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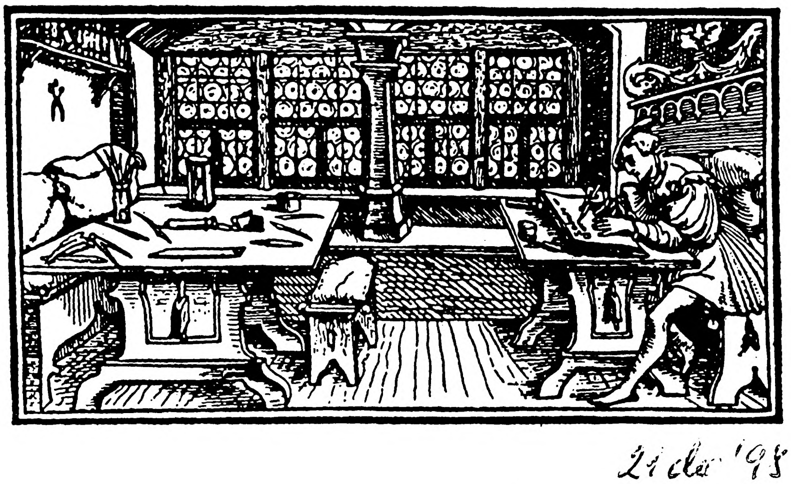

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STUDIA UNIV "BABEŞ-BOLYAI", MATHEMATICA, Volume XLI, Number 4, December 1996

PROFESSOR IOAN A. RUS AT HIS 60TH ANNIVERSARY

PAUL SZILÁGYI

1. CURRICULUM VITAE

Professor Ioan A. Rus was born in August 28, 1936, in Ianoşda (Bihor), Romania. He attended primary school in Ianoşda, secondary school in Ianoşda and Oradea (Liceul "Emanuil Gojdu"), then universitary studies (1955-1960) at the Faculty of Mathematics, Babeş-Bolyai University of Cluj-Napoca. In 1968 he obtained the title of doctor in mathematical sciences with the thesis "Dirichlet problem for strongly elliptic systems" (under the guidance of Professor D.V. Ionescu). He worked at the Babeş-Bolyai University of Cluj-Napoca as assistant professor (1960-1967), lecturer (1967-1972), associate professor (1972-1977), full professor (since 1977).

Administrative appointments: Vice-Dean of the Faculty of Mathematics (1973-1976), Vice-Rector of Babeş-Bolyai University (1976-1984 and 1992-1996), Head of the Chair of Differential Equations (since 1985).

Editorial positions: Editorial board of: Mathematica, Revue d'Analyse Numerique et de Théorie de l'Approximation, Studia Univ. Babeş-Bolyai, Pure Mathematics Manuscripts.

Ph.D. Supervisor since 1990.

Seminar activity: Chairman of the Seminar on Fixed Point Theory

Teaching Activity: The basic course on Differential Equations and many other special courses (Qualitative theory of differential equations, Fixed point theory, Nonlinear operators, Mathematical modelling). Author of five books on Differential equations and six books on Fixed point theory.

Scientific activity: More than 100 papers in the field of ODE, PDE, Integral Equations and Fixed Point Theory. The main results in:

Sturm separation and comparation theorems (see "List of publications"): [6], [16], [17], [18], [19], [20], [21], [22], [25], [85], [129].

Maximum principles in the theory of ODE and PDE: [6], [26], [27], [28], [29], [30], [44], [90], [94], [96], [98], [101], [110], [117].

Green functions: [6], [23], [24], [33].

Metrical fixed point theory: [8], [10], [11], [15], [35], [37]-[42], [45]-[56], [58], [63]-[65], [67], [68], [76], [91], [106], [115], [119].

Fixed point structures: [7], [8], [11], [57], [60], [75], [76], [79], [81], [86], [88], [89], [92], [95], [99], [100], [102], [105], [112], [114], [116], [120], [121], [122].

Applications of the fixed point theory: [6], [7]-[15], [59], [62], [66], [83], [93], [109], [113], [118], [128].

Travels: 1966-1967 (october 1, 1966 - june 1, 1967), Lund; 1966 - Göteborg; 1971 (october - december) - Moscow; 1972 - Brno; 1973 Oberwalvach; 1976 - Zagreb; 1983 - Budapesta; 1991 - Vadul lui Vodă; 1992 - Chişinău; 1993 - Tübingen; 1993 - Florenţa; 1994 - Budapesta.

Membership: Romanian Mathematical Society (1962-), American Mathematical Society (1971-), Japonesse Association of Mathematical Sciences (1995-), International Federation of Nonlinear Analist (1995-).

2. LIST OF PUBLICATIONS

A. Textbooks

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B. Books and monographs

7. Teoria punctului fix I, Teoria punctului fix în structuri algebrice, Babeş-Bolyai University, 1971.
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C. Scientific Papers

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D. Other Publications

123. Mulţimi, aplicaţii şi ecuaţii, Lucrările Seminarului Didactica Matematicii, 1(1984/1985), 83-90.

124. Principii de punct fix, Sem. Did. Mat., 2(1985/1986), 172-179.

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127 (with P. Mocanu and M. Ţarină) Creativitatea în matematică, Sem. Did. Mat., 5(1988/1989), 177-190.

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Sem. Did. Mat., 7(1990/1992), 117-122.

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APPROXIMATION THEOREM IN L_p FOR A CLASS OF OPERATORS CONSTRUCTED BY WAVELETS

OCTAVIAN AGRATINI

Dedicated to Professor Ioan A. Rus on his 60^{th} anniversary

Abstract. In this paper we deal with a linear operator of Baskakov-type which has been constructed in [1] by using wavelets. Now, we estimate the order of approximation in L_p-spaces $(1 < p \leq \infty)$ for smooth functions.

1. Introduction

Recently, "wavelets" have become a versatile tool in both theoretical and applied mathematics. Certain families of functions generated by dilations and translations of a single function ψ, i.e. given by

$$\psi_{a,b}(x) = |a|^{-1/2}\psi(ax - b), \quad a, b \in \mathbf{R},\ a \neq 0,$$

have been studied in many works, see [3], [4], [6]. By using wavelets it is possible to construct classes of operators which are useful in the approximation theory. In [5] H.H. Gonska and Ding-Xuan Zhou introduced a class of Szász-type operators by means of Daubechies'compactly supported wavelets, which have the advantage that they can be used for L_p-approximation $(1 < p \leq \infty)$. Following the same idea, in [1] was presented Baskakov-type operators. These operators are defined as

$$(L_n f)(x) = n\sum_{k=0}^{\infty} b_{n,k}(x)\int_{\mathbf{R}} f(t)\psi(nt-k)dt = \sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\lambda} f\left(\frac{t+k}{n}\right)\psi(t)dt, \quad (1)$$

where

$$b_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}, \quad x \geq 0,$$

and ψ belongs to $L_\infty(\mathbf{R})$ such as supp$\psi \subset [0, \lambda]$ with $0 < \lambda < \infty$. Also, for ψ we require the following conditions:

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(i) its first r moments vanish:

$$\int_{\mathbf{R}} t^k \psi(t)dt = 0, \quad 1 \leq k \leq r, \tag{2}$$

and

(ii)

$$\int_{\mathbf{R}} \psi(t)dt = 1. \tag{3}$$

The condition (i) implies that our operators have the same moments as Baskakov operators in an arbitrarily chosen number. When $\psi = \chi_{[0,1)}$ L_n are exactly the Baskakov-Kantorovich operators B_n^* given by:

$$(B_n^* f)(x) = n \sum_{k=0}^{\infty} b_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(u)du.$$

The main goal in this paper is to give an approximation theorem in L_p $(1 < p \leq \infty)$ for the operators introduced by (1).

2. Results

Firstly, we recall the Hardy-Littlewood maximal function M of a locally integrable function g, that is $g \in L_1^{loc}$. It is a sublinear operator of the kind

$$(Mg)(x) = \sup_{t \neq x} \left| \frac{1}{t-x} \int_x^t |g(u)|du \right|. \tag{4}$$

Obviously $\|Mg\|_\infty \leq \|g\|_\infty$. Further, an application of Marcinkiewicz's theorem (see [2], page 80) leads to the relation

$$\|Mg\|_p \leq \gamma(p)\|g\|_p \quad 1 < p \leq \infty, \tag{5}$$

where $\gamma(p)$ is a constant and $\|\cdot\|_p$ indicates the norm of the Banach space L_p. In our investigation we shall use the function φ,

$$\varphi(x) = \sqrt{x(x+1)}, \quad x \geq 0, \tag{6}$$

which represents the step-weight function related to the operators of Baskakov and Baskakov-Kantorovich.

Also, we need the first moments of Baskakov operators:

$$\mu_{n,0}(x) = 1, \quad \mu_{n,1}(x) = 0, \quad \mu_{n,2}(x) = \frac{\varphi^2(x)}{n} \tag{7}$$

where, generally

$$\mu_{n,m}(x)=\sum_{k=0}^{\infty}b_{n,k}(x)\left(\frac{k}{n}-x\right)^m,\quad m\in\mathbf{N}.$$

Finally, we note with $A.C._+^{loc}$ the space of all functions absolutely continuous in every closed finite and positive interval.

Now, we mention some results in the form of lemmas which will be used in the sequel.

Lemma 1. *If f and f' belong to $A.C._+^{loc}$ then, for any $x\in\left[0,\frac{1}{\sqrt{n}}\right]$, the following inequality*

$$|(L_nf)(x)-f(x)|\le\frac{\lambda(\lambda+3\sqrt{n})}{2n}\|\psi\|_\infty(Mf')(x)$$

holds.

Proof. Starting from the identity $f(t)=f(x)+\int_x^t f'(u)du$, and using the relations (1) and (4) we can write successively:

$$|(L_nf)(x)-f(x)| = \left|L_n\left(\int_x^t f'(u)du;x\right)\right|\le\sum_{k=0}^{\infty}b_{n,k}(x)\int_0^{\lambda}\left|\int_x^{\frac{t+k}{n}}|f'(u)|du\right|dt\|\psi\|_\infty\le$$

$$\le\sum_{k=0}^{\infty}b_{n,k}(x)\int_0^{\lambda}\left|\frac{t+k}{n}-x\right|dt\|\psi\|_\infty(Mf')(x). \tag{8}$$

Further, we have:

$$\int_0^{\lambda}\left|\frac{t+k}{n}-x\right|dt\le\frac{\lambda^2}{2n}+\lambda\left|\frac{k}{n}-x\right|. \tag{9}$$

Applying Cauchy's inequality and taking into account the relations (7), we obtain for any $x\in\left[0,\frac{1}{\sqrt{n}}\right]$:

$$\sum_{k=0}^{\infty}b_{n,k}(x)\left|\frac{k}{n}-x\right|\le\mu_{n,2}^{1/2}(x)<\frac{3}{2\sqrt{n}}. \tag{10}$$

Substituting (9) and (10) in relation (8) we arrive at the desired result. □

Lemma 2. *Let $x>\frac{1}{\sqrt{n}}$. If we define:*

$$A_n(x,\lambda)=\frac{1}{x(x+1)}\sum_{k=0}^{\infty}b_{n,k}(x)\int_0^{\lambda}\left(x-\frac{t+k}{n}\right)^2dt,$$

$$B_n(x,\lambda)=\frac{1}{x}\sum_{k=0}^{\infty}b_{n,k}(x)\int_0^{\lambda}\frac{\left(x-\frac{t+k}{n}\right)^2}{1+\frac{t+k}{n}}dt,$$

then

$$A_n(x,\lambda) < \left(\frac{\lambda^3}{3}+\lambda\right)\frac{1}{n}, \quad n \geq 1, \tag{11}$$

and

$$B_n(x,\lambda) < \left(\frac{\lambda^2}{2}+4\lambda\right)\frac{1}{\sqrt{n}}, \quad n \geq 2. \tag{12}$$

Proof. According to (7) we have:

$$A_n(x,\lambda) = \frac{1}{x(x+1)}\int_0^\lambda \left(\mu_{n,2}(x) - \frac{2}{n}t\mu_{n,1}(x) + \frac{t^2}{n^2}\right)dt = \frac{\lambda}{n} + \frac{\lambda^3}{3n^2x(x+1)}.$$

The assumption $\sqrt{n}x > 1$ implies the relation (11).

To estimate $B_n(x,\lambda)$ we notice that

$$\begin{aligned} B_n(x,\lambda) &= \frac{n}{x}\sum_{k=0}^{\infty} b_{n,k}(x)\int_{1+\frac{k}{n}}^{1+\frac{\lambda+k}{n}} \frac{(x+1-y)^2}{y}dy = \\ &= \frac{n(x+1)^2}{x}\sum_{k=0}^{\infty} b_{n,k}(x)\ln\left(1+\frac{\lambda}{n+k}\right) + \frac{\lambda^2}{2nx} - \lambda\frac{x+1}{x}. \end{aligned} \tag{13}$$

From the inequality $\ln(1+t) < t$, $t > 0$, and (7) it follows for every $n \geq 2$ that:

$$\begin{aligned} n\sum_{k=0}^{\infty} b_{n,k}(x)\ln\left(1+\frac{\lambda}{n+k}\right) &< \frac{\lambda}{1+x}\sum_{k=0}^{\infty}\frac{(n+k-2)!}{k!(n-2)!}\frac{x^k}{(1+x)^{n+k-1}}\frac{n+k-1}{n+k}\frac{n}{n-1} < \\ &< \frac{\lambda}{1+x}\mu_{n-1,0}(x)\frac{n}{n-1} = \frac{\lambda}{1+x}\frac{n}{n-1}. \end{aligned}$$

Returning to (13) we deduce:

$$B_n(x,\lambda) < \frac{\lambda^2}{2nx} + \lambda\left(1+\frac{1}{x}\right)\frac{1}{n-1} < \frac{\lambda(\lambda+8)}{2\sqrt{n}},$$

because $\frac{1}{x} < \sqrt{n}$. Hence (12) holds. □

Lemma 3. *If f and $\varphi^2 f''$ belong to $A.C._+^{loc}$ then, for any $x > \frac{1}{\sqrt{n}}$ and $n \geq 2$, the following inequality*

$$|(L_nf)(x) - f(x)| \leq \left(\frac{\lambda^3}{3}+\frac{\lambda^2}{2}+5\lambda\right)\frac{1}{\sqrt{n}}\|\psi\|_\infty M(\varphi^2 f'')(x)$$

holds.

Proof. We use the Taylor formula

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t (t-u)f''(u)du$$

and considering (1) we obtain:

$$(L_nf)(x) - f(x) = f'(x)L_n(\tau_x; x) + L_n(R_{2,x}f; x),$$

where $\tau_x(t) = t - x$ and

$$(R_{2,x}f)(t) = \int_x^t (t-u)f''(u)du.$$

The conditions (2) and (3) ensure $L_n(\tau_x; x) = 0$, consequently we get:

$$|(L_nf)(x) - f(x)| \leq \sum_{k=0}^{\infty} b_{n,k}(x) \int_0^{\lambda} \left| \int_x^{\frac{t+k}{n}} \left| \frac{t+k}{n} - u \right| |f''(u)| du \right| dt \|\psi\|_{\infty}. \tag{14}$$

We use the fact that:

$$\frac{|v-u|}{u(1+u)} \leq \frac{|x-v|}{x(1+u)} \leq \frac{|x-v|}{x} \left(\frac{1}{1+x} + \frac{1}{1+v} \right)$$

for every u between x and v ($x > 0,\ v > 0$), and choosing $v = \frac{t+k}{n}$ we can write:

$$\left| \frac{t+k}{n} - u \right| |f''(u)| \leq \frac{|x - \frac{t+k}{n}|}{x} \left(\frac{1}{1+x} + \frac{1}{1+\frac{k+t}{n}} \right) |\varphi^2(u)f''(u)|,$$

where φ is defined in (6).

We place this above inequality in (14) and taking into account both the definition of Hardy-Littlewood operator from (4) and the notations which were introduced in Lemma 2, we have:

$$|(L_nf)(x) - f(x)| \leq (A_n(x,\lambda) + B_n(x,\lambda))\|\psi\|_{\infty} M(\varphi^2 f'')(x).$$

Recalling now (11) and (12) the proof of Lemma 3 is complete. □

Combining the cases of Lemma 1 and Lemma 3 after a new increase, we have for $x \in [0,\infty)$ and $n \geq 2$:

$$|(L_nf)(x) - f(x)| \leq \left(\frac{\lambda^3}{3} + \lambda^2 + \frac{13}{2}\lambda \right) \frac{\|\psi\|_{\infty}}{\sqrt{n}} ((Mf')(x) + M(\varphi^2 f'')(x)).$$

This implies for $1 < p \leq \infty$ and $f', \varphi^2 f'' \in L_p[0,\infty)$:

$$\|L_nf - f\|_p \leq \frac{c_\lambda \|\psi\|_{\infty}}{\sqrt{n}} (\|M(f')\|_p + \|M(\varphi^2 f'')\|_p),$$

where c_λ is a constant.

If we use the relation (4) in this above inequality, we are able to state our main result.

Theorem. *Let $1 < p \leq \infty$ and L_n the operators defined in (1). If f, f' and $\varphi^2 f''$ belong to $A.C._{+}^{loc} \cap L_p[0, \infty)$, then we have*

$$\|L_n f - f\|_p \leq \frac{M}{\sqrt{n}}(\|f'\|_p + \|\varphi^2 f''\|_p)\|\psi\|_\infty,$$

where M is a constant which depends on λ and p.

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THE GENERALIZATION OF FIXED POINT THEOREMS IN ULTRAMETRIC SPACES

ANTAL BEGE

Dedicated to Professor Ioan A. Rus on his 60th anniversary

Abstract. In this paper we prove some fixed point theorems in partially ordered sets which generalize the results states in ultrametric spaces .

1. Introduction

Let $(P, \leq)$ be an ordered set with $0 \notin P$ and $0 < p$ for every $p \in P$, we do not assume that the order is total.

Let X be a nonempty set . A mapping $d : X \times X \longrightarrow P \cup \{0\}$ will be called an ultrametric distance (and (X, d, P) an ultrametric space) , if the following properties hold for all $x, y, z \in X, p \in P$:

$$1) d(x,y) = 0 \quad \text{if and only if} \quad x = y$$
$$2) d(x,y) = d(y,x)$$
$$3) \text{ If } \quad d(x,y) \leq p \ , \ d(y,z) \leq p \quad \text{then} \quad d(x,z) \leq p$$

Example 1

For each prime p,the p-adic valuation v_p on the field Q of rational number is defined as follows:

if $m = p^t \cdot l$ (where $t \geq 0$, $(p,l) = 1$) let $v_p(m) = t$

if $\frac{m}{n} \in Q$, $(m,n) = 1, v_p\left(\frac{m}{n}\right) = v_p(m) - v_p(n)$

and $v_p(0) = \infty$.

If we associate the p-adic distance $d_p : Q \times Q \longrightarrow R_{\geq 0}$,

$$d_p(x,y) = p^{-v_p(x-y)}$$

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then d_p is ultrametric.

Example 2.

Let V be a nonempty set.The $(\mathcal{P}(V), d, \mathcal{P}(V))$ is ultrametric space,where

$$d(U,W) = U\Delta W = (U - W) \cup (W - U)$$

(the symetric difference of U and $W \in \mathcal{P}(V)$).

Let X be an ultrametric space , let $\alpha \in P$ and $a \in X$. The set:

$$B_\alpha(a) = \{x \in X / \ d(x,a) \leq \alpha\}$$

will be called a ball with centre a and radius α.

The ultrametric spa^e X will be called spherically complete when every non-empty set $\mathcal{B}$ of balls of X , which is totally ordered by inclusion , has a non-empty intersection . If $(P, \leq)$ is totally ordered , the relation between the concepts of spherical completeness , of pseudo-convergence and completeness was examinate in [1].

Example 3.

The $(\mathcal{P}(V), d, \mathcal{P}(V))$ is spherically complete ultrametric spaces.(see example 2)

Example 4.

If $(P, \leq)$ is artinian and narow , then X is spherically complete.

A map $\varphi : X \longrightarrow X$ is said to be contracting,when $x, y \in X$, $x \neq y$ implies

$$d(\varphi(x), \varphi(y)) < d(x,y)$$

In [1] and [2] the authors states some theorems which same to the Banach Fixed point Theorem.The general form is the following:

Theorem 1.1.(S.Priess-Crampe,P.Ribenboim [2])

If X is spherically complete and $\varphi : X \longrightarrow X$ is contracting, then φ has exactly one fixed point.

In this paper we generalize Theorem 1.1. in ordered sets.

2. The fixed point theorem

Theorem 2.1.

Let $X \neq \emptyset$ be a set,$(P, \leq)$ be a poset.Let $\varphi : X \longrightarrow X$ and $A : X \longrightarrow P$ be a maps such that the cardinality of $\{x \ / \ A(x) = A(x_0)\}$ is finite and

$$A\varphi(x) \leq Ax \qquad \forall x \in X$$

Let $\mathcal{B} = \{B_x \ /x \in X\}$ and $B_x = \{y \ / \ Ay \leq A\varphi(x)\}$.If every $X_1 \subset X$,

$$\bigcap_{x \in X_1} B_x \neq \emptyset$$

then exist n natural number such that

$$\varphi^n(x) = x$$

for some $x \in X$.

Proof

We observe that $B_x \neq \emptyset \quad \forall x \in X$. $(\varphi(x) \in Bx)$

Let C be the maximal chain by inclusion in $\mathcal{B}$ .It follows that exists $z \in \bigcap_C B_x$.

We prove that

$$B_z = \{y \ / \ Ay \leq A\varphi(z)\} \subseteq B_x \quad \forall B_x \in C$$

If $z \in B_x$ implied $Az \leq A\varphi(x)$.For every $y \in B_z$

$$Ay \leq A\varphi(z) \leq Az \leq A\varphi(x)$$

which implied $y \in B_x$.

Since C is a maximal chain in $\mathcal{B}$ then B_z is the smallest element of C .But $y \in B\varphi(z)$ implies

$$Ay \leq A\varphi^2(z) \leq A\varphi(z) \ , \ y \in B_z$$

which implies $B\varphi(z) \subseteq Bz$ and by the minimality of B_z , $B\varphi(z) = Bz$.
If $A\varphi^2(z) < A\varphi(z)$ follows that $B\varphi(z) \subset B_z$ which contradiction.Then $A\varphi^2(z) = A\varphi(z)$
.
Similarly we can prove that $B\varphi^n(z) \subseteq B\varphi^{n-1}(z)$ every $n \in N^*$ which implies $B\varphi^n(z) = B\varphi^{n-1}(z)$ and $A\varphi^{n+1}(z) = A\varphi^n(z)$ .Thus:

$$A\varphi(z) = A\varphi^2(z) = \ldots = A\varphi^n(z) = \ldots$$

But cardinality of $\{A\varphi^n(z) \ / \ n \in N^*\}$ is finite ,follows that exists $n_1 > n_2$ natural numbers such that:

$$\varphi^{n_1}(z) = \varphi^{n_2}(z)$$

which implies

$$\varphi^{n_1-n_2}(\varphi^{n_2}(z)) = \varphi^{n_2}(z).$$

Corollary 2.2.
Let $X \neq \emptyset$ be a set, $(P, \leq)$ be a poset.Let $\varphi : X \longrightarrow X$ and $A : X \longrightarrow P$ be a maps such that A injective and

$$A\varphi(x) \leq Ax \quad \forall x \in X.$$

If every $X_1 \subset X$,

$$\bigcap_{x \in X_1} B_x \neq \emptyset$$

then there exist x fixed point of φ .

Proof
If A injective the cardinality of $\{x \ / \ Ax = Ax_0\}$ one , and if we apply the proof of Theorem 1.1. we find that

$$\varphi(z) = \varphi^2(z)$$

which implies that $\varphi(z)$ the fixed point of φ .

Theorem 2.3.

Let $X \neq \emptyset$ be a set, $(P, \leq)$ be a poset and $0 \notin P$ such that $0 < p$ for every $p \in P$. Let $\varphi : X \longrightarrow X$, $A : X \longrightarrow P \cup \{0\}$ such that

$$A\varphi(x) < Ax \quad \forall x \in X \text{ (for which } Ax \neq 0)$$

Let $\mathcal{B} = \{B_x \ / \ x \in X\}$, $B_x = \{y \ / \ Ay \leq A\varphi(x), y \in X\}$ such that

$$\forall X_1 \subset X \quad \bigcap_{x \in X_1} B_x \neq \emptyset .$$

Then exists $x_0 \in X$ such that $Ax_0 = 0$.

Proof

Assume that $Ax \neq 0 \quad \forall x \in X$ .The set $\mathcal{B} = \{B_x \ / \ x \in X\}$ is ordered by inclusion.Let C be a maximal chain in $\mathcal{B}$ .There exists an element

$$z \in \bigcap_{x \in C} B_x .$$

It follows that $B_z \subset B_x \quad \forall x \in X, (Ay \leq A\varphi(z) < Az \leq A\varphi(x))$ which implies that B_z an element in $\mathcal{B}$,contradicts the maximality of $\mathcal{B}$.

Then there exists x_0 such that $Ax_0 = 0$.

Corollary 2.3.

If X spherically complete and $\varphi : X \longrightarrow X$ is contracting,then φ has a fixed point.

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SEQUENCES OF OPERATORS AND FIXED POINTS IN QUASIMETRIC SPACES

VASILE BERINDE

Dedicated to Professor Ioan A. Rus on his 60^{th} *anniversary*

Abstract. For a sequence of certain selfoperators (f_n) of a quasimetric space X, uniformly convergent to a φ-contraction f, we establish a convergence theorem for the sequence of fixed points of $f_n, (x_n^*)$, to the unique fixed point of f.

This paper is in continuation with our investigations concerning the generalized contraction mapping principle in quasimetric spaces [2]. Its main goal is to extend a result of S.B.NADLER [3] for contractions in usual metric spaces to generalized contractions in quasimetric spaces. In order to state Theorem 3 (the main result of this paper) we need some definitions , examples and results from [1]-[4] which we summarize here.

A quasimetric space is a nonempty set X endowed with a **quasimetric**, that is a function $d : X \times X \to \mathbf{R}_+$,satisfying the following conditions:

d1) $d(x,y) = 0$ if and only if $x = y$,

d2) $d(x,y) = d(y,x), \forall x,y \in X$;

d3) $d(x,z) \le a[d(x,y) + d(y,z)], \forall x,y,z \in X$,

where $a \ge 1$ is a given real number, see [1].

Obviously, when $a = 1$ we obtain the usual notion of **metric (space)**.

Example 1. [1] The space $l_p (0 < p < 1)$,

$$l_p = \left\{ (x_n) \subset \mathbf{R} \;/\; \sum_{n=1}^{\infty} |x_n|^p < \infty \right\},$$

together with the function $d : l_p \times l_p \to \mathbf{R}$,

$$d(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p},$$

where $x = (x_n), y = (y_n) \in l_p$, is a quasimetric space. Indeed, by an elementary calculation we obtain

$$d(x,z) \le 2^{\frac{1}{p}} [d(x,y) + d(y,z)],$$

hence $a = 2^{\frac{1}{p}} > 1$ in this case.

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Example 2.[1]. The space $L_p(0 < p < 1)$ of all real functions $x(t)$, $t \in [0,1]$,such that

$$\int_0^1 |x(t)|^p \, dt < \infty,$$

becomes a quasimetric space if we take

$$d(x,y) = \left(\int_0^1 |x(t) - y(t)|^p \, dt\right)^{1/p}, \text{for each } x, y \in L_p.$$

The constant a is the same as in the previous example, $a = 2^{\frac{1}{p}}$.

DEFINITION 1. A mapping $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ is called **comparison function** if

(i) φ is monotone increasing;

(ii) $\varphi^n(t) \to 0$, as $n \to \infty$;for each $t \in \mathbf{R}_+$.

Example 3. The mapping $\varphi(t) = \alpha t, t \in \mathbf{R}_+$,where $0 \le \alpha < 1$, is a comparison function.

DEFINITION 2. Let (X,d) be a quasimetric space. A mapping $f : X \to X$ is called **φ-contraction** if there exists a comparison function φ such that

$$d(f(x), f(y)) \le \varphi(d(x,y)), \forall x, y \in X. \tag{1}$$

Remark. A $\varphi-$contraction with φ as in Example 3 is an usual contraction.

DEFINITION 3. A mapping $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ which satisfies:

(i) φ is monotone increasing (isotone);

(ii) There exist a convergent series of positive terms $\sum_{n=0}^{\infty} v_n$ and a real number $\alpha, 0 \le \alpha < 1$, such that

$$\varphi^{k+1}(t) \le \alpha\varphi^k(t) + v_k, \text{ for each t} \in \mathbf{R}_+ \text{ and n} \ge \text{N (fixed)} \tag{2}$$

is called **(c)-comparison function** (φ^k stands for the k the iterate of φ).

Remark.1) Using a generalization of the ratio test [5],[7], it results that if φ is a (c)-comparison function then the series

$$\sum_{k=0}^{\infty} \varphi^k(t) \tag{3}$$

converges for each $t \in \mathbf{R}_+$,therefore

$$\varphi^k(t) \to 0, \text{ as } k \to \infty,$$

that is any (c)-comparison function is a comparison function too.

2) If we denote by $s(t)$ the sum of the series (4) then s is monotone increasing and continuous at zero.

Example 4. The function given in example 3 is a (c)-comparison function but, generally, a comparison function is not a (c)-comparison function, see [2]-[3]..

THEOREM 1.([2]) *Let (X,d) be a complete quasimetric space and $f : X \to X$ a $\varphi-$contraction. Then f has a unique fixed point if and only if there exists $x_0 \in X$, such that the sequence $(x_n)_{n\in\mathbf{N}}$ of the succesive approximations,*

$$x_n = f(x_{n-1}), n \in \mathbf{N},$$

is bounded.

In order to establish a generalized fixed point principle which furnishes an approximation method to the fixed point,we have to consider a stronger concept than that of comparison function.

DEFINITION 4. Let $a \geq 1$ be a given real number. A mapping $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$is called **a-comparison function** if f satisfies (i) and
(iii) there are a convergent series of positive terms $\sum\limits_{n=0}^{\infty} v_n$ and a real number $\alpha, 0 \leq \alpha < 1$ such that

$$a^{k+1}\varphi^{k+1}(t) \leq \alpha \cdot a^k\varphi^k(t) + v_k, \text{for t} \in \text{R}_+ \text{ and for k} \geq \text{N (fixed)} \quad (4)$$

Remark. In view of the generalized ratio test [2] , the series

$$\sum_{k=0}^{\infty} a^k\varphi^k(t) \quad (5)$$

converges for each $t \in \mathbf{R}_+$ and its sum, denoted by $s_a(t)$, is monotone increasing and continuous at zero.

Obviously , for $a = 1$ (that is, d is a metric on X) such an a-comparison function is a actually (c)-comparison function.

Based on this concept, in [2] we established the following generalized contraction principle in quasimetric spaces

THEOREM 2. *Let (X,d) be a complete quasimetric space , $f : X \to X$ a φ-contraction with φ an a-comparison function. If $x_0 \in X$ is such that the sequence (x_n),*

$$x_n = f(x_{n-1}), n \in N^*,$$

is bounded and $F_f = \{x^*\}$, then we have

$$d(x_n, x^*) \le a \cdot s_a(d(x_n, x_{n+1})), n \ge 0, \tag{6}$$

where $s_a(t)$ is the sum of the series (5).

Remark. 1) If $a = 1$, Theorem 2 is just the generalized contraction principle, given in [2]-[4];

2) When $\varphi(t)$ is as in example 3, the condition (5) is satisfied if $\alpha \in [0;1[$ is such that

$$\alpha a < 1.$$

In concrete problems we need to compute the fixed point x^*. An usual way is to seek for a sequence of operators (f_n) which approximates (uniformly) the given operator f such that $F_{f_n} \neq \emptyset$, for each $n \in \mathbf{N}^*$ and the set F_{f_n} is easier to be determined.

The following question then arise: in what conditions over f and f_n , from $x_n^* \in F_{f_n}$, it results that

$$x_n^* \to x^*, \text{ as } n \to \infty \ ?$$

If the answer is in the affirmative, then x^* may be approximated by x_n^*, for n sufficiently large.

The main result of this paper is given by.

THEOREM 3. *Let (X,d) be a complete quasimetric space, f, $f_n : X \to X (n \in \mathbf{N}^*)$ such that*

1) f satisfies the assumptions of Theorem 2 and, in addition, φ is subadditive;

2) (f_n) converges uniformly to f on X;

3) $x_n^* \in F_{f_n}$, for each $n \in \mathbf{N}$.

Then (x_n^*) converges to x^*, the unique fixed point of f.

Proof. From (d_3) we have

$$d(x_n^*, x^*) = d(f_n(x_n^*), f(x^*)) \le a[d(f(x_n^*), f(x^*) + d(f_n(x_n^*), f(x_n^*))],$$

and, in view of the contraction condition, that is,

$$d(f(x_n^*), f(x^*) \le \varphi(d(x_n^*, x^*)),$$

we obtain

$$d(x_n^*, x^*) \leq a\varphi(d(x_n^*, x^*)) + ad(f_n(x_n^*), f(x_n^*)).$$

Based on the subadditivity of φ we then obtain by induction

$$d(x_n^*, x^*) \leq a^{k+1}\varphi^{k+1}(d(x_n^*, x^*)) + \sum_{i=0}^{k} a^i\varphi^i(d(f_n(x_n^*), f(x_n^*))), k \geq 0 \qquad (7)$$

But φ is a-comparison function, hence the series (3) is convergent. This implies

$$a^{k+1}\varphi^{k+1}(d(x_n^*, x^*)) \to 0, \text{ as } k \to \infty$$

and then from (7) we obtain

$$d(x_n^*, x^*) \leq s_a(d(f_n(x_n^*), f(x_n^*))).$$

But s_a is continuous at zero and then,from 2) we deduce that

$$d(f_n(x_n^*), f(x_n^*)) \to 0, \text{ as } n \to \infty,$$

therefore

$$d(x_n^*, x^*) \to 0, \text{ as } n \to \infty$$

that is $x_n^* \to x^*, n \to \infty$, as required.

Remark. 1) For φ as in Example 3, from Theorem 3 we obtain a theorem of Nadler type in quasimetric spaces;

2) For $a = 1$ and φ as in Example 3, from Theorem 3 we obtain a result of Nadler [3].

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ON THE SEPARATION OF VARIABLES IN THE GEODESIC HAMILTON-JACOBI EQUATION FOR A SPHERICALLY-SYMMETRIC DILATON BLACK HOLE

CRISTINA BLAGA AND PAUL A. BLAGA

Dedicated to Professor Ioan A. Rus, on his 60^{th} Anniversary

Abstract. The aim of this letter is to show the geodesic Hamilton–Jacobi equation for a new spherical symmetric exact solution of the Einstein–dilaton equation is separable and to construct its complete integral.

It is known ([1]) that the metric of a spherically-symmetric, massless dilaton black hole in four dimensions can be written in the following form, using the curvature coordinates:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r\left(r - \frac{Q^2}{M}\right)\left[d\theta^2 + \sin^2\theta d\varphi^2\right] \tag{1}$$

The covariant components of the metric tensor can be read off from the equation (1). They are

$$(g_{ij}) = \begin{bmatrix} -\left(1-\frac{2M}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1-\frac{2M}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r\left(r-\frac{Q^2}{M}\right) & 0 \\ 0 & 0 & 0 & r\left(r-\frac{Q^2}{M}\right)\sin^2\theta \end{bmatrix} \tag{2}$$

while the contravariant ones are

$$(g^{ij}) = \begin{bmatrix} -\frac{1}{\left(1-\frac{2M}{r}\right)} & 0 & 0 & 0 \\ 0 & 1-\frac{2M}{r} & 0 & 0 \\ 0 & 0 & \frac{1}{r\left(r-\frac{Q^2}{M}\right)} & 0 \\ 0 & 0 & 0 & \frac{1}{r\left(r-\frac{Q^2}{M}\right)\sin^2\theta} \end{bmatrix} \tag{3}$$

The geodesic Hamilton–Jacobi equation for a pseudo–Riemannian space is written on the form

$$-\frac{\partial S}{\partial \lambda} = \frac{1}{2} g^{ij} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^j} \tag{4}$$

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In our case, the Hamilton–Jacobi equation reads (see [2]):

$$
\begin{aligned}
2\frac{\partial S}{\partial \lambda} &= -\frac{1}{\left(1-\frac{2M}{r}\right)}\left(\frac{\partial S}{\partial t}\right)^2+\left(1-\frac{2M}{r}\right)\left(\frac{\partial S}{\partial r}\right)^2+\frac{1}{r\left(r-\frac{Q^2}{M}\right)}\left(\frac{\partial S}{\partial \theta}\right)^2\\
&+\frac{1}{r\left(r-\frac{Q^2}{M}\right)\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2
\end{aligned}
\tag{5}
$$

Now, to show that the equation (5) is separable, we could show that the following necessary and sufficient condition of Levi-Cività is fulfilled ([2]):

$$
\partial^i H\partial^j H\partial_{ij}H+\partial_i H\partial_j H\partial^{ij}H-\partial^i H\partial_j H\partial_i^j H-\partial^j H\partial_i H\partial_j^i H=0, \tag{6}
$$

where we have used the notations:

$$
\partial_i H=\frac{\partial H}{\partial q_i},\quad \partial^i H=\frac{\partial H}{\partial p_i}\quad \text{and so on.}
$$

We mention that the canonical coordinates are numbered as follows: $q^1 = r$, $q^2 = \theta$, $q^3 = \phi$, $q^4 = t$, and p_i are their canonical conjugate momenta.

Nevertheless, in this particular case is simpler to look directly to the Hamilton-Jacobi equation. Thus, since, clearly, t and ϕ are ignorable coordinates, we shall seek a solution of the form

$$
S=\frac{1}{2}\delta_1\lambda-Et+L_z\varphi+S_r(r)+S_\theta(\theta), \tag{7}
$$

where $S_r(r)$ and $S_\theta(\theta)$ are functions only of the indicated variables (see the book of Chandrasekhar [3] for a justification of the notations E and L_z; actually, E is to be identified with the energy of a test particle moving freely along the geodesics, while L_z is its angular momentum).

Now, if S is of this form the the equation (5) becomes:

$$
\begin{aligned}
\delta_1 r\left(r-\frac{Q^2}{M}\right) &= -\frac{1}{\left(1-\frac{2M}{r}\right)}E^2+\frac{1}{\sin^2\theta}L_z^2+\\
&+\left(1-\frac{2M}{r}\right)r\left(r-\frac{Q^2}{M}\right)\left(\frac{dS_r}{dr}\right)^2+\left(\frac{dS_\theta}{d\theta}\right)^2,
\end{aligned}
\tag{8}
$$

which is manifestly separable and gives

$$
\begin{aligned}
\left(1-\frac{2M}{r}\right)r\left(r-\frac{Q^2}{M}\right)\left(\frac{dS_r}{dr}\right) &= \delta_1 r\left(r-\frac{Q^2}{M}\right)+\frac{1}{\left(1-\frac{2M}{r}\right)}E^2-Q\\
\left(\frac{dS_\theta}{d\theta}\right)^2 &= Q-\frac{1}{\sin^2\theta}L_z^2,
\end{aligned}
\tag{9}
$$

where Q is a separation constant.

Let us make the following notations:

$$\Delta = \left(1 - \frac{2M}{r}\right) r \left(r - \frac{Q^2}{M}\right),$$

$$R = \frac{1}{\Delta}\left(\delta_1 r \left(r - \frac{Q^2}{M}\right) + \frac{1}{\left(1 - \frac{2M}{r}\right)} E^2 - Q\right)$$

and

$$\Theta = Q - \frac{1}{\sin^2\theta} L_z^2.$$

With this notations, using the equations (9), the complete integral of the Hamilton–Jacobi equation (5) reads:

$$S = \frac{1}{2}\delta_1\lambda - Et + L_z\phi + \int^r \frac{\sqrt{R}}{\Delta} dr + \int^\theta \sqrt{\Theta} d\theta. \tag{10}$$

The integral (10) will allow us to perform a complete investigation of the geodesics of this spacetime. This investigation is left for a forthcoming paper.

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DATA DEPENDENCE THEOREMS ON COINCIDENCE PROBLEMS

ADRIANA BUICA

Dedicated to Professor Ioan A. Rus on his 60^{th} anniversary

Abstract. Data dependence results for coincidence problems are established using the contraction fixed point technique. Applications to differential equations and complementary problems are given.

1. Introduction.

There are various techniques in the coincidence theory (for more details see [6] and the references cited therein). One of them concern the equivalence, under additional assumptions, with a fixed point problem. This paper gives some data dependence results on coincidence problems using the above remark.

Several authors dealt with this subject. Among them we remind the results of Goebel [5] and Rus [6] which are needed further on.

Let X and Y be two nonempty sets and $F, G : X \to Y$, $H : X \to X$ be some mappings. We denote by :

$$F_H = \{x \in X : H(x) = x\}$$

the fixed points set of the mapping H and by :

$$C(F,G) = \{x \in X : F(x) = G(x)\}$$

the coincidence points set of the pair of the mappings F and G.

Theorem 1. (Goebel) *Let X be a nonempty set and (Y, ρ) be a complete metric space. Let $F, G : X \to Y$ be two mappings such that following conditions are fullfiled:*

(i) G is surjective;

(ii) there exists $a \in (0,1)$ such that $\rho(F(x), F(y)) \leq a\rho(G(x), G(y))$ for all $x, y \in X$;

Then the pair of mappings F and G has a coincidence point.

Moreover, if G is bijective then the coincidence point is unique.

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A map F which satisfies the condition (ii) is called, by some authors, G-contraction mapping. (see [2], [3])

Theorem 2. (Rus) *Let (X,d) be a complete metric space, (Y,ρ) be a metric space and $F,G: X \to Y$ be two mappings. We suppose that:*

(i) there exist $a_1 > 0$, $a_2 > 0$, $a_2 a_1^{-1} < 1$, such that:

$$\rho(G(x),G(y)) \geq a_1 d(x,y) \text{ for all } x,y \in X,$$
$$\rho(F(x),F(y)) \leq a_2 d(x,y) \text{ for all } x,y \in X;$$

(ii) $F(X) \subset G(X)$.

Then the pair of mappings F and G has an unique coincidence point.

Another result we use is the classical data dependence theorem in metric spaces.

Theorem 3. *Let (X,d) be a complete metric space and $f,h: X \to X$ be two mappings such that:*

(i) there exists $a \in (0,1)$ such that $d(f(x),f(y)) \leq ad(x,y)$ for all $x,y \in X$;

(ii) there exists $\eta > 0$ such that $d(f(x),h(x)) \leq \eta$ for all $x \in X$;

Then we have the following estimation:

$$d(x^*,z^*) \leq \frac{\eta}{1-a}.$$

where x^* is the fixed point of f and z^* is a fixed point of h.

In the end our results are applied for some initial value problems and complementary problems.

2. Main results.

Theorem 4. *Let (X,d) be a metric space, (Y,ρ) be a complete metric space, $F,G,T,H: X \to Y$ be some mappings such that:*

(i) G and H are surjectives;

(ii) there exists $a \in (0,1)$ such that $\rho(F(x),F(y)) \leq a\rho(G(x),G(y))$ for all $x,y \in X$;

(iii) for each $x,y \in X$ such that $H(x) = H(y)$ we have $T(x) = T(y)$;

(iv) there exist $\eta_1,\eta_2 > 0$ such that $\rho(F(x),T(x)) \leq \eta_1$, $\rho(G(x),H(x)) \leq \eta_2$ for all $x \in X$;

Then we have the following estimation:

$$\rho(G(x^*), H(z^*)) \leq \frac{\eta_1 + a\eta_2}{1-a}$$

where x^* is a coincidence point for F and G and z^* is for T and H.

If, in addition, we have:

(v) there exists $L > 0$ such that $\rho(G(x), G(y)) \geq Ld(x,y)$ for all $x, y \in X$ then:

$$d(x^*, z^*) \leq \frac{\eta_1 + \eta_2}{L(1-a)}$$

Proof. G is surjective, thus there exists G_r^{-1}, a right inverse of G.

From **theorem 1** it follows that $F \circ G_r^{-1} : Y \to Y$ is a contraction mapping with constant a and $F_{F \circ G_r^{-1}} = \{G(x^*)\}$.

Since H is surjective, there exists $H_r^{-1} : Y \to X$ a right inverse of H.

Let us denote by $H(z^*)$ a fixed point for $T \circ H_r^{-1} : Y \to Y$.

For each $y \in Y$ we have:

$$\begin{aligned} \rho\left(F \circ G_r^{-1}(y), T \circ H_r^{-1}(y)\right) &\leq \rho\left(F(G_r^{-1}(y)), F(H_r^{-1}(y))\right) + \rho\left(F(H_r^{-1}(y)), T(H_r^{-1}(y))\right) \leq \\ &\leq a\rho\left(G(G_r^{-1}(y)), G(H_r^{-1}(y))\right) + \eta_1 \leq a\rho\left(H(z), G(z)\right) + \eta_1 \leq \\ &\leq \eta_1 + a\eta_2 \end{aligned}$$

where we denoted $z = H_r^{-1}(y)$.

Conditions of **theorem 3** are fullfiled, so we have the estimation:

$$\rho(G(x^*), H(z^*)) \leq \frac{\eta_1 + a\eta_2}{1-a}.$$

Using this relation and conditions (iv) and (v) we obtain the estimation for the distance between solutions of two coincidence problems:

$$d(x^*, y^*) \leq \frac{\eta_1 + \eta_2}{L(1-a)}$$

□

Theorem 5. *Let (X, d) be a complete metric space, (Y, ρ) be a metric space and $F, G, T, H : X \to Y$ be some mappings such that:*

(i) there exist $a_1 > 0$, $a_2 > 0$, $a_2 a_1^{-1} < 1$, such that

$$\begin{aligned} \rho(G(x), G(y)) &\geq a_1 d(x,y) \text{ for all } x, y \in X, \\ \rho(F(x), F(y)) &\leq a_2 d(x,y) \text{ for all } x, y \in X; \end{aligned}$$

(ii) $F(X) \subset G(X)$;

(iii) H *is injective and* $T(X) \subset H(X)$;

(iv) there exist $\eta_1, \eta_2 > 0$ *such that* $\rho(F(x), T(x)) \leq \eta_1$, $\rho(G(x), H(x)) \leq \eta_2$ *for all* $x \in X$.

Then we have the following estimation:

$$d(x^*, z^*) \leq \frac{\eta_1 + \eta_2}{a_1 - a_2}.$$

where x^* *is the coincidence point for* F *and* G *and* z^* *is a coincidence point for* T *and* H.

Proof. The condition (i) assure that G is injective. Thus there exists G_l^{-1}, a left inverse of G.

From theorem 2 it follows that $G_l^{-1} \circ F : X \to X$ is a contraction mapping with constant $a_2 a_1^{-1}$ and $F_{G_l^{-1} \circ F} = \{x^*\}$.

Since H is injective, there exists $H_l^{-1} : Y \to X$ a left inverse of H.

Let us denote by z^* a fixed point for $H_l^{-1} \circ T : X \to X$.

For each $x \in X$ we have:

$$\begin{aligned} d(G_l^{-1} \circ F(x), H_l^{-1} \circ T(x)) &\leq \frac{1}{a_1} \rho(G(G_l^{-1}(F(x))), G(H_l^{-1}(T(x)))) = \\ &= \frac{1}{a_1} \rho(F(x), G(H_l^{-1}(T(x)))) \leq \\ &\leq \frac{1}{a_1} \left[\rho(F(x), T(x)) + \rho(T(x), G(H_l^{-1}(T(x))))\right] = \\ &= \frac{1}{a_1} \left[\rho(F(x), T(x)) + \rho(H(z), G(z))\right] \leq \frac{\eta_1 + \eta_2}{a_1} \end{aligned}$$

where we denoted $z = H_l^{-1}(T(x))$.

Conditions of **theorem 3** are fullfiled, so we have the estimation:

$$d(x^*, z^*) \leq \frac{\eta_1 + \eta_2}{a_1 - a_2}$$

3. An application to differential equations.

Let us consider the Cauchy problem:

$$(1) \begin{cases} x'(t) = f(t, x(t)), \ t \in [a, b) \\ x(a) = x_0 \end{cases}$$

We intend to find sufficient conditions for existence and uniqueness of classical solution on the entire interval [a,b). The idea is to use a "smoothness" function. So, let $p \in C^1[a,b)$ be a mapping such that $p' > 0$, $p(a) = 1$ and $\lim_{t \nearrow b} p(t) = +\infty$.

We are looking for solution in the following set:

$$M = \{x \in C[a,b) : \lim_{t \nearrow b} \frac{x(t)}{p(t)} \text{ exists and it is a real number}\}$$

We suppose that $f \in C[a,b) \times R$ and for each $x \in M$ there exists $\lim_{t \nearrow b} \frac{f(t,x(t))}{p'(t)}$, it is a real number, and, also, $\int_a^b f(t,x(t))dt$ exists. All these conditions are basic for our work.

An existence theorem and a data dependence theorem we shall establish.

Theorem 6. (Existence) *Assume that:*

$$|f(t,u) - f(t,\overline{u})| \leq L(t)\,|u - \overline{u}| \ \textit{for all } u, \overline{u} \in R,\ t \in [a,b);$$

with $L \in C[a,b)$ such that $\int_a^b L(t)p(t)dt$ exists. Then there exists an unique $x \in M$ solution of the problem (1).

Proof. We shall define two mappings such that any solution of the problem (1) is a coincidence point for them and conversely.

For each $x \in M$, $t \in [a,b)$ we denote:

$$\begin{aligned} Fx(t) &= \frac{x_0 + \int_a^t f(s,x(s))ds}{p(t)} e^{-2\int_a^t L(s)p(s)ds} \\ Gx(t) &= \frac{x(t)}{p(t)} e^{-2\int_a^t L(s)p(s)ds}. \end{aligned}$$

We get, according to our hypothesis, two mappings $F, G : M \to C[a,b]$ where $Fx(b) = \lim_{t \nearrow b} Fx(t)$ and $Gx(b) = \lim_{t \nearrow b} Gx(t)$. We intend to use **theorem 1**.

We have:

$$\|Fx - F\overline{x}\| \leq \frac{1}{2}\|Gx - G\overline{x}\| \ \textit{for all } x, \overline{x} \in M.$$

We also have that G is bijective and $(C[a,b], d_C)$ is a complete metric space, where d_C is the metric which correspond to the ussualy max-norm. **Theorem 1** assures that the pair of mappings F and G has an unique coincidence point, which is the unique $x \in M$ solution of the Cauchy problem (1). □

Let us consider two Cauchy problems:

$$(1)\begin{cases} x' = f(t,x) \\ x(a) = x_0 \end{cases} \text{ and } (2)\begin{cases} z' = g(t,z) \\ z(a) = z_0 \end{cases}$$

such that basic conditions are fullfiled for both of our problems.

Theorem 7. (Data dependence theorem) *Assume that:*

(i) $|f(t,u) - f(t,\overline{u})| \leq L(t)\,|u - \overline{u}|$ *for all* $t \in [a,b)$, $u, \overline{u} \in R$ *with* $L \in C[a,b)$ *such that* $\int_a^b L(t)p(t)dt$ *exists;*

(ii) there exists $\eta > 0$ *such that*

$$|f(t,u) - g(t,u)| \leq \eta \text{ for all } u \in R, t \in [a,b)$$

Then we have the following estimation:

$$|x^*(t) - z^*(t)| \leq 2[|x_0 - z_0| + \eta(b-a)p(t)e^{2\int_a^b L(s)p(s)ds}, \text{ for every } t \in [a,b).$$

where x^* *is the solution of problem (1) and* z^* *is a solution of problem (2).*

Proof. We shall use **theorem 4**. We have $F, G, T : M \to C[a,b]$, where

$$\begin{aligned} Fx(t) &= \frac{x_0 + \int_a^t f(s,x(s))ds}{p(t)} e^{-2\int_a^t L(s)p(s)ds}, \text{ for all } t \in [a,b); \\ Tx(t) &= \frac{z_o + \int_a^t g(s,z(s))ds}{p(t)} e^{-2\int_a^t L(s)p(s)ds}, \text{ for all } t \in [a,b); \\ Gx(t) &= \frac{x(t)}{p(t)} e^{-2\int_a^t L(s)p(s)ds}, \text{ for all } t \in [a,b). \end{aligned}$$

G is bijective.

x^* is the unique coincidence point of the pair of mappings F and G.

z^* is a coincidence point of the pair of mappings T and G.

It is easy now to obtain the estimation. □

We shall give a simple example in order to ilustrate the above theorems.

Let us consider the Cauchy problems:

$$(1)\begin{cases} x' = \frac{x}{t-1}, t \in [0,1) \\ x(0) = -1 \end{cases} \quad (2)\begin{cases} z' = \frac{z}{t-1} + \eta, t \in [0,1) \\ z(0) = z_0 \end{cases}$$

We consider:

$$\begin{aligned} p : [0,1) \to R \;\; p(t) &= \frac{1}{1-t} \\ M &= \{x \in C[0,1) : \lim_{t \nearrow 1}(1-t)x(t) \in R\} \end{aligned}$$

The basic conditions are fullfiled and hyphothesis of theorem 6 are satisfie

$$L : [0,1) \to (0,+\infty); L(t) = \frac{1}{1-t}$$

The unique solution of problem (1) is $x(t) = t - 1$.

We apply **theorem 7** and we get:

If there exists a solution $z^* \in M$ of the problem (2) then we have the fo estimation:

$$|z^*(t) - t + 1| \leq 2(|z_0 + 1| + \eta)\frac{1}{1-t}e^{\frac{2t}{1-t}},\ t \in [0,1)$$

4. An application to complementary problems

We consider in this section the complementary problem and its relation coincidence theory. The complementary problem is one of the interesting an problems defined after 1964 and much studied in the last fifteen years. It variety of practical problems arising in: mathematical programming, optimiza economic equilibrium theory, structural mechanics and elasticity theory (see [ ] refrences cited therein).

In [4] G. Isac proved that if $(H, <\cdot,\cdot>)$ is a Hilbert space ordered convex cone $K \subset H$ and $f, g : H \to H$ are two mappings, then a solution coincidence problem

$$g(x) = P_K\left(g(x) - \tau f(x)\right)$$

is a solution of the complementary problem

(1) find $x_* \in H$ such that $g(x_*) \in K$ $f(x_*) \in K^*$ and $< g(x_*), f(x_*) >$

Here P_K denotes the projection onto K, and K^* is the dual cone of K.

Theorem 8. (Existence) *Let H be a Hilbert space and let $K \subset H$ be a closed co cone. Given the mappings $f, g : H \to H$ we assume that:*

(i) there exist $\tau > 0$, $a \in (0,1)$ such that

$$\|g(x) - g(y) - \tau[f(x) - f(y)]\| \leq a\,\|g(x) - g(y)\| \text{ for all } x, y \in H$$

(ii) g is surjective.

Then the complementary problem has at least a solution.

Proof. We apply **theorem 1** for $F, G : H \to H$, $F = P_K \circ (g - \tau f)$, $G = g$. □

We consider another complementary problem:

(2) find $z_* \in H$ such that $h(x_*) \in K$, $t(x_*) \in K^*$ and $< h(x_*), t(x_*) >= 0$.

Theorem 9. (Data dependence theorem) *We assume that:*

(i) there exist $\tau > 0$, $a \in (0,1)$ such that

$$\|g(x) - g(y) - \tau[f(x) - f(y)]\| \leq a \|g(x) - g(y)\| \text{ for all } x, y \in H;$$

(ii) g and h are surjectives;

(iii) there exists $L > 0$ such that $\|g(x) - g(y)\| \geq L \|x - y\|$;

(iv) there exist $\eta_1, \eta_2 > 0$ such that $\|f(x) - t(x)\| \leq \eta_1$, $\|g(x) - h(x)\| \leq \eta_2$ for all $x \in H$.

Then we have the estimation:

$$\|x_* - z_*\| \leq \frac{\tau\eta_1 + 2\eta_2}{L(1-a)}$$

where x^* is the solution of problem (1) and z^* is a solution of problem (2).

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EQUIVALENCE BETWEEN IMPLICIT FUNCTION THEOREMS

DOMOKOS ANDRÁS

Dedicated to Professor Ioan A. Rus on his 60th anniversary

Abstract. We prove an equivalence between implicit function theorems. The importance of this fact is that the classical implicit function theorem, its nonsmooth generalizations and results on stability of the solutions of parametric variational inequalities from [4], [5], [6], in Hilbert spaces are consequences of an implicit function theorem for monotone mappings.

In this paper we will show equivalence between two kind of implicit function theorems. One of these is based on generalizations of the strong regularity and strong approximation concepts. These concepts appeared in the papers of S.M. Robinson [5],[6], A.L.Dontchev-W.W.Hager [4], A.Domokos [3]. The other kind of theorem appeared in [1],[2] and is based on monotonicity conditions.

Theorem 1. *Let X be a Hilbert space, let Y be a normed space (Y could be only a topological space),let $T : X \times Y \to X$ be a map, let $(x_0, y_0) \in X \times Y$, let X_0 be a neighborhood of x_0 and let Y_0 be a neighborhood of y_0. Let us assume that:*
i) $0 = T(x_0, y_0)$.
ii) T is continuous at (x_0, y_0) and the mappings $T(\cdot, y)$ are continuous on X_0, for each $y_0 \in Y_0$.
iii) There exists an increasing function $\varphi : I\!R_+ \to I\!R_+$, with $\varphi(r) > 0$ for $r > 0$, such that

$$\langle T(x_1,y) - T(x_2,y), x_1 - x_2 \rangle \geq \varphi(\|x_1 - x_2\|)\|x_1 - x_2\|,$$

$\forall y \in Y_0, \forall x_1, x_2 \in X_0$.
Then, there exists a constant $r > 0$ and a unique mapping $x : B(y_0, r) \to X_0$, continuous at y_0 such that $x(y_0) = x_0$, $T(x(y), y)) = 0$, $\forall y \in B(y_0, r)$.

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Using Theorem 1 we will prove the following theorem.We mention it that this theorem could be proven independently from Theorem 1 using the methods from the papers [3],[4].

Theorem 2. *Let X be a Hilbert space, let Y be a normed space, let $f : X \times Y \to X$ be a map, let $F : X \times Y \rightsquigarrow X$ be a set-valued map, let $(x_0, y_0) \in X \times Y$ such that $0 \in f(x_0, y_0) + F(x_0, y_0)$ and let ε_1, ε_2, > 0. We assume that:*

i) $f(x_0, \cdot)$ is continuous at y_0.

ii) There exist functions $\alpha, \gamma : I\!R_+ \to I\!R_+$, $\beta : B(y_0, \varepsilon_1) \to I\!R_+$ with γ monotone increasing, $Id - \gamma \circ \alpha : I\!R_+ \to I\!R_+$ increasing with $r - \gamma \circ \alpha(r) > 0$ for $r > 0$ and

$$\lim_{r \to 0} \alpha(r) = 0 \, , \; \lim_{y \to y_0} \beta(y) = 0 \, , \; \lim_{r \to 0} \gamma(r) = 0 \, .$$

iii) There exists a mapping $g : B(x_0, \varepsilon_1) \to X$ such that

$$\| f(x_1, y) - g(x_1) - f(x_2, y) + g(x_2) \| \leq \alpha(\|x_1 - x_2\|) \, ,$$

$\forall\, x_1, x_2 \in B(x_0, \varepsilon_1)$, $\forall\, y \in B(y_0, \varepsilon_1)$.

iv) For $z_0 = g(x_0) - f(x_0, y_0)$ the mappings

$$\Psi(\cdot, y) = [g + F(\, , y)]^{-1}(\cdot) : B(z_0, \varepsilon_2) \to X$$

are single-valued for all $y \in B(y_0, \varepsilon_2)$ and

$$\| \Psi(z_1, y) - \Psi(z_2, y) \| \leq \gamma(\|z_1 - z_2\|) \, ,$$

$$\| \Psi(z_0, y_0) - \Psi(z, y) \| \leq \gamma(\|z - z_0\|) + \beta(y) \, ,$$

$\forall\ z, z_1, z_2 \in B(z_0, \varepsilon_2)$, $\forall\ y \in B(y_0, \varepsilon_2)$. *Then there exists a constant $r > 0$ and a unique mapping $x : B(y_0, r) \to X_0$, continuous at y_0, such that $x(y_0) = x_0$ and $0 \in f(x(y), y) + F(x(y), y)$, $\forall\, y \in B(y_0, r)$.*

Proof of Theorem 2. We can choose $\varepsilon > 0$ such that $g(x) - f(x, y) \in B(z_0, \varepsilon_2)$, when $x \in B(x_0, \varepsilon)$, $y \in B(y_0, \varepsilon)$ and $\varepsilon \leq \min\{\varepsilon_1, \varepsilon_2\}$. We define the mapping $T : B(x_0, \varepsilon) \times B(y_0, \varepsilon) \to X$ by

$$T(x, y) = x - \Psi(g(x) - f(x, y) \, , y) \, .$$

Then, $0 \in T(x, y)$ if and only if $0 \in f(x, y) + F(x, y)$.

We have $0 \in T(x_0, y_0)$

Let $x_1, x_2 \in B(x_0, \varepsilon)$, $y \in B(y_0, \varepsilon)$. Then

$$\| T(x_1, y) - T(x_2, y) \| =$$

$$= \| x_1 - \Psi(g(x_1) - f(x_1, y), y) - x_2 + \Psi(g(x_2) - f(x_2, y), y) \| \leq$$

$$\leq \|x_1 - x_2\| + \| \Psi(g(x_1) - f(x_1, y), y) - \Psi(g(x_2) - f(x_2, y), y) \| \leq$$

$$\leq \|x_1 - x_2\| + \gamma(\| f(x_1, y) - g(x_1) - f(x_2, y) + g(x_2) \|) \leq$$

$$\leq \|x_1 - x_2\| + \gamma \circ \alpha(\|x_1 - x_2\|) .$$

Using the continuity of γ and α at 0 we obtain the continuity of $T(\cdot, y)$.
Let $x \in B(x_0, \varepsilon)$ and $y \in B(y_0, \varepsilon)$. Then

$$\| T(x, y) - T(x_0, y_0) \| \leq$$

$$\leq \|x_1 - x_2\| + \gamma(\|g(x) - f(x, y) - g(x_0) + f(x_0, y_0)\|) + \beta(y) \leq$$

$$\leq \|x_1 - x_2\| + \gamma(\|x - x_0\| + \|f(x_0, y_0) - f(x_0, y)\|) + \beta(y) .$$

Using the continuity of α and γ at 0 and the continuity of β at y_0 we obtain the continuity of T at (x_0, y_0).
Let $x_1, x_2 \in B(x_0, \varepsilon)$, $y \in B(y_0, \varepsilon)$. Then

$$\langle T(x_1, y) - T(x_2, y) , x_1 - x_2 \rangle =$$

$$= \|x_1 - x_2\|^2 - \langle \Psi(g(x_1) - f(x_1, y), y) - \Psi(g(x_2) - f(x_2, y), y) , x_1 - x_2 \rangle \geq$$

$$\geq \|x_1 - x_2\|^2 - \|\Psi(g(x_1) - f(x_1, y), y) - \Psi(g(x_2) - f(x_2, y), y)\| \cdot \|x_1 - x_2\| \geq$$

$$\geq (\|x_1 - x_2\| - \gamma \circ \alpha(\|x_1 - x_2\|))\|x_1 - x_2\| .$$

Using the fact that $Id - \gamma \circ \alpha$ is an increasing function, with $r - \gamma \circ \alpha(r) > 0$ for $r > 0$, we can see that the assumption *iii)* of Theorem 1 is satisfied and the proof is complete.

Remark 1. If we change the assumption *iii)* with:
iii)' The mappings $T(\cdot, y)$ are strongly-monotone for all $y \in Y_0$ i.e.

$$\langle T(x_1, y) - T(x_2, y) , x_1 - x_2 \rangle \geq c \|x_1 - x_2\|^2 , \forall x_1, x_2 \in X_0 ,$$

then we can prove this theorem using Theorem 2 in a same way as is proved Theorem 4.1 from [3].

Indeed, we can take an $\varepsilon > 0$ sufficiently small, such that $B(x_0,\varepsilon) \subset X_0$, $B(y_0,\varepsilon) \subset Y_0$ and we can define

$$f(x,y) = 0,\ F(x,y) = T(x,y) + N_{B(x_0,\varepsilon)},\ g(x) = \varepsilon x,$$

$$\alpha(r) = \varepsilon r,\ \gamma(r) = \frac{1}{c+\varepsilon} r,\ \beta(y) = \|T(x_0,y_0) - T(x_0,y)\| .$$

We can use Theorem 2 to find a constant $r > 0$ and a mapping $x : B(y_0,r) \to B(x_0,r)$ continuous at y_0 such that

$$x(y_0) = x_0 \text{ and } 0 \in T(x(y),y) + N_{B(x_0,\varepsilon)}(x(y)) .$$

But, if $\|y - y_0\|$ is sufficiently small,then

$$\|x(y) - x_0\| < \varepsilon,\ N_{B(x_0,\varepsilon)}(x(y)) = \{0\} \text{ and } T(x(y),y) = 0 .$$

We can now prove Theorem 1 using Theorem 2:

Proof of Theorem 1. Let $\varepsilon > 0$ sufficiently small to $B(x_0,\varepsilon) \subset X_0$, $B(y_0,\varepsilon) \subset Y_0$. Let

$$F(x,y) = T(x,y) + N_{B(x_0,\varepsilon)},\ f(x,y) = 0,\ g(x) = \varepsilon x,\ \alpha(r) = \varepsilon r .$$

In this case F is maximal-monotone, $g + F(\ ,y)$ is surjective, strongly- monotone and $\Psi(\cdot,y) = (g + F(\ ,y))^{-1}(\cdot)$ is single-valued for all $y \in B(y_0,\varepsilon)$. Let $x = \Psi(z,y)$. Then

$$0 \in \varepsilon x + T(x,y) - z + N_{B(x_0,\varepsilon)}(x) \ = \ T_1(x,y,z) + N_{B(x_0,\varepsilon)}(x) .$$

$T_1(\cdot,y,z)$ is strongly-monotone, $0 \in T_1(x_0,y_0,z_0) + N_{B(x_0,\varepsilon)}$, so we can use Remark 1 to find an $\varepsilon_2 > 0$ such that if $y \in B(y_0,\varepsilon_2)$ and $z \in B(z_0,\varepsilon_2)$ then $x \in B(x_0,\varepsilon)$.
Let $z_1, z_2 \in B(z_0,\varepsilon_2)$, $y \in B(y_0,\varepsilon_2)$, $x_1 = \Psi(z_1,y)$, $x_2 = \Psi(z_2,y)$. Then $z_1 \in \varepsilon x_1 + F(x_1,y)$, $z_2 \in \varepsilon x_2 + F(x_2,y)$ and

$$\varphi(\|x_1 - x_2\|)\|x_1 - x_2\| \ \le \langle z_1 - \varepsilon x_1 - z_2 + \varepsilon x_2\, ,\, x_1 - x_2\rangle \ =$$

$$= \ \langle z_1 - z_2\, ,\, x_1 - x_2\rangle \ - \ \varepsilon\|x_1 - x_2\|^2 \ \le$$

$$\le \ \|z_1 - z_2\|\,\|x_1 - x_2\| \ - \ \varepsilon\|x_1 - x_2\|^2 .$$

Then $\varphi(\|x_1 - x_2\|) + \varepsilon\|x_1 - x_2\| \ \le \ \|z_1 - z_2\|$.
The multivalued function $\varphi_1 : I\!R_+ \rightsquigarrow I\!R_+$, $\varphi_1(r) = [\varphi(r-0)+\varepsilon r, \varphi(r+0)+\varepsilon r]$ is maximal-monotone, surjective, and has a single-valued continuous, increasing inverse $\varphi_1^{-1} : I\!R_+ \to I\!R_+$ with $\lim_{r\to 0}\varphi_1^{-1}(r) = 0$.

Then $\|x_1 - x_2\| \leq \varphi_1^{-1}(\|z_1 - z_2\|)$. In this case we can take $\gamma(r) = \varphi_1^{-1}(r)$.
Indeed, let $r_1, r_2 > 0$. Then

$$(r_1 - \varphi_1^{-1}(\varepsilon r_1) - r_2 + \varphi_1^{-1}(\varepsilon r_2))(r_1 - r_2) =$$

$$= (r_1 - r_2)^2 - (\varphi_1^{-1}(\varepsilon r_1) - \varphi_1^{-1}(\varepsilon r_2))(r_1 - r_2) \geq$$

$$= (|r_1 - r_2| - |\varphi_1^{-1}(\varepsilon r_1) - \varphi_1^{-1}(\varepsilon r_2)|)|r_1 - r_2| .$$

If we note $t_1 = \varphi_1^{-1}(\varepsilon r_1)$, $t_2 = \varphi_1^{-1}(\varepsilon r_2)$,then $\varepsilon r_1 \in \varphi_1(t_1)$, $\varepsilon r_2 \in \varphi(t_2)$, hence

$$\varepsilon r_1 \in [\varphi(t_1 - 0), \varphi(t_1 + 0)] + \varepsilon t_1,\ \varepsilon r_2 \in [\varphi(t_2 - 0), \varphi(t_2 + 0)] + \varepsilon t_2$$

and

$$0 \leq (\varepsilon r_1 - \varepsilon t_1 - \varepsilon r_2 + \varepsilon t_2)(t_1 - t_2) = \varepsilon(r_1 - r_2)(t_1 - t_2) - (t_1 - t_2)^2 .$$

Then $|t_1 - t_2| \leq |r_1 - r_2|$ and so we have proved that $Id - \varphi_1^{-1} \circ \alpha$ is an increasing function.
Because of $\lim_{r\to 0} (r - \varphi_1^{-1}(\alpha(r))) = 0$, $Id - \varphi_1^{-1} \circ \alpha$ has positive values.
Let us prove that $r - \varphi_1^{-1}(r) > 0$ when $r > 0$. Let $r \geq 0$. Then

$$0 = r - \varphi_1^{-1}(\varepsilon r) \Leftrightarrow \varepsilon r \in [\varphi(r-0), \varphi(r+0)] ,$$

so $r = 0$, because $\varphi(r) > 0$ for $r > 0$.
Let $z \in B(z_0, \varepsilon_2)$, $y \in B(y_0, \varepsilon_2)$. Then

$$\|\Psi(z,y) - \Psi(z_0,y_0)\| \leq \|\Psi(z,y) - \Psi(z_0,y)\| + \|\Psi(z_0,y) - \Psi(z_0,y_0)\| \leq$$

$$\leq \gamma(\|z - z_0\|) + \|\Psi(z_0,y) - \Psi(z_0,y_0)\| .$$

We define $\beta(y) = \|\Psi(z_0,y) - \Psi(z_0,y_0)\|$.
If we note $x = \Psi(z_0,y)$, then $0 \in \varepsilon x + F(x,y) - z_0$ and we can use Remark 1 to see that $\|x - x_0\| \to 0$ when $\|y - y_0\| \to 0$. Then β is continuous at y_0 and the assumtions of Theorem 2 are satisfied.

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ON THE BEHAVIOUR OF A THIN LIQUID LAYER FLOWING DUE TO GRAVITY AND A SURFACE TENSION GRADIENT

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Dedicated to Professor Ioan A. Rus on his 60^{th} anniversary

Abstract. A thin liquid layer flowing due to gravity and a surface tension gradient is taken into account. On the liquid/gas interface one of the boundary conditions reduces to the fact that the normal stress equals the atmospheric pressure. This is the main difference between our study and those where the same boundary condition expresses the fact that the normal stress is proportional to the curvature. In these, by using the standard lubrication theory, a fourth-order nonlinear parabolic equation for the fluid film height is obtained. In ours, by using the same theory, a nonlinear conservation law with a nonconvex flux function is deduced for the same variable. For this equation a similarity solution is carried out. It shows that the behaviour of the liquid layer depends essentially upon the gradient of surface tension and is quite insensitive to the viscosity of the liquid. "Viscous" and weak formulations for the conservation law are also carried out. An entropy condition to pick out physically relevant weak solutions is used.

1. Introduction

The thin film theory (lubrication theory) and similarity methods are used to determine the behaviour of the free surface (the liquid / gas interface) of a thin liquid layer flowing due to gravity and a gradient of surface tension. This gradient acts on the liquid / gas interface (the upper surface of the liquid layer). The surface tension σ at each point of the interface is related to the local surfactant concentration Γ through an empirically determined equation of state $\sigma(\Gamma(x))$. The gradient in Γ, and thus in σ, along the interface induces a shear stress at the surface of the underlying liquid, and thus a Marangoni flow in the substrate. If the liquid substrate is thin, and if diffusion of the surfactant on the surface of the layer is sufficiently slow, and consequently negligible, that shear stress induces large deformations in the layer of liquid. From the mathematical point of view this gradient of surface tension behaves like an advancing rigid plate. Thus, if the initial

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gradients in surface tension are sufficiently large, the deformations of the liquid layer may be severe enough leading the film to rupture.

In order to refine the similarity solutions, a "viscous" and also a weak equation for the evolution of the interface $z = h(x,t)$ are deduced.To peak out the physically relevant weak solution an entropy condition is displayed. Numerical solutions starting from both "viscous" and weak formulations will be the aim of a following work.

The dynamics of thin liquid layers is important in many industrial process, from painting a car-body to coating a microchip ([5], [6]) and also in medicine in the development of the respiratory distress syndrome of many prematurely born infants ([3], [6] and [7]).

The last two quoted works represent a very keen analysis on the existence of shock profiles. They also give a continuous dependence result for the initial value problem encountered in flows described above.

Our analysis is eventually orientated towards numerical results.

2. The model

The model to be investigated here has been described in details in our previous work [2] and accordingly only a brief summary is given here. We will consider a thin liquid layer of a viscous incompressible Newtonian fluid flowing on a rigid inclined plane (α is the slope). A monolayer of insoluble surfactant creates a gradient of surface tension which acts at the upper surface of the layer. Thus, this gradient of surface tension can act along or against gravity.

The variables of the flow are scaled as follows. Let U be a typical velocity corresponding to undisturbed height d of the layer. We consider $U = \rho g d^2 \sin\alpha/\mu$ as the average velocity of the undisturbed flow, where ρ is the density assumed constant, g gravitational acceleration and μ is fluid's dynamic viscosity. According to what we reported in [2], the aspect ratio $\varepsilon = d/L \ll 1$, where L is the initial length of the layer, thus the thin film theory ([1], p.239), may be used.

From the equation of mass conservation we are led to scale the vertical velocity by $\varepsilon\ U$. We choose to scale time by $h/\varepsilon\ U$ and the pressure by $\rho\ U^2$. We also suppose that the Reynolds number $Re = \rho U d/\mu$ is sufficiently small so that the leading order inertial terms in momentum equation, of $O(\varepsilon^2 Re)$, are negligible.

The surface tension σ at each point of monolayer is related to the local surfactant concentration Γ through an empirically determined equation of state $\sigma = \sigma(\Gamma(x))$. The gradient in Γ, and thus in σ, along the monolayer induces a shear stress at the surface of underlying liquid, and thus a Marangoni flow in the substrate. If the liquid substrate is thin (as we assume), and if diffusion of the surfactant on the upper surface of the layer is sufficiently slow, the flow induces large deformations in the layer (Jensen & Grotberg [3]).

Neglecting surface diffusivity of the surfactant, our aim is to analyse these deformations.

A very elaborate discussion on the dependence of σ on Γ can be found in [3]. In the expression of the gradient of surface tension $d\sigma/dx = \frac{d\sigma}{d\Gamma}\frac{d\Gamma}{dx}$, $d\sigma/d\Gamma$ is in general nonlinear, although a linear law is predominantly used in literature. Our analysis remains chiefly qualitative and mathematically orientated so we do not pay more attention to these aspects.

If σ is scaled by σ_0, the higher surface tension on the liquid / gas interface, and if we take the coordinates (x, z), with z vertical to plane and x downwards the plane, scaled by d, the corresponding velocity field is $(u(x,z,t), w(x,z,t))$. The upper surface of the layer is at $z = h(x,t)$.

We notice that in practice it is highly unlikely that gravitational and intermolecular forces (van der Waals forces) would ever be of the same order. As in the work of Jensen & Grothberg [3], the influence of intermolecular forces is deeply analysed, our intention is to concentrate on the dependence of the behaviour (deformations) of the liquid layer upon the competition between gravity and the surface tension gradient.

Thus, the equations of momentum and mass conservation for the layer of liquid are

$$0 = -p_x + \frac{1}{Re}u_{zz} + \frac{\sin\alpha}{F^2} \tag{1}$$

$$0 = -p_z - \frac{\cos\alpha}{F^2} \tag{2}$$

$$u_x + w_z = 0 \tag{3}$$

where $F = U/(gh)^{1/2}$ is the Froude number and subscripts denote differentiation with respect to that variable.

On integrating the second of these,

$$p = -\frac{\cos\alpha}{F^2} z + f(x,t).$$

On the liquid / gas interface, $z = h(x,t)$, the condition that the normal stress be equal to atmospheric pressure p_0 reduces essentially to $p = p_0$, so

$$p(x,z,t) = \frac{\cos\alpha}{F^2}\left[h(x,t) - z\right] + p_0 \tag{4}$$

On this boundary condition we will comment at the end of the paper.

The tangential stress condition at $z = h$ reads

$$u_z = Ca \frac{d\sigma}{d\Gamma}\frac{d\Gamma}{dx} \tag{5}$$

where $Ca = \sigma_0/U\mu$ is the capillary number.

With (2.4) the equation of motion (2.1) becomes

$$\frac{1}{Re}u_{zz} = -\frac{\sin\alpha}{F^2} + \frac{\cos\alpha}{F^2}h_x.$$

Now, h_x is small, by virtue of the thin film approximation. Thus, unless α is very small, the last term may be neglected and with the boundary condition (2.5) and the no - slip condition at $z = 0$, we find

$$u = -\frac{z^2}{2} + \left(h + Ca\frac{d\sigma}{d\Gamma}\frac{d\Gamma}{dx}\right) z. \tag{6}$$

Taking into account that the quantities $d\sigma/d\Gamma$ and $d\Gamma/dx$ are given, the incompressibility condition (2.3) gives

$$w_z = -u_x = -z\, h_x.$$

On integration and application again of the no - slip boundary condition we have

$$w = -\frac{z^2}{2}h_x. \tag{7}$$

The final consideration is the purely kinematic condition at the free surface. In dimensionless form it reads:

$$w = \varepsilon h_t + u h_x,$$

or with u and w form (2.6), (2.7)

$$\varepsilon\, h_t + \left(h^2 + Ca\frac{d\sigma}{d\Gamma}\frac{d\Gamma}{dx}h\right) h_x = 0. \tag{8}$$

The evolution equation for $h\,(x,t)$ is therefore (2.8).

3. Similarity and weak solutions

The solution of this last equation is by similarity

$$h = f\left[x - \frac{1}{\varepsilon}\left(h^2 + Ca\frac{d\sigma}{d\Gamma}\frac{d\Gamma}{dx}h\right)t\right] \tag{9}$$

where f is an arbitrary function of a single variable, so any particular value of h propagates up or down with speed

$$h^2 + Ca\frac{d\sigma}{d\Gamma}\frac{d\Gamma}{dx}h \tag{10}$$

depending on the sign of this quantity. Thus for some negative values of the gradient of surface tension, waves travelling upward are possible. This fact is in accordance with our observations reported in [2] and the works quoted there. Plainly, h remains constant if $x - \frac{1}{\varepsilon}\left(h^2 + Ca\frac{d\sigma}{d\Gamma}\frac{d\Gamma}{dx}h\right)t$ does. The function f could be determined by adding to (2.8) appropriate initial conditions.

If no gradient of surface tension acts ($d\sigma/dx = 0$), from (3.1) the results of Acheson [1], p.247, are confirmed.

Due to the form of (3.2) an explicit similarity solution of (2.8) writes

$$h = \frac{-Ca\frac{d\sigma}{dx} \pm \sqrt{(Ca\sigma_x)^2 + 4\frac{\varepsilon x}{t}}}{2}. \tag{11}$$

Thus, as time goes on, the main part of the perturbation, denoted by h, approaches this simple similarity solution more or less depending upon the initial conditions.

However, the evolution equation for h, (2.8), can be written as a nonlinear conservation law ([4]):

$$h_t + F\,(h)_x = 0, \tag{12}$$

where the flux function

$$F\,(h) = \frac{1}{\varepsilon}h^3 + \frac{1}{2}Ca\sigma_x h^2$$

is a nonconvex one.

The corresponding "viscous" equation for (3.4) reads as follows:

$$h_t + F(h)_x = \gamma h_{xx}, \quad 0 < \gamma \ll 1. \tag{13}$$

Here γ is the "viscous" parameter and a solution of this, for vanishing γ, is called an entropy solution or a vanishing viscosity solution.

A wealk formulation for (3.4) is obtained in a straightforward manner. Multiplying the equation by a smooth "test function" $\Phi \in C_0^1(\mathbf{R} \times \mathbf{R})$ - the space of functions that are continuously differentiable with "compact support", we obtain the problem:

$$\begin{cases} \text{find } u \in L^3(\mathbf{R} \times \mathbf{R}) \text{ such that} \\ \int\limits_0^\infty \int\limits_{-\infty}^\infty [u\Phi_t + F(u)\Phi_x]\, dt dx = -\int\limits_{-\infty}^\infty \Phi(x,0)\, u(x,0)\, dx, \forall \Phi \in C_0^1(\mathbf{R} \times \mathbf{R}). \end{cases} \tag{14}$$

Thus a solution of (3.6), named a weak solution, if it exists, involves no derivative on u and hence requires less smoothness than the corresponding solutions of the "viscous" equation (3.5) or even "inviscid" equation (2.8). Unfortunately, weak solutions are often not unique, and so an additional question is to identify which weak solution is the physically correct vanishing viscosity solution. In order to avoid working with the "viscous" equation directly, we will formulate another condition on weak solutions which is easier to check, and which will also pick out the physically relevant solutions. This is the so called entropy condition (due to Oleinik, [4], p.36) which reads as follows: $h(x,t)$ is the entropy solution if all discontinuities propagating with speed s given by $F(h_l) - F(h_r) = s(h_l - h_r)$ have the property that

$$\frac{F(h) - F(h_l)}{h - h_l} \geq s \geq \frac{F(h) - F(h_r)}{h - h_r} \tag{15}$$

for all h between h_l and h_r.

Finally we observe that the case of F nonconvex is more complicated mathematically than that of F convex and more important the entropy solution might involve both a shock or a rarefaction wave.

4. Concluding remarks

The similarity solution (3.3) can be interpreted as follows:

for $t \to \infty$,

$$h = \begin{cases} |Ca\sigma_x|, & \sigma_x < 0 \\ 0, & \sigma_x \geq 0. \end{cases}$$

This means that a negative gradient of surface tension could sustain a liquid layer of height $|Ca\sigma_x|$ and a positive gradient does not. It is phisically plausible and is in fact a linear theory of thin liquid layer (thin liquid film) rupture. Moreover, this result is quite insensitive to the viscosity of the liquid. It depends essentially upon the sign of the gradient of surface tension, confirming the fact that this gradient drives the system. In [9] one could find a nonlinear theory of film rupture for a horizontal liquid film. There the surface tension is assumed to be constant, London / van der Waals forces are included, but double - layer forces are neglected.

The "viscous" equation for evolution equation, (3.5), and the weak formulation of that, (3.6), with entropy condition, (3.7), create a fine background on which numerical methods could work. Such numerical results could refine the rough information given by similarity solution.

They will be the aim of a following paper.

On the importance of the boundary condition for pressure on the liquid/gas interface we have the following comment. If one take the Laplace-Young equation (the normal stress due to surface tension is proportional to curvature) as a boundary condition, instead of $p = p_0$, $z = h$, which is physically motivated, he obtains an equation for $h(x,t)$ which is similar to the lubrication one from [5]. It reads:

$$\varepsilon h_t + \frac{1}{3}\left[h^3\left(Re h_{xxx} + 1\right) + \frac{1}{2} Ca\sigma_x h^2\right]_x = 0.$$

A proper comparison between these two type of boundary conditions and their implications on the theory of flows where the surface tension is a driving mechanism remains an open problem.

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PERIODIC SOLUTIONS FOR PERTURBED HAMILTONIAN SYSTEMS WITH SUPERLINEAR GROWTH AND IMPULSIVE EFFECTS

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Dedicated to Professor Ioan A. Rus on his 60th anniversary

Abstract. The aim of this paper is to prove the existence of periodic piecewise continuous solutions for planar systems of impulsive differential equations having the form of a perturbed Hamiltonian. The proof relies on a continuation method introduced in [1] and adapted for impulsive equations in [2].

1. Introduction

In this paper we shall prove the existence of at least one piecewise continuous solution for the periodic boundary value problem:

$$\begin{cases} x'(t) = J \operatorname{grad} V(x(t)) + q(t, x(t)) & \text{for a.e. } t \in [0,1], \\ x(t_k^+) = \psi^k(x(t_k)) & \text{for } k \in \overline{1,m}, \\ x(0) = x(1), \end{cases} \tag{1}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $0 < t_1 < t_2 < \ldots < t_m < 1$, are fixed points, in which the solutions are subject to impulsive effects, and we have denoted $x(t^+) := \lim_{s \searrow t} x(s)$. In what follows $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ will be the euclidean scalar product, respective the euclidean norm in $\mathbf{R}^2$, and we shall suppose that:

(h1) $V : \mathbf{R}^2 \to \mathbf{R}$ is of class C^1 with the properties:

$$\lim_{\|z\| \to \infty} | \langle \operatorname{grad} V(z), z \rangle | / \| z \|^2 = +\infty, \tag{2}$$

and

$$\| \operatorname{grad} V(z) \| \leq A \mid V(z) \mid + B, \quad \text{for all } z \in \mathbf{R}^2, \tag{3}$$

for some fixed $A,\ B \in \mathbf{R}_+$;

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(h2) $q : [0,1] \times \mathbf{R}^2 \to \mathbf{R}^2$ is a L^1–Carathéodory function, i.e. q is Carathéodory and

$$\| q(t,z) \| \leq Q(t) \quad \text{for a.e. } t \in [0,1] \text{ and all } z \in \mathbf{R}^2,$$

for a fixed function $Q \in L^1(0,1;\mathbf{R}_+)$;

(h3) $\psi^k : \mathbf{R}^2 \to \mathbf{R}^2$ are continuous for every $k \in \overline{1,m}$ and there exist $n \in \mathbf{N}^*$, $r > 0$ such that

$$\frac{n}{\pi}\sum_{k=1}^{m}[\arg z_k - \arg \psi^k(z_k)] \not\equiv 1 \mod(2)$$

for all $(z_k)_{k=\overline{1,m}} \in (\mathbf{R}^2)^m$ with $\| z_k \| \geq r$ and $\| \psi^k(z_k) \| \geq r$, $k \in \overline{1,m}$.

We point out that the superlinear character of (1) is given by condition (h1). Such problem have been already studied in [1], where no impulses were considered, and in [2, Example 3].Our main goal is to improve the result from [2, Example 3] using more deeply the properties of the planar Hamiltonian systems, namely the uniform rotation around the origin of the solutions. Those properties will allow us to relax the assumption on impulses from [2, Example 3] using (h3) instead. In order to obtain the existence of at least one piecewise continuous solution for (1) we shall need an abstract result proved in [4] in the frame of the topological transversality theory. For convenience we shall state this theorem.

Let X be a real Banach space, $K \subset X$ a convex set and $H : K \times [0,1] \to K$ a completely continuous map. Denote

$$S = \{(x,\lambda) \in K \times [0,1];\ H(x,\lambda) = x\}$$

and for any fixed $x_0 \in K$, let

$$S(x_0) = \{x \in K;\ (1-\mu)x_0 + \mu H(x,0) = x, \text{ for some } \mu \in [0,1]\}.$$

Also consider a continuous functional $\Phi : K \times [0,1] \to \mathbf{R}$. Then, we have the following theorem [4, Corollary 2.]:

Theorem 1. *Assume*

(i1') *Φ is proper on S;*

(i2') *Φ is bounded bellow on S and there is a sequence ${c_j}_{j\in\mathbf{N}}$ of real numbers such that $c_j \to \infty$ and $c_j \notin \Phi(S)$ for all $j \in \mathbf{N}$;*

(i3') *there is $x_0 \in K$ such that $S(x_0)$ is bounded.*

Then, for each $\lambda \in [0,1]$, there exists at least one fixed point of $H(\cdot,\lambda)$ in K.

2. Main result

In this section we shall prove the existence of solutions for (1) in the following space of functions

$$C_T = \{x : [0,1] \to \mathbf{R}^2;\ x \text{ is everywhere continuous except, eventually, the points } t_1, t_2, \dots, t_m \text{ of discontinuity of first type at which } x \text{ is left continuous}\}.$$

Theorem 2. *If* (h1)-(h3) *are satisfied, then the periodic boundary value problem* (1) *has at least one solution in C_T.*

Proof. Let us consider the family of periodic boundary value problems:

$$\begin{cases} x'(t) = J \operatorname{grad} V(x(t)) + \lambda q(t, x(t)) & \text{for a.e. } t \in [0,1], \\ x(t_k^+) = \lambda \psi^k(x(t_k)) & \text{for } k \in \overline{1,m}, \\ x(0) = x(1). \end{cases} \tag{4}$$

In order to apply the Theorem 1 we choose $X := C_T$ endowed with the usual C norm, $\| x \|_C = \sup\{| x(t) |;\ t \in [0,1]\}$. Notice that C_T can be identified with the real Banach space $\prod_{k=0}^m C[t_k, t_{k+1}]$, where $t_0 := 0$ and $t_{m+1} := 1$. Thus, C_T is a Banach space too. Also, we choose $K := \{x \in C_T;\ x(0) = x(1)\}$ the convex subset of C_T.

To construct the completely continuous map $H : K \times [0,1] \to K$ we define

$$W_p^{1,1} = \{x \in K;\ x \text{ is absolutely continuous on each }]t_k, t_{k+1}[,\ k = 0, 1, \dots, m\}.$$

and the linear map

$$L : W_p^{1,1} \to \mathrm{L}^1(0,1;\mathbf{R}^2) \times (\mathbf{R}^2)^m$$

$$L(x) = (x', \{x(t_k^+)\}_{1 \le k \le m}).$$

This map is invertible and to get its inverse

$$L^{-1} : \mathrm{L}^1 \times (\mathbf{R}^2)^m \to K$$

we have to solve $m+1$ initial value problems:

$$\begin{cases} x'(t) = y(t) & \text{for a.e. } t \in [t_k, t_{k+1}], \\ x(t_k) = u_k, \end{cases}$$

for $1 \le k \le m$, (recall that $t_{m+1} := 1$) and

$$\begin{cases} x'(t) = y(t) & \text{for a.e. } t \in [0, t_1], \\ x(0) = x(1), \end{cases}$$

where $y \in L^1$ and $u = \{u_k\}1 \le k \le m \in (\mathbf{R}^2)^m$. Thus, the unique solution $x \in K$ to $L(x) = (y, u)$ is the function:

$$\begin{array}{rcl} x(t) & = & u_k + \int_{t_k}^{t} y(s)ds \quad \text{for } t_k < t \le t_{k+1}, \ 1 \le k \le m, \\ x(t) & = & x(1) + \int_0^t y(s)ds \quad \text{for } 0 \le t \le t_1. \end{array} \tag{5}$$

We also define the nonlinear map

$$N : K \times [0,1] \to L^1(0,1;\mathbf{R}^2) \times (\mathbf{R}^2)^m$$

$$N(x,\lambda) = (J \operatorname{grad} V(x) + \lambda q(\cdot, x); \lambda \psi^k(x(t_k))).$$

Then, under assumptions (h1) and (h2), N is well-defined, continuous and bounded. Moreover, by (5) and Ascoli-Arzela's theorem, the map $L^{-1}N : K \times [0,1] \to K$ is completely continuous and we can choose

$$H = L^{-1}N.$$

Finally, we consider $\Phi : K \times [0,1] \to \mathbf{R}_+$ given by

$$\Phi(x,\lambda) = \frac{1}{2\pi} \left| \int_0^1 \frac{\langle Jx(t), J \operatorname{grad} V(x(t)) + \lambda q(t, x(t)) \rangle}{\max\{r, \| x(t) \|^2\}} dt \right|$$

where $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the euclidean scalar product, respective the euclidean norm in $\mathbf{R}^2$. This functional is a modification of the classical map which counts the number of rotations around the origin of the continuous integral curves of a planar system (see [3]) and, clearly, Φ is continuous on $K \times [0,1]$.

Now, if we can prove that the hypothesis (i1')-(i3') are satisfied, then, applying theorem1, we will deduce that $H(\cdot, 1)$ has at least one fixed point in K, which is equivalent with the existence of at least one solution in C_T for the periodic boundary value problem (1).

Check of (i1'): Due to the completely continuity of H, Φ is proper on $S = \{(x,\lambda) \in K \times [0,1];\ H(x,\lambda) = x\}$ if, for every $j \in \mathbf{N}$, $\Phi^{-1}([0,j]) \cap S$ is bounded. Indeed, let us consider an arbitrary compact set $P \subset \mathbf{R}_+$, then P is bounded, so, there exists $j \in \mathbf{N}$ such that $P \subseteq [0,j]$. Hence, $\Phi^{-1}(P) \cap S \subseteq \Phi^{-1}([0,j]) \cap S$ and $\Phi^{-1}(P) \cap S$ is bounded as a subset of the bounded set $\Phi^{-1}([0,j]) \cap S$. Since H is completely continuous we have that $H(\Phi^{-1}(P)\cap S)$ is relatively compact. Moreover, from the definition of S, we deduce $\Phi^{-1}(P)\cap S\subseteq H(\Phi^{-1}(P)\cap S)\times[0,1]$ and, by Tychonov's theorem, $\Phi^{-1}(P)\cap S$ is relatively compact. Now, $\Phi^{-1}(P)$ and S are closed, because of the continuity of the maps Φ respectively H. Consequently, $\Phi^{-1}(P)\cap S$ is compact. Since P was an arbitrary compact subset of $\mathbf{R}_+$, we have just proved that the restriction of Φ on S is proper provided that $\Phi^{-1}([0,j])\cap S$ is bounded for all $j\in\mathbf{N}$. So, it remains to verify that $\Phi^{-1}([0,j])\cap S$ is bounded for all $j\in\mathbf{N}$ and we shall do this in two steps

In the first step we shall show that for each $j\in\mathbf{N}$ there exists $r_j>0$ such that, for every $(x,\lambda)\in S$, $\Phi(x,\lambda)\le j$ implies $\inf\{\| x(t)\|;\ t\in[0,1]\}\le r_j$, while, in the second step, we shall show that for each $r_j>0$ there is $R_j\ge r_j$ such that, for every $(x,\lambda)\in S$, $\inf\{\| x(t)\|;\ t\in[0,1]\}\le r_j$ implies $\sup\{\| x(t)\|;\ t\in[0,1]\}\le R_j$. Clearly, from these two steps, we shall have that $\Phi^{-1}([0,j])\cap S$ is bounded for all $j\in\mathbf{N}$, and the check of (i1') will be complete.

For the first step, we fix $j\in\mathbf{N}$ arbitrarily, and we choose

$$r_j=\max\{r,(q+1)/2\pi,r_j'\}$$

where

$$q=\int_0^1 Q(t)dt$$

is the L^1-norm of Q and r_j' is such that:

$$|\langle\operatorname{grad}V(z),z\rangle|/\| z\|^2\ge 2\pi(j+1) \tag{6}$$

for all $z\in\mathbf{R}^2$ with $\| z\|>r_j'$. Note that the existence of r_j' with the above property is a consequence of the superlinear growth of V (see the relation (2) from (h1)).

In order to prove that for every $(x,\lambda)\in S$, $\Phi(x,\lambda)=j$ implies $\inf\{\| x(t)\|;\ t\in[0,1]\}\le r_j$, let us suppose contrary, i.e. there is $(x,\lambda)\in S$, such that $\Phi(x,\lambda)=j$ and $\inf\{\| x(t)\|;\ t\in[0,1]\}>r_j$. Then, by the definition of the functional Φ, we have successively:

$$\begin{aligned}\Phi(x,\lambda) &= \frac{1}{2\pi}\left|\int_0^1\frac{\langle Jx(t),J\operatorname{grad}V(x(t))+\lambda q(t,x(t))\rangle}{\max\{r,\| x(t)\|^2\}}dt\right| = \\ &= \frac{1}{2\pi}\left|\int_0^1\frac{\langle x(t),\operatorname{grad}V(x(t))\rangle}{\| x(t)\|^2}+\lambda\frac{\langle x(t),q(t,x(t))\rangle}{\| x(t)\|^2}dt\right|\ge \\ &\ge \frac{1}{2\pi}\left|\int_0^1\frac{\langle x(t),\operatorname{grad}V(x(t))\rangle}{\| x(t)\|^2}dt\right|-\frac{\lambda}{2\pi}\left|\int_0^1\frac{\langle x(t),q(t,x(t))\rangle}{\| x(t)\|^2}dt\right|\end{aligned}$$

Since $\| x(t) \| > r_j \geq r'_j$, for all $t \in [0,1]$, we can apply, in the first term of the relation, the inequality (6). Using also the Cauchy-Schwarz inequality for the ... product in the second term, we get:

$$\Phi(x,\lambda) \geq j+1-\lambda \int_0^1 \frac{\| q(t,x(t)) \|}{2\pi \| x(t) \|} dt$$

Now, from $2\pi \| x(t) \| > 2\pi r_j \geq q+1$, for all $t \in [0,1]$, and (h2) using also the fact that is the L^1−norm of Q, we obtain:

$$\Phi(x,\lambda) \geq j+1-\lambda \int_0^1 \frac{Q(t)}{q+1} dt = j+1-\lambda \frac{q}{q+1} > j$$

which contradicts $\Phi(x,\lambda) = j$.

For the second step we shall prove the more general result:

Lemma 1. *If* (h1)-(h3) *are satisfied, then, for each* $\widetilde{r} \geq 0$, *there exists* $\widetilde{R} > \widetilde{r}$ *su... for every* $(x,\lambda) \in S$ *with* $\inf\{\| x(t) \|;\ t \in [0,1]\} \leq \widetilde{r}$, *we have* $\sup\{\| x(t) \|;\ t \in$ ... $\widetilde{R}$.

Proof. First of all, using the Cauchy-Schwarz inequality and relation (2), we dedu... $\lim_{\|z\| \to \infty} \| \operatorname{grad} V(z) \| = \infty$, hence, by (3), we have $\lim_{\|z\| \to \infty} |V(z)| = \infty$. So... exists $r' \geq 0$ such that

$$|V(z)| > 0$$

whenever $\| z \| \geq r'$.

Now, let us fix $\widetilde{r} \geq 0$, and $(x,\lambda) \in S$ such that $\inf\{\| x(t) \|;\ t \in [0$ ... $x \in C_T$, the restriction of x on each interval $]t_k, t_{k+1}]$, $0 \leq k \leq m$, ca... continuous function x_k on $[t_k, t_{k+1}]$. Consequently, there... with the property that

$$|x_k(s)| = \inf\{\| x(t) \|;\ t \in [0,1]\}.$$

In what follows we shall construct an upper bound for the ... $]t_k, t_{k+1}]\}$. We have only two possibilities: either $\| x(t) \| \leq r'$, for all $t \in$ ... is an $t \in]t_k, t_{k+1}]$ such that $\| x(t) \| > r'$. In the second case we can find ...

with the properties

$$\begin{aligned}\|x_k(s_0)\| &= \max\{\widetilde{r}, r'\},\\ \|x_k(s_1)\| &= \sup\{\| x(t) \|;\ t \in]t_k, t_{k+1}]\},\\ \|x_k(t)\| &\geq r' \quad \text{for all } t \in [s_0, s_1] \text{ (or } [s_1, s_0]).\end{aligned}$$

If we denote

$$v(t) = \ln \mid V(x_k(t)) \mid \quad \text{for all } t \in [s_0, s_1] \text{ (or } [s_1, s_0]),$$

we can write

$$v(s_1) = v(s_0) + \int_0^1 v'(t)dt \leq v(s_0) + \operatorname{sgn}(s_1 - s_0) \int_0^1 |v'(t)|\, dt. \tag{7}$$

But, using the fact that(x, λ) verifies (2), we have for every $t \in [s_0, s_1]$ (or $[s_1, s_0]$):

$$\begin{aligned}|v'(t)| &= \left|\frac{\langle \operatorname{grad} V(x(t)), x'(t)\rangle}{V(x(t))}\right| = \left|\frac{\langle \operatorname{grad} V(x(t)), J \operatorname{grad} V(x(t)) + \lambda q(t, x(t))\rangle}{V(x(t))}\right| =\\ &= \lambda \left|\frac{\langle \operatorname{grad} V(x(t)), q(t, x(t))\rangle}{V(x(t))}\right|.\end{aligned}$$

Hence, by the Cauchy-Schwarz inequality, relation (3) and assumption (h2), we obtain

$$|v'(t)| \leq \lambda\, |q(t, x(t))| \frac{|\operatorname{grad} V(x(t))|}{|V(x(t))|} \leq \lambda Q(t)(A + B).$$

Replacing the last inequality in (7), we get

$$v(s_1) \leq v(s_0) + q(A + B).$$

Now, the properness of the function $\ln \circ |V|$ (recall that $\lim_{\|z\| \to +\infty} |V(z)| = \infty$) implies that there exists an $R_1 \geq 0$, depending only on $v(s_0) + \lambda q(A + B)$, such that $\|x_k(s_1)\| = \sup\{\| x(t) \|;\ t \in]t_k, t_{k+1}]\} \leq R_1$.

Anyway, choosing $\widetilde{R}_1 \doteq \max\{r', R_1\}$, we have

$$\sup\{\| x(t) \|;\ t \in]t_k, t_{k+1}]\} \leq \widetilde{R}_1$$

where $\widetilde{R}_1$ depends only on r (and the fixed constants r', q, A and B) regardless the two possible cases. Consequently, $\|x(t_{k+1})\| \leq \widetilde{R}_1$ which implies $\|x(t_{k+1})\| = \|x(0)\| \leq \widetilde{R}_1$, if $k = m$ (recall that $t_{m+1} := 1$) or, using the continuity of the functions ψ^k, $1 \leq k \leq m$, $\|x(t_{k+1}^+)\| \leq \sup_{k \in \overline{1,m}} \sup \left\{ \|\psi^k(z)\|;\ \|z\| \leq \widetilde{R}_1 \right\}$. Anyway,

$$\inf\{\| x(t) \|;\ t \in]t_{k+1}, t_{k+2}]\} \leq \widetilde{r}_1$$

where $\widetilde{r}_1 = \max\left\{\widetilde{R}_1, \sup_{k\in\overline{1,m}} \sup\left\{\left\|\psi^k(z)\right\|;\ \|z\| \le \widetilde{R}_1\right\}\right\}$ depends only on $\widetilde{r}$ and if $k = m$, then $]t_{m+1}, t_{m+2}]$ must be interpreted as the interval $]0,1]$.

The same arguments on $]t_{k+1}, t_{k+2}]$ as on $]t_k, t_{k+1}]$, where $\widetilde{r}$ will be replaced by $\widetilde{r}_1$, will allow us to construct the positive numbers $\widetilde{R}_2$ and $\widetilde{r}_2$ such that

$$\sup\{\| x(t) \|;\ t \in]t_{k+1}, t_{k+2}]\} \le \widetilde{R}_2$$

and

$$\inf\{\| x(t) \|;\ t \in]t_{k+2}, t_{k+3}]\} \le \widetilde{r}_2$$

where $\widetilde{R}_2$ and $\widetilde{r}_2$ depends only on $\widetilde{r}$ by the intermediate number $\widetilde{r}_1$, and if $k = m$, then $]t_{m+1}, t_{m+2}]$ must be interpreted as the interval $]0,1]$. Continuing in the same way we can construct the finite sequence $\widetilde{R}_1, \widetilde{R}_2, \dots, \widetilde{R}_{m+1}$ which depends only on $\widetilde{r}$, and clearly $\widetilde{R} = \max\left\{\widetilde{R}_1, \widetilde{R}_2, \dots, \widetilde{R}_{m+1}\right\}$ will have the property

$$\sup\{\| x(t) \|;\ t \in [0,1]\} \le \widetilde{R}.$$

and will depend only on $\widetilde{r}$.

Thus, the lemma is completely proved. □

The lemma covers also the purpose of the second step, so, as we saw at the beginning, the check of (i1') is now complete.

Check of (i2'): Clearly, Φ is bounded bellow by 0, since Φ takes values only in $\mathbf{R}_+$.

By the Lemma 1, there is a positive number R such that $\sup\{\| x(t) \|;\ t \in [0,1]\} \le R$ whenever $(x,\lambda) \in S$ and $\inf\{\| x(t) \|;\ t \in [0,1]\} \le r$. Then, from the completely continuity of H and Tychonov's theorem we deduce that $\{(x,\lambda) \in S;\ \| x(t) \| \le R$ for all $t \in [0,1]\}$ is relatively compact in $X \times [0,1]$. Consequently, the continuity of Φ on $X \times [0,1]$ implies that the set $\Phi(\{(x,\lambda) \in S;\ \| x(t) \| \le R$ for all $t \in [0,1]\})$ is bounded in $\mathbf{R}$. Let $j_0 \in \mathbf{N}$ be such that

$$(j_0 + 1/2)/n > \Phi(\{(x,\lambda) \in S;\ \| x(t) \| \le R \text{ for all } t \in [0,1]\}).$$

where $n \in \mathbf{N}^*$ is given by the hypothesis (h3).

In order to verify that (i2') holds we can choose

$$c_j = (j + j_0 + 1/2)/n, \quad \text{for all } j \in \mathbf{N}$$

Thus, $\lim_{j\to\infty} c_j = \infty$ and it remains to check that $\Phi(S) \cap \{c_j;\ j \in \mathbf{N}\} = \emptyset$. Suppose contrary, that there exist $j \in \mathbf{N}$ and $(x, \lambda) \in S$ such that $c_j = \Phi(x, \lambda)$. Then

$$\inf\{\| x(t) \|;\ t \in [0,1]\} > r,$$

otherwise, by the construction of R, we should have $\sup\{\| x(t) \|;\ t \in [0,1]\} \leq R$ and, consequently, $\Phi(x, \lambda) < (j_0 + 1/2)/n \leq c_j$ which contradicts $\Phi(x, \lambda) = c_j$.

We define the real function $\theta : [0,1] \to \mathbf{R}$ in the following way

$$\begin{aligned} \theta(0) &= \arg x(0), \\ \theta(t_k^+) &= \theta(t_k) + \arg x(t_k^+) - \arg x(t_k), \text{ for } k \in \overline{1,m}, \\ \theta(t) &\in \operatorname{Arg} x(t), \text{ for } t \in [0,1] \end{aligned}$$

and θ is continuous at 0 and on each interval $]t_k, t_{k+1}]$, $0 \leq k \leq m$. Clearly, θ is uniquely determinate by the above properties, moreover, since x is absolutely continuous on each interval $]t_k, t_{k+1}]$, $0 \leq k \leq m$, as a solution of (2) for some $\lambda \in [0,1]$, we also have that θ is absolutely continuous on each interval $]t_k, t_{k+1}]$, $0 \leq k \leq m$ and

$$\begin{aligned} \Phi(x, \lambda) &= \left| \int_0^1 \theta'(t)dt \right| /2\pi = \left| \sum_{k=0}^{m} \int_{t_k^+}^{t_{k+1}} \theta'(t)dt \right| /2\pi = \\ &= \left| \theta(1) - \theta(0) + \sum_{k=1}^{m} \left[\arg x(t_k) - \arg x(t_k^+)\right] \right| /2\pi. \end{aligned}$$

Since $x(0) = x(1)$, there is an integer i such that $\theta(1) - \theta(0) = 2\pi i$. Hence

$$(j + j_0 + 1/2)/n = \Phi(x, \lambda) = \left| i + \sum_{k=1}^{m} \left[\arg x(t_k) - \arg x(t_k^+)\right]/2\pi \right|$$

or, equivalently,

$$\frac{n}{\pi} \sum_{k=1}^{m} \left[\arg x(t_k) - \arg x(t_k^+)\right] = \pm(2j + 2j_0 - 2ni + 1)$$

which contradicts the assumption (h3).

Thus, we have proved that (i2') holds.

Check of (i3'): Let $x_0 \in K$ be the null function. Then

$$S(x_0) = \{x \in K;\ \mu H(x, 0) = x, \text{ for some } \mu \in [0,1]\}.$$

Now, by the definition of H, we have that each $x \in S(x_0)$ verifies

$$\begin{cases} x'(t) = \mu J \operatorname{grad} V(x(t)) & \text{for a.e. } t \in [0,1], \\ x(t_k^+) = 0 & \text{for } k \in \overline{1,m}, \\ x(0) = x(1). \end{cases}$$

for some $\mu \in [0,1]$. Hence

$$\langle x'(t), \operatorname{grad} V(x(t)) \rangle = 0 \quad \text{for a.e. } t \in [0,1],$$

or, equivalently,

$$[V(x(t))]' = 0 \quad \text{for a.e. } t \in [0,1]$$

Tacking into account that $V(x(t_k^+)) = V(0)$ and that the function $V(x(\cdot))$ is absolutely continuous on $[0,1]$ except, eventually, the points $t_1, t_2, \ldots t_m$ in which it is left continuous and admits right limits, we deduce that $V(x(t)) = V(0)$ for all $t \in [0,1]$. Since V is a proper function on $\mathbf{R}^2$ (see the proof of Lemma 1), we can conclude that $S(x_0)$ is bounded.

So, (i3') also holds, and the theorem is completely proved. □

Remark 1. *The main difference between the assumption* (h3) *and the correspondent assumptions on impulses from* [2] *is that* (h3) *allows the impulsive effects to compensate each other. For example, let us consider* $m = 2$, ψ^1, *respectively* ψ^2, *one of the continuous branch of the multivalued complex function* $z \to z^{3/2}$, *respectively* $z \to z^{1/2}$, *(we have identified* $\mathbf{R}^2$ *with* $\mathbf{C}$*). Then,* (h3) *is verified for* $n = 1$ *and* $r = 1$ *while the condition on impulses from* [2, Example 3] *is not.*

Unfortunately, the technique used in this paper can not be extended at other planar differential systems different from a perturbed Hamiltonian. The difficulty arises from the nonuniform rotation around the origin of the solutions for such systems. More precisely, even if (h3) *is satisfied for a well chosen* n *(in order to agree with the symmetry of the system), it may be possible that the impulses move integral curves from high angular speed zones to the low ones and, consequently, the functional* Φ *may not be proper anymore.*

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BOUNDED SOLUTIONS AND PERIODIC SOLUTIONS FOR CERTAIN SYSTEMS OF DIFFERENTIAL EQUATIONS

NICOLAIE LUNGU

Dedicated to Professor Ioan A. Rus on his 60^{th} *anniversary*

1. Introduction

The study of the existence of bounded solutions and periodic solutions for certain systems of differential equations have large applications in technical problems. In this paper one gives existence conditions for bounded solutions and periodic solutions, for one system which generalizes the systems of Liénard type.

2. Bounded Solutions and Periodic Solutions

Let be the system:

$$\begin{cases} \dot{x}_1 & = u_1(x_1)v_1(x_2)g_1(x_1,x_2) + f_1(x_1,x_2)w_1(x_1)z_1(x_2) \\ \dot{x}_2 & = u_2(x_2)v_2(x_1)g_2(x_1,x_2) + f_2(x_1,x_2)w_2(x_2)z_2(x_1) \end{cases}$$

and suppose that $f_i, g_i : \mathbf{R}^2 \to \mathbf{R}$, $u_i, v_i, w_i, z_i : \mathbf{R} \to \mathbf{R}$, $i \in \{1,2\}$ are continuous functions. Also, suppose that the system (1) has unique solutions for any Cauchy problem formulated about it.

We shall prove that in certain conditions there exists a bounded, closed and simply connected domain $\mathbf{D} \subset \mathbf{R}^2$, such that any trajectory of the system (1) , which meets the frontier of the domain $\mathbf{D}$ at a moment t_0, remains in $\mathbf{D}$ for any $t \geq t_0$.

Theorem 1. *Suppose that:*

1. There exists bounded sets $\mathbf{D} \subset \mathbf{R}^2$, $i \in \{1,2,3\}$, *with the following property:*

$$f_1(x_1,x_2) \leq 0 \ , \ (x_1,x_2) \notin \mathbf{D}_1,$$

$$f_2(x_1,x_2) \leq 0 \ , \ (x_1,x_2) \notin \mathbf{D}_2,$$

2. $z_1(x_2), z_2(x_1) \geq 0 \ , \ \forall x_1, x_2 \in \mathbf{R}$,

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3.

$$u_1(x_1)sgn\,x_1 > 0 \qquad , u_2(x_2)sgn\,x_2 > 0$$

$$-v_2(x_1)sgn\,x_1 > 0 \qquad , v_1(x_2)sgn\,x_2 > 0$$

$$w_1(x_1)sgn\,x_1 > 0 \quad , w_2(x_2)sgn\,x_2 > 0, \quad \forall x_1, x_2 \in \mathbf{R} \setminus \{0\}$$

Then there exists a bounded, closed and simply connected domain $D \subset R$ such that any trajectory of the system (1), which meets the frontier of D at a moment t_0 cannot leave D for any $t \geq t_0$.

Proof. Let be the function $V_i : \mathbf{R} \to \mathbf{R}$, $i \in \{1,2\}$, defined by:

$$V_1(x_1) = -\int_0^{x_1} \frac{v_2(s)}{u_1(s)w_1(s)} ds, \quad V_1(x_1) = -\int_0^{x_1} \frac{v_1(y)}{u_2(y)w_2(y)} dy$$

and the function $V : \mathbf{R}^2 \to \mathbf{R}$, $V(x_1, x_2) = V(x_1) + V(x_2)$. Let C be a pozitive number, great enough such that the set $D_1 \cup D_2 \cup D_3$ is contained inside the domain bounded by the curve $V(x_1, x_2) = C$. We consider the following domain:

$$\mathbf{D} = \{(x_1, x_2) \in \mathbf{R}^2 | V(x_1, x_2) \leq C\} \tag{1}$$

We prove that this is the domain we search for.

Let $x_0 = x_{10}, x_{20})$ be an point of the curve $V(x_1, x_2) = C$, $t_0 \geq 0$ and $x = (x_1, x_2)$ a solution of system (1) verifying the initial conditions $x_1(t_0) = x_{10}$, $x_2(t_0) = x_{20}$, and T_t^+ the point of $\mathbf{R}^2$ in which lies, at the time t, the corresponding positive half-trajectory. We shall show that for any time $t \geq t_0$, at which the solution x is defined, we have $T_{t_1}^t \in \mathbf{D}$. Indeed, otherwise there would be a time $t_1 > t_0$ at which the solution x is defined and $T_{t_1}^+ \notin \mathbf{D}$. We may assume, without loss of generality, that $T_t^+ \notin \mathbf{D}$ for all $t \in [t_0, t_1]$. Then,

$$V(x_1(t_1), x_2(t_2)) > C, \quad C = V(x_{10}, x_{20}). \tag{2}$$

BuT V, as t function, is nonincreasing on the interval $[t_0, t_1]$ along the solution x, since, by (a), (b), (c) and (e), the derivative of V by virtue of the system (1) is

$$\dot{V}(x_1, x_2) = v_1(x_2)v_2(x_1)\left[\frac{g_2(x_1, x_2)}{w_2(x_2)} - \frac{g_1(x_1, x_2)}{w_1(x_2)}\right] - \frac{v_2(x_1)z_1(x_2)}{u_2(x_1)} \times$$
$$\times f_1(x_1, x_2) + \frac{v_1(x_1)z_2(x_2)}{u_2(x_2)} f_2(x_1, x_2) \leq 0$$

whenever the point (x_1, x_2) is so that $V(x_1, x_2) \geq C$. Thus, for any $t \in [t_0, t_1]$ we obtain $\dot{V}(x_1(t), x_2(t)) \leq 0$, whence

$$V(x_1(t_1), x_2(t_1)) \leq V(x_1(t_0), x_2(t_0))$$

and that is a contradiction to (2).

Remark. The domain **D** may be chosen so large that it contains every given point of $\mathbf{R}^2$ and all trajectories of system (1) are bounded.

Theorem 1. *Suppose that all requirements of Theorem 1 are fulfilled, that (1) is such as to guarantee the uniqueness of any initial problem for this system, and that* $(0,0)$ *is the single singular point of (1). If, moreover,* $f_1(0,0) > 0$, $f_2(0,0) > 0$, $\left[\frac{g_2(x_1,x_2)}{w_2(x_2)} - \frac{g_1(x_1,x_2)}{w_1(x_1)}\right] sgn x_1 sgn x_2$ 0 *in a neighbourhood of* $(0,0)$, *then the system (1) has at least one distinct from* $90,0)$ *closed trajectory, or limit cycle.*

Proof. Choose a positive number r so small that the circle $x_1^2 + x_2^2 = r^2$is included in D, and $f_1(x_1, x_2) > 0$,$f_2(x_1, x_2) > 0$ for $x_1^2 + x_2^2 \leq r^2$. We can see that the positive half trajectories, starting from points of the circle $x_1^2 + x_2^2 = r^2$, enter into the ring i limited by the circle and the curve $V(x_1, x_2) = C$, with $C > 0$ as in the proof of Theorem 1. Indeed, for points (x_1, x_2) with $x_1^2 + x_2^2 \leq r^2$, it holds that $\dot{V}(x_1, x_2) > 0$.

If it is the moment when the trajectory crossing the circle $x_1^2 + x_2^2 = r^2$ and $t_1 > t_0$, and because V is an increasing function of t, we have $V(x_1(t_1), x_2(t_1)) \geq V(x_1(t_0), x_2(t_0))$, but that is in contradiction with the fact that for one point by inside the circle , we have $V(x_1(t), x_2(t)) \leq V(x_1(t_0), x_2(t_0))$, therefore the trajectories crossing the circle , go out fro the circle and enter in the ring I, therefore owing to Poincaré-Bendixson theorem, the ring I contains at least one closed trajectory.

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A POLYNOMIAL SPLINE APPROXIMATION METHOD FOR SOLVING VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

GHEORGHE MICULA AND AHMET AYAD

Dedicated to Professor Ioan A. Rus on his 60^{th} anniversary

Abstract. In this paper, a new method for approximating the solution of the nonlinear n^{th} order Volterra integro-differential equations is presented by using a polynomial spline functions. Error estimation and convergence as well as the stability of the method were investigated.

1. Introduction

Consider the nonlinear n^{th} order Volterra integro-differential equation of the n^{th} order form

$$\begin{array}{ll} y^{(n)}_{(x)} = f\left(x, y(x), \int_0^x k(x,t,y(t))dt\right), & 0 \le x \le a \\ y^{(i)}_{(0)} = y_0^{(i)}, & i = 0(1)n-1 \end{array} \tag{1}$$

where f and k are given functions and y is the unknown function to be found. There are a number of important problems and phenomena which are modelled using such kind integrao-differential equation, therefore their numerical treatment is desired.

Recently many authors [5,8,10] have proposed methods to approximate the solution of first-order Fredholm integro-differential equations.

In this paper, is proposed a polynomial spline approximation method for solving Volterra integro-differential equations. The purpose of the present study is to extend some results from ordinary case to the n^{th} order Volterra one. We use a polynomial spline function for finding the approximate solution. The method is one step method $O(h^{r+\alpha})$ in $y^{(i)}$, and $O(h^{r+q})$ in $y^{(n+q)}$ assuming that $f \in C^r([0,a] \times \mathbf{R}^2)$ and the modulus of continuity of $y^{(n)}$ is $O(h^\alpha)$, where $i = 0(1)n-1$, $q = 0(1)r$, $r \in \mathbf{N}$ and $0 < \alpha \le 1$. It is also shown that the method is stable.

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2. Description of the method

Following [5] we shall write problem (1) in the following form:

$$\begin{aligned} &y^{(n)}(x) = f(x, y(x), z(x)), \; y^{(i)}(0) = y_0^{(i)}, \; i = 0(1)n-1, \quad 0 \le x \le a \\ &z(x) = \int_0^x k(x, t, y(t))dt \end{aligned} \tag{2}$$

and suppose that $f : [0, a] \times \mathbf{R}^2 \to \mathbf{R}$ is defined and continuous together with its r^{th} derivatives, $r \in \mathbf{N}$ satisfying the Lipschitz condition

$$|f^{(q)}(x, y_1, z_1) - f^{(q)}(x, y_2, z_2)| \le L_1\{|y_1 - y_2| + |z_1 - z_2|\} \tag{3}$$

$\forall\ (x, y_1, z_1), (x, y_2, z_2) \in [0, a] \times \mathbf{R}^2$ and $q = 0(1)r$.

Also assume that Kernel $k : [0, a] \times [0, a] \times \mathbf{R} \to \mathbf{R}$ is a smooth bounded function satisfying the Lipschitz condition:

$$|k(x, t, y_1) - k(x, t, y_2)| \le L_2|y_1 - y_2| \tag{4}$$

$\forall\ (x, t, y_1), (x, t, y_2) \in [0, a] \times [0, a] \times \mathbf{R}$.

These conditions assure the existence of a unique solution y of problem (2).

Let Δ be a uniform partition of the interval $[0, a]$ defined by Δ:

$$0 = x_0 < x_1 < x_2 < \cdots < x_k < x_{k+1} < \cdots < x_N = a, \quad x_k = kh,$$

and $h = a/N$. Assume the function $y^{(n+r)}$ has a modulus of continuity $\omega(y^{(n+r)}, h) = \omega(h) = O(h^\alpha)$, $0 < \alpha \le 1$. For $x \in [x_k, x_{k+1}]$, $k = 0(1)N-1$, we define the polynomial spline functions approximating the solution $y(x)$ by $s_\Delta(x)$, where

$$\begin{aligned} s_\Delta(x) = s_k(x) &= \sum_{i=0}^{n-1} \frac{s_{k-1}^{(i)}(x_k)}{i!}(x - x_k)^i + \sum_{j=0}^{r} \frac{M_k^{(j)}}{(j+n)!}(x - x_k)^{j+n} \\ M_k^{(j)} &= f^{(j)}(x_k, S_{k-1}(x_k), \int_0^{x_k} k(x_k t, S_{k-1}(t)dt), \; j = 0(1)n-1 \end{aligned} \tag{5}$$

where $\underline{S}_1(t) = y_0$, $\underline{S}_1^{(i)}(0) = y_0^{(i)}$ and $S_{k-1}^{(i)}(x_k)$, $i = 0(1)n-1$ are the left hand limits of the derivatives $S_{k-1}^{(i)}(x)$ as $x \to x_k$ of the segment $s_\Delta(x)$ defined on $[x_{k-1}, x_k]$ obviously such $S_\Delta(x) \in C^{n-1}[0, a]$ exists and unique.

3. Error estimation and convergence

To estimate the error, for all $x \in [x_k, x_{k+1}]$, $k = 0(1)N-1$, the exact solutions can be written by using Taylor's expansion in the following form:

$$y(x) = \sum_{i=0}^{n+r-1} \frac{y_k^{(i)}(x_k)}{i!}(x - x_k)^i + \frac{y^{(r+n)}(\xi_k)}{(r+n)!}(x - x_k)^{r+n} \tag{6}$$

where $y_k^{(i)} = y^{(i)}(x_k)$, $i = 0(1)n + r - 1$ and $\xi_k \in (x_k, x_{k+1})$. Moreover we denote the estimated error of $y(x)$ at any point $x \in [0, a]$ by:

$$e^{(i)}(x) = |y^{(i)}(x) - s_\Delta^{(i)}(x)|, \quad e_k^{(i)} = |y_k^{(i)} - s_\Delta^{(i)}(x_k)|, \quad i = 0(1)n - 1. \tag{7}$$

Lemma 3.1 [7 p.25].

Let α and β be nonnegative real numbers, $B \neq 1$ and $\{A_i\}^k$ be a sequence satisfying $A_0 \geq 0$ and $A_{i+1} \leq \alpha + B_i A_i$ for $i = 0(1)k$ then:

$$A_{k+1} \leq \beta^{k+1} A_0 + \alpha \frac{[B^{k+1} - 1]}{B - 1}. \tag{8}$$

By using direct calculs and math induction, it is easy to prove the following lemma:

Lemma 3.2. *If $y(x) = \sum_{i=0}^{n} \frac{y_k^{(i)}}{i!}(x - x_k)^i$ then for $p = 0(1)n - 1$*

$$y^{(p)}(x) = \sum_{i=0}^{n-p} \frac{y_k^{(i+p)}}{i!}(x - x_k)^i. \tag{9}$$

Lemma 3.3. *Let $e^{(i)}(x)$, $i = 0(1)n-1$ be defined as in (7), then for $p = 0(1)n-1$ the following inequality holds:*

$$e^{(p)}(x) \leq \sum_{i=0}^{n-p-1} \frac{e_k^{(i+p)}}{i!} h^i + hbe_k + \frac{\omega(h) h^{n+r-p}}{(n+r-p)!} \tag{10}$$

where $b = (L_1 + L_1 L_2 k)e$ is constant independent of h.

Proof. Using (6), (5), (3), (4) and (7), we get:

$$\begin{aligned}
e^{(p)}(x) &= |y^{(p)}(x) - S_k^{(p)}(x)| = \Bigg| \sum_{i=0}^{n+r-p-1} \frac{y_k^{(i+p)}(x-x_k)^i}{i!} + \frac{y^{(n+r)}(\xi_k)}{(n+r-p)!}(x-x_k)^{n+r-p} - \\
&- \sum_{i=0}^{n-p-1} \frac{S_{k-1}^{(i+p)}}{i!}(x-x_k)^i + \sum_{j=0}^{r} \frac{M_k^{(j)}}{(n-p+j)}(x-x_k)^{n-p+j} \Bigg| \\
&= \leq \sum_{i=0}^{n-p-1} \frac{|y_k^{(i+p)} - S_{k-1}^{(i+p)}(x_k)|}{i!}|x-x_k|^i + \sum_{i=0}^{r-1} \frac{|y_k^{(n+i)} - M_k^i|}{(n-p+i)!}|x-x_k|^{n-p+i} \\
&+ \frac{|y^{(n+r)}(\xi_k) - M_k^{(r)}|}{(n+p-i)!}|x-x_k|^{n+r-p} \leq \sum_{i=0}^{n-p-1} \frac{e_k^{(i+p)}}{i!}h^i + \\
&+ \sum_{i=0}^{r-1} \frac{V_{ki}}{(n-p+i)!}h^{n-p+i} + \frac{V_r}{n+r-p}h^{n+r-p} \qquad (11)
\end{aligned}$$

where

$$\begin{aligned}
V_{ki} &= |y_k^{(n+i)} - M_k^{(i)}| = \Big| f^{(i)}(x_k, y_k, \int_0^{x_k} K(x_k, t, y(t)dt) - \\
&- f^{(i)}(x_k, S_{k-1}(x_k)), \int_0^{x_k} K(x_k, t, S_{k-1}(t)dt \Big| \leq \\
&\leq L_1 \left\{ |y_k - S_{k-1}(x_k)| + L_2 \int_0^{x_k} |y(t) - S_{k-1}(t)|dt \right\}
\end{aligned}$$

but for $t \in [x_{k-1}, x_k] e(t) = |y(t) - S_{k-1}(t)| \to e_k$ as $t \to x_k$.

Hence

$$V_{ik} \leq L_1 \left(e_k + L_2 e_k \int_0^{x_k} dt \right) \leq (L_1 + L_1 L_2 k) e_k \qquad (12)$$

$$\begin{aligned}
V_{rk} &= |y^{(n+r)}(\xi_k) - M_k^{(r)}| \leq |y^{(n+r)}(\xi_k) - y_k^{(n+r)}| + |y_k^{(n+r)} - M_k^{(r)}| \leq \\
&\leq \omega(y^{(n+r)}, h) + (L_1 + L_1 L_2 k) e_k \leq \omega(h) + (L_1 + L_1 L_2 k) e_k
\end{aligned}$$

using (12) and (13) in (14), we get:

$$\begin{aligned} e^{(p)}(x) &\leq \sum_{i=0}^{n-p-1} \frac{e_k^{(i+p)}}{i!} h^i + \left[(L_1 + L_1 L_2 k) \sum_{i=0}^{r} \frac{h^{n-p-i}}{(n-p+i)!} \right] e_k + \\ &+ \frac{h^{n-p+r}}{(n+r-p)!} \leq h \left[(L_1 + L_1 L_2 k) \sum_{i=0}^{r} \frac{1}{(n-p+i)!} \right] e_k + \\ &+ \sum_{i=0}^{n-p-1} \frac{e_k^{(i+p)}}{i!} h^i + \frac{h^{n+r-p}}{(n+r-p)!} \omega(h) + \\ &+ hbe_k + \sum_{i=0}^{n-p-1} \frac{e_k^{(i+p)}}{i!} h^i + \frac{h^{n+r-p}}{(n+r-p)!} \omega(h) \end{aligned}$$

where

$$b = (L_1 + L_1 L_2 k) \sum_{i=0}^{r} \frac{1}{n-p+i} \leq (L_1 + L_1 L_2 k) e$$

is a constant independent of h.

Definition 3.1. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices of the same order. Then we say that $A \leq B$ iff

(i) both a_{ij} and b_{ij} are nonnegative

(ii) $a_{ij} \leq b_{ij} \ \forall\, i, j$.

In view of this definition and if we use a matrix notions

$$E(x) = (e(x) e^{(1)}(x) e^{(2)}(x) \ldots e^{(n-1)}(x))^T, \quad E_k = (e_k e_k^{(i)} \ldots e_k^{(n_1)})^T,$$

then from Lemma 3.4, we can write

$$E(x) \leq (I + hA) E_k + h^{r+1} \omega(h) B \tag{14}$$

where

$$A = \begin{pmatrix} b & 1 & \frac{1}{1!} & \frac{1}{2!} & \cdots & \frac{1}{n-1} \\ b & 0 & 1 & \frac{1}{2!} & \frac{1}{3} & \frac{1}{(n-2)!} \\ \vdots & \vdots & 0 & 0 & \frac{1}{1!} & \frac{1}{n-3} \\ \vdots & \vdots & \vdots & \vdots & 1 & \frac{1}{1!} \\ \vdots & \vdots & \vdots & \vdots & \vdots & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{n+1} \\ \frac{1}{n+r-1} \\ \vdots \\ \\ \vdots \\ \frac{1}{(r+1)!} \end{pmatrix},$$

and I is the identity of order n.

Definition 3.2. Let $T = [t_{ij}]$ be an $m \times n$ matrix, then we define the norm of T by

$$\|T\| = \max_i \sum_{j=0}^{n} |t_{ij}|.$$

In view of this definition an inequality (14) yield

$$\|E(x)\| \leq (1 + h\|A\|)\|E_k\| + \|B\|h^{r+1}\omega(h).$$

This inequality holds for any $x \in [0, a]$. Setting $x = x_{k+1}$ then

$$\|E_{k+1}\| \leq (1 + h\|A\|)\|E_k\| + \|B\|h^{r+1}\omega(h).$$

Using Lemma 3.1 and noting that $\|E_0\| = 0$, we get:

$$\begin{aligned}\|E(x) &\leq \|B\|h^{r+1}\omega(h)\frac{[(1 + h\|A\|)^{k+1} - 1]}{1 + h\|A\| - 1} \leq \\ &\leq \frac{\|B\|}{\|A\|}h\omega(h)\left[\left(1 + \frac{\|A\|a}{1 + h\|N\|}\right)^{N} - 1\right] \leq \\ &\leq \frac{\|B\|}{\|A\|}(e^{\|A\|a} - 1)h^{r}\omega(h) = b_1 h^{r}\omega(h)\end{aligned}$$

where $b_1 = \frac{\|B\|}{\|A\|}(e^{\|A\|a} - 1)$ is a constant independent of h. Using Definition 3.2 we get:

$$e^{(i)}(x) \leq b_1 h^{r}\omega(h) \quad \text{for} \quad i = 0(1)n - 1. \tag{15}$$

We now estimate $|y^{(n+q)}(x) - S_{\Delta}^{(n+q)}(x)|$, $q = 0(1)r$. For this purpose, using (6), (5), (3), (4), (7), (9) and (15), we obtain

$$|y^{(n+q)}(x) - S_{\Delta}^{(n+q)}(x)| \leq b_2 h^{r-q}\omega(h) \tag{16}$$

where $b_2 = \frac{1}{(r-a)!} + bb_1$ is a constant independent of h. Thus, we proved the following result.

Theorem 3.1. *Let $y(x)$ be the exact solution to the problem (1). If $S_{\Delta}(x)$, given by (5) is the approximate solution for the problem and $f \in C^{r}([0,a] \times \mathbf{R}^2)$ then the following inequalities*

$$|y^{(q)}(x) - S_{\Delta}^{(q)}(x)| \leq b_3 h^{r}\omega(h), \quad q = 0(1)n - 1$$

and

$$|y^{(n+q)}(x) - S_{\Delta}^{(n+q)}(x)| \leq b_4 h^{r-q}\omega(h), \quad q = 0(1)r$$

hold for all $x \in [0, a]$, where $r \in \mathbf{N}$, B_3 and b_4 are constants independent of h.

4. Stability of the method

The stability concept for a one-step method means that "a small changes in the starting value only produce changes in the numerical values provided by the method". To study the method given by (5), we change $S_\Delta(x)$ by $W_\Delta(x)$, where

$$\begin{aligned} W_\Delta(x) = W_k(x) &= \sum_{i=0}^{n-1} \frac{W_{k-1}^{(i)}(x_k)}{i!}(x-x_k) + \sum_{j=0}^{r} \frac{N_k^{(j)}}{(n+j)!}(x-x_k)^{j+n} \\ N_k^{(j)} &= f^{(j)}\left(x_k, W_{k-1}(x_k), \int_{k-1}^{x_k} K(x_k,t,W_{k-1}(t)dt\right), \quad j=0(1)r \end{aligned} \tag{17}$$

where $W_{-1}(t) = y_0^*$, $W_{-1}^{(j)}(0) = y_0^{*(i)}$ and $W_{k-1}^{(j)}(x_k)$, $i = 0(1)n-1$ are the left hand limits of the derivatives $S_{k-1}^{(i)}$ as $x \to x_k$ of the segment $W_\Delta(x)$ defined on $[x_{k-1}, x_k]$. Moreover we use the following notations:

$$e^{*(i)}(x) = |S_\Delta^{(i)}(x) - W_\Delta^{(i)}(x)|, \quad e_k^{*(i)} = |S_\Delta^{(i)}(x_k) - W_\Delta^{(i)}(x_k)|, \quad i = 0(1)n-1. \tag{18}$$

Lemma 4.1. *Let $e^{*(i)}(x)$, $i = 0(1)n-1$ be defined as in (18), then for $p = 0(1)n-1$, the following inequality holds*

$$e^{*(p)}(x) \le hbe_k^* + \sum_{i=0}^{n-p-1} \frac{e_k^{*(i+p)}}{i!} \tag{19}$$

where b is defined as in Lemma 3.4.

Using (5), (17), (3), (4), (18) and (19) the proof is similar to the proof of Lemma 3.4.

Using Definition 3.1 and the matrix notations

$$E^*(x) = (e^*(x)e^{*(1)}(x)\dots e^{*(n-1)}(x)),\ E_k^*(e_k^*e_k^{*(1)}\dots e_k^{*(1)})^T$$

then from lemma (4.1) we write

$$E^*(x) \le (I + hA)E_k^* \tag{20}$$

where A and I are matrices defined as in an inequality (14). Use Definition 3.2 we obtain:

$$\|E^*(x)\| \le (1 + h\|A\|)\|E_k^*\|.$$

Setting $x = x_{k+1}$ and use Lemma 3.1 we get

$$\|E^*(x)\| \le (1+h\|A\|)^{k+1}\|E_0^*\| \le \left(1 + \frac{a\|A\|}{N}\right)^N \|E_0^*\| \le e^{\|A\|a}\|E_0^*\| \le \|E_0^*\| \le b_4\|E_0^*\|.$$

Using Definition 3.2 we get:

$$e^{*(i)}(x) \le b_4\|E_4^*\| \quad \text{for} \quad i = 0(1)n-1 \tag{21}$$

where $\|E_0^*\| = \max_{0\le i\le n-1}\{|y^{(i)} - y_0^{*(i)}|\}$.

We now estimate $|S_\Delta^{(n+q)}(x) - W_\Delta^{(n+q)}(x)|$, $q = 0(1)r$, $r \in \mathbf{N}$. For this purpose (5), (17), (3), (4), (8), (9) and (21), we obtain:

$$|S_\Delta^{(n+q)}(x) - W_\Delta^{(n+q)}(x)| \le b_5\|E_0^*\| \tag{22}$$

where $b_5 = bb_4$ is a constant independent of h.

Thus, we could prove the following result.

Theorem 4.1. *Let $S_\Delta(x)$, given by (5) be the approximate solution of problem (1) with initial conditions $y^{(i)}(0) = y_0^{(i)}$, $i = 0(1)n-1$ and let $W_\Delta(x)$, given by (17) be the approximate solution of the same problem with initial conditions $y^{(i)}(0) = y_0^{*(i)}$, $i = 0(1)n-1$. Then the following inequality*

$$|S_\Delta^{(q)}(x) - W_\Delta^{(q)}(x)| \le b_6\|E_0^*\|$$

holds for all $x \in [0, a]$, $q = 0(1)n + r$, where b_6 is a constant independent of h.

Numerical examples

The method is tested using the following four examples in the interval $[0, 1]$ with step size $h = 0.1$ when $r = 0$ and 1. The tabulated below are evaluated at the point $x = 0.3$. To test the stability of the method we do change in the starting value by adding 0.000001 to the initial condition and solve the same problems.

Example 1. Consider the Volterra integro-differential equation

$$y' = y^2 + \int_0^x y(t)dt - 1 - e^{-2x}, \quad y(0) = 1.$$

The exact solution is $y = e^{-x}$.

To test the stability we solve the same problem with $y(0) = 1.000001$.

		First APP. S	Absolute error	Second APP. S	First APP – Second APP
y	$r = 0$	0.724525804	0.16×10^{-1}	0.724527352	1.55×10^{-6}
	$r = 1$	0.740976819	0.16×10^{-3}	0.740978547	1.73×10^{-6}
y'	$r = 0$	-0.841011197	0.10019	-0.841008843	2.35×10^{-6}
	$r = 1$	-0.739987835	0.83×10^{-3}	0.739984877	2.96×10^{-6}
y''	$r = 1$	0.827559033	0.87×10^{-1}	0.827562344	3.3×10^{-6}

Example 2. Consider the Volterra integro-differential equation

$$y''(x) = y^2(x) + \int_0^x y(t)dt - e^{-2x} + 1, \quad y(0) = y'(0) = 1$$

The exact solution is $y = e^x$.

To test the stability we solve the same problem with $y(0) = y'(0) = 1.000001$.

		First APP. S	Absolute error	Second APP. S	First APP - Second APP
y	$r=0$	1.347694525	0.22×10^{-2}	0.134695929	1.4×10^{-6}
	$r=1$	1.37987429	0.41×10^{-4}	1.349818845	1.42×10^{-6}
y'	$r=0$	1.332932704	0.17×10^{-1}	1.332934486	1.78×10^{-6}
	$r=1$	1.349810499	0.48×10^{-4}	1.349812379	1.88×10^{-6}
y''	$r=0$	1.224538134	0.125321	1.224541394	3.26×10^{-6}
	$r=1$	1.34872591	0.11×10^{-2}	1.349829983	4.1×10^{-6}
y'''	$r=1$	1.220426686	0.129432	1.220434674	7.99×10^{-6}

Example 3. Consider the Volterra integro-differential equation

$$y^{(3)}(x) = y^2 + \int_0^x y(t)dt - 1 - e^{-2x}, \quad y(0) = y^2(0) = 1, \quad y'(0) = -1.$$

The exact solution is $y = e^{-x}$.

To test the stability we solve the same problem with

$$y(0) = y^{(2)}(0) = 1.000001, \quad y'(0) = -0.999999.$$

		First APP. S	Absolute error	Second APP. S	First APP - Second APP
y	$r=0$	0.740640432	0.18×10^{-3}	0.740641795	1.36×10^{-6}
	$r=1$	0.740822838	0.46×10^{-5}	0.740824193	1.36×10^{-6}
$y^{(1)}$	$r=0$	-0.742690124	0.19×10^{-2}	-0.742688695	1.43×10^{-6}
	$r=1$	-0.740783236	0.35×10^{-4}	-0.740781841	1.4×10^{-6}
$y^{(2)}$	$r=0$	0.727166402	0.14×10^{-1}	0.727167937	1.54×10^{-6}
	$r=1$	0.74093608	0.25×10^{-4}	0.740795266	1.66×10^{-6}
$y^{(3)}$	$r=0$	-0.823491072	0.83×10^{-1}	-0.823488835	2.24×10^{-6}
	$r=1$	-0.741530348	0.71×10^{-3}	-0.741527994	2.35×10^{-6}
$y^{(4)}$	$r=1$	0.818769243	0.78×10^{-1}	0.818770495	1.25×10^{-6}

Example 4. Consider the Volterra integro-differential equation

$$y^{(4)}(x) = y^2(x) - \int_0^x y(t)dt - e^{-2x} + 1 \quad 0 \le x \le 1$$

$$y(0) = y^{(2)}(0) = 1, \quad y^{(1)}(0) = y^{(3)}(0) = -1.$$

The exact solution is $y = e^{-x}$.

To test the stability we solve the same problem with $y(0) = y^{(2)}(0) = 1.000001$, $y^{(1)}(0) = y^{(3)}(0) = -0.999999$.

	First APP. S		Absolute error	Second APP. S	First APP - Second APP
y	$r=0$	0.740830815	0.13×10^{-4}	0.740832166	1.35×10^{-6}
	$r=1$	0.740817966	0.25×10^{-6}	0.740819252	1.29×10^{-6}
$y^{(1)}$	$r=0$	-0.740640452	0.18×10^{-3}	-0.740639099	1.35×10^{-6}
	$r=1$	-0.740819677	0.15×10^{-5}	-0.740821169	1.49×10^{-6}
$y^{(2)}$	$r=0$	0.742689635	0.19×10^{-2}	0.742691015	1.38×10^{-6}
	$r=1$	0.740783261	0.35×10^{-4}	0.740784046	0.79×10^{-6}
$y^{(3)}$	$r=0$	-0.72717479	0.14×10^{-1}	-0.72713368	1.42×10^{-6}
	$r=1$	-0.740791128	0.27×10^{-4}	-0.740793566	2.44×10^{-6}
$y^{(4)}$	$r=0$	0.823414525	0.83×10^{-1}	0.823414856	0.33×10^{-6}
	$r=1$	0.741536336	0.72×10^{-3}	0.741537924	0.59×10^{-6}
$y^{(5)}$	$r=1$	-0.818732957	0.78×10^{-1}	-0.818733468	0.51×10^{-6}

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DIFFERENTIAL INEQUALITIES AND BOUNDEDNESS PRESERVING INTEGRAL OPERATORS

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Abstract. Let p be analytic in the unit disc U and let $L > 0$, $P, Q, R : U \to \mathbb{C}$. The authors determine an appropriate $M > 0$ such that

$$|P(z) \cdot zp'(z) + Q(z) \cdot p(z) + R(z)| < L \;\Rightarrow\; |p(z)| < M.$$

They also consider integral operators of the form

$$I[f](z) = z^{-\gamma}\phi(z)^{-1} \int_0^z [f(t) + w(y)]\psi(t)t^{\gamma-1}dt,$$

and show, with appropriate conditions, that these operators preserve boundedness.

1. Introduction

Let $\mathbf{H}_0$ denote the set of functions f that are analytic in the unit disc U with $f(0) = 0$. If $\in \mathbf{H}_0$ and $|g(z)| < 1$, for $z \in U$, then by using the Schwarz lemma it can be shown that the function p defined by

$$p(z) = z^{-1/2} \int_0^z g(t)t^{-1/2}dt$$

is in $\mathbf{H}_0$ and satisfies $|p(z)| < 2/3$, for $z \in U$. This result can be written in terms of derivatives as

$$|zp'(z) + p(z)/2| < 1, \text{ for } z \in U, \quad \Rightarrow \quad |p(z)| < 2/3, \text{ for } z \in U. \tag{1}$$

All of the inequalities in this article involving functions of z, such as (1), hold uniformly in U. The condition "$z \in U$" will be omitted in the remainder, although it is understood to hold.

In this article we consider generalizations of (1) of the form

$$|P(z) \cdot zp'(z) + Q(z) \cdot p(z) + R(z)| < L \quad \Rightarrow \quad |p(z)| < M, \tag{2}$$

where $L > 0$, $M > 0$ and P, Q, R are functions defined on U. In particular, for a given P, Q and R we shall determine an $M = M(P, Q, R, L)$ such that (2) holds. These results

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will be presented in Section 2. Integral analogues of (2), which deal with integral operators that preserve boundedness, will be presented in Section 3.

2. Differential inequalities

We begin with a lemma that will be used in the proof of the main theorem of this section. A more general form of this lemma appears in [1, p. 158].

Lemma 1. *Let $p \in \mathbf{H}_0$, and let $q(z) = Mz$, where $M > 0$. If $|p(z)| \not< M$, then there exist points $z_0 \in U$, $\zeta \in \partial U$, and $m \geq 1$ such that $p(|z| < |z_0|) \subset q(U)$,*

(a) $p(z_0) = q(\zeta) = M\zeta$, and

(b) $z_0 p'(z_0) = m\zeta q'(\zeta) = mM\zeta$.

Theorem 1. *Let $L < 0$, $M > 0$ and let $P, Q, R : U \to \mathbf{C}$, with $P(z) \neq 0$. In addition, suppose that*

(i) $\mathrm{Re}[Q(z)/P(z)] \geq -1$ and

(ii) $|P(z) + Q(z)| \geq [L + |R(z)|]/M$.

If $p \in \mathbf{H}_0$ and

$$|P(z) \cdot zp'(z) + Q(z) \cdot p(z) + R(z)| < L, \tag{3}$$

then $|p(z)| < M$.

Proof. Note that (3) requires that $|R(0)| < 1$. Assume that $|p(z)| \not< M$ and let

$$W(z) \equiv P(z) \cdot zp'(z) + Q(z) \cdot p(z) + R(z). \tag{4}$$

According to Lemma 1 there exist points $z_0 \in U$ and $\zeta \in \partial U$, and an $m \geq 1$ such that $p(z_0) = M\zeta$ and $z_0 p'(z_0) = mM\zeta$. Using these conditions and (4) we obtain

$$\begin{aligned} |W(z_0)|^2 &= |mM\zeta P(z_0) + M\zeta Q(z_0) + R(z_0)|^2 = \\ &= |M[mP(z_0) + Q(z_0)] + \overline{\zeta} R(z_0)|^2 = \\ &\geq (M|mP(z_0) + Q(z_0)| - |R(z_0)|)^2. \end{aligned} \tag{5}$$

Since $m \geq 1$ and $P(z) \neq 0$, by condition (i) we have

$$|m + Q(z)/P(z)| \geq |1 + Q(z)/P(z)|$$

sau

$$|mP(z) + Q(z)| \geq |P(z) + Q(z)|.$$

Using this last result and condition (ii) in (5) we deduce that

$$|W(z_0)|^2 \geq (M|P(z_0) + Q(z_0)| - |R(z_0)|)^2 \geq L^2,$$

and $|W(z_0)| \geq L$. Since this contradicts (3) we obtain the desired result $|p(z)| < M$. □

Instead of prescribing the constant M in Theorem 1, in some cases we can use condition (ii) to determine an appropriate $M = M(P, Q, R, L)$ so that (3) implies $|p(z)| < M$. This can be accomplished by solving (ii) for M, and by then taking the supremum of the resulting function over U. If this supremum exists we have the following version of Theorem 1.

Corollary 1. *Let $P, Q, R : U \to \mathbf{C}$ with $P(z) \neq 0$ and*

$$\operatorname{Re}[Q(z)/P(z)] \geq -1. \tag{6}$$

If

$$M = \sup_{|z|<1} \left\{ \frac{L + |R(z)|}{|P(z) + Q(z)|} \right\} \tag{7}$$

and $p \in \mathbf{H}_0$ then

$$|P(z) \cdot zp'(z) + Q(z) \cdot p(z) + R(z)| < L \quad \Rightarrow \quad |p(z)| < M.$$

Note that for the example given in (1) we have $P(z) = 1$, $Q(z) = 1/2$ and $R(z) = 0$. In this case (7) leads to $M = 2/3$ as expected. A simple example using this corollary is as follows:

$$|(1 + z)zp'(z) + (1 - z)p(z) + e^z - 1| < 1 \;\Rightarrow\; |p(z)| < e/2.$$

Several, more interesting, examples will be given in the next section.

In the special case when $R(z) \equiv 0$, Theorem 1 and Corollary 1.1 can still be used. However, we can replace conditions (i) and (ii) of Theorem 1 by a single condition as follows.

Theorem 2. *Let $L > 0$, $M > 0$ and let $P, Q : U \to \mathbf{C}$, with $P(z) \neq 0$. If*

$$|\operatorname{Im} Q(z)/P(z)| \geq L/[M|P(z)|], \tag{8}$$

and $p \in \mathbf{H}_0$ then,

$$|P(z) \cdot zp'(z) + Q(z) \cdot p(z)| < L \;\Rightarrow\; |p(z)| < M.$$

Proof. Following the proof of Theorem 1, from (5) we obtain

$$|W(z_0)|^2 - L^2 \geq (M^2|P(z_0)|^2 m^2 + (2M^2 \operatorname{Re} \overline{P(z_0)}Q(z_0))m + (M^2|Q(z_0)|^2 - L^2).$$

Condition (8) can be used to show that this quadratic in m has a non-positive discriminant. Since the coefficient of m^2 is positive we must have $|W(z_0)|^2 - L^2 \geq 0$, or $|W(z_0)| \geq L$. The rest of the proof follows as in Theorem 1. □

We can use (8) to determine a bound M on $|p(z)|$. In analogue with the case $R \not\equiv 0$ (Corollary 1.1) we have

Corollary 2. *Let $L > 0$ and let $P, Q : U \to \mathbf{C}$, with $P(z) \neq 0$. If*

$$M = \sup_{|z|<1} \left\{ \frac{L}{|P(z)||\operatorname{Im} Q(z)/P(z)|} \right\} \tag{9}$$

and $p \in \mathbf{H}_0$, then

$$|P(z) \cdot zp'(z) + Q(z) \cdot p(z)| < L \Rightarrow |p(z)| < M.$$

When $R \equiv 0$, condition (8) of Theorem 2 implies condition (ii) of Theorem 1 since

$$|P + Q|/|P| \geq |\operatorname{Im} Q/P| \geq L/(M|P|). \tag{10}$$

It does not imply condition (i) of Theorem 1. Also in this case we can see from (10) that the bound M obtained from (9) will be greater than or equal to the bound M obtained from (7). However, (7) requires that condition (6) be satisfied, while (9) requires no such condition. As a simple example consider the differential inequality

$$|zp'(z) + (3i + 2z)p(z)| < 1.$$

Corollary 1 can not be applied since $\operatorname{Re} Q(z)/P(z) = \operatorname{Re}[3i+2z] \not> -1$. However, Corollary 2 can be applied to obtain

$$M = \sup_{|z|<1} \left\{ \frac{1}{|\operatorname{Im}[3i + 2z]|} \right\} = 1,$$

and $|p(z)| < 1$.

3. Boundedness preserving integral operators

In this section we obtain integral analogues of the results of Section 2.

Theorem 3. *Let γ be a complex number and $w \in \mathbf{H}_0$. Let ϕ and ψ be analytic in U with $\phi(z)/z^n \neq 0$, $\psi(z)/z^n/ne0$ (n a non-negative integer) and*

$$\mathrm{Re}[z\phi'(z)/\phi(z) + \gamma + 1] > 0. \tag{11}$$

If $f \in |bfH_0$, and if the function F is defined by

$$F(z) = z^{-\gamma}\phi(z)^{-1}\int_0^z [f(y) + w(t)]\psi(t)t^{\gamma-1}dt, \tag{12}$$

then $F \in \mathbf{H}_0$, and

$$|f(z)| < L \;\Rightarrow\; |F(z)| < M,$$

where

$$M = M(\gamma,\phi,\psi,w,L) = \sup_{|z|<1}\left\{\left|\frac{\psi(z)}{\phi(z)}\right|\frac{L + |w(z)|}{|z\phi'(z)/\phi(z) + \gamma + 1|}\right\}. \tag{13}$$

Proof. Note that (11) requires that the constant γ satisfies $\mathrm{Re}\,\gamma > -n-1$. This restriction on γ together with the conditions on ϕ, ψ, w and f imply that F is analytic in U and $F(0) = 0$.

If we let $P(z) = \phi(z)/\psi(z)$, $Q(z) = [\gamma\phi(z) + z\phi'(z)]/\psi(z)$ and $R(z) = -w(z)$, then $P(z) \neq 0$. Conditions (11) and (13) are equivalent to conditions (6) and (7) respectively of Corollary 1. Since $|f(z)| < L$, if we differentiate (12) we obtain

$$|P(z)\cdot zF'(z) + Q(z)\cdot F(z) + R(z)| = |f(z)| < L.$$

Hence all conditions of Corollary 1 are satisfied with $p = F$ and we obtain $|F(z)| < M$. □

Note that by Schwarz's lemma the conclusion of Theorem 3 can be rewritten as

$$|f(z)| \leq L|z| \;\Rightarrow\; |F(z)| \leq M|z|.$$

We next refine the conclusion of Theorem 3 by carefully selecting the functions ϕ, ψ and w. If $g \in \mathbf{H}_0$ and if we let $\phi(x) = g(z)/z$ and $\psi(z) = g'(z)$ in Theorem 3 we obtain the following corollary.

Corollary 3. *Let γ be a complex constant and $w \in \mathbf{H}_0$. Let $g \in \mathbf{H}_0$ with $g'(z) \neq 0$ and*

$$\mathrm{Re}[zg'(z)/g(z) + \gamma] > 0. \tag{14}$$

If $f \in \mathbf{H}_0$ and if the function F is defined by

$$F(z) = I[f](z) = z^{1-\gamma}g(z)^{-1}\int_0^z [zf(t) + w(t)]g'(t)t^{\gamma-1}dt, \tag{15}$$

then $F \in \mathbf{H}_0$, and $|f(z)| < L$ implies $|F(z)| < M$, where

$$M = \sup_{|z|<1}\left\{\frac{L + |w(z)|}{|1 + \gamma g(z)/(zg'(z)|}\right\}. \tag{16}$$

Condition (14) requires that $g(z)/z \neq 0$ and that $\operatorname{Re}\gamma > -1$. The conclusion of this corollary can be written as

$$|f(z)| \leq |z| \;\Rightarrow\; \left|\int_0^z [f(t) + w(t)]g'(t)t^{\gamma-1}dt\right| \leq M|z^\gamma g(z)|.$$

Example 1. Let γ be real, with $\gamma > 0$, and $w(z) \equiv 0$. Let $g(z) = ze^{\lambda z}$, with $|\lambda| \leq 1$. In this case $g'(z) \neq 0$,

$$\operatorname{Re}[zg'(z)/g(z) + \gamma] = \operatorname{Re}[1 + \lambda z + \gamma] > 0,$$

and from (16) we deduce

$$M = \sup_{|z|<1}\left|\frac{L + \lambda z}{1 + \gamma + \lambda z}\right| = \frac{L + |\lambda|}{1 + \gamma + |\lambda|}. \tag{17}$$

Hence for $|\lambda| \leq 1$ and $\gamma > 0$, from (15) and Corollary 3.1, we have

$$|f(z)| \leq L|z| \;\Rightarrow\; \left|z^{-\gamma}e^{-\lambda z}\int_0^z f(t)(1 + \lambda t)e^{\lambda t}t^{\gamma-1}dt\right| \leq M|z|,$$

where M is given by (17). In the special case $L = \gamma = \lambda = 1$ this simplifies to

$$|f(z)| \leq |z| \;\Rightarrow\; \left|z^{-1}e^{-z}\int_0^z f(t)(1 + t)e^t dt\right| \leq \frac{2}{3}|z|.$$

For our next result we let $\gamma = 1$ and $\psi \equiv \phi$ in Theorem 3 to obtain the following corollary.

Corollary 4. *Let w and ϕ be analytic in U with $w(0) = 0$, $\phi(z) \neq 0$ for $z \neq 0$, and*

$$\operatorname{Re}[z\phi'(z)/\phi(z)] > -2.$$

If $f \in \mathbf{H}_0$ and if F is given by

$$F(z) = I[f](z) = z^{-1}\phi(z)^{-1}\int_0^z [f(z) + w(t)]\phi(t)dt \tag{18}$$

then $F \in \mathbf{H}_0$ and

$$|f(z)| \leq L|z| \;\Rightarrow\; |F(z)| \leq M|z|,$$

where

$$M = \sup_{|z|<1} \left\{ \frac{L + |w(z)|}{2 + z\phi'(z)/phi(z)|} \right\}. \tag{19}$$

Example 2. Let $\phi(z) = e^{\lambda z^2}$, with $|\lambda| < 1$ and let $w(z) = \mu z/(1 + \lambda z^2)$. Then $\mathrm{Re}[z\phi'(z)/\phi(z)] = 2\,\mathrm{Re}[\lambda z^2] > -2$ and from (19) we obtain

$$M = \sup_{|z|<1} \left\{ \frac{L + |\mu z|/|1 + \lambda z^2|}{2|1 + \lambda z^2|} \right\} = \frac{L - |\lambda| + |\mu|}{2(1 - |\lambda|)^2}. \tag{20}$$

Hence for $|\lambda| < 1$ we obtain

$$|f(z)| \leq L|z| \;\Rightarrow\; \left| e^{-\lambda z^2} \int_0^z [f(t) + \mu t/(1 + \lambda t^2)] e^{\lambda t^2} dt \right| \leq M|z|^2,$$

where M is given by (22). For $\lambda = \mu = 1/2$ and $L = 1$ we have

$$|f(z)| \leq |z| \;\Rightarrow\; \left| e^{-z^2/2} \int_0^z [f(t) + t/(2_t^2)] e^{t^2/2} dt \right| \leq 2|z|^2.$$

Just as Theorem 3 is the integral analogue of Corollary 1, we state, without proof, the integral analogue of Corollary 2.

Theorem 4. *Let ϕ and ψ be analytic in U with $\phi(z)/z^n \neq 0$ and $\psi(z)/z^n \neq 0$ (n a non-negative integer), and let γ be a complex number such that* $\mathrm{Re}\,\gamma > -n-1$. *If $f \in \mathbf{H}_0$, and if the function F is defined by*

$$F(z) = I[f](z) = z^{-\gamma}\phi(z)^{-1} \int_0^z f(t)\psi(t)t^{\gamma-1} dt,$$

then $F \in \mathbf{H}_0$ and

$$|f(z)| < L \;\Rightarrow\; |F(z)| < M,$$

where

$$M = M(\gamma, \phi, \psi) = \sup_{|z|<1} \left\{ \left| \frac{\psi(z)}{\phi(z)} \right| \frac{L}{|\,\mathrm{Im}[\gamma + z\phi'(z)/\phi(z)]|} \right\}. \tag{21}$$

If we compare Theorem 4 with Theorem 3 when $w \equiv 0$, we see that the constant M as given by (13) will be less than or equal to the constant M as given by (23). However, in order to use (13) we need to also satisfy condition (11). The following example illustrates a case when condition (11) is not satisfied, but in which we can apply Theorem 4.

Example 3. Let $f \in \mathbf{H}_0$ and consider the function

$$F(z) = I[f](z) = z^{-\gamma} e^{-\lambda z} \int_0^z f(t) e^{\lambda t} t^{\gamma-1} dt,$$

where $\gamma = \alpha + i\beta$, $\beta > 0$ and $0 < 1 + \alpha < |\lambda| < \beta$. The function F is well-defined with $F \in |bfH_0$. If we try to apply Theorem 3 or Theorem 4 to this operator we need to take $\phi(z) = \psi(z) = e^{\lambda z}$. Unfortunately, Theorem 3 can not be used because condition (11) is not satisfied, as can be seen from

$$\operatorname{Re}[z\phi'(z)/\phi(z) + \gamma + 1] = \operatorname{Re}[\lambda z] + 1 + \alpha \not> 0.$$

However, Theorem 4 can be applied since there is no such supplementary condition required. In this case, from (21) we obtain

$$M = \sup_{|z|<1} \left\{ \frac{1}{|\operatorname{Im}(\lambda z) + \beta|} \right\} = \frac{L}{\beta - |\lambda|}.$$

Hence

$$|f(z)| \le L|z| \Rightarrow \left| z^{-\gamma} e^{-\lambda z} \int_0^z f(t) t^{\gamma - 1} e^{\lambda t} dt \right| \le L|z|/(\beta - |\lambda|).$$

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NEWTON'S METHOD FOR NONLINEAR DIFFERENTIAL EQUATIONS WITH LINEAR DEVIATING ARGUMENT

VIORICA MUREŞAN AND DAMIAN TRIF

Dedicated to Professor Ioan A. Rus on his 60th anniversary

Abstract. In this paper we describe and we show the convergence of an algorithm for solving some nonlinear equations with delay.

1. Introduction

J. Terjéki in [2], 1995, gives an algorithm for solving a differential linear homogeneous equation with delay and with variable coefficients.

Terjéki's method can be extended to the differential linear nonhomogeneous equations and so it can be included in a Newton type algorithm for solving some nonlinear equations with delay.

In the present paper we describe and we show the convergence of such algorithms and we give a numerical example in that the linear differential problems are solved with the power series method. We also give the graphics of the solutions and of the approximation error.

2. An algorithm for linear homogeneous equations with delay [2]

Let be the equation

$$x'(t) = A(t)x(t) + B(t)x(\theta(t)), t \geq 0 \tag{1}$$

where A, B are continuous (complex or real) $n \times n$ matrix functions on $\mathbb{R}_+$, the lag function θ is continuous and $0 \leq \theta(t) \leq t$. The following fundamental result takes place:

Theorem 1. *If y_0 denote an arbitrary solution of equation*

$$y'(t) = A(t)y(t)$$

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and the sequence $(y_n)_{n\geq 1}$ *is defined by*

$$\begin{aligned} y_n'(t) &= A(t)y_n(t) + B(t)y_{n-1}(\theta(t)) \\ y_n(0) &= 0 \end{aligned}$$

then x, *where*

$$x(t) = \sum_{n=0}^{\infty} y_n(t)$$

is a solution of equation (1). Moreover, this series absolutely and uniformly converges on every finite subinterval of $[0,\infty[$.

3. An algorithm for linear nonhomogeneous equations with delay

Let us consider the following Cauchy problem

$$\begin{aligned} y'(x) + p(x)y(x) + q(x)y(g(x)) &= v(x), x \in [0,T] \\ y(0) &= c \end{aligned} \tag{2}$$

We have

Theorem 2. *Suppose that the following conditions hold*

(i) $p,q,g,v \in C[0,T]$ *and* $0 \leq g(x) \leq x$ *for every* $x \in [0,T]$

(ii) y_0 *is the solution of the problem*

$$\begin{aligned} y'(x) + p(x)y(x) &= v(x), x \in [0,T] \\ y(0) &= c \end{aligned} \tag{3}$$

(iii) the sequence $(y_n)_{n\geq 1}$ *is defined by solutions of the following problems:*

$$\begin{aligned} y_n'(x) + p(x)y_n(x) + q(x)y_{n-1}(g(x)) &= 0, x \in [0,T] \\ y_n(0) &= 0 \end{aligned} \tag{4}$$

Then the function y *defined by* $y(x) = \sum_{n=0}^{\infty} y_n(x)$ *is the solution of the problem (2), the series being absolutely and uniformly convergent on* $[0,T]$.

Proof. The solution of the problem (3) is given by

$$y_0(x) = ce^{-\int_0^x p(s)ds} + \int_0^x Y(x,s)v(s)ds$$

where

$$Y(x,s) = e^{-\int_0^x p(u)du + \int_0^s p(u)du}$$

We define the operator $Y_* : C[0,T] \to C[0,T]$ by

$$Y_* v(x) = \int_0^x Y(x,s)v(s)ds$$

and then we have

$$y_0(x) = ce^{-\int_0^x p(s)ds} + Y_* v(x)$$

The solutions of the problems (4) are given by

$$y_n(x) = Y_*\left(-q(x)y_{n-1}(g(x))\right), n = 1,2,...$$

We denote $M_0 = \max_{x\in[0,T]} |y_0(x)|, Y_0 = \max_{x,s\in[0,T]} |Y(x,s)|, Y_1 = \max_{x,s\in[0,T]} \left|\frac{\partial Y}{\partial x}(x,s)\right|,\ P = \max_{x\in[0,T]} |p(x)|, Q = \max_{x\in[0,T]} |q(x)|$ and we have

$$|y_n(x)| \le M_0 (Y_0 Q)^n \frac{x^n}{n!} \le M_0 (Y_0 Q)^n \frac{T^n}{n!}$$

for every $x \in [0,T]$ and $n = 0,1,....$

Similarly it can be shown that

$$|y_n'(x)| \le M_0 (Y_1 Q)^n \frac{T^n}{n!} + QM_0 (Y_0 Q)^{n-1} \frac{T^{n-1}}{(n-1)!}, n = 1,2,...$$

It follows that the series $\sum_{n=0}^{\infty} y_n(x)$ and $\sum_{n=0}^{\infty} y_n'(x)$ are absolutely and uniform convergent on $[0,T]$. Therefore

$$y'(x) = y_0'(x) + \sum_{n=1}^{\infty} y_n'(x) = -p(x)y_0(x) + v(x) - \sum_{n=1}^{\infty} p(x)y_n(x) -$$
$$- \sum_{n=1}^{\infty} q(x)y_{n-1}(g(x)) = -p(x)y(x) - q(x)y(g(x)) + v(x)$$

□

4. An algorithm for nonlinear equations with linear deviating of the argument

Let $(Y, \|.\|)$ be a Banach space and T a diferentiable operator in Y. We consider the equation $Ty = 0$ with a solution y^*. We search a sufficiently good approximation for this solution.

Newton's sequence associated to the equation $Ty = 0$ is defined by

$$y_{m+1} = y_m - [T'(y_m)]^{-1} T(y_m), m = 0,1,...$$

where $y_0 \in Y$ and we suppose that $[T'(y_m)]^{-1}$ exists for $m = 0,1,....$

Since T' is a linear operator, the above relation can be written as follows

$$T'(y_m)(y_{m+1} - y_m) = -T(y_m), m = 0,1,...$$

Kantorovitch's theorem gives us sufficient conditions for the convergence of Newton's sequence to the exact solution y^* and also an evaluation of the approximation error:

Theorem 3. *(Kantorovitch) We suppose that*

(i) T' *is continuous and* $[T'(y_0)]^{-1}$ *exists for an* $y_0 \in Y$ *and* $\|[T'(y_0)]^{-1}\| \leq B_0$

(ii) $\|T''(y)\| \leq K$ *for* $y \in \overline{B}(y_0, r)$

(iii) $\|y_1 - y_0\| \leq \eta_0$, *where* $y_1 = y_0 - [T'(y_0)]^{-1} T(y_0)$

(iv) $B_0 K \eta_0 \leq \frac{1}{2}$

Then Newton's sequence converges to a solution y^* *of the equation* $Ty = 0$ *that belongs to* $\overline{B}(y_0, r_1)$, *where*

$$r_1 \geq \frac{1 - \sqrt{1 - 2B_0 K \eta_0}}{B_0 K}$$

We apply Newton's method to a nonlinear differential equation with linear deviating of the argument. Let us consider the following Cauchy problem

$$\begin{aligned} &y'(x) = f(x, y(x), y(\lambda x)), x \in [0, T], 0 < \lambda < 1 \\ &y(0) = c \end{aligned} \tag{5}$$

where $f \in C([0,T] \times \mathbb{R} \times \mathbb{R}), f = f(x, y, u)$ and it has continuous first and second order partial derivatives in y and u and $c \in \mathbb{R}$ is a given number.

This problem has an unique solution on $C^1[0,T]$. We consider the operator T : $C^1[0,T] \to C[0,T]$ defined by

$$T(y)(x) = \frac{dy}{dx} - f(x, y(x), y(\lambda x))$$

Let be $y_0 \in C^1[0,T]$. The Gateaux derivative $T'(y_0)$ is given by

$$T'(y_0)(h) = \lim_{t \to 0} \frac{T(y_0 + th) - T(y_0)}{t}$$

and we obtain

$$T'(y_0)(h)(x) = \frac{dh}{dx} - \frac{\partial f}{\partial y}(x, y_0(x), y_0(\lambda x)) h(x) - \frac{\partial f}{\partial u}(x, y_0(x), y_0(\lambda x)) h(\lambda x)$$

The operator $T'(y_0) : C^1[0,T] \to C[0,T]$ is linear and continuous and from Section 3 it follows that it is invertible with bounded inverse. Newton's sequence associated to the equation $Ty = 0$ is defined by

$$\begin{aligned} &T'(y_m)(y_{m+1} - y_m) = -T(y_m), m = 0, 1, \ldots \\ &y_m(0) = c \end{aligned}$$

where $y_0 \in C^1[0,T]$ is the first approximation of the solution. We usually consider $y_0(x) = c$.

We denote $u_m(x) = y_{m+1}(x) - y_m(x), p_m(x) = -\frac{\partial f}{\partial y}(x, y_m(x), y_m(\lambda x))$, $q_m(x) = -\frac{\partial f}{\partial u}(x, y_m(x), y_m(\lambda x))$, $v_m(x) = -T(y_m)(x) = -\frac{dy_m}{dx}(x) + f(x, y_m(x), y_m(\lambda x))$. Then we obtain the following Cauchy problems

$$\begin{aligned} &\tfrac{d}{dx}u_m(x) + p_m(x)u_m(x) + q(x)u_m(\lambda x) = v_m(x), x \in [0,T], 0 < \lambda < 1 \\ &u_m(0) = 0 \end{aligned} \tag{6}$$

for $m = 0, 1, \ldots$.

We can apply Theorem 2 to these problems and we have the following result:

If $u_{m,0}$ is solution of the problem

$$\begin{aligned} &\tfrac{d}{dx}u_{m,0} + p_m(x)u_{m,0}(x) = v_m(x), x \in [0,T] \\ &u_{m,0}(0) = 0 \end{aligned}$$

and the terms of the sequence $(u_{m,k})_{k\geq 1}$ are solutions of the following problems

$$\begin{aligned} &\tfrac{d}{dx}u_{m,k} + p_m(x)u_{m,k}(x) + q_m(x)u_{m,k-1}(\lambda x) = 0, x \in [0,T] \\ &u_{m,k}(0) = 0 \end{aligned}$$

for $k = 0, 1, \ldots$. Then the function u_m defined by

$$u_m(x) = \sum_{k=0}^{\infty} u_{m,k}(x)$$

is a solution of the problem 6 for $m = 0, 1, \ldots$, the series being absolutely and uniformly convergent on $[0,T]$. If we know the sequence $(u_m)_{m\geq 0}$ we can find Newton's sequence $(y_m)_{m\geq 0}$.

5. Numerical application

Let be the problem

$$\begin{aligned} &y'(x) = -y^2(x) - \tfrac{1}{4}y\left(\tfrac{x}{3}\right) + \tfrac{1}{6}\ln(x+1), x \in [0,1] \\ &y(0) = 0 \end{aligned} \tag{7}$$

Considering $y_0(x) \equiv 0$, the linear equations from Newton's method are

$$\begin{aligned} &\tfrac{d}{dx}u_m(x) + 2y_m(x)u_m(x) + \tfrac{1}{4}u_m\left(\tfrac{x}{3}\right) = v_m(x), x \in [0,1] \\ &u_m(0) = 0 \end{aligned}$$

where

$$v_m(x) = -\frac{d}{dx}y_m(x) - y_m^2(x) - \frac{1}{4}y_m\left(\frac{x}{3}\right) + \frac{1}{5}\ln(x+1)$$

By the power series method for these problems, with $O(x^{30})$ cut-off, the $y_1, \ldots, y_5$ from Newton's method were computed (Fig. 1). We remark that y_1, y_2, y_3, y_4, y_5 coincide in the picture. We obtained

$$\begin{aligned}
y_5(x) = {} & -0.000251204011558687x^{29} + 0.000270054521825534x^{28} - \\
& 0.000291113735062186x^{27} + 0.000314741477048363x^{26} - \\
& 0.000341374133306388x^{25} + 0.000371545114687104x^{24} - \\
& 0.000405912019237038x^{23} + 0.000445293122630196x^{22} - \\
& 0.000490717054542795x^{21} + 0.000543491412125040x^{20} - \\
& 0.000605299051459985x^{19} + 0.000678335579358926x^{18} - \\
& 0.000765509439599829x^{17} + 0.000870739740404521x^{16} - \\
& 0.000999409435061762x^{15} + 0.00115906824229100x^{14} - \\
& 0.00136057811310449x^{13} + 0.00162004039311335x^{12} - \\
& 0.00196176987753260x^{11} + 0.00242413713889046x^{10} - \\
& 0.00307396125494130x^{9} + 0.00401519041392206x^{8} - \\
& 0.00540841469240449x^{7} + 0.00781070136094290x^{6} - \\
& 0.0120103370188318x^{5} + 0.0167459705075447x^{4} - \\
& 0.0342592592592592x^{3} + 0.100000000000000x^{2}
\end{aligned}$$

The replacement error for y_5 into the equation (7) is

$$\begin{aligned}
err = {} & -0.00702790033127195x^{29} - 1.0\times10^{-17}x^{28} + 2.3\times10^{-17}x^{27} - \\
& 2.5\times10^{-17}x^{26} - 2.8\times10^{-17}x^{24} + 1.0\times10^{-17}x^{23} - 1.4\times10^{-17}x^{22} - \\
& 1.0\times10^{-17}x^{21} + 7.6\times10^{-17}x^{20} + 1.0\times10^{-16}x^{19} - 3.4\times10^{-17}x^{18} - \\
& 1.0\times10^{-16}x^{16} - 3.4\times10^{-17}x^{14} + 1.33\times10^{-16}x^{13} + 3.2\times10^{-16}x^{12} + \\
& 5.0\times10^{-17}x^{11} + 6.0\times10^{-17}x^{10} + 2.6\times10^{-16}x^{9} - 8.0\times10^{-17}x^{8} + \\
& 1.8\times10^{-16}x^{7} + 1.8\times10^{-16}x^{6} + 1.6\times10^{-16}x^{5} + 1.0\times10^{-16}x^{4} + \\
& 4.0\times10^{-16}x^{3}
\end{aligned}$$

and its graphic is given in fig. 2.

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NEWTON'S METHOD FOR NONLINEAR DIFFERENTIAL EQUATIONS

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CONTINUOUS SELECTIONS FOR MULTIVALUED OPERATORS WITH DECOMPOSABLE VALUES

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Dedicated to Professor Ioan A. Rus on his 60^{th} anniversary

1. Introduction

Consider $(T, \mathcal{A}, \mu)$ a complete σ-finite and nonatomic measure space, i.e. T is an arbitrary set, $\mathcal{A}$ is a complete σ-algebra of subsets of T and μ is a positive σ-finite and nonatomic measure on $\mathcal{A}$ (see [1]). If $(E, \|\cdot\|)$ is a Banach space, let $L^1(T, E)$ be the Banach space of all functions $u : T \to E$ which are Bochner μ-integrable [6].

We call a set $K \subset L^1(T, E)$ decomposable if for all $u, v \in K$ and $A \in \mathcal{A}$

$$u\chi_A + v\chi_{T\setminus A} \in K, \text{ for all } u, v \in K, \tag{1}$$

where χ_A stands for the characteristic function of set A.

Denote:

$P(E)$ the space of all nonempty subset of E;

$P_{cl}(E)$ the space of all nonepmty closed subsets of E;

$P_{cv}(E)$ the space of all nonempty convex subsets of E;

$P_{dec}(L^1(T, E))$ the space of all nonempty decomposable subsets of $L^1(T, E)$.

A classical theorem make use of a convexity assumption.

Theorem 1.1. ([1], p. 83). *Let F be a lower semicontinuous mapping with closed, convex values from a paracompact space X to a Banach space Y. Let $G : X \to P_{cv}(Y)$ be a multivalued mapping with open graph. If $F(x) \cap G(x) \neq \emptyset$, for all $x \in X$ then there exists a continuous selection of $F \cap G$.*

The purpose of this paper is to show that in the case where $Y = L^1(T, E)$ then an analogue of the above theorem holds with the convexity assumption remplaced by decomposability.

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The first result concerning the existence of a continuous selection for a Hausdorff-continuous multifunction with decomposable values is due to Antosiewicz and Cellina [2]. Their selection therem yields the existence of solutions for the differential inclusions $x' \in F(t,x)$ with Hausdorff continuous right-hand side.

In 1983 A. Fryszkowski stated a general selection theorem for lower semicontinuous multivalued mapping from a compact metric space X to the Banach space $L^1(T,E)$ with decomposable values [4]. More recently A. Bressan and A. Cellina extended this result to the case of a lower semi-continuous multivalued mapping defined on a separable metric space X.

Their theorem will be considered here:

Theorem 1.2 ([3]). *Let (X,d) be a separable metric space and let $F : X \to P_{cl,dec}(L^1(T,E))$ be a lower semi-continuous multivalued mapping. Then F has a continuous selection.*

The main result of this paper is given in Section 3. Section 2 contains notations and basic definitions.

2. Notations and basic definitions

Throughout this paper $(T,\mathcal{A},\mu)$ denotes a measure space, such that $\mathcal{A}$ is a complete σ-algebra of subsets of T and μ is a positive σ-finite and nonatomic measure on $\mathcal{A}$. If $(E,\|\cdot\|)$ is a Banach space, $L^1(T,E)$ denotes the Banach space of Bochner μ-integrable functions $u : T \to E$ with norm $\|u\|_1 = \int_T \|u\| d\mu$. Let (X,d) be a separable metric space.

Definition 2.1 ([1]). A multivalued operator $F : X \to P(E)$ is lower semicontinuous (l.s.c.) iff the set $\{x \in X : F(x) \subset C\}$ is closed for every closed set $C \subset Y$.

Definition 2.2 ([1]). A multivalued operator $F : X \to P(E)$ is locally selectionable at $x_0 \in X$ if for all $y_0 \in F(x_0)$ there exist an open neighborhood $N(x_0)$ of x_0 and a continuous map $f : N(x_0) \to E$ such that

$$f(x_0) = y_0 \text{ and for all } x \in N(x_0) \qquad f(x) \in F(x) \tag{2}$$

F is said to be locally selectionable if it is locally selectionable at every $x_0 \in X$.

Definition 2.3 ([1]). Let $F : X \to P(E)$ be a multivalued operator. We say that a singlevalued mapping $f : X \to E$ is a continuous selection for F iff f is continuous and

$$f(x) \in F(x), \text{ for every } x \in X. \tag{3}$$

Remark 2.1 ([1]). Any locally selectionable multivalued operator is l.s.c.

3. Basic results

We begin with the following:

Lemma 3.1. *Let (X,d) be a separable metric space, $(T,\mathcal{A},\mu)$ be a complete σ-finite and nonatomic measure space and $(E,\|\cdot\|)$ be a Banach space. Let $F : X \to P_{dec}(L^1(T,E))$ be a locally selectionable multivalued operator. Then F has a continuous selection.*

Proof. We associate with any $y \in X$ and $z \in F(y)$ an open neighborhood $N(y)$ and a local continuous selection $f_y : N(y) \to L^1(T,E)$ satisfying $f_y(y) = z$ and $f_y(x) \in F(x)$, when $x \in N(y)$.

Since X is separable metric space, there exists a countable locally finite open refinement of the open covering $\{N(y) : y \in X\}$.

We denote by $\{V_n : n \geq 1\}$ this refinement. Using Theorem 2 p.10 from [1] it follows that there exists a continuous partition of unity $\{\psi_n : n \geq 1\}$ associated with the countable locally finite refinement $\{V_n : n \geq 1\}$.

Then, for each $n \geq 1$ there exist an element $y_n \in X$ such that $V_n \subset N(y_n)$ and a continuous function $f_{y_n} : N(y_n) \to L^1(T,E)$ with $f_{y_n}(y_n) = z_n$, $f_{y_n}(x) \in F(x)$, for all $x \in N(y_n)$. We define $\lambda_0(x) = 0$ and $\lambda_n(x) = \sum_{m \leq n} \psi_m(x)$; $n \geq 1$. Let $g \in L^1(T,\mathbf{R}_+)$ be a function defined by $g(t) = \|z_n(t) - z_m(t)\|$, for each $m,n \geq 1$.

We can arrange these functions into a sequence $\{g_k : k \geq 1\} \subset L^1(T,\mathbf{R}_+)$. Consider the function $\tau(x) = \sum_{m,n\geq 1} \psi_m(x)\cdot\psi_n(x)$.

Using Lemma 1 from [3] it results that there exist a family $\{\phi(\tau,\lambda)\}$ of measurable subsets of T with the properties:

(a) $\phi(\tau,\lambda_1) \subseteq (\tau,\lambda_2)$ if $\lambda_1 \leq \lambda_2$

(b) $\mu(\phi(\tau_1,\lambda_1)\Delta\phi(\tau_2,\lambda_2)) \leq |\lambda_1 - \lambda_2| + 2|\tau_1 - \tau_2|$

(c) $\int_{\phi(\tau,\lambda)} g_n d\mu = \lambda \int_T g_n d\mu, \ \forall\, n \leq \tau$

for all $\lambda,\lambda_1,\lambda_2 \in [0,1]$, $\tau,\tau_1,\tau_2 \geq 0$.

Define $f_n(x) = f_{y_n}(x)$, $\forall\, n \geq 1$ and $\chi_n(x) = \chi_{\phi(\tau(x),\lambda_n(x))\setminus\phi(\tau(x),\lambda_{n-1}(x))}$, $\forall\, n \geq 1$.

Construct a singlevalued mapping $f : X \to L^1(T,E)$ defined by $f(x) = \sum_{n\geq 1} f_n(x)\chi_n(x)$, for all $x \in X$. Clearly f is continuous, because the above summation is locally finite. Moreover, by the decomposability assumption we have that $f(x) \in F(x)$, for all $x \in X$. □

Lemma 3.2 ([1], p.81). *Let $F : X \to P(E)$ be locally selectionable at $x_0 \in X$. Let $G : X \to P(E)$ a multivalued mapping with open graph.*

If $F(x_0) \cap G(x_0) \neq \emptyset$ then $F \cap G$ is locally selectionable at x_0.

The main result of this paper is the following:

Theorem 3.1. *Let (X, d) be a separable metric space, $F : X \to P_{cl,dec}(L^1(T, E))$ be a lower semicontinuous multivalued operator and $G : X \to P_{dec}(L^1(T, E))$ be with open graph.*

If $F(x) \cap G(x) \neq \emptyset$, for each $x \in X$ then there exists a continuous selection of $F \cap G$.

Proof. Let $x_0 \in X$ an arbitrary element. For each $y_0 \in F(x_0)$ we consider the multivalued operator given by:

$$F_0(x) = \begin{cases} \{y_0\}, & \text{if } x = x_0 \\ F(x), & \text{if } x \neq x_0. \end{cases}$$

Obviously $F_0 : X \to P_{cl,dec}(L^1(T, E))$ is l.s.c. Using Theorem 1.2, we find a continuous selection of F_0, i.e. $f_0(x_0) = y_0$ and $f_0(x) \in F(x)$, for all $x \in X$ with $x \neq x_0$. Using now Lemma 3.2, it follows that $F \cap G$ is locally selectionable at x_0, with decomposable values.

Lema 3.1 implies the existence of a continuous selections of $F \cap G$. □

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CONTINUATION THEOREMS FOR MAPPINGS OF CARISTY TYPE

RADU PRECUP

Dedicated to Professor Ioan A. Rus on his 60^{th} *anniversary*

Abstract. In this paper we prove some continuation theorems for mappings of Caristi type. Our results generalize Caristi's fixed point theorem and the continuation theorem for contractions.

1. Preliminaries

We first recall Ekeland's variational principle.

Theorem 1. *Let M be a complete metric space and let $\Phi : M \to]-\infty, +\infty]$ be a lower semicontinuous function, bounded from below and not identical to $+\infty$. Let $\varepsilon > 0$ be given and $x \in M$ be such that*

$$\Phi(x) \le \inf_M \Phi + \varepsilon.$$

Then there exists $y \in M$ such that

$$\Phi(y) \le \Phi(x), \quad d(x,y) \le 1$$

and, for each $z \neq y$ in M,

$$\Phi(z) \;>\; \Phi(y) - \varepsilon d(y,z).$$

For the proof see, for example, [4]. The following famous fixed point theorem due to Caristi [1] (see also [4], [7]), is a simple consequence of Ekeland's principle.

Theorem 2. *Let M be a complete metric space, $\varphi : M \to R_+$ a lower semicontinuous function and $T : M \to M$ a mapping such that*

$$d(x, T(x)) \le \varphi(x) - \varphi(T(x)) \tag{1}$$

for each $x \in M$. Then T has at least one fixed point.

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For the proof, apply Theorem 1 with $\varepsilon = 1/2$ to get $y \in M$ such that

$$(1/2)d(y, T(y)) \geq \varphi(y) - \varphi(T(y)).$$

This, by (1), yields $(1/2)d(y, T(y)) \geq d(y, T(y))$ whence $T(y) = y$.

Remark 1. *If $T : M \to M$ is a contraction, that is*

$$d(T(x), T(y)) \leq \alpha d(x, y)$$

for some $\alpha \in [0, 1[$ and all $x, y \in M$, then T satisfies (1) with $\varphi(x) = (1-\alpha)^{-1}d(x, T(x))$. Thus, Caristi's theorem is a generalization of Banach's fixed point theorem. Nevertheless, a mapping satisfying (1) can be not continuous. For an example, take $M = R_+$, $\varphi(x) = x$, $T(x) = x$ for $0 \leq x < 1$ and $T(x) = x - 1$ for $1 \leq x < +\infty$.

Recently, Granas [3] and Frigon-Granas [2] proved some continuation theorems of Leray-Schauder for contractions in complete metric spaces. Also, in [5], we have obtained some improvements of the continuation principle for nonexpansive mappings, while in [6], we have presented a very general continuation principle. Motivated by these results, in this paper we shall state and prove continuation theorems for mappings of Caristi type.

2. Main results

Theorem 3. *Let M be a complete metric space, $X \subset M$ a closed nonempty set, $\psi : X \times [0, 1] \to R_+$ a lower semicontinuous function and $N : X \times [0, 1] \to M$ a mapping. Let X_λ be the biggest subset invariated by $N_\lambda = N(., \lambda)$, i.e.*

$$X_\lambda = \cap\{(N_\lambda^k)^{-1}(X);\ k = 1, 2, \ldots\}.$$

Suppose that

(i) $d(x, N_\lambda(x)) \leq \psi_\lambda(x) - \psi_\lambda(N_\lambda(x))$ *for all $x \in X_\lambda$ and $\lambda \in [0, 1]$, where $\psi_\lambda = \psi(., \lambda)$;*

(ii) *there is a closed nonempty set $S \subset \{(x, \lambda) \in X \times [0, 1],\ x \in X_\lambda\}$ such that:*

if $(x_0, \lambda_0) \in S$ and $\lambda_0 < 1$, then there exists $(x, \lambda) \in S$ such that

$$\lambda_0 < \lambda, \quad d(x_0, x) \leq \psi_{\lambda_0}(x_0) - \psi_\lambda(x), \tag{2}$$

$$(N_1(x_0), 1) \in S \text{ whenever } (x_0, 1) \in S. \tag{3}$$

Then, if N_0 has a fixed point x with $(x, 0) \in S$, N_1 also has a fixed point.

Proof. We define an order relation on S, namely

$$(x,\lambda) \preceq (y,\eta) \text{ if } \lambda \leq \eta \text{ and } d(x,y) \leq \psi_\lambda(x) - \psi_\eta(y).$$

Let us show that Zorn's lemma is applicable. Suppose $S_0 \subset S$ is a totally ordered set and denote

$$\psi^* = \inf\{\psi_\lambda(x);\ (x,\lambda) \in S_0\}.$$

Consider a sequence $(x_n,\lambda_n) \in S_0$ such that $\psi_{\lambda_n}(x_n)$ decreases to ψ^* as $n \to \infty$. Then, since S_0 is totally ordered, we have

$$(x_1,\lambda_1) \preceq (x_2,\lambda_2) \preceq \ldots \preceq (x_n,\lambda_n) \preceq \ldots .$$

From

$$d(x_n, x_{n+p}) \leq \psi_{\lambda_n}(x_n) - \psi_{\lambda_{n+p}}(x_{n+p}) \to 0 \text{ as } n \to \infty,$$

uniformly with respect to p, it follows that there exists $x^* \in X$ such that $x_n \to x^*$. Denote $\lambda_* = \lim \lambda_n$. Since ψ is lower semicontinuous, we then have $\psi_{\lambda_*}(x^*) = \psi^*$. S being closed, $(x^*,\lambda_*) \in S$. In addition, $(x_n,\lambda_n) \preceq (x^*,\lambda_*)$. Two cases are possible:

Case 1. There is no $(x,\lambda) \in S_0$ with $(x^*,\lambda_*) \prec (x,\lambda)$. Then (x^*,λ_*) is a upper bound for S_0. Indeed, let (x,λ) be any element of S_0.

a) if $(x,\lambda) \preceq (x_n,\lambda_n)$ for some n, then, since $(x_n,\lambda_n) \preceq (x^*,\lambda_*)$, it clearly follows $(x,\lambda) \preceq (x^*,\lambda_*)$.

b) if $(x_n,\lambda_n) \prec (x,\lambda)$ for some n, then obviously, $\psi_\lambda(x) = \psi^*$ and $\lambda_* \leq \lambda$. If we would have $\lambda_* < \lambda$, then $(x^*,\lambda_*) \prec (x,\lambda)$, which has been excluded by the beginning. Hence $\lambda_* = \lambda$. On the other hand, $d(x_n,x) \leq \psi_{\lambda_n}(x_n) - \psi_\lambda(x) \to 0$, and so $x^* = x$. Therefore $(x,\lambda) = (x^*,\lambda_*)$.

Case 2. There is $(x,\lambda) \in S_0$ with $(x^*,\lambda_*) \prec (x,\lambda)$. Then, $x = x^*$. Let

$$\lambda^* = \sup\{\lambda;\ (x^*,\lambda) \in S_0,\ (x^*,\lambda_*) \preceq (x^*,\lambda)\}.$$

We have $\lambda_* < \lambda^* \leq 1$. Let us consider a sequence $(x^*,\lambda'_n) \in S_0$ such that λ'_n increases to λ^* and $(x^*,\lambda_*) \preceq (x^*,\lambda'_n)$. The element (x^*,λ^*) is an upper bound for S_0. Indeed, let $(x,\lambda) \in S_0$.

a) if $(x,\lambda) \preceq (x^*,\lambda'_n)$ for some n, then clearly, $(x,\lambda) \preceq (x^*,\lambda^*)$.

b) if $(x^*,\lambda'_n) \prec (x,\lambda)$ for every n, then $x = x^*$ and $\lambda > \lambda'_n$, whence $\lambda \geq \lambda^*$. Consequently, $\lambda = \lambda^*$.

Therefore we can apply Zorn's lemma and obtain a maximal element $(x_0, \lambda_0) \in S$. According to (ii), $\lambda_0 = 1$ and $x_0 \in X_1$. Now, using (i) and (3), we get $(x_0, 1) \preceq (N_1(x_0), 1)$ whence, due to the maximality of $(x_0, 1)$, $x_0 = N_1(x_0)$. □

Theorem 3 together with Theorem 2 immediately yield the following result for continuous mappings N.

Corollary 1. *Let M be a complete metric space, $X \subset M$ a closed set, $\psi : X \times [0,1] \to R_+$ a lower semicontinuous function and $N : X \times [0,1] \to M$ a continuous mapping. Suppose*

1) $d(x, N_\lambda(x)) \leq \psi_\lambda(x) - \psi_\lambda(N_\lambda(x))$ for all $x \in X_\lambda$ and $\lambda \in [0,1]$;

2) if $N_{\lambda_0}(x_0) = x_0$ and $\lambda_0 < 1$, there exists $\lambda \in]\lambda_0, 1[$ such that $x_0 \in X_\lambda$ and $\psi_\lambda(x_0) \leq \psi_{\lambda_0}(x_0)$.

Then, if $X_0 \neq \emptyset$, each mapping N_λ, $\lambda \in [0,1]$, has at least one fixed point.

Proof. In order to apply Theorem 3, take

$$S = \{(x, \lambda) \in X \times [0,1];\ N_\lambda(x) = x\}.$$

Since N is continuous, the sets S and X_λ are closed. Hence X_0 is a closed nonempty subset of M. In addition, $N_0(X_0) \subset X_0$. Consequently, by Theorem 2, there exists x with $N_0(x) = x$. It remains to show (2). For this, suppose $(x_0, \lambda_0) \in S$ and $\lambda_0 < 1$. By 2), $x_0 \in X_\lambda$ for some $\lambda \in]\lambda_0, 1[$. Further, by 1), the sequence $(N_\lambda^k(x_0))$ is fundamental and so convergent to some x. Clearly, $(x, \lambda) \in S$ and

$$d(x_0, x) \leq \psi_\lambda(x_0) - \psi_\lambda(x) \leq \psi_{\lambda_0}(x_0) - \psi_\lambda(x).$$

□

Remark 2. *For $X = M$, $N_\lambda = T$ continuous and $\psi_\lambda = \varphi$ for all $\lambda \in [0,1]$, Corollary 4 reduces to Caristi's theorem for continuous mappings.*

A simple consequence of Corollary 4 is the following result by Granas.

Corollary 2. ([3]) *Let M be a complete metric space, $U \subset M$ an open set, and $N : \overline{U} \times [0,1] \to M$ a mapping such that the following conditions hold:*

(h1) *$N(x, \lambda) \neq x$ for all $x \in \partial U$ and $\lambda \in [0,1]$;*

(h2) *there is $\alpha \in [0,1[$ such that*

$$d(N(x, \lambda), N(y, \lambda)) \leq \alpha d(x, y)$$

for all $x, y \in \overline{U}$ *and* $\lambda \in [0,1]$;

(h3) *there is a nondecreasing lower semicontinuous function* $\omega : [0,1] \to R$ *such that*

$$d(N(x,\lambda), N(x,\eta)) \leq | \omega(\lambda) - \omega(\eta) |$$

for all $\lambda, \eta \in [0,1]$ *and* $x \in \overline{U}$.

Then N_1 *has a fixed point if and only if* N_0 *has one.*

Proof. Apply Corollary 4 to $X = \overline{U}$ and

$$\psi_\lambda(x) = (1-\alpha)^{-1} \left[d(x, N_\lambda(x)) + \omega(1) - \omega(\lambda) \right].$$

□

We finish with a continuation theorem for not necessarily continuous mappings of Caristi type.

Theorem 4. *Let* M *be a complete metric space,* $X \subset M$ *a closed set,* $\psi : M \times [0,1] \to R_+$ *a lower semicontinuous function, and* $N : X \times [0,1] \to M$ *a mapping. Suppose that the following conditions hold:*

(i) X_λ *is closed for every* $\lambda \in [0,1]$;

(ii) $d(x, N_\lambda(x)) \leq \psi_\lambda(x) - \psi_\lambda(N_\lambda(x))$ *for all* $x \in X$ *and* $\lambda \in [0,1]$;

(iii) $\psi_\lambda(x) \leq d(x, \partial X)$ *for all* $\lambda \in [0,1]$ *and whenever* $N_\eta(x) = x$ *for some* $\eta \in [0,1]$.

Then, if $X_0 \neq \emptyset$*, each mapping* N_λ*,* $\lambda \in [0,1]$*, has at least one fixed point.*

Proof. Since X_0 is a closed nonempty set and $N_0(X_0) \subset X_0$, by Theorem 2, there exists $x_0 \in X$ such that $N_0(x_0) = x_0$. Further, by (ii) and (iii),

$$d(x_0, N_\lambda^k(x_0)) \leq \psi_\lambda(x_0) - \psi_\lambda(N_\lambda^k(x_0)) \leq \psi_\lambda(x_0) \leq d(x_0, \partial X),$$

whence $N_\lambda^k(x_0) \in X$ for all $k \in$ N. Consequently, $x_0 \in X_\lambda$ for every $\lambda \in [0,1]$. Hence, for each $\lambda \in [0,1]$, $X_\lambda \neq \emptyset$ and we can apply Theorem 2. □

Remark 3. *In particular, if* $X = M$*,* $N_\lambda = T$ *and* $\psi_\lambda = \varphi$ *for all* $\lambda \in [0,1]$*, Theorem 6 reduces to Theorem 2. Indeed, in this case, we have* $X_\lambda = M$*,* $\partial X = \emptyset$ *and* $d(x, \partial X) = +\infty$.

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EXTREMAL SOLUTIONS FOR THE DISCONTINUOUS DELAY-EQUATIONS

ALEXANDRU TĂMĂŞAN

Dedicated to Professor Ioan A. Rus on his 60^{th} anniversary

Abstract. We prove the existence of the extremal solutions of the initial value problem, for short IVP: $y'(t) = f(t, y(t), y(\theta(t)))$, $y(0) = y_0$ with f discontinuous, using some monoton iterative technique.

1. Introduction

The subject matter of the present article is the delay-differential equation :

$$\begin{cases} y'(t) = f(t, y(t), y(\theta(t))), a.e. t \in [0, T] \\ y(0) = y_0 \end{cases} \tag{1}$$

with f satisfying some Caratheodory's type conditions and monotony. The lag θ is an absolutely continuous function with $\theta(0) = 0$, $0 \le \theta(t) \le t$ a. e. $t \in [0, T]$.

This problem is dealt with in several papers (see [3], [4], [5]). The main merit of this paper consists of allowing discontinuous right hand side. The idea is to treat the delay term as a new variable. We are looking for solutions in the space of absolutely continuous functions denoted by $AC[0, T]$.

The monotone iterative method used in section 2 is the one presented for ODE in [1]. In particular we use the following two results:

Proposition 1. *(Theorem 1.5.1, [1]) Consider the IVP $x'(t) = f(t, x)$, $x(0) = x_0$, where $f : [0, T] \times \mathbf{R} \longrightarrow \mathbf{R}$. Let $\alpha, \beta \in AC[0, T]$ be a lower and respectively an upper solution, such that $\alpha \le \beta$. Consider f is a Caratheodory function in $\Omega = \{(t, x) : \alpha(t) \le x \le \beta(t), t \in [0, T]\}$. If there exists an $m \in L^1([0, T], \mathbf{R}_+)$ such that: $|f(t, x)| \le m(t)$ for all $x \in [\alpha(t), \beta(t)]$ and a. e. $t \in [0, T]$ then the IVP has the extremal solutions in the order interval $[\alpha, \beta]$.*

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Proposition 2. *(Proposition 1.4.4, [1]) Given a nonempty order interval $[\alpha,\beta] \subset AC[0,T]$, a nondecreasing mapping $G : [\alpha,\beta] \longrightarrow [\alpha,\beta]$ and assume there exists $v \in L^1([0,T],\mathbf{R}_+)$ such that $|(Gx)'(s)| \leq v(t)$, $x \in [\alpha,\beta]$, a. e. $t \in [0,T]$ then the chain $\{G^n\alpha : n \in \mathcal{N}\}$ has a maximum x_* and the chain $\{G^n\beta : n \in \mathcal{N}\}$ has a minimum x^* and $x_* = \min\{x : Gx \leq x\}$ and $x^* = \max\{x : x \leq Gx\}$. In particular x_*, x^* are the extremal fixed points of G.*

We call a lower solution of (1) a function $\alpha \in AC[0,T]$ which satisfies:

$$\begin{cases} \alpha'(t) \leq F(t,\alpha(t),\alpha(\theta(t)))a.e.t \in [0,T] \\ \alpha(0) \leq y_0. \end{cases}$$

By duality we get an upper solution.

In the beginning we assume the existence of a lower and a upper solution but later on we shall drop it under some additional conditions on F. The monotony dependence on data of the extremal solutions is also pointed out.

In the last paragraph we apply the results to a discontinuous pantograph- like equation.

2. Existence of the extremal solutions

Consider the IVP (1) with $f : [0,T] \times \mathbf{R}^2 \longrightarrow \mathbf{R}$ obeying the followings:

$(H1)$ there exist $\alpha, \beta \in AC[0,T]$ a lower and a upper solution for (1) with $\alpha(t) \leq \beta(t)$, $t \in [0,T]$;

$(H2)$ there exist a $N \in L^1([0,T],\mathbf{R}_+)$ such that $|f(t,x,y)| \leq N(t)$ a. e. $t \in [0,T]$, $x \in [\alpha(t),\beta(t)]$ and $y \in [\alpha(\theta(t)),\beta(\theta(t))]$;

$(H3)$ $f(\cdot,x,y(\cdot))$ is measurable for all $x \in \mathbf{R}$ and $y \in AC[0,T]$;

$(H4)$ $f(t,\cdot,y)$ is continuous a. e. $t \in [0,T]$ and $y \in \mathbf{R}$;

$(H5)$ $f(t,x,\cdot)$ is nondecreasing a. e. $t \in [0,T]$ and $x \in \mathbf{R}$.

We are able now to state the following:

Theorem 1. *If the hypothesis $(H1)$ to $(H5)$ hold then the problem (1) has the extremal solutions in the order interval $[\alpha,\beta]$ for each $y_0 \in [\alpha(0),\beta(0)]$. The minimal solution $y_* = \max\{G^n\alpha : n \in \mathcal{N}\} = \min\{y \in [\alpha,\beta] : Gy \leq y\}$ and the maximal solution $y^* = \min\{G^n\beta : n \in \mathcal{N}\} = \max\{y \in [\alpha,\beta] : Gy \geq y\}$.*

Proof Let $y_0 \in [\alpha(0), \beta(0)]$ and $y \in [\alpha, \beta]$ be given. Consider the following IVP:

$$\begin{cases} x'(t) = F_y(t, x(t)) \\ x(0) = y_0 \end{cases} \tag{2}$$

where $F_y(t,x) = f(t,x,y(\theta(t)))$.

It is easy that α and β are a lower solution and respectively an upper solution for (2). Also we have

$$|F_y(t,x)| = |f(t,x,y(\theta(t))| \leq N(t)$$

for $x \in [\alpha(t), \beta(t)]$ and $F_y(t,x)$ is a Caratheodory function on $\Omega = \{(t,x) : \alpha(t) \leq x \leq \beta(t),\ t \in [0,T]\}$. Using Proposition 1 the IVP (2) has for each $y \in [\alpha, \beta]$ the extremal solutions in $[\alpha, \beta]$. We set $G : [\alpha, \beta] \longrightarrow [\alpha, \beta]$ by $Gy = x$, where x is the maximal solution of (2) for each $y \in [\alpha, \beta]$. Since $(Gy)'(t) = F_y(t, Gy(t))$ for all $y \in [\alpha, \beta]$ and a. e. $t \in [0,T]$ we get the bounding condition $|(Gy)'(t)| \leq N(t)$. Moreover G is a nondecreasing operator. Indeed let $y_1, y_2 \in [\alpha, \beta]$ be such that $y_1 \leq y_2$ and $x_i = Gy_i,\ i = 1,2$. Since

$$x_1' = F_{y_1}(t, x_1) = f(t, x_1, y_1(\theta(t))) \leq f(t, x_1, y_2(\theta(t))) = F_{y_2}(t, x_1)$$

we have x_1 is a lower solution for $x' = F_{y_2}(t,x)$. But x_2 is a maximal solution of it whence (cf. [1]) $x_1 \leq x_2$ or $Gy_1 \leq Gy_2$. Thus G satisfies the hypothesis of Proposition 2. Therefore $x_* = \max_{n \in \mathbf{N}} G^n \alpha$ is the minimal solution and $x^* = \min_{n \in b/N} G^n \beta$ is the maximal solution of (1).

Remark (i) If we use a criteria to solve the problem (2) and so to build up the operator G one can approach the extremal solutions by successive iteration starting from any lower and respectively upper solution;

(ii) Assuming

$$|f(t,x,y)| \leq H(t, |x|, |y|) \text{ for } x, y \in \mathbf{R} \text{ and a. a. } t \in [0,T] \tag{3}$$

where $H : [0,T] \times \mathbf{R}_+^2 \longrightarrow \mathbf{R}_+$ is a nondecreasing on the second and the third variable and the IVP: $w' = H(t,w,w),\ w(0) = |x_0|$ has an upper solution w^* then $-w^*$ and $+w^*$ are lower and upper solution for (1);

(iii) The existence of the extremal solutions are within the order interval $[\alpha, \beta]$ provided $y_0 \in [\alpha(0), \beta(0)]$. We do not know anything about solutions for $y_0 \notin [\alpha(0), \beta(0)]$.

The main assumption we made is the existence of a lower and an upper solution. Under an additional assumption we can avoid this inconvenient. The same arguments as given in [1] work out for our delay differential equation and give us a sufficient condition to ensure lower and upper solutions for all $y_0 \in \mathbf{R}$ as follows:

$(H6)$ $|f(t,x,y)| \leq p(t)h(|x|,|y|)$ a. e. $t \in [0,T]$ and $x,y \in \mathbf{R}$ where $p \in L^1([0,T],\mathbf{R}_+)$, $h : [0,\infty) \times [0,\infty) \longrightarrow (0,\infty)$ is a nondecreasing function in both of its arguments and

$$\int_0^\infty \frac{du}{h(u,u)} = \infty$$

Suppose f confined to $(H6)$, then a lower and an upper solutions for (1) are given by:

$$\begin{aligned} \alpha(t) &= y_0 + |y_o| - w(t) \\ \beta(t) &= y_0 - |y_o| + w(t) \end{aligned} \tag{4}$$

where $w \in AC[0,T]$ is the only solution of

$$\begin{cases} w' = p(t)h(w,w) \\ w(0) = |y_0| \end{cases}$$

and moreover all the solutions of (1) will lie within these lower and upper solutions. We conclude:

Theorem 2. *Consider the IVP (1) with f satisfying $(H2)$ to $(H6)$. Then for each $y_0 \in \mathbf{R}$ the IVP (1) has the extremal solutions which lie together with all the other solutions in the order interval $[\alpha,\beta]$ where α,β are given by (4).*

3. Monotony dependence on data

The result stated in **Theorem 1** can be used in studying the dependence of the extremal solutions on the initial value y_0 and on f. We emphasize that we refer to extremal solution only within the order interval $[\alpha,\beta]$.

Theorem 3. *Let $f,\widetilde{f} : [0,T] \times \mathbf{R}^2 \longrightarrow \mathbf{R}$ be satisfying $(H1)$ to $(H5)$ with $f(t,x,y) \leq \widetilde{f}(t,x,y)$ for all $t \in [0,T]$ and $x,y \in \mathbf{R}$ and $y_0,\widetilde{y}_0 \in [\alpha(0),\beta(0)]$ be such that $y_0 \leq \widetilde{y}_0$. Consider the two corresponding IVP:*

$$\begin{cases} y'(t) = f(t,y(t),y(\theta(t))), a.e. t \in [0,T] \\ y(0) = y_0 \end{cases} \tag{5}$$

and

$$\begin{cases} \tilde{y}'(t) = \tilde{f}(t, \tilde{y}(t), \tilde{y}(\theta(t))), a.e.t \in [0,T] \\ \tilde{y}(0) = \tilde{y}_0 \end{cases} \tag{6}$$

If y_*, y^* respectively $\tilde{y}_*, \tilde{y}^*$ are the extremal solutions of the above problems within the order interval $[\alpha, \beta]$ then $y_* \leq \tilde{y}_*$ and $y^* \leq \tilde{y}^*$.

Proof It is obvious that y_* is a lower solution for (6) but a lower solution is less then any solution hence is less then the minimal solution $\tilde{y}_*$. The dual fact holds similarly.

4. Application

Let us consider the IVP of the discontinuous Pantograph-like equation:

$$\begin{cases} y'(t) = ay(t) + [y(qt)]_*, \ \ t \in [0,T] \\ y(0) = y_0 \end{cases} \tag{7}$$

where $[x]_*$ is the integer part of x, a is a real constant and $0 < q < 1$.

For $y_0 = 0$ the only continuous solution of (5) is the null one. Whereas for $y_0 \neq 0$ we have

$$\alpha(t) = y_0 + |y_0| - |y_0|e^{2ct}$$
$$\beta(t) = y_0 - |y_0| + |y_0|e^{2ct}$$

with $c = \max\{|a|, 1\}$ are respectively some lower and upper solutions. Since the above IVP satisfies $(H1)$ to $(H5)$ we guarantee the existence of the extremal solutions and moreover the minimal solution $y_*(t) = \lim_{n\to\infty} G^n\alpha(t)$ and the maximal solution $y^*(t) = \lim_{n\to\infty} G^n\beta(t)$.

For this simple example we can compute analyticaly some iteration of α through G. We have $G\alpha$ is the only solution of:

$$\begin{cases} y'(t) = ay(t) + [y_0 + |y_0| - |y_0|e^{2cqt}]_*, \ \ t \in [0,T] \\ y(0) = y_0 \end{cases} \tag{8}$$

For the sake of simplicity let us set $\lambda = -\frac{[y_0]_*}{|y_0|} + \frac{y_0}{|y_0|} + 1$. For $0 \leq \frac{1}{2cq}\ln(\lambda + \frac{1}{|y_0|})$ the only solution of (8) is the solution of

$$\begin{cases} y'(t) = ay(t) + [y_0]_*, \ \ t \in [0,T] \\ y(0) = y_0 \end{cases} \tag{9}$$

$$\text{i. e. } y_1(t) = e^{at}\left(y_0 + \frac{[y_0]_*}{a}\right) - \frac{[y_0]_*}{a}.$$

Iteratively we get for

$$\frac{1}{2cq}\ln(\lambda+\frac{i}{|y_0|})\le t<\frac{1}{2cq}\ln(\lambda+\frac{i+1}{|y_0|})$$

the solution of (8) is the solution of

$$\begin{cases} y'(t)=ay(t)+[y_0]_*-i, & t\in[0,T] \\ y(a_i)=y_i(a_i-0) \end{cases} \tag{10}$$

where

$$a_i=\frac{1}{2cq}\ln(\lambda+\frac{i}{|y_0|})$$

and y_i is the solution of the previous problem (10) with i is replaced by $i-1$.

It is worth noting that the null set of non-differentiable points changes on each iteration.

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A NOTE ON LINKING PROBLEMS IN EQUIVARIANT CASE

CSABA VARGA AND VIORICA VARGA

1. Introduction

In the article [4] is defined the limit of relative category and this notion is used for establish multiplicity theorems and the authors resolve the V.I. Arnold conjecture. In paper [5] the authors study linking problems for differentiable functionals with the aid of limit of relative category. These problems were studied by many authors, see [8], [6] and [1]. A generalization of the lusternik-Schnirelmann category and the relative category appear in the papers of M. Clapp, D. Puppe, see [2], [3]. In the previous papers the authors have studied the notion of $\mathcal{A}$-category and the relative $\mathcal{A}$-category. Using the notion of relative $\mathcal{A}$-category we introduce the limit of relative $\mathcal{A}$-category.

In the following we suppose that G is a compact Lie group. Let (X, X') be a G-pair, i.e. X a G-space and X' a G-subspace of X. Let $(X, X'), (Y, Y')$ be two G-pairs and $f : (X, X') \to (Y, Y')$ an application of G-pairs, i.e. $f : X \to Y$ is a G-equivariant such that $f(X') \subseteq Y'$. We consider $\mathcal{A}$ a class of G-space for example:

(i) $\mathcal{A}$ is the class $\mathcal{G}$ of omogen G-space, i.e. G-space of the form G/H, where H is a closed subgroup of G.

(ii) $\mathcal{A}$ is the class $\mathcal{G}^0$, i.e. the disjoint union of spaces from $\mathcal{G}$.

(iii) $\mathcal{A}$ is the class $\mathcal{G}^0_{fin}$, i.e. the finit disjoint union of space from $\mathcal{G}$.

Definition 1.1. The $\mathcal{A}$-category of a G-map $f : (X, X') \to (Y, Y')$, $\mathcal{A} - cat(f)$, is the smallest number k such that X can be covered by $k+1$ open G-subspaces $X_0, X_1, \dots, X_k$ with the following properties:

(i) $X' \subset X_0$ and there is a G-homotopy $\varphi_r : (X_0, X') \to (Y, Y')$, $t \in [0, 1]$ such that $\varphi_0(x) = f(x)$ and $\varphi_1(x) \in Y'$ for all $x \in X_0$.

(ii) For every $i = 1, \dots, k$ there exist G-maps $\alpha_i : X_i \to A_i$ and $\beta_i : A_i \to Y$ with $A_i \to \mathcal{A}$ such that the restriction of f to X_i is G-homotopic to the composition $\beta_i \alpha_i$.

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If no such number exists, we define $\mathcal{A} - cat(f) = \infty$. If X is a G-space and X', Y are G-subspace of X we use the notation

$$\mathcal{A} - cat_{X,X'}Y = \mathcal{A} - cat((Y, Y \cap X')) \to (X, X')$$

$$\mathcal{A} - cat^{X}Y = \mathcal{A} - cat_{X,\emptyset}Y.$$

Let G be a compact topological group and X a G-space. We consider a sequence $(X_n)_{n\geq 1}$ a G-invariant subspaces of X. Suppose that for every $n \in N^*$ exists $r : X \to X_n$ a G-equivariant retraction, i.e., $r : X \to X_n$ is a G-equivariant continuous function such that $r \circ i_n = 1_{X_n}$, where $i_n : X_n \to X$ is the inclusion. If $A \subseteq X$ we note $A_n = A \cap X_n$.

Remark 1.1. A subset of X_n is a $\mathcal{A}$-contractible in X_n is and only if is $\mathcal{A}$-contractible in X.

For this, let B_n a subset of X_n, which is contractible in X_n. Then exists $A_n \in \mathcal{A}$, $\alpha : B_n \to A_n$, $\beta_n : A_n \to X_n$ and a G-homotopie $h : B_n \times [0,1] \to X_n$, such that $h(x,0) = (\beta_n \alpha_n)(x)$ and $h(x,1) = x$ for every $x \in B_n$. If we consider the homotopies $H : B_n \times [0,1] \to X$, given by $H(x,t) = h(i_n(x),t)$, then H satisfied the following conditions:

(a) $H(x,0) = (\beta_n\alpha_n)(i_n(x)) = (\beta_n\alpha_n)(x)$, for every $x \in B$.

(b) $H(x,1) = h(i_n(x),1) = i_n(x) = x$, for every $x \in B$.

(c) H is a G-equivariant homotopie.

Now, we suppose that $B_n \subseteq X_n$ is $\mathcal{A}$-contractible in X, i.e. exists $A_m \in \mathcal{A}$, $\alpha_n : B_n \to A_N$, $\beta_n : A_n \to X$ and a G-equivariant homotopie $h : B_n \times [0,1] \to X$ such that $h(x,0) = (\beta_n\alpha_n)(x)$ and $h(x,1) = x$ for every $x \in B_n$. If we consider the homotopie $H : B_n \times [0,1] \to X_n$, given by $H(x,t) = r(h(x,t))$ satisfied the desired properties.

Definition 1.3. Let Y, A closed G-subset of the G spaces X, such that $Y \subseteq A$. The limit of relative $\mathcal{A}$-category of A in X relative to Y with respect the sequence $(X_n)_{n\geq 1}$ is from definitions: $\mathcal{A} - cat^{\infty}_{X,Y}(A) = \lim_{n\to\infty} \sup \mathcal{A} - cat_{X_n,Y_n}(A_n)$.

Definition 1.4. If Y, Z and X' are closed G-subset of the G-space X, we write $Y \prec_{X'} Z$ if is G-deformable in Z *mod* X.

By definition we put $Y \prec^{\infty}_{X'} Z$ with respect the $(X_n)_{n\geq 1}$, is and only if for n sufficiently large $Y_n <_{X'_n} Z_n$.

Proposition 1.1. $\mathcal{A} - cat^{\infty}_{(X,X')}$ *have the following properties:*

1) $\mathcal{A} - cat^{\infty}_{(X,X')}Y$ [illegible] $Y = \emptyset$.

2) If $Y <^{\infty}_{X'} Z$, then $\mathcal{A} - cat^{\infty}_{(X,X')}(Y) \le \mathcal{A} - cat^{\infty}_{(X,X')}(Z)$.

3) If X is a $G - ANR$ and Y, Z are G-subspace of X such that $z \cap X' = \emptyset$, the we have:

$$\mathcal{A} - cat^{\infty}_{(X,X')}(Y \cup Z) \le \mathcal{A} - cat^{\infty}_{(X,X')}(Y) + \mathcal{A} - car_X Z.$$

2. The main deformation lemma

In this paragraph we generalize a deformation lemma of Willem in equivariant case for locally Lipschitz invariant continuous functions. For locally Lipschitz continuous functions this result is proved in [9].

Let G be a compact Lie group which action isometric on the reflexive Banach space X, i.e. the function $G \times X \to X$ is differentiable and for every $g \in G$, $g : X \to X$ is an isometrie.

On the dual space X^* the action is defined by $(gx^*)(x) = x^*(gx)$, for evey

Indeed $\|gx^*\| = \sup_{\|x\|=1} |(gx^*)(x)| = \sup_{\|x\|=1} |x^*(gx)|$, and using the fact that the action of G is isometric on X, we get $g(S(0,1)) = S(0,1)$, i.e. $\|gx^*\| = \sup_{\|x\|=1} |x^*(x)| = \|x^*\|$, where $S(0,1)$ is the unit sphere of Banach space X.

We suppose that $f : X \to \mathbf{R}$ is a locally Lipschitz G-invariant function, i.e. $f(gx) = f(x)$ for every $g \in G$ and $x \in X$. From the paper [KrMa] we have the relation: $g\partial f(x) = \partial f(gx) = \partial f(x)$ for every $g \in G$ and $x \in X$, where $\partial f(x)$ denote the generalized gradient (see [7]).

In conclusion we have that, the subset $\partial f(x) \subset X^*$ is G-invariant.

The function $\lambda(u) = \inf_{w \in \partial f)x} \|w\|$ is G-invariant.

Lemma 2.1. *Let X be reflexive G-Banach space and $f : X \to \boldsymbol{R}$ a locally Lipschitz continuous G-invariant function and S a G-invariant subset of X. We consider the number $x \in \boldsymbol{R}$ and $\varepsilon, \delta > 0$ such that for every $u \in f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap S_{2\delta}$ we have $\lambda(u) \ge \frac{4\varepsilon}{\delta}$. In this conditions exists a G-equivariant vector field $\hat{v} : f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap S_{2\delta} \to X$ which satisfies the following:*

(α) $\|\hat{v}(x)\| \le 1$ *for every $x \in f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap S_{2\delta}$.*

(β) $\langle x^*, \hat{v}(x) \rangle > \frac{2\varepsilon}{\delta}$ *for every $x^* \in \partial f(x)$ and $x \in f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap S_{2\delta}$.*

Proof. Using the standard method (see [1]), let $v : f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap S_{2\delta} \to X$ the vector field constructed in Lemma 2.1 (see [9]). We define the vector field $\hat{v} : f^{-1}([c -$

$2\varepsilon, c+2\varepsilon]) \cap S_{2\delta} \to X$, given by $\hat{v}(x) = \int_G g^{-1} v(g(x)) d\mu$ where $d\mu$ is the right Haar measure on G with $\int_G d\mu = 1_G$. The vector field $\hat{v}$ is well defined, because $f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap S_{2\delta}$ is invariant subset of X.

We proof that $\hat{v}$ is G-equivariant.

Indeed, $\hat{v}(g'x) = \int_G g^{-1} v(gg'x) d\mu = g' \int_G (gg')^{-1} v(gg'x) d\mu = g' \hat{v}(x)$ for every $g' \in G$.

Let $x, y \in f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap S_{2\delta}$ and M_0 a positive real number such that $\|v(x) = v(y)\| \leq M\|x-y\|$, then we have

$$\|\hat{v}(x) - \hat{v}(y)\| = \left\| \int_G g^{-1}[v(gx) - v(gy)] d\mu \right\| \leq \int_G \|g^{-1}[v(gx) - v(gy)]\| d\mu =$$

$$= \int_G \|v(gx) - v(gy)\| d\mu \leq M\|gx - gy\| = M\|x - y\|.$$

The condition (α) results from the fact that the action of G is isometric and (β) from the property that the subset $\partial f(x) \subset X^*$ is G-invariant.

We have the following results:

Theorem 2.1. *Let X be a reflexive Banach space and G a compact Lie group which actions on X is isometric. We consider a locally Lipschitz G-invariant function $f : X \to \mathbf{R}$ and a G-invariant subset $S \subseteq X$.*

Now consider the real numbers $c \in \mathbf{R}$, $\varepsilon, \delta > 0$ such that for every $x \in f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap S_{2\delta}$ we have $\lambda(u) \geq \frac{4\varepsilon}{\delta}$. In these condition exists a continuous function $\eta : [0,1] \times X \to X$ such that the following properties holds:

1) $\eta(0, u) = u$ for every $u \in X$.

2) $\eta(t, \cdot)$ is a G-homeomorphism of the X for every $t \in [0,1]$.

3) $\eta(t, u) = u$ for every $u \notin A$ and $t \in [0,1]$.

4) $\|\eta(t, u) - u\| \leq \delta$ for every $u \in X$ and $t \in [0,1]$.

5) $f(x) - f(\eta(t, x)) \geq 2\varepsilon t$ for every $t \in [0,1]$ with $(t, x) \in A$.

The proof is analogous with the proof of Lemma 2.2 (see [9]) if we replace $v(x)$ with $\hat{v}(x)$.

3. Applications in critical point theory

We suppose that X is a reflexive Banach space on which G acts free and isometric and let $(X_n)_{n\geq 1}$ a sequence of closed G-invariant subspaces such that every subspaces have a closed complement.

Let $f : X \to \mathbf{R}$ be a G-invariant locally Lipschitz function and Y a G-invariant, closed subset of X. For every $j \geq 1$ we define the sets $\mathcal{A}_j = \{A \subseteq X |\ A-$closed $G-$subset, $Y \subseteq A$ and $\mathcal{A} - cat^{\infty}_{(X,Y)}(A) \geq j\}$ and the numbers $c_j = \inf_{A\in A_j} \sup_{x\in X} f(x)$.

Theorem 3.1. *Suppose that the following conditions holds:*

a) $\sup_{x\in Y} f(x) = c_k = c_{k+1} = \cdots = c_{k+m} = c < +\infty$.

b) The function f satisfied the $(PS)^*_c$ condition with respect the sequence $(X_n)_{n\in N}$.

In these conditions c is a critical value for f and $A - cat_X(K_c) \geq m + 1$.

Proof. The fact that c is a critical value of f results from Lemma 1.2. Using the continuity property of $\mathcal{A} - cat_X$ exists an open G-invariant neighbourhood $N \in V_X(K_c)$ such that $N \cap Y = \emptyset$.

From the conditions a) and b) follow that the existence of a real number $f^{x+\epsilon} \setminus \mathrm{N} <^{\infty}_{r} f^{c-\epsilon}$.

From the definition of the number $c = c_m = c_{m+k}$ we have:

$$k+m \leq \mathcal{A} - cat^{\infty}_{X,Y}(f^{c+\epsilon}) \leq \mathcal{A} - cat^{\infty}_{X,Y}(f^{c+\epsilon} \setminus N) + \mathcal{A} - cat_X(\overline{N}) \leq k - 1 + \mathcal{A} - cat_X(K_c),$$

$$\mathcal{A} - cat_X(K_c) \geq m + 1.$$

Theorem 3.2. *Let $f : X \to \boldsymbol{R}$ be a G-invariant locally Lipschitz continuous function and let F, G be two nonvoid, closed and G-invariant subset of X. We consider the set:*

$$\mathcal{A} = \{A \subset Z |\ A \textit{ is a closed, } G - \textit{invariant subset, } Y \subseteq A \textit{ and, } \mathcal{A} - cat^{\infty}_{X,Y} = 1\}.$$

We define the numbers:

$$c = \inf_{A\in\mathcal{A}} \sup_{u\in A} f(u).$$

We suppose that the following conditions holds:

(α) For every $n \in \boldsymbol{N}$ and for all closed G-subset B_n of X_n, for which $Y_n \subseteq B_n$, $B_n \cap F = \emptyset$ we have $\mathcal{A} - cat_{X_n,Y_n}(B_n) = 0$.

(β) $dist(F, Y) > 0$.

(γ) $-\infty < x - \inf_Y f$.

Then for every $j \in \boldsymbol{N}$, $\varepsilon > 0$, $\delta \in]0, dist(F, Y)/2[$ *and* $A \in \mathcal{A}$ *such that* $\sup_A \varphi < c + \varepsilon$, *exists* $n \in \boldsymbol{N}$, $n \geq j$ *and* $u \in X_n$ *for which the following conditions holds:*

a) $c - 2\varepsilon \leq \varphi(u) \leq c + 2\varepsilon$.

b) $dist(u, F_n \cap A_\delta) \leq 2\delta$.

c) $\|\lambda_{X_n}(u)\| \leq \frac{4\varepsilon}{\delta}$.

Proof. Let j, ε, δ and $A \in \mathcal{A}$ such that the conditions $(\alpha), (\beta)$ and (δ) holds we suppose that the conclusion a), b) and c) of the Theorem is not true.

If we apply Theorem 2.1 for $X := X_n$, $f := -f$ and $S := F_n \cap A_\delta$, then for every $n \geq j$ exists a deformation η_n which satisfied the conclusion (1)-(6) from Theorem 2.1.

For every $n \geq j$, we define $B_n = \{u \in X_n | \eta)n(1, u) \in A_n\}$. From conclusion (3) follow that $\eta_n(1, y) = y$ for every $y \in Y_n$, consequently $\mathcal{A} - cat_{X_n,Y_n}(A_n) = \mathcal{A} - cat_{X_n,Y_n}(B_n)$.

If for every $n \geq j$ we have $B_n \cap H = \emptyset$, then we have:

$$1 = \mathcal{A} - cat^{\infty}_{X,Y}(A) = \lim_{n \to \infty} \sup \mathcal{A} - cat_{X_n,Y_n}(A_n) = \lim_{n \to \infty} \sup \mathcal{A} - cat_{X_n,Y_n}(B_n) = 0.$$

Contradiction.

Therefore, exists a $n \geq j$ such that $B_n \cap F \neq \emptyset$. Consequently exists an element $u \in F_n$ for which $\eta_n(1, u) \in A_n$.

From the condition $c = \inf_F f$ we obtain that $u \in F \subset f^{-c}$ and from the inequality $\|\eta(t, u) - u\| \leq \delta$ follows that $dist(u, A_n) \leq \delta$. Therefore $u \in S \cap (-f)^{-c}$ and from relation $\eta(1, f^{c+\varepsilon} \cap S) \subseteq f^{c-\varepsilon}$ we obtain $c + \varepsilon \leq f_n(\eta_n(1, u)) \leq \sup_{A_n} f_n < c + \varepsilon$, and these is a contradiction.

Definition 3.2. Let $c \in \mathbf{R}$ be a real number $f : X \to \mathbf{R}$ G-invariant locally Lipschitz continuous function and F a closed nonvoid G-invariant set. Let $(X_n)_{n \geq 1}$ be a sequence of G-invariant subspaces of X such that each X_n has a closed complement. We say the function satisfies the $(P.S.)^{*}_{F,c}$ condition if for every sequence $(u_k)_{k \in \mathbf{N}} \subset X$ satisfying: $k \to \infty$, $u_k \in X_k$, $dist(u_k, F_k) \to 0$, $\varphi(u_k) \to c$, $\lambda|(x_k) \to 0$ possesses a subsequence which converges in X to a critical point of f.

Theorem 3.3. *Let* $f : X \to \boldsymbol{R}$ *be a locally Lipschitz, G-invariant function and* F, Y *two closed G-invariant subset of* X. *We suppose that* f, F *and* Y *satisfied the conditions* $(\alpha) - (\gamma)$ *from Theorem 3.2 and* f *satisfied the* $(P.S.)^{*}_{F,c}$ *condition where* $c = \inf_{A \in \mathcal{A}} \sup_{x \in A} f(x)$. *Then* $F \cap f^{-1}(c) \neq \emptyset$.

The proof results immediately from Theorem 3.2.

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philosophie (semestriellement)
sociologie (semestriellement)
politique (annuel)
ephemerides (annuel)
études européennes (semestriellement)
affaires (semestriellement)
psychologie - pédagogie (semestriellement)
études économiques (semestriellement)
études juridiques (semestriellement)
histoire (trois apparitions per année)
philologie (trimestriellement)
théologie orthodoxe (semestriellement)
théologie catholique (annuel)
éducation physique (annuel)