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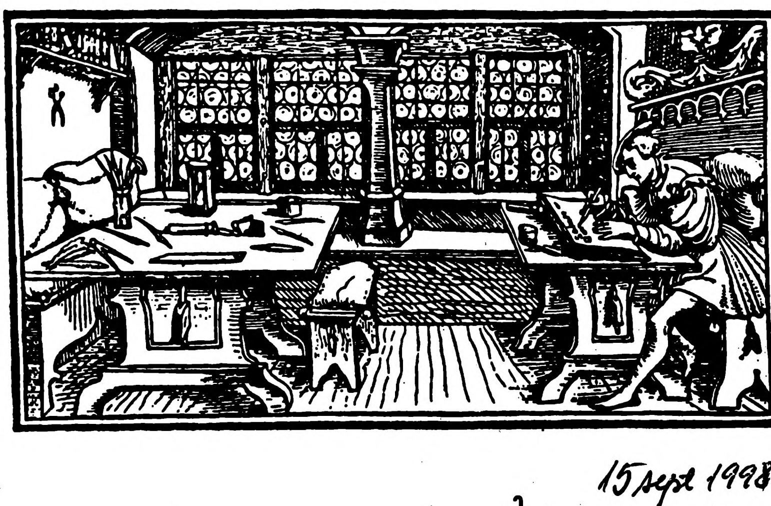

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STUDIA UNIV. "BABEŞ-BOLYAI", MATHEMATICA, Volume XLI, Number 2, June 1996

PROFESSOR GHEORGHE COMAN AT HIS 60^{TH} ANNIVERSARY

D.D. STANCU

Professor Gheorghe Coman is the chief of the Chair of Numerical and Statistical Calculus, Faculty of Mathematics and Informatics, University Babeş-Bolyai, Cluj-Napoca, Romania. He was born on January 24, 1936 in Grindeni-Mureş, Romania. After attending the primary school in his native village, he entered the secondary school in Luduş -Mureş, and after 3 years moved to Cluj-Napoca, where he obtained the school certificate. In the period 1956-1961 he studied at the Faculty of Mathematics and Physics, University of Cluj. In 1970 received his doctoral degree in mathematics under the direction of the distinguished Romanian mathematician D.V.Ionescu. After his graduation, in 1961, he began his academic career at the University of Cluj as an assistant (1961-1970), lecturer (1970-1977), associate professor (1977-1990) and since 1990 he is full professor.

During his teaching career, professor Coman has given courses on numerical analysis, complexity of algorithms, optimal numerical methods, programming languages, approximation of functions of several variables.

Since 1975 he is a member of American Mathematical Society and a reviewer at Mathematical Reviews. He was visiting at the University of Moscow, Russia (1968) and at the University of Wisconsin, Madison, Wisconsin, USA (1973-1974).

Between 1988 and 1996 he was the dean of the Faculty of Mathematics and Informatics. He is a member of the editorial board of the jurnals: Studia Univ. Babeş -Bolyai, Mathematica (Cluj), Revue d'analyse numérique et la théorie de l'approximation. Since 1990 professor Gheorghe Coman became a scientific guide of doctorands.

Professor Gheorghe Coman has obtained important scientific results in various areas of the following domains (see "List of publications, Scientific papers"): numerical integration of functions of one and several variables, with emphasis on optimal formulas with regard to the error and efficiency (3-19, 21 , 27, 29, 30, 39, 56, 59, 73), the approximation of functions (20, 22-26, 28, 31, 32, 34, 38, 42, 44, 46, 49, 53, 57, 58, 61, 62, 65-68, 71, 72, 77, 78, 79, 80, 82-85), the complexity of the numerical methods (33, 36, 37, 41, 47, 51, 63), parallel numerical methods (54, 55, 60, 64, 70).

We join the members of the family of Professor Gheorghe Coman, his colleagues and students, congratulating him on his 60^{th} anniversary, wishing him good health and happiness. May he be granted with many more years with an active life and with new satisfactions in his scientific research work.

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FORMULES OPTIMALES DE QUADRATURE ATTACHÉ À LA FORMULE DE QUADRATURE DU TRAPÈZE ET À LA FORMULE DE SIMPSON

DUMITRU ACU

Dedicated to Prof. Gheorghe Coman at his 60^{th} anniversary

Abstract. In the paper one obtains optimal quadrature formulas attached to the trapezoid's formula and to the Simpson's formula.

1. Introduction

Soit $I[0,1]$ la classe des fonctions f définies sur l'intervale $[0,1]$ et intégrable dans le sens de Lebesgue sur cet intervalle.

On considère la formule de quadrature

$$\int_0^1 f(x)dx = \sum_{i=0}^{n-1} A_i f(x_i) + \gamma_n^{(0)}(f), \tag{1}$$

ayant l'évaluation exacte pour le reste

$$\gamma_n^{(0)} = \sup_{f \in J[0,1]} \left|\gamma_n^{(0)}(f)\right|. \tag{2}$$

On se pose le problème d'obtenir une formule de quadrature du type:

$$\int_0^1 f(x)dx = \sum_{i=0}^{n-1} A_i f(x_i) + \sum_{k=0}^{m-1} B_k f(y_k) + R_{n+m}^{(0)}(f) \tag{3}$$

de sorte qu'elle soit optimale sur $I[0,1]$ c'est à dire de déterminer les coefficients B_k et les noeuds:

$$0 < y_0 < y_1 < \dots < y_{m-1} < 1$$

de telle manière que:

$$R_{n+m}^{(0)} = \sup_{f \in I[0,1]} \left|R_{m+n}^{(0)}(f)\right| \tag{4}$$

soit minime.

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La formule de quadrature obtenue ainsi s'appelle la *formule optimale attachée à la formule* (1), pour la classe de fonctions $I[0,1]$.

Pour une formule de quadrature donnée (1) on peut obtenir des variantes du problème posé, si on cherche la formule optimale de quadrature attachée à la formule (1) parmi les formules du type (2) où les coefficients $B_k, k = \overline{0, m-1}$ et les noeuds $y_k, k = \overline{0, m-1}$ ne sont pas arbitraires, mais soumis à des liaisons déjà données.

Les formules optimales attachées à une formule de quadrature ont été introduites par M.Levin [6].

Sois $W_0^{(1)}L_2(M;0,1)$ l'ensemble des fonctions f, définies sur l'intervale [0,1] absolument continues et qui satisfait les conditions $f(0) = 0, \| f' \|_{L_2} \leq M$. La classe $W^{(1)}L_2(M;0,1)$ de fonctions est formée des fonctions $f : [0,1] \to R$ absolument continues et qui satisfait la condition $\| f' \|_{L_2} \leq M$.

Dans [1] on a fait l'étude des formules des quadratures optimales attachées à la formule de quadrature optimale sur les classes de fonctions $W_0^{(1)}L_2(M;0,1)$ et $W^{(1)}L_2(M;0,1)$ ([4], [5]):

$$\int_0^1 f(x)dx = \frac{2}{2n+1}\sum_{k=0}^{n-1} f\left(\frac{2k+2}{2n+1}\right) + r_n^{(0)}(f),$$

avec

$$r_n^{(0)} = \frac{M}{(2n+1)\sqrt{3}},$$

quand $m = n$ et les noeuds $y_0, y_1, \ldots, y_{n-1}$ sont considérés ains.

$$y_0 = r,\ y_1 = \frac{2}{2n+1} + r, \ldots, y_{n-1} = \frac{2n-2}{2n+1} + r,$$

où r est un nombre qui appartient à l'intervale $(0,2/(2n+1))$.

Nous allons présenter dans ce travail les formules de quadrature optimales attachées à la formule des trapèzes et à la formule le Simpson sur certaines classes de fonctions ([1], [2], [3]).

2. L'optimisation de la formule du trapèze.

On considère d'abord $f \in W_0^{(1)}L_2(M;0,1)$. La formule du trapèze a la forme ([1])

$$\int_0^1 f(x)dx = \frac{1}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + \left(\frac{n-1}{n}\right) + \frac{1}{2}f(1)\right] + r_n^{(0)}(f) \qquad (5)$$

avec

$$r_n^{(0)} = \sup_{f \in W_0^{(1)} L_2(M;0,1)} \left| r_n^{(0)}(f) \right| = \frac{M}{2n\sqrt{3}}. \tag{6}$$

Puis nous allons considérer $n = m$ et les nœuds $y_k, k = \overline{0, m-1}$, qui ont la représentation suivante:

$$y_0 = a, y_1 = y_1 + a, \ldots, y_{n-1} = x_{n-1} + a, 0 < a \leq x_r \tag{7}$$

Le problème posé ci-dessus revient dans le cas où on détermine les coefficients $B_k, k = \overline{0, n-1}$, de sorte que:

$$\int_0^1 f(x)dx = \frac{1}{n}\left[\frac{1}{2}f(1) + \sum_{i=1}^{n-1} f\left(\frac{i}{n}\right)\right] + \sum_{k=0}^{n-1} B_k f(y_k) + R_n^{(0)}(f, a) \tag{8}$$

$0 < a \leq \frac{1}{n}$, soit optimale sur la classe $W_0^{(1)} L_2(M; 0, 1)$.

Si on agit comme dans (1) on trouve:

$$R_n^{(0)}(a) = \sup_{f \in W_0^{(1)} L_2(M;0,1)} \left| R_n^{(0)}(f, a) \right| = M \left(\int_0^1 K^2(t)dt \right)^{\frac{1}{2}} \tag{9}$$

où

$$K(t) = 1 - t - \frac{1}{n}\sum_{i=1}^{n-1}\left(\frac{i}{n} - t\right)_+ - \frac{1}{2n}(1-t)_+ - \sum_{k=0}^{n-1} B_k (y_k - t)_+$$

avec

$$u_+ = \begin{cases} 0, & \text{pour } u \leq 0 \\ 1, & \text{pour } u > 0. \end{cases}$$

En utilisant $I = \int_0^1 K^2(t)dt$, alors $\frac{\partial I}{\partial B_l} = 0, l = \overline{0, n-1}$, et on obtient le système

$$\sum_{k=0}^{l} B_k y_k + y_k \sum_{k=l+1}^{n-1} B_k = \frac{a}{2}\left(\frac{l}{n} - a\right), l = \overline{0, n-1}$$

qui admet la solution unique

$$B_0 = \frac{1}{2}\left(\frac{1}{n} - a\right), \; B_1 = B_2 = \ldots = B_{n-1} = 0. \tag{10}$$

Avec (10) on trouve:

$$I = \frac{1}{12n^2}\left[1 - 3(1 - na)^2 a\right]$$

et donc:

$$R_n^{(0)}(a) = \frac{M}{2n\sqrt{3}} \tag{11}$$

Par conséquence, on a:

Théorème 1. *De toutes les formules (8), avec $0 < a \leq \frac{1}{n}$, la formule optimale attachée à la formule des trapèzes sur la classe $W_0^{(1)}L_2(M;0,1)$ est:*

$$\int_0^1 f(x)dx = \frac{1}{n}\left[\sum_{i=1}^{n-1} f\left(\frac{i}{n}\right) + \frac{1}{2}f(1)\right] + \frac{1}{2}\left(\frac{1}{n} - a\right)f(a) + R_n^{(0)}(a) \tag{12}$$

avec l'évaluation pour le reste donnée par (11).

Si on agit comme dans (1) on démonstre qu'il y a les théorèmes suivants:

Théorème 2. *De toutes les formules (12) la meilleure est obtenue pour $a = 1/3n$ c'est à dire la formule:*

$$\int_0^1 f(x)dx = \frac{1}{n}\left[\sum_{i=1}^{n-1} f\left(\frac{i}{n}\right) + \frac{1}{2}f(1) + \frac{1}{3}f\left(\frac{1}{3n}\right)\right] + R_n^{(0)}\left(\frac{1}{3n}\right) \tag{13}$$

avec

$$R_n^{(0)}\left(\frac{1}{3n}\right) = \frac{M}{2n\sqrt{3}}\sqrt{1 - \frac{4}{9n}}.$$

Théorème 3. *De toutes les formules du type*

$$\int_0^1 f(x)dx = \frac{1}{n}\left[\frac{1}{2}f(1) + \sum_{i=1}^{n-1} f\left(\frac{i}{n}\right)\right] + Cf(0) + \sum_{k=0}^{n-1} B_k f(y_k) + R_n(f,a)$$

avec $y_k, k = \overline{0, n-1}$, donneé (7), $0 < a \leq \frac{1}{n}$, la formule optimale attachée à la formule des trapèzes sur la classe $W^{(1)}L_2(M;0,1)$ est

$$\int_0^1 f(x)dx = \frac{1}{n}\left[\frac{1}{2}f(1) + \sum_{i=1}^{n-1} f\left(\frac{i}{n}\right)\right] + \frac{a}{2}f(0) + \frac{1}{2}\left(\frac{1}{n} - a\right)f(a) + R_n(a) \tag{14}$$

avec l'évaluation du reste $R_n(a) = R_n^{(0)}(a)$, étant donnée par (11). La meilleure des formules (4) est obtenue pour $a = 1/3\ n$.

Remarque 1. Pour $a = 1/n$ de (14) résulte la formule de quadrature étudiée par Gh.Coman en [4]. Dans ce cas la formule de quadrature obtenue a le degré algebrique égal à 1.

3. L'optimisation de la formule de Simpson

Pour $f \in W_0^{(1)} L_2(M;0,1)$ la formule de Simpson a la forme ([6]):

$$\int_0^1 f(x)dx = \frac{1}{6n}\left[4\sum_{i=1}^{n} f\left(\frac{2i-1}{2n}\right) + 2\sum_{i=1}^{n-1} f\left(\frac{i}{n}\right) + f(1)\right] + r_{2n+1}(f) \tag{15}$$

avec

$$r_{2n+1}\left(W_{0L_2}^1(M;0,1)\right) = \frac{M}{6n}\sqrt{5-\frac{4}{n}}.$$

On se propose de trouver la formule optimale de quadrature attachée à la formule (15) sur la classe $W_{0L_2}^{(1)}(M;0,1)$ qui a la forme

$$\begin{aligned}\int_0^1 f(x)dx &= \frac{1}{6n}\left[4\sum_{i=1}^{n} f\left(\frac{2i-1}{2n}\right) + 2\sum_{i=1}^{n-1} f\left(\frac{i}{n}\right) + f(1)\right] + \\ &+ \sum_{k=1}^{2n} B_k f(y_k) + r_{2n+1}(f,a)\end{aligned} \tag{16}$$

où

$$y_k = \frac{2k-1}{2n} + a, \ k = \overline{1,2n} \tag{17}$$

a étant un nombre réel fixé de l'interval $\left(0, \frac{1}{2n}\right)$.

Comme dans le paragraphe 2, on demontre:

Théorème 4. *La formule optimale de quadrature du type (16) attachée à la formule de Simpson (15) sur la classe fonctions $W_{0L_2}^{(1)}$ est*

$$\int_0^1 f(x)dx = \frac{1}{6n}\left[4\sum_{i=1}^{n} f\left(\frac{2i-1}{n}\right) + 2\sum_{i=1}^{n-1} f\left(\frac{i}{n}\right) + f(1)\right] + \left(\frac{1}{4n} - \frac{5}{6}a\right) f(a) +$$

$$+\frac{2}{3}\left(a-\frac{1}{4n}\right)\sum_{i=1}^{n-1} f\left(\frac{2i-1}{2n}+a\right) + \frac{2}{3}\left(\frac{1}{4n}-a\right)\sum_{i=2}^{n} f\left(\frac{i-1}{n}+a\right) +$$

$$+\frac{1}{3}\left(a-\frac{1}{4n}\right) f\left(\frac{2n-1}{2n}+a\right) + R_{2n+1}(f,a),$$

avec l'évaluation optimale du reste donnée par:

$$R_{2n+1}\left(W_{0L_2}^{(1)}(M;0,1);a\right) = M\sqrt{I},$$

où

$$I=\frac{1}{36n^2}\left(5-\frac{4}{n}\right)+\left(\frac{1}{4n}\right)+\left(\frac{1}{4n}-\frac{5a}{6}\right)\left(\frac{a}{6}-\frac{1}{12n}\right)a+$$
$$+\frac{4}{9}\left(a-\frac{1}{4n}\right)^2\left(\frac{2n-1}{8n}+\frac{a}{4}\right)+$$
$$+\frac{4}{3}\left(a-\frac{1}{4n}\right)\left[\frac{a^2}{6}-\frac{4n-3}{24n}+\frac{2n-1}{48n^2}\right],\ a\in\left(0,\frac{1}{2n}\right) \qquad (18)$$

Théorème 5. *De toutes les formules de quadrature*

$$\int_0^1 f(x)dx=\frac{1}{6n}\left[4\sum_{i=1}^{n}f\left(\frac{2i-1}{2n}\right)+2\sum_{i=1}^{n-1}f\left(\frac{i}{n}\right)+f(1)\right]+Cf(0)+$$
$$+\sum_{k=1}^{2n}B_kf(y_k)+R_{2n+1}(f,a),$$

avec $y_k, k=\overline{1,2n}$, *données par (17),* $0<a<1/2n$ *la formule optimale attachée à la formule de Simpson sur la classe* $W_{L_2}^{(1)}(M;0,1)$ *est:*

$$\int_0^1 f(x)dx=\frac{1}{6n}\left[4\sum_{i=1}^{n}f\left(\frac{2i-1}{n}\right)+2\sum_{i=1}^{n-1}f\left(\frac{i}{n}\right)+f(1)\right]+\frac{1}{6}\left(\frac{1}{n}-a\right)f(0)+$$
$$+\left(\frac{1}{4n}-\frac{5}{6}a\right)f(a)+\frac{2}{3}\left(a-\frac{1}{4n}\right)\cdot\sum_{i=1}^{n-1}f\left(\frac{2i-1}{2n}+a\right)+\frac{2}{3}\left(\frac{1}{4n}-a\right)\cdot$$
$$\cdot\sum_{i=2}^{n}f\left(\frac{i-1}{n}+a\right)+\frac{1}{3}\left(a-\frac{1}{4n}\right)f\left(\frac{2n-1}{2n}+a\right)+R_{2n+1}(f,a),$$

avec l'évaluation optimale du reste donnée par

$$R_{2n+1}\left(W_{L_2}^{(1)}(M;0,1);a\right)=R_{2n+2}\left(M_{0L_2}^{(1)}(M;0,1);a\right).$$

Remarque 2. Pour $a=1/4n$ les théorèmes 4 et 5 conduisent aux résultats donnés par M.Levin [6].

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ON THE MONOTONICITY OF A SEQUENCE OF STANCU-BERNSTEIN TYPE OPERATORS

OCTAVIAN AGRATINI

Dedicated to Prof. Gheorghe Coman at his 60^{th} anniversary

Abstract. One makes a study of a sequence of Bernstein type operators, introduced and studied in [9]. These are depending on two parameters a and b, $0 \leq a \leq b$. First, one deduces a representations by divided differences for the difference of two consecutive terms of the sequence of polynomials obtained by applting these operators to a function $f \in C[0,1]$. Using this representation, one enounces several sufficient conditions for the monotony of the sequence of Stancu-Bernstein polynomials.

1. Introduction

In 1969 D.D.Stancu [9] considered and studied the following generalization of the Bernstein polynomial:

$$\left(S_n^{a,b} f\right)(x) = \sum_{k=0}^{n} P_{n,k}(x) f\left(\frac{k+a}{n+b}\right), \tag{1}$$

where

$$P_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \tag{2}$$

and a,b are real parameters, independent of n, such that $0 \leq a \leq b$. This is an interpolatory type polynomial characterized by the fact that it uses equally spaced nodes $x_k = \frac{k+a}{n+b}$ $(k = 0,1,\ldots,n)$. If $ab \neq 0$ and $a \neq b$ then it does not coincide at any node with the function f; if $a = 0$ and ' $\neq 0$ then it coincides with f at $x_0 = 0$, while if $a = b \neq 0$ then it coincides with f at $x_n = 1$. When $a = b = 0$ one obtains the classical Bernstein polynomial.

It was proved that for $f \in C[0,1]$ the sequence of the polynomials (1) converges uniformly to f on [0,1]. Then the corresponding order of approximation was evaluated by using the modulus of continuity of f; also there were deduced expressions for the

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remainder term of the approximation formula

$$f(x) = \left(S_n^{a,b} f\right)(x) + \left(R_n^{a,b} f\right)(x)$$

and, finally, the author presented a theorem of Voronovskaja type giving an asymptotic estimation for the remainder term. In this paper we shall investigate the monotonicity properties of this sequence.

2. The basic theorem

In order to study the monotonicity of the sequence $\left(S_n^{a,b} f\right)$ we shall establish a useful formula for the difference of two consecutive terms of the Stancu-Bernstein polynomials. The following theorem holds:

Theorem 1. *The difference between the polynomials* $\left(S_{n+1}^{a,b} f\right)(x)$ *and* $\left(S_n^{a,b} f\right)(x)$ *can be expressed under the form:*

$$\left(S_{n+1}^{a,b} f\right)(x) - \left(S_n^{a,b} f\right)(x) == -\frac{nx(1-x)}{(n+b)(n+b+1)} \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot$$
$$\cdot \left(\frac{1}{n+b} \left[\frac{k+a}{n+b}, \frac{k+1+a}{n+1+b}, \frac{k+1+a}{n+b}; f \right] + \frac{a}{k+1} \left[\frac{k+1+a}{n+1+b}, \frac{k+1+a}{n+b}; f \right], \right.$$
$$\left. - \frac{b-a}{n-k} \left[\frac{k+a}{n+b}, \frac{k+1+a}{n+1+b}; f \right] \right) x^k (1-x)^{n-k-1} + \left(U_n^{a,b} f\right)(x), \tag{3}$$

where the brackets represent the symbol for divided differences and

$$\begin{aligned} \left(U_n^{a,b} f\right)(x) &= \left(f\left(\frac{a}{n+1+b}\right) - f\left(\frac{a}{n+b}\right) \right)(1-x)^{n+1} + \\ &+ \left(f\left(\frac{n+1+a}{n+1+b}\right) - f\left(\frac{n+a}{n+b}\right) \right) x^{n+1} \end{aligned} \tag{4}$$

Proof. First we write:

$$\begin{aligned} \left(S_{n+1}^{a,b} f\right)(x) &= \sum_{k=0}^{n+1} \binom{n+1}{k} x^k (1-x)^{n+1-k} f\left(\frac{k+a}{n+1+b}\right) = \\ &= \sum_{k=1}^{n} \binom{n+1}{k} x^k (1-x)^{n+1-k} f\left(\frac{k+a}{n+1+b}\right) + \\ &+ (1-x)^{n+1} f\left(\frac{a}{n+1+b}\right) + x^{n+1} f\left(\frac{n+1+a}{n+1+b}\right) \end{aligned} \tag{5}$$

Then let us consider the relation:

$$\left(S_n^{a,b}f\right)(x) = \sum_{k=0}^{n} xP_{n,k}(x)f\left(\frac{k+a}{n+b}\right) + \sum_{k=0}^{n}(1-x)P_{n,k}(x)f\left(\frac{k+a}{n+b}\right)$$

In the first sum from the right-hand side of this equality we set $i = k+1$ and then denote again the summation index by k we obtain:

$$\begin{aligned}
\left(S_n^{a,b}f\right)(x) &= \sum_{k=1}^{n+1} xP_{n,k-1}(x)f\left(\frac{k-1+a}{n+b}\right) + \sum_{k=0}^{n}(1-x)P_{n,k}(x)f\left(\frac{k+a}{n+b}\right) = \\
&= \sum_{k=1}^{n}\binom{n}{k-1}x^k(1-x)^{n-k+1}f\left(\frac{k-1+a}{n+b}\right) + x^{n+1}f\left(\frac{n+a}{n+b}\right) + \\
&+ (1-x)^{n+1}f\left(\frac{a}{n+b}\right) + \sum_{k=1}^{n}\binom{n}{k}x^k(1-x)^{n+1-k}f\left(\frac{k+a}{n+b}\right) \qquad (6)
\end{aligned}$$

By using (5) and (6) we get:

$$\begin{aligned}
&\left(S_{n+1}^{a,b}f\right)(x) - \left(S_n^{a,b}f\right)(x) = \sum_{k=1}^{n}\left(\binom{n+1}{k}f\left(\frac{k+a}{n+1+b}\right) -\right. \\
&\left. - \binom{n}{k-1}f\left(\frac{k-1+a}{n+b}\right) - \binom{n}{k}f\left(\frac{k+a}{n+b}\right)\right)x^k(1-x)^{n-k+1} \\
&+ \left(f\left(\frac{a}{n+1+b}\right) - f\left(\frac{a}{n+b}\right)\right)(1-x)^{n+1} + \\
&+ \left(f\left(\frac{n+1+a}{n+1+b}\right) - f\left(\frac{n+a}{n+b}\right)\right)x^{n+1}
\end{aligned}$$

If we use the identities:

$$\binom{n}{k-1} = \frac{k}{n+1-k}\binom{n}{k}, \quad \binom{n+1}{k} = \frac{n+1}{n+1-k}\binom{n}{k}$$

and the notation defined at (4), we can write:

$$\begin{aligned}
&\left(S_{n+1}^{a,b}f\right)(x) - \left(S_n^{a,b}f\right)(x) = -\sum_{k=1}^{n}\binom{n}{k}\left(f\left(\frac{k+a}{n+b}\right) + \frac{k}{n-k+1}f\left(\frac{k-1+a}{n+b}\right) -\right. \\
&\left. -\frac{n+1}{n-k+1}f\left(\frac{k+a}{n+1+b}\right)\right)x^k(1-x)^{n-k+1} + \left(U_n^{a,b}f\right)(x)
\end{aligned}$$

If we make the change $k = j+1$ and then denote again the summation index by k we have:

$$\left(S_{n+1}^{a,b}f\right)(x) - \left(S_{n}^{a,b}f\right)(x) =$$

$$= -\sum_{k=0}^{n-1} \binom{n}{k+1} \left(f\left(\frac{k+1+a}{n+b}\right) + \frac{k+1}{n-k} f\left(\frac{k+a}{n+b}\right) - \right.$$

$$\left. - \frac{n+1}{n-k} f\left(\frac{k+1+a}{n+1+b}\right)\right) x^{k+1}(1-x)^{n-k} +$$

$$+ \left(U_n^{a,b}f\right)(x) = -x(1-x)\sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{n}{k+1} f\left(\frac{k+1+a}{n+b}\right) + \frac{n}{n-k} f\left(\frac{k+a}{n+b}\right) - \right.$$

$$\left. - \frac{n(n+1)}{(k+1)(n-k)} f\left(\frac{k+1+a}{n+1+b}\right)\right) x^k(1-x)^{n-k-1} + \left(U_n^{a,b}f\right)(x) \tag{7}$$

since

$$\binom{n}{k+1} = \frac{n}{k+1}\binom{n-1}{k}.$$

According to the fact that:

$$\frac{n(n+1)}{(k+1)(n-k)} f\left(\frac{k+1+a}{n+1+b}\right) = n\left(\frac{1}{k+1} + \frac{1}{n-k}\right) f\left(\frac{k+1+a}{n+1+b}\right),$$

we can write successively:

$$\frac{n}{k+1} f\left(\frac{k+1+a}{n+b}\right) + \frac{n}{n-k} f\left(\frac{k+a}{n+b}\right) - \frac{n(n+1)}{(k+1)(n-k)} f\left(\frac{k+1+a}{n+1+b}\right) =$$

$$= \frac{n}{k+1}\left(f\left(\frac{k+1+a}{n+b}\right) - f\left(\frac{k+1+a}{n+1+b}\right)\right) + \frac{n}{n-k}\left(f\left(\frac{k+a}{n+b}\right) - f\left(\frac{k+1+a}{n+1+b}\right)\right) =$$

$$= \frac{n}{(n+b)(n+b+1)} \left\{ \frac{k+1+a}{k+1} \left[\frac{k+1+a}{n+1+b}, \frac{k+1+a}{n+b}; f\right] - \right.$$

$$\left. - \frac{(n-k)+(b-a)}{n-k} \left[\frac{k+a}{n+b}, \frac{k+1+a}{n+1+b}; f\right] \right\} =$$

$$= \frac{n}{(n+b)(n+b+1)} \left\{ \left[\frac{k+1+a}{n+1+b}, \frac{k+1+a}{n+b}; f\right] - \left[\frac{k+a}{n+b}, \frac{k+1+a}{n+1+b}; f\right] + \right.$$

$$+\frac{a}{k+1}\left[\frac{k+1+a}{n+1+b},\frac{k+1+a}{n+b};f\right]-\frac{b-a}{n-k}\left[\frac{k+a}{n+b},\frac{k+1+a}{n+1+b};f\right]\Bigg\}=$$

$$=\frac{n}{(n+1)(n+b+1)}\left\{\frac{1}{n+b}\left[\frac{k+a}{n+b},\frac{k+1+a}{n+1+b},\frac{k+1+a}{n+b};f\right]+\right.$$

$$\left.+\frac{a}{k+1}\left[\frac{k+1+a}{n+1+b},\frac{k+1+a}{n+b};f\right]-\frac{b-a}{n-k}\left[\frac{k+a}{n+b},\frac{k+1+a}{n+1+b};f\right]\right\} \tag{8}$$

Taking (8) into account, the equality (7) leads us to the desired formula (3).

□

We notice that if $a=b=0$ our result becomes:

$$\left(S_{n+1}^{0,0}f\right)(x)-\left(S_{n}^{0,0}f\right)(x)=$$

$$=-\frac{x(1-x)}{n(n+1)}\sum_{k=0}^{n-1}\binom{n-1}{k}\left[\frac{k}{n},\frac{k+1}{n+1},\frac{k+1}{n};f\right]x^k(1-x)^{n-k+1}=$$

$$=-\frac{x(1-x)}{n(n+1)}\sum_{k=0}^{n-1}P_{n-1,k}(x)\left[\frac{k}{n},\frac{k+1}{n+1},\frac{k+1}{n};f\right]$$

This formula was established by D.D.Stancu in the paper [8].

3. Sufficient conditions to ensure the monotonicity of sequence

The following definition of the notion of higher-order convex functions is known (see [7]).

Definition 1. A real-valued function on an interval I is called convex of order n on I if all its divided differences of order $n+1$, on $n+2$ distinct points of I, are positive. The function f is said to be non-concave of order n on the interval I is all its divided differences of order $n+1$, on any $n+2$ points of I, are non-negative.

We shall consider the next particular cases:

A. If we choose $a=b>0$ in Theorem 1 we obtain

$$\left(S_{n+1}^{a,a}f\right)(x)-\left(S_{n}^{a,a}f\right)(x)=$$

$$=-\frac{nx(1-x)}{(n+a)(n+a+1)}\sum_{k=0}^{n-1}\binom{n-1}{k}\left\{\frac{1}{n+a}\left[\frac{k+a}{n+a},\frac{k+a+1}{n+a+1},\frac{k+a+1}{n+a};f\right]+\right.$$

$$+\frac{a}{k+1}\left[\frac{k+a+1}{n+a+1},\frac{k+a+1}{n+a};f\right]\right\}x^k(1-x)^{n-k-1}-$$

$$-\frac{a}{(n+a)(n+a+1)}\left[\frac{a}{n+a+1},\frac{a}{n+a};f\right](1-x)^{n+1}.$$

Since on the interval [0,1] we have $P_{n-1,k}(x)\geq 0$ $(k=0,1,\ldots,n-1)$, from this identity and Definition 1 there follows:

Theorem 2. *If the function f is convex of first order on the interval [0,1] and increasing on [0,1] then the sequence* $(S_n^{a,b}f)$ *is decreasing on (0,1), that is* $(S_n^{a,a}f)>(S_{n+1}^{a,a}f)$ *on (0,1) for n = 1,2,...*

Analysing similarly it is easy to state:

Corollary 1. *If the function f is concave of first order on the interval [0,1] and decreasing on [0,1] then the sequence* $(S_n^{a,a}f)$ *is increasing on the interval (0,1).*

B. If we take $a=0$ and $b>0$ in Theorem 1 we have:

$$\left(S_{n+1}^{0,b}f\right)(x)-\left(S_n^{0,b}f\right)(x)=$$

$$=-\frac{nx(1-x)}{(n+b)(n+b+1)}\sum_{k=0}^{n-1}\binom{n-1}{k}\left\{\frac{1}{n+b}\left[\frac{k}{n+b},\frac{k+1}{n+b+1},\frac{k+1}{n+b};f\right]-\right.$$

$$\left.-\frac{b}{n-k}\left[\frac{k}{n+b},\frac{k+1}{n+b+1};f\right]\right\}x^k(1-x)^{n-k-1}+\frac{b}{(n+b)(n+b+1}\left[\frac{n+1}{n+b+1},\frac{n}{n+b};f\right]$$

Now we can state the following proposition:

Theorem 3. *If the function f is convex of first order on the interval [0,1] and decresing on [0,1] then sequence* $(S_n^{0,b}f)$ *is decreasing on the interval (0,1), that is*

$$(S_n^{0,b}f)>\left(S_{n+1}^{0,b}f\right) \tag{9}$$

on (0,1) and n = 1,2,...

Analogously we formulate:

Corollary 2. *If the function f is concave of first order on the interval [0,1] and increasing on [0,1] then the sequence* $(S_n^{0,b}f)$ *is decreasing on the interval (0,1).*

To conclude, we mention that different liniar approximation operators introduced and studied by D.D.Stancu (mainly by probabilistic methods) have been the object of other investigations, made by many other researchers - see [1], [2], [3], [4], [5], [6].

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ON THE APPROXIMATION BY INTEGRAL FAVARD-SZASZ TYPE OPERATORS

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Dedicated to Prof. Gheorghe Coman at his 60^{th} anniversary

Abstract. In this paper we study a modified Favard-Szasz type operator for the approximation of integrable functions on [0,∞). We give an asymptotic result of Voronovskaia type. We also prove that the studied operator has variation-diminishing properties.

I. Introduction In our paper [2], we have introduced a new operator for the approximation of integrable functions on $[0,\infty)$. We have modified the operator of A.Jakimovski and D.Leviatan [4]. Let us remind this operator. One considers $g(z) = \sum_{n=0}^{\infty} a_n z^n$ an analytic function in the disk $|z| < R$, $R > 1$ and supposes $g(1) \neq 0$. One defines the Appell polynomials $p_k(x)$, $k \geq 0$ by

$$g(u)e^{ux} \equiv \sum_{k=0}^{\infty} p_k(x)u^k \tag{1}$$

To each function f defined in $[0,\infty)$, A.Jakimovski and D.Leviatan [4], associated the operators:

$$(P_n f)(x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right) \tag{2}$$

If $g(z) \equiv 1$, one obtains $p_k(x) = \frac{x^k}{k!}$ and the operators P_n becomes the well known Favard-Szasz operators:

$$(S_n f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \tag{3}$$

B.Wood [8] proved that the operators defined at (2) are positive if and only if $\frac{a_n}{g(1)} \geq 0, n = 0, 1, \ldots$

We have modified the operator P_n by replacing $f\left(\frac{k}{n}\right)$ by a positive linear functional

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defined for integrable functions on $[0,\infty)$ by

$$A_k(f)=\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^\infty e^{-nt}t^{\lambda+k}f(t)dt,\ \lambda\geq 0$$

and thus we have obtained the operators

$$(L_nf)(x)=\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty}p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^\infty e^{-nt}t^{\lambda+k}f(t)dt,\ x\geq 0 \qquad (4)$$

This operators are linear and positive, i.e. $\frac{a_n}{g(1)}\geq 0,\ n=0,1,\ldots$

Remark: for $g(z)\equiv 1$ and $\lambda=0$ one obtains the operators

$$(S_n^*f)(x)=e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\int_0^\infty e^{-nt}\frac{(nt)^k}{k!}f(t)dt \qquad (5)$$

The operators S_n^* has been introduced by S.M.Mazhar and V.Totik [5]. S.P.Singh and M.Tiwari [7], Zhu-Rui Guo and Ding-Xuan Zhou [9] also have studied the properties of the operators S_n^*.

II. In this section we give a basic result for the approximation of integrable functions by means of the sequence (L_n). The next lemma gives the values of the operator P_n for the test functions.

Lemma 2.1. [2] *For all $x\geq 0$, we have*

$$(L_ne_0)(x)=1$$

$$(L_ne_1)(x)=x+\frac{1}{n}\left(\lambda+1+\frac{g'(1)}{g(1)}\right)$$

$$(L_ne_2)(x)=x^2+\frac{2x}{n}\left(\lambda+2+\frac{g'(1)}{g(1)}\right)+\frac{1}{n^2}\left[(\lambda+1)(\lambda+2)+(2\lambda+3)\frac{g'(1)}{g(1)}+\frac{g''(1)+g'(1)}{g(1)}\right]$$

where $e_i(x)=x^i, i\in 0,1,2$.

Theorem.2. *Let f be integrable on $[0,\infty)$ and bounded. Then,* $\lim_{n\to\infty}(L_nf)(x)=f(x)$,*almost everywhere on* $[0,\infty)$.

Proof. Let $F(x) = \int_0^x f(t)dt$ be, where f is integrable on $(0,\infty)$. Because $F'(a) = f(a)$ almost everywhere on $(0,\infty)$, we will prove that the sequence $(L_n f)(a)$ converges to $f(a)$.

We will denote $\Phi_{n,k}(t) = t^{\lambda+k}e^{-nt}$. For all k, $0 \leq k < \infty$, the function $\Phi_{n,k}(t)F(t)$ is absolutely continuous and we have, for all $t \in (0,\infty)$:

$$\Phi_{n,k}(t)F(t) = \int_0^t \Phi'_{n,k}(y)F(y)dy + \int_0^t \Phi_{n,k}(y)f(y)dy$$

Hence, we can write the operator L_n as follows:

$$(L_n f)(x) = -\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}\Phi'_{n,k}(t)F(t)dt$$

The function F is differentiable and we have

$$F(t) = F(a) + (t-a)F'(a) + (t-a)\varepsilon(t-a),$$

where $\varepsilon(u) \to 0$ when $u \to 0$ and $\varepsilon(u)$ is bounded.

So, we obtain:

$$(L_n f)(x) = -\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}\Phi'_{n,k}(t)[F(a)+$$
$$+(t-a)F'(a) + (t-a)\varepsilon(t-a)]dt$$

Next, we will use the linearity of the operator and it must consider two cases;
(i) If $\lambda > 0$ we have $\lim_{t\to\infty}\Phi_{n,k}(t) = 0$ and $\Phi_{n,k}(0) = 0, k = 0,1,\ldots$
Hence

$$(L_n f)(a) = \frac{e^{-na}}{g(1)}\sum_{k=0}^{\infty} p_k(na)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}F'(a)\int_0^{\infty}\Phi_{n,k}(t)dt -$$

$$-\frac{e^{-na}}{g(1)}\sum_{k=0}^{\infty} p_k(na)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}(t-a)\varepsilon(t-a)\Phi_{n,k}n,k'(t)dt =$$

$$= F'(a) - \frac{e^{-na}}{g(1)}\sum_{k=0}^{\infty} p_k(na)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}(t-a)\varepsilon(t-a)\Phi_{n,k}n,k'(t)dt$$

(ii) If $\lambda = 0$, we also have $\lim_{n\to\infty}\Phi_{n,k}(t) = 0, \Phi_{n,k}(0) = 0$ for $k = 1,2,\ldots$ and $\Phi_{n,0}(0) = 1$
So, we obtain

$$(L_nf)(a) = \frac{e^{-na}}{g(1)}p_0(na)nF(a) - \frac{e^{-na}}{g(1)}p_0(na)nF'(a)a + F'(a) -$$

$$-\frac{e^{-na}}{g(1)}\sum_{k=0}^{\infty}p_k(na)\frac{n^{k+1}}{k!}\int_0^{\infty}(t-a)\varepsilon(t-a)\Phi'_{n,k}(t)dt$$

Now, to prove that $\lim_{n\to+\infty}(L_nf)(a) = F'(a)$ almost everywhere on $(0,\infty)$, it is suffices to show that $R_n(a) \to 0$ as $n \to \infty$, where we have denoted

$$R_n(a) = \frac{e^{na}}{g(1)}\sum_{k=0}^{\infty}p_k(na)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}(t-a)\varepsilon(t-a)\Phi'_{n,k}(t)dt$$

To continue the proof, let us remark that

$$\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\Phi'_{n,k}(t) = n\left(-\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\Phi_{n,k}(t) + \frac{n^{\lambda+k}}{\Gamma(\lambda+k)}\Phi_{n,k-1}(t)\right)$$

If we denote $T_{n,k}(t) = \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\Phi_{n,k}(t)$, we can write that

$$\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\Phi'_{n,k}(t) = n\left(-T_{n,k}(t) + T_{n,k-1}(t)\right)$$

Now, we can write successively:

$$R_n(a) = \frac{e^{-na}}{g(1)}\sum_{k=0}^{\infty}p_k(na)\int_0^{\infty}n\left(-T_{n,k}(t) + T_{n,k-1}(t)\right)(t-a)\varepsilon(t-a)dt =$$

$$= \frac{e^{-na}}{g(1)}\sum_{k=0}^{\infty}(p_k(na) - p_{k+1}(na))\,n\int_0^{\infty}T_{n,k}(t)(a-t)\varepsilon(t-a)dt$$

By making use the Schwarz's inequality for summation, we have

$$R_n^2(a) \le n^2\frac{e^{-na}}{g(1)}\sum_{k=0}^{\infty}(p_k(na) - p_{k+1}(na))\cdot$$

$$\cdot\frac{e^{-na}}{g(1)}\sum_{k=0}^{\infty}(p_k(na) - p_{k+1}(na))\left(\int_0^{\infty}T_{n,k}(t)(a-t)\varepsilon(t-a)dt\right)^2 = n^2S_1(n,a)\cdot S_2(n,a)$$

Again, by using the Schwarz's inequlity for integral, we can write that

$$\left(\int_0^{\infty}T_{n,k}(t)(a-t)\varepsilon(t-a)dt\right)^2 \le \int_0^{\infty}T_{n,k}(t)dt\cdot\int_0^{\infty}T_{n,k}(t)(a-t)^2\varepsilon^2(t-a)dt =$$

$$= \int_0^{\infty}T_{n,k}(t)(a-t)^2\varepsilon^2(t-a)dt$$

Therefore, we have

$$S_2(n,a) \leq \frac{e^{-na}}{g(1)} \sum_{k=0}^{\infty} (p_k(na) - p_{k+1}(na)) \int_0^{\infty} T_{n,k}(t)(a-t)^2 \varepsilon^2(t-a)dt =$$

$$= \frac{e^{-na}}{g(1)} \sum_{k=0}^{\infty} (p_k(na) - p_{k+1}(na)) \left\{ \int_{|a-t|<\delta} T_{n,k}(t)(a-t)^2 \varepsilon^2(t-a)dt + \right.$$

$$\left. + \int_{|a-t|\geq\delta} T_{n,k}(t)(a-t)^2 \varepsilon^2(t-a)dt \right\} = I_1 + I_2$$

Because $\varepsilon(u) \to 0$ for $u \to 0$, it results that there exists to each $\varepsilon > 0$ a $\delta > 0$ such that $|\varepsilon(t-a)| < \varepsilon$ whenever $|t\text{-}a| < \delta$.

It follows that

$$I_1 = \frac{e^{-na}}{g(1)} \sum_{k=0}^{\infty} (p_k(na) - p_{k+1}(na)) \int_{|a-t|<\delta} T_{n,k}(t)(a-t)^2 \varepsilon^2(t-a)dt \leq$$

$$\leq \varepsilon^2 \frac{e^{-na}}{g(1)} \sum_{k=0}^{\infty} (p_k(na) - p_{k+1}(na)) \int_0^{\infty} T_{n,k}(t)(a-t)^2 dt$$

We have

$$\int_0^{\infty} T_{n,k}(t)(a-t)^2 dt = \frac{1}{n^2} \left[(\lambda+1)(\lambda+2) + (2\lambda+3)k + k^2 \right] - \frac{2a}{n}(\lambda+k+1) + a^2$$

and so we obtain

$$I_1 \leq \varepsilon^2 \frac{e^{-na}}{g(1)} \sum_{k=0}^{\infty} (p_k(na) - p_{k+1}(na)) \,.$$

$$\cdot \left\{ \frac{1}{n^2} \left[(\lambda+1)(\lambda+2) + (2\lambda+3)k + k^2 \right] - \frac{2a}{n}(\lambda+k+1) + a^2 \right\} = J_1 - J_2$$

To calculate J_1 and J_2 we will use the values of the operator P_n for the monomials

$$e_i(x) = x^i, i \in \{0,1,2\}:$$

$$(P_n e_0)(x) = 1$$

$$(P_n e_1)(x) = x + \frac{1}{n} \cdot \frac{g'(1)}{g(1)}$$

$$(P_n e_2)(x) = x^2 + \frac{x}{n} \left(1 + 2\frac{g'(1)}{g(1)} \right) + \frac{1^2}{n} \cdot \frac{g''(1) + g'(1)}{g(1)}$$

We have

$$J_1 = \varepsilon^2\frac{e^{-na}}{g(1)}\sum_{k=0}^{\infty} p_k(na)\left\{\frac{1}{n^2}\left[(\lambda+1)(\lambda+2)+(2\lambda+3)k+k^2\right]-\frac{2a}{n}(\lambda+k+1)+a^2\right\}=$$

$$=\frac{\varepsilon^2}{n^2}\left[(\lambda+1)(\lambda+2)+(2\lambda+3)\frac{g'(1)}{g(1)}+\frac{g''(1)+g'(1)}{g(1)}\right]+\frac{\varepsilon^2}{n}\cdot 2a$$

To calculate

$$J_2=\varepsilon^2\frac{e^{-na}}{g(1)}\sum_{k=0}^{\infty} p_{k+1}(na)\cdot$$

$$\cdot\left\{\frac{1}{n^2}\left[(\lambda+1)(\lambda+2)+(2\lambda+3)k+k^2\right]-\frac{2a}{n}(\lambda+k+1)+a^2\right\}$$

we substitute into it $k+1=i$, and in according with

$$(P_nf)(x)=\frac{e^{-nx}}{g(1)}a_0f(0)+\frac{e^{-nx}}{g(1)}\sum_{i=1}^{\infty}p_i(nx)f\left(\frac{i}{n}\right),$$

we obtain

$$J_2=\varepsilon^2\frac{e^{-na}}{g(1)}\sum_{i=1}^{\infty}p_i(na)\left\{\frac{1}{n^2}[(\lambda+1)(\lambda+2)+(2\lambda+3)(i-1)+(i-1)^2]-\frac{2a}{n}(\lambda+i)+a^2\right\}$$

$$=\frac{\varepsilon^2}{n^2}\left[\left(1-a_0\frac{e^{-na}}{g(1)}\right)(\lambda^2+\lambda)+(2\lambda+3)\frac{g'(1)}{g(1)}+\frac{g''(1)+g'(1)}{g(1)}\right]+$$

$$+\frac{\varepsilon^2}{n}2a\left(1+\lambda_0\frac{e^{-na}}{g(1)}\right)-\varepsilon^2a^2a_0\frac{e^{-na}}{g(1)}$$

Thus, it results that

$$I_1\le J_1-J_2=\frac{\varepsilon^2}{n^2}\left[\left(a_0\frac{e^{-na}}{g(1)}-1\right)(\lambda^2+\lambda)+2\frac{g'(1)}{g(1)}+(\lambda+1)(\lambda+2)\right]-$$

$$-\frac{\varepsilon^2}{n}2a\lambda_0\frac{e^{-na}}{g(1)}+\varepsilon^2a^2a_0\frac{e^{-na}}{g(1)}=\varepsilon^2\mathcal{O}\left(\frac{1}{n^2}\right)$$

For $|a\text{-}t|\ge\delta$, the boundedness of $\varepsilon(t-a)$ on $[0,\infty)$ implies that $|\,(t\text{-}a)|\le M \le M\frac{|a-t|}{\delta}$. Using this inequality, similarly we obtain

$$I_2=\frac{e^{-na}}{g(1)}\sum_{k=0}^{\infty}(p_k(na)-p_{k+1}(na))\int_{|a-t|\ge\delta}T_{n,k}(t)(a-t)^2\varepsilon^2(t-a)dt\le$$

$$\leq \frac{M^2}{\delta^2}\frac{e^{-na}}{g(1)}\sum_{k=0}^{\infty}(p_k(na) - p_{k+1}(na))\int_0^{\infty} T_{n,k}(t)(a-t)^4 dt = \frac{M^2}{\delta^2}\mathcal{O}\left(\frac{1}{n^3}\right)$$

Thus, we get to

$$S_2(n,a) \leq I_1 + I_2 \leq \left(\varepsilon^2 + \frac{M^2}{n\delta^2}\right)\mathcal{O}\left(\frac{1}{n^2}\right)$$

and so,

$$R_n^2(a) \leq n^2\frac{e^{-na}}{g(1)}\sum_{k=0}^{\infty}(p_k(na) - p_{k+1}(na)) \cdot S_2(n,a) \leq$$

$$\leq n^2\frac{e^{-na}}{g(1)}\sum_{k=0}^{\infty}(p_k(na) - p_{k+1}(na)) \cdot \left(\varepsilon^2 + \frac{M^2}{n^2\delta^2}\right)\mathcal{O}\left(\frac{1}{n^2}\right) =$$

$$= n^2\left(\varepsilon^2 + \frac{M^2}{n^2\delta^2}\right) \cdot a_0 \cdot \frac{e^{-na}}{g(1)} \cdot \mathcal{O}\left(\frac{1}{n^2}\right)$$

It results that $\lim_{n\to\infty} R_n^2(a) = 0$. This completes the proof.

III. In this section we establish an asymptotic estimate of the remainder, which corresponds to a result of Voronovskaia about Bernstein polynomials.

Theorem 3.1 *Let f be integrable and bounded on $[0,\infty)$. If f'' exists at a point x $[0,\infty)$, then*

$$\lim_{n\to\infty} n\left[(L_n f)(x) - f(x)\right] = \left(\lambda + 1 + \frac{g'(1)}{g(1)}\right)f'(x) + xf''(x)$$

Proof. Because f'' exists at a point $x \in [0,\infty)$, we may write the Taylor's expansion

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2!}f''(x) + (t-x)^2\varepsilon(t-x),$$

where $\varepsilon(u) \to 0$ as $u \to 0$.

Hence, there exists to each $\varepsilon > 0$ a $\delta > 0$ such that $|\varepsilon(t-x)| < \varepsilon$ whenever $|t\text{-}x| < \delta$, and also there exists a positive number M such that $|\varepsilon(t-x)| \leq M$.

Now we introduce the function $\lambda_\delta(t)$ defined by

$$\lambda_\delta(t) = \begin{cases} 1 & \text{whenever } |t-x| \geq \delta \\ 0 & \text{whenever } |t-x| < \delta \end{cases}$$

Then the inequality $|\varepsilon(t-x)| \leq \varepsilon + M \cdot \lambda_\delta(t)$ holds everywhere on $[0,\infty)$. Since the operators L_n are linear it follows that

$$(L_n f)(x) - f(x) = f'(x)\left(L_n(t-x)\right)(x) + \frac{1}{2}f''(x)\left(L_n(t-x)^2\right)(x) + E(t,x),$$

where $E(t,x) = \left(L_n(t-x)^2\varepsilon(t-x)\right)(x)$

By making use of lemma 2.1, we can write:

$$n\left[(L_n f)(x) - f(x)\right] = \left(\lambda + 1 + \frac{g'(1)}{g(1)}\right) f'(x) +$$

$$+\frac{1}{2}\left\{2x + \frac{1}{n}\left[(\lambda+1)(\lambda+2) + (2\lambda+3)\frac{g'(1)}{g(1)} + \frac{g''(1)+g'(1)}{g(1)}\right]\right\} f''(x) + E(t,x)$$

To prove the enunciated asymptotic it is sufficient to show that

$$n \cdot E(t,x) = H_n(t,x) \to 0 \text{ as } n \to \infty$$

We have

$$|H_n(t,x)| \leq n\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^\infty e^{-nt}t^{\lambda+k}(t-x)^2\,|\varepsilon(t-x)|\,dt =$$

$$= \varepsilon \cdot n\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^\infty e^{-nt}t^{\lambda+k}(t-x)^2 dt +$$

$$+M \cdot n\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^\infty e^{-nt}t^{\lambda+k}(t-x)^2\lambda_\delta(t)dt = I_1 + I_2$$

Obvious, in view of lemma 2.1, we have

$$I_1 = \varepsilon \cdot n\left(L_n(t-x)^2\right)(x) =$$

$$= \varepsilon \cdot \left\{2x + \frac{1}{n}\left[(\lambda+1)(\lambda+2) + (2\lambda+3)\frac{g'(1)}{g(1)} + \frac{g''(1)+g'(1)}{g(1)}\right]\right\} = \varepsilon \cdot \mathcal{O}(1)$$

If $|t\text{-}x| \geq \delta$, it results that $1 \leq \frac{(t-x)^2}{\delta^2}$ and so, we have

$$I_2 \leq \frac{M}{\delta^2} \cdot n \cdot \frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^\infty e^{-nt}t^{\lambda+k}(t-x)^4 dt = \mathcal{O}\left(\frac{1}{n^2}\right)$$

Combining the estimates of I_1 and I_2, due to arbitrariness of $\varepsilon > 0$, it results that $H_n(t,x) \to 0$ for sufficiently large n.

IV. Variation-diminishing properties. I.J.Schoenberg has introduced [6] the concept of a variation-diminishing operator. Such an operator has the property $V[L_n(f)] \leq V[f]$, where $V[f]$ is the variation of f, defined as the number of changes of sign of the function f as x varies its domain.

The next lemma is due to E.W.Cheney and A.Sharma [1].

Lemma 4.1. *Let* $f(x) = \sum_{k=0}^{\infty} a_k x^k$ *be uniformly convergent on* $[0,a]$. *Then* $V[f] \leq V[\{\alpha_k\}]$..

The following lemma is obvious.

Lemma 4.2. *Let* $\beta_\nu = \sum_{k=\nu}^{\infty} \alpha_k$, $\nu = 0,1,2,...$ *Then* $V[\{\beta_\nu\}] \leq V[\{\alpha_\nu\}]$.

Lemma 4.3. [3] *Let* $A : C[0,a] \to R, a \leq \infty$ *be a positive functional and* $e_k(x) = x^k$. *If* $\operatorname{sgn} A(f \cdot e_{k_i}) = (-1)^i, i = \overline{1,n}, k_1 < k_2 < ..., k_n$, *then* f *changes the sign in at least* n-1 *points of* $[0,a]$.

Theorem 4.4. *If* f $C[0,a]$ *and* $| f(t) | \leq e^{At}, t \geq 0$ *and some finite* A, *then* L_n *is variation-diminishing,* i.e. $V[L_n(f)] \leq V[f]$.

Proof. We have proved [2] that if $f \in C[0,a]$ and $| f(t) | \leq e^{At}, t \geq 0$, A finite, then we have uniformly $\lim_{n\to\infty}(L_n f)(x) = f(x), x \in [0,a]$.

We can write the operator as follows:

$$(L_n f)(x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} \left(\frac{n^k}{k!} \sum_{\nu=k}^{\infty} a_{\nu-k} A_\nu(f) \right) \cdot x^k$$

where the series are uniformly convergent on $[0,a]$, for any $a > 0$. Since L_n is positive in $[0,\infty)$, we have $a_\nu \geq 0, \nu = 0,1,...,$ or $a_\nu \leq 0, \nu = 0,1,...$

Let us denote $\phi(x) \equiv g(1)e^{nx}(L_n f)(x)$. The lemmas 4.1, 4.2 and 4.3 imply

$$V[L_n(f)] = V[\phi(x)] \leq V\left[\left\{ \frac{n^k}{k!} \sum_{\nu=k}^{\infty} a_{\nu-k} A_\nu(f) \right\}\right] \leq$$

$$\leq V[\{a_{\nu-k} A_\nu(f)\}] = V[\{A_\nu(f)\}] \leq V[f].$$

This completes the proof.

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QUADRATURES BASED ON QUASI-INTERPOLATING SPLINE-PROJECTORS FOR PRODUCT SINGULAR INTEGRATION

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Dedicated to Prof. Gheorghe Coman at his 60^{th} *anniversary*

Abstract. In this paper we propose product quadrature rules based on quasi-interpolating spline-projectors and we prove convergence results for bounded integrands. Convergence results are proved for sequences of Cauchy Principal Value Integrals of these quasi-interpolating spline-projectors.

1. Introduction.

In several applications we have to deal with integrals of the form

$$I(Kf) = \int_{-1}^{1} K(x)f(x)dx \tag{1}$$

where K is singular, but absolutely integrable, and f is bounded in $\mathcal{I} \equiv [-1,1]$, or of the form

$$J(uf;\lambda) = \int\!\!\!\!\!\!-_{-1}^{1} u(x)\frac{f(x)}{x-\lambda}dx, \quad -1 < \lambda < 1 \tag{2}$$

where

$$J(uf;\lambda) = \lim_{\epsilon\to 0}\left\{\int_{-1}^{\lambda-\epsilon} + \int_{\lambda+\epsilon}^{1}\right\}\frac{u(x)f(x)}{x-\lambda}dx$$

is the Cauchy Principal Value (CPV) of $u\,f$, with u and f such that $J(uf;\lambda)$ exists.

The majority of numerical methods proposed for (1) and (2) are global methods, usually based on orthogonal polynomial approximations in [-1,1].

It is well known that the above methods have good convergence properties, if f is smooth over the entire interval. However since they can present some difficulties associated with their implementation (see [11]). For instance, since the node points are generally chosen as the zeros of an orthogonal polynomial, global methods are not appropriate for integrals with input functions f that behave *badly* in some subinterval $[\alpha,\beta]$ of [-1,1].

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For such integrals a numerical method with no restriction on the choice of node points would have to be used in order to concentrate the nodes in $[\alpha,\beta]$. Generally this is not possible with global methods.

In contrast, local methods permit a flexible choice of the node points. Therefore some authors (see [1]-[6]) have proposed quadratures for (1) and (2) based on the approximation of f by a spline of order k of the form

$$S(x) = \sum_{i=1-k}^{N-1} \alpha_i B_{ik}(x)$$

where $\{B_{ik}(x)\}$ is the set of normalized B-splines forming a basis for a given spline space $S_{k,\pi}$ and α_i are chosen so that S interpolates f at some convenient set of points. Since the convergence theory for interpolatory splines is incomplete and imposes restrictions both on the spline spaces $S_{k,\pi}$ and the interpolation points, the theory of corresponding integration rules is not completely satisfactory (see [1],[6] and [13]).

For such reason, recently, some papers have tried to remedy the disadvantages of the above local schemes by using local schemes, based on quasi-interpolatory (q-i) splines (see [7]-[10] and [13]). Such strategies approximate f by q-i splines $Q_N f$ of the form

$$Q_N f(x) = \sum_{i=1-k}^{N-1} \left(\sum_{j=1}^{\ell} \alpha_{ij} [\tau_{i1}, \tau_{i2}, \dots, \tau_{ij}] f \right) B_{ik}(x) \tag{3}$$

that is defined for any $f \in C[-1,1]$ and where $[z_0,z_1,\dots,z_p]f$ is the p-th divided difference based on the points $z_0,z_1,\dots,z_p$, $\ell \leq k$, the sets $\{\tau_{i1},\tau_{i2},\dots,\tau_{i\ell}\}$, $i = 1$-k, ..., N-1, are chosen in a suitable way in [-1,1] and the α_{ij} are such that

$$Q_N g = g \tag{4}$$

for all $g \in P_\ell$, the set of polynomials of order ℓ or degree $\leq \ell - 1$.

If the τ_{ij} are distinct for each i, the divided differences in (3) involve only function values. In this case, since

$$[\tau_{i1}, \tau_{i2}, \dots, \tau_{ij}] f = \sum_{r=1}^{j} \frac{f(\tau_{ir})}{\prod_{\substack{s=1 \\ s \neq r}}^{j} (\tau_{ir} - \tau_{is})} \tag{5}$$

we can obtain the following expression for $Q_N f$

$$Q_N f = \sum_{i=1-k}^{N-1} B_{ik}(x) \sum_{j=1}^{\ell} v_{ij} f(\tau_{ij}) \tag{6}$$

where

$$v_{ij} = \sum_{r=j}^{\ell} \frac{\alpha_{ir}}{\prod_{\substack{s=1 \\ s\neq j}}^{r} (\tau_{ij} - \tau_{is})} \tag{7}$$

Once we choose a spline space $S_{k,\pi}$ and the set of points $\{\tau_{ij}\}$ we replace f by $Q_N f$ in (1) and obtain

$$I(Kf) \simeq I(KQ_N f) = \sum_{i=1-k}^{N-1} \sum_{j=1}^{\ell} \mu_{ij} f(\tau_{ij}) \tag{8}$$

where

$$\mu_{ij} = v_{ij} I(KB_{ik}). \tag{9}$$

Similarly, for CPV integral (2), we obtain

$$J(uf;\lambda) \simeq J(uQ_N f;\lambda) = \sum_{i=1-k}^{N-1} \sum_{j=1}^{\ell} \nu_{ij}(\lambda) f(\tau_{ij}) \tag{10}$$

where

$$\nu_{ij}(\lambda) = v_{ij} J(uB_{ik};\lambda). \tag{11}$$

We recall that the quadratures (8) and (10), proposed up to now in the literature (see [7]-[10] and [14]) and based on q.i splines (3), have interesting convergence properties but, contrary to those based on interpolatory splines, they are not exact for splines belonging to $S_{k,\pi}$.

In this paper we propose and study quadratures for (1) and (2) based on an approximation method of the form (6) with $\ell=k$, that is a projector, i.e. reproducing splines of order k.

In Section 2 we define the q.i spline-projector (s-p) operators and we emphasize some of their properties.

In Section 3 we propose product quadratures based on a q.i spline-projector and we prove convergence results. We remark that they can be generalized to other q.i spline-projectors.

2. On quasi-interpolating spline-projectors.

Let Y_n: $-1 = y_0 < y_1 < \ldots < y_n = 1$ and a corresponding sequence of positive integers $\{d_i\}_{i=1}^{n-1}$ be given. We write π_N for the non decreasing sequence $\{x_i\}_{i=0}^{N}$ obtained from $\{y_i\}_{i=0}^{n}$ by repeating y_i exactly d_i times (thus $N = \sum_{i=1}^{n-1} d_i + 1$). If k is an integer with $k > d_i$, $i = 1,2,\ldots,n-1$, we define the class of polynomial splines of order k and knots of multiplicity d_i at the points $\{y_i\}$:

$$
\begin{aligned}
S_{k,\pi_N} = \quad & \{g : g|_{(y_i,y_{i+1})} \in P_k, \quad i = 0,1,\ldots,n-1 \\
& g^{(j)}(y_i^+) = g^{(j)}(y_i^-), \quad j = 0,1,\ldots,k-d_i-1 \\
& \qquad\qquad\qquad\qquad i = 1,2,\ldots,n-1\}
\end{aligned}
$$

where P_k is the set of polynomials of degree less than k.

We say that, the sequence of partitions $\{Y_n\}$ is locally uniform (l.u), if

$$
\frac{y_{i+1} - y_i}{y_{j+1} - y_j} \leq A, \quad \text{for all } 0 \leq i < n \text{ and } j = i \pm 1, \tag{12}
$$

where $A \geq 1$.

To define a basis for S_{k,π_N}, let $\pi_N = \{x_i\}_{i=0}^{N}$ be extended to a non decreasing sequence

$$
\pi_{N_e} = \{x_i\}_{i=1-k}^{N+k-1}
$$

where

$$
\begin{aligned}
& x_i < x_{i+k}, \quad i = 1-k, \ldots, N-1 \\
& x_{1-k} = \ldots = x_{-1} = x_0 = -1, \\
& x_{N+k-1} = \cdots = x_{N+1} = x_N = 1.
\end{aligned} \tag{13}
$$

With

$$
G_k(t; x) = (t - x)_+^{k-1},
$$

we define

$$
B_{ik}(x) = (x_{i+k} - x_i)[x_i, \ldots, x_{i+k}]G_k(\cdot, x), \quad i = 1-k, \ldots, N-1. \tag{14}
$$

The functions B_{ik} satisfy

$$\begin{cases} 0 < B_{ik}(x) \leq 1 & \text{for all } x \in (x_i, x_{i+k}) \\ B_{ik}(x) = 0 & \text{otherwise,} \end{cases} \tag{15}$$

except that $B_{1-k,k}(-1) = B_{N-1,k}(1) = 1$;

$$\sum_{i=1-k}^{N-1} \xi_i^{(\mu)} B_{ik}(x) = U_\mu(x) = x^{\mu-1}, \; \mu = 1, 2, ..., k \tag{16}$$

with

$$\xi_i^{(\mu)} = (-1)^{\mu-1} \frac{(\mu-1)!}{(k-1)!} \psi_i^{(k-\mu)}(0) = \frac{\text{symm}_{\mu-1}(x_{i+1}, ..., x_{i+k-1})}{\begin{pmatrix} k-1 \\ \mu-1 \end{pmatrix}}$$

where $\text{symm}_{\mu-1}(x_{i+1}, ..., x_{i+k-1})$ and ψ_i are defined by

$$\begin{aligned} \psi_i(x) &= (x - x_{i+1})...(x - x_{i+k-1}) = \\ &= \sum_{\mu=1}^{k} (-1)^{\mu-1} x^{k-\mu} \text{symm}_{\mu-1}(x_{i+1}, ..., x_{i+k-1}). \end{aligned}$$

Let $\mathcal{F}$ be a linear space of real-valued functions on $\mathcal{I}$ and let $\{\lambda_i\}_{i=1-k}^{N-1}$ be a set of linear functionals $\lambda_i : \mathcal{F} \to \mathrm{R}$. Given $f \in \mathcal{F}$ we construct an approximation for f (see [12]):

$$Q_N f(x) = \sum_{i=1-k}^{N-1} \lambda_i f B_{ik}(x) \tag{17}$$

that defines a linear operator mapping $\mathcal{F}$ into S_{k,π_N}.

For each i = 1-k, ..., N-1 we consider a set $\{\lambda_{ij}\}_{j=1}^{k}$ of linear functionals defined on $\mathcal{F}$ such that

$$\det(\lambda_{ij} U_\mu)_{j,\mu=1}^{k} \neq 0. \tag{18}$$

Let $\{\alpha_{ij}\}_{j=1}^{k}$ be the solution of

$$\sum_{j=1}^{k} \alpha_{ij} \lambda_{ij} U_\mu = \xi_i^{(\mu)}, \; \mu = 1, 2, ..., k. \tag{19}$$

We assume

$$\lambda_i = \sum_{j=1}^{k} \alpha_{ij} \lambda_{ij} \tag{20}$$

in the definition (17) of Q_N.

We recall (see [12]) that the operator (17) is a projector for all $S \in S_{k,\pi_N}$ if and only if $\{\lambda_i\}_{i=1-k}^{N-1}$ is a dual basis to $\{B_{ik}\}_{i=1-k}^{N-1}$, i.e.

$$\lambda_i B_{jk} = \delta_{ij},\ i,j = 1-k,...,N-1.$$

The following Theorem (see [12]) gives a sufficient condition to assure that Q_N is a projector.

Theorem 2.1. *For $i = 1-k,...,N-1$, let $\{\lambda_{ij}\}_{j=1}^{k}$ satisfy (18), and suppose $\{\lambda_{ij}\}_{j=1}^{k}$ all have support in one subinterval $[x_{\nu_i}, x_{\nu_i+1}]$ of $[x_i, x_{i+k}]$. Then with $\{\alpha_{ij}\}_{j=1}^{k}$ given by (19) the set $\{\lambda_i\}_{i=1-k}^{N-1}$ is a dual set to $\{B_{ik}\}_{i=1-k}^{N-1}$.*

If we define

$$\lambda_{ij} f = [\tau_{i1}, \tau_{i2}, ..., \tau_{ij}] f \tag{21}$$

with $\{\tau_{ij}\}_{j=1}^{k}$ distinct and chosen from intervals $[x_{\nu_i}, x_{\nu_i+1}] \subset [x_i, x_{i+k}]$, then the hypotheses of Theorem 2.1 are verified and the corresponding operator defined in (17) is a q.i spline-projector, defined for any $f \in C(\mathcal{I})$.

We remark that in this case $\{\lambda_{ij}\}_{j=1}^{k}$ satisfy (18) and, moreover, the hypotheses of Theorem 3.3 of [12] hold. Therefore, from Lemma 4.3 of [12] the real values $\{\alpha_{ij}\}_{j=1}^{k}$ are given by

$$\begin{cases} \alpha_{i1} = 1 \\ \alpha_{ij} = \frac{(k-j)!}{(k-1)!} \sum (x_{\mu 1} - \tau_{i1}) ... (x_{\mu_{j-1}} - \tau_{i,j-1}),\ j = 2,...,k \end{cases} \tag{22}$$

where the sum is taken over all choices of distinct $\mu_1, ..., \mu_{j-1}$ from $i+1,...,i+k-1$.

Now we consider how well the above q.i spline-projector defined by (17), (20),(21) approximates smooth functions. In particular we are interested in the quantities

$$E_{rs}(t) = \begin{cases} D^r (f - Q_N f)(t), & 0 \le r < s \\ D^r Q_N f(t), & s \le r < k. \end{cases} \tag{23}$$

where D^r is the r-th derivative operator and s is an integer with $1 \le s \le k$.

Let $\{Y_n\}$ be such that

$$\max_{0 \le i \le n-1} (y_{i+1} - y_i) \to 0 \text{ as } n \to \infty \tag{24}$$

and for a fixed -1 $\leq t \leq 1$ let m be such that $x_m \leq t < x_{m+1}$, with $m = 0,...,N$-1. Let $I_m = [x_{m+1-k}, x_{m+k}]$. In our discussion we need some parameters that we define as follows:

$$\sigma_{is} = \min_{1\leq\mu\leq\ell-s}(\tau_{i,\mu+s} - \tau_{i\mu}),\ 1 \leq s \leq \ell - 1 \tag{25}$$

where the nodes τ_{ij} are ordered,

$$\bar{\Delta}_m = \max_{m+1-k\leq i\leq m+k-1}(x_{i+1} - x_i) \tag{26}$$

$$\bar{\Delta}_N = \max_{1-k\leq i\leq N+k-2}(x_{i+1} - x_i) \tag{27}$$

$$\Delta_{m,k-r} = \min_{m+1-k+r\leq i\leq m}(x_{i+k-r} - x_i),\ r = 0, 1, ..., k-1 \tag{28}$$

$$\Delta_{k-r} = \min_{0\leq m\leq N-1} \Delta_{m,k-r}. \tag{29}$$

Finally we recall that the modulus of continuity of a function $g \in C(\mathcal{I}^*)$, where $\mathcal{I}^*$ is an arbitrary interval, is given by

$$\omega(g;\Delta;\mathcal{I}^*) = \max_{x,x+h\in\mathcal{I}^*;\ 0\leq h\leq\Delta} |g(x+h) - g(x)| \tag{30}$$

Now we are ready to present the following convergence Theorem, which is a special case of Theorem 5.2 in [12], with $q = \infty$.

Theorem 2.2. *Let* $1 \leq s \leq \ell \leq k$. *If* $f \in C^{s-1}(I_m)$, *then for* $0 \leq r < k$

$$\max_{x_m\leq t<x_{m+1}} |\ E_{rs}(t)\ | \leq K_m \bar{\Delta}_m^{s-r-1} \omega(D^{s-1}f; \bar{\Delta}_m; I_m) \tag{31}$$

where

$$K_m = \frac{k^{s+1}\Gamma_{kr}}{(s-1)!}\left(\frac{\bar{\Delta}_m}{\Delta_{m,k-r}}\right)^r \cdot \left[2^{s-1} + \sum_{j=s+1}^{\ell}(2\rho_m)^{j-s}\right] \tag{32}$$

with

$$\rho_m = \max_{m+1-k\leq i\leq m} \frac{(x_{i+k} - x_i)}{\sigma_{is}} \tag{33}$$

and

$$\Gamma_{kr} = \frac{(k-1)!}{(k-r-1)!}\begin{pmatrix} r \\ [r/2] \end{pmatrix}. \tag{34}$$

Corollary 2.3. *If ρ_m is uniformly bounded for all m and for all N, then, for all $f \in C[-1,1]$, if $\bar{\Delta}_N \to 0$ as $N \to \infty$,*

$$\| f - Q_N f \|_\infty \to 0 \text{ as } N \to \infty. \tag{35}$$

Furthermore if, in addition, $\bar{\Delta}_m / \Delta_{m,k-1}$ is uniformly bounded for all N, then

$$\max_{x_m \le t < x_{m+1}} | DQ_N f(t) | \le C_1 \bar{\Delta}_m^{-1} \omega(f, \bar{\Delta}_m; I_m).$$

We also recall in the following Theorem, an interesting result about the modulus of continuity of $Q_N f$ and its derivatives (see [14]).

Theorem 2.4. *Let $0 \le r < k$ and $\mathcal{I} \equiv [-1,1]$. Let $f \in C^r(\mathcal{I})$ and consider any spline space S_{k,π_N} such that π_N is locally uniform with constant A. If any spline $S \in S_{k,\pi_N}$ satisfies*

1. $S \in C^r(\mathcal{I})$,

2. $| D^r(f - S)(t) | \le C_1 \omega(D^r f; \bar{\Delta}_m; I_m)$, $x_m \le t < x_{m+1}$,

3. $| D^{r+1} S(t) | \le C_2 \bar{\Delta}_m^{-1} \omega(D^r f; \bar{\Delta}_m; I_m)$, $x_m < t < x_{m+1}$,

then

$$\omega(D^r S; \bar{\Delta}_N; \mathcal{I}) \le C_3 \omega(D^r f; \bar{\Delta}_N; \mathcal{I}),$$

where

$$C_3 = \max(4C_2, 1 + 2C_4) \text{ and } C_4 = [A^k] C_1.$$

3. On product quadratures based on a q.i spline-projector

We consider product integration rules for (1) and (2), based on the approximation of f by the spline-projector defined by (17), (20), (21). These quadratures have the form (8) and (10) respectively, with $\ell = k$.

Since the nodes $\{\tau_{ij}\}_{j=1}^k$ are chosen from nondegenerate intervals $[x_{\nu_i}, x_{\nu_i+1}] \subset [x_i, x_{i+k}]$, then for i = 1-k, ..., N-1 we have to mark out an interval formed by two consecutive distinct knots of the B_{ik} support.

Such an interval always exists, because we suppose that we repeat every point y_i, d_i times, where $1 \le d_i \le k$, for i = 0, 1, ..., n, with $d_0 = d_n = k$ and $d_i < k$, i = 1, ..., n-1. Moreover, we choose it as the "most central" interval among those effective in the support $[x_i, x_{i+k}]$.

Let $P = \{p_i\}_{i=1-k}^{N+k-1}$ be a set of integer numbers, such that p_i is equal to the index of the element of Y_n repeated d_i times, i.e.

$$P = \left\{ \underbrace{0,0,\dots,0}_{k \text{ times}}, \underbrace{1,1,\dots,1}_{d_1 \text{ times}} \dots, \underbrace{n-1,n-1,\dots,n-1}_{d_{n-1} \text{ times}}, \underbrace{n,n,\dots,n}_{k \text{ times}} \right\}.$$

For i = 1-k, ..., N-1 it results that

$$[x_i, x_{i+k}] = [y_{p_i}, y_{p_{i+k}}]\,. \tag{36}$$

Then we will assume

$$[x_{\nu_i}, x_{\nu_i+1}] = [y_{p_i+[k1/2]}, y_{p_i+[k1/2]+1}] \tag{37}$$

or

$$[x_{\nu_i}, x_{\nu_i+1}] = [y_{p_i+[(k1-1)/2]}, y_{p_i+[(k1-1)/2]+1}] \tag{38}$$

with $k1 = p_{i+k} - p_i$.

We remark that if $k1$ is odd, then both the choices (37) and (38) give rise to the same interval, if $k1$ is even, then the choice (37) gives rise to a right central interval, the choice (38) gives rise to a left central interval among those effective in the support $[x_i, x_{i+k}]$.

Now, for all i = 1-k, ..., N-1 we propose to choose the points $\tau_{i1}, \tau_{i2}, \dots, \tau_{ik}$ equally spaced throughout $[x_{\nu_i}, x_{\nu_i+1}]$, i.e.

$$\tau_{ij} = x_{\nu_i} + \frac{x_{\nu_i+1} - x_{\nu_i}}{k-1}(j-1),\ i = 1-k, \dots, N-1,\ j = 1,2,\dots,k \tag{39}$$

and we study the convergence of $I(K\,Q_N f)$ to $I(Kf)$ and of $J(uQ_N f);\ \lambda)$ to $J(uf;\ \lambda)$.

In our discussion we need the following Lemma.

Lemma 3.1. *Let the partition Y_n be locally uniform with constant A and $1 \le s \le k$. Then*

$$\rho_m = \max_{m+1-k \le i \le m} \frac{x_{i+k} - x_i}{\sigma_{is}} < \rho \tag{40}$$

where

$$\rho = (k-1)\left[1 + 2\left(A + A^2 + \dots + A^{[k/2]}\right)\right] \tag{41}$$

and for $0 \le r \le k$

$$\frac{\bar{\Delta}_m}{\Delta_{m,k-r}} \le \alpha \tag{42}$$

with

$$\alpha = A^{k-1}. \tag{43}$$

Proof. From the definition (39) of points $\{\tau_{ij}\}$ it follows that

$$\sigma_{is} \ge \sigma_{i1} = \frac{x_{\nu_i+1} - x_{\nu_i}}{k-1} \tag{44}$$

From (44) and from the hypothesis of local uniformity of the partition Y_n, we can easily prove the thesis (40), with ρ equal to (41). Now, for the partition π_N we can write

$$\max_{m+1-k \le i \le m+k-1} (x_{i+1} - x_i) \le A^{k-1}(x_{m+1} - x_m) \tag{45}$$

and

$$\min_{m+1-k+r \le i \le m} (x_{i+k-r} - x_i) \ge x_{m+1} - x_m, \; r = 0, 1, \dots, k-1. \tag{46}$$

From (45) and (46) we deduce (42) and (43). □

Theorem 3.2. *Let the partition* Y_n *be locally uniform with constant* A *and* $k \in L_1(\mathcal{I})$. *If* $\bar{\Delta}_N \to 0$ *as* $N \to \infty$ *then*

$$I(KQ_N f) \to I(Kf) \text{ for all } f \in \mathcal{R}(\mathcal{I}), \tag{47}$$

where $\mathcal{R}(\mathcal{I})$ *is the set of all (bounded) Riemann integrable functions on* $\mathcal{I}$.

Proof. From Theorem 2.2 and Lemma 3.1 where we put $r = 0$ and $s = 1$, we can deduce that

$$I(KQ_N f) \to I(Kf) \text{ for all } f \in C(\mathcal{I}). \tag{48}$$

The weights $\{\mu_{ij}\}$ of the quadrature are defined by (9) and they are such that

$$| \mu_{ij} | \le | v_{ij} | | I(KB_{ik}) | . \tag{49}$$

Moreover

$$\prod_{\substack{s=1 \\ s \ne j}}^{r} (\tau_{ij} - \tau_{is}) \ge (\tau_{ij} - \tau_{i,j-1})^{r-1} = \frac{(x_{\nu_i+1} - x_{\nu_i})^{r-1}}{(k-1)^{r-1}} \tag{50}$$

and, from (22), it results that

$$|\alpha_{ij}| \leq (x_{i+k} - x_i)^{j-1}. \tag{51}$$

Therefore

$$|v_{ij}| \leq \sum_{r=j}^{k} \left(\frac{x_{i+k} - x_i}{x_{\nu_i+1} - x_{\nu_i}} \right)^{r-1} (k-1)^{r-1}. \tag{52}$$

From the hypothesis of local uniformity of Y_n we can obtain

$$|v_{ij}| \leq \sum_{r=j}^{k} \gamma^{r-1}(k-1)^{r-1} \tag{53}$$

where

$$\gamma = 1 + 2\left(A + A^2 + \ldots + A^{[k/2]}\right). \tag{54}$$

Now, from (48), (49), (53) we can proceed as in [7] and [13] to show (47). □

The following Corollary provides a bound for the quadrature error

$$R_N(Kf) = I(Kf) - I(KQ_Nf). \tag{55}$$

Corollary 3.3. *Let $1 \leq s \leq k$, then for all $f \in C^{s-1}(\mathcal{I})$ and for $K \in L_1(\mathcal{I})$*

$$|R_N(Kf)| = O(\bar{\Delta}_N^{s-1}\omega(D^{s-1}f; \bar{\Delta}_N; \mathcal{I})). \tag{56}$$

Proof. From Theorem 2.2, where we put $r = 0$ and from Lemma 3.1 the thesis follows. □

Remark. The choice above proposed for $[x_{\nu_i}, x_{\nu_i+1}]$ is better numerically, but it requires, for the convergence, that the spline spaces are locally uniform. A different choice enables us to avoid the requirement of local uniformity. In fact, if we assume

$$[x_{\nu_i}, x_{\nu_i+1}] = \max_{0 \leq \mu \leq k-1}(x_{i+\mu+1} - x_{i+\mu})$$

it is easy to show that, if the q-i points τ_{ij} are equally spaced in the subinterval $[x_{\nu_i}, x_{\nu_i+1}]$, then the sequence of q-i splines will converge for every continuous function and the product quadratures $I(kQ_Nf)$ will converge for all $f \in \mathcal{R}(\mathcal{I})$.

Theorem 3.4. *Let the partition Y_n be locally uniform with constant A. Let $u \in L_1(\mathcal{I}) \cap DT(N_\delta(\lambda))$ and $f \in C(\mathcal{I}) \cap DT(N_\delta(\lambda))$ for some $\lambda \in (-1,1)$ and some δ such that $N_\delta(\lambda) \equiv [\lambda - \delta, \lambda + \delta] \subset (-1,1)$. If $\bar{\Delta}_N \to 0$ as $N \to \infty$, then*

$$J(uQ_Nf; \lambda) \to J(uf; \lambda). \tag{57}$$

Proof. For our approximation scheme $Q_N f$ the hypotheses of Theorem 2.4 hold. In fact, for all $f \in C(\mathcal{I})$ and $x_m \leq t < x_{m+1}$ with $0 \leq m \leq N$-1, from Theorem 2.2 and from Lemma 3.1 it follows that

$$| (f - Q_N f)(t) | = | E_{0,1}(t) | \leq C_1 \omega(f; \bar{\Delta}_m; I_m) \tag{58}$$

and

$$| DQ_N f(t) | = | E_{1,1}(t) | \leq C_2 \bar{\Delta}_m^{-1} \omega(f; \bar{\Delta}_m; I_m) \tag{59}$$

with

$$C_1 = k^2 \left[1 + \sum_{j=2}^{k} (2\rho)^{j-1} \right] \tag{60}$$

and

$$C_2 = A^{k-1} k^2 \Gamma_{k1} \left[1 + \sum_{j=2}^{k} (2\rho)^{j-1} \right]. \tag{61}$$

Therefore from Theorem 2.4, where we put $r = 0$, we can write

$$\omega(Q_N f; \bar{\Delta}_N; \mathcal{I}) \leq C_3 \omega(f; \bar{\Delta}_N; \mathcal{I}) \tag{62}$$

where $C_3 = \max\{4C_2, 1{+}2C_4\}$ and $C_4 = C_1[A^k]$. Now the proof of (57) is similar to that of Theorem 8 in [13], therefore we do not report it here. □

The following Corollary provides a bound for the quadrature error

$$E_N(uf; \lambda) = J(uf; \lambda) - J(uQ_N f; \lambda). \tag{63}$$

Corollary 3.5. *Let* $2 \leq s \leq k$, *then for all* $f \in C^{s-1}(\mathcal{I})$

$$| E_N(uf; \lambda) | = O(\bar{\Delta}_N^{s-2} \omega(D^{s-1} f; \bar{\Delta}_N, \mathcal{I})). \tag{64}$$

Proof. From Theorem 2.2, where we put $r = 0$ and $r = 1$, and from Lemma 3.1, we can proceed as in [7] to show the thesis (64). □

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BEST PIECEWISE CONVEX UNIFORM APPROXIMATION

Z. FINTA

Dedicated to Professor Gh. Coman on his 60^{th} anniversary

Abstract. In this paper there is given a constructive proof of the existence of the best uniform approximation of a continous function by piecewise convex functions.

1. Introduction.

The problem considered is that of finding a best uniform approximation to a real function $f \in C[a,b]$ from the class of piecewise convex functions. The existence and nonuniqueness of best approximations are established.

Let $I = [a,b]$ be a compact real interval and $B = B(I)$ (resp. $C = C(I)$) be the Banach space of all bounded (resp. continuous) real function f on I with the uniform norm $\|f\| = \sup\ \{|f(x)|: x \in I\}$. For any integer $n \geq 1$, let $\Lambda = \{p = (p_0, p_1, ..., p_n) \in R^{n+1}: a = p_0 \leq p_1 \leq ... \leq p_n = b\}$. Then, Λ is compact in R^{n+1}. Given a $p \in \Lambda$ define intervals $I_j = [p_{j-1}, p_j)$, for $1 \leq j \leq n$-1, and $I_n = [p_{n-1}, p_n]$. Let $K(p) = \{ \varphi \in B: \varphi$ is convex on I_j, $1 \leq j \leq n\}$. $K(p)$ is called the set of all n-piecewise convex functions with the vector p. Next, let $K = \cup\{K(p): p \in \Lambda \}$ the set of piecewise convex functions. We denote by Conv I the set of all convex and continuous functions on the interval I.

2. Best approximations from Conv I.

Since the set Conv I is nonlinear, we cannot apply the general theory of linear approximation to obtain the existence of the best convex approximation in the space C. So we shall prove its existence directly.

Theorem 2.1. *Let $f \in C$. Then there exists a function $\bar{\varphi} \in$ Conv I such that $\| f - \bar{\varphi} \| \leq \| f - \varphi \|$ for all $\varphi \in$ Conv I.*

Proof. If f is a convex function, then we put $\bar{\varphi} = f$. Otherwise, we consider the following sets: $[f] = \{(x, z) \in [a, b] \times R : z \geq f(x)\}$ the epigraph of f and $\text{conv}[f] = \cap\{M : [f] \subseteq$

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$M \subset R^2$, M is convex}, respectively. Obviously, conv $[f]$ is a convex set, so bd (conv $[f]$) is non-empty and card ({$\alpha(x,y)$ + (1-α)(x_0,y_0): $\alpha \geq 0$} $\cap$ bd (conv $[f]$)) ≤ 1 for every (x_0,y_0)$\in$ int (conv $[f]$) and $(x,y) \in R^2$. Hence {(x,z): z } $\cap$ bd (conv $[f]$) = {(x,z_x)} for every $x \in (a,b)$. Now, we can define the function $\bar{f} : [a,b] \to R$ by

$$\bar{f}(x) = \begin{cases} f(a), & \text{if} \quad x = a \\ z_x, & \text{if} \quad a < x < b \\ f(b), & \text{if} \quad x = b \end{cases} \tag{1}$$

We assert that $\bar{f} \in$ Conv $I \cap C$. Indeed, we have $(\lambda x_1 + (1-\lambda)x_2, \lambda z_{x1} + (1-\lambda)z_{x2}) = \lambda \cdot (x_1, z_{x1}) + (1-\lambda) \cdot (x_2, z_{x2}) \in$ int (conv $[f]$) for every $\lambda \in (0,1)$ and $(x_1, z_{x1}), (x_2, z_{x2}) \in$ conv $[f]$. So $\{\alpha(\lambda x_1 + (1-\lambda)x_2, 0) + (1-\alpha)(\lambda x_1 + (1-\lambda)x_2, \lambda z_{x1} + (1-\lambda)z_{x2}) : \alpha \in [0,1]\} \cap$ bd (conv$[f]$) $= \{(\lambda x_1 + (1-\lambda)x_2, z_{\lambda x_1 + (1-\lambda)x_2})\}$, i.e. $\bar{f}(\lambda x_1 + (1-\lambda)x_2) = z_{\lambda x1 + (1-\lambda)x2} = \lambda z_{x1} + (1-\lambda)z_{x2} = \lambda \bar{f}(x_1) + (1-\lambda)\bar{f}(x_2)$. Thus $\bar{f} \in$ Conv I. On the other hand it results, by $f \in C$ and {$\alpha(x,z_x)$ + (1-α)(x,0): $\alpha \geq 0$} $\cap$ {$(x,f(x))$: x $[a,b]$} $\neq \emptyset$, that $\bar{f} \in C$. We observe, if $g \in$ Conv I and $g \leq f$ on I then $g \leq \bar{f}$ on I (1).

Let $G(g) = \{(x, g(x)) : x \in [a,b]\}$ be the graph of $g : [a,b] \to R$ and the constant function $f_\alpha : [a,b] \to R$, $f_\alpha(x) = \alpha$, respectively. Next, let $\alpha_0 = \sup\{\alpha \geq 0 : d(G(f + f_\alpha), G(\bar{f} - f_\alpha)) = 0\}$, where $d(A,B)$ denotes the distance between the sets $A \subset R^2$ and $B \subset R^2$. Then there exists $x_0 \in [a,b]$ such that $\bar{f}(x_0) + \alpha_0 = f(x_0) - \alpha_0$.

We shall show that $\bar{\varphi} = \bar{f} + f_{\alpha 0}$. Indeed, $\| f - \bar{\varphi} \| = \alpha_0$ and $f - f_{\alpha 0} \leq \bar{\varphi} \leq f + f_{\alpha 0}$ on I. So it is enough to show $\|f - \varphi\| \geq \alpha_0$ for every $\varphi \in$ Conv I with condition $f - f_{\alpha 0} \leq \varphi \leq f + f_{\alpha 0}$ on I. Because $\varphi \in$ Conv$I$ and $\varphi \leq f + f_{\alpha 0}$, therefore $\varphi \leq \bar{f} + f_{\alpha 0} = \bar{\varphi}$ on I by (1). At the same time $f(x_0) + \alpha_0 - f(x_0) - \alpha_0 \leq \varphi(x_0)$ and be $\varphi(x_0) = \bar{f}(x_0) + \alpha_0$. So $| f(x_0) - \varphi(x_0) | = | f(x_0) - (\bar{f}(x_0) + \alpha_0) | = \alpha_0$ by (2). Consequently $\| f - \varphi \| \geq | f(x_0) - \varphi(x_0) | = \alpha_0 = \| f - \bar{\varphi} \|$.

Proposition 2.2. *The best uniform approximation to a real function $f \in C$ from class* Conv (I) *is nonunique.*

Proof. We show our assertion by means of exemple.

Let $f, \bar{\varphi}_k : \left[0, \frac{5}{2}\right] \to \mathrm{R}$ be two function defined by

$$f(x) = \begin{cases} x, & \text{if} \quad x \in [0,1) \\ 2 - x, & \text{if} \quad x \in [1,2) \\ x - 2, & \text{if} \quad x \in \left[2, \frac{5}{2}\right] \end{cases}$$

and

$$\bar{\varphi}_k(x) = \begin{cases} \frac{1}{2}, & \text{if} \quad x \in [0,2) \\ (2k-1)x - \frac{1}{2}(8k-5), & \text{if} \quad x \in \left[2, \frac{5}{2}\right], \end{cases}$$

respectively, where $k \in \left[\frac{1}{2}, 1\right]$. Then every function φ_k satisfies the conditions of theorem 2.1. $\left(\bar{\varphi} = \bar{\varphi}_1 \text{ and } \| f - \bar{\varphi}_k \| = \frac{1}{2}\right)$. □

3. Best approximation from $K(p)$ and K.

Theorem 3.1. *If $p \in \Lambda$ then there exists $\bar{\psi}_p \in K(p)$ such that $\| f - \bar{\psi}_p \| \leq \| f - \psi_p \|$ for every $\psi \in K(p)$.*

Proof. Apply theorem 2.1. upon every intervals $I_j \cup p_j$, $1 \leq j \leq n$. Then there exist $\bar{\varphi}_j, 1 \leq j \leq n$, such that $\| f - \bar{\varphi}_j \|_j \leq \| f - \psi_p \|_j$ for each $j \in 1, 2, ..., n$ and $\psi_p \in K(p)$, where $\| g \|_j = \sup\{| g(x) | : x \in I_j \cup p_j\}$. If $\bar{\psi}_p(x) = \varphi_j(x)$ for $x \in I_j$ and $j \in 1, 2, ..., n$, then we have $\| f - \bar{\psi}_p \| \leq \| f - \psi_p \|$ for all $\psi_p \in K(p)$. □

Theorem 3.2. *There exists a function $\bar{\psi} \in K$ such that $\| f - \bar{\psi} \| \leq \| f - \psi \|$ for every $\psi \in K$.*

Proof. Let $h : \Lambda \to K$ be a function defined by $h(p) = \bar{\psi}_p$, where $\bar{\psi}_p$ appears in the statement of theorem 3.1. Because the function f and φ (see theorem 2.1.) are continuous, we obtain the continuity of h. But Λ is compact in R^{n+1}, therefore there exists $\bar{p} \in \Lambda$ such that $h(\bar{p}) = min\{h(p) : p \in \Lambda\}$. So there exists $\bar{\psi}_{\bar{p}}$ by theorem 3.1. Let $\bar{\psi} = \bar{\psi}_{\bar{p}}$. Then $\| f - \bar{\psi} \| \leq \| f - \psi \|$ for every $\psi \in K$. □

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INTERPOLATION BY CUBIC SPLINE WITH FIXED POINTS

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Dedicated to Prof. Gheorghe Coman on his 60^{th} aniversary

Abstract. In this note we study an interpolation problem by cubic spline functions with a fixed point. We give a method for the construction of a function similar to to that from [19] and we obtain some inequalities in the case where tha dates are convex.

1. Introduction

In the last years, knowing a considerable development, both theoretical and practical in a vest literature (e.g. [11], [13], [14]), spline functions have been constructed using interpolation conditions, fitting and smoothing conditions of data by a functional minimization (see: [1]-[5], [10], etc.), at which it has been added joining conditions, periodicity or other end conditions, least squares method, [12], convexity etc.

The goal of this note is to study a cubic spline function, s, of the form such in the [19], at which the conditions $s'(x_0) = s''(x_0) = 0$ are replaced by the condition that the spline function, s, and first derivative, s', has a fixed point.

It is known, [15], that if X be a nonempty set and f: $X \to X$, a mapping, then $x \in X$ is called fixed point, for f, if $f(x) = x$.

Let f: $[a,b] \to$ be a function and let

$$\Delta_n : a = x_0 < x_1 < ... < x_n = b, \; n \in \mathrm{N}, \; n > 1 \tag{1.1}$$

be a partition of the interval $[a,b]$.

We know of the function f only his values $y_i = f(x_i), i = 0, 1, ..., n$.

Let $S_p(3, \Delta_n)$ be the set of the cubic splines for the partition Δ_n, and having the properties:

(i) $s\mid_{[x_{i-1},x_i]} \in P_3$

(ii) $s \in C^2[a,b], i = \overline{1,n}$.

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In [19] we shown that there exist and it is unique a cubic spline functions $s \in S_p(3, \Delta_n)$ of the form:

$$s(x) = \frac{M_i - M_{i-1}}{6h_i}(x - x_{i-1})^3 + \frac{M_{i-1}}{2}(x - x_{i-1})^2 + m_{i-1}(x - x_{i-1}) + y_{i-1}, \tag{1.2}$$

for $x \in [x_{i-1}, x_i], i = 1, 2, ..., n$, where

$$\begin{cases} M_k = s''(x_k) \\ m_k = s'(x_k), \quad k = 0, 1, ..., n \\ h_k = x_k - x_{k-1}, \quad k = 1, 2, ..., n \end{cases} \tag{1.3}$$

if the following conditions are satisfied

$$\begin{cases} s(x_i) = y_i \\ s'(x_i) = m_i, \quad i = \overline{0, n} \end{cases} \Leftrightarrow \begin{cases} M_i = 6\frac{y_i - y_{i-1}}{h_i^2} - 6\frac{m_{i-1}}{h_i} - 2M_{i-1} \\ m_i = 3\frac{y_i - y_{i-1}}{h_i} - 2m_{i-1} - \frac{M_{i-1}}{2}h_i \end{cases} \tag{1.4}$$

and

$$m_0 = M_0 = 0. \tag{1.4'}$$

2. Spline functions with fixed points.

Now, one considers the following spline interpolation problem: *find the cubic spline $s \in S_p(3, \Delta_n)$ that interpolates the data $(x_i, y_i), i = \overline{0, n}$, when the data are convex on interval $[a, b]$.*

Theorem 1. *For this problem there exists a unique cubic spline $s \in S_p(3, \Delta_n)$ of the form (1.2) in the following conditions:*

$$\begin{cases} M_i = 6\frac{y_i - y_{i-1}}{h_i^2} - 6\frac{m_{i-1}}{h_i} - 2M_{i-1} \\ m_i = 3\frac{y_i - y_{i-1}}{h_i} - 2m_{i-1} - \frac{M_{i-1}}{2}h_i, \quad i = \overline{1, n} \\ s(x_1^*) = x_1^* \\ s'(x_2^*) = x_2^* \end{cases} \tag{2.1}$$

and there exists a following inequality

$$M_{i+1}h_{i+1} + M_i(2h_{i+1} - h_i) - 2M_{i-1}h_i + 6(m_i - m_{i-1}) \geq 0, \; i = 1, ..., n-1. \tag{2.2}$$

Proof. In (2.1) for $i = 1$, we have

$$\begin{cases} M_1 = 6\frac{y_1-y_0}{h_1^2} - 6\frac{m_0}{h_1} - 2M_0 \\ m_1 = 3\frac{y_1-y_0}{h_1} - 2m_0 - \frac{M_0}{2}h_1 \\ M_1 + 5M_0 + 24\frac{m_0}{h_1} = 48\frac{x_1^*-y_0}{h_1^2}, \quad \text{for} \quad x_1^* = \frac{x_0+x_1}{2} \\ m_0 = x_0, \quad \text{for} \quad x_2^* = x_0 \end{cases} \tag{2.3}$$

This system has a unique solution:

$$\begin{cases} m_0 = x_0 \\ m_1 = \frac{x_0(h_1-4)-4x_1+10y_0-2y_1}{h_1} \\ M_0 = \frac{2x_0(4-3h_1)+8x_1-14y_0-2y_1}{h_1^2} \\ M_1 = \frac{-2x_0(8-3h_1)-16x_1+22y_0+10y_1}{h_1^2} \end{cases} \tag{2.4}$$

The next step is to put $i = 2$ in (2.1). Thus we have:

$$\begin{cases} M_2 = 6\frac{y_2-y_1}{h_2^2} - 6\frac{m_1}{h_2} - 2M_1 \\ m_2 = 3\frac{y_2-y_1}{h_2} - 2m_1 - \frac{M_1}{2}h_2 \end{cases}$$

which determines the values of M_2 and m_2. So, the system (2.1) can be recursively solved starting from the solution (2.4) and the existence and the uniqueness of $s \in S_p(3, \Delta_n)$ is completely proved. □

When the data (x_i, y_i), $i = \overline{0,n}$ are convex on $[a,b]$, we can write the following inequality

$$\frac{y_i - y_{i-1}}{h_i} \le \frac{y_{i+1} - y_i}{h_{i+1}}, \; i = 1, 2, ..., n-1 \tag{2.5}$$

which is equivalent with the inequality (see (1.2)-(1.4)):

$$\frac{\frac{M_i - M_{i-1}}{6h_i}h_i^3 + \frac{M_{i-1}}{2}h_i^2 + m_{i-1}h_i}{h_i} \le \frac{\frac{M_{i+1}-M_i}{6h_{i+1}}h_{i+1}^3 + \frac{M_i}{2}h_{i+1}^2 + m_i h_{i+1}}{h_{i+1}}$$

$$\Leftrightarrow M_i\frac{i}{6}h_i - \frac{M_{i-1}}{6}h_i + \frac{M_{i-i}}{2}h_i + m_{i-1} \le \frac{M_{i+1}}{6}h_{i+1} - M_i\frac{i}{6}h_{i+1} + M_i\frac{i}{2}h_{i+1} + m_i$$

$$\Leftrightarrow \frac{M_i}{6}h_i + 2\frac{M_{i-1}}{6}h_i + m_{i-1} \le \frac{M_{i+1}}{6}h_{i+1} + 2\frac{M_i}{6}h_{i+1} + m_i$$

and from this we have (2.2).

Consider now the *fundamental spline function* $s_j \in S_p(3, \Delta_n), j = \overline{0,n}$, satisfying the following conditions

$$s_j(x_i) = \delta_{ij} = \begin{cases} 0 & \text{for} \quad i \neq j \\ 1 & \text{for} \quad i = j \end{cases} \tag{2.6}$$

and

$$\begin{cases} M_1 + 5M_0 + 24\frac{m_0}{h}{}_1 = 48\frac{x_0+x_1-2y_0}{2h_1^2} \\ m_0 = x_0 \end{cases} \tag{2.7}$$

Using the technique from [19] we can construct fundamentals cubic spline functions relating to above data and for $x \in [x_0, x_1]$, one obtain:

$$s_0(x) = 1 + [2x_0(h_1 - 2) - 4x_1 + 6y_0 + 2y_1]\left[\frac{x - x_0}{h_1}\right]^3 + \tag{2.8}$$

$$+ [x_0(4 - 3h_1) + 4x_1 - 7y_0 - y_1]\left[\frac{x - x_0}{h_1}\right]^2 + x_0(x - x_0).$$

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EXISTENCE AND CONTINUOUS DEPENDENCE ON DATA OF THE POSITIVE SOLUTIONS OF AN INTEGRAL EQUATION FROM BIOMATHEMATICS

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Dedicated to Prof. Gheorghe Coman at his 60^{th} anniversary

1. Introduction

We deal with the nonlinear integral equation:

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))ds \tag{1}$$

which is a model for the spread of certain infectious diseases with a contact rate that varies seasonally (see [1]). In this equation $x(t)$ is the proportion of infectives in a population at time t, τ is the lenght of time individual remains infectious and $f(t, x(t))$ represents the proportion of new infectives per unit time.

The existence of periodic solutions for Eq (1) is studied in [1], [3], [5], [9-11], in the case of periodic contact rate ($f(t+\omega, x) = f(t, x)$, $f(t, 0) = 0$), while the papers [7] and [8] deal with the initial value problem for Eq (1).

In the next section we shall give sufficient conditions, different from those in [7] and [8], for the existence of at least one positive and continuous solution $x(t)$ of (1), for $-\tau \leq t \leq \mathcal{T}$, when we know the proportion $\phi(t)$ of infectives in the population for $-\tau \leq t \leq 0$, i.e.

$$x(t) = \phi(t), \quad \text{for} \quad -\tau \leq t \leq 0. \tag{2}$$

Clearly, we must suppose that ϕ satisfies the condition

$$b = \phi(0) = \int_{-\tau}^{0} f(s, \phi(s))ds. \tag{3}$$

Section 3 and 4 deal with the continuous dependence on f, ϕ and τ of the solutions for (1)-(2) under the assumption of Section 2.

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2. Existence Results

Let us list our assumptions:

(H_1) $f(t,x)$ is nonnegative and continuous for $-\tau \leq t \leq \mathcal{T}$ and $x \geq 0$;

(H_2) there exist the real numbers $c > 0$ and $\gamma > 1$ such that

$$\sup_{t\in[-\tau,\mathcal{T}]} \frac{f(t,c)}{c} < \frac{1}{\gamma\tau}$$

$$\frac{\gamma}{\gamma-1}c \leq \inf\left\{x > c \,|\, \text{there exists } t \in [-\tau, \mathcal{T}] \text{ such that } f(t,x) = \frac{x}{\gamma\tau}\right\} \tag{4}$$

(where, if the set is emply, we mean $\inf \emptyset = +\infty$);

(H_3) there exist a real number $a > 0$ and an integrable function $g(t)$ such that

$$f(t,x) \geq g(t), \quad \text{for} \quad -\tau \leq t \leq \mathcal{T} \quad \text{and } a \leq x \leq \frac{\gamma}{\gamma-1}c \tag{5}$$

and

$$\int_{t-\tau}^{t} g(s)ds \geq a, \quad \text{for } 0 \leq t \leq \mathcal{T}$$

(H_4) $\phi(t)$ is continuous and

$$0 < a \leq \phi(t), \quad \text{for } -\tau \leq t \leq 0. \tag{6}$$

Now, we state our first result:

Theorem 2.1. *Suppose (H_1)-(H_4) hold. Then the problem (1)-(2) has at least one continuous solution $x(t)$, $-\tau \leq t \leq \mathcal{T}$, with the property:*

$$a \leq x(t) \leq \frac{\gamma}{\gamma-1}c \quad for \ -\tau \leq t \leq \mathcal{T} \tag{7}$$

provided that

$$c \leq \phi(t) \leq \frac{\gamma}{\gamma-1}c \quad for \ -\tau \leq t \leq 0. \tag{8}$$

Proof. Let E be the Banach space of all continuous functions $x(t)$, $0 \leq t \leq \mathcal{T}$ with norm

$$\|x\| = \max_{0\leq t\leq \mathcal{T}} |x(t)|$$

let $K = \{x \in E\text{:a } x(0) = b \text{ and } x(t) \geq a \text{ for } 0 \leq t \leq \mathcal{T}\}$, let $U = \left\{x \in K; \| x \| < \frac{\gamma}{\gamma-1}c\right\}$. Clearly, K is a convex and closed subset of E and U is a nonempty, open, bounded subset of K.

For each $\lambda \in [0,1]$ we define the nonlinear integral operator:

$$T_\lambda : \bar{U} \to K$$

$$T_\lambda x(t) = (1-\lambda)b + \lambda \int_{t-\tau}^{t} f(s,x(s))ds, \quad 0 \le t \le \mathcal{T}$$

where $f(s,x(s)) = f(s,\phi(s))$ for $-\tau \le s \le 0$. It is easy to show, using conditions (H_1), (H_3), (H_4) and Ascoli-Arzelá's Theorem, that the operators T_λ are well-defined and completely continuous from $\bar{U}$ into K. Now we shall prove that, for every $\lambda \in [0,1]$, the operator T_λ has no fixed points on the boundary ∂U of U with respect to K.

Suppose contrary, there exist $\lambda_0 \in [0,1]$ and $x_0 \in K$ such that $\| x_0 \| = \frac{\gamma}{\gamma-1}c$ and $T_{\lambda_0}x_0 = x_0$. Then there exists at least one $t_0 \in [0,\mathcal{T}]$ such that $x_0(t_0) = \frac{\gamma}{\gamma-1}c$, and by $T_{\lambda_0}x_0 = x_0$ we obtain

$$\frac{\gamma}{\gamma-1}c = x_0(t_0) = T_{\lambda_0}x_0(t_0) = (1-\lambda_0)b + \lambda_0 \int_{t_0-\tau}^{t_0} f(s,x_0(s))ds \tag{9}$$

If $\lambda_0 = 0$, then (5) implies $\frac{\gamma}{\gamma-1}c = b = \phi(0)$ which contradicts (8). Therefore we have $\lambda_0 > 0$.

Now, (H_2) and the continuity of $f(t,x)$ imply that there exists $\delta > 0$ such that:

$$f(t,x) \le \frac{x}{\gamma\tau} \le \frac{1}{\gamma\tau}\cdot\frac{\gamma}{\gamma-1}c, \text{ for } -\tau \le t \le \mathcal{T} \text{ and } c-\delta \le x \le \frac{\gamma}{\gamma-1}c \tag{10}$$

If $x_0(s) > c\text{-}\delta$ for all $s \in [0,t_0]$, then from $\| x_0 \| = \frac{\gamma}{\gamma-1}c$ we get $c\text{-}\delta \le x_0(s) \le \frac{\gamma}{\gamma-1}c$, $s \in [0,t_0]$ and, using (10) and (8), (9) implies:

$$\frac{\gamma}{\gamma-1}c \le (1-\lambda_0)b + \lambda_0 \cdot \frac{1}{\gamma\tau}\cdot\frac{\gamma}{\gamma-1}c\int_{t_0-\tau}^{t_0} ds < (1-\lambda_0)b + \lambda_0\frac{\gamma}{\gamma-1}c$$

and, by $b = \phi(0) < \frac{\gamma}{\gamma-1}c$, we deduce $\lambda_0 > 1$, which contradicts $\lambda \in [0,1]$.

If there exists $s_0 \in [0,t_0]$ such that $x_0(s_0) < c\text{-}\delta$, then, by the continuity of x_0, there exists at least one $t_1 \in]s_0,t_0[$ such that $x_0(t_1) = c \in]x_0(s_0),x_0(t_0)[$. Choosing now

$$t_1 = \sup\{t \in [0,t_0]; x_0(t) = c\}$$

it is easy to show that

$$\begin{aligned} 0 < t_1 &< t_0, \\ x_0(t_1) &= c, \end{aligned}$$

and using $\| x_0 \| = \frac{\gamma}{\gamma-1}c$,

$$c < x_0(s) \leq \frac{\gamma}{\gamma-1}c, \text{ for all } s \in]t_1, t_0].$$

By $x_0(t_1) = c$, $0 < t_1 < t_0$ and $T_{\lambda_0}x_0 = x_0$ we get

$$c = x_0(t_1) = T_{\lambda_0}x_0(t_1) = (1-\lambda_0)b + \lambda_0 \int_{t_1-\tau}^{t_1} f(s, x_0(s))ds$$

From the last equality and (9), using the nonnegativity of function $f(t,x)$, we have:

$$\begin{aligned} \frac{\gamma}{\gamma-1}c - c &= \lambda_0 \left[\int_{t_0-\tau}^{t_0} f(s, x_0(s))ds - \int_{t_1-\tau}^{t_1} f(s, x_0(s))ds \right] = \\ &= \lambda_0 \left[\int_{t_1}^{t_0} f(s, x_0(s))ds - \int_{t_1-\tau}^{t_0-\tau} f(s, x_0(s))ds \right] \leq \\ &\leq \lambda_0 \int_{t_1}^{t_0} f(s, x_0(s))ds. \end{aligned} \tag{11}$$

By $c < x_0(s) \leq \frac{\gamma}{\gamma-1}c$, $s \in]t_1, t_0[$, using (10) we find

$$\int_{t_1}^{t_0} f(s, x_0(s))ds < \frac{1}{\gamma\tau}\frac{\gamma}{\gamma-1}c \int_{t_1}^{t_0} ds < \frac{c}{\gamma-1}.$$

Since $\lambda_0 \in]0, 1]$, the last inequality and (11) imply

$$\frac{\gamma}{\gamma-1}c - c < \frac{c}{\gamma-1}, \quad \text{a contradiction.}$$

Therefore, T_λ defines an admissible compact homotopy on $\bar{U}$ which is fixed point free on the boundary of U with respect to K. Now $T_0 \equiv b$ is essential because the function $x(t) \equiv b$, $0 \leq t \leq T$ is an element of U (see [2, Theorem 2.2]). By the topological transversality theorem (see [2, Theorem 2.5]), T_1 is essential too. This implies that T_1 has at least one fixed point $x(t) \in U$. Clearly

$$a \leq x(t) < \frac{\gamma}{\gamma-1}c \text{ for } 0 \leq t \leq T$$

and

$$x(0) = b.$$

So, $\tilde{x}(t) = \begin{cases} x(t), & 0 < t \leq T \\ \phi(t), & -\tau \leq t \leq 0 \end{cases}$

is a solution of problem (1)-(2) and satisfies (7). The proof is complete. □

Remark 2.1. From (10) we see that Theorem 2.1 remains true if we replace condition (8) with:

$$c - \delta \le \phi(t) < \frac{\gamma}{\gamma - 1}c, \text{ for } -\tau \le t \le 0. \tag{8'}$$

If we suppose that $b \le c$, then we can use instead of (H_2) a weaker condition:

(H_2') there exist $c \ge b$ and $\gamma > 1$ such that

$$\sup_{t\in[0,\mathcal{T}]} \frac{f(t,c)}{c} < \frac{1}{\gamma\tau},$$

$$\frac{\gamma}{\gamma - 1}c \le \inf\left\{x > c \mid \text{ there exists } t \in [0,\mathcal{T}] \text{ such that } f(t,x) = \frac{x}{\gamma\tau}\right\}. \tag{4'}$$

Our next result is:

Theorem 2.2. *Suppose (H_1), (H_2'), (H_3) and (H_4) hold. Then problem (1)-(2) has at least one continuous solution such that*

$$a \le x(t) < \frac{\gamma}{\gamma - 1}c \text{ for } 0 \le t \le \mathcal{T} \tag{7'}$$

Proof. Let E, K, U, T_λ be as in the previous theorem. If we prove that T_λ has no fixed points on the boundary ∂U of U with respect to K, the conclusion follows as in the proof of Theorem 2.1.

Suppose contrary, there exists $\lambda_0 \in [0,1]$, $x_0 \in \partial U$, and $t_0 \in [0,\mathcal{T}]$, such that $T_{\lambda_0}x_0 = x_0$ and $x_0(t_0) = \frac{\gamma}{\gamma-1}c$. As in Theorem 2.1 we can have $\lambda_0 = 0$ because of $b \le c < \frac{\gamma}{\gamma-1}c$. Now from $b = x(0) \le c < x(t_0)$, and the continuity of $x(t)$, we that the set $\{t \in [0,t_0], x(t) = c\}$ is nonempty. Therefore we can choose $t_1 = \sup\{t \in [0,t_0]; x(t) = c\}$. Argueing now as in the proof of Theorem 2.1 we obtain $\frac{\gamma}{\gamma-1}c - c < \frac{c}{\gamma-1}$ even in the case that $t_1 = 0$. Thus, we have $\frac{c}{\gamma-1} < \frac{c}{\gamma-1}$, a contradiction. Therefore, the conclusion follows from the topological transversality theorem. The proof is complete. □

Remark 2.2. If $\mathcal{T} < \gamma\tau \ln\frac{\gamma}{\gamma-1}$, then Theorem 2.2 can be derived by using Theorem 1 from [7], with a suitable choise of function $h(x)$. Nevertheless, we note that Theorem 2.1 and 2.2 are completely different from those in [7] and [8].

In order to study the continuous dependence on f, ϕ, τ of the solutions of problem (1)-(2) we need an existence result which holds in a "neighourhood" of f, ϕ, τ. This is the goal of our next results.

Let η be a real positive number. Consider a continuous extension to $[-\tau-\eta,\mathcal{T}]\times\mathbb{R}_+$ of $f(t,x)$ and a continuous extension to $[-\tau-\eta,0]$ of $\phi(t)$. These two new functions will be also denoted by $f(t,x)$ and $\phi(t)$. From the biological point of view this means that we must know the functions $f(t,x)$ and $\phi(t)$ on some interval larger than $[-\tau,\mathcal{T}]$.

Now, let us consider a number ε , $0<\varepsilon\leq\eta$, a positive real number τ_ε, and the continuous and nonnegative functions $f_\varepsilon(t,x)$, $\phi_\varepsilon(t)$, such that:

$$\begin{cases} |\tau_\varepsilon-\tau|<\varepsilon & \\ |\phi_\varepsilon(t)-\phi(t)|<\varepsilon & \text{for all } t\in[-\tau_\varepsilon,0] \\ |f_\varepsilon(t,x)-f(t,x)|<\varepsilon & \text{for all } t\in[-\tau_\varepsilon,\mathcal{T}] \text{ and } x\in\left[a,\frac{\gamma}{\gamma-1}c\right] \\ b_\varepsilon=\phi_\varepsilon(0)=\int_{-\tau_\varepsilon}^{0} f_\varepsilon(s,\phi_\varepsilon(s))ds. & \end{cases} \quad (12)$$

Then for the problem

$$x(t)=\int_{t-\tau_\varepsilon}^{t} f_\varepsilon(s,x(s))ds, \text{ for } 0\leq t\leq\mathcal{T} \quad (1_\varepsilon)$$

$$x(t)=\phi_\varepsilon(t), \text{ for } -\tau_\varepsilon\leq t\leq 0, \quad (2_\varepsilon)$$

we have the following existence result:

Corollary 2.3. *Suppose that for problem (1)-(2), conditions (H_1)-(H_4) are satisfied. Then, there exists $0<\varepsilon_0\leq\eta$ such that each problem (1_ε)-(2_ε), with $\varepsilon\in[0,\varepsilon_0]$, has at least one continuous solution, provided that conditions (12) and (8) hold and inequalities (4), (5) and (6) are strict. Moreover, for each $\varepsilon\in[0,\varepsilon_0]$ and each solution $x_\varepsilon(t)$ of (1_ε)-(2_ε), we have*

$$a<x_\varepsilon(t)<\frac{\gamma}{\gamma-1}c, \text{ for } -\tau_\varepsilon\leq\tau\leq\mathcal{T}.$$

Proof. We choose ε_0 sufficiently small such that for the problems (1_ε)-(2_ε) with $\varepsilon\in[0,\varepsilon_0]$ conditions (H_1)-(H_4) and (8') hold and, consequently, we can apply Remark 2.1 and Theorem 2.1.

In order to do this let us remember that (H_2) with strict inequality in (4) implies that there exists $\delta>0$ such that

$$f(t,x)<\frac{x}{\gamma\tau} \text{ for all } t\in[-\tau,\mathcal{T}] \text{ and } x\in\left[c-\delta,\frac{\gamma}{\gamma-1}c\right]$$

Therefore

$$0 < d = \inf\left\{\frac{x}{\gamma\tau} - f(t,x);\ t \in [-\tau, T],\ x \in \left[c-\delta, \frac{\gamma}{\gamma-1}c\right]\right\}.$$

Now we can choose ε_1 such that if $|\ \tau_\varepsilon - \tau\ | < \varepsilon \le \varepsilon_1$, we have $\left|\frac{x}{\gamma\tau_\varepsilon} - \frac{x}{\gamma\tau}\right| < \frac{d}{3}$ for all $x \in \left[c-\delta, \frac{\gamma}{\gamma-1}c\right]$; also using the continuity of $f(t,x)$ we can choose ε_1' such that if $|\ \tau_\varepsilon - \tau\ | < \varepsilon \le \varepsilon_1'$, we have $\min\{|f(s,x) - f(t,x)|;\ s \in [-\tau, T]\} < \frac{d}{3}$ for all $t \in [-\tau_\varepsilon, T]$ and $x \in \left[c-\delta, \frac{\gamma}{\gamma-1}c\right]$. If we choose $\varepsilon_1 = \min\left\{\varepsilon_1, \varepsilon_1', \frac{d}{3}\right\}$ then we have:

$$\begin{aligned}\frac{x}{\gamma\tau_\varepsilon} - f_\varepsilon(t,x) &= \frac{x}{\gamma\tau_\varepsilon} - \frac{x}{\gamma\tau} + \frac{x}{\gamma\tau} - f(t_1,x) + f(t_1,x) - f(t,x) + \\ &+\ f(t,x) - f_\varepsilon(t,x) > d - \frac{d}{3} - \frac{d}{3} - \frac{d}{3} = 0\end{aligned}$$

for all $\varepsilon \in [0,\varepsilon_1]$, $t \in [-\tau_\varepsilon, T]$ and $x \in \left[c-\delta, \frac{\gamma}{\gamma-1}c\right]$, where $t_1 \in [-\tau, T]$ is such that $|f(t_1,x) - f(t,x)| = \min\{|f(s,x) - f(t,x)|;\ s\ [-\tau, T]\}$

In a similar way we can choose ε_2 such for all $\varepsilon \in [0,\varepsilon_2]$, we have

$$f_\varepsilon(t,x) > g(t) \tag{5_ε}$$

for $t \in [-\tau_\varepsilon, T]$ and $x \in \left[a, \frac{\gamma}{\gamma-1}c\right]$; ε_3 such that for all $\varepsilon \in [0,\varepsilon_3]$, we have

$$a < \phi_\varepsilon(t) \tag{6_ε}$$

for $t \in [-\tau, 0]$; and ε_4 such that

$$c - \delta \le \phi_\varepsilon(t) < \frac{\gamma}{\gamma-1}c \tag{$8_\varepsilon'$}$$

for all $\varepsilon \in [0,\varepsilon_4]$, $t \in [-\tau_\varepsilon, 0]$.

Now, for $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$, knowing (12) and the continuity of $f_\varepsilon(t,x)$ and $\phi_\varepsilon(t)$, we dedu e that conditions (H_1)-(H_4) and (8') hold for (1_ε)-(2_ε), $\varepsilon \in [0,\varepsilon_0]$. Applying Remark 2.1 and Theorem 2.1 we conclude that each problem (1_ε)-(2_ε), $\varepsilon \in [0,\varepsilon_0]$, has at least one continuous solution.

Moreover, let fix us $\varepsilon \in [0,\varepsilon_0]$ and consider an arbitrary solution $x_\varepsilon(t)$ of the problem (1_ε)-(2_ε). If we suppose that there exists $t \in [0,T]$ such that $x_\varepsilon(t) \ge \frac{\gamma}{\gamma-1}c$, then, choosing $t_0 = \inf\left\{t \in [0,T]\ |\ x_\varepsilon(t_0) \ge \frac{\gamma}{\gamma-1}c\right\}$, we have $x_\varepsilon(t_0) < \frac{\gamma}{\gamma-1}c$ for $-\tau_\varepsilon \le t < t_0$ and $x(t_0) = \frac{\gamma}{\gamma-1}c$. Thus, we can follow the same arguments as in the proof of Theorem 2.1 to obtain a contradiction. On the other hand, if we suppose that there exists $t \in [0,T]$ such that $x_\varepsilon(t) \leq a$, then, choosing $t_0 = \inf\{t \in [0,\mathcal{T}], x(t) \leq a\}$, we have $a < x_\varepsilon(t)$ for $-\tau_\varepsilon \leq t < t_0$ and $x_\varepsilon(t_0) = a$. Now, from (H$_3$) and (5_ε) we deduce

$$a = x_\varepsilon(t_0) = \int_{t_0-\tau}^{t_0} f_\varepsilon(s, x_\varepsilon(s))ds > \int_{t_0-\tau}^{t_0} g(s)ds \geq a.$$

Thus, we obtain $a > a$, a contradicton.

So, we can conclude that for each $\varepsilon \in [0, \varepsilon_0]$ and each solution $x_\varepsilon(t)$ of problem (1_ε)-(2_ε) we have

$$a < x_\varepsilon(t) < \frac{\gamma}{\gamma - 1}c \text{ for } -\tau_\varepsilon \leq t \leq \mathcal{T}.$$

Now the proof is complete.

□

In a similar way we obtain the following corollary of Theorem 2.2:

Corollary 2.4. *Suppose that for problem (1)-(2),conditions (H_1), (H_2'), (H_3) and (H_4) are satisfied. Then, there exists $0 < \varepsilon_0 \leq \eta$ such that each problem (1_ε)-(2_ε), with $0 \leq \varepsilon \leq \varepsilon_0$, has at least one continuous solution, provided that (12) holds and inequalities $b \leq c$, (4'), (5) and (6) are strict. Moreover, for each $\varepsilon \in [0, \varepsilon_0]$ and each solution $x_\varepsilon(t)$ of problem (1_ε)-(2_ε), we have:*

$$a < x_\varepsilon(t) < \frac{\gamma}{\gamma - 1}c, \textit{ for } 0 \leq \tau \leq \mathcal{T}.$$

We omit here the proof since it is similar with that of Corollary 2.3.

3. Uniqueness and Continuous Dependence on f, ϕ, τ.

Let us suppose that the assumption of Corollary 2.3 or 2.4 hold for (1)-(2). Consider the continuous and nonnegative functions $f_n(t,x)$, $\phi_n(t)$, $n \geq 1$ and the positive real numbers τ_n, $n \geq 1$ such that $b_n = \phi_n(0) = \int_{-\tau_n}^0 f_n(s, \phi_n(s))ds, \tau_n \to \tau$, $\phi_n \,' \phi$ and $f_n \,' f$ as $n \to \infty$, i.e. for each $0 < \varepsilon \leq \eta$ there exists $n_0 \geq 1$ such that for every $n \geq n_0$, we have $|\tau_n - \tau| < \varepsilon$, $| \phi_n(t) - \phi(t) | < \varepsilon$ for all $t \in [-\tau_n, 0]$ and $| f_n(t,x) - f(t,x) | < \varepsilon$ for all $t \in [-\tau_n, \mathcal{T}]$ and $x \in \left[a, \frac{\gamma}{\gamma-1}c\right]$.

Finally, let us consider the problems

$$x(t) = \int_{t-\tau_n}^{t} f_n(s, x(s))ds, \text{ for } 0 \leq t \leq \mathcal{T} \qquad (1_n)$$

$$x(t) = \phi_n(t), \text{ for } -\tau_n \le t \le 0. \tag{2_n}$$

By Corollary 2.3 or Corollary 2.4 we deduce that for sufficiently large n, problem (1_n)-(2_n) has at least one continuous solution, and every solution $x_n(t)$ of (1_n)-(2_n) satisfies

$$a < x_n(t) < \frac{\gamma}{\gamma - 1}c, \text{ for } -\tau_n \le t \le \mathcal{T} \tag{7_n}$$

respectively

$$a < x_n(t) < \frac{\gamma}{\gamma - 1}c, \text{ for } 0 \le t \le \mathcal{T}. \tag{$7'_n$}$$

The question is if we choose an arbitrary sequence $x_n(t))_{n\ge1}$ of solutions of problems (1_n)-(2_n) respectively, then $x_n(t))_{n\ge1}$ converges uniformly to a solution of (1)-(2)? In the case that (1)-(2) has at most one continuous solution the answer is given by the main result of this section:

Theorem 3.1. *Suppose that problem (1)-(2) has at most one continuous solution and let the conditions of Corollary 2.3 or Corollary 2.4, respectively, hold. Then, the continuous solution $x(t)$ of (1)-(2) depends continuously on f, ϕ, τ, i.e. $(x_n(t))_{n\ge1}$ converges uniformly to $x(t)$.*

Proof. From (7_n) or $(7'_n)$, respectively, we deduce that the sequence $(x_n(t))_{n\ge1}$ is uniformly bounded. In order to apply Arzelá's Theorem (see [6, p.138]) we derive (1_n) with $x(t) = x_n(t)$ and we obtain

$$x'_n(t) = f_n(t, x_n(t)) - f_n(t - \tau_n, \phi_n(t - \tau_n)), \text{ for } 0 \le t \le \mathcal{T}.$$

From the continuity of f and ϕ we deduce that there exist M, $N > 0$ such that

$$| f(t,x) | \le M \text{ for all } t \in [0, \mathcal{T}] \text{ and } x \in \left[a, \frac{\gamma}{\gamma - 1}c\right]$$

$$| f(t, \phi(t) | \le N \text{ for all } t \in [-\tau - \eta, 0].$$

Now, using that f_n ′ f, ϕ_n ′ ϕ and $\tau_n \to \tau$, we get for n sufficiently large

$$| f_n(t,x) | \le M + 1 \text{ for all } t \in [0, \mathcal{T}] \text{ and } x \in \left[a, \frac{\gamma}{\gamma - 1}c\right]$$

$$| f_n(t, \phi_n(t)) | \le N + 1 \text{ for all } t \in [-\tau_n, 0].$$

Consequently

$$| x_n'(t) |=| f_n(t, x_n(t)) - f_n(t - \tau_n, \phi_n(t - \tau_n)) |\leq M + N + 2 \text{ for } 0 \leq t \leq \mathcal{T}.$$

So $(x_n'(t))_{n\geq 1}$ is uniformly bounded. By Arzelá's Theorem we can choose a subsequence $(x_{n_k}(t))_{k\geq 1}$ such that $x_{n_k} \rightrightarrows x_0$ on $[0, \mathcal{T}]$, where x_0 is a continuous function on $[0, \mathcal{T}]$. But, from (1_n) we have

$$x_{n_k}(t) = \int_{t-\tau_{n_k}}^{t} f_{n_k}(s, x_{n_k}(s))ds \text{ for } 0 \leq t \leq \mathcal{T}$$

and taking the limit as $k \to \infty$, we obtain

$$x_0(t) = \int_{t-\tau}^{t} f(s, x_0(s))ds \text{ for } 0 \leq t \leq \mathcal{T}$$

where we have used the hypothesis $f_{n_k} \rightrightarrows f$, $\phi_{n_k} \rightrightarrows \phi$ and $\tau_{n_k} \to \tau$. Hence $x_0(t)$ is a solution of (1)-(2)

Now the uniqueness of solutions of (1)-(2) implies that $x_0 \equiv x$. Moreover the entire sequence $(x_n(t))_{n\geq 1}$ converges uniformly to $x(t)$. The proof is now complete. □

An immediate consequence of Theorem 3.1 is:

Corollary 3.2. *Suppose the conditions of Corollary 2.3 or 2.4 hold for (1)-(2) and there exists $k \geq 0$ such that*

$$| f(t, x_1) - f(t, x_2) |\leq k | x_1 - x_2 | \quad \textit{for all } t \in [0, \mathcal{T}] \textit{ and } x_1, x_2 \in \left[a, \frac{\gamma}{\gamma - 1}c\right[.$$

Then problem (1)-(2) has an unique continuous solution which depends continuously on f, ϕ, τ.

Proof. Using Corollary 2.3 or 2.4 respectively and Gronwall's Lemma it is easy to obtain the uniqueness of solutions for (1)-(2). Now, we can apply Theorem 3.1 and the proof is complete. □

Another result can be derived by using the "one-sided generalization of Nagumo's criterion" (see [4. p.35]).

Corollary 3.3. *Suppose that the conditions of Corollary 2.3 or 2.4 hold for problem (1)-(2) and*

$$[f(t,x_1) - f(t,x_2)](x_1 - x_2) \le \frac{(x_1 - x_2)^2}{t} \quad \textit{for} \quad 0 < t \le \mathcal{T} \quad \textit{and}$$
$$x_1, x_2 \in \left[a, \frac{\gamma}{\gamma - 1}c\right[. \tag{13}$$

Then (1)-(2) has an unique solution which depends continuously on f, ϕ, τ.

Proof. Let us observe that (1)-(2) under condition (3) is equivalent with the Cauchy problem:

$$\begin{cases} x'(t) = h(t, x(t)), & \text{for } 0 \le t \le \mathcal{T}, \\ x(0) = b, \end{cases} \tag{14}$$

where $h(t, x(t)) = f(t, x(t)) - f(t - \tau, \phi(t - \tau))$. From (13) we obtain that

$$[h(t,x_1) - h(t,x_2)](x_1 - x_2) \le \frac{(x_1 - x_2)^2}{t} \quad \text{for} \quad 0 < t \le \mathcal{T} \quad \text{and}$$
$$x_1, x_2 \in \left[a, \frac{\gamma}{\gamma - 1}c\right[,$$

and consequently, (14) has an unique solution (see [4, p.35]). Therefore, problem (1)-(2) has an unique solution and we can apply Theorem 3.1 to obtain the conclusion. □

If we suppose that $f(t,x)$ is nonincreasing with respect to $x \in \left[a, \frac{\gamma}{\gamma-1}c\right[$, for $t \in [0, \mathcal{T}]$, then (13) is obviously satisfied. So we proved the following result:

Corollary 3.4. *Suppose that the assumptions of Corollary 2.3 or 2.4 are satisfied and*

$$f(t,x_1) \ge f(t,x_2) \textit{ whenever } x_1 \le x_2, \; x_1, x_2 \in \left[a, \frac{\gamma}{\gamma - 1}c\right[\textit{ and } t \in [0, \mathcal{T}].$$

Then problem (1)-(2) has an unique solution which depends continuously on f, ϕ, τ.

4. Continuous Dependence on f, ϕ, τ of the Minimal and Maximal Solutions.

What happens when we have no uniqueness? Let us note that problem (1)-(2) is equivalent with the Cauchy problem (14). So, we locally have a minimal $x_*(t)$ and a maximal $x^*(t)$ solution of (14) (see [4, p.25]). Suppose that the assumptions of Theorem 2.1 or 2.2 hold. Then we have at least one continuous solution $x(t)$ of problem (14) on $[0,\mathcal{T}]$, with the property (7'), hence $a \le x(t) \le x^*(t)$ and using the apriori bound $\frac{\gamma}{\gamma-1}c$ obtained in the proofs of Theorems 2.1 and 2.2 we deduce that $x^*(t)$ is defined on $[0,\mathcal{T}]$

and $a \le x^*(t) < \frac{\gamma}{\gamma-1}c$ for $t \in [0,\mathcal{T}]$. On the other hand, if we suppose that $f(t,0) = 0$ for $t \in [-\tau,\mathcal{T}]$ then considering only nonnegative solutions for (14), we have $0 \le x_*(t) \le x(t)$ for $t \in [0,\mathcal{T}]$. Consequently $x_*(t)$ is defined on $[0,\mathcal{T}]$. So we have a minimal and a maximal solution for (1)-(2) on $[0,\mathcal{T}]$. It is well known that in general we have no continuous dependence on the data for the maximal and minimal solutions. Moreover the result from [6, p.176] can not be applied for problem (1)-(2) because the variation of b in (14) implies, by (3) the variation of ϕ which implies the variation of h in (14).

In order to obtain the continuous dependence on f, ϕ, τ of the minimal and maximal solutions we suppose that

(H_5) $f(t,x)$ is nondecreasing in x for all t $[-\tau,\mathcal{T}]$ and $x \in \left[a, \frac{\gamma}{\gamma-1}c\right[$.

As in the previous section we consider the family of continuous functions $f_n(t,x)$, $\phi_n(t)$, $n \ge 1$ and the sequence of positive real numbers τ_n, $n \ge 1$ such that $b_n = \phi_n(0) = \int_{-\tau_n}^0 f_n(s,\phi_n(s))ds$, $\tau_n \to \tau$, ϕ_n ' ϕ and f_n ' f as $n \to \infty$ and the problems (1_n)-(2_n). Moreover, we suppose that for each $n \ge 1$, $f_n(t,x)$ satisfies (H_5) with $\tau = \tau_n$.

We need the following comparison lemma:

Lemma 4.1. *Let us consider the functions $f_i(t,x)$, $\phi_i(t)$, $i = 1,2$ and the real positive numbers τ_i, $i = 1,2$ such that (H_1)-(H_5) hold for (1_i)-(2_i), $i = 1,2$. Also suppose that $f_1(t,x) \le f_2(t,x)$ for all t $[-\tau_1,T] \cap [-\tau_2,T]$ and $x \in \left[a, \frac{\gamma}{\gamma-1}c\right[$, $\phi_1(t) \le \phi_2(t)$ for all $t \in [-\tau_1,0] \cap [-\tau_2,0]$, and $\phi_i(0) = \int_{-\tau_i}^0 f_i(s,\phi_i(s))ds$, $c \le \phi_i(t) < \frac{\gamma}{\gamma-1}c$ for all t $[-\tau_i,0]$, $i = 1,2$. Then (1_i)-(2_i), $i = 1,2$ have the minimal $x_{i*}(t)$ and the maximal $x_i^*(t)$ solutions such that*

$$a \le x_{i*}(t) \le x_i^*(t) < \frac{\gamma}{\gamma-1}c \quad for \ t \in [-\tau_i,\mathcal{T}],\ i = 1,2$$

and

$$x_{1*}(t) \le x_{2*}(t),\ x_1^*(t) \le x_2^*(t) \quad for \ t \in [-\tau_1,\mathcal{T}] \cap [-\tau_2,\mathcal{T}].$$

Proof. E_i, K_i, U_i be as in the proof of Theorem 2.1, $i = 1,2$ (with $\tau = \tau_i$). Define

$$T_i : U_i \to K_i$$

$$T_i x(t) = \int_{t-\tau_i}^t f_i(s,x(s))ds \ \text{ for } \ t \in [0,\mathcal{T}],$$

where $f_i(s, x(s)) = \phi_i(s)$ for $s \in [-\tau_i, 0]$. Also consider the sequences $(A_i^n)_{n\geq 1}, (B_i^n)_{n\geq 1} \subset \bar{U}_i$ such that

$$A_i^1(t) \equiv a \text{ for } t \in [0, \mathcal{T}],$$
$$A_i^{n+1} = T_i(A_i^n) \text{ for } n \geq 1$$
$$B_i^1 \equiv \frac{\gamma}{\gamma - 1}c \text{ for } t \in [0, \mathcal{T}]$$
$$B_i^{n+1} = T_i(B_i^n).$$

Following the proofs of Theorem 2 and 3 from [8] we obtain that

$$a = A_i^1 \leq A_i^2 \leq ... \leq x_{i*} \leq x_i^* \leq ... \leq B_i^2 < B_i^1 = \frac{\gamma}{\gamma - 1}c, \ i = 1, 2$$

and $(A_i^n), (B_i^n)$ converge uniformly on $[0, \mathcal{T}]$ to x_{i*} and x_i^*, respectively, the minimal and maximal solution for (1_i)-(2_i), $i = 1,2$.

Moreover, it is easy to prove that if $x_i \in \bar{U}_i$ and $x_1(t) \leq x_2(t)$ for all $t \in [0, \mathcal{T}]$ then

$$(T_1x_1)(t) \leq (T_2x_2)(t) \text{ for all } t \in [0, \mathcal{T}].$$

Consequently, $A_1^n(t) \leq A_2^n(t)$, $B_1^n(t) \leq B_2^n(t)$, for all $n \geq 1$ and $t \in [0, \mathcal{T}]$.

Taking the limits as $n \to \infty$, we obtain that $x_{1*}(t) \leq x_{2*}(t)$, $x_1^*(t) \leq x_2^*(t)$ for $t \in [0, \mathcal{T}]$.

Thus, the proof is complete. □

First we shall deal with the continuous dependence of the minimal solution:

Theorem 4.2. *Suppose that $f_n \nearrow f$, $\phi_n \nearrow \phi$, $\tau_n \to \tau$ as $n \to \infty$, (H_5) holds for each $n \geq 1$ and (H_1)-(H_5) hold for problem (1)-(2) with strict inequalities in (4), (5) and (6). Then, for sufficiently large n problem (1_n)-(2_n) has a minimal solution $x_{n*}(t)$ such that $a < x_{n*}(t) < \frac{\gamma}{\gamma-1}c$ for $t \in [0, \mathcal{T}]$ provided that (8) holds. Moreover, if for each $n \geq 1$, $f_n(t, x) \leq f(t, x)$, $\phi_n(t, x) \leq \phi(t, x)$ and $\tau_n \leq \tau$ for all $t \in [-\tau_n, \mathcal{T}]$ and $x \in \left[a, \frac{\gamma}{\gamma-1}c\right]$, then $x_{n*} \nearrow x_*$.*

Proof. From Corollary 2.3 we deduce that Lemma 4.1 applies to problems (1_n)-(2_n), (1)-(2) for sufficiently large n. Hence

$$a \leq x_{n*} \leq x_* < \frac{\gamma}{\gamma - 1}c.$$

As in the proof of Theorem 3.1 we can choose an uniformly convergent subsequence of $(x_{n*}(t))_{n\geq 1}$ to x_0. Taking the limit in (1_n) we obtain that x_0 is a solution of (1)-(2). But

x_* is the minimal solution of (1)-(2) in U, and by Corollary 2.3, each solution of (1)-(2) is in U, hence $x_* \leq x_0$ and using the fact that x_* is un upper bound for the sequence $(x_{n^*}(t))_{n\geq 1}$, we deduce that $x_* = x_0$. Now, the uniqueness of the minimal solution x_* implies that the entire sequence $(x_{n^*}(t))_{n\geq 1}$ converges uniformly to x_*. □

The corresponding result for the maximal solution is:

Theorem 4.3. *Suppose that $f_n \nearrow f$, $\phi_n \nearrow \phi$, $\tau_n \to \tau$ as $n \to \infty$, (H_5) holds for each n ≥ 1 and (H_1)-(H_5) hold for problem (1)-(2) with strict inequalities in (4), (5) and (6). Then, for sufficiently large n problem (1_n)-(2_n) has a maximal solution $x_n^*(t)$ such that $a \leq x_n^*(t) < \frac{\gamma}{\gamma-1}c$ for $t \in [0,T]$ provided that (8) holds. Moreover, if for each $n \geq 1$, $f_n(t,x) \geq f(t,x)$, $\phi_n(t,x) \leq \phi(t,x)$ and $\tau_n \geq \tau$ for all $t \in [-\tau_n, T]$ and$x \in \left[a, \frac{\gamma}{\gamma-1}c\right[$, then $x_n^* \nearrow x^*$.*

The proof is similar with that of Theorem 4.3.

Remark 4.1. In Theorem 4.2 we can give up the condition (8) and use (H_2') instead of (H_2). But this is not possible for Theorem 4.3 because, in this case, Theorem 3 from [8] and Lemma 4.1 are no more true.

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LOW ORDER SPLINES IN SOLVING NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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Dedicated to Prof. Gheorghe Coman at his 60^{th} *anniversary*

Abstract. Some low order splines (linear, quadratic, hermite cubic) and spline recurrences are used in algorithms for the numerical solution of initial value problems in neutral delay differential equations.

Key words: splines, numerical solution of retarded differential equations

1. Introduction

The mathematical models of retarded processes in many fields of applied sciences lead to delay differential equations. We are interested in the numerical solution of the initial value problem

$$y'(t) = f(t, y(t), y(d(t)), y'(d(t))),\ t \in [a, b],\ \alpha \leq d(t) \leq t, \tag{1}$$

$$y(t) = g(t) \ \text{ for } t \in [\alpha, a],\quad \alpha \leq a < b, g \in C^1 \tag{2}$$

Suppose that the functions f, g, d satisfy certain conditions which guarantee the existence and uniqueness of the solution of the IVP (1) (see e.g. [4],[7],[1]).
The validity of the condition

$$g'(a) = f(a, g(a), v, w),\ \text{with } v = g(u), w = g'(u), u = d(a) \tag{3}$$

eliminates the jump discontinuities in the first derivative of the solution caused by the delay function $d(t)$. These discontinuities appear generally at the points ξ_i which are the roots of the equation (see [1], [17], [4], [7])

$$d(\xi_i) = \xi_{i-1},\ \xi_0 = a. \tag{4}$$

These points can now be the the points of discontinuity of y'' (y') and we shall count them to the obligatory knots of the spline used to the approximation of the solution. In practical computations with given discrete values $g(t_i)$ we can arrange the validity of the condition (3) by approximation of $g'(t_i)$ by unique spline $S_{21}(x)$ or $s_{32}(x)$) interpolating the starting values $g(t_i)$ and fulfilling the condition (3).

In case that the condition (3) does not hold we recommend to restart our algorithms at such points. The simplest case $d(t) < t$ for $t > a$ with monotone increasing delay function $d(t)$ is quite natural in many applications.

2. Low order splines

Let us have the spline knot set on the interval $[\alpha, b]$

$$(\Delta x) := x_0 < x_1 < ... < x_n < x_{n+1} \text{ with stepsizes } h_i = x_{i+1} - x_i. \quad (5)$$

Definition 1 *A function $s(x) = s_{k,d}(x)$ is called a spline of the degree k and with the defect d on the knotset (Δx) if*

1. $s(x) \in C^{k-d}[x_0, x_{n+1}]$,
2. *$s(x)$ is a polynomial of the degree k on each interval $[x_i, x_{i+1}]$,* $i = 0(1)n$.

In this contribution we shall use the following low degree splines (see e.g.[2], [10],[12], [18])

- linear splines (polygons) $s_{11} \in C^0$,
- quadratic splines $s_{21} \in C^1$,
- the local Hermite splines $s_{32} \in C^1$, $s_{53} \in C^2$.

We use the local parameters $s_i = s(x_i)$, $m_i = s'(x_i)$ in appropriate local representations with the local parameter $q = (x - x_i)/h_i$

$$s_{11}(x_i + qh_i) = (1 - q)s_i + qs_{i+1}, \quad (6)$$

$$s_{21}(x_i + qh_i) = (1 - q^2)s_i + q^2 s_{i+1} + h_i q(1 - q)m_i, \quad (7)$$

$$= s_i + h_i q[(1 - q/2)m_i + (q/2)m_{i+1}], \quad (8)$$

$$= (1 - q)^2 s_i + q(2 - q)s_{i+1} + h_i q(q - 1)m_{i+1}. \quad (9)$$

We can use some function values $g_i = s(t_i)$ with $x_i < t_i < x_{i+1}, d_i = (t_i - x_i)/h_i$ in the local representations

$$s_{21}(x) = (1 - q)(1 - q/d_i)s_i + q[1 - (1 - q)/(1 - d_i)]s_{i+1} + [(q/d_i)(1 - q)/(1 - d_i)]g_i,$$

$$s_{21}(x) = g_i + h_i(q - d_i)[(2 - q - d_i)m_i + (q + d_i)m_{i+1}]/2. \quad (10)$$

With the local parameters $s'(t_i)$ we can use the representation

$$s_{21}(x) = a(q)s_i + b(q)s_{i+1} + h_i c(q)s'(t_i), \quad d_i \neq 1/2 \quad (11)$$

with $b(q) = -q(q - 2d_i)/(2d_i - 1)$, $a(q) = 1 - b(q)$, $c(q) = q(q-1)/(2d_i - 1)$,

$$s_{21}(x) = s_i + h_i m_i q + h_i q^2(s'(t_i) - m_i), \quad d_i = 1/2. \quad (12)$$

For the local Hermite cubic spline we use the local representation (see e.g.[2], [10], [12])

$$s_{32}(x) = (1-q)^2(1+2q)s_i + q^2(3-2q)s_{i+1} + h_i q(1-q)[(1-q)m_i - qm_{i+1}]. \quad (13)$$

The local representations of splines enables us to compute the function value or the value of the derivative in any point of the retarded argument from known local parameters. In the predictor formulas we use some of the recurrence relations between local parameters $s_i = s(x_i)$, $m_i = s'(x_i)$, $g_i = s(t_i)$,
$x_i < t_i < x_{i+1}$ of the approximating spline $s_{21}(x)$:

$$m_i + m_{i+1} = 2(s_{i+1} - s_i)/h_i, \quad (14)$$

$$a_i m_{i-1} + b_i m_i + c_i m_{i+1} = 2(g_i - g_{i-1}) \quad (15)$$

with $a_i = h_{i-1}(1 - d_{i-1})^2$, $c_i = d_i^2 h_i$, $b_i = h_i d_i(2 - d_i) + h_{i-1}(1 - d_{i-1}^2)$.
On the equidistant knotset with the stepsize $h_i = h$, $d_i = 1/2$ the last recurrence reads

$$(m_{i-1} + 6m_i + m_{i+1})/8 = (g_i - g_{i-1})/h \quad (16)$$

Similar recurrences hold for the values of the second derivative (which is constant over the local intervals) - see e.g.[10] and section 5.2 .

3. Simple explicit algorithms

3.1. Spline s_{11} in Euler's method. Let us have the problem (1)-(2) with known points ξ_i of discontinuities of $y'(y'')$. These points should be used as obligatory knots of the approximating spline, with so many additional inner knots as the accuracy requires.
Let us denote in the following t_j the knots of spline s_{11} used,
$u_j = d(t_j) \leq t_j$, $v_j = y(u_j)$, $w_j = y'(u_j)$.
The modified Euler explicit algorithm known from ODE's (see [3]) and [5] for the problem

(1)-(2) with given functions f, d, g can be described as follows:

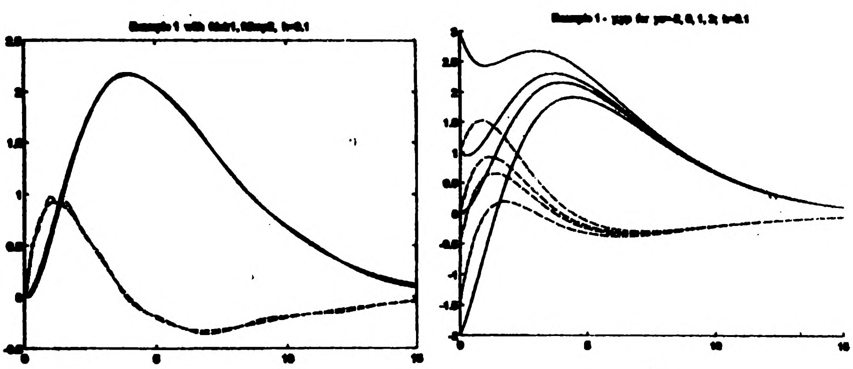
Algorithm FDELR1(h,t,ys,yps) for computing y'_i, y_{i+1}:

1. Let us have the knotset $t = [t_i]$ containing the points ξ_i with stepsizes $h = [h_i]$, constructed a priori or during computations; the starting points are included in our knotset. The input vectors ys, yps contain the starting values $ys(i) = g(t_i), yps(i) = g'(t_i)$ of the solution and its derivative - in the amount which at least corresponds to the needs of the problem solved. We do not suppose to have the analytic expression for the starting function $g(t)$ generally, although it is easily possible to use it and obtain so quite naturally the starting data on the knotset. The prescribed accuracy parameter may be added for controlling accuracy strategy, which will be not discused here.
2. In case $\alpha = a,\ d(a) = a,\ y_1 = g(a)$ we compute
   $y_2 = y_1 + h_1 f(a, y_1, y_1, g'(a))$
   to have the initial vector ys with at least two components;
   in the general case, given analytic form of function $g(t)$,
   we compute $d(a)$ and corresponding values of the function g, g'
   to calculate the value y_2.
3. In the general step in the knot $t = t_i$:
   find $u_i = d(t_i)$;
   find index j such that $t_{j-1} < u_i \le t_j$;
   compute $q = (u_i - t_{j-1})/h_{j-1}$; $v_i = (1-q)y_{j-1} + qy_j$;
4. if $j < i - 1$, then $w_i = (1-q)y'_{j-1} + qy'_j$;
   else $w_i = (1-q)y'_{i-1} + qf(t_i, y_i, v_i, y'_{i-1})$; end
   $y'_i = f(t_i, y_i, v_i, w_i)$,
   $y_{i+1} = y_i + h_i y'_i$.

Example 1 The equation with the general delay function $d = d(t)$

$$y'(t) = -\frac{1}{2}[y(t) + y(d(t))] - y'(d(t)) + 2[t.exp(-t/2) + d.exp(-d/2)]$$

has the exact solution $y(t) = t^2.exp(-t/2)$ with $y(0) = 0,\ y'(0) = 0$.
For the delay function $d(t) = 3t/4$ with $d(0) = 0$ the conditions (3) and $y(0) = 0$ give $y'(0) = 0$. For the general delay function with $p = d(0) \le 0$ the condition (3) reads $y'(0) = -\frac{1}{2}[y(0) + y(p)] - y'(p) + 2p.exp(-p/2)$. For the delay function $d(t) = 3t/4$ we



a) b)

Fig. 1.

can see plots of the numerical solution and its first derivative computed on the equidistant set with h=0.1 on the Fig. 1a. On the Fig.1b there are plotted the solutions corresponding to the values $y(0) = -2, 0, 1, 3$ and to the condition (3).

3.2. Linear and quadratic spline algorithm. We describe now some slightly more sofisticated algorithm which uses some quadratic spline relations. We will keep the notation u_j, v_j, w_j from 3.1 for the delayed values of t, y, y' .

Algorithm FDESP21(h,t,ys,yps)

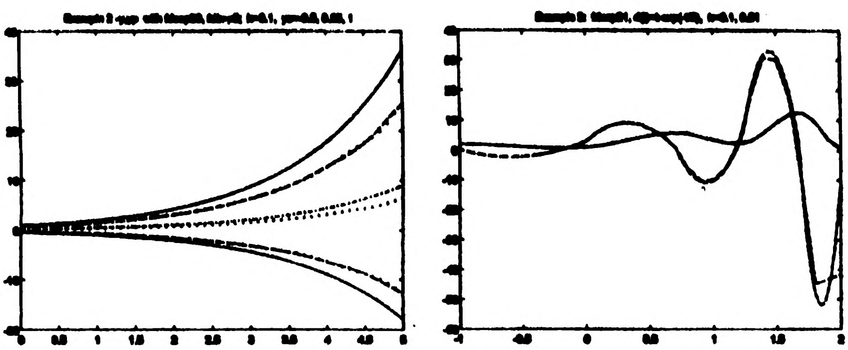
1. In case of $d(a) = a$ with given $y_1 = y(a) = g(a)$ we find the next value in two steps with halved stepsize (and possible extrapolation):

$$y_{3/2} = y(a + h_1/2) = y_1 + h_1 y'(a)/2;\ u = d(a + h_1/2),\ q = (u - a)/(h_1/2);$$

$$v = y(a)(1 - q^2) + y_{3/2}q^2 + h_1 q(1 - q)y_1'/2;\ w = 2q(y_{3/2} - y_1)/(h_1/2) + (1 - 2q)y_1';$$

$$y_{3/2}' = f(a + h_1/2, y_{3/2}, v, w); \qquad y_2 = y_{3/2} + h_1 y_{3/2}'/2.$$

2. In the general step with at least two preceding points t_{i-1}, t_i with yet known values y_{i-1}, y_i, y_{i-1}'
compute $u_i = d(t_i)$ and find the index j such that $t_{j-1} < u_i \leq t_j$;
3. compute $v_i = s_{21}(u_i)$, $w_i = s_{21}'(u_i)$ using the local parameters y_{j-1}, y_j, y_{j-1}' or y_{j-1}, y_j, y_j' (e.g. in case $j = i$).
4. compute $y_i' = f(t_i, y_i, v_i, w_i)$,
choose t_{i+1} according to acurracy needs (or $t_{i+1} \leq \xi_k$) and finally compute $y_{i+1} = y_i + h_i y_i'$ (spline s_{11} used)



a) b)

Fig. 2.

or $y_{i+1} = p_i^2 y_{i-1} + (1 - p_i^2) y_i + (h_{i-1} + h_i) p_i y'_i$ (spline s_{21} used, $p_i = h_i / h_{i-1}$),
or $y_{i+1} = y_i + h_i p_i [-p_i y'_{i-1} + (2 + p_i) y'_i]/2$ (the most stable formula, $p = 2$).

Example 2 The equation with unknown exact solution

$$y'(t) = y(t) + y(d(t)) - 2y'(d(t))$$

was solved with delay function $d(t) = t(1 - exp(-t/2))$ with $d(0) = 0$, $d(t) < t$ for $t > 0$. For the initial value $y(0) = a$ the condition (3) reads $y'(0) = 2a/3$. The computed results for $a = -0.5, 0.25, 1$, are plotted on Fig.2a . For the delay function $d(t) = t - exp(-t/2)$, $d(0) = -1$ the condition (3) reads now $y'(0) = y(0) + y(-1) - 2y'(-1)$ and the initial function $g(t)$ can be constructed e.g. as the Hermite interpolant s_{32} or quadratic spline interpolant s_{21}. The results for $h = 0.1, 0.01$ are plotted on Fig.2b.

3.3. **Explicit algorithm based on splines s_{11}, s_{21}, s_{32}.** We can combine the above mentioned splines to use more pieces of information at hand and to obtain more accurracy in that way.

Algorithm FDESP23 (computing y'_i, y_{i+1} from y_{i-1}, y'_{i-1}, y_i) :

1. Let us keep the notation from foregoing sections and suppose that we have the starting interval with known values at least y_{i-1}, y'_{i-1}, y_i at our disposal; in case of $d(a) = a$ we compute y_1 as in Alg. FDESP12.
2. In the general situation at $t = t_i$ we can compute the value y'_i by computing subsequently
   $u_i = d(t_i)$; find index j such that $t_{j-1} \le u_i < t_j$, $q = (u_i - t_{j-1})/h_{j-1}$;

if $j < i$, (s_{32} interpolant)

$v = (1-q)^2(1+2q)y_{j-1} + q^2(3-2q)y_j + h_{j-1}q(1-q)[(1-q)y'_{j-1} - qy'_j]$;

$w = 6q(1-q)(y_j - y_{j-1})/h_{j-1} + (1-q)(1-3q)y'_{j-1} + q(3q-2)y'_j$;

else (s_{21} interpolant)

$$v = y_{i-1}(1-q^2) + y_i q^2 + h_{i-1}q(1-q)y'_{i-1}; \tag{17}$$

$$w = 2q(y_i - y_{i-1})/h_{i-1} + (1-2q)y'_{i-1}; \tag{18}$$

end .

3. $y'_i = f(t_i, y_i, v, w)$;
4. $y_{i+1} = p_i^2 y_{i-1} + (1-p_i^2)y_i + (h_{i-1}+h_i)p_i y'_i, \quad p_i = h_i/h_{i-1}$

(extrapolation with s_{21} used - with midpoint rule as a result in equidistant case)

- or with more stability

(the generalization of LMM $y_{i+1} = y_i + h_i[3y'_i - y'_{i-1}]/2$)

$$y_{i+1} = y_i + h_i p_i[-p_i y'_{i-1} + (2+p_i)y'_i]/2.$$

Remark The weak stability of resulting midpoint formula is known from ODE's and can be recognized in computations. Extrapolation from cubic s_{32} spline leads to the formula

$$y_{i+1} = 5y_{i-1} - 4y_i + 2h(y'_{i-1} + 2y_i)$$

which is unstable.

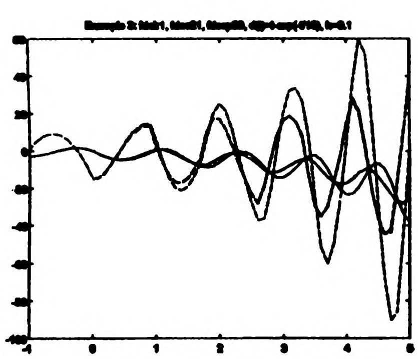
Example 3 The equation

$$y'(t) = 4y(d(t)) - 2y(t) - y'(d(t)), \quad t \in [0,5]$$

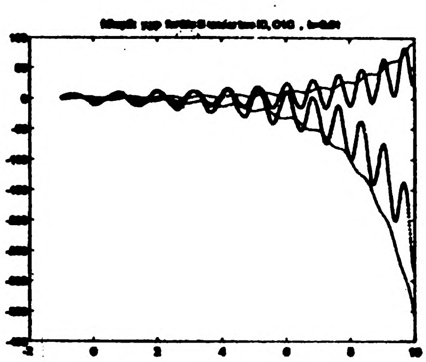
with the delay function $d(t) = t - exp(-t/10)$, $d(0) = -1, d(t) < t$ for $t > 0$ has unknown exact solution. The condition (3) for the starting function $g(t)$ reads now

$$y'(0) = 4g(-1) - 2g(0) - g'(-1) - 2.$$

We have constructed this function by Hermite interpolation with given values $g(-1), g'(-1)$ and $y'(0)$ computed from this condition. The results for the equidistant knot set with $h = 0.1$ are plotted in Fig.3a. The solution on the interval $[0,10]$ for $g(-1) = 0, -2, g'(-1) = 1, g(0) = 2$ is plotted on Fig.3b.



a)



b)

Fig. 3.

4. Simple implicit algorithms

The implicit linear multistep methods for ODE's are known to have better precision and stability properties. Combined with some explicit methods they are used in predictor-corrector algorithms. We will demonstrate similar idea for the algorithm of the predictor-corrector method for the problem (1).

Algorithm FDIMP2 (computing y_{i+1}, y'_{i+1}) from y_i, y'_i):

1. Let us suppose that we have computed yet the discrete approximation of the solution to the point t_i of the knotset with at least two known values $y(t_j) = y_j$, $y'(t_j) = y'_j, j \leq i$ and with computed values
   $u_j = d(t_j)$, $v_j = s(u_j)$, $w_j = s'(u_j)$.
   The starting interval -if needed - is computed similarly as in foregoing algorithms.
2. Choose t_{i+1} according to accuracy needs; $h_i = t_{i+1} - t_i$.
   Compute $u_{i+1} = d(t_{i+1})$ and find index j such that $t_j \leq u_{i+1} < t_{j+1}$.
3. If $u_{i+1} < t_i$ then compute $v_{i+1} = s_{32}(u_{i+1})$, $w_{i+1} = s'_{11}(u_{i+1})$ from the spline representations (13),(6).
   Predict the slope $m^0_{i+1} = (y_i - y_{i-1})/(t_i - t_{i-1})$ and iterate
   $y^{k+1}_{i+1} = y_i + \frac{1}{2}h_i(y'_i + m^k_{i+1})$,
   $m^{k+1}_{i+1} = f(t_{i+1}, y^{k+1}_{i+1}, v_{i+1}, w_{i+1}), k = 0, 1, \ldots$ to convergence.
4. Else in case $u_{i+1} > t_i$ predict the following parameters by extrapolation
   $v^0_{i+1} = s_{21}(u_{i+1})$ with the local parameters y_{i-1}, y_i, y'_i,

$y_{i+1}^0 = s_{21}(t_{i+1})$ with local parameters y_i, y_i', v_{i+1};

$w_{i+1}^0 = s_{21}'(u_{i+1})$ with local parameters y_i, y_{i+1}^0, y_i' ;

$m_{i+1}^0 = f(t_{i+1}, y_{i+1}^0, v_{i+1}^0, w_{i+1}^0)$.

5. Correct the predicted values - iterate

$y_{i+1}^{k+1} = y_i + \frac{1}{2}h_i(y_i' + m_{i+1}^k)$,

$v_{i+1}^{k+1} = s_{21}(u_{i+1})$ -local parameters y_i, y_i', y_{i+1}^{k+1},

$w_{i+1}^{k+1} = s_{21}'(u_{i+1})$ - local parameters y_{i-1}, y_i, y_i' ,

$m_{i+1}^{k+1} = f(t_{i+1}, y_{i+1}^{k+1}, v_{i+1}^{k+1}, w_{i+1}^{k+1})$ (collocation)

to convergence ; then $y_{i+1}' = m_{i+1}$.

Remark: We recognize the trapezoidal formula in the correction step; it is known to be the convergent method of the order $p = 2$ (see e.g. [1],[3]).

Examples 4, 5. For the equation

$$y'(t) = \frac{1}{2}[3t - y'(d(t))] - \frac{3}{4}[y(t) + y(d(t))]$$

with the delay function $d(t) = t - exp(-t/10)$, $d(0) = -1$ the continuity condition (3) for the starting function reads now

$$y'(0) = -\frac{1}{2}g'(-1) - \frac{3}{4}[g(0) + g(-1)];$$

we have constructed again the function $g(t)$ by the Hermite interpolation from data in $t = -1, 0$; the results for five such data are plotted in Fig. 4a . On Fig. 4b we can see the result of numerical solution of the equation

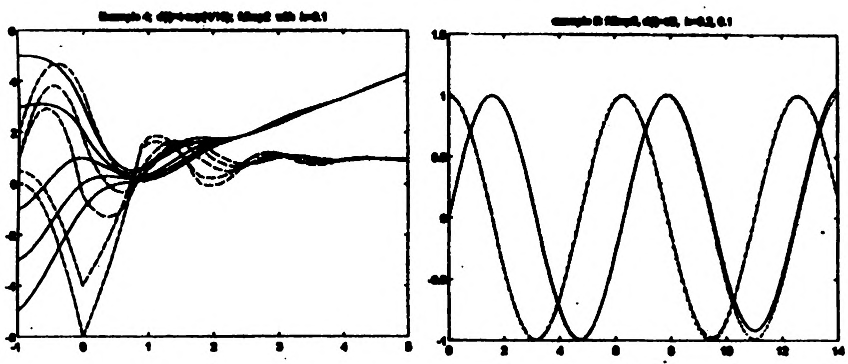
$y'(t) = 1 - 2y^2(t/2)$ with $y(0) = 0$, $y'(0) = 0$, $t \in [0, 14]$

and with the exact solutin $y(t) = sin(t)$; the starting values were taken from the exact solution.

5. Higher accuracy methods

We can try to obtain the higher accuracy in case of the solution smooth enough by correction with the second derivative, obtained from the values of the first derivative or from the recurrence relations for the splines s_{21}, s_{32} .

5.1. *Algorithm FDIMP3.*



a) b)

Fig. 4.

1. Let us realize the steps for computing at least two starting values y, y' similarly as in foregoing algorithms.
2. In the general step, given the values y_{i+1}, y'_{i+1} we can compute the local value of the second derivative $M_i = s''_{21}(t_i + 0)$ as $M_i = (y'_{i+1} - y'_i)/h_i$ and then make the final correction to $y_{i+1} = s_{21}(t_{i+1})$ using the formula

$$y_{i+1} = y_i + \frac{1}{3}h_i(2y'_i + y'_{i+1}) + \frac{1}{6}h_i^2 M_i. \tag{19}$$

The corrected value is now computed from the formula, which is related to the stable two step formula in ODE's with the order $p = 3$ and the error constant $C_4 = -1/72$ (see [8]). But substituting for M_i we find once more the trapezoidal formula from FDIMP2. In our tests the differences in the computed values y_i with $h = 0.1$ ranged around the level of 2.10^{-4}.

5.2. **Algorithms based on another formulas.** For a quadratic spline $s_{21}(x)$ interpolating at midpoints $t_i = \frac{1}{2}(x_i + x_{i+1})$ of an equidistant knot set $[x_i]$ the values of the function $y \in C^4$ the following asymptotic error estimation formula is known (see [11]):

$$y'(x_{i+1}) = [s(t_{i+1}) - s(t_i)]/h - \frac{1}{24}h[s''(t_{i+1}) - s''(t_i)] + O(h^4). \tag{20}$$

The spline recurrences for the local parameters on such a knotset

$$s(x_{i+1}) = \frac{1}{2}[s(t_i) + s(t_{i+1})] - \frac{1}{16}h^2[s''(t_i) + s''(t_{i+1})], \tag{21}$$

$$s''(t_{i+1}) + 6s''(t_i) + s''(t_{i-1}) = 8[s(t_{i+1}) - 2s(t_i) + s(t_{i-1})]/h^2 \tag{22}$$

together with the condition of collocation at $t = x_{i+1}, y_{i+1} = y(x_{i+1}),$)

$$\frac{1}{h}[s(t_{i+1}) - s(t_i)] - \frac{1}{24}h[s''(t_{i+1}) - s''(t_i)] = f(x_{i+1}, y_{i+1}, v, w), \tag{23}$$

with $u = d(x_{i+1})$, $v = s(u)$, $w = s'(u)$ and delay function $d(t)$ offers to be the formulas for computing stepwise the new local parameters on the interval $[t_i, t_{i+1}]$; unfortunately these recursions correspond to the unstable LMM formulas, what is the case of many similar formulas at hand.

After some experiments we propose to use the formulas described below in the Algorithm FDIMP4 for computing local parameters $[y, y', y'']$ at the points of the equidistant knotset $[t_i]$ (using the values of y, y' computed in the midpoints).

Algorithm FDIMP4

1. Suppose we have given the starting interval (or computed it with some similar -or more precise, with iterations to the given tolerance - starting procedure) containing at minimum the points $t_1 < x_1 < t_2$ of the equidistant knotset $[t_i]$ with values of y_j, y'_j, y''_j at $t = t_j$, $j \le i$.
2. We compute the function value of $y(x_{i+1})$ at the midpoint $x_{i+1} = t_i + h/2$ with the stable formula

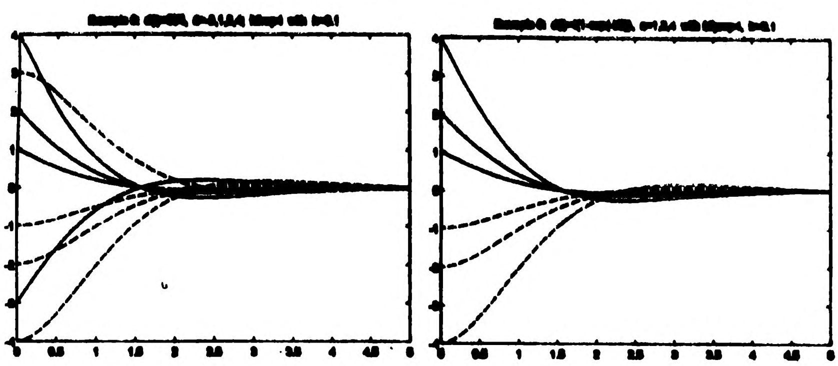
$$y(x_{i+1}) = [y(t_{i-1}) + 3y(t_i) + 3hy'(t_i)]/4, \tag{24}$$

   the delayed values of $u = d(x_{i+1}), v = y(u), w = y'(u)$ and then we compute the value of $y'(x_{i+1})$ by collocation to FDE.
3. We predict the value $y(t_{i+1})$ from the stable formula $(p = 3)$

$$y(t_{i+1}) = [y(t_{i-1}) + y(t_i) + 3hy'(x_{i+1})]/2 - h^2[y''(t_{i-1}) + 7y''(t_i)]/16, \tag{25}$$

   and compute the delayed values $u = d(t_{i+1}), v = y(u), w = y'(u)$ using interpolation with s_{32} in case $u \le t_i$, or from s_{21}.
4. We correct the value $y(t_{i+1})$ and compute the values $y'(t_{i+1}), y''(t_{i+1})$ to the given tolerance using Hermite interpolant s_{53} with y_j, y'_j, y''_j, $j = i, i+1$

$$y(t_{i+1}) = y(t_i) + h[y'(t_i) + y'(t_{i+1})]/2 + h^2[y''(t_i) - y''(t_{i+1})]/12, \tag{26}$$



a) b)

Fig. 5.

the collocation to FDE for computing $y'(t_{i+1})$. The value $y''(t_{i+1})$ we compute from s_{21} by the formula

$$y''(t_{i+1}) = 2[(y(t_{i+1}) - y(t_i))/h - y'(t_i)]/h, \tag{27}$$

or from the simplest formula $y''(t_{i+1}) = [y'(t_{i+1}) - y'(t_i)]/h$.

Remark The computation of the second derivatives seems to be very sensitive to errors. The recurrence relation for the second derivatives is highly unstable when used to compute M_{i+1} from M_i, M_{i-1}. Unsatisfactory results were obtained also with computing y_i'' from the Hermite interpolant to $y_{i-1}, y'_{i-1}, y_i, y'_i$

Example 6 The equation with the delay function $d = d(t)$

$$y'(t) = -y(t) - y(d(t)) - y'(d(t)) - a[sin(t).exp(-t) + sin(d).exp(-d)]$$

has the exact solution $y(t) = a.cos(t).exp(-t)$, $y(0) = a$, $y'(0) = -a$
The results from numerical computation with $d(t) = 3t/4$ are plotted
for $a = -3, 1, 2, 4$ on Fig. 5a.
The results for the case $d(t) = t(1 - exp(-t/2))$ are given in Fig. 5b.

Remark All examples were computed with MATLAB facilities and special M-files worked out by the first (corresponding) author.

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GROUP-GRADED ALGEBRAS, ADJOINT FUNCTORS AND THE GREEN CORRESPONDENCE

ANDREI MARCUS

Dedicated to Prof. Gheorghe Coman at his 60^{th} anniversary

Abstract. Let G be a finite group and k a commutative ring. In the context of fully G-graded k-algebras versions of the Green correspondence and of the Burry-Carlson-Puig theorem, generalizing recent results of M. Auslander and M. Kleiner.

1. Introduction

In [AK], M. Auslander and M. Kleiner have proved two categorical versions of the Green correspondence and two versions of the Burry-Carlson-Puig theorem. These results were obtained as applications of some general theorems concerning adjoint functors and quotient categories. It is well-known that the classical theorems mentioned above hold in the context of fully graded algebras (see [D], [N] or [B]). The aim of this note is to show that the results of [AK] also hold for fully-graded algebras.

We begin by recalling some basic facts about group-graded algebras and graded modules. The main references are [N] and [NV]. Let k be a commutative noetherian ring, G a group and R a fully G-graded k-algebra. We shall assume that G is finite and that R is a finitely generated k-module, and denote by $\mathcal{S}(G)$ the set of subgroups of G and by R-mod the category of finitely generated (left) R-modules. If H is a subgroup of G, then an R-module M is called G/H-graded if $M = \oplus_{x\in G/H} M_x$ (as additive subgroups) and $R_g M_x \subseteq M_{gx}$ for all $g \in G$ and $x \in G/H$. We denote by $(G/H, R)$-Gr the subcategory of R-mod consisting of G/H-graded R-modules and and grade-preserving linear maps. Notice that for $H = G$ we obtain the category R-mod, and for $H = \{1\}$ we obtain the category R-gr of finitely generated G-graded R-modules.

A direct summand of a G/H-graded R-module will be called a *H-projective R-module*, and we denote by (R, H)-mod the full subcategory of R-mod consisting of H-projective R-modules. If $\mathcal{F}$ is a subset of $\mathcal{S}(G)$, then an R-module M is called *$\mathcal{F}$-projective*

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if there is $H \in \mathcal{F}$ such that M is H-projective, and $(R, \mathcal{F})$-mod will denote the full subcategory of R-mod consisting of $\mathcal{F}$-projective R-modules. Finally, we denote by $(G/\mathcal{F}, R)$-mod the full subcategory of R-mod consisting of objects M for which there is $H \in \mathcal{F}$ such that $M \in (G/H, R)$-Gr.

If $\mathcal{A}$ is an additive category and $M, N \in \mathcal{A}$, denote for short $\mathcal{H}(M, N) = \mathrm{Hom}_{\mathcal{A}}(M, N)$. Let $\mathcal{B}$ be a full additive subcategory of $\mathcal{A}$, and let $\mathcal{B}(M, N)$ be the subgroup of $\mathcal{A}(M, N)$ consisting of morphisms between M and N which factor through an object of $\mathcal{B}$. Then one can define the *quotient category* $\mathcal{A}/\mathcal{B}$, whose objects are the objects of $\mathcal{A}$ and $(\mathcal{A}/\mathcal{B})(M, N) = \mathcal{A}(M, N)/\mathcal{B}(M, N)$. If $f\, M \to N$ and $g\, N \to P$ are morphisms in $\mathcal{A}$, then the composition is defined as $(g + \mathcal{B}(N, P)) \circ (f + \mathcal{B}(M, N)) = g \circ f + \mathcal{B}(M, P)$.

Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories, $\mathcal{X}$ and $\mathcal{Y}$ full additive subcategories of $\mathcal{A}$, and $T\, \mathcal{A} \to \mathcal{B}$ an additive functor. We denote by $T\mathcal{X}$ the full subcategory of $\mathcal{B}$ consisting of all finite direct sums of objects $T(X)$ with $X \in \mathcal{X}$. Finally, we say that $\mathcal{X}$ *divides* $\mathcal{Y}$ if every object of $\mathcal{X}$ is a direct summand of an object of $\mathcal{Y}$.

2. Induction, Coinduction and Restriction

The main ingredients in [AK] were the properties of adjoint functors. We present here the constructions over G-graded algebras (see also for instance [NRV] and [M]). The assumptions are those of the first section.

2.1. Let H be a subgroup of G and consider the subalgebra $R_H = \oplus_{h \in H} R_h$. We have two functors from R_H-mod to $(G/H, R)$-Gr. The first is the *induction* functor

$$\mathrm{Ind}_H^G = R \otimes_{R_H} -\; R_H\text{-mod} \to (G/H, R)\text{-Gr},$$

where for $N \in R_H$-mod, $(\mathrm{Ind}_H^G N)_x = R_x \otimes_{R_H} N$ for all $x \in G/H$, where $R_x = \oplus_{g \in x} R_g$. The second is the *coinduction* functor

$$\mathrm{Coind}_H^G = \mathrm{Hom}_{R_H}(R, -)\; R_H\text{-mod} \to (G/H, R)\text{-Gr}.$$

For $x \in G/H$ and $N \in R_H$-mod,

$$\mathrm{Coind}_H^G(N)_x = \{f \in \mathrm{Hom}_{R_H}(R, N) \mid f(R_{y^{-1}}) = 0 \text{ for } y \in G/H,\ y \neq x\},$$

where $y^{-1} = \{g^{-1} \mid g \in y\}$.

2.2. The truncation functor $(-)_H\, (G/H, R)\text{-Gr} \to R_H$-mod is clearly a left inverse for both Ind_H^G and Coind_H^G. If R is fully graded, that is, $R_g R_h = R_{gh}$ for all $g, h \in G$, then it is well-known that these functors are equivalences of categories, and therefore Ind_H^G and Coind_H^G are naturally isomorphic. We give here an explicit isomorphism between these two functors. If $N \in R_H$-mod, let

$$\zeta\ R \otimes_{R_H} N \to \mathrm{Hom}_{R_H}(R, N),$$

$\zeta(r_g \otimes_{R_H} n)(r'_h) = r'_h r_g n$ if $hg \in H$ and 0 otherwise, where $r_g \in R_g$, $r'_h \in R_h$, $g, h \in G$. We may also write ζ as follows: for $r' \in R$,

$$\zeta(r_g \otimes_{R_H} n)(r') = r'_{Hg^{-1}},$$

where $r'_{Hg^{-1}} = \sum_{h \in H} r'_{hg^{-1}}$. It is straightforward to see that ζ is a natural isomorphism in $(G/H, R)$-Gr. To define the inverse of ζ, we need to choose a system $\{g_1, \dots, g_l\}$ of representatives for the left cosets of H in G, and for each g_i, let $r_1^i, \dots, r_{t_i}^i \in R_{g_i}$, $s_1^i, \dots, s_{t_i}^i \in R_{g_i^{-1}}$ such that $\sum_{j=1}^{t_i} r_j^i s_j^i = 1$. Then $\zeta^{-1}\ \mathrm{Hom}_{R_H}(R, N) \to R \otimes_{R_H} N$ is given by $\zeta^{-1}(f) = \sum_{i=1}^{l} \sum_{j=1}^{t_i} r_j^i \otimes_{R_H} f(s_j^i)$.

We shall be interested to regard Ind_H^G and Coind_H^G as functors from R_H-mod to R-mod, and we shall also consider the restriction functor $\mathrm{Res}_H^G\ R\text{-mod} \to R_H\text{-mod}$.

2.3. The functor Ind_H^G is a left adjoint of Res_H^G. For $M \in R$-mod and $N \in R_H$-mod, we have the natural isomorphism

$$\alpha_{N,M}\ \mathrm{Hom}_R(R \otimes_{R_H} N, M) \to \mathrm{Hom}_{R_H}(N, M)$$

defined by $\alpha_{N,M}(f)(n) = f(1 \otimes n)$, with inverse $\alpha_{N,M}^{-1}(f')(r \otimes n) = rf'(n)$, for all $n \in N$, $r \in R$. The unit of this adjunction is

$$\eta_N\ N \to \mathrm{Res}_H^G(R \otimes_{R_H} N), \quad \eta(n) = 1 \otimes n;$$

the counit is $\mu_M\ R \otimes_{R_H} \mathrm{Res}_H^G(M) \to M$, $\mu_M(r \otimes m) = rm$.

2.4. The functor Coind_H^G is a right adjoint of Res_H^G. The natural isomorphism

$$\gamma_{M,N}\ \mathrm{Hom}_{R_H}(\mathrm{Res}_H^G M, N) \to \mathrm{Hom}_R(M, \mathrm{Hom}_{R_H}(R, N))$$

is defined by $\gamma_{M,N}(f)(m)(r) = f(rm)$, and its inverse by $\gamma_{M,N}^{-1}(f')(m) = f'(m)(1)$, for all $r \in R$, $m \in M$. The unit $\nu_M\ M \to \mathrm{Coind}_H^G(\mathrm{Res}_H^G M)$ is defined by $\nu_M(m)(r) = rm$, and the counit $\delta_N\ \mathrm{Res}_H^G \mathrm{Coind}_H^G N \to N$ by $\delta_N(f) = f(1)$.

2.5. If R is fully graded, it follows that Ind_H^G is both a left and a right adjoint of Res_H^G. Using the above definition of ζ we may give explicitely the definition of the isomorphism

$$\beta_{M,N}\ \mathrm{Hom}_{R_H}(\mathrm{Res}_H^G M, N) \to \mathrm{Hom}_R(M, R \otimes_{R_H} N).$$

We have that

$$\beta_{M,N}(f)(m) = \sum_{i=1}^{l} \sum_{j=1}^{t_i} r_j^i \otimes_{R_H} f(s_j^i m),$$

where for $i = 1, \ldots, l$, $r_1^i, \ldots, r_{t_i}^i$, $s_1^i, \ldots, s_{t_i}^i$ are chosen as in (2.2). The inverse is $\beta^{-1}(f')(m) = f'(m)_H$, for all $m \in M$. Then the unit associated to β is $\tau_M\ M \to R \otimes_{R_H} M$, $\tau_M(m) = 1 \otimes_{R_H} m$, and the counit is $\epsilon_N\ \mathrm{Res}_H^G(R \otimes_{R_H} N) \to N$, $\epsilon_N(r \otimes_{R_H} n) = r_H n$, where $r_H = \sum_{h \in H} r_h \in R_H$ and $n \in N$. We have used here that $(R \otimes_{R_H} N)_H = R_H \otimes_{R_H} N \simeq N$.

Finally, we record Mackey's formula (cf. [D], [B] or [N]).

Theorem 2.6. *Let K and H be subgroups of G, and $g_1, \ldots, g_s$ be representatives for the double cosets of (K, H) in G. Assume that R is fully graded and that N is an R_H-module. Then*

$$\mathrm{Res}_K^G(R \otimes_{R_H} N) \simeq \bigoplus_{i=1}^{s} R_K \otimes_{R_{K \cap {}^{g_i}H}} (R_{g_i H} \otimes_{R_H} N).$$

3. The Green Correspondence

In this section, k will be a commutative ring, R a fully G-graded k-algebra, and we shall preserve the conventions and notations of the preceding sections.

We fix the subgroups $D \leq H$ of G, and consider the categories $\mathcal{G} = R$-mod, $\mathcal{H} = R_H$-mod, $\mathcal{D} = R_D$-mod, and the functors $S = \mathrm{Ind}_H^G$, $S' = \mathrm{Ind}_D^H$, $T = \mathrm{Res}_H^G$, $T' = \mathrm{Res}_D^H$.

Let $g_1 = 1, g_2, \ldots, g_s$ be representatives for the double cosets of (H, H) in G. Then, as an R_H-bimodule, we have that $R = \bigoplus_{i=1}^{s} R_{Hg_iH}$. Define the functor

$$U = \bigoplus_{i=2}^{s} R_{Hg_iH} \otimes_{R_H} - \ \mathcal{H} \to \mathcal{H}.$$

We need only to check that the assumptions made in [AK] on these functors are satisfied.

We have already seen that S (respectively S') is a left and a right adjoint of T (respectively T').

From the fact that R_H is an R_H-bimodule summand of R, one easily can deduce that the following assertions hold.

3.1.a. $T \circ S = id_{\mathcal{H}} \oplus U$.

3.1.b. The unit $\eta\ id_{\mathcal{H}} \to T \circ S$ induces the functorial morphisms $\eta_1\ id_{\mathcal{H}} \to id_{\mathcal{H}}$ and $\eta_U\ id_{\mathcal{H}} \to U$ such that η_1 is an isomorphism.

3.2.c. The counit $\epsilon\ T \circ S \simeq id_{\mathcal{H}} \oplus U \to id_{\mathcal{H}}$ induces the morphisms $\epsilon_1\ id_{\mathcal{H}} \to id_{\mathcal{H}}$ and $\epsilon_U\ U \to id_{\mathcal{H}}$ such that ϵ_1 is an isomorphism.

We need now to look for a subcategory $\mathcal{Y}$ of $\mathcal{H}$ such that $S' \circ T'\mathcal{Y}$ divides $\mathcal{Y}$ and $U^{-1}\mathcal{Y}$.

We chose a subset Y of $\mathcal{S}(H)$ closed under subgroup and conjugation in H, let $\mathrm{X} = \{D \cap Y \mid Y \in \mathrm{Y}\}$, and let $\mathcal{Y} = (H/\mathrm{Y}, R_H)$-mod (or with the same effect, $\mathcal{Y} = (R_H, \mathrm{Y})$-mod).

If $g \in G$ and $N \in R_H$-mod, denote ${}^gH = gHg^{-1}$ and ${}^gN = R_g \otimes_{R_1} N \simeq R_{gH} \otimes_{R_H} N$, which is clearly an $R_{{}^gH}$-module. With these notations we have:

Proposition 3.2. *a)* $S' \circ T'\mathcal{Y} \subseteq (H/\mathrm{X}, R_H)$-mod. *b)* $(H/\mathrm{X}, R_H)$-mod *divides* $S' \circ T'\mathcal{Y}$. *c) The quotient categories* R_H-mod$/S' \circ T'\mathcal{Y}$ *and* R_H-mod$/(H/\mathrm{X}, R_H)$-mod *coincide. d)* $S' \circ T'\mathcal{Y} \subseteq \mathcal{Y}$.

Proof. c) and d) are consequences of a) and b). For a) let $N = \mathrm{Ind}_Y^H W$, where $Y \in \mathrm{Y}$ and W is an R_Y-module. By Mackey's formula we have that

$$T'N \simeq \mathrm{Res}_D^H \mathrm{Ind}_Y^H W \simeq \bigoplus_h \mathrm{Ind}_{D\cap {}^hY}^D \mathrm{Res}_{D\cap {}^hY}^{{}^hY} {}^hW,$$

where h runs over a system of representatives for the double cosets of (D, Y) in H. We have that ${}^hY \in \mathrm{Y}$, so $T'N \in (D/\mathrm{X}, R_D)$-mod. Consequently, $S'T'\mathcal{Y} \subseteq (H/\mathrm{X}, R_H)$-mod.

Let now $Y \in \mathrm{Y}$ and $W \in R_{D\cap Y}$-mod. Since $T \circ S = id \bigoplus U$, again by Mackey's formula it follows that $\mathrm{Ind}_{D\cap Y}^D W$ is a summand of $T'N$, where $N = \mathrm{Ind}_Y^H \mathrm{Ind}_{D\cap Y}^D W$. It follows that $\mathrm{Ind}_{D\cap Y}^H W$ is a direct summand of $S' \circ T'N$, hence $(H/\mathrm{X}, R_H)$-mod divides $S' \circ T'\mathcal{Y}$. □

We need some additional notation. if $\mathrm{F} \subseteq \mathcal{S}(H)$ denote $\mathrm{F}' = \{H \cap {}^gF \mid F \in \mathrm{F},\ g \in G \setminus H\}$, and by Z the largest subset of $\mathcal{S}(D)$ satisfying $\mathrm{Z}' \subseteq \mathrm{Y}$.

Proposition 3.3. *Assume that* F *is a subset of* $\mathcal{S}(H)$. *Then:*

a) $U((H/\mathrm{F}, R_H)\text{-mod}) \subseteq (H/\mathrm{F}', R_H)$-mod.

b) $(H/\mathrm{F}', R_H)$-mod *divides* $U((H/\mathrm{F}, R_H)\text{-mod})$.

Proof. Proof a) Let $F \in \mathcal{F}$, $V \in R_F$-mod and $N = \mathrm{Ind}_F^H V$. We have that $U(N) = \bigoplus_{i=2}^{s} R_{Hg_iH} \otimes_{R_H} N$, so it is enough to prove that $R_{HgH} \otimes_{R_H} N \in (H/\mathcal{F}', R_H)$-mod for $g \in G \setminus H$. We have that

$$R_{HgH} \otimes_{R_H} N \simeq R_H \otimes_{R_{H\cap {}^gH}} ({}^gN) \simeq \mathrm{Ind}_{H\cap {}^gH}^H \mathrm{Res}_{H\cap {}^gH}^{{}^gH} \mathrm{Ind}_{{}^gF}^{{}^gH}({}^gV)$$

By Mackey's formula we obtain

$$\mathrm{Res}_{H\cap {}^gH}^{{}^gH} \mathrm{Ind}_{{}^gF}^{{}^gH}({}^gV) \simeq \bigoplus_h \mathrm{Ind}_{H\cap {}^gH\cap {}^{hg}F}^{H\cap {}^gH} \mathrm{Res}_{H\cap {}^gH\cap {}^{hg}F}^{{}^{hg}F}({}^{hg}V),$$

where h runs over a system of representatives for the double cosets of $(H\cap {}^gH, {}^gF)$ in ${}^gH$.

There is $t \in H$ such that $hg = gt$, so $H \cap {}^gH \cap {}^{hg}F = H \cap {}^{gt}F$, hence $U(N)$ is a direct sum of modules of the form $P_{g,t} = \mathrm{Ind}_{H\cap {}^{gt}F}^H \mathrm{Res}_{H\cap {}^{gt}F}^{{}^{gt}F}({}^{gt}V)$, with $g \in G \setminus H$, $t \in H$. Since $gt \in G \setminus H$, the assertion follows.

b) If $g \in G \setminus H$, $t \in H$ and $V \in R_F$-mod, we have that $P_{g,t}$ above divides $U((H/\mathcal{F}, R_H)$-mod), since we can regard g as a representative for HgH, $t \in H$ as $t = g^{-1}hg$ for some $h \in {}^gH$ which, in turn, is regarded as a representative for $(H \cap {}^gH)h^gF$.

We take $W = \mathrm{Ind}_{H\cup {}^gF}^H Q \in (H/\mathcal{F}', R_H)$-mod, where $g \in G \setminus H$ and $Q \in (H \cap {}^gF)$-mod, and let $t = 1$ and $V = \mathrm{Ind}_{g^{-1}H\cap F}^F({}^{g^{-1}}Q)$. Mackey's formula again gives us that

$$\mathrm{Ind}_{H\cap {}^gF}^H \mathrm{Res}_{H\cap {}^gF}^{{}^gF} \mathrm{Ind}_{H\cap {}^gF}^{{}^gF} Q \simeq \mathrm{Ind}_{H\cap {}^gF}^H(Q \oplus \dots) \simeq W \oplus \dots$$

Since the modules of the form $P_{g,t}$ divide $U((H/\mathcal{F}, R_H)$-mod), it folows that $(H/\mathcal{F}', R_H)$-mod divides $U((H/\mathcal{F}, R_H)$-mod). □

Corollary 3.4. *a) If* $\mathrm{X}' \subseteq \mathrm{Y}$, *then* $S' \circ T'\mathcal{Y}$ *divides* $\mathcal{Y}$ *and* $U^{-1}\mathcal{Y}$.

b) Let $\mathrm{Y} = \{Y \mid \exists g \in G \setminus H$ *such that* $Y \le H \cap {}^gD\}$. *Then* $\mathrm{X} = \{X \mid \exists g \in G \setminus H$ *such that* $X \le D \cap {}^gD\}$ *and* $\mathrm{X}' \subseteq \mathrm{Y}$.

c) Assume that $E \trianglelefteq H$ *and* $D \cap E \trianglelefteq G$, *and let* $\mathrm{Y} = \mathcal{S}(E)$. *Then* $\mathrm{X} = \mathcal{S}(D \cap E)$ *and* $\mathrm{X}' \subseteq \mathrm{Y}$.

Since we have generalized to the case of fully graded algebras all the results of [AK, Section 3], we may now state the main results.

Theorem 3.5. *Assume that* $\mathrm{X}' \subseteq \mathrm{Y}$. *Then:*

a) The functor Ind_H^G (R_H, Z)-mod$/(R_H, \mathrm{X})$-mod $\to (R, \mathrm{Z})$-mod$/(R, \mathrm{X})$-mod *is an equivalence.*

b) The functor Res_H^G (R, Z)-mod$/(R, \mathrm{X})$-mod $\to (R_H, \mathrm{Z})$-mod$/(R_H/\mathrm{Y})$-mod *is an equivalence.*

c) $\mathrm{Res}_H^G \circ \mathrm{Ind}_H^G$ (R_H, Z)-mod$/(R_H, \mathrm{X})$-mod $\to$ (R_H, Z)-mod$/(R_H, \mathrm{Y})$-mod *is an equivalence isomorphic to the natural projection.*

In order to state the classical Green correspondence and the variants of the Burry-Carlson-Puig theorem, we shall assume that the commutative ring k is local, noetherian and complete. This implies that for all $H \in \mathcal{S}(G)$, R_H-mod is a Krull-Schmidt category.

Theorem 3.6. *Assume that k local and complete and that* $\mathrm{X}' \subseteq \mathrm{Y}$. *Let* $M \in (R, \mathrm{Z})$-mod$\setminus$ (R, X)-mod *and* $N \in (R_H, \mathrm{Z})$-mod $\setminus$ (R_H, X)-mod *be indecomposable modules. Then:*

a) $\mathrm{Ind}_H^G N$ *has a unique indecomposable direct summand* $g(N) \in (R, \mathrm{Z})$-mod $\setminus$ (R, X)-mod.

b) $\mathrm{Res}_H^G M$ *has a unique indecomposable direct summand* $f(M) \in (R_H, \mathrm{Z})$-mod $\setminus$ (R_H, Y)-mod.

c) $f(g(N)) \simeq N$.

d) $g(f(M)) \simeq M$.

Theorem 3.7. *Let* Z, Y *and k be as above, and assume that* $M \in (R, \mathrm{Z})$-mod$\setminus(R, \mathrm{X})$-mod *and* $N \in (R_H, \mathrm{Z}) \setminus (R_H, \mathrm{X})$-mod *are indecomposable modules.*

a) If B is an indecomposable R_H*-module such that* $U(B)$ *is relatively* Y*-projective and M is a direct summand of* $\mathrm{Ind}_H^G B$, *then* $B \simeq f(M)$.

b) If A is an indecomposable R-module such that N is a direct summand of $\mathrm{Res}_H^G A$, *then* $A \simeq g(N)$.

Remark 3.8. Using the duality theorem of M. Cohen and S. Montgomery, J. Haefner has obtained in [H] a version for graded algebras (not assumed fully graded) of the Green correspondence We hope to deal with this situation in a forthcoming paper.

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A NOTE ON THE REMAINDER IN A POLYNOMIAL APPROXIMATION FORMULA

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Dedicated to Professor Gh. Coman on his 60^{th} anniversary

Abstract. In this note one uses the divided differences as basic tool for expressing the remainder in an approximation formulaof a function $f \in C[0,1]$ by means of a Bernstein type polynomial, introduced in 1994 in the paper [5], defined by the formula (1) from the paper, where r and s are integer, nonnegative parameters, subject to the condition $2sr < m$, where m is a natural number. For the remainder of the approximation formula (8) we have obtained a convex combination of second order divided differences, given in (18)-(19). It generalizes the formula (21), established in 1964 in the author's paper [3].

1. In this note we use the divided differences as a basic mathematical tool for expressing the remainder in the approximation formula of a function $f \in C[0,1]$ by means of an associated generalized Bernstein operator, introduced in 1984 in our paper [5], which is defined by the following formula

$$(S_{m,r,s}f)(x) := \sum_{k=0}^{m-sr} p_{m-sr,k}(x) \left\{ \sum_{j=0}^{s} p_{s,j}(x) f\left(\frac{k+jr}{m}\right) \right\}, \tag{1}$$

where r and s are nonnegative integer parameters satisfying the condition: $2sr < m$, while

$$p_{s,j}(x) := \binom{s}{j} x^j (1-x)^{s-j}. \tag{2}$$

It is easy to see that we have $S_{m,r,0} = S_{m,0,s} = B_m$ -the Bernstein operator, derfined by

$$(B_m f)(x) := \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right), \tag{3}$$

while $S_{m,r,1} = L_{m,r}$ -the operator of Stancu [4], defined by

$$(L_{m,r}f)(x) := \sum_{k=0}^{m-r} p_{m-r,k}(x) \left\{ (1-x) f\left(\frac{k}{m}\right) + x f\left(\frac{k+r}{m}\right) \right\}. \tag{4}$$

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One observes that the Bernstein type polynomial $S_{m,r,s}f$ is interpolatory at both sides of the interval [0,1], that is

$$(S_{m,r,s}f)(0) = f(0),\ (S_{m,r,s}f)(1) = f(1). \tag{5}$$

By a straightforward calculation one can verify that if we consider the monomials $e_j(t) = t^j$ $(j = 0,1,2)$, where $t \in [0,1]$, then we obtain

$$S_{m,r,s}e_0 = e_0,\ S_{m,r,s}e_1 = e_1 \tag{6}$$

and

$$(S_{m,r,s}e_2)(x) = x^2 + \left[1 + s\frac{r(r-1)}{m}\right] \cdot \frac{x(1-x)}{m}. \tag{7}$$

Therefore the degree of exactness of the approximation formula

$$f(x) = (S_{m,r,s}f)(x) + (R_{m,r,s}f)(x) \tag{8}$$

is equal with one.

2. We now proceed to investigate the remainder of the approximation formula (8). The main result of this paper is represented by

Theorem 1. *The remainder of the approximation formula (8) can be expressed, by means of the second-order divided differences, in the following form*

$$(R_{m,r,s}f)(x) = \frac{x(x-1)^2}{m} \cdot (C_{m,r,s}f)(x), \tag{9}$$

where

$$(C_{m,r,s}f)(x) =$$

$$= \left\{(m-sr)\sum_{k=0}^{m-sr-1} p_{m-sr-1,k}(x)\left(\sum_{j=0}^{s} p_{s,j}(x)\left[x, \frac{k+jr}{m}, \frac{k+jr+1}{m}; f\right]\right)\right.$$
$$\left. + sr^2 \sum_{k=0}^{m-sr} p_{m-sr,k}(x)\left(\sum_{j=0}^{s-1} p_{s-1,j}(x)\left[x, \frac{k+jr}{m}, \frac{k+jr+r}{m}; f\right]\right)\right\}. \tag{10}$$

Proof. It is easy to see that we can write

$$(R_{m,r,s}f)(x) = f(x) - (S_{m,r,s}f)(x) =$$

$$= \sum_{k=0}^{m-sr} p_{m-sr,k}(x) \left\{ \sum_{j=0}^{s} p_{s,j}(x) \left(f(x) - f\left(\frac{k+jr}{m}\right)\right)\right\}.$$

By using the first order divided difference, we have

$$f(x) - f\left(\frac{k+jr}{m}\right) = \left(x - \frac{k+jr}{m}\right)\left[x, \frac{k+jr}{m}; f\right]$$

and the expression of the remainder becomes

$$(R_{m,r,s}f)(x) =$$

$$= \sum_{j=0}^{s} p_{s,j}(x) \left\{ \frac{1}{m} \sum_{k=0}^{m-sr} p_{m-sr,k}(x)(mx - k - jr)\left[x, \frac{k+jr}{m}; f\right]\right\}.$$

The key of the proof of the theorem consists in using the following important simple identity:

$$mx - k - jr = (m - sr - k)x - k(1 - x) + r(s - j)x - jr(1 - x).$$

Further we split the expression of the remainder in two parts

$$(R_{m,r,s}f)(x) = (P_{m,r,s}f)(x) + (Q_{m,r,s}f)(x), \tag{11}$$

where

$$(P_{m,r,s}f)(x) =$$

$$= \frac{1}{m} \sum_{j=0}^{s} p_{s,j}(x) \left\{ x(V_{m,r,s}f)(x) - (1 - x)(W_{m,r,s}f)(x)\right\}, \tag{12}$$

with

$$(V_{m,r,s}f)(x) =$$

$$= \sum_{k=0}^{m-sr-1} (m - sr - k) \binom{m-sr}{k} x^k (1 - x)^{m-sr-k} \left[x, \frac{k+jr}{m}; f\right]$$

and

$$(W_{m,r,s}f)(x) =$$

$$= \sum_{k=1}^{m-sr} k \binom{m-sr}{k} x^k (1 - x)^{m-sr-k} \left[x, \frac{k+jr}{m}; f\right],$$

while

$$(Q_{m,r,s}f)(x) =$$

$$= \frac{r}{m}\sum_{k=0}^{m-sr} p_{m-sr,k}(x)\left\{x(T_{m,r,s}f)(x) - (1-x)(U_{m,r,s}f)(x)\right\},$$

with

$$(T_{m,r,s}f)(x) = \sum_{j=0}^{s-1}(s-j)p_{s,j}(x)\left[x, \frac{k+jr}{m}; f\right], \tag{13}$$

$$(U_{m,r,s}f)(x) = \sum_{j=1}^{s} jp_{s,j}(x)\left[x, \frac{k+jr}{m}; f\right], \tag{14}$$

By using some combinatorial relations we can establish the equalities

$$(V_{m,r,s}f)(x) =$$

$$= (m-sr)(1-x)\sum_{k=0}^{m-sr-1} p_{m-sr-1,k}(x)\left[x, \frac{k+jr}{m}; f\right],$$

$$(W_{m,r,s}f)(x) =$$

$$= (m-sr)x\sum_{k=0}^{m-sr-1} p_{m-sr-1,k}(x)\left[x, \frac{k+jr+1}{m}; f\right].$$

It follows that we can write

$$x(V_{m,r,s}f)(x) - (1-x)(V_{m,r,s}f)(x) =$$

$$= (m-sr)\frac{x(1-x)}{m}\sum_{k=0}^{m-sr-1} p_{m-sr-1,k}(x)\left[k, \frac{k+jr}{m}, \frac{k+jr+1}{m}; f\right], \tag{15}$$

since, according to the recurrence relation of divided differences, we have

$$\left[x, \frac{k+jr}{m}; f\right] - \left[x, \frac{k+jr+1}{m}; f\right] = -\frac{1}{m}\left[x, \frac{k+jr}{m}, \frac{k+jr+1}{m}; f\right].$$

If we replace this result in (12) we obtain

$$(P_{m,r,s}f)(x) =$$

$$= \frac{x(x-1)}{m^2}(m-sr)\sum_{k=0}^{m-sr-1} p_{m-sr-1,k}(x)\left(\sum_{j=0}^{s} p_{s,j}(x)\left[x, \frac{k+jr}{m}, \frac{k+jr+1}{m}; f\right]\right), \tag{16}$$

By using a similar technique, one can prove that

$$(Q_{m,r,s}f)(x) =$$

$$= sr^2\frac{x(x-1)}{m^2}\sum_{k=0}^{m-sr} p_{m-sr,k}(x)\left(\sum_{j=0}^{s-1} p_{s-1,j}(x)\left[x,\frac{k+jr}{m},\frac{k+jr+r}{m};f\right]\right). \tag{17}$$

If we replace (16) and (17) in (11), we obtain just the representation of the remainder given at (9)-(10). □

3. By using the result established above we can state and prove

Theorem 2. *The remainder of the approximation formula (8) can be expressed by means of a convex combination of second-order divided differences.*

Proof. We can see that the expression of the remainder given at (9)-(10) can be represented by means of a linear functional $D_{m,r,s}$, composed by a convex combination of second-order divided differences, namely

$$(R_{m,r,s}f)(x) = \left[1+s\frac{r(r-1)}{m}\right]\frac{x(x-1)}{m}\,(D_{m,r,s}f)(x), \tag{18}$$

where

$$(D_{m,r,s}f)(x) =$$

$$= \frac{1}{m-sr+sr^2}\left\{(m-sr)\sum_{k=0}^{m-sr-1} p_{m-sr-1,k}(x)\left(\sum_{j=0}^{s} p_{s,j}(x)\left[x,\frac{k+jr}{m},\frac{k+jr+1}{m};f\right]\right)\right.$$

$$\left. + sr^2\sum_{k=0}^{m-sr} p_{m-sr,k}(x)\left(\sum_{j=0}^{s-1} p_{s-1,j}(x)\left[x,\frac{k+jr}{m},\frac{k+jr+r}{m};f\right]\right)\right\}. \tag{19}$$

It is obvious that all the coefficients of this linear functional are positive and their sum equals $(D_{m,r,s}e_2)(x) = 1$, for any $x \in [0,1]$. Hence it is made up by a convex combination of the second-order divided differences evidenced in formula (19). □

The equality (18) tells us that the remainder of the approximation formula (8) can be represented under the following form

$$(R_{m,r,s}f)(x) = (R_{m,r,s}e_2)(x)\cdot(D_{m,r,s}f)(x). \tag{20}$$

4. Now let us consider some special cases.

If $s = 0$ or $r = 0$, then we obtain the Bernstein operator B_m, defined at (3). In this case the representation given at (9)-(10) reduces to the following expression for the remainder in the classical Bernstein approximation formula

$$(R_m f)(x) = \frac{x(x-1)}{m} \sum_{k=0}^{m-1} p_{m-1,k}(x) \left[x, \frac{k}{m}, \frac{k+1}{m}; f\right], \tag{21}$$

established first in 1964 in our paper [3].

For $s = 1$ we get the Bernstein type operator $L_{m,r}$, introduced and investigated in the paper [4], which is defined at (4). In this case the expression of the remainder is

$$(R_{m,r,1}f)(x) = \frac{x(x-1)^2}{m} \left\{(m-r) \sum_{k=0}^{m-r-1,k} p_{m-r-1,k}(x)(1-x) \left(\left[x, \frac{k}{m}, \frac{k+1}{m}; f\right]\right.\right.$$

$$\left.\left. +x\left[x, \frac{k+r}{m}, \frac{k+r+1}{m}; f\right]\right) + r^2 \sum_{k=0}^{m-r} p_{m-r,k}(x) \left[x, \frac{k}{m}, \frac{k+r}{m}; f\right]\right\},$$

which was first obtained in our paper [4].

Now let us state some consequences of our theorems.

Corollary 1. *We have $(R_{m,r,s}f)(0) = (R_{m,r,s}f)(1) = 0$ and $R_{m,r,s}f = 0$ if and only if f is a linear function.*

Corollary 2. *If all the second-order divided differences of the function f are bounded on [0,1], then we can write*

$$|(R_{m,r,s}f)(x)| \le \left[1 + s\frac{r(r-1)}{m}\right] \cdot \frac{x(1-x)}{m} \cdot M_2(f) \tag{22}$$

where $M_2(f)$ is the least upper bound of the absolute values of the second-order divided differences of the function f on [0,1].

Corollary 3. *If f is convex (resp. concave) on [0,1], without being linear, then we have $S_{m,r,s}f > f$ (resp. $S_{m,r,s}f < f$) on the interval (0,1).*

Corollary 4. *If $f \in C[0,1]$ and x is any fixed point of [0,1], then there exist on the interval three distinct points u_m, v_m and w_m, which might depend upon f, such that*

$$(R_{m,r,s}f)(x) = \left[1 + s\frac{r(r-1)}{m}\right] \cdot \frac{x(x-1)}{m} [u_m, v_m, w_m; f]. \tag{23}$$

This statement needs to be demonstrated.

From the expression obtained for the remainder we can see that the degree of exactness of the approximation formula (8) is one and also that $R_{m,r,s}f \neq 0$ when f is a convex function. By using a known theorem of T.Popoviciu [1] we can state that there exist three distinct points u_m, v_m and w_m on [0,1] such that

$$(R_{m,r,s}f)(x) = (R_{m,r,s}e_2)(x) \cdot [u_m, v_m, w_m; f].$$

By using formula (7) we arrive just to the representation (23), which permits to give a simple proof of the inequality (22).

Corollary 5. *If $f \in C^2[0,1]$ then there exists a point $\xi \in (0,1)$ such that the remainder can be expressed under the Cauchy form*

$$(R_{m,r,s}f)(x) = \left[1 + s\frac{r(r-1)}{m}\right] \cdot \frac{x(x-1)}{2m} f''(\xi). \tag{24}$$

In the case $s = r = 0$ this formula was established first in our paper [2].

Finally, we want to mention that details and complete proofs of our theorems will appear elsewhere.

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BOOK REVIEWS

D i r k W e r n e r, Functionalanalysis, Springer Lehrbuch, Springer-Verlag, Berlin Heildeberg, 1995, 446 pp.

The author is a leading specialist in geometric functional analysis with substantial contributions to the study of M-ideals in Banach spaces, mainly in spaces of operators. For a good account on results in this area we recommend the book by P.Harmand, D.Werner and W.Werner, M-Ideals in Banach spaces and Banach Algebras, Lecture Notes in Math. vol. 1547, Springer-Verlag, Berlin Heidelberg 1993. As it is natural to expect, the present book reflects, by the choice of the topics and by their presentation, author's interest in the geometric aspects of the theory.

The central theme of the book is the theory of Banach and Hilbert spaces and of operators acting on them. The author postpones as much as possible the use of locally convex spaces which are introduced only in the eighth chapter. For this reason, weak and weak* properties of Banach spaces are considered first in their sequential versions.

The book is divided into nine chapters and two appendices - A.Measure and Integration Theory and B.Metric and Topological Spaces.

Pass now to the detailed presentation of the book. The first chapter "Normed Spaces" contains the basic facts about normed spaces as well as classical Banach spaces of sequences or of functions. The second chapter "Functional and Operators" is concerned with continuous and compact operators between Banach spaces. The conjugate spaces to classical Banach spaces are calculated. The chapter ends with a proof of Riesz-Thorin interpolation theorem for L^p spaces (real and complex forms).

Ch. III. "Hahn-Banach Theorem and its Applications" is dealing with various versions of Hahn-Banach extension theorem and its applications to separation of convex sets in normed spaces. Some weak sequential properties of reflexive spaces are proved. The chapter ends with a study of adjoint operators including Schauder compactness theorem.

Ch. IV. "Fundamental Theorems for Operators on Banach Spaces" contains the basic principles of Banach space theory relying on Baire category theorem, i.e. Banach-Steinhaus principle, open mapping and closed graph theorems. As applications, one proves Korovkin's theorem (the trigonometric case) and Féjér's theorem on Cesàro summability of Fourier series.

Ch. V. "Hilbert Spaces" is dealing with fundamental properties of Hilbert spaces and of operators acting on them with applications to Fourier transform and Sobolev spaces.

The spectral theory of compact operators on Banach and Hilbert spaces is developed in the sixth chapter "Spectral Theory for Compact Operators", with applications to integral equations. Nuclear and Hilbert-Schmidt operators are also considered.

Ch. VII. "Spectral decompositions of Selfadjoint Operators" contains a presentation of continuous and measurable functional calculi for such operators. The case of unbounded operators on Hilbert spaces is also included.

Ch. VIII. "Locally Convex Spaces" is devoted to the study of basic properties of locally convex spaces. Weak and weak* topologies on normed spaces are defined now in full generality and some fundamental results on Banach spaces such as Alaoglu-Bourbaki, Krein-Smulyan, Krein-Milman, Banach-Dieudonné and Lyapunov theorems are proved. A brief (but relatively thorough) introduction to distribution theory is provided.

Each chapter contains a set of well chosen exercises illuminating, illustrating or completing the questions in the main text. A section, containing remarks and comments of historical nature as well as references to related results, is included at the end of each chapter.

Symbol and key indexes are included. The bibliography at the end of the book contains only books on functional analysis and related fields. References to original papers are given in the comment section of each chapter.

The result is a fine book which we a recommend warmly to all people desiring to learn or to teach (or both) functional analysis. In reviewer opinion, the book fully deserves an English version which will make it accesible to a larger audience.

S. COBZAŞ

W o l f g a n g W a l t e r: Analysis, 1-2, Springer-Verlag, 1992 ISBN 3-540-55234-0; ISBN 3-540-55385-1

The two volumes of this mathematical analysis textbook written by Professor Wolfgang Walter from the University of Karlsruhe, has been first published as the third and fourth volumes of the series "Grundwissen Mathematik", Springer Verlag, 1985 (vol.1) and 1990 (vol.2), respectively. This third edition contains only a few changes with respect to the first ones.

The book is addressed, first of all, to those students who wish to profound their knowledge in mathematical analysis, but - especially by its conception - it is useful also for those who teach mathematical analysis, for those who use it as an instrument, or simply wish to know something about the scientifical (and cultural) importance of this discipline in a more general context. It is a lucid book, accessibly written, without excessive formalism. By the presentation of some critical situations from the history of mathematical analysis, the author succeeds to capture reader's attention and, in this way, the lecture of the book becomes more agreable. From each part of this book comes out the idea that the understanding of the environing world, from a rational point of view, and the revealing of its laws is the most fascinant activity which has ever been contrived.

Let us mention some subjects and ideas treated in the book which distinguishes it from the majority of books on mathematical analysis. The first volume is devoted to functions of one variable; from complex analysis only the power series are presented. It contains topics, usually taught in Germany in the first semester, to students from the faculties of mathematics, informatics and physics, but, in some places, supplementary topics are included too.

The real numbers are the basis on which the mathematical analysis is built. They are defined axiomatically, as the elements of a totally ordered field where the supremum axiom is verified. Approximately, one half of the first volume is dedicated to differential and integral calculus of functions of one real variable. It starts with integral calculus followed by the differential calculus, by paying a special attention to applications. The improper integrals, linear differential equations of second order with constant coefficients, the Stirling formula, Dini derivatives and an example of continuous nowhere differentiable function are given at the end of the first volume in a section entitled "Complements".

The second volume is concerned with the analysis of functions of several variables. The concepts of limit and continuity are treated in metric spaces. Banach and Hilbert spaces, orthogonality, hperplanes, convex sets and convex functions are shortly treated. In the paragraph concerning differential calculus, beside the usual topics, the local classification of smooth functions (Morse's lemma) is treated. The integral is introduced on the basis of Moore and Smith's nets, defined by the refinements of divisions. As an application of Sard's lemma, a new proof for the formula of changing of variables is given. This is applied to the convolution product and to potential theory. Weierstrass'theorem concerning approximation of continuous functions by polynomials is also proved using the convolution product. Lebesgue measure is defined by the method of Carathéodory. The Lebesgue integral is introduced as the limit of a net with order defined by the refinement of divisions.

The L^p spaces and absolutely continuous functions finish the part dedicated to integral calculus. The second volume ends with a paragraph concerning Fourier series and their generalizations to Hilbert spaces. The ideas of Chernoff and Redheffer are followed in order to develop this topic.

A lot of examples, exercises and applications (for instance those from physics and astronomy) complete the 22 paragraphs of the book.

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K o n r a d K ö n i g s b e r g e r, **Analysis**, vol.1 (3rd edition), 392 pp., 1995, vol.2, 365 pp., 1993, Springer-Verlag, Berlin-Heidelberg-New York (in German).

This is a course on mathematical analysis for students in mathematics and informatics or at technical universities. A characteristic feature of this books is that the author tries not merely to explain what mathematical analysis is but also to show what it is good for (except promoting semestrial exams). For this reason, a wealth of examples, most of them classical and which are customary neglected in the modern treatises on mathematical analysis, are included. As examples, we mention the fundamental theorem of algebra (a proof given by Argand in 1914 and based on the properties of continuous functions), the irrationality of π (Niven's proof), Walles and Stirling's formulae, a study of the Riemann function ζ and Euler function Γ, Kepler's laws, the isoperimetric problem (Hurwitz's proof). Differential equations are studied in two chapters - 10 and 13.

The existence of natural numbers is accepted a priori (with a quotation of Kronecker's famous "God created natural numbers, the rest is man's work"). Real numbers are introduced, as usual, axiomatically, with Cantor property on descending sequences of compact intervals taken as axiom of completeness. Complex numbers are considered too, allowing the author to study power series and elementary functions in complex setting. For instance, transcedental functions (exponential, trigonometric, hyperbolic and their inverses) are defined through their power series expansions. This way, their basic properties can be derived in a rapid and elegant manner.

Some .esults in differential calculus are developed in the more general context of functions which are derivable excepting a countable subset of their interval of definition. Properties of convex functions are studied with applications to some classical inequalities.

Integral calculus is developed for the class of regular functions, (uniform limits of step functions). This is a particular class of Riemann integrable functions but sufficiently large for applications. At the same time, this approach prepares the reader for the study of Lebesgue integral given in the second volume via Stone's definition (using L^1 - limits of integrable step functions). Improper integrals are also considered.

Local (Taylor polynomials and series) and global (Weierstrass'theorems) approximation of functions are studied. The first volume ends with a chapter on Fourier series and approximation of periodic functions (including Bessel inequality and Parseval identity).

The first chapter of the second volume is dealing with topological properties of R^n and properties of integrals depending on a parameter. The second chapter develops the differential calculus for complex valued functions defined on domains in R^n. The study of differentiable applications (functions of n variables with values in R^m) or vector functions, is given in the third chapter where the fundamental theorems of differential calculus are proved - local inversion, implicit function theorem. This chapter contains also an introduction to differentiable manifolds in R^n with applications to constrained extrema.

Chapters 4 to 7 are devoted to Lebesgue integral in R^n. The study is relatively complete, including convergence theorems, the theorems of Fubini and Tonelli and change of variables formula. Differentiation theory is not considered.

Integral calculus is applied to convolution product in $L^1((R^n)$ and to the Fourier transform (including a proof of the inversion formula).

The fundamental integral theorems of vector analysis (Gauss, Stokes, Green) are proved in chapter 10, which contains also a study of harmonic functions.

The last chapter of the book, Ch. 11, is dealing with Pfaff forms and path integrals.

The book is clearly written, the topics are carefully chosen, providing the reader with a solid background in mathematical analysis. Each chapter is followel by a set of interesting and well chosen exercises, of various degrees of difficulty, completing the main text. Numerous comments of historical character are spreaded through the books.

Altogether, this is a very good introductory text in mathematical analysis, preparing the reader for further investigations. We recommend it to all interested in mathematical analysis and its applications.

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