$

where $f_1, f_2 \in C^r([0,1]) \times \mathbf{R}^4)$, $i = 0(1)2$.

Let Δ be the partition:

$$\Delta: \; 0 = x_0 < x_1 < \cdots < x_k < x_{k+1} < \cdots < x_n = 1$$

where $x_{k+1} - x_k = h < 1$ and $k = 0(1)n-1$.

Let L_1 and L_2 be the Lipschitz constants satisfied by the functions $f_1^{(q)}$, $f_2^{(q)}$ respectively, i.e.,

$$|f_i^{(q)}(x, y_1, y_1', z_1, z_1') - f_i^{(q)}(x, y_2, y_2', z_2, z_2')| \leq L_i\{|y_1 - y_2| + |y_1' - y_2'| +$$

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$$+|z_1 - z_2| + |z_1' - z_2'|\}, \quad i = 1, 2 \tag{1.3}$$

for all $x, y_1, y_1', z_1, z_1')$, $(x, y_2, y_2', z_2, z_2')$ in the domain of definition of the functions $f_1^{(q)}$, $f_2^{(q)}$ where $q = 0(1)r$.

The functions $f_1^{(q)}$, $i = 1, 2$ and $q = 1(1)r$ are functions of x, y, y', z, z' only and they are given from the following Algorithm:

Set $f_i^{(0)} = f_o(x, y, y', z, z')$ and if $f_i^{(q-1)}$ are defined, then

$$f_i^{(q)} = \frac{\partial f_i^{(q-1)}}{\partial x} + \frac{\partial f_i^{(q-1)}}{\partial y} y' + \frac{\partial f_i^{(q-1)}}{\partial y'} y'' + \frac{\partial f_i^{(q-1)}}{\partial z} z' + \frac{\partial f_i^{(q-1)}}{\partial z'} z''.$$

Then, we define the Spline functions approximating $y(x)$ and $z(x)$ by $S_\Delta(x)$ and $\overline{S}_\Delta(x)$ where

$$S_\Delta(x) = S_k(x) = S_{k-1}(x_k) + S_{k-1}'(x_k)(x - x_k) + S_{k-1}''(x_k)\frac{(x - x_k)^2}{2!} +$$

$$+ \sum_{j=0}^{r} f_1^{(j)}[x_k, S_{k-1}(x_k), S_{k-1}'(x_k), \overline{S}_{k-1}(x_k), \overline{S}_{k-1}'(x_k)] \frac{(x - x_k)^{j+3}}{(j+3)!} \tag{1.4}$$

and

$$\overline{S}_\Delta(x) = \overline{S}_k(x) = \overline{S}_{k-1}(x_k) + \overline{S}_{k-1}'(x_k)(x - x_k) + \overline{S}_{k-1}''(x_k)\frac{(x - x_k)^2}{2!} +$$

$$+ \sum_{j=0}^{r} f_2^{(j)}[x_k, S_{k-1}(x_k), S_{k-1}'(x_k), \overline{S}_{k-1}(x_k), \overline{S}_{k-1}'(x_k)] \frac{(x - x_k)^{j+3}}{(j+3)!} \tag{1.5}$$

where $S_{-1}^{(i)}(x_0) = y_0^{(i)}$, $\overline{S}_{-1}^{(i)}(x_0) = z_0^{(i)}$, $i = 0(1)2$.

By construction, it is clear that $S_\Delta(x)$, $\overline{S}_\Delta(x) \in \mathbf{C}^2([0, 1] \times \mathbf{R}^4)$.

2. Error estimations and convergence

For all $x \in x_k, x_{k+1}]$, $k = 0(1)n - 1$, let the exact solution of (1.1) and (1.2) be written in the following forms:

$$y(x) = \sum_{j=0}^{r+2} \frac{y_k^{(j)}}{j!}(x - x_k)^j + y^{(r+3)}(\xi_k)\frac{(x - x_k)^{r+3}}{(r+3)!} \tag{2.1}$$

and

$$z(x) = \sum_{j=0}^{r} \frac{z_k^{(j)}}{j!}(x - x_k)^j + z^{(r+3)}(\eta_k)\frac{(x - x_k)^{r+3}}{(r+3)!} \tag{2.2}$$

where ξ_k, $\eta_k \in (x_k, x_{k+1})$ and $k = 0(1)n - 1$.

Before we proceed to discuss the convergence of these Spline approximants, we state first the following notations:

$$e(x) = |y(x) - S_\Delta(x)|,$$

$$e_k = |y_k - S_\Delta(x_k)|,$$

$$\overline{e}(x) = |z(x) - \overline{S}_\Delta(x)|,$$

$$\overline{e}_k = |z_k - \overline{S}_\Delta(x_k)|,$$

$$f_{1,k}^{(j)} = f_1^{(j)}[x_k, S_{k-1}(x_k), S'_{j-1}(x_k), \overline{S}_{k-1}(x_k), \overline{S}'_{k-1}(x_k)], \tag{2.3}$$

$$f_{2,k}^{(j)} = f_2^{(j)}[x_k, S_{k-1}(x_k), S'_{j-1}(x_k), \overline{S}_{k-1}(x_k), \overline{S}'_{k-1}(x_k)],$$

$$f_{1,k}^{*(j)} = f_1^{(j)}[x_k, y_k, y'_k, z_k, z'_k]$$

and

$$f_{2,k}^{*(j)} = f_2^{(j)}[x_k, y_k, y'_k, z_k, z'_k]$$

where $j = 0(1)r$ and $k = 0(1)n-1$.

Throughout this work, we will consider the general subinterval $I_k = [x_k, x_{k-1}]$, $k = 0(1)n-1$.

First, we estimate $|y(x) - S_k(x)|$.

Using (1.4), (2.1), the Lipschitz condition (1.3) and the notations (2.3) we get:

$$e(x) \le |y_k - S_{k-1}(x_k)| + |y'_k - S'_{k-1}(x_k)||x - x_k| + |y''_{k-1}(x_k) - S''_{k-1}(x_k)|\frac{|x - x_k|^2}{2!} +$$

$$+ \sum_{j=0}^{r-1} |y_k^{j+3} - f_{1,k}^{(j)}| \frac{|x - x_k|^{j+3}}{(j+3)} + |y^{r+3}(\xi_k) - f_{1,k}^{(r)}| \frac{|x - x_k|^{r+3}}{(r+3)!} \le$$

$$\le e_k + he'_k + \frac{h^2}{2!}e''_k + \sum_{j=0}^{r-1} |y_k^{(j+3)} - f_{1,k}^{(j)}| \frac{h^{j+3}}{(j+3)!} + |y^{(r+3)}(\xi_k) - f_{1,k}^{(r)}| \frac{h^{r+3}}{(r+3)!} \tag{2.4}$$

If we let

$$P = |y_k^{(j+3)} - f_{1,k}^{(j)}|$$

then, using (1.3) and (2.3), we get:

$$P \le L_1(e_k + e'_k + \overline{e}_k + \overline{e}'_k). \tag{2.5}$$

Also, let

$$\hat{P} = |y^{(r+3)}(\xi_k) - f_{1,k}^{(r)}|.$$

Then, using (1.3) and (2.3) we get:

$$\hat{P} \le \omega(y^{(r+3)}, h) + L_1(e_k + e'_k + \bar{e}_k + \bar{e}'_k) \tag{2.6}$$

where $\omega(y^{(r+3)}, h)$ is the modulus of continuity of the function $y^{(r+3)}$.

Using (2.5) and (2.6) and noting that

$$\sum_{j=0}^{r-1} \frac{h^{j+2}}{(j+3)!} < e^h - 2 < e$$

we can easily get:

$$e(x) \le (1 + c_0 h)e_k + c_0 h\bar{e}_k + (1 + c_0)he'_k + c_0 h\bar{e}'_k + \frac{h^2}{2!}e''_k + \frac{h^{r+3}}{(r+3)!}\omega(y^{(r+3)}, h) \tag{2.7}$$

where $c_0 = L_1\left(e + \frac{1}{(r+3)!}\right)$ is a constant independent of h.

In a similar manner, using (1.5), (2.2), the Lipschitz condition (1.3) and the notations (2.3), it can be easily shown that:

$$\bar{e}(x) \le c_1 he_k + (1 + c_1 h)\bar{e}_k + c_1 he'_k + (1 + c_1)h\bar{e}'_k + \frac{h^2}{2!}\bar{e}''_k + \frac{h^{(r+3)}}{(r+3)!}\omega(z^{(r+3)}, h) \tag{2.8}$$

where $\omega(z^{(r+3)}, h)$ is the modulus of continuity of the function $z^{(r+3)}$ and $c_1 = L_2\left(1 + \frac{1}{(r+3)!}\right)$ is a constant independent of h.

Now, we are going to estimate $|y'(x) - S'_k(x)|$ and $|z'(x) - \bar{S}'_k(x)|$.

Using (1.3)-(2.3) and noting that

$$\sum_{j=0}^{r} \frac{h^{j+1}}{(j+2)!} < e - 1 < e$$

we can easily get:

$$e'(x) \le c_2 he_k + c_2 h\bar{e}_k + (1 + c_2 h)e'_k + c_2 h\bar{e}'_k + he''_k + \frac{h^{r+2}}{(r+2)!}\omega(y^{(r+3)}, h) \tag{2.9}$$

and

$$\bar{e}'(x) \le c_3 he_k + c_3 h\bar{e}_k + c_3 he'_k + (1 + c_3 h)\bar{e}'_k + h\bar{e}''_k + \frac{h^{r+2}}{(r+2)!}\omega(z^{(r+3)}, h) \tag{2.10}$$

where $c_2 = L_1\left(e + \frac{1}{(r+2)!}\right)$ and $c_3 = L_2\left(e + \frac{1}{(r+2)!}\right)$ are constants independent of h.

We now estimate $|y''(x) - S''_k(x)|$ and $|z''(x) - \bar{S}''_k(x)|$.

Using equations (1.3)-(2.3) and utilizing the inequality

$$\sum_{j=0}^{r-1} \frac{h^j}{(j+1)!} < e$$

we can see that:

$$e''(x) \leq c_4 h e_k + c_4 h \overline{e}_k + c_4 h e'_k + c_4 h \overline{e}'_k + e''_k + \frac{h^{r+1}}{(r+1)!}\omega(y^{r+3}, h) \tag{2.11}$$

and

$$\overline{e}''(x) \leq c_5 h e_k + c_5 h \overline{e}_k + c_5 h e'_k + c_5 h \overline{e}'_k + \overline{e}''_k + \frac{h^{r+1}}{(r+1)!}\omega(z^{r+3}, h) \tag{2.12}$$

where $c_4 = L_1\left(e + \frac{1}{(r+1)!}\right)$ and $c_5 = L_2\left(e + \frac{1}{(r+1)!}\right)$ are constants independent of h.

To complete the convergence proof, we introduce the following definition of the matrix inequality:

Definition 1. Let $A = [a_{i,j}]$, $B = [b_{i,j}]$ be two matrices of the same order, then we say that $A \leq B$ iff:

(i) $a_{i,j}$ and $b_{i,j}$ are non negative,

(ii) $a_{i,j} \leq b_{i,j}$, $\forall\ i, j$.

In view of this definition, and if we use the matrix notations:

$$E(x) = (e(x) \quad \overline{e}(x) \quad e'(x) \quad \overline{e}'(x) \quad e''(x) \quad \overline{e}''(x))^T$$

and

$$E_k = (e_k \quad \overline{e}_k \quad e'_k \quad \overline{e}'_k \quad e''_k \quad \overline{e}''_k)^T, \quad k = 0(1)n-1$$

then, we can write the estimations (2.7)-(2.12) in the following form:

$$E(x) \leq (I + hA)E_k + h^{r+1}\omega(h)B \tag{2.13}$$

where

$$A = \begin{bmatrix} c_0 & c_0 & 1+c_0 & c_0 & 1/2! & 0 \\ c_1 & c_1 & c_1 & 1+c_1 & 0 & 1/2! \\ c_2 & c_2 & c_2 & c_2 & 1 & 0 \\ c_3 & c_3 & c_3 & c_3 & 0 & 1 \\ c_4 & c_4 & c_4 & c_4 & 0 & 1 \\ c_5 & c_5 & c_5 & c_5 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1/(r+3)! \\ 1/(r+3)! \\ 1/(r+2)! \\ 1/(r+2)! \\ 1/(r+1)! \\ 1/(r+1)! \end{bmatrix},$$

I is the identity matrix of order 6 and

$$\omega(h) = \max\{\omega(y^{(r+3)}, h), \omega(z^{(r+3)}, h)\}.$$

Next, we give the following definition of the matrix norm.

Definition 2. Let $T = [\tau_{ij}]$ be an $m \times n$ matrix, then we define

$$\|T\| = \max_i \sum_{j=1}^{n} |\tau_{ij}|.$$

According to this definition, we get:

$$\|E(x)\| = \max\{e(x), \bar{e}(x), e'(x), \bar{e}'(x), e''(x), \bar{e}''(x)\}. \tag{2.14}$$

Since (2.13) is valid for all $x \in [x_k, x_{k+1}]$, $k = 0(1)n-1$, then the following inequalities hold true:

$$\|E(x)\| \leq (1 + h\|A\|)\|E_k\| + h^{r+1}\omega(h)\|B\|$$

$$(1 + h\|A\|)\|E_k\| \leq (I + h\|A\|)^2\|E_{k-1}\| + h^{r+1}\omega(h)\|B\|(1 + h\|A\|)$$

$$(1 + h\|A\|)^2\|E_{k-1}\| \leq (I + h\|A\|)^3\|E_{k-2}\| + h^{r+1}\omega(h)\|B\|(1 + h\|A\|)^2$$

$$\vdots$$

$$(1 + h\|A\|)^k\|E_1\| \leq (I + h\|A\|)^{k+1}\|E_0\| + h^{r+1}\omega(h)\|B\|(1 + h\|A\|)^k.$$

Adding L.H.S. and R.H.S. of these inequalities and noting that $\|E_0\| = 0$, we get:

$$\|E(x)\| \leq c_6 h^r \omega(h)$$

where $c_6 = (e^{\|A\|} - 1)\frac{\|B\|}{\|A\|}$ is a constant independent of h.

Thus using (2.14), we get:

$$\begin{aligned} e^{(i)}(x) &\leq c_6 h^r \omega(h) = O(h^{\alpha+r}) \\ \bar{e}^{(i)}(x) &\leq c_6 h^r \omega(h) = O(h^{\alpha+r}) \end{aligned} \tag{2.15}$$

where $i = 0(1)2$.

We are going to estimate $|y^{(q)}(x) - S_k^{(q)}(x)|$ where $q = 3(1)r+2$.

Using (1.3), (1.4), (2.1), (2.3), (2.5),(2.6) and (2.15), we get:

$$|y^{(q)}(x) - S_k^{(q)}(x)| \leq \sum_{j=q-3}^{r-1} |y_k^{(j+3)} - f_{1,k}^{(j)}| \frac{|x - x_k|^{j+3-q}}{(j+3-q)!} +$$

$$+|y^{(r+3)}(\xi_k) - f_{1,k}^{(r)}| \frac{|x - x_k|^{r+3-q}}{(r+3-q)!} \leq c_7 h^{r+3-q}\omega(h) = O(h^{\alpha+r+3-q})$$

where $c_7 = 4L_1 c_6 \left(1 + \frac{1}{(r+3-q)!}\right) + \frac{1}{(r+3-q)!}$ is a constant independent of h.

Similarly, using (1.3), (1.5), (2.2), (2.3), (2.5), (2.6) and (2.15), it can be shown that:

$$|z^{(q)}(x) - \bar{S}_k^{(q)}(x)| \leq c_8 h^{r+3-q}\omega(h) = O(h^{\alpha+r+3-q})$$

where $q = 3(1)r+2$ and $c_8 = 4L_2c_6\left(e + \frac{1}{(r+3-q)!}\right) + \frac{1}{(r+3-q)!}$ is a constant independent of h.

For the case $q = r+3$, we have:

$$|y^{(r+3)}(x) - S_k^{(r+3)}| = |y^{(r+3)}(x) - f_{1,k}^{(r)}| \leq$$

$$\leq |y^{(r+3)} - y_k^{(r+3)}| + |f_{1,k}^{*(r)} - f_{1,k}^{(r)}| \leq c_9\omega(h) = O(h^\alpha).$$

Similarly,

$$|z^{(r+3)}(x) - \overline{S}_k^{(r+3)}| \leq c_{10}\omega(h) = O(h^\alpha)$$

where $c_9 = 1 + 4L_1c_6$ and $c_{10} = 1 + 4L_2c_6$, are constants independent of h.

Thus, we have proved the following theorem:

Theorem 1. *Let $S_\Delta(x)$ and $\overline{S}_\Delta(x)$ be the approximate solutions to problem (1.1)-(1.2) given by the equations (1.4)-(1.5), and let $f_1 f_2 \in \mathbf{C}^r([x_0, x_n] \times \mathbf{R}^4)$, then for all $x \in [x_k, x_{k+1}]$, $k = 0(1)n-1$, we have:*

$$|y^{(i)}(x) - S_k^{(i)}(x)| \leq Ch^r\omega(h),$$

$$|z^{(i)}(x) - \overline{S}_k^{(i)}(x)| \leq Ch^r\omega(h),$$

$$|y^{(j)}(x) - S_k^{(j)}(x)| \leq Kh^{r+3-j}\omega(h)$$

and

$$|z^{(j)}(x) - \overline{S}_k^{(j)}(x)| \leq K^* h^{r+3-j}\omega(h)$$

where $i = 0(1)2$, $j = 3(1)r+3$, C, K and K^* are constanis independent of h.

3. Stability of the method

The stability concept for a one-step method means that small perturbations in the initial data for the numerical method will result in small changes in the numerical values, independent of the grid size h of the numerical method.

To study the stability of the method given by (1.4)-(1.5), we change $S_\Delta(x)$ by $W_\Delta(x)$ and $\overline{S}_\Delta(x)$ by $\overline{W}_\Delta(x)$, where

$$W_\Delta(x) \equiv W_k(x) = W_{k-1}(x_k) + W'_{k-1}(x_k)(x - x_k) + W''_{k-1}(x_k)\frac{(x-x_k)^2}{2!} +$$

$$\sum_{j=0}^{r} f_1^{(j)}\{x_k, W_{k-1}(x_k), W'_{k-1}(x_k), \overline{W}_{k-1}(x_k), \overline{W}'_{k-1}(x_k)\}\frac{|x-x_k|^{j+3}}{(j+3)!} \qquad (3.1)$$

and

$$\overline{W}_\Delta(x) \equiv \overline{W}_k(x) = \overline{W}_{k-1}(x_k) + \overline{W}'_{k-1}(x_k)(x - x_k) + \overline{W}''_{k-1}(x_k)\frac{(x - x_k)^2}{2!} +$$

$$\sum_{j=0}^{r} f_2^{(j)}\{x_k, W_{k-1}(x_k), W'_{k-1}(x_k), \overline{W}_{k-1}(x_k), \overline{W}'_{k-1}(x_k)\}\frac{|x - x_k|^{j+3}}{(j+3)!} \quad (3$$

where $W_{-1}^{(i)}(x_0) = y_0^{*(i)}$, $\overline{W}_{-1}^{(i)}(x_0) = z_0^{*(i)}$, $i = 0(1)2$.

We define the following notations:

$$\varepsilon(x) = |W_\Delta(x) - S_\Delta(x)|, \quad \varepsilon_k = |W_\Delta(x_k) - S_\Delta(x_k)|,$$

$$\overline{\varepsilon}(x) = |\overline{W}_\Delta(x) - \overline{S}_\Delta(x)|, \quad \overline{\varepsilon}_k = |\overline{W}_\Delta(x_k) - \overline{S}_\Delta(x_k)|, \quad (\text{;}$$

$$\hat{f}_{1,k}^{(j)} = f_1^{(j)}[x_k, W_{k-1}(x_k), W'_{k-1}(x_k), \overline{W}_{k-1}(x_k), \overline{W}'_{k-1}(x_k)]$$

and

$$\hat{f}_{2,k}^{(j)} = f_2^{(j)}[x_k, W_{k-1}(x_k), W'_{k-1}(x_k), \overline{W}_{k-1}(x_k), \overline{W}'_{k-1}(x_k)].$$

For all $x \in [x_k, x_{k+1}]$, $k = 0(1)n - 1$, by using (1.4), (3.1), we get:

$$|W_\Delta(x) - S_\Delta(x)| \leq |W_{k-1}(x_k) - S_{k-1}(x_k)| + |W'_{k-1}(x_k) - S'_{k-1}(x_k)||x - x_k| +$$

$$+|W''_{k-1}(x_k) - S''_{k-1}(x_k)|\frac{|x - x_k|^2}{2!} + \sum_{j=0}^{r}|\hat{f}_{1,k}^{(j)} - f_{1,k}^{(j)}|\frac{|x - x_k|^{j+3}}{(j+3)!} \quad ($$

Now, let

$$\hat{V}_1 = |\hat{f}_{1,k}^{(j)} - f_{1,k}^{(j)}| \quad ($$

Then, from (2.3), (3.3) and the Lipschitz condition (1.3), we get:

$$\hat{V}_1 \leq L_1(\varepsilon_k + \varepsilon'_k + \overline{\varepsilon}_k + \overline{\varepsilon}'_k). \quad ($$

Thus, (3.4) gives:

$$\varepsilon(x) \leq (1 + d_0 h)\varepsilon_k + d_0 h\overline{\varepsilon}_k + (1 + d_0)h\varepsilon'_k + d_0 h\overline{\varepsilon}'_k + \frac{h^2}{2!}\varepsilon''_k \quad ($$

where $d_0 = L_1 e$ is a constant independent of h.

In a similar manner, by using (1.4), (1.5), (3.1)-(3.3) and the Lipschitz condi (1.3), it can be shown that:

$$\overline{\varepsilon}(x) \leq d_1 h\varepsilon_k + (1 + d_1 h)\overline{\varepsilon}_k + d_1 h\varepsilon'_k + (1 + d_1)h\overline{\varepsilon}'_k + \frac{h^2}{2!}\overline{\varepsilon}''_k,$$

$$\varepsilon'(x) \le d_0 h\varepsilon_k + d_0 h\bar{\varepsilon}_k + (1 + d_0 h)\varepsilon'_k + d_0 h\bar{\varepsilon}'_k + h\varepsilon''_k, \tag{3.8}$$

$$\bar{\varepsilon}'_x \le d_1 h\varepsilon_k + d_1 h\bar{\varepsilon}_k + d_1 h\varepsilon'_k + (1 + d_1 h)\bar{\varepsilon}'_k + h\bar{\varepsilon}''_k,$$

$$\varepsilon''(x) \le d_0 h\varepsilon_k + d_0 h\bar{\varepsilon}_k + d_0 h\varepsilon'_k + d_0 h\bar{\varepsilon}'_k + \varepsilon''_k$$

and

$$\bar{\varepsilon}''(x) \le d_1 h\varepsilon_k + d_1 h\bar{\varepsilon}_k + d_1 h\varepsilon'_k + d_1 h\bar{\varepsilon}'_k + \bar{\varepsilon}''_k$$

where $d_1 = L_2 e$, is a constant independent of h.

If we put:

$$\hat{E}(x) = (\varepsilon(x) \quad \bar{\varepsilon}(x) \quad \varepsilon'(x) \quad \bar{\varepsilon}'(x) \quad \varepsilon''(x) \quad \bar{\varepsilon}''(x))^T \tag{3.9}$$

and

$$\hat{E}_k = (\varepsilon_k \quad \bar{\varepsilon}_k \quad \varepsilon'_k \quad \bar{\varepsilon}'_k \quad \varepsilon''_k \quad \bar{\varepsilon}''_k)^T, \quad k = 0(1)n-1.$$

Then, from (3.7)-(3.9), we get the following inequality:

$$\hat{E}(x) \le (I + h\hat{A})\hat{E}_k \tag{3.10}$$

where

$$\hat{A} = \begin{bmatrix} d_0 & d_0 & 1+d_0 & d_0 & 1/2! & 0 \\ d_1 & d_1 & d_1 & 1+d_1 & 0 & 1/2! \\ d_0 & d_0 & d_0 & d_0 & 1 & 0 \\ d_1 & d_1 & d_1 & d_1 & 0 & 1 \\ d_0 & d_0 & d_0 & d_0 & 0 & 0 \\ d_1 & d_1 & d_1 & d_1 & 0 & 0 \end{bmatrix}$$

and I is the identity matrix of order 6.

Since (3.10) is valid for all $x \in [x_k, x_{k+1}]$, $k = 0(1)n-1$, then the following inequalities hold true:

$$||\hat{E}(x)|| \le (1 + h)||\hat{A}||)||\hat{E}_k||$$

$$(1 + h||\hat{A}||)||\hat{E}_k|| \le (1 + h||\hat{A}||)^2||\hat{A}_{k-1}||$$

$$(1 + h||\hat{A}||)^k||\hat{E}_1|| \le (1 + h||\hat{A}||)^{k+1}||\hat{E}_0||$$

Adding L.H.S. and R.H.S. of these inequalities, we can easily get:

$$||E(x)|| \le c_1||\hat{E}_0|| \tag{3.11}$$

where $c_1 = e^{||\hat{A}||}$ is a constant independent of h.

Applying definition 2, we get:

$$\varepsilon^{(i)}(x) \le c_1 \|\hat{E}_0\| \tag{3.12}$$

and

$$\overline{\varepsilon}^{(i)}(x) \le c_1 \|\hat{E}_0\|$$

where $\|\hat{E}_0\| = \max\{|y_0 - y_0^*|, |y_0' - y_0^{*\prime}|, |y_0'' - y_0^{*\prime\prime}|, |z_0 - z_0^*|, |z_0' - z_0^{*\prime}|, |z_0'' - z_0^{*\prime\prime}|\}$ and $i = 0(1)2$.

We are going to estimate $|W_\Delta^{(q)}(x) - S_\Delta^{(q)}(x)|$ where $q = 3(1)r + 3$.

Using (1.4), (3.1), (3.6) and (3.11), we get:

$$|W_\Delta^{(q)}(x) - S_\Delta^{(q)}(x)| \le \sum_{j=q-3}^{r} |\hat{f}_{1,k}^{(j)} - f_{1,k}^{(j)}| \frac{|x - x_k|^{j+3-q}}{(j+3-q)!} \le d^* \|\hat{E}_0\| \tag{3.13}$$

where $d^* = 4L_1 e c_1$ is a constant independent of h.

In a similar manner, using (1.5), (3.2), (3.11), it can be shown that:

$$|\overline{W}_\Delta^{(q)}(x) - \overline{S}_\Delta^{(q)}(x)| \le \overline{d}^* \|\hat{E}_0\| \tag{3.14}$$

where $\overline{d}^* = 4L_2 e c_1$ is a constant independent of h and $q = 3(1)r + 3$.

Thus, we have proved the following theorem:

Theorem 2. *Let $(S_\Delta, \overline{S}_\Delta(x))$ given by (1.4)-(1.5) be the approximate solution to problem (1.1)-(1.2) with the initial conditions $y^{(i)}(x_0) = y_0^{(i)}$ and $z^{(i)}(x_0) = z_0^{(i)}$, and let $(W_\Delta(x), \overline{W}_\Delta(x))$ given by (3.1)-(3.2) be the approximate solution for the same problem with the initial conditions $y^{(i)}(x_0) = y_0^{*(i)}$, $z^{(i)}(x_0) = z_0^{*(i)}$, $i = 0(1)2$, then the inequalities*

$$|W_\Delta^{(q)}(x) - S_\Delta^{(q)}(x)| \le \overline{c} \|\hat{E}_0\|$$

and

$$|\overline{W}_\Delta^{(q)}(x) - \overline{S}_\Delta^{(q)}(x)| \le \overline{k} \|\hat{E}_0\|$$

hold true for all $x \in [x_k, x_{k+1}]$, $k = 0(1)n - 1$ and $q = 0(1)r + 3$, $r \in I^+$ where $\overline{c}, \overline{k}$ are constants independent of h and

$$\|\hat{E}_0\| = \max\{|y_0^{(i)} - y_0^{*(i)})|, |z_0^{(i)} - z_0^{*(i)}|\}, \quad i = 0(1)2.$$

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ON THE POSITIVE SOLUTION OF SOME SEMILINEAR ELLIPTIC EQUATIONS ON A BOUNDED DOMAIN

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Abstract. In this note we study the existence and some properties of positive solutions of the problem

$$\begin{cases} -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}(x)\right) + a_0(x)u(x) = \lambda f(x,u) & \text{in } \Omega \\ \alpha(x)u(x) + \beta(x)\frac{\partial u}{\partial v}(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a sufficiently smooth bounded domain in $\mathbf{R}^n$ $(N \geq 3)$, $\lambda \in \mathbf{R}_+$ and the functions $a_{ij}(\cdot), a_0(\cdot), \alpha(\cdot), \beta(\cdot)$ and $f(\cdot,\cdot)$ have certain properties.

Let Ω be a sufficiently smooth bounded domain in $\mathbf{R}^N$ $(N \geq 3)$ and let L be the uniformly elliptic, self-adjoint, second order operator

$$Lu(x) = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}(x)\right) + a_0(x)u(x).$$

Suppose that the coefficients $a_{ij}(x) = a_{ji}(x)$ are continuously differentiable and $a_0(x) > 0$ is continuous. The boundary conditions will be taken as

$$Bu(x) = \alpha(x)u(x) + \beta(x)\frac{\partial u}{\partial v}(x) = 0, \quad x \in \partial\Omega,$$

where $\alpha(x) \geq 0$ $(\not\equiv 0)$, $\beta(x) \geq 0$. Here $\frac{\partial}{\partial v}$ is the conormal derivative

$$\frac{\partial u}{\partial v}(x) = \sum_{i,j}^{N} v_i(x)a_{ij}(x)\frac{\partial u}{\partial x_j}(x),$$

where $v(x) = (v_1(x), \ldots, v_N(x))$ is the outer unit normal to $\partial\Omega$ at x.

Under certain conditions of $f(x,u)$ we seek that for all $\lambda \in \mathbf{R}_+^*$ the boundary value problem

$$\begin{cases} Lu = \lambda f(x,u), & x \in \Omega \\ Bu = 0, & x \in \partial\Omega \end{cases} \tag{1}$$

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has a unique positive solution $u(x) > 0$, for all $x \in \Omega$.

We call the set $\{\lambda\}$ of real values of λ, for which positive solutions of 1 exist, th spectrum of the problem 1.

The conditions to be imposed on f will be the following

F-0: $f(x,\varphi)$ is continuous on the $N+1$ dimensional half-cylinder $\Omega \times \mathbf{R}_+$;

F-1: $f(x,0) = f_0(x) > 0$ on Ω;

F-2: $0 < f(x,\varphi) < f(x,\psi)$ on Ω if $\varphi > \psi > 0$ (i.e. f is strictly decreasing in tl second variable on $\mathbf{R}_+$);

F-3: Exist $0 < \alpha < 1$ such that

$$t^\alpha f(x,tu) \le f(x,u)$$

on $\Omega \times \mathbf{R}_+$, for every $t \in (0,1)$.

The problem 1, in a different conditions about the function f, has been studi by various authors (see, for example, [2] and [3]). Under **F-0**, 1 and

F'-2: $f(x,\varphi) > f(x,\psi)$ on Ω if $\varphi > \psi \ge 0$ (i.e. f is strictly increasing in t second variable on $\mathbf{R}_+$);

it has been shown by Keller and Cohen in [2] that if $\lambda^* > 0$, where

$$\lambda^* = \sup\{\lambda : \text{ the problem 1 has a positive solution}\},$$

the positive solutions exist for all $\lambda \in (0, \lambda^*)$. If in addition f is concave in the seco variable, it was shown that $\lambda^* < \infty$ and that the positive solutions do not exist : $\lambda = \lambda^*$. Moreover it was shown that the positive solutions are unique and stable. Th results are extended by Keener and Keller in [3].

We state for future reference

Lemma 1. *(Positivity Lemma in [2]). Let $\rho(x)$ be positive and continuous on Ω and $\varphi(x)$ be twice continuously differentiable and satisfy*

$$\begin{cases} L\varphi(x) - \lambda\rho(x)\varphi(x) > 0, & x \in \Omega \\ B\varphi(x) = 0, & x \in \partial\Omega. \end{cases}$$

Then $\varphi(x) > 0$ on Ω if and only if $\lambda < \mu_1$, where μ_1 is the principal (i.e. least) eigenva of

$$\begin{cases} L\psi(x) - \mu\rho(x)\psi(x) > 0, & x \in \Omega \\ B\psi(x) = 0, & x \in \partial\Omega. \end{cases}$$

Theorem 2. *(Theorem 3.1 in [2]). Let $f(x,\varphi) > 0$ on Ω if $\varphi > 0$ and satisfy* **F-0**. *The only positive λ can be in the spectrum of 1.*

Our result in this note is the following

Theorem 3. *i) For every $\lambda > 0$ the problem 1 has exactly one positive solutions $u(\lambda,\cdot)$ on Ω. Moreover*

$$\lim_{n\to\infty} \|u_n - u(\lambda,\cdot\|_\infty = 0$$

where $u_0(x) = 0$ and

$$\begin{cases} Lu_n(x) = \lambda f(x, u_{n-1}(x)) & \text{in } \Omega \\ Bu_n(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

ii) $u(\lambda,\cdot)$ is strictly increasing in λ.

Proof. Let $\lambda > 0$ and $u_0(x) \equiv 0$. Define u_n such that

$$\begin{cases} Lu_n(x) = \lambda f(x, u_{n-1}(x)) & \text{in } \Omega \\ Bu_n(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Then it follows, from Positivity Lemma and **F-1**, that

$$\begin{cases} Lu_1(x) = \lambda f(x, 0) & \text{in } \Omega \\ Bu_1(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence $u_1 > 0 \equiv u_0$. Suppose that

$$u_0 < u_2 < \cdots < u_{2p-2} < u_{2p-1} < u_{2p-3} < \cdots < u_1. \tag{3}$$

We see from **F-2** and 3 that

$$\begin{cases} L(u_{2p} - u_{2p-2}) = \lambda[f(x - u_{2p-1}) - f(x, u_{2p-3})] > 0 & \text{in } \Omega \\ B(u_{2p} - u_{2p-2}) = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

Then, from Positivity Lemma, $u_{2p} > u_{2p-2}$.

$$\begin{cases} L(u_{2p-1} - u_{2p}) = \lambda[f(x, u_{2p-2}) - f(x, u_{2p-1})] > 0 & \text{in } \Omega \\ B(u_{2p-1} - u_{2p}) = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

Then, from Positivity Lemma, $u_{2p-1} > u_{2p}$.

$$\begin{cases} L(u_{2p+1} - u_{2p}) = \lambda[f(x, u_{2p}) - f(x, u_{2p-1})] > 0 & \text{in } \Omega \\ B(u_{2p+1} - u_{2p}) = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $u_{2p+1} > u_{2p}$.

$$\begin{cases} L(u_{2p-1} - u_{2p+1}) = \lambda[f(x, u_{2p-2}) - f(x, u_{2p})] > 0 & \text{in } \Omega \\ B(u_{2p-1} - u_{2p+1}) = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence $u_{2p-1} > u_{2p+1}$.

Thus, from 4, 5 and 6, we have

$$u_{2p-2} < u_{2p} < u_{2p+1} < u_{2p-1}.$$

Hence it follows by induction that

$$u_0 < u_2 < \cdots < u_{2p} < u_{2p+1} < \cdots < u_1, \quad p = 0, 1, \ldots$$

Since $\{u_{2p}\}_{p\geq 0}$ is a monotone sequence, it follows that $u_{2p} \nearrow u_*$ as $p \to$ Similarly $u_{2p+1} \searrow u^*$ as $p \to \infty$. It can be seen easily from 7 that

$$u_0 < u_2 < \cdots < u_{2p} < u_* \leq u^* < u_{2p+1} < \cdots < u_1, \quad p = 0, 1, \ldots$$

We now employ the Green's function $G_0(x, \xi)$ for L on Ω subject to BG_0 for $x \in \partial\Omega$ to write the iteration scheme 2 in the equivalent form

$$u_0(x) = 0$$

$$u_n(x) = \lambda \int_\Omega G_0(x, \xi) f(\xi, u_{n-1}(\xi)) d\xi, \quad n = 1, 2, \ldots$$

It is now a simple matter to show that

$$u_*(x) = \lambda \int_\Omega G_0(x, \xi) f(\xi, u^*(\xi)) d\xi$$

and

$$u^*(x) = \lambda \int_\Omega G_0(x, \xi) f(\xi, u_*(\xi)) d\xi.$$

We next show that $u_* = u^*$.

Let

$$M_n = ||u_n||_\infty \quad \text{and} \quad \varepsilon_n = \min_{x\in\Omega} \frac{f(x, u_n(x))}{f(x, u_{n-1}(x))}, \quad n = 1, 2, \ldots$$

We observe that $0 < \varepsilon_n < 1$ is n is odd and $\varepsilon_n > 1$ if n is even. We have

$$\begin{aligned} u_*(x) &> u_{2p}(x) = \lambda \int_\Omega G_0(x,\xi) f(\xi, u_{2p-1}(\xi)) d\xi \geq \\ &\geq \varepsilon_{2p-1} \lambda \int_\Omega G_0(x,\zeta) f(\xi, u_{2p}(\xi)) d\xi \geq \\ &\geq \varepsilon_{2p-1} u_{2p+1} > \\ &> \varepsilon_{2p-1} u^*. \end{aligned} \tag{9}$$

Set

$$A = \{\varepsilon > 0 : \ u_* \geq \varepsilon u^*\}.$$

From 9, A is nonempty. Let $\varepsilon_0 = \sup A$ and we claim that $\varepsilon_0 \geq 1$.

Suppose that $0 < \varepsilon_0 < 1$. Then

$$\begin{aligned} u^*(x) &= \lambda \int_\Omega G_0(x,\xi) f(\xi, u_*(\xi)) d\xi \leq \\ &\leq \lambda \int_\Omega G_0(x,\xi) f(\xi, \varepsilon_0 u^*(\xi)) d\xi \leq \\ &\leq \frac{1}{\varepsilon_0^\alpha} \lambda \int_\Omega G_0(x,\xi) f(\xi, u^*(\xi)) d\xi = \frac{1}{\varepsilon_0^\alpha} u_*(x), \end{aligned}$$

which implies that $u_* \geq \varepsilon_0^\alpha u^*$. Since $\varepsilon_0^\alpha > \varepsilon_0$, this contradicts the choice of ε_0. Thus we have $u_* \geq u^*$. From this and from $u_* \leq u^*$ we obtain

$$u_* = u^* = u(\lambda, \cdot).$$

Thus

$$u(\lambda, x) = \lambda \int_\Omega G_0(x,\xi) f(\xi, u(\lambda,\xi)) d\xi.$$

It follows that $u(\lambda, \cdot)$ is a positive solution of 1. Note that, since $u(\lambda, \cdot)$ is continuous, then $\lim_{n\to\infty} u_n(x) = u(\lambda, x)$ uniformly in Ω (see Dini's Theorem).

Finally, if v is another solution of 1, we have $v > 0$. Thus

$$u_0 \equiv 0 \leq v = \lambda \int_\Omega G_0(x,\xi) f(\xi, v(\xi)) d\xi \leq u_1.$$

Continuing this process, we obtain

$$u_{2p} \leq v \leq u_{2p+1}, \quad p = 0, 1, \ldots,$$

which implies $u(\lambda, \cdot) = v$. Thus we have proved that for any $\lambda > 0$ 1 admits a unique positive solution $u(\lambda, \cdot)$ on Ω and

$$\lim_{n\to\infty} \|u_n - u(\lambda, \cdot)\|_\infty = 0,$$

where $u_0 \equiv 0$ and

$$\begin{cases} Lu_n(x) = \lambda f(x, u_{n-1}(x)) & \text{in } \Omega \\ Bu_n(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

ii) By employing the uniform convergence of the iterates $\{u_n\}$ to the posit solution $u(\lambda, \cdot)$, it is not difficult to deduce that

$$\frac{\partial u}{\partial \lambda}(\lambda, x) \equiv (\lambda, x)$$

exists, is continuous in λ and satisfies the variational system

$$\begin{cases} Lv - \lambda f_u(x, u)v = f(x, u) & x \in \Omega \\ Bv = 0 & x \in \partial\Omega. \end{cases}$$

Thus we have

$$\begin{cases} Lv > 0 & x \in \Omega \\ Bv = 0 & x \in \partial\Omega. \end{cases}$$

Then $v(\lambda, x) \geq 0$ on Ω. If $v(\lambda, x) = 0$ at some point $x \in \Omega$, we have

$$\frac{\partial v}{\partial x_i} = 0, \quad i = 1, \ldots, N$$

and at which the matrix

$$\left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)_{1 \leq i,j \leq N}$$

must be positive semi-definite. We obtain

$$-\sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 v}{\partial x_i \partial x_j}(x, u) = f(x, u) > 0,$$

which contradicts the fact that $(a_{ij}(x))_{1 \leq i,j \leq N}$ is positive definite. Thus $v(\lambda, x) >$ (Ω and this ends the proof. □

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ON THE INEQUALITY OF HERMITE-HADAMARD

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Abstract. One considers a notion of convexity with respect of a function h, called h-convexity. One improves the Hermite-Hadamard inequality for functions with an h-convex inverse, generalizing a result of. H.-J. Seiffert.

1. Introduction

We consider a notion of convexity with respect to a function h, called h-convexity. We improve Hermite-Hadamard's inequality for functions with h-convex inverse. For $h(x) = x^r$ we obtain the result from [9] which includes the results of H.-J. Seiffert from [8] and that of H. Alzer from [1]. The proof is like that of [1] and not like those of [8] and [9].

To formulate them we need more definitions on functionals and means.

2. Functionals

Let E be a nonempty set and $F(E)$ be a linear space of real-valued functions defined on E. A functional $T : F(E) \to \mathbf{R}$ is linear if:

$$T(tf + sg) = tT(f) + sT(g), \quad \forall\, t, s \in \mathbf{R},\ f, g \in F(E).$$

It is isotonic if:

$$T(f) \geq 0, \quad \forall\, f \in F(E),\ f \geq 0.$$

We shall suppose also that $T(1) = 1$, where the first "1" denotes the constant function $f(x) = 1$, $\forall\, x \in E$. Here as in what follows, if we use $T(f)$, we assume that $f \in F(E)$.

Common examples of such functionals are given by:

$$T(f) = \int_E f\, dm / \int_E dm$$

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and

$$T(f) = \sum_{k=1}^{n} p_k f(x_k) / \sum_{k=1}^{n} p_k$$

where m is a positive measure on E and $p_k > 0$, $x_k \in E$ for $k = 1, \ldots, n$.

A. Lupas has generalized in [4] Hermite-Hadamard's inequality for isotonic linear functionals but we need it in a more general form given in [2].

Theorem 0. *If the function f is convex on $[c,d]$ and the functional T is isotonic and linear on $F(E)$, with $T(1) = 1$, then for every function $g : E \to [c,d]$ we have $T(g) \in [c,d]$ and:*

$$f(T(g)) \leq T(f(g) \leq [(d - T(g))f(c) + (T(g) - c)f(d)]/(d - c). \tag{1}$$

For $E = [a,b] = [c,d]$ and $g(x) = x$ we get the result from [4]. In the special case when T is the integral arithmetic mean W, defined for a continuous on $[a,b]$ function f by:

$$W(f; a, b) = \frac{1}{b-a} \int_a^b f(x)dx$$

the inequality (1) becomes Hermite-Hadamard's inequality:

$$f((a+b)/2) \leq W(f; a, b) \leq [f(a) + f(b)]/2. \tag{2}$$

3. Means

In what follows we use some quasi-arithmetic means. If h is a positive strictly monotone function defined on the set of positive numbers and $t \in [0,1]$, we denote:

$$A_{h,t}(x,y) = h^{-1}(th(x) + (1-t)h(y)).$$

If h is the identity function we get the usual weighted arithmetic mean A_t, which for $t = 1/2$ becomes the arithmetic mean A.

For $h(x) = e_r(x) = x^r$, $r \neq 0$, we have the power means:

$$P_{r,t}(x,y) = (tx^r + (1-t)y^r)^{1/r}.$$

For $r = 0$ one takes $h(x) = e_0(x) = \log x$, getting the (weighted) geometric mean:

$$P_{0,t}(x,y) = G_t(x,y) = x^t y^{1-t}.$$

It is easy to verify (see [3]) that:

$$A_{h,t}(x,y) \leq A_{g,t}(x,y), \quad \forall\, x,y>0,\ 0,t\in[0,1] \tag{3}$$

if and only if:

i) g is increasing and $g(h^{-1})$ is convex, or:

ii) g is decreasing and $g(h^{-1})$ is concave.

As shown by J.G. Mikusinski (see [3], p.31) if g and h are twice differentiable and g', h' are never zero, then the above conditions hold if and only if:

$$g''/g' \geq h''/h'.$$

In the special case of the power means we see that they are increasing, that is:

$$P_{r,t}(x,t) < P_{s,t}(x,y) \quad \text{if} \quad r<s,\ t\in(0,1);\ x\neq y.$$

We use also the family of generalized logarithmic means defined for r different from -1 and 0 by:

$$L_r(x,y) = [(y^{r+1}-x^{r+1})/((r+1)(y-x))]^{1/r}$$

but

$$L_0(x,y) = I(x,y) = (1/e)(y^y/x^x)^{1/(y-x)}$$

is the identical mean, and

$$L_{-1}(x,y) = L(x,y) = (y-x)/(\log y - \log x)$$

the logarithmic mean. For $y = x$ all the means have the value x. This family is also increasing:

$$L_r(x-y) < L_s(x,y) \quad \text{if} \quad r<s,\ x\neq y. \tag{4}$$

4. Generalized convexity

Using the quasi-arithmetic means we can define a notion of convexity generalizing the logarithmic convexity.

Definition. The positive function $f\in C[a,b]$ is called h-convex if:

$$f(A_t(x,y)) \leq A_{h,t}(f(x),f(y)), \quad \forall\, x,y\in[a,b].$$

In addition to the usual convexity (with $h = e_1$) and the logarithmic convexit (where $h = \log = e_0$), C. Das has considered in his Ph. D. Thesis (see [5]) the case harmonic convexity by taking $h = e_{-1}$. The notion of e_r-convexity was considered in under the name of r-convexity.

Of course, the function f is h-convex if and only if $h(f)$ is convex for h increasin and concave for h decreasing. So (3) holds if and only if h^{-1} is g-convex. Thus, the abo definition is in concordance with that of logarithmic convexity but differs from a definitic accepted in [3, pp.30-31].

From the above remarks we deduce that every h-convex function is also g-conv if and only if h^{-1} is g-convex. In the special case of the power means it follows that $r < s$ every e_r-convex function is also e_s-convex. This generalizes the relation betwe logarithmic convexity and convexity.

5. A result of Seiffert

In what follows we suppose that $0 < a < b$. In [8] H.-J. Seiffert proved that $f' \in C[a,b]$ is strictly increasing and f^{-1} is log-convex then:

$$W(f,a,b) \leq f(I(a,b)). \qquad ($$

We remark that if f^{-1} is log-convex then f is also concave but (5) improves t corresponding inequality from (2) because by (4):

$$I = L_0 < L_1 = A.$$

Also, H. Alzer proved in [1] a related result: if $f \in C[a,b]$ is strictly increasi and $1/f^{-1}$ is convex, then:

$$W(f;a,b) \geq f(L(a,b)).$$

The result of H.-J. Seiffert is related to e_0-convexity and that of H. Alzer e_{-1}-concavity. In what follows we shall generalize these results.

Theorem. *If the function $f : [a,b] \to [c,d]$ and $h : [a,b] \to \mathbf{R}$ are strictly increasi f^{-1} is h-convex and the functional T is isotonic and linear on $F([a,b])$, with $T(1) =$ then:*

$$h(a) \leq T(h) \leq h(b)$$

and

$$\frac{f(a)[h(b)-T(h)]+f(b)[T(h)-h(a)]}{h(b)-h(a)} \leq T(f) \leq f(h^{-1}(T(h))). \tag{6}$$

Proof. In (1) we put $c = f(a)$, $d = f(b)$, $h(f^{-1})$ for f and f for g obtaining:

$$h(f^{-1}(T(f))) \leq T(h) \leq \frac{[f(b)-T(f)]h(a)+[T(f)-f(a)]h(b)}{f(b)-f(a)}.$$

Extracting from each inequality $T(f)$ we get (6). □

Remark 1. If h^{-1} is g-convex, (6) gives:

$$\frac{T(h)-h(a)}{h(b)-h(a)} \geq \frac{T(g)-g(a)}{g(b)-g(a)}$$

and

$$h^{-1}(T(h)) \leq g^{-1}(T(g)).$$

So if we pass from g-convexity to h-convexity the class of functions for which (6) is valid is diminished but the evaluations are improved.

Consequence 1. *If the function $f \in C[a,b]$ is strictly increasing and f^{-1} is* log*-convex then:*

$$\frac{f(a)[L(a,b)-a]+f(b)[b-L(a,b)]}{b-a} \leq W(f;a,b) \leq f(I(a,b)). \tag{7}$$

Proof. We have $h = \log$ and

$$W(\log;a,b) = \frac{b\log b - a\log a}{b-a} = 1$$

so that (6) gives (7).

We remark that (7) offers a companion inequality to Seiffert's inequality (5). □

Consequence 2. *If the function $f \in C[a,b]$ is strictly increasing and f^{-1} is e_r-convex, with $r \neq 0$, then:*

$$\frac{f(a)[b^r - L_r^r(a,b)]+f(b)[L_r^r(a,b)-a^r]}{b^r-a^r} \leq W(f;a,b) \leq f(L_r(a,b)).$$

Remark 2. This result was proved otherwise in [9]. As we have shown there, the conditions of the consequences are satisfied by twice differentiable functions f is and only if:

$$f'(x) > 0 \quad \text{and} \quad 1 + \frac{xf''(x)}{f'(x)} \leq r, \ \forall\, x \in [a,b]. \tag{8}$$

We obtain so a class of functions which can be very interesting because
second relation of (8) is analogous with that satisfied by complex convex functions (
[7], pp.255-256).

Also the relation (8) shows that an inequality of J.D. Kečkic and I.B. Lacko
(see [6], pp.367-368) can be deduced from (6).

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APPROXIMATION PROPERTIES OF A CLASS OF BIVARIATE OPERATORS OF D.D. STANCU

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Abstract. In 1984 D.D. Stancu [15] introduced and studied a two-parameters class of linear positive operators, generalizing the Bernstein operators. They have been investigated by probabilistic methods. In this paper we consider a two dimensional generalization of these operators and we investigate their approximation properties, we give a convergence theorem and we evaluate the rest and the order of approximation.

1. In 1984 D.D. Stancu [15] has introduced and investigated a class of positive linear operators, useful in constructive approximation theory: $\left(S_{n,r}^{(\alpha)}\right)$, where $m \in \mathbf{N}$, r being a non-negative integer parameter, such that $2r < m$, while α is a non-negative parameter which may depend on m. To each function $f : [0,1] \to \mathbf{R}$, there was associated the operator $S_{m,r}^{(\alpha)}$, defined by

$$\left(S_{m,r}^{(\alpha)} f\right)(x) := \sum_{k=0}^{m-r} p_{m-r,k}^{(\alpha)}(x) \left\{ [1 - x + (m-r-k)x] f\left(\frac{k}{m}\right) + (x + k\alpha) f\left(\frac{k+r}{m}\right) \right\} \tag{1}$$

where, in terms of factorial powers

$$y^{[n,h]} := y(y-h)\dots(y-(n-1)h), \quad y^{[0,h]} := 1,$$

we have

$$p_{m-r,k}^{(\alpha)}(x) = \binom{m-r}{k} \frac{x^{[k,-\alpha]}(1-x)^{[m-r-k,-\alpha]}}{(1+\alpha)^{[m-r,-\alpha]}}. \tag{2}$$

These operators were further investigated by D.D. Stancu in the paper [17], where he indicated a probabilistic way for their construction.

The operators corresponding to the case $\alpha = 0$ have been investigated in detail in a paper by the same author [14].

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In this paper we present a bivariate extension of these operators and investigat how they can be used in the constructive theory of functions.

For any real-valued function, defined on the standard unit square $D = \{(x, y) | 0 \leq x,\ y \leq 1\}$, and any given non-negative integers r $(2r < m)$ and s $(2s < n)$ we define th D.D.Stancu bivariate positive linear operator $S_{m,n,r,s}^{(\alpha,\beta)}$ by the following formula

$$\left(S_{m,n,r,s}^{(\alpha,\beta)} f\right)(x, y) = \sum_{k=0}^{m-r} \sum_{j=0}^{n-s} p_{m-r,k}^{(\alpha)}(x) p_{n-s,j}^{(\beta)}(y) F_{m,n,r,s}^{f}(x, y; \alpha, \beta), \qquad (3$$

where α and β are non-negative parameters, while

$$F_{m,n,r,s}^{f}(x, y; \alpha, \beta) = [1 - x + (m - r - k)\alpha][1 - y + (n - s - j)\beta] f\left(\frac{k}{m}, \frac{j}{n}\right) +$$

$$+[1 - x + (m - r - k)\alpha](y + j\beta) f\left(\frac{k}{m}, \frac{j+s}{n}\right) +$$

$$+(x + k\alpha)[1 - y + (n - s - j)\beta] f\left(\frac{k+r}{m}, \frac{j}{n}\right) + (x + k\alpha)(y_j\beta) f\left(\frac{k+r}{m}, \frac{j+s}{n}\right). \qquad (4$$

It is easy to see that the polynomial defined at (3)-(4) is interpolatory at th vertices of the square D.

Now we want to mention two special cases of the operators introduced above.

(i) If $\alpha = \beta = 0$, then it reduces to the two dimensional extension, given in [16 of the operator $S_{m,r} = S_{m,r}^{(0)}$, introduced and investigated in detail by D.D. Stancu i 1983 in the paper [14]. The extension $S_{m,n,r,s} = S_{m,n,r,s}^{(0,0)}$ given in [16] is the following

$$(S_{m,n,r,s} f)(x, y) = \sum_{k=0}^{m-r} \sum_{j=0}^{n-s} p_{m-r,k}(x) p_{n-s,j}(y) F_{m,n,r,s}^{f}(x, y), \qquad (5$$

where

$$F_{m,n,r,s}^{f}(x, y) = (1 - x)(1 - y) f\left(\frac{k}{m}, \frac{j}{n}\right) + (1 - x) y f\left(\frac{k}{m}, \frac{j+s}{n}\right) +$$

$$+x(1 - y) f\left(\frac{k+r}{m}, \frac{j}{n}\right) + xy f\left(\frac{k+r}{m}, \frac{j+s}{n}\right). \qquad (6$$

(ii) For $r = 0$ or $r = 1$ and $s = 1$ we obtain from (3)-(4) the operator considere in [13], representing the bivariate extension of the classical operator of D.D. Stancu $S_m^{(\alpha)}$ introduced in 1968 in [11] and investigated further in [12], [9], [7], [8], [5], [4], [1], [6].

2. By using the formulas (3)-(4) we can find at once the values of the Stancu bivariate operator, considered above, for the test functions $e_{i,j}$, defined for any $(x,y) \in D$ by $e_{i,j}(x,y) = x^i y^j$ $(0 \le i+j \le 2)$. We have

$$S_{m,n,r,s}^{\langle\alpha,\beta\rangle} e_{i,j} = e_{i,j} \quad (0 \le i+j \le 1), \tag{7}$$

while

$$\begin{aligned} \left(S_{m,n,r,s}^{\langle\alpha,\beta\rangle} e_{2,0}\right)(x,y) &= \tfrac{1}{1+\alpha}\left\{x(x+\alpha) + \left[1 + \tfrac{r(r-1)}{m}\right]\tfrac{x(1-x)}{m}\right\} \\ \left(S_{m,n,r,s}^{\langle\alpha,\beta\rangle} e_{0,2}\right)(x,y) &= \tfrac{1}{1+\beta}\left\{y(y+\beta) + \left[1 + \tfrac{s(s-1)}{n}\right]\tfrac{y(1-y)}{n}\right\} \end{aligned} \tag{8}$$

If we assume that

$$\begin{cases} 0 \le \alpha = \alpha(m) \to 0 \text{ as } m \to \infty, \\ 0 \le \beta = \beta(n) \to 0 \text{ as } n \to \infty, \end{cases} \tag{9}$$

then we have uniformly on the square D:

$$\lim_{m,n\to\infty} S_{m,n,r,s}^{\langle\alpha,\beta\rangle} e_{i,j} = e_{i,j} \quad (0 \le i+j \le 2). \tag{10}$$

By virtue of the Bohman-Korovkin-Volkov convergence criterion we can state

Theorem 1. *If the conditions (9) are satisfied, then for any function $f \in C(D)$ we have*

$$\lim_{m,n\to\infty} S_{m,n,r,s}^{\langle\alpha,\beta\rangle} f = f,$$

uniformly on D.

3. Since $S_{m,n,r,s}^{\langle 0,0\rangle} = S_{m,n,r,s}$ are given at (5), while the operator defined at (3)-(4) is interpolatory at the vertices of D, next we consider that $\alpha > 0$, $\beta > 0$, $0 < x < 1$ and $0 < y < 1$.

It is easy to see that the operator defined at (3)-(4) can be represented as an average of the operator given at (5)-(6). We can state

Theorem 2. *If $(x,y) \in (0,1) \times (0,1)$ and $\alpha > 0$, $\beta > 0$, then we have the following integral representation*

$$\left(S_{m,n,r,s}^{\langle\alpha,\beta\rangle} f\right)(x,y) =$$
$$= C^{\langle\alpha,\beta\rangle}(x,y) \int\int_D t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} z^{\frac{y}{\beta}}(1-z)^{\frac{1-y}{\beta}-1} \left(S_{m,n,r,s} f\right)(t,z)\,dt\,dz, \tag{11}$$

where

$$C^{\langle\alpha,\beta\rangle}(x,y) = \left[B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right) B\left(\frac{y}{\beta}, \frac{1-y}{\beta}\right)\right]^{-1},$$

by $B(a,b)$ denoting the beta function

$$B(a,b) = \int_0^1 y^{a-1}(1-y)^{b-1}dy \quad (a,b>0).$$

For proving this theorem we can use the relation between the beta and gamn functions:

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

where

$$\Gamma(c) = \int_0^\infty x^{c-1}e^{-x}dx \quad (c>0).$$

We need also to apply the well known formula

$$\Gamma(c+n) = c(c+1)\dots(c+n-1)\Gamma(c).$$

4. If we take into account formula (6.8) from the paper [10], which correspon to the extension to two variables of a Peano-Milne integral representation formula o linear functional having a certain algebraic degree of exactness, we can state

Theorem 3. *If the function $f : D \to \mathbf{R}$ has continuous partial derivatives of sec order on the unit square D, then the remainder of the approximation formula*

$$f(x,y) = \left(S^{(\alpha,\beta)}_{m,n,r,s}f\right)(x,y) + \left(R^{(\alpha,\beta)}_{m,n,r,s}f\right)(x,y)$$

can be expressed, for any $(x,y) \in D$, by definite integrals as follows

$$\left(R^{(\alpha,\beta)}_{m,n,r,s}f\right)(x,y) = \int_0^1 G^{(\alpha,\beta)}_{m,r}(t,x)f^{(2,0)}(t,y)dt+$$

$$+\int_0^1 H^{(\beta)}_{n,s}(y,z)f^{(0,2)}(y,z)dz - \int_0^1\int_0^1 G^{(\alpha)}_{m,r}(t,x)H^{(\beta)}_{n,s}(y,z)f^{(2,2)}(t,z)dt\,dz, \qquad ($$

where

$$G^{(\alpha)}_{m,r}(t,x) = \left(R^{(\alpha)}_{m,r}\varphi_x(t)\right), \quad \varphi_x(t) = (x-t)_+,$$

$$H^{(\beta)}_{n,s}(y,z) = \left(R^{(\beta)}_{n,s}\psi_y(z)\right)(z), \quad \psi_y(z) = (y-z)_+,$$

$R^{(\alpha)}_{m,r}$ and $R^{(\beta)}_{n,s}$ being the one dimensional remainder terms corresponding to the lir positive operators $L^{(\alpha)}_{m,r}$, $L^{(\beta)}_{n,s}$ introduced in the paper [15].

It is easy to see that $G^{(\alpha)}_{m,r}(t,x) \le 0$, $H^{(\beta)}_{n,s}(y,z) \le 0$ on the unit square.

By applying the mean value theorem to the integrals in (12) we get

$$\left(R^{(\alpha,\beta)}_{m,n,r,s}\right)(x,y) = f^{(2,0)}(\xi,y)\int_0^1 G^{(\alpha)}_{m,r}(t,x)dt +$$

$$+f^{(0,2)}(x,\eta)\int_0^1 H^{(\beta)}_{n,s}(y,z)dz - f^{(2,2)}(\xi,\eta)\int_0^1\int_0^1 G^{(\alpha)}_{m,r}(t,x)H^{(\beta)}_{n,s}(y,z)dt\,dz.$$

But we have

$$\int_0^1 G^{(\alpha)}_{m,r}(t,x)dt = -\frac{1}{2}\left[1+\alpha m+\frac{r(r-1)}{m}\right]\frac{x(1-x)}{m(1+\alpha)}$$

and

$$\int_0^1 H^{(\beta)}_{n,s}(y,z)dz = -\frac{1}{2}\left[1+\beta n+\frac{s(s-1)}{n}\right]\frac{y(1-y)}{n(1+\beta)}.$$

Consequently we can state

Theorem 4. *If the function $f(x,y)$ has continuous partial derivatives of second order on D, then there exists a point $(\xi,\eta)\in D$ such that we have*

$$\left(R^{(\alpha,\beta)}_{m,n,r,s}f\right)(x,y) = \left[1+\alpha m+\frac{r(r-1)}{m}\right]\frac{x(1-x)}{2m(1+\alpha)}f^{(2,0)}(\xi,y)+$$

$$+\left[1+\beta n+\frac{s(s-1)}{n}\right]\frac{y(1-y)}{2n(1+\beta)}f^{(0,2)}(x,\eta)-$$

$$-\left[1+\alpha m+\frac{r(r-1)}{m}\right]\left[1+\beta n+\frac{s(s-1)}{n}\right]\frac{x(x-\eta)y(y-\eta)}{4mn(1+\alpha)(1+\beta)}f^{(2,2)}(\xi,\eta). \qquad (13)$$

5. The order of approximation of a function $f\in C(D)$ by means of the operator defined at (3)-(4) can be evaluated by means of the two dimensional modulus of continuity, defined by

$$\omega(f;\delta_1,\delta_2) = \sup|f(x'',y'')-f(x',y')|,$$

where (x',y') and (x'',y'') are points from D such that: $|x''-x'|\le\delta_1$, $|y''-y'|\le\delta_2$, δ_1 and δ_2 being given positive numbers.

By using standard procedures one can prove

Theorem 5. *If $f\in C(D)$ then for any point $(x,y)\in D$ we have*

$$\left|f(x,y)-\left(S^{(\alpha,\beta)}_{m,n,r,s}f\right)(x,y)\right| \le [1+A_{m,r}(\alpha)+A_{n,s}(\beta)]\,\omega\left(f;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right), \qquad (14)$$

where

$$A_{m,r}(\alpha) = \frac{1}{2\sqrt{1+\alpha}}\sqrt{1+\alpha m+\frac{r(r-1)}{m}}$$
$$A_{n,s}(\beta) = \frac{1}{2\sqrt{1+\beta}}\sqrt{1+\beta n+\frac{s(s-1)}{n}} \quad (1$$

Sketch of the proof. In order to prove this theorem we have to take into account that o operator is a positive linear operator on D and that it reproduces the linear functio On the other side we have to use the following properties of the modulus of continuit

$$|f(x'',y'')-f(x',y')| \leq \omega(f;|x''-x'|,|y''-y'|),$$

$$\omega(f;\lambda_1\delta_1,\lambda_2\delta_2) \leq (1+\lambda_1+\lambda_2)\omega(f;\delta_1,\delta_2).$$

One obtains

$$\left|f(x,y)-\left(S_{m,n,r,s}^{(\alpha,\beta)}f\right)(x,y)\right| \leq$$

$$\leq \left(1+\frac{1}{\delta_1}\sum_{k=0}^{m-r} p_{m-r,k}^{(\alpha)}(x)\left|x-\frac{k}{m}\right|+\frac{1}{\delta_2}\sum_{j=0}^{n-s} p_{n-s,j}^{(\beta)}(y)\left|y-\frac{j}{n}\right|\right)\omega(f;\delta_1,\delta_2). \quad ($$

According to the Cauchy inequality we can write

$$\sum_{k=0}^{m-r} p_{m-r,k}^{(\alpha)}(x)\left|x-\frac{k}{m}\right| \leq \left[\sum_{k=0}^{m-r} p_{m-r,k}^{(\alpha)}(x)\left(x-\frac{k}{m}\right)^2\right]^{1/2} =$$

$$= \left[\left(L_{m,r}^{(\alpha)}e_2\right)(x)-x^2\right]^{1/2} = \sqrt{\frac{x(1-x)}{m(1+\alpha)}}\sqrt{1+\alpha m+\frac{r(r-1)}{m}} \leq \frac{1}{\sqrt{m}}A_{m,r}(\alpha), \quad ($$

where

$$A_{m,r}(\alpha) = \frac{1}{2\sqrt{1+\alpha}}\sqrt{1+\alpha m+\frac{r(r-1)}{m}}.$$

Similarly we obtain the inequality

$$\sum_{j=0}^{n-s} p_{n-s,j}^{(\beta)}(y)\left|y-\frac{j}{n}\right| \leq \frac{1}{\sqrt{n}}A_{n,s}(\beta). \quad ($$

By taking $\delta_1=\frac{1}{\sqrt{m}}$, $\delta_2=\frac{1}{\sqrt{n}}$ and using (17) and (18), the estimations (16) le us to the T. Popoviciu type inequality (14).

In the special case $\alpha=\beta=0$ it reduces, for any $(x,y)\in D$, to the following

$$|f(x,y)-(L_{m,n,r,s}f)(x,y)| \leq \left(1+\frac{1}{2}\sqrt{1+\frac{r(r-1)}{m}}+\frac{1}{2}\sqrt{1+\frac{s(s-1)}{n}}\right)\omega\left(f;\frac{1}{\sqrt{m}},\right.$$

obtained in [16].

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