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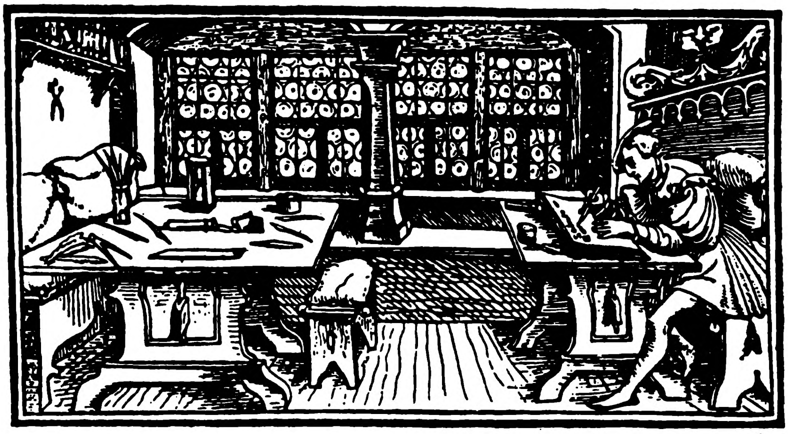

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1

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INDUCTIVELY CLOSED FAMILIES IN SEMIGROUPS

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Dedicated to Professor I. Kolumbán on his 60th anniversary

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REZUMAT. - Familii inductiv închise în semigrupuri. Folosind noţiunea de familie inductiv închisă de mulţimi, se arată că un semigrup X, în care există o familie inductiv închisă $\mathcal{P}$ de submulţimi nevide ale lui X astfel încât $X \in \mathcal{P}$ şi $aX \in \mathcal{P}$, $Xa \in \mathcal{P}$ pentru fiecare $a \in X$, posedă ideale stângi minimale şi ideale drepte minimale, iar acestea aparţin toate lui $\mathcal{P}$. Un asemenea semigrup are atunci un ideal minimal şi elemente idempotente.

0. From the theory of topological semigroups (see [2]) it is well-known that a compact semigroup has idempotents, compact minimal left ideals and compact minimal right ideals. This fact has determined us to search general conditions that allow the deduction of some properties of semigroups that, besides the algebraic structure, are endowed with an additional structure (for example, with a topology or an order relation). The main tool to realize this goal is the notion of an inductively closed family of sets. By means of this notion we show in the present paper that every semigroup X that has an inductively closed family $\mathcal{P}$ of nonempty subsets of X satisfying $X \in \mathcal{P}$, on the one hand, and $aX \in \mathcal{P}$, $Xa \in \mathcal{P}$ for each $a \in X$, on the other hand, has minimal left ideals and

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and minimal right ideals and that these ideals lie in $\mathcal{P}$. Such a semigroup has also a minimal ideal and idempotents.

1. An element F_0 of a family $\mathcal{F}$ of sets is called a *minimal set* if $F \in \mathcal{F}$ and $F \subseteq F_0$ imply that $F = F_0$.

The minimal sets play an important part in all the fields of mathematics. In the present paper we apply this concept in the theory of semigroups. We recall that a *semigroup* is an ordered pair $(X,\cdot)$, where X is a nonempty set and $\cdot$ is an associative composition law on X. The set of idempotents of a semigroup $(X,\cdot)$ is denoted by $E(X)$.

Let $(X,\cdot)$ be a semigroup. A subset Y of X is called:

i) a *left* (*right*) *ideal* if $Y \neq \varnothing$ and $XY \subseteq Y$ ($YX \subseteq Y$);

ii) an *ideal* if it is both a left and a right ideal.

Let $I_l(X)$, $I_r(X)$ and $I(X)$ be respectively the family of all left ideals, right ideals and ideals of the semigroup $(X,\cdot)$. The minimal sets of these families are called respectively *minimal left ideal, minimal right ideal* and *minimal ideal.* We point out that a semigroup may have more than one minimal left (right) ideal, but one minimal ideal at most. It is easy to verify that if the set

$$M(X) = \cap \{I \mid I \in I(X)\}$$

is not empty, then it is the unique minimal ideal of $(X,\cdot)$.

The set of all minimal left ideals of $(X,\cdot)$ is denoted by $\mathcal{L}(X)$, and that of all minimal right ideals is denoted by $\mathcal{R}(X)$. If these sets are not empty, then the following statements are true.

PROPOSITION 1.1 [2, COROLLARY 3.17]. *Let* $(X,\cdot)$ *be a semigroup with* $\mathcal{L}(X) \neq \varnothing$ $(\mathcal{R}(X) \neq \varnothing)$. *Then* $M(X) \neq \varnothing$ *and the equality*

$$M(X) = \cup \{L \mid L \in \mathcal{L}(X)\} \ (M(X) = \cup \{R \mid R \in \mathcal{R}(X)\})$$

holds.

PROPOSITION 1.2 [2, LEMMA 3.18]. *Let* $(X,\cdot)$ *be a semigroup, let* $L \in \mathcal{L}(X)$, *and let* $R \in \mathcal{R}(X)$. *Then the following assertions are true:*

1° $(L \cap R,\cdot)$ *is a group and* $E(X) \neq \varnothing$.

2° *If* e *is the identity for the group* $(L \cap R,\cdot)$, *then* $R \cap L = eXe$.

The aim of the present paper is to establish conditions that assure not only the existence of minimal left (right) ideals of a semigroup, but also that these ideals satisfy a supplementary property (for example, a topological or an order property). The main auxiliary instrument suitable for this aim is the notion of inductively closed family of sets, which will be introduced in section 2. With the aid of this notion and that of Zorn's lemma we will establish in section 2 the main results of our paper. In the last section there will be mentioned some applications of these results. They will show that some results from the theory of topological semigroups, presented in [2], are valid in a more general background.

2. A family $\mathcal{F}$ of sets is called:

i) a *tower* if it is not empty and if for each $F_1, F_2 \in \mathcal{F}$ we have either $F_1 \subseteq$

F_2 or $F_2 \subseteq F_1$;

ii) *inductively closed* if it is not empty and if $\cap\{F_i |\ i \in I\} \in \mathcal{F}$ for each tower $(F_i)_{i\in I}$ of sets from $\mathcal{F}$.

It results from Zorn's lemma that any inductively closed family $\mathcal{F}$ of sets has a minimal set. Using this result, we can establish the following theorems.

THEOREM 2.1. *Let $(X,\cdot)$ be a semigroup, and let $\mathcal{P}$ be an inductively closed family of nonempty subsets of X such that:*

(i) $X \in \mathcal{P}$;

(ii) $aY \in \mathcal{P}$ *for every* $a \in X$ *and* $Y \in \mathcal{P}$;

(iii) $Ya \in \mathcal{P}$ *for every* $a \in X$ *and* $Y \in \mathcal{P}$.

Then $E(X) \neq \varnothing$.

Proof. The family

$$\mathcal{F} = \{S \in \mathcal{P} | S \text{ subsemigroup of } (X,\cdot)\}$$

is inductively closed, since $\mathcal{P}$ is inductively closed and the intersection of a family of subsemigroups is also a subsemigroup, in case this intersection is not empty. Hence there exists a minimal set S_0 of $\mathcal{F}$. We fix any $x \in S_0$. In view of (ii), (iii) and of the fact that S_0 is a subsemigroup of $(X,\cdot)$, it follows that $xS_0 \in \mathcal{F}$ and $S_0x \in \mathcal{F}$. On the other hand we have $xS_0 \subseteq S_0$ and $S_0x \subseteq S_0$. Consequently the minimality of S_0 implies $xS_0 = S_0 = S_0x$. So $(S_0,\cdot)$ is a group. The identity of this group is obviously an element of $E(X)$. □

THEOREM 2.2. *Let $(X,\cdot)$ be a semigroup, and let $\mathcal{P}$ be an inductively closed family of nonempty subsets of X such that:*

(i) $X \in \mathcal{P}$;

(ii) $aX \in \mathcal{P}$ *for every* $a \in X$;

(iii) $Xa \in \mathcal{P}$ *for every* $a \in X$.

Then the following assertions are true:

1° $\mathcal{L}(X) \neq \varnothing$ *and* $\mathcal{L}(X) \subseteq \mathcal{P}$.

2° $\mathcal{R}(X) \neq \varnothing$ *and* $\mathcal{R}(X) \subseteq \mathcal{P}$.

Proof. 1° From (i) it follows that the family $\mathcal{F} = \mathcal{P} \cap I_l(X)$ is not empty. It is even inductively closed, because $\mathcal{P}$ is inductively closed and the intersection of a family of left ideals is also a left ideal, in case this intersection is not empty. Hence there exists a minimal set L_0 of $\mathcal{F}$. We prove that L_0 is a minimal left ideal of $(X,\cdot)$.

Let L be a left ideal of $(X,\cdot)$ such that $L \subseteq L_0$. Since $L \neq \varnothing$, we can choose an $a \in L$. Note that $Xa \subseteq L$, and hence $Xa \subseteq L_0$. On the other hand, (iii) implies that $Xa \in \mathcal{F}$. In view of the minimality of L_0 it follows that $Xa = L_0$. But, we have $Xa \subseteq L \subseteq L_0$. Thus $L = L_0$, i.e., L_0 is a minimal left ideal of $(X,\cdot)$ and so $\mathcal{L}(X) \neq \varnothing$.

Now let $L \in \mathcal{L}(X)$. Since $L \neq \varnothing$, we can choose an $a \in L$. From the fact that L is a minimal left ideal of $(X,\cdot)$ it follows that $Xa = L$. Combined with (iii), it results that $L \in \mathcal{P}$ and so the inclusion $\mathcal{L}(X) \subseteq \mathcal{P}$ is proved.

2° This assertion is obtained by analogy to 1°, with the difference that instead of condition (iii) one uses condition (ii). □

THEOREM 2.3. *Let* $(X,\cdot)$ *be a cancellative semigroup, and let* $\mathcal{P}$ *be an* *inductively closed family of nonempty subsets of X such that:*

(i) $X \in \mathcal{P}$;

(ii) $aX \in \mathcal{P}$ *for every* $a \in X$;

(iii) $Xa \in \mathcal{P}$ *for every* $a \in X$;

(iv) $aXb \in \mathcal{P}$ *for every* $a, b \in X$.

Then $(X,\cdot)$ *is a group.*

Proof. In view of theorem 2.2 and assertion 1° from proposition 1.2 we have $E(X) \neq \emptyset$. Let $e \in E(X)$. For each $x \in X$ we have then $e(ex) = ex$ and $(xe)e = xe$. Since $(X,\cdot)$ is cancellative, one obtains $ex = x$ and $xe = x$ for each $x \in X$, i.e., e is an identity for $(X,\cdot)$. Since a semigroup has at most one identity, it follows that $E(X) = \{e\}$.

We fix now an arbitrary element $x \in X$. After that we denote

$$X^* = xX \text{ and } \mathcal{P}^* = \{P \in \mathcal{P} \mid P \subseteq X^*\}.$$

Then $(X^*,\cdot)$ is a semigroup and $\mathcal{P}^*$ is an inductively closed family of nonempty subsets of X^* satisfying the following conditions:

(j) $X^* \in \mathcal{P}^*$;

(jj) $aX^* \in \mathcal{P}^*$ for each $a \in X^*$;

(jjj) $X^*a \in \mathcal{P}^*$ for each $a \in X^*$.

Condition (jj) results from (ii) and (jjj) from (iv). Applying once more theorem 2.2 and assertion 1° from proposition 1.2, we conclude that $E(X^*) \neq \emptyset$, i.e., $E(X) \cap xX \neq \emptyset$. Since $E(X) = \{e\}$, there exists an element $x^{-1} \in X$ such that $e = xx^{-1}$. Thus $(X,\cdot)$ is a group. □

We recall that a semigroup $(X,\cdot)$ is said to be *completely simple* if it has no proper ideals and if the sets $\mathcal{L}(X)$ and $\mathcal{R}(X)$ are not empty. Taking into account this definition, we state the following theorem.

THEOREM 2.4. *Let* $(X,\cdot)$ *be a semigroup such that* $M(X) \neq \emptyset$, *and let* $\mathcal{P}$ *be an inductively closed family of nonempty subsets of* $M(X)$ *such that:*

(i) $M(X) \in \mathcal{P}$;

(ii) $aM(X) \in \mathcal{P}$ *for every* $a \in M(X)$;

(iii) $M(X)a \in \mathcal{P}$ *for every* $a \in M(X)$.

Then the following assertions are true:

1° $(M(X),\cdot)$ *is a completely simple semigroup and* $M(X) \cap E(X) \neq \emptyset$.

2° $(eXe,\cdot)$ *is a group for every* $e \in M(X) \cap E(X)$.

Proof. 1° $(M(X),\cdot)$ is a semigroup, since $M(X)$ is an ideal of $(X,\cdot)$. This semigroup has no proper ideals. Indeed, if I is an ideal of $(M(X),\cdot)$, then $M(X)IM(X) \subseteq I \subseteq M(X)$. Since $M(X)IM(X)$ is an ideal of $(X,\cdot)$, the minimality of $M(X)$ implies $M(X)IM(X) = M(X)$. Thus we must have $I = M(X)$, i.e., the semigroup $(M(X),\cdot)$ has no proper ideals.

Applying theorem 2.2 to the semigroup $(M(X),\cdot)$, we conclude that $\mathcal{L}(M(X)) \neq \emptyset$ and $\mathcal{R}(M(X)) \neq \emptyset$. In conclusion $(M(X),\cdot)$ is a completely simple semigroup.

Applying now assertion 1° from proposition 1.2, it results that $E(M(X)) \neq \emptyset$, i.e., $M(X) \cap E(X) \neq \emptyset$.

2° Let $e \in M(X) \cap E(X)$. Since $M(X)$ is an ideal of $(X,\cdot)$, we have eXe

$\subseteq e(eXe)e \subseteq eM(X)e$. Notice that $eM(X)e \subseteq eXe$ is obvious. Thus we have $eXe = eM(X)e$.

On the other hand, the semigroup $(M(X),\cdot)$ has no proper ideals and so $M(M(X)) = M(X)$. Therefore we have $e \in M(M(X))$. In virtue of proposition 1.1, there exist an $L \in \mathcal{L}(M(X))$ and an $R \in \mathcal{R}(M(X))$ such that $e \in L \cap R$. Hence, in view of assertion 1° from proposition 1.2, $(L \cap R,\cdot)$ is a group. Since $e \in L \cap R$ and $e^2 = e$, it follows that e is the identity for this group. From assertion 2° of proposition 1.2 it follows that $L \cap R = eM(X)e$. Thus the pair $(eXe,\cdot)$ is a group. □

3. In the theorems stated in the previous section the existence of some inductively closed families of nonempty subsets of the semigroups that occured was assumed. In the present section we show, by giving three concrete examples, that such families really exist. By applying for each of these concrete examples theorem 2.2, we will deduce both the existence of minimal left (right) ideals and the fact that these ideals satisfy some topological or order properties.

THEOREM 3.1. *Let $(X,\cdot)$ be a semigroup, and let $\mathcal{T}$ be a topology on X such that the following conditions are satisfied:*

(i) *$(X,\mathcal{T})$ is a T_2 and compact space;*

(ii) *aX is compact for every $a \in X$;*

(iii) *Xa is compact for every $a \in X$.*

Then the following assertions are true:

1° $\mathcal{L}(X) \neq \emptyset$ *and each* $L \in \mathcal{L}(X)$ *is compact.*

2° $\mathcal{R}(X) \neq \emptyset$ *and each* $R \in \mathcal{R}(X)$ *is compact.*

Proof. Let $\mathcal{P}$ be the family of all compact nonempty subsets of X. This family is not empty, since $X \in \mathcal{P}$. Let $(P_i)_{i\in I}$ be a tower of sets from $\mathcal{P}$. Put $P = \cap \{P_i | i\in I\}$. We claim that $P \neq \emptyset$. To prove this we assume that $P = \emptyset$. Then, since the sets P_i ($i\in I$) are closed and $(X,\mathcal{T})$ is a compact space, there exist $i_1,...,i_n \in I$ such that

$$\cap \{P_{i_j} | j\in \{1,...,n\}\} = \emptyset.$$

But $(P_i)_{i\in I}$ is a tower. Thus there exists a $k \in \{1,...,n\}$ such that $P_{i_k} = \cap \{P_{i_j} | j\in \{1,...,n\}\}$. Consequently, we have $P_{i_k} = \emptyset$, which is absurd. This contradiction shows that $P \neq \emptyset$, as claimed. Since P is closed and X compact, it results that P is compact. In conclusion, $P \in \mathcal{P}$. Therefore $\mathcal{P}$ is an inductively closed family. Applying theorem 2.2, we obtain the two assertions of the theorem. □

COROLLARY 3.2. *If* $(X,\cdot,\mathcal{T})$ *is a compact topological semigroup, then the following assertions are true:*

1° $\mathcal{L}(X) \neq \emptyset$ *and each* $L \in \mathcal{L}(X)$ *is compact.*

2° $\mathcal{R}(X) \neq \emptyset$ *and each* $R \in \mathcal{R}(X)$ *is compact.*

Remark. Corollary 3.2 is already known in the theory of topological semigroups. A direct proof for it is given in the theorems 1.28 and 1.29 from [2].

Before stating the next theorem we remember that if X is a nonempty set and $\leq$ is a quasi-order on X (i.e., a reflexive and transitive binary relation), then the family $(B_x)_{x\in X}$, where $B_x = \{y \in X \mid y \leq x\}$, is the basis for a topology on X, denoted by $\mathcal{T}_L(X)$ and called the *left topology induced on X by $\leq$* (see [1, page 142, Exercice 2a]).

THEOREM 3.3. *Let $(X,\cdot)$ be a semigroup, let x_0 be an element of X. Further let $\leq$ be a quasi-order on X such that $x_0 \leq x$ for each $x \in X$ and such that the left topology $\mathcal{T}_L(X)$ induced on X by $\leq$ satisfies the following conditions:*

(i) *$aX \in \mathcal{T}_L(x)$ for every $a \in X$;*

(ii) *$Xa \in \mathcal{T}_L(x)$ for every $a \in X$.*

Then the following assertions are true:

1° *$\mathcal{L}(X) \neq \varnothing$ and $\mathcal{L}(X) \subseteq \mathcal{T}_L(X)$.*

2° *$\mathcal{R}(X) \neq \varnothing$ and $\mathcal{R}(X) \subseteq \mathcal{T}_L(X)$.*

Proof. Let $\mathcal{P} = \mathcal{T}_L(X)\setminus\{\varnothing\}$. Since $X \in \mathcal{T}_L(X)$, the family $\mathcal{P}$ is not empty. Let $(P_i)_{i\in I}$ be a tower of sets from $\mathcal{P}$. Then $P = \cap\{P_i \mid i \in I\}$ belongs to $\mathcal{P}$. To prove this, first we note that $x_0 \in P$, because $x_0 \in P_i$ for each i $\in I$. So P is not empty. On the other hand, taking into account that $x \in B_x$ for every $x \in X$, it results that

$$P \subseteq \cup \{B_x \mid x \in P\}. \tag{3.1}$$

But the inclusion

$$\cup \{B_x \mid x \in P\} \subseteq P \tag{3.2}$$

is also valid. To see this, let $y \in \cup \{B_x \mid x \in P\}$. Then there exists an $x \in P$ such that $y \in B_x$, and hence $y \leq x$. Let now $i \in I$ be arbitrary. Since $P_i \in \mathcal{T}_L(X)$, there exists a subset T of X such that

$$P_t = \bigcup\{B_t \mid t \in T\}.$$

But $x \in P_t$, hence there exists a $t \in T$ for which $x \in B_t$, i.e., $x \leq t$. From $y \leq x$ and $x \leq t$ we deduce that $y \leq t$, i.e., $y \in B_t \subseteq P_t$. This shows that (3.2) is valid. From (3.1) and (3.2) we obtain

$$P = \bigcup\{B_x \mid x \in P\}.$$

This means that $P \in \mathcal{T}_L(X)$. In conclusion, we have $P \in \mathcal{P}$. Consequently $\mathcal{P}$ is an inductively closed family. Applying theorem 2.2, we obtain the two assertions of the theorem. □

Let X be a nonempty set endowed with a quasi-order $\leq$. If a and b are elements from X such that $a \leq b$, then we define $[a,b]$ by

$$[a,b] = \{x \in X \mid a \leq x \text{ and } x \leq b\}.$$

Every subset Y of X for which there exist a, $b \in X$ such that $a \leq b$ and $Y = [a,b]$ is called a *closed interval.*

THEOREM 3.4. *Let $(X,\cdot)$ be a semigroup, and let $\leq$ be a binary relation on X such that $(X,\leq)$ is a complete lattice with the following properties:*

(i) *aX is a closed interval for every $a \in X$;*

(ii) *Xa is a closed interval for every $a \in X$.*

Then the following assertions are true:

1° *$\mathcal{L}(X) \neq \varnothing$ and each $L \in \mathcal{L}(X)$ is a closed interval.*

2° *$\mathcal{R}(X) \neq \varnothing$ and each $R \in \mathcal{R}(X)$ is a closed interval.*

Proof. Let $\mathcal{P}$ be the family of all closed intervals of X. Since $X = [\inf X, \sup X]$, it follows that $X \in \mathcal{P}$. Let $(P_i)_{i\in I}$ be a tower of sets from $\mathcal{P}$. Put $P =$

$\cap\{P_i \mid i \in I\}$. We assume that $P_i = [a_i, b_i]$ for every $i \in I$ and denote

$$a = \sup \{a_i \mid i \in I\} \text{ and } b = \inf \{b_i \mid i \in I\}.$$

We claim that $a \leq b$. To prove this, we notice that if $i, j \in I$ are arbitrarily chosen, then we have $P_i \subseteq P_j$ or $P_j \subseteq P_i$, $a_i \in P_i$ and $b_j \in P_j$; thus it results that $a_i \leq b_j$. So we must have $a \leq b$. From $a_i \leq a \leq b \leq b_i$ for every $i \in I$, it follows that $[a,b] \subseteq P$. On the other hand, if $x \in P$, then $a_i \leq x \leq b_i$ holds for every $i \in I$; hence $a \leq x \leq b$. This result shows that $P \subseteq [a,b]$. Consequently we have $P = [a,b]$; thus $P \in \mathcal{P}$. Therefore $\mathcal{P}$ is an inductively closed family. Applying theorem 2.2, we obtain the two assertions of the theorem. □

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ON α - DERIVATION OF PRIME RINGS

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REZUMAT. - Asupra α-derivării inelelor prime. În lucrare sunt demonstrate proprietăţile i) - iii) privind α-derivata pe un inel prim.

Abstract. Let R be a prime ring, char $R \neq 2$, U lie ideal of R and $0 \neq d$: $R \to R$ is α-derivation. In this paper we prove the following results.

i) If $d(U) \subset U$ and $d^2(U) = 0$, then $U \subset C(R)$, where $C(R) = \{c \in R: cx = xc, \forall x \in R\}$ is the center of R.

ii) If $[d(u),u]_\alpha \in C_\alpha(R)$ and $u^2 \in U$, then $[d(u),u]_\alpha = 0$.

iii) If $[d(u),u]_\alpha \in C_\alpha(R)$, then $[[d(r),u]_\alpha, u]_\alpha \in C_\alpha(R)$. If $\forall\ u \in U$, $r \in R[d(u),u]_\alpha = 0$, then $[[d(r),u]_\alpha, u]_\alpha = 0$.

1. Introduction. Let R be a ring and d is an additive mapping of R. We say that d is a α-derivation of R if $d(xy) = d(x)\alpha(y) + xd(y)$ for all $x,y \in R$, where α is automorphism of R. We will set $C_\alpha(R) = \{c \in R: c\alpha(x) = xc$ for all

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$x \in R\}$ and $[x,y]_\alpha = x\alpha(y) - yx$.

Let R be a prime ring, char $R \neq 2$ and U lie ideal of R. If d is a non zero derivation of R such that $d^2(U) = (0)$, then $U \subset C(R)$. This was proved in [4]. Our aim in this paper is to extend the above mentioned result to a more general situation using the α-derivation.

2. Some Lemmas.

LEMMA 1. *Every non zero weak - α-derivation of a prime ring is a non zero α-derivation* [2].

LEMMA 2. *Let d_1: $R \to R$ be a g-derivation and d_2: $R \to R$ α-derivation and $d_2\alpha = \alpha d_2$. If $d_1 d_2(R) = 0$, then $d_1 = 0$ or $d_2 = 0$* [6].

LEMMA 3. i) *If U is a non zero right ideal of R and $d(U) = 0$, then $d =$ 0.* ii) *If U is a non zero ideal of R and $ad(U) = 0$ for an element a of R, then $a = 0$ or $d = 0$.* iii) *If $d(R)a = 0$ for an element a of R, then $a = 0$ or $d = 0$* [2].

LEMMA 4. *If U is a non zero lie ideal of R, $0 \neq d$: $R \to R$ α-derivation and $d(U)t = 0$ for an element t of R, then $t = 0$* [6].

Throunghout the present paper, R will represent a prime ring of characteristic not 2 with center $C(R)$, U a non zero lie ideal of R and d a α-derivation of R.

3. Main Theorems.

THEOREM 1. *Let d be a non zero α-derivation of R such that $d\alpha = \alpha d$ and U a non zero lie ideal of R such that $d(U) \subset U$. If $d^2(U) = 0$, then $U \subset C(R)$.*

Proof. Suppose that $U \not\subset C(R)$, $V = [U,U]$ is a noncentral lie ideal of R, by [4]. So, if we prove that $V \subset C(R)$, then the theorem is proved.

There exists an ideal M of R such that $[M,R] \subset U$ and $[M,R] \not\subset C(R)$, by [4]. If $m \in [M,R] \subset U \cap M$ and $u \in U$, then $d(u) \in d(V) \subset U$ and so $d^2(u) = 0$, for any element y of $R, 0 = d^2([mu,y]) = 2d(m)d(\alpha[u,y])) + 2d([m,y])d(\alpha(u))$ since, char $R \neq 2$ it follows that:

$$d(m)d(\alpha([u,y])) + d([m,y])d(\alpha(u)) = 0 \text{ for all } m \in [M,R],\, y \in R. \quad (3.1)$$

Replacing u by $d(u)$ in (3.1), we have

$$d([M,R])\alpha(d([d(u),y])) = 0 \text{ for all } y \in R,\, u \in U. \quad (3.2)$$

But $[MR]$ is a noncentral lie ideal of R. From Lemma 4 and (3.2) we have $d([d(U),y]) = 0$ for all $y \in R$, $u \in U$. Thus, $0 = d([d(u),y]) = d^2(u)\alpha(y) + d(u)d(y) - d(y)\alpha(d(u)) - yd^2(u) = -[d(y)d(u)]_\alpha$ gives us

$$[d(y), d(u)]_\alpha = 0 \text{ for all } y \in R,\, u \in U. \quad (3.3)$$

Replacing y by $yd(v)$, $v \in U$ in (3.3), we have

$$[d(y), d(u)]\, d(\alpha(v)) = 0 \text{ for all } y \in R,\, u,v \in U. \quad (3.4)$$

But $\alpha(U)$ is a non zero lie ideal of R. Then by [2] and (3.4) we have

$$[d(y),d(u)] = 0 \text{ for all } y \in R,\ u \in U. \qquad (3.5)$$

Replacing y by $xd(y)$, $x \in R$ in (3.5), we have

$$[x,d(u)]d^2(y) = 0 \text{ for all } x,y \in R,\ u \in U. \qquad (3.6)$$

Replacing x by $xz, z \in R$ in (3.6), we have

$$[x,d(u)]Rd^2(y) = 0 \text{ for } x,y \in R,\ u \in U. \qquad (3.7)$$

So since R is a prime ring, $d(U) \subset C(R)$ or $d^2(R) = 0$.

If $d^2(R) = 0$, then from lemma 2, $d = 0$. This is a contradiction. Thus $d(U) \subset C(R)$. Now, we take $v = ur - ru$, $u \in U$, $r \in R$. Since $uv \in U$, using the hypothesis, we obtain $0 = d^2(uv)$. Then $d(u)\alpha(d(v)) = 0$. We know that $d(u) \in C(R)$, so $d(u) = 0$ or $d(v) = 0$. Hence,

$$d(u) = 0 \text{ or } d(ur - ru) = 0 \text{ for all } u \in U,\ r \in R. \qquad (3.8)$$

Let $K = \{u \in U: d(u) = 0\}$ and $L = \{u \in U: d(ur - ru) = 0 \text{ for all } r \in R\}$. Then $U = K \cup L$, where K and L are subgroups of U and $U \neq K$. From Brauer trick, $U = L$. Thus, $d(V) = 0$. Since V is a lie ideal, using [2] we have arrived to $V \subset C(R)$. That is $[U,U] \subset C(R)$. Then, by [3], $U \subset C(R)$. This is a contradiction so the proof is finished.

THEOREM 2. *If for each* $u \in U$, $[d(u),u]_\alpha \in C_\alpha$ *and* $u^2 \in U$, *then* $[d(u),u]_\alpha = 0$.

Proof. If the hypothesis is made linear, then we have $[d(u),u^2]_\alpha + [d(u^2),u]_\alpha \in C_\alpha$. Thus, if we make some needed operations on these equations at the end of these operations we get

$$u[d(u),u]_\alpha \in C_\alpha.$$

Since $[d(u),u]_\alpha \in C_\alpha$, for every $r \in R$,

$$[u,r]\, R\, [d(u),u]_\alpha = 0.$$

So,

$$[u,r] = 0 \text{ or } [d(u),u]_\alpha = 0 \text{ for all } r \in R,\ u \in R. \tag{3.9}$$

Let $u \in C(R)$. Since R is a prime ring, by $[d(u),u]_\alpha \in C_\alpha$, $[d(u),u]_\alpha = 0$ or $\alpha(u) = u$. If $c(u) = u$, $d(ux - xu) = 0$, then $[d(u),x]_\alpha = 0$. Hence, $d(u) \in C_\alpha$. If $d(u) \in C_\alpha$, then $[d(u),u]_\alpha = 0$.

THEOREM 3. *For each $r \in R$ and for every $u \in U$, if $[d(u),u]_\alpha \in C_\alpha$, then $[[d(r),u]_\alpha, u]_\alpha \in C_\alpha$. Also, if $[d(u),u]_\alpha = 0$, then $[[d(r),u]_\alpha, u]_\alpha = 0$.*

Proof. Since $[d(u),u]_\alpha \in C_\alpha$, if we take $u + [u,r]$ instead of u, for each $r \in R$, $u \in U$, then this implies that $[d(u),[u,r]]_\alpha + [d([u,r]), u]_\alpha \in C_\alpha$. Thus,

$$[d(u),[u,r]]_\alpha + [[d(u),r]_\alpha, u]_\alpha - [[d(r),u]_\alpha \in C_\alpha. \tag{3.10}$$

If we use the equation $[x,[y,z]]_\alpha + [[x,z]_\alpha, y]_\alpha - [[x,y]_\alpha, z]_\alpha = 0$, then we have $[[d(u),u]_\alpha, r]_\alpha$ for the first and second terms of equation ((3.10). Therefore, we arrive at the following form of equation (3.10).

$$[[d(u),u]_\alpha, r]_\alpha - [[d(r),u]_\alpha, u]_\alpha \in C_\alpha.$$

Thus, $[[d(r),u]_\alpha, u]_\alpha \in C_\alpha$. If $[d(u),u]_\alpha = 0$, then $[[d(r),u]_\alpha, u]_\alpha = 0$ can be done similar.

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ON CERTAIN CLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS. II

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Dedicated to Professor I. Kolumbán on his 60[th] anniversary

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REZUMAT. - Asupra unei clase de funcţii analitice cu coeficienţi negativi. În lucrare sunt studiate unele subclase ale unei clase de funcţii analitice cu coeficienţi negativi. Sunt obţinute teoreme de caracterizare, de deformare, delimitări ale coeficienţilor, sunt determinate punctele extremale, sunt evidenţiate relaţii între diferite clase etc. Rezultatele obţinute sunt exacte (în sensul că nu mai pot fi îmbunătăţite).

Abstract. Let T denote the class of functions of the form

$$f(z) = a_1 z - \sum_{k=2}^{\infty} a_k z^k \quad (a_1 > 0;\ a_k \geq 0)$$

which are analytic and univalent in the unit disc $U = \{z\colon |z| < 1\}$. Let $T_m(z_0)$ denote the subclasses of T satisfying $D^m f(z_0) = z_0$ $(0<z_0<1)$, where $D^m f(z) = D(D^{m-1}f(z))$ and $D^1 f(z) = zf'(z)$. The object of this paper is to study some properties of certain subclasses of $T_m(z_0)$. Also the extreme points of these subclasses are determined. All results obtained in this paper are sharp.

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1. Introduction. Let A denote the class of functions $f(z)$ which are analytic and univalent in the unit disc $U = \{z: |z| < 1\}$ with $f(0) = 0$. For a function $f(z)$ in A, we define

$$D^0 f(z) = f(z), \tag{1.1}$$

$$D^1 f(z) = Df(z) = zf'(z), \tag{1.2}$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \quad (n \in \mathbf{N} = \{1,2,...\}). \tag{1.3}$$

The differential operator D^n was introduced by Sălăgean [2]. With the help of the differential operator D^n, we say that a function $f(z)$ belonging to A is in the class $S_n(\alpha)$ if and only if

$$\mathrm{Re}\left\{\frac{D^{n+1} f(z)}{D^n f(z)}\right\} > \alpha \quad (n \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}) \tag{1.4}$$

for some α $(0 \le \alpha < 1)$, and for all $z \in U$. The class $S_n(\alpha)$ was defined by Sălăgean [2].

Let T denote the subclass of A consisting of functions of the form

$$f(z) = a_1 z - \sum_{k=2}^{\infty} a_k z^k \quad (a_1 > 0;\ a_k \ge 0). \tag{1.5}$$

Further, we define the class $T^{**}(n,\alpha)$ by

$$T^{**}(n,\alpha) = S_n(\alpha) \cap T.$$

We note that when $a_1 = 1$, the class $T^{**}(n,\alpha) = T^*(n,\alpha)$ was studied by Hur and Oh [1] and by Sălăgean [4], [5].

For a given real number z_0 $(0 < z_0 < 1)$ let $T_m(z_0)$ $(m \in \{0,1,\ldots,n+1\})$ be the subclasses of T satisfying

$$D^m f(z_0) = z_0 \tag{1.6}$$

We say that this function has two fixed points. Let

$$T_m^{**}(n,\alpha,z_0) = T^{**}(n,\alpha) \cap T_m(z_0) \tag{1.7}$$

We note that the classes $T_m^{**}(n,\alpha,z_0)$ were introduced in [3] and the classes $T_0^{**}(0,\alpha,z_0)$, $T_1^{**}(0,\alpha,z_0)$, $T_0^{**}(1,\alpha,z_0)$ and $T_1^{**}(1,\alpha,z_0)$ were studied by Silverman [6].

In this paper we obtain coefficient estimates, distortion theorems, closure theorems and radius of convexity of order ρ $(0 \le \rho < 1)$ for the classes $T_m^{**}(n,\alpha,z_0)$ $(m \in \{0,1,\ldots,n+1\})$. Further we determine a necessary and sufficient condition that a subset B of the real interval $(0,1)$ should satisfy the property that $\bigcup_{z_i \in B} T_m^{**}(n,\alpha,z_i)$ forms a convex family. For these classes the extreme points are also determined.

2. Characterisation Theorem and Coefficient Estimates

THEOREM 1. *Let the function $f(z)$ be defined by (1.5). Then $f(z)$ is in the class $T^{**}(n,\alpha)$ if and only if*

$$\sum_{k=2}^{\infty} k^n(k-\alpha)\, a_k \le (1-\alpha)a_1. \tag{2.1}$$

The result is sharp for the function $f(z)$ defined by

$$f(z) = a_1 z - \frac{(1-\alpha)a_1}{k^n(k-\alpha)} z^k \ (k \geq 2). \tag{2.2}$$

Proof. The proof follows exactly on the lines of the proof of Theorem 1 in [1]: hence the details are omitted.

THEOREM 2. *Let the function $f(z)$ be defined by* (1.5) *and* (1.6). *Then $f(z) \in T_m^{**}(n,\alpha,z_0)$ if and only if*

$$\sum_{k=2}^{\infty} \left\{ k^n(k-\alpha) - k^m(1-\alpha)z_0^{k-1} \right\} a_k \leq (1-\alpha). \tag{2.3}$$

Proof. Since $f(z) \in T_m^{**}(n,\alpha,z_0)$, we have $D^m f(z_0) = z_0 = a_1 z_0 - \sum_{k=2}^{\infty} k^m a_k z_0^k$ ($a_1 > 0$: $a_k \geq 0$) which gives

$$a_1 = 1 + \sum_{k=2}^{\infty} k^m a_k z_0^{k-1}. \tag{2.4}$$

Substituting the value of a_1 in Theorem 1, we have the theorem.

Remark. A proof of Theorem 2, without the use of Theorem 1, may be found in [2].

COROLLARY 2.1. *Let the function $f(z)$ defined by* (1.5) *be in the class $T_m^{**}(n,\alpha,z_0)$. Then*

$$a_k \leq \frac{1-\alpha}{k^n(k-\alpha) - k^m(1-\alpha)z_0^{k-1}}, \quad k \in \{2,3,\dots\}. \tag{2.5}$$

The equality in (2.5) *is attained for the function $f_k(z)$ given by*

$$f_k(z) = \frac{k^n(k-\alpha)z - (1-\alpha)z^k}{k^n(k-\alpha) - k^m(1-\alpha)z_0^{k-1}}, \quad k \in \{2,3,\dots\}. \tag{2.6}$$

COROLLARY 2.2. *Let the function $f(z)$ defined by* (1.5) *be in the class $T_m^{**}(n,\alpha,z_0)$.*

Then

$$a_1 \le \frac{2-\alpha}{2-\alpha-2^{m-n}(1-\alpha)z_0}. \tag{2.7}$$

The result is sharp.

Proof. We have

$$a_1 = 1 + \sum_{k=2}^{\infty} k^m a_k z_0^{k-1} \le 1 + z_0 \sum_{k=2}^{\infty} k^m a_k. \tag{2.8}$$

From (2.3) we obtain

$$1 \ge \sum_{k=2}^{\infty} k^m \left(k^{n-m} \frac{k-\alpha}{1-\alpha} - z_0^{k-1} \right) a_k \ge \sum_{k=2}^{\infty} k^m \left(2^{n-m} \frac{2-\alpha}{1-\alpha} - z_0 \right) a_k$$

and we deduce

$$\sum_{k=2}^{\infty} k^m a_k \le 1 \Big/ \left(2^{n-m} \frac{2-\alpha}{1-\alpha} - z_0 \right), \tag{2.9}$$

because $m \le n+1$ and

$$2^{n-m} \frac{2-\alpha}{1-\alpha} > 1 > z_0.$$

From (2.8) and (2.9) we obtain (2.7). The extremal function is $f_2(z)$ given by (2.6).

3. Distortion Theorems

THEOREM 3. *Let the function* $f(z)$ *defined by* (1.5) *be in the class* $T_m^{**}(n,\alpha,z_0)$. *Then*

$$|f(z)| \le \left\{ \frac{2^n(2-\alpha)+(1-\alpha)r}{2^n(2-\alpha)-2^m(1-\alpha)z_0} \right\} r \tag{3.1}$$

and

$$|f'(z)| \le \{\frac{2^n(2-\alpha)+2(1-\alpha)r}{2^n(2-\alpha)-2^m(1-\alpha)z_0}\} \tag{3.2}$$

for $|z| = r < 1$. *The result is sharp.*

Proof. It follows from Theorem 2 that

$$\sum_{k=2}^{\infty} a_k \le \frac{(1-\alpha)}{\{2^n(2-\alpha)-2^m(1-\alpha)z_0\}} \tag{3.3}$$

and

$$\frac{1}{2}\left\{2^n(2-\alpha)-2^m(1-\alpha)z_0\right\}\sum_{k=2}^{\infty} k\,a_k$$
$$\le \sum_{k=2}^{\infty}\left\{k^n(k-\alpha)-k^m(1-\alpha)z_0^{k-1}\right\}a_k \le (1-\alpha). \tag{3.4}$$

which implies

$$\sum_{k=2}^{\infty} k\,a_k \le \frac{2(1-\alpha)}{\{2^n(2-\alpha)-2^m(1-\alpha)z_0\}}. \tag{3.5}$$

Further from (2.7) we have

$$a_1 \le \frac{2^n(2-\alpha)}{2^n(2-\alpha)-2^m(1-\alpha)z_0}. \tag{3.6}$$

Hence we have

$$|f(z)| \le a_1 r + r^2\sum_{k=2}^{\infty} a_k \le \left\{\frac{2^n(2-\alpha)+(1-\alpha)r}{2^n(2-\alpha)-2^m(1-\alpha)z_0}\right\} r \tag{3.7}$$

by using (3.3) and (3.6). Further

$$|f'(z)| \le a_1 + r\sum_{k=2}^{\infty} k a_k \le \left\{\frac{2^n(2-\alpha)+2(1-\alpha)r}{2^n(2-\alpha)-2^m(1-\alpha)z_0}\right\} \tag{3.8}$$

by using (3.5) and (3.6). Finally the result is sharp for the function $f_2(z)$ given by (2.6).

4. Closure Theorems. Let the functions $f_i(z)$ be defined, for $i = 1,2,\dots,l$, by

$$f_i(z) = a_{1,i}z - \sum_{k=2}^{\infty} a_{k,i}z^k \quad (a_{1,i} > 0;\ a_{k,i} \geq 0) \tag{4.1}$$

for $z \in U$.

We shall prove the following results for the closure of functions in the classes $T_m^{**}(n,\alpha,z_0)$ $(m \in \{1,2,\dots,n+1\})$.

THEOREM 4. *Let the functions* $f_i(z)$ $(i = 1,2,\dots,l)$ *defined by* (4.1) *be in the class* $T_m^{**}(n,\alpha,z_0)$. *Then the function* $h(z)$ *defied by*

$$h(z) = \sum_{i=1}^{l} d_i f_i(z) \quad (d_i \geq 0) \tag{4.2}$$

is also in the same class $T_m^{**}(n,\alpha,z_0)$, *where*

$$\sum_{i=1}^{l} d_i = 1. \tag{4.3}$$

Proof. According to the definition of $h(z)$ we can write

$$h(z) = b_1 z - \sum_{k=2}^{\infty} b_k z^k, \tag{4.4}$$

where

$$b_1 = \sum_{i=1}^{l} d_i a_{1,i} \text{ and } b_k = \sum_{i=1}^{l} d_i a_{k,i} \ (k = 2,3,\dots). \tag{4.5}$$

Since $f_i(z)$ are in $T_m^{**}(n,\alpha,z_0)$ $(i = 1,2,\dots,l)$, by means of Theorem 2, we have

$$\sum_{k=2}^{\infty} \left\{k^n(k-\alpha) - k^m(1-\alpha)z_0^{k-1}\right\} a_{k,i} \leq (1-\alpha) \tag{4.6}$$

for every $i = 1,2,\dots,l$. Therefore we have

$$\sum_{k=2}^{\infty} \left\{k^n(k-\alpha) - k^m(1-\alpha)z_0^{k-1}\right\} \left(\sum_{i=1}^{l} d_i a_{k,i}\right)$$

$$= \sum_{i=1}^{l} d_i \left(\sum_{k=2}^{\infty} \left\{ k^n(k-\alpha) - k^m(1-\alpha) z_0^{k-1} \right\} a_{k,i} \right)$$

$$\leq \left(\sum_{i=1}^{l} d_i \right) (1-\alpha) = (1-\alpha)$$

which shows that $h(z) \in T_m^{**}(n,\alpha,z_0)$. Thus we have the theorem.

COROLLARY 4.1. *Let the functions $f_i(z)$ ($i \in \{1,2,\ldots,l\}$) defined by* (4.1) *be the class $T_m^{**}(n,\alpha,z_0)$. Then the function $h(z)$ defined by*

$$h(z) = b_1 z - \sum_{k=2}^{\infty} b_k z^k \tag{4.7}$$

also belongs to the class $T_m^{**}(n,\alpha,z_0)$, where

$$b_1 = \frac{1}{l} \sum_{i=1}^{l} a_{1,i} \text{ and } b_k = \frac{1}{l} \sum_{i=1}^{l} a_{k,i},\ k \in \{2,3,\ldots\}. \tag{4.8}$$

Proof. We have

$$h(z) = \frac{1}{l} \left\{ \sum_{i=1}^{l} a_{1,i} z - \sum_{i=1}^{l} a_{k,i} z^k \right\} = \sum_{i=1}^{l} \frac{1}{l} f_i(z)$$

and now we can use Theorem 4 with $d_i = 1/l$, $i \in \{1,2,\ldots,l\}$.

COROLLARY 4.2. *The class $T_m^{**}(n,\alpha,z_0)$ is closed under convex linear combination.*

Proof. Let the functions $f_i(z)$, $i \in \{1,2\}$ defined by (4.1) be in the class $T_m^{**}(n,\alpha,z_0)$. Then it is sufficient to show that the function $h(z)$ defined by

$$h(z) = \lambda f_1(z) + (1-\lambda) f_2(z),\ 0 \leq \lambda \leq 1 \tag{4.9}$$

is also in the class $T_m^{**}(n,\alpha,z_0)$. But $h \in T_m^{**}(n,\alpha,z_0)$ by Theorem 4.

THEOREM 5. *Define $f_1(z) = z$ and*

$$f_k(z) = \frac{k^n(k-\alpha)z - (1-\alpha)z^k}{k^n(k-\alpha) - k^m(1-\alpha)z_0^{k-1}},\ k \in \{2,3,\ldots\}. \tag{4.10}$$

Then $f(z)$ is in the class $T_m^{**}(n,\alpha,z_0)$ if and only if it can be expressed in the form

$$f(z)=\sum_{k=1}^{\infty}\lambda_k f_k(z), \tag{4.11}$$

where $\lambda_k \geq 0$ $(k \in \mathbf{N})$ *and*

$$\sum_{k=1}^{\infty}\lambda_k = 1. \tag{4.12}$$

Proof. Assume that

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty}\lambda_k f_k(z) = \lambda_1 f_1(z) + \sum_{k=2}^{\infty}\lambda_k f_k(z) = \\ &= \lambda_1 z + \sum_{k=2}^{\infty}\frac{k^n(k-\alpha)z-(1-\alpha)z^k}{k^n(k-\alpha)-k^m(1-\alpha)z_0^{k-1}}\lambda_k = \\ &= \left(\lambda_1 + \sum_{k=2}^{\infty}\frac{k^n(k-\alpha)\lambda_k}{k^n(k-\alpha)-k^m(1-\alpha)z_0^{k-1}}\right)z - \\ &- \sum_{k=2}^{\infty}\frac{(1-\alpha)\lambda_k}{k^n(k-\alpha)-k^m(1-\alpha)z_0^{k-1}}z^k. \end{aligned} \tag{4.13}$$

Then it follows that

$$\begin{aligned} &\sum_{k=2}^{\infty}\left\{k^n(k-\alpha)-k^m(1-\alpha)z_0^{k-1}\right\}\cdot\frac{(1-\alpha)\lambda_k}{k^n(k-\alpha)-k^m(1-\alpha)z_0^{k-1}} \\ &= (1-\alpha)\sum_{k=2}^{\infty}\lambda_k = (1-\alpha)(1-\lambda_1) \leq (1-\alpha). \end{aligned}$$

Also by definition we have $D^m f_k(z_0) = z_0$. Therefore

$$D^m f(z_0) = \sum_{k=1}^{\infty}\lambda_k D^m f_k(z_0) = \sum_{k=1}^{\infty}\lambda_k z_0 = z_0\sum_{k=1}^{\infty}\lambda_k = z_0.$$

This implies $f(z)\in T_m(z_0)$. So by Theorem 2, $f(z)\in T_m^{**}(n,\alpha,z_0)$.

Conversely, assume that the function $f(z)$ defined by (1.5) belongs to the class $T_m^{**}(n,\alpha,z_0)$. Then

$$a_k \leq \frac{(1-\alpha)}{k^n(k-\alpha) - k^m(1-\alpha)z_0^{k-1}} \quad (k \geq 2).$$

Set

$$\lambda_k = \frac{k^n(k-\alpha) - k^m(1-\alpha)z_0^{k-1}}{1-\alpha} a_k \quad (k \geq 2),$$

and

$$\lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k.$$

Hence we can see that $f(z)$ can be expressed in the form (4.11). This complet

the proof of the theorem.

THEOREM 6. *The extreme points of* $T_m^{**}(n,\alpha,z_0)$ *are the functions* f

$k \in \mathbf{N}$, *defined by* (4.10).

Proof. We show that f_k cannot be expressed as a convex combin

functions in $T_m^{**}(n,\alpha,z_0)$.

We suppose that there exist g_1 and g_2 in $T_m^{**}(n,\alpha,z_0)$ such that

$$f_k(z) = \lambda g_1(z) + (1-\lambda)g_2(z).$$

From Theorem 5 we know that

$$g_i(z) = \sum_{j=1}^{\infty} \lambda_{i,j} f_j(z),\ \lambda_{i,j} \geq 0,\ \sum_{j=1}^{\infty} \lambda_{i,j} = 1,\ i \in \{1,2\}$$

and then

$$f_k(z) = \sum_{j=1}^{\infty} \left[\lambda\lambda_{1,j} + (1-\lambda)\lambda_{2,j}\right] f_j(z) = \sum$$

We obtain

$$\lambda_j = \lambda\lambda_{1,j} + (1-\lambda)\lambda_{2,j} = 0,\ j \in \{2,3$$

and this implies $\lambda_{1,j} = \lambda_{2,j} = 0$, $j \in \{2,3,...,k-1,k+1,...\}$.

Hence we have

$$f_k(z) = [\lambda\lambda_{1,1} + (1-\lambda)\lambda_{2,1}]z + $$
$$+ [\lambda\lambda_{1,k} + (1-\lambda)\lambda_{2,k}]f_k(z)$$

which implies $\lambda_{1,1} = \lambda_{2,1} = 0$ and then $g_1 = g_2 = f_k$.

We obtained that the functions f_k, $k \in \{1,2,3,...\}$ are extreme points for $T_m^{**}(n,\alpha,z_0)$ and by using again Theorem 5 we deduce that all function in $T_m^{**}(n,\alpha,z_0)$ can be expressed as a finite convex combination of the functions f_k, $k \in \{1,2,3,...\}$, or as a limit of a sequence of functions which are finite convex combinations of the functions f_k.

5. Radius of Convexity. In this section we determine the radius of convexity of order $\rho(0 \le \rho < 1)$ for the classes $T_m^{**}(n,\alpha,z_0)$, $m \in \{0,1,...,n+1\}$.

THEOREM 7. *Let the function f(z) defined by* (1.5) *be in the class* $T_m^{**}(n,\alpha,z_0)$, *let j be an integer,* $0 \le j \le n+1$ *and let* ρ *be a real number,* $0 \le \rho < 1$; *then f satisfies*

$$\mathrm{Re}\ \frac{D^{j+1}f(z)}{D^j f(z)} > \rho, \text{ for } |z| < r^*, \textit{ where}$$

$$r^* = \inf_k \left\{ \frac{(k-\alpha)(1-\rho)}{(1-\alpha)(k-\rho)} k^{n-j} \right\}^{\frac{1}{k-1}}, \ k \in \{2,3...\} \tag{5.1}$$

The result is sharp.

Proof. It is sufficient to show that

$$\left|\frac{D^{j+1}f(z)}{D^j f(z)} - 1\right| < 1 - \rho \quad \text{for } |z| < r^*.$$

We have

$$\left|\frac{D^{j+1}f(z)}{D^j f(z)} - 1\right| \le \frac{\sum_{k=2}^{\infty} k^j (k-1) a_k |z|^{k-1}}{a_1 - \sum_{k=2}^{\infty} k^j a_k |z|^{k-1}}.$$

Thus (5.2) holds if

$$\sum_{k=2}^{\infty} k^j (k-1) a_k |z|^{k-1} \le (1-\rho)\left(a_1 - \sum_{k=2}^{\infty} k^j a_k |z|^{k-1}\right)$$

By using (2.4) we find that the inequality (5.2) is equivalent to

$$\sum_{k=2}^{\infty}\left[k^j (k-\rho)|z|^{k-1} - k^m (1-\rho) z_0^{k-1}\right] a_k \le 1 - \rho$$

But Theorem 2 ensures that

$$\sum_{k=2}^{\infty} (1-\rho)\left[\frac{k^n (k-\alpha)}{1-\alpha} - k^m z_0^{k-1}\right] a_k \le 1 - \rho$$

Hence (5.3) holds if

$$k^j (k-\rho)|z|^{k-1} \le (1-\rho)\frac{k^n (k-\alpha)}{1-\alpha}, \; k \in \{2, 3, \ldots\},$$

or if $|z| \le r^*$.

The sharpness follows by taking the function f_k given by

$$\frac{D^{j+1} f_k(z)}{D^j f_k(z)} = \frac{k^n (k-\alpha) z - k^{j+1}(1-\alpha) z^k}{k^n (k-\alpha) z - k^j (1-\alpha) z^k} =$$

$$= \frac{1 - k\beta z^{k-1}}{1 - \beta z^{k-1}} = h_k(z), \; \beta = \frac{1-\alpha}{k-\alpha} k^{j-n}$$

and

$$\inf_{|z| \le r} \operatorname{Re} h_k(z) = h_k(r).$$

From $h_k(r) = \rho$ we obtain $r^{k-1} = \dfrac{(1-\rho)(k-\alpha)}{(1-\alpha)(k-\rho)} k^{n-j}$.

Remarks. 1). The conclusion of Theorem 7 is independent of the point z_0.

2). If we choose $n = 0$ and $j = n+1$, then we obtain that a starlike function of order α is convex of order ρ in the disc $\{z; |z| < r^*\}$, where

$$r^* = \inf_{k \in \{2,3,\dots\}} \left\{ \frac{(k-\alpha)(1-\rho)}{(1-\alpha)(k-\rho)k} \right\}^{\frac{1}{k-1}}.$$

If $\alpha = \rho$, then $\inf_{k \in \{2,3,\dots\}} \{1/k\}^{1/(k-1)} = 1/2$ and we obtain that a starlike function with negative coefficients is convex in the disc $\{z; |z| < 1/2\}$ (and the result is sharp).

THEOREM 8. *Let j be an integer, $0 \le j \le n-1$; then $T_m^{**}(n,\alpha,z_0) \subset T_m^{**}(j,\beta,z_0)$, where*

$$\beta = \beta(n,j,\alpha) = 1 - \frac{1-\alpha}{2^{n-j}(2-\alpha) - (1-\alpha)}.$$

The result is sharp.

Proof. Let f be a function in $T_m^{**}(n,\alpha,z_0)$. By using a method similar to that used in the proof of Theorem 7 we obtain (see (5.4) for $\rho=\beta$ and $|z| = 1$)

$$k^j(k-\beta) \le (1-\beta)\frac{k^n(k-\alpha)}{1-\alpha}, \quad k \in \{2,3,\dots\},$$

or equivalently,

$$\beta \le g(k) = \frac{k^{n+1-j} - \alpha k^{n-j} - (1-\alpha)k}{k^{n+1-j} - \alpha k^{n-j} - (1-\alpha)}, \quad k \in \{2,3,\dots\} \tag{5.5}$$

and it is sufficient to find the largest β which satisfies (5.5) for all k.

Let $g(x)$ be a real function, $x \in [2,\infty)$ and $g(x) = \dfrac{\varphi(x) - (1-\alpha)x}{\varphi(x) - (1-\alpha)}$, where

$\varphi(x) = x^{n+1-j} - \alpha x^{n-j}$; then

$$g'(x) = \frac{\alpha - 1}{[\varphi(x) - (1-\alpha)]^2} \cdot \psi(x),$$

where $\psi(x) = \varphi(x) + \varphi'(x) - x\varphi'(x) + \alpha - 1$ and $\psi'(x) = (1-x)\varphi''(x)$.

But $\varphi''(x) > 0$ for $x \in [2,\infty)$, hence $\psi'(x) < 0$ and ψ is a decreasing function and then $\psi(x) \le \psi(2) < 0$.

We obtain $g'(x) > 0$, hence $g(2) \le g(x)$, $x \in [2,\infty)$ and $\beta(n,j,\alpha) = g(2) \le g(k)$, $k \in \{2,3,...\}$.

The extremal function is

$$f(z) = \frac{2^n(2-\alpha)z - (1-\alpha)z^2}{2^n(2-\alpha) - 2^m(1-\alpha)z_0}.$$

Remarks. 1) If $j = n$-1, then $\beta(n,n\text{-}1,\alpha) = 2/(3\text{-}\alpha)$. For $n = 1$ we deduce that if a function with negative coefficients and two fixed points is convex of order α, then it is also starlike of order $2/(3\text{-}\alpha)$.

2). If $n = 1$ and $\alpha < 0$, the conclusion of Theorem 8 holds too. For instance, if there exist $z_0 \in [0,1)$ and $\alpha \in (-\infty,1)$ such that $f(z_0) = z_0$ and

$$\text{Re}\left\{ \frac{zf''(z)}{f'(z)} + 1 \right\} > \alpha,$$

then f is starlike of the positive order $2/(3\text{-}\alpha)$.

6. Convex Families. Suppose B is a nonempty subset of the real interval $(0,1)$. We define the family $T_m^{**}(n,\alpha,B)$ by

$$T_m^{**}(n,\alpha,B) = \bigcup_{z_i \in B} T_m^{**}(n,\alpha,z_i).$$

If B has only one element, then $T_m^{**}(n,\alpha,B)$ is a convex family (Theorem 4 and Corollary 4.2). It is interesting to investigate this class for other subsets B. We shall make use of the following:

LEMMA 1. *If* $f(z) \in T_m^{**}(n,\alpha,z_0) \cap T_m(n,\alpha,z_1)$, *where* z_0 *and* z_1 *are distinct positive numbers, then* $f(z) = z$.

Proof. Let $f(z) \in T_m^{**}(n,\alpha,z_0) \cap T_m(n,\alpha,z_1)$ and let $f(z) = a_1 z - \sum_{k=2}^{\infty} a_k z^k$, $a_1 > 0$; $a_k \geq 0$. Then

$$a_1 = 1 + \sum_{k=2}^{\infty} k^m a_k z_0^{k-1} = 1 + \sum_{k=2}^{\infty} k^m a_k z_1^{k-1}.$$

Since $a_k \geq 0$, $z_0 > 0$ and $z_1 > 0$, this implies $a_k = 0$ for $k \geq 2$.

Remark. The fact that if $f(z) \in T_m^{**}(n,\alpha,z_0)$ and $f(z)$ is odd, then $f(z) \in T_m^{**}(n,\alpha,-z_0)$ shows that the conclusion of the lemma doen't have to follow if we relax the condition that the fixed points be positive.

THEOREM 9. *If B is contained in the interval* $(0,1)$, *then* $T_m^{**}(n,\alpha,B)$ *is a convex family if and only if B is connected.*

Proof. Let B be connected. Suppose $z_0, z_1 \in B$ with $z_0 \leq z_1$. To prove $T_m^{**}(n,\alpha,B)$ is a convex family it suffices to show, for $f(z) = a_1 z - \sum_{k=2}^{\infty} a_k z^k \in T_m^{**}(n,\alpha,z_0)$, $g(z) = b_1 z - \sum_{k=2}^{\infty} b_k z^k \in T_m^{**}(n,\alpha,z_1)$ and $0 \leq \lambda \leq 1$, that there exists z_2 $(z_0 \leq z_2 \leq z_1)$ such that $h(z) = \lambda f(z) + (1-\lambda)g(z)$ is in $T_m^{**}(n,\alpha,z_2)$. Since $f(z)$

$\in T_m^{**}(n,\alpha,z_0)$ and $g(z) \in T_m^{**}(n,\alpha,z_1)$, we have $a_1 = 1 + \sum_{k=2}^{\infty} k^m a_k z_0^{k-1}$ and $b_1 = 1 + \sum_{k=2}^{\infty} k^m b_k z_1^{k-1}$. Therefore we have

$$\begin{aligned} t(z) &= \frac{D^m h(z)}{z} = \lambda a_1 + (1-\lambda) b_1 - \lambda \sum_{k=2}^{\infty} k^m a_k z^{k-1} - (1-\lambda) \sum_{k=2}^{\infty} k^m b_k z^{k-1} \\ &= 1 + \lambda \sum_{k=2}^{\infty} k^m a_k \left(z_0^{k-1} - z^{k-1}\right) + (1-\lambda) \sum_{k=2}^{\infty} k^m b_k \left(z_1^{k-1} - z^{k-1}\right). \end{aligned} \tag{6.1}$$

$t(z)$ is continuous function of z and is real when z is real with $t(z_0) \geq 1$ and $t(z_1) \leq 1$. Hence $t(z_2) = 1$ for some z_2, $z_0 \leq z_2 \leq z_1$. This implies $D^m h(z_2) = z_2$ for some z_2, $z_0 \leq z_2 \leq z_1$, that is $h(z) \in T_m(z_2)$. Now, from (6.1) and $h(z_2) = z_2$, we have

$$\begin{aligned} &\sum_{k=2}^{\infty} \left\{k^n(k-\alpha) - k^m(1-\alpha) z_2^{k-1}\right\} \{\lambda a_k + (1-\lambda) b_k\} \\ &= \lambda \sum_{k=2}^{\infty} \left\{k^n(k-\alpha) - k^m(1-\alpha) z_0^{k-1}\right\} a_k + (1-\lambda) \sum_{k=2}^{\infty} \{k^n(k-\alpha) \\ &- k^m(1-\alpha) z_1^{k-1}\Big\} b_k + \lambda(1-\alpha) \sum_{k=2}^{\infty} \left\{z_0^{k-1} - z_2^{k-1}\right\} k^m a_k \\ &+ (1-\lambda)(1-\alpha) \sum_{k=2}^{\infty} \left\{z_1^{k-1} - z_2^{k-1}\right\} k^m b_k \\ &= \lambda \sum_{k=2}^{\infty} \left\{k^n(k-\alpha) - k^m(1-\alpha) z_0^{k-1}\right\} a_k + (1-\lambda) \sum_{k=2}^{\infty} \{k^n(k-\alpha) \\ &- k^m(1-\alpha) z_1^{k-1}\Big\} b_k \leq (1-\alpha)\lambda + (1-\alpha)(1-\lambda) = (1-\alpha) \end{aligned}$$

by Theorem 2, since $f(z) \in T_m^{**}(n,\alpha,z_0)$ and $g(z) \in T_m^{**}(n,\alpha,z_1)$. Hence we have $h(z) \in T_m^{**}(n,\alpha,z_2)$, by Theorem 2. Since z_0, z_1 and λ are arbitrary, the family $T_m^{**}(n,\alpha,B)$ is convex.

Conversely, if B is not connected, then there exists z_0, z_1 and z_2 such that $z_0, z_1 \in B$, $z_2 \notin B$ and $z_0 < z_2 < z_1$. Assume $f(z) \in T_m^{**}(n,\alpha,z_0)$ and

$g(z) \in T_m^{**}(n,\alpha,z_1)$ are not both the identity function. Then, for fixed z_2 and $0 \le \lambda \le 1$, we have from (6.1).

$$\begin{aligned} t(z_2) = s(\lambda) &= 1 + \lambda \sum_{k=2}^{\infty} k^m a_k \left(z_0^{k-1} - z_2^{k-1}\right) \\ &+ (1-\lambda) \sum_{k=2}^{\infty} k^m b_k \left(z_1^{k-1} - z_2^{k-1}\right). \end{aligned}$$

Since $s(0) > 1$ and $s(1) < 1$, there must exist λ_0, $0 < \lambda_0 < 1$ such that $s(\lambda_0) = 1$ or $h(z_2) = z_2$, where $h(z) = \lambda_0 f(z) + (1-\lambda_0)g(z)$. Thus $h(z) \in T_m^{**}(n,\alpha,z_2)$. From Lemma 1 we have $h(z) \notin T_m^{**}(n,\alpha,B)$, since $z_2 \notin B$ and $h(z) \neq z$. This implies that the family $T_m^{**}(n,\alpha,B)$ is not convex which is a contradiction.

THEOREM 10. *Let* $[z_0, z_1] \subset (0,1)$. *Then the extreme points of* $T_m^{**}(n,\alpha,[z_0,z_1])$ *are* z,

$$f_k(z) = \frac{k^n(k-\alpha)z - (1-\alpha)z^k}{\{k^n(k-\alpha) - k^m(1-\alpha)z_0^{k-1}\}} \quad (k = 2,3,\dots),$$

and

$$g_k(z) = \frac{k^n(k-\alpha)z - (1-\alpha)z^k}{\{k^n(k-\alpha) - k^m(1-\alpha)z_1^{k-1}\}} \quad (k = 2,3,\dots).$$

Proof. Since $T_m^{**}(n,\alpha,[z_0,z_1])$ is convex, a function $h(z) \in T_m^{**}(n,\alpha,z_2)$, $z_0 \le z_2 \le z_1$, can only be an extreme point of $T_m^{**}(n,\alpha,[z_0,z_1])$ if $h(z)$ is an extreme point of $T_m^{**}(n,\alpha,z_2)$. Therefore to prove the theorem it suffices to show, when $h(z)$ is an extreme point of $T_m^{**}(n,\alpha,z_2)$, that $h(z)$ is an extreme point of $T_m^{**}(n,\alpha,[z_0,z_1])$ if and only if $z_2 = z_0$ or $z_2 = z_1$. Let $h_k(z)$ be an extreme point of $T_m^{**}(n,\alpha,z_2)$. Then

$$h_k(z) = \frac{k^n(k-\alpha)z - (1-\alpha)z^k}{\{k^n(k-\alpha) - k^m(1-\alpha)z_2^{k-1}\}} \quad (k = 2,3,...).$$

Define

$$p_k(\lambda,z) = \lambda\left(\frac{k^n(k-\alpha)z - (1-\alpha)z^k}{\{k^n(k-\alpha) - k^m(1-\alpha)z_0^{k-1}\}}\right)$$
$$+ (1-\lambda)\left(\frac{k^n(k-\alpha)z - (1-\alpha)z^k}{\{k^n(k-\alpha) - k^m(1-\alpha)z_1^{k-1}\}}\right).$$

When $z_0 < z_2 < z_1$, we have $p_k(1,z) < h_k(z) < p_k(0,z)$ for z real and positive. Hence there exists a λ_0, $0 < \lambda_0 < 1$ such that $p_k(\lambda_0,z) = h_k(z)$. This implies the coefficients of $p_k(\lambda_0,z)$ agree with the coefficients of $h_k(z)$ for λ_0 so that

$$\frac{\lambda_0}{k^n(k-\alpha) - k^m(1-\alpha)z_0^{k-1}} + \frac{1-\lambda_0}{k^n(k-\alpha) - k^m(1-\alpha)z_1^{k-1}}$$
$$= \frac{1}{k^n(k-\alpha) - k^m(1-\alpha)z_2^{k-1}}.$$

That is, $p_k(\lambda_0,z) = h_k(z)$ throughout the unit disc U when

$$\lambda_0 = \frac{k^n(k-\alpha) - k^m(1-\alpha)z_0^{k-1}}{\{k^n(k-\alpha) - k^m(1-\alpha)z_2^{k-1}\}} \cdot \left(\frac{z_1^{k-1} - z_2^{k-1}}{z_1^{k-1} - z_0^{k-1}}\right).$$

This shows that $h_k(z)$ is expressed as a linear combination of $f_k(z)$ and $g_k(z)$ when $z_0 < z_2 < z_1$. Hence $h_k(z)$ can not be an extreme point of $T_m^{**}(n,\alpha,[z_0,z_1])$ when $z_0 < z_2 < z_1$ and $h_k(z) \in T_m^{**}(n,\alpha,z_2)$.

Now we have only to show that $f_k(z)$ and $g_k(z)$ cannot be expressed respectively, as a linear combination of extreme functions in $T_m^{**}(n,\alpha,z)$, $z_0 <$ $z \leq z_1$ and in $T_m^{**}(n,\alpha,z)$, $z_0 \leq z < z_1$. This really follows from the following I

z positive and $0 \leq \lambda \leq 1$, we have

$$f_k(z) < \lambda\left(\frac{k^n(k-\alpha)z-(1-\alpha)z^k}{\{k^n(k-\alpha)-k^m(1-\alpha)z_3^{k-1}\}}\right)$$

$$+(1-\lambda)\left(\frac{k^n(k-\alpha)z-(1-\alpha)z^k}{\{k^n(k-\alpha)-k^m(1-\alpha)z_4^{k-1}\}}\right)$$

$$(z_0 < z_3 \leq z_1,\ z_0 < z_4 \leq z_1)$$

and

$$g_k(z) > \lambda\left(\frac{k^n(k-\alpha)z-(1-\alpha)z^k}{\{k^n(k-\alpha)-k^m(1-\alpha)z_5^{k-1}\}}\right)$$

$$+(1-\lambda)\left(\frac{k^n(k-\alpha)z-(1-\alpha)z^k}{\{k^n(k-\alpha)-k^m(1-\alpha)z_6^{k-1}\}}\right)$$

$$(z_0 \leq z_5 < z_1,\ z_0 \leq z_6 < z_1).$$

Hence the proof is completed.

Using the method of proof in Theorem 16, we obtain the following:

COROLLARY 10.1. *If* $0 < z_0 < z_1 < 1$, *the closed convex hull of* $T_m^{**}(n,\alpha,\{z_0,z_1\})$ *is* $T_m^{**}(n,\alpha,[z_0,z_1])$.

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A SIMPLE CRITERION FOR CONVEXITY

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REZUMAT. - Un criteriu de convexitate. În această notă, utilizând tehnica subordonărilor diferenţiale, vom obţine un criteriu simplu de convexitate pentru funcţii analitice.

1. Introduction. Let $\mathcal{H}(U)$ be the set of functions that are analytic in the unit disc U. We denote by $\mathcal{H}[a,n]$, $\mathcal{A}_n$, $n \geq 1$, the following classes:

$$\mathcal{H}[a,n] = \left\{f \in \mathcal{H}(U),\ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots,\ z \in U\right\}$$

$$\mathcal{A}_n = \left\{f \in \mathcal{H}(U),\ f(z) = z + a_{n+1} z^{n+1} + \ldots,\ z \in U\right\}.$$

Let f be analytic in the unit disc. The function f, with $f'(0) \neq 0$ is convex if and only if $\operatorname{Re}(zf''(z)/f'(z) + 1) > 0$ in U. The function f, with $f(0) = 0$ and $f'(0) \neq 0$ is starlike if and only if $\operatorname{Re}(zf'(z)/f(z)) > 0$ in U.

If f and g are analytic in U, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$ if g is univalent in U, $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

The purpose of this note is to establish a sufficient condition for convexity, concerning functions of class $\mathcal{A}_n$. We use the differential

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subordination technique.

This convexity criterion involves the second and third derivative of considered functions, and it may be useful in the cases in which these derivatives have a simply form.

2. Preliminary. We will need the following results:

LEMMA 1 [4]. *Let $a \in \mathbf{C}$, $p \in \mathcal{H}[a,n]$, $h \in \mathcal{H}(U)$. If the function h is starlike in U and $zp'(z) \prec h(z)$ then*

$$p(z) \prec q(z) = p(0) + \frac{1}{n}\int_0^z \frac{h(t)}{t}\,dt.$$

LEMMA 2. *Let the function ψ: $\mathbf{C}^2 \times U \to \mathbf{C}$ satisfy the condition:*

(A) Re $\psi(ix,y;z) \le 0$, *for all real x and y with $y \le -\frac{n}{2}(1+x^2)$ and for all $z \in U$.*

If p belongs to the class $\mathcal{H}[1,n]$ and $\operatorname{Re}\psi(p, zp', z) > 0$ for $z \in U$ then Re $p(z) > 0$, $z \in U$.

More general forms of this lemma may be found in [1] and in [2].

3. Main results.

THEOREM. *Let $f \in \mathcal{A}^n$ and*

$$0 < \alpha_n \le \frac{n+2}{2\left(1 + \frac{4}{n}\ln 2 + \frac{n+2}{n^2}\cdot\frac{\pi^2}{12}\right)} \qquad (1)$$

If

$$\operatorname{Re}\,[z^2 f'''(z) + zf''(z)] > -\alpha_n,\ z \in U \tag{2}$$

then f is convex.

Proof. If we note $k(z) = zf''(z)$, condition (2) can be written $\operatorname{Re} zk'(z) > -\alpha_n$ which is equivalent to $zk'(z) \prec \frac{2\alpha_n z}{1-z}$, $z \in U$. Since the function $h(z) = \frac{2\alpha_n z}{1-z}$ is starlike, by Lemma 1 we deduce

$$k(z) \prec q(z) = -\frac{2\alpha_n}{n}\log(1-z)$$

and hence

$$zf''(z) \prec q(z) = -\frac{2\alpha_n}{n}\log(1-z) \tag{3}$$

Since q given by (3) is convex and q has real coefficients then Re $q(z) > q(-1)$ and so

$$\operatorname{Re} zf''(z) > \beta_n = -\frac{2\alpha_n}{n}\ln 2,\ z \in U. \tag{4}$$

If we let now $k_1 = f'$, then $k_1 \in \mathcal{H}[1,n]$ and using again Lemma 1 we deduce that

$$k_1(z) \prec Q(z) = 1 - \frac{2\alpha_n}{n^2}\int_0^z \frac{\log(1-t)}{t}\,dt.$$

The function Q is convex (we used Alexander's integral operator), it has real coefficients and so

$$\operatorname{Re} Q(z) > Q(-1) = 1 - \frac{2\alpha_n}{n^2}\int_0^{-1}\frac{\log(1-t)}{t}dt = 1 - \frac{2\alpha_n}{n^2}\int_0^{1}\frac{\log(1-t)}{t}dt = 1 - \frac{\alpha_n}{n^2}\cdot\frac{\pi^2}{6}$$

Hence

$$\operatorname{Re} f'(z) > \gamma_n = 1 - \frac{\alpha_n}{n^2}\cdot\frac{\pi^2}{6},\ z \in U. \tag{5}$$

A simple calculation, according to (1), shows that $\gamma_n > 0$ and

$$\operatorname{Re} f'(z) > 0 \text{ for } z \in U. \tag{6}$$

If we let

$$p(z) = \frac{zf''(z)}{f'(z)} + 1 \text{ for } z \in U, \tag{7}$$

then p is an analytic in U, because of (6).

From (7) we obtain first

$$f'(p-1) = zf''. \tag{8}$$

Derivating (8), multiplying by z and substituting the expression of zf'' from (8) we deduce

$$f'[p^2 - 1 + zp'] = z^2 f''' + 3zf''. \tag{9}$$

Taking the real parts in (9) and using (2) and (4) we obtain

$$\operatorname{Re}[f'(p^2 - 1 + zp')] > -\alpha_n + 2\beta_n. \tag{10}$$

Let

$$\psi(p, zp'; z) = f'(z)(p^2(z) - 1 + zp'(z)) \tag{11}$$

We will show that (1) is fulfilled then the function ψ defined by (11) satisfies condition (A) of Lemma 2.

Indeed, if x and y are real, $y \le -\frac{n}{2}(1 + x^2)$ by using (1), (4) and (5) we obtain, after a short calculation, that

$$\operatorname{Re} \psi(ix, y; z) = \operatorname{Re}[f'(x^2 - 1 - y)] \le -\left(\frac{n}{2} + 1\right)\gamma_n \le -\alpha_n + 2\beta_n.$$

Hence the conclusion of Lemma 2 is true and we deduce that f is a convex function.

COROLLARY 1. *Let* $f \in \mathcal{A}_1$ *and*

$$0 < \alpha_1 \leq \frac{3}{2\left(1 + 4\ln 2 + \frac{\pi^2}{4}\right)} = 0.240385\ldots$$

If

$$\mathrm{Re}\,[z^2 f'''(z) + zf''(z)] > -\alpha_1,\ z \in U$$

then f is convex.

COROLLARY 2. *Let* $f \in \mathcal{A}_2$ *and*

$$0 < \alpha_2 \leq \frac{2}{1 + 2\ln 2 + \frac{\pi^2}{12}} = 0.6232\ldots$$

If

$$\mathrm{Re}\,[z^2 f'''(z) + zf''(z)] > -\alpha_2,\ z \in U$$

then f is convex.

We close this note by considering the following example:

Example. If λ is a real number which satisfies $0 < \lambda < 0.6232\ldots$ and if f is defined by

$$f(z) = z - \lambda \int_0^z \int_0^t \frac{\log(1 + u^2)}{u}\, du\, dt$$

then f is convex.

Indeed, f belongs to $\mathcal{A}_2$

$$z^2 f'''(z) + zf''(z) = -\frac{2\lambda z^2}{1 + z^2}$$

and

$$\mathrm{Re}\left(-\frac{2\lambda z^2}{1+z^2}\right) > -\lambda > -0.6232...$$

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SOME SUFFICIENT CONDITIONS FOR UNIVALENCE IN THE UPPER HALF-PLANE

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Dedicated to Professor I. Kolumbán on his 60^{th} anniversary

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REZUMAT. - Condiţii suficiente de univalenţă în semiplanul superior. Lucrarea se ocupă de un criteriu de univalenţă şi extensibilitate cvaziconformă a unei funcţii analitice f în semiplanul superior.

Introduction. Let $G \subset \mathbf{C}$ be a Jordan domain with quasiconformal boundary curve, let $\lambda: G \to \bar{\mathbf{C}} \setminus \bar{G}$ be a quasiconformal reflection in ∂G, and let Δ be the upper half-plane, i.e. $\Delta = \{z \in \mathbf{C} \,/\, \mathrm{Im}\, z > 0\}$.

This paper is concerned with a criterion for univalence and quasiconformal extensibility of an analytic function f in Δ.

We shall need the following theorem to prove our results:

THEOREM 1 [3] *Let λ be generated by the q - chain $h(z,t)$, $q \in (0,1)$; let $z_0 = h(0,0)$. Let a,b,c,d be analytic in G, $a(z)d(z) - b(z)c(z) \neq 0$, $z \in G$ and*

$$\phi(z,w) = \frac{a(z) + (w-z)\cdot b(z)}{c(z) + (w-z)\cdot d(z)}, \; z \in G, \; w \in \bar{\mathbf{C}} \tag{1}$$

Let f be meromorphic and locally univalent in G, $f(z) = \phi(z,z) = a(z)/c(z)$, $z \in G$ and suppose that

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$$\left| \frac{\phi_z'}{\phi_w'}(z,\lambda(z)) + \lambda_z'(z) \right| \le k \left| \lambda_{\bar z}'(z) \right| \text{ a.e. } in\ G,\ k \le 1 \tag{2}$$

Then f is univalent; if k < 1 f has a quassiconformal extension F to $\bar{\mathbf{C}}$ given by

$$F(\lambda(z)) = \phi(z,\lambda(z)),\ z \in G \tag{3}$$

The Theorem 1 can be transferred to unbounded domains G as well by means of linear transformations.

COROLLARY 1 [3] *Let λ_0 be a Löewner reflection in ∂G_0 (that is, generated by a Löewner chain), and let T: $\bar{\mathbf{C}} \to \bar{\mathbf{C}}$ be a linear transformation $G = T(G_0)$, $\lambda = T \circ \lambda_0 \circ T^{-1}$. Then the conclusion of the Theorem 1 also holds for G and λ.*

Mains results. The Theorem 1 and the Corollary 1 cover, in particular, the standard cases $G = U = \{z \in \mathbf{C}/|z|<1\}$, $\lambda(z) = 1/\bar z$, $G = H = \{z \in \mathbf{C}/\mathrm{Re} z > 0\}$, $\lambda(z) = -\bar z$ and $G = \Delta = \{z \in \mathbf{C}/\mathrm{Im}\, z > 0\}$, $\lambda(z) = \bar z$. If we consider the case $G = \Delta$ and $\lambda(z) = \bar z$, we obtain the following corollary of the Theorem 1:

COROLLARY 2. *Let a,b,c,d: $\Delta \to \mathbf{C}$ be analytic functions in Δ such that $a(z)\cdot d(z) - b(z)\cdot c(z) \neq 0$, $z \in \Delta$ and*

$$\phi(z,w) = \frac{a(z) + (w-z)\cdot b(z)}{c(z) + (w-z)\cdot d(z)},\ z \in \Delta,\ w \in \bar{\mathbf{C}}$$

Let f: $\Delta \to \mathbf{C}$ be an analytic and locally univalent function in Δ, $f(z) = \phi(z,z)$ and suppose that

$$\left| \frac{\phi_z'}{\phi_w'}(z,\bar z) \right| \le k,\ \text{a.e } in\ \Delta,\ k \le 1 \tag{4}$$

Then f is univalent in Δ. If k < 1, f has a quasiconformal extension F to $\bar{\mathbf{C}}$ given

by
$$F(\bar{z}) = \phi(z,\bar{z}),\ z \in \Delta \tag{5}$$

Using Corollary 2 and suitable choices of $\phi(z,w)$ we can obtain the next univalence criteria.

THEOREM 2. *Let f, h, τ: $\Delta \to \mathbf{C}$ be analytic functions in Δ, such that $f'(z)\cdot h'(z)\cdot \tau'(z) \neq 0$, $z \in \Delta$. If*

$$\left| \frac{(\operatorname{Im} z)^2}{\tau'(z)} \cdot S_h(z) - \frac{(\operatorname{Im} z)^2}{\tau'(z)} \cdot \frac{f''(z)}{f'(z)} \cdot \frac{h''(z)}{h'(z)} - (i \cdot \operatorname{Im} z) \left[\frac{f''(z)}{f'(z)} + \frac{h''(z)}{h'(z)} - \frac{\tau''(z)}{\tau'(z)} \right] + \frac{\tau'(z) - 1}{2} \right| \leq \frac{k}{2},\ z \in \Delta,\ k \leq 1 \tag{6}$$

then f is an univalent function in Δ and if $k < 1$ f has a quasiconformal extension to $\bar{\mathbf{C}}$

Proof. The conclusion of this theorem follow by applying Corollary 2 to the function

$$\phi(z,w) = f(z) + \frac{(w-z)f'(z)}{\tau'(z) + \dfrac{w-z}{2} \cdot \dfrac{h''(z)}{h'(z)}},\ z \in \Delta,\ w \in \bar{\mathbf{C}}$$

If $k < 1$, the quasiconformal extension F to $\bar{\mathbf{C}}$ of the function f is given by

$$F(\bar{z}) = \phi(z,\bar{z}) = f(z) - \frac{(2i \cdot \operatorname{Im} z) \cdot f'(z)}{\tau'(z) - (i \cdot \operatorname{Im} z) \cdot \dfrac{h''(z)}{h'(z)}},\ z \in \Delta.$$

Remarks.

1) If $\tau' = 1$ and $h(z) = z$ in (6), we obtain the univalence condition of Becker [2] for the half-plane Δ:

$$\operatorname{Im} z \cdot \left| \frac{f''(z)}{f'(z)} \right| \leq \frac{k}{2},\ z \in \Delta,\ k \leq 1. \tag{7}$$

2) If $\tau' = 1$ and $h' = (f')^{-1}$, (6) gives the univalence condition of Nehari [1] for the half-plane Δ:

$$(\operatorname{Im} z)^2 \cdot |S_f(z)| \le \frac{k}{2}, \; z \in \Delta, \; k \le 1 \tag{8}$$

3) If $h' = g' \cdot (f')^{-1}$, where $g: \Delta \to \mathbf{C}$ is an analytic function in Δ and $g'(z) \neq 0$, $z \in \Delta$ in (6), we obtain the univalence condition of Epstein [4] for the halfplane Δ:

$$\left| \frac{2}{\tau'(z)} \cdot (\operatorname{Im} z)^2 \cdot [S_f(z) - S_g(z)] + 2i \operatorname{Im} z \cdot \left[\frac{g''(z)}{g'(z)} - \frac{\tau''(z)}{\tau'(z)} \right] + 1 - \tau'(z) \right| \le k, \; z \in \Delta, \; k \le 1 \tag{9}$$

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VARIATIONAL METHODS FOR SOME OPERATOR EQUATIONS IN HILBERT SPACES

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REZUMAT. - Metode variaţionale pentru ecuaţii operatoriale în spaţii Hilbert. Sunt considerate două clase de ecuaţii asociate unor operatori liniari pe un spaţiu Hilbert şi se pun în evidenţă unele relaţii între soluţiile lor şi punctele de minim ale unor funcţionale pătratice asociate.

Abstract. In this paper we shall consider two classes of equations associated to nondense defined linear operators on a Hilbert space and we shall point out some natural connections between the solutions of these equations and the elements which minimize some quadratic functionals.

0. Introduction. Let $(H;(\, , \,))$ be a (real) complex Hilbert space. The following variational theorem is well-known (see [4] or [1], p.125):

THEOREM 0.1. *Assume that* $\mathcal{D}(A) \subset H$ *is dense in* H, A: $\mathcal{D}(A) \to H$ *is a (symmetric) strictly positive linear operator on* $\mathcal{D}(A)$ *and* f *is a given element in* H. *Then the following statements are true*:

(i) *If the equation*:

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$(A;f)$ $$Au = f,\ u \in \mathcal{D}(A),$$

has a solution, then this solution is unique and minimizes the following quadratic functional:

$$F(u) := (Au,u) - 2\,\mathrm{Re}\,(f,u),\ u \in \mathcal{D}(A). \tag{0.1}$$

(ii) *If $u_0 \in \mathcal{D}(A)$ minimizes the functional F on $\mathcal{D}(A)$, then u_0 is the unique solution of the equation $(A;f)$.*

For some developements of this fact to K-positive definite operators or non-linear operators with K-symmetrical and K-positive definite differential see [3], [5-6] and [2] or the monography [1] where further references are given.

In the sequel we shall study the equation $(A;f)$ for some classes of nondense defined linear operators in a Hilbert space. Applications for special equations associated to some linear operators in Hilbert spaces are also given.

1. A variational method for nondense defined and positive operators. Let $(H;(\ ,\ ))$ be a (real) complex Hilbert space, $\mathcal{D}(A)$ be a given linear subspace in H, nondense in general, and $A: \mathcal{D}(A) \to H$ be a (symmetric) positive operator on $\mathcal{D}(A)$.

Further on, we shall consider the next operator equation:

$(A;f)$ $$Au = f,\ u \in \mathcal{D}(A) \text{ and } f \in H,$$

and we shall study this equation in connection to other equation in terms of variational methods.

Put, for a given operator A as above:

$$L(A) := \{u \in \mathcal{D}(A) \mid (Au, u) = 0\}$$

and

$$Q(A) := \{u \in \mathcal{D}(A) \mid Au \in \mathcal{D}(A)^{\perp}\}.$$

Then $L(A)$ and $Q(A)$ are linear subspaces in $\mathcal{D}(A)$ and the following inclusion holds.

LEMMA 1.1. *If* $A: \mathcal{D}(A) \subset H \to H$ *is as above, then:*

$$\mathrm{Ker}(A) \subseteq L(A) \subseteq Q(A). \tag{1.1}$$

Proof. The first inclusion is obvious. Let prove the second. Suppose $u_0 \in L(A)$, i.e., $(Au_0, u_0) = 0$.

For all $\lambda \in K$ and $u \in \mathcal{D}(A)$ we have:

$$(\lambda Au + Au_0, \lambda u + u_0) \geq 0$$

what is equivalent to:

$$|\lambda|^2 (Au, u) + 2 \operatorname{Re} [\lambda (Au, u)] \geq 0 \text{ for all } \lambda \in K, \tag{1.2}$$

since $(Au_0, u_0) = 0$.

Put $\lambda = t \in \mathbf{R}$. Then by (1.2) we derive:

$$t^2 (Au, u) + 2t \operatorname{Re} (Au, u_0) \geq 0 \text{ for all } t \in \mathbf{R},$$

what gives $\operatorname{Re} (Au, u_0) = 0$.

Put $\lambda = - it$, $t \in \mathbf{R}$. Then by (1.2) we also have:

$$t^2 (Au, u) + 2t \operatorname{Im} (Au, u_0) \geq 0 \text{ for all } t \in \mathbf{R},$$

what implies $\operatorname{Im} (Au, u_0) = 0$.

Consequently:

$$0 = (Au, u_0) = (u, Au_0) \text{ for all } u \in \mathcal{D}(A), \text{ i.e., } Au_0 \in \mathcal{D}(A)^{\perp},$$

and the lemma is proved.

Remark 1.2. By the symmetrie of the operator A we have:

$$Q(A) = \{u \in \mathcal{D}(A) \mid u \in R(A)^{\perp}\},$$

where $R(A)$ denotes the range of the operator A.

COROLLARY 1.3. *If A is as above and dense defined, then we have:*

$$\mathrm{Ker}(A) = L(A) = Q(A). \tag{1.3}$$

Proof. If $u_0 \in Q(A)$, then $(Au_0,u) = 0$ for all $u \in \mathcal{D}(A)$ and by the density of $\mathcal{D}(A)$ in H we derive that $(Au_0,z) = 0$ for all $z \in H$, i.e., $Au_0 = 0$ and $Q(A) \subseteq \mathrm{Ker}(A)$.

Now, let $S_{(A;f)}$ be the set of all solutions of the operator equation $(A;f)$ for a given element f in H. Then the following proposition holds:

PROPOSITION 1.4. *If u_0 is a solution of $(A;f)$ then:*

$$S_{(A;f)} = u_0 + \mathrm{Ker}(A) \subseteq u_0 + L(A) \subseteq u_0 + Q(A). \tag{1.4}$$

The proof is obvious and we omit the details.

Let also consider the following equation associated to the positive operator A:

$(A;f;\mathcal{D}(A)^{\perp})$ $\qquad Au \in f + \mathcal{D}(A)^{\perp}$, $u \in \mathcal{D}(A)$ and $f \in H$.

PROPOSITION 1.5. *If u_0 is a solution of equation $(A;f;\mathcal{D}(A)^{\perp})$, then the set $S_{(A;f;\mathcal{D}(A)^{\perp})}$ of all solutions of this equation is given by:*

$$S_{(A;f;\mathcal{D}(A)^{\perp})} = u_0 + Q(A). \tag{1.5}$$

Proof. "$\supseteq$". Let $u_1 = u_0 + w_0$ with $w_0 \in Q(A)$. Then for all $u \in \mathcal{D}(A)$ we have:

$$(Au_1 - f, u) = (Au_0 - f, u) + (Aw_0, u) = (Au_0 - f, u)$$

since $w_0 \in Q(A)$, then $u_1 \in \mathcal{S}_{(A;f;\mathcal{D}(A)^\perp)}$.

"$\subseteq$". Let u_1 be another solution of $(A;f;\mathcal{D}(A)^\perp)$. Then for all u we have:

$$(Au_0 - f, u) = 0, (Au_1 - f, u) = 0$$

what implies that:

$$(A(u_1 - u_0),u) = 0 \text{ for all } u \in \mathcal{D}(A),$$

i.e., $u_1 - u_0 \in Q(A)$ and then $u_1 \in u_0 + Q(A)$.

The proof is finished.

Remark 1.6. If u_0 is a solution of $(A;f;\mathcal{D}(A)^\perp)$, then:

$$u_0 + \mathrm{Ker}(f) \subseteq u_0 + L(A) \subseteq \mathcal{S}_{(A;f;\mathcal{D}(A)^\perp)}.$$

COROLLARY 1.7. *If A is as above and dense defined on H, then we have:*

$$\mathcal{S}_{(A;f;\mathcal{D}(A)^\perp)} = u_0 + \mathrm{Ker}(A) = u_0 + L(A) = u_0 + Q(A). \quad (1.7)$$

The proof is obvious by Proposition 1.5 and Corollary 1.3 and we shall omit the details.

To equation $(A;f)$ we can associate, as in the case of strictly positive or positive definite operators on a dense subspace of H (see for example [4] or [1] p.125) the following quadratic functional:

$$F_{(A;f)} : \mathcal{D}(A) \to K,\ F_{(A;f)}(u) := (Au,u) - 2\,\mathrm{Re}(f,u) \quad (1.8)$$

which will be called *energetic functional* associated to equation $(A;f)$.

It is easy to see that, for all $u,v \in \mathcal{D}(A)$ and $\lambda \in [0,1]$ we have (see for example [1], p.126):

$$\lambda F_{(A;f)}(u) + (1 - \lambda)F_{(A;f)}(v) - F_{(A;f)}(\lambda u + (1 - \lambda)v)$$
$$= \lambda(1 - \lambda)(A(u - v),u - v)$$

what implies that $F_{(A;f)}$ is convex if A is positive, strictly convex if A is strictly positive and uniformly convex if A is positive definite.

The following lemma is important in the sequel.

LEMMA 1.8. i. *If* $u_0 \in \mathcal{D}(A)$ *is a solution of equation* $(A;f)$ *then* u_0 *minimizes the energetic functional* $F_{(A;f)}$.

ii. *If* $u_0 \in \mathcal{D}(A)$ *minimizes the energetic functional* $F_{(A;f)}$ *then* u_0 *is a solution of equation* $(A;f;\mathcal{D}(A)^{\perp})$.

Proof. i. If $Au = f$, then for all $u \in \mathcal{D}(A)$ we have (as in [1] p.126):

$F_{(A;f)}(u) - F_{(A;f)}(u) = (Au,u) - 2\ \mathrm{Re}(f,u) - (A(u_0),u_0) +$

$2\ \mathrm{Re}(f,u_0) = (Au,u) - 2\ \mathrm{Re}(Au_0,f) - (Au_0,u_0) + 2\ \mathrm{Re}(Au_0,u_0) =$

$(A(u - u_0),u - u_0) \geq 0,$

i.e., u_0 minimizes the energetic functional $F_{(A;f)}$.

ii. Let $u_0 \in \mathcal{D}(A)$ minimize the energetic functional $F_{(A;f)}$. Then for all $u \in \mathcal{D}(A)$ and for all $\lambda \in K$ we have:

$$F_{(A;f)}(\lambda u + u_0) \geq F_{(A;f)}(u_0).$$

By simple computations we obtain:

$$|\lambda|^2(Au,u) + 2\ \mathrm{Re}[\lambda(u,Au_0 - f)] \geq 0 \qquad (1.10)$$

for all $u \in \mathcal{D}(A)$ and $\lambda \in K$.

Put $\lambda = t \in \mathbf{R}$. Then by (1.10) we get:

$$t^2(Au,u) + 2t\ \mathrm{Re}(u,Au_0 - f) \geq 0 \text{ for all } t \in \mathbf{R},$$

what implies $\text{Re}(u, Au_0 - f) = 0$.

Put $\lambda = -it$, $t \in \mathbf{R}$. Then by (1.10) we also derive:

$$t^2(Au,u) + 2t \text{ Im } (u, Au_0 - f) \geq 0 \text{ for all } t \in \mathbf{R},$$

what gives $\text{Im } (u, Au_0 - f) = 0$.

Consequently, $(u, Au_0 - f) = 0$ for all $u \in \mathcal{D}(A)$, i.e., $Au \in f + \mathcal{D}(A)^\perp$ and the lemma is proved.

Now, let denote by $\mathcal{M}_{(A;f)}$ the set of all elements in $\mathcal{D}(A)$ which minimize the energetic functional $F_{(A;f)}$. Then the following result of representation holds:

PROPOSITION 1.9. *If* $u_0 \in \mathcal{D}(A)$ *minimize the functional* $F_{(A;f)}$, *then we have:*

$$\mathcal{M}_{(A;f)} = u_0 + Q(A). \tag{1.11}$$

Proof. "$\subseteq$". If $u_0 \in \mathcal{D}(A)$ minimize $F_{(A;f)}$, then $u_0 \in \mathcal{S}_{(A;f;\mathcal{D}(A)^\perp)}$ and since $\mathcal{M}_{(A;f)} \subseteq \mathcal{S}_{(A;f;\mathcal{D}(A)^\perp)}$ and $\mathcal{S}_{(A;f;\mathcal{D}(A)^\perp)} = u_0 + Q(A)$, the inclusion is proved.

"$\supseteq$". Let $u_1 \in u_0 + Q(A)$. Then for all $u \in \mathcal{D}(A)$ we have

$$F_{(A;f)}(u) - F_{(A;f)}(u_1) = (Au,u) - 2 \text{ Re}(u,f) -$$
$$- (Au_1,u_1) + 2 \text{ Re}(u_1,f) = (Au - Au_1, u - u_1) +$$
$$2 \text{ Re } (u, Au_1 - f) + 2 \text{ Re } (u_1, f - Au_1).$$

Since $u_1 \in \mathcal{S}_{(A;f;\mathcal{D}(A)^\perp)}$ we conclude that $(u, Au_1 - f) = 0$ and $(u_1, f - Au_1) = 0$ what implies that:

$$F_{(A;f)}(u) - F_{(A;f)}(u_1) = (Au - Au_1, u - u_1) \geq 0,$$

i.e., u_1 minimize the functional $F_{(A;f)}$.

Remark 1.10. If $u_0 \in \mathcal{D}(A)$ minimize the functional $F_{(A;f)}$ then:

$$u_0 + \mathrm{Ker}(A) \subseteq u_0 + L(A) \subseteq \mathcal{M}_{(A;f)}. \tag{1.12}$$

COROLLARY 1.11. *If A is as above and $\mathcal{D}(A)$ is dense in H, then for all u_0 which minimize the functional $F_{(A;f)}$ we have:*

$$\mathcal{M}_{(A;f)} = u_0 + \mathrm{Ker}(A) = u_0 + L(A) = u_0 + Q(A). \tag{1.13}$$

Consequently, we have the following main result:

THEOREM 1.12. *Let $A : \mathcal{D}(A) \subset H \rightarrow H$ be a (symmetric) positive operator on $\mathcal{D}(A)$ and f a given element in H. Then we have the inclusion:*

$$\mathcal{S}_{(A;f)} \subseteq \mathcal{M}_{(A;f)} \subseteq \mathcal{S}_{(A;f;\mathcal{D}(A)^{\perp})}. \tag{1.14}$$

The proof is obvious from the previous considerations and we shall omit the details.

COROLLARY 1.13. *Let A be as in Theorem* 1.12 *and $\mathcal{D}(A)$ is dense in H. Then we have:*

$$\mathcal{S}_{(A;f)} = \mathcal{M}_{(A;f)} = \mathcal{S}_{(A;f;\mathcal{D}(A)^{\perp})}. \tag{1.15}$$

Proof. **Let u_0 be a solution of $(A;f;\mathcal{D}(A)^{\perp})$. Then we have $(Au_0 - f,u) = 0$ for all $u \in \mathcal{D}(A)$. Since $\mathcal{D}(A)$ is dense in H we conclude that $(Au_0 - f,z) = 0$ for all $z \in H$, i.e., $u_0 \in \mathcal{S}_{(A;f)}$.**

2. A variational method for general linear operators. Further on, H will be a Hilbert space over the real or complex number field and A: $\mathcal{D}(A) \subset H \rightarrow H$ be a linear operator on linear subspace $\mathcal{D}(A)$ which is nondense, in general, in H.

Let reconsider the next operator equation:

$(A;f)$ $\quad Au = f$, where $u \in \mathcal{D}(A)$ and f is given in H.

The following simple proposition holds.

PROPOSITION 2.1. *If u_0 is a solution of $(A;f)$ then the set of solutions of $(A;f)$ is given by:*

$$S_{(A;f)} = u_0 + \mathrm{Ker}(A). \tag{2.1}$$

We also consider the following equation:

$(A;f;R(A)^{\perp})$ $\quad Au \in f + R(A)^{\perp}$, $u \in \mathcal{D}(A)$ and $f \in H$,

where $R(A)$ denotes the range of the operator A.

The next proposition is also valid.

PROPOSITION 2.2. *If u_0 is a solution of the operator equation $(A;f;R(A)^{\perp})$ then the set $S_{(A;f;R(A)^{\perp})}$ of all solutions of this equation is given by:*

$$S_{(A;f;R(A)^{\perp})} = u_0 + \mathrm{Ker}(A). \tag{2.2}$$

Proof. "$\supseteq$". It's obvious.

"$\subseteq$". If u_0 is a solution of $(A;f;R(A)^{\perp})$ then for all $u \in \mathcal{D}(A)$ we have:

$$(Au_0 - f, Au) = 0.$$

If u_1 is another solution of $(A;f;R(A)^{\perp})$ then

$$(Au_1 - f, Au) = 0 \text{ for all } u \in \mathcal{D}(A).$$

Then we obtain:

$$(Au_0 - Au_1, Au) = 0 \text{ for all } u \in \mathcal{D}(A),$$

what implies: $A(u_1) - A(u_0) = 0$, i.e., $u_1 - u_0 \in \mathrm{Ker}(A)$ and the proposition is proved.

To equation $(A;f)$ we can also associate, as in the case of positive operators, the following quadratic functional:

$$\bar{F}_{(A;f)}: \mathcal{D}(A) \to K,\ \bar{F}_{(A;f)}(u) := \|Au\|^2 - 2\,\mathrm{Re}(Au,f),$$

which will be called the *energetic functional* associated to $(A;f)$. A simple calculus gives:

$$\lambda\bar{F}_{(A;f)}(u_2) + (1-\lambda)\bar{F}_{(A;f)}(u_1) - \bar{F}_{(A;f)}(\lambda u_2 + (1-\lambda)u_1) =$$
$$= \lambda(1-\lambda)\,\|A(u_2 - u_1)\|^2, \tag{2.4}$$

for all $\lambda \in [0,1]$ and $u_1,u_2 \in \mathcal{D}(A)$, i.e., the mapping $\bar{F}_{(A;f)}$ is convex and if there exists a constant $k > 0$ so that:

$$\|Ax\| \geq k\,\|x\| \text{ for all } x \in \mathcal{D}(A), \tag{2.5}$$

then $\bar{F}_{(A;f)}$ is uniformly convex. We also remark that $\bar{F}_{(A;f)}$ is strictly convex if A is an injective operator.

The following lemma is important in the sequel.

LEMMA 2.3. *Let A: $\mathcal{D}(A) \subset H \to H$ be a linear operator on $\mathcal{D}(A)$ and f a given element in H. The following statements are true:*

(i) *If u_0 is a solution of $(A;f)$ then u_0 minimizes the energetic functional $\bar{F}_{(A;f)}$.*

(ii) *If $u_0 \in \mathcal{D}(A)$ minimizes the functional $\bar{F}_{(A;f)}$ then u_0 is a solution of the equation $(A;f;R(A)^{\perp})$.*

Proof. (i). Let $u_0 \in \mathcal{D}(A)$ be a solution of $(A;f)$. Then we have:

$$\bar{F}_{(A;f)}(u) - \bar{F}_{(A;f)}(u_0) = \|Au\|^2 - 2\,\mathrm{Re}(Au, Au_0)$$
$$+ 2\,\mathrm{Re}(Au_0, Au_0) = \|Au_0 - Au\|^2 \geq 0,$$

i.e., u_0 minimizes the functional $\bar{F}_{(A;f)}$.

(ii). Let $u_0 \in \mathcal{D}(A)$ minimize the energetic functional $\bar{F}_{(A;f)}$. Then for all $u \in \mathcal{D}(A)$ and $\lambda \in K$ we have:

$$\bar{F}_{(A;f)}(\lambda u + u_0) \geq \bar{F}_{(A;f)}(u_0).$$

By simple computations we obtain:

$$|\lambda|^2 \|Au\|^2 + 2\,\mathrm{Re}[\lambda(Au, Au_0 - f)] \geq 0$$

for all $u \in \mathcal{D}(A)$ and for all $\lambda \in K$.

Put $\lambda = t \in \mathbf{R}$. Then we obtain:

$$t^2\|Au\|^2 + 2t\,\mathrm{Re}(Au, Au_0 - f) \geq 0 \text{ for all } t \in \mathbf{R},$$

what implies that:

$$\mathrm{Re}(Au, Au_0 - f) = 0.$$

Put $\lambda = -it$, $t \in \mathbf{R}$. Then we derive:

$$t^2\|Au\|^2 + 2t\,\mathrm{Im}\,(Au, Au_0 - f) \geq 0 \text{ for all } t \in \mathbf{R},$$

what gives:

$$\mathrm{Im}\,(Au, Au_0 - f) = 0.$$

consequently, $(Au, Au_0 - f) = 0$ for all $u \in \mathcal{D}(A)$, i.e., $Au_0 - f \in R(A)^{\perp}$ and the lemma is proved.

Now, let $\mathcal{M}_{(A;f)}$ be the set of all elements in $\mathcal{D}(A)$ which minimize the energetic functional $\bar{F}_{(A;f)}$. Then the following result of representation holds.

PROPOSITION 2.4. *If $u_0 \in \mathcal{D}(A)$ minimize the functional $F_{(A;f)}$ then:*

$$\mathcal{M}_{(A;f)} = u_0 + \mathrm{Ker}(A). \tag{2.5}$$

Proof. "$\subseteq$". If $u_0 \in \mathcal{D}(A)$ minimize $\bar{F}_{(A;f)}$ then $u_0 \in \mathcal{S}_{(A;f;R(A)^{\perp})}$ and since

$\mathcal{M}_{(A;f)} \subseteq \mathcal{S}_{(A;f;R(A)^{\perp})} = u_0 + \mathrm{Ker}(A)$, the inclusion is proved.

"$\supseteq$". Let $u_1 \in u_0 + \mathrm{Ker}(f)$. Then for all $u \in \mathcal{D}(A)$ we have

$\bar{F}_{(A;f)}(u) - \bar{F}_{(A;f)}(u_1) = \|Au\|^2 - 2\mathrm{Re}(Au,f) - \|Au_1\| +$

$2\mathrm{Re}(Au_1,f) = \|Au - Au_1\|^2 + 2\mathrm{Re}(Au, Au_1 - f) + 2\mathrm{Re}(Au_1, f - Au_1)$.

Since $u_1 \in \mathcal{S}_{(A;f;R(A)^{\perp})}$ we conclude that $(Au, Au_1 - f) = 0$ and $(Au_1, f - Au_1)$

0 what implies that:

$$\bar{F}_{(A;f)}(u) - \bar{F}_{(A;f)}(u_1) = \|Au - Au_1\|^2 \geq 0,$$

i.e., u_1 minimize the functional $\bar{F}_{(A;f)}$.

In conclusion, we have the following main result:

THEOREM 2.5. *Let A: $\mathcal{D}(A) \subset H \to H$ be a linear operator on $\mathcal{D}(A)$ and f be a given element in H. Then:*

$$\mathcal{S}_{(A;f)} \subseteq \mathcal{M}_{(A;f)} \subseteq \mathcal{S}_{(A;f;R(A)^{\perp})}. \tag{2.6}$$

The proof is obvious from Lemma 2.3 and by the above proposition. We shall omit the details.

COROLLARY 2.6. *Let A: $\mathcal{D}(A) \subset H \to H$ be a linear operator such that $R(A)$ is dense in H. Then for all $f \in H$ we have:*

$$\mathcal{S}_{(A;f)} = \mathcal{M}_{(A;f)} = \mathcal{S}_{(A;f;R(A)^{\perp})}. \tag{2.7}$$

Proof. Let $u_0 \in \mathcal{S}_{(A;f;R(A)^{\perp})}$. Then $(Au, Au_0 - f) = 0$ for all $u \in \mathcal{D}(A)$. Since $R(A)$ is dense in H we deduce that $Au_0 = f$ and the corollary is proved.

COROLLARY 2.7. *Let A: $\mathcal{D}(A) \subset H \to H$ be a closed dense defined linear operator on $\mathcal{D}(A)$ and suppose that A^* is injective. Then for all $f \in H$ relation* (2.7) *holds.*

Proof. To prove this fact, we need the following well-known result from operator theory (see for example [7], p. 105):

LEMMA 2.8. *Let T be a closed dense defined operator on Hilbert space H. Then T^* is dense defined and:*

(i) $\operatorname{Ker}(T^*) = R(T)^{\perp}$;

(ii) $\overline{R(T^*)} = \operatorname{Ker}(T)^{\perp}$.

If A^* is injective, then it follows that $R(A)^{\perp} = \{0\}$, i.e.,

$$\overline{R(A)} = \overline{\operatorname{Sp}(R(A))} = (R(A)^{\perp})^{\perp} = \{0\}^{\perp} = H,$$

and by Corollary 2.6 we obtain the desired result.

COROLLARY 2.9. *Let A: $\mathcal{D}(A) \subset H \to H$ be an injective self-adjoint operator on dense linear subspace $\mathcal{D}(A)$. Then (2.7) is also valid and there exists at most one element $u_0 \in \mathcal{D}(A)$ such that $u_0 \in S_{(A;f)}$.*

The proof is obvious by the above corollary and we omit the details.

3. Applications to linear operators in Hilbert spaces

1. Let H be a (real) complex Hilbert space and A: $\mathcal{D}(A) \subset H \to H$ be a (symmetric) positive definite operator on dense linear subspace $\mathcal{D}(A)$. Put:

$$i(A) := \inf \{(Ax,x)/\|x\|^2, x \in \mathcal{D}(A), x \neq 0\},$$

then $i(A) > 0$.

By the use of Corollary 1.3 for the positive operator $A - i(A)I$ the element $u_0 \in \mathcal{D}(A)$ is a solution of the equation:

$$Au_0 = i(A)u_0,$$

if and only if u_0 is a solution of the scalar equation

$$(Au,u) = i(A)\|u\|^2.$$

The following result is also valid.

PROPOSITION 3.1. *Let $f \in H$ and $u_0 \in \mathcal{D}(A)$. Then the following statements are equivalent:*

(i) *u_0 is a solution of the equation:*

$$u = 1/i(A)\, Au + f;$$

(ii) *u_0 minimizes the functional $G_{(A;f)}: \mathcal{D}(A) \to K$,*

$$G_{(A;f)}(u) := (Au,u) + 2\,i(A)Re(f,u) - i(A)\|u\|^2$$

The proof follows by Corollary 1.13 for the positive operator $A - i(A)I$ and for the element $-i(A)f$.

We also remark that the positive operator $A - i(A)I$ has the following properties:

(i) $A - i(A)I$ is injective iff for every $f \in H$ there exists at most one element $u_0 \in \mathcal{D}(A)$ which minimize the functional:

$$H_{(A;f)}: \mathcal{D}(A) \to K,\ H_{(A;f)}(u) = (Au,u) - 2\,\mathrm{Re}(f,u) - i(A)\|u\|^2,$$

(ii) $A - i(A)I$ is surjective iff for every $f \in H$ there exists at least one element $u_0 \in \mathcal{D}(A)$ which minimize the above functional $H_{(A;f)}$,

(iii) $A - i(A)I$ is bijective iff for every $f \in H$ there exists a unique element $u_0 \in \mathcal{D}(A)$ such that u_0 minimize $H_{(A;f)}$.

2. Let $A: \mathcal{D}(A) \subset H \to H$ be a symmetric operator on dense subspace $\mathcal{D}(A)$ and suppose that A is bounded on $\mathcal{D}(A)$. Put

$$\|A\| := \sup \{(Ax,x)/\|x\|^2, x \in \mathcal{D}(A)\backslash\{0\}\}.$$

By the use of Corollary 1.3, we derive that $u_0 \in \mathcal{D}(A)$ is a solution of equation:

$$u = 1/\|A\| \, Au$$

if and only if u_0 is a solution of the scalar equation:

$$(Au,u) = \|A\| \, \|u\|^2.$$

The following proposition holds.

PROPOSITION 3.2. *Let $f \in H$ and $u_0 \in \mathcal{D}(A)$. Then the following statements are equivalent:*

(i) *u_0 is a solution of the equation:*

$$u = 1/\|A\| \, Au + f,$$

(ii) *u_0 minimizes the following energetic functional:*

$$L_{(A;f)}: \mathcal{D}(A) \to K, \; L_{(A;f)}(u) := \|u\|^2\|A\| - 2\|A\| \operatorname{Re}(f,u) - (Au,u).$$

The proof follows from Corollary 1.13 for the positive operator $\|A\|I - A$ and for the element $\|A\| \, f$ and we omit the details.

3. Let $A: H \to H$ be a bounded linear operator on H. Then the following identity holds:

$$\operatorname{Ker}(A^*A) = \operatorname{Ker}(A) \; (\operatorname{Ker}(AA^*) = \operatorname{Ker}(A^*)).$$

Indeed, by Corollary 1.3 for the positive operator A^*A we have:

$$\operatorname{Ker}(A^*A) = \{u \in H | \; (A^*Au,u) = 0\} = \{u \in H| \; \|Au\| = 0\} = \operatorname{Ker}(A).$$

This fact implies that the operator A^*A (AA^*) is injective if and only if the operator $A(A^*)$ is injective.

PROPOSITION 3.3. *Let $f \in H$ and $u_0 \in H$. Then the following sentences* *are equivalent:*

(i) u_0 *is a solution of the equation:*

$$A^*A\,u = f \quad (AA^*u = f);$$

(ii) u_0 *minimizes the following functional:*

$$J_{(A;f)}\left(J^*_{(A;f)}\right): H \to K$$

$$J_{(A;f)}(u) = \|Au\|^2 - 2\mathrm{Re}(f,u) \quad \left(J^*_{(A;f)}(u) = \|A^*u\|^2 - 2\mathrm{Re}(f,u)\right).$$

The proof follows by Corollary 1.13 for the positive operator A^*A (AA^*) and for the element $f \in H$.

Remark 3.4. If A: $H \to H$ is a self-adjoint operator on H, then:

$$\mathrm{Ker}(A^2) = \mathrm{Ker}(A),$$

and for a given element $f \in H$ the equation:

$$A^2u = f$$

has the solution u_0 if and only if u_0 minimizes the energetic functional $J_{(A;f)}$.

4. Let A: $H \to H$ be a self-adjoint operator on Hilbert space H and put:

$$\omega_A := \inf\{(Ax,x) \mid \|x\| = 1\},\ \Omega_A := \sup\{(Ax,x) \mid \|x\| = 1\}.$$

Then

$$\omega_A\|x\|^2 \le (Ax,x) \le \Omega_A\|x\|^2 \text{ for all } x \in H,$$

and it is well-known that $\sigma(A) \subset [\omega_A, \Omega_A]$ and $\omega_A, \Omega_A \in \sigma(A)$.

By the use of Corollary 1.3 we observe that $u_0 \in H$ is a solution of equation:

$$Au = \omega_A u \quad (Au = \Omega_A u),$$

if and only if u_0 is a solution of the scalar equation:

$$(Au,u) = \omega_A\|u\|^2 \left((Au,u) = \Omega_A\|u\|^2\right).$$

The following proposition is valid too.

PROPOSITION 3.5. *Let $f \in H$ and $u_0 \in H$. Then the following sentences are equivalent:*

(i) *u_0 is a solution of the equation:*

$$Au = \omega_A u + f \ (Au = \Omega_A u + f)$$

(ii) *u_0 minimizes the quadratic functional:*

$K_{(A;f)}\left(\bar{K}_{(A;f)}\right): H \to K$ *given by*

$$K_{(A;f)}(u) := (Au,u) - \omega_A\|u\|^2 - 2\mathrm{Re}(f,u)$$

$$\left(\bar{K}_{(A;f)}(u) = \Omega_A\|u\|^2 + 2\mathrm{Re}(f,u) - (Au,u)\right).$$

5. Let A: $\mathcal{D}(A) \subset H \to H$ be a closed linear operator on Hilbert space H and consider the following operator equation:

$(A;\lambda;f)$ $\quad Au = \lambda u + f,\ u \in \mathcal{D}(A), f \in H$ and $\lambda \in C$.

PROPOSITION 3.6. *Suppose that $\sigma_c(A)$ is nonvoid (where $\sigma_c(A)$ denotes the continuous spectrum of A) and $\lambda \in \sigma_c(A)$. Then the following statements are equivalent:*

(i) *$u_0 \in \mathcal{D}(A)$ is the unique solution of $(A;\lambda;f)$;*

(ii) *$u_0 \in \mathcal{D}(A)$ is the unique element which minimize the quadratic $J_{(A;\lambda;f)}$: $\mathcal{D}(A) \to K$ where:*

$$J_{(A;\lambda;f)}(u) := \|Au - \lambda u\|^2 - 2\mathrm{Re}(Au - \lambda u, f).$$

Proof. If $\lambda \in \sigma_c(A)$, then the operator $\lambda I - A$ is injective and $\overline{R(\lambda I - A)} = X$. Applying Corollary 2.6 the proposition is proved.

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PERFECT DUALITY FOR K-CONVEXLIKE PROGRAMMING PROBLEMS*

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REZUMAT. - Dualitatea perfectă a problemelor de programare *K*-convexă. Se demonstrează o teoremă de alternativă de tip Farkas, ce generalizează o serie de rezultate obţinute în această direcţie. Se obţine un rezultat de dualitate perfectă pentru probleme de programare *K*-convexă.

Abstract. In this paper a Farkas' type lemma and a perfect duality theorem is given for *K*-convexlike programming problems. Our result (Theorem 2.2.) is more general than Hayashi and Komiya's Lemma 2.1. ([7]), since the class of *K*-convexlike functions includes that of convexlike (in the sense of [7]). A similar result under different regularity condition has been obtained by Gwinner and Jeyakumar ([6]). Their regularity condition and the structure of the defined problem as well, allowed them to get an ε-duality theorem ($\varepsilon > 0$), while we prove a perfect duality theorem (where $\varepsilon = 0$; Theorem 3.1.).

The results of this paper are presented in finite dimensional Euclidean spaces ordered by there positive orthant, but it has to be mentioned that these

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results can be extended with the same proofs to topological vector spaces ordered by arbitrary conex cones. In this sense the class of problems considered here is more general than that of [6,7].

1. Introduction. Let X be a nonempty subset of $\mathbf{R}^n$, and let

$$f_0: X \to \mathbf{R},\ f: X \to \mathbf{R}^m,\ g: X \to \mathbf{R}^k$$

Consider the problem (P):

$$\min_{x \in X} f_0(x) \tag{1}$$

$$f(x) \leq 0 \tag{2}$$

$$g(x) = 0. \tag{3}$$

In this paper a condition related to generalized convexity is given in order to obtain a perfect duality theorem between problem (P) and its Lagrangian dual.

Let F: $X \to \mathbf{R}^p$, where p is an arbitrary positive integer and $K \subset \mathbf{R}^p$ be a nonempty cone*. The function F is said to be K - *convexlike* if there exists $a \in (0,1)$ such that for each $x_1,x_2 \in X$ there exists $x_3 \in X$ with

$$aF(x_1) + (1-a)F(x_2) - F(x_3) \in K.$$

Concerning problem (P) we take

$$p = 1 + m + k,\ F = (f_0, f, g)$$

and

$$K = \mathbf{R}_+ \times \mathbf{R}_+^m \times \{0\} \subset \mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^k.$$

(The constraint (2) can be reformulated as $-f(x) \in \mathbf{R}_+^m$).

Our extended version of Farkas' lemma (Theorem 2.2.) is proved for a K-

* *Throughout this paper cones always have their vertices at the origin, as usual.*

convexlike function and it is based on the separation theorem of two disjoint convex sets. Applying Theorem 2.2. for F and K defined above we obtain a perfect duality result for problem (P) (Theorem 3.1.).

There are several generalizations of Farkas' lemma (see for instance M.Hayashi and H.Komiya [7], B.D.Craven, J.Gwinner and V.Jeyakumar [5], V.Jeyakumar [5], V.Jeyakumar [9], C.Zălinescu [15], L.Blaga and J.Kolumbán [2], T.Illés and G.Kassay [8], G.Kassay and J.Kolumbán [10]). Most of these papers deal only with inequality constraints.

In paper [6] of J.Gwinner and V.Jeyakumar, theorems of the alternative of Motzkin (Theorem 3.1.), Farkas (Theorem 3.2., 3.4.) and Gordan type (Theorem 3.5.) are given for a K-convexlike function $F: X \to \mathbf{R}^Y$ where $Y \neq \varnothing$ and $K = \mathbf{R}^Y_+$. Since their results are based on the separation theorem of disjoint closed convex sets, their regularity condition is different from our's, which is of Slater type. They also obtained an ε- duality theorem (with $\varepsilon > 0$; in case of perfect duality one has $\varepsilon = 0$).

The paper is organized as follows. In section 2 we prove a new version of Farkas' lemma (Theorem 2.1.), which allows us in section 3 to prove a perfect duality result for K-convexlike programming problems. As it is well known, saddle point theorems give another method in proving perfect duality results (see for instance Zeidler [16]). In order to obtain a similar perfect duality theorem for K-convexlike functions by using saddle point techniques, we need a minimax (for instance König's [11]) theorem. The disadvantage of these techniques is that we have to suppose the lower- semicontinuity of the Lagrangian. However, in our Theorem 3.1. we do not need this additional assumption. Hence, it seems that the method based on Farkas' lemma is more efficient for K-convexlike programs than that of saddle point.

2. Farkas' lemma for K-convexlike functions. We first introduce some notations and recall some well-known results.

A subset M of $\mathbf{R}^p$ is said to be *nearly convex* if there exists $a \in (0,1)$ such that for each $y_1, y_2 \in M$ we have $ay_1 + (1-a)y_2 \in M$. If M is nearly convex then the interior ($intM$) and the closure (clM) of M are convex sets. Furthermore the set $B = \{b \in [0,1] \mid \forall y_1, y_2 \in M\colon by_1 + (1-b)y_2 \in M\}$ is dense in $[0,1]$ (see for instance Aleman [1]).

Let X be a nonempty set, $\varphi\colon X \to \mathbf{R}^{p_1}$, $\psi\colon X \to \mathbf{R}^{p_2}$ be given functions and let $K = K_1 \times K_2$, where $K_i \subset \mathbf{R}^{p_i}$ $(i = 1,2)$ are convex cones, with $intK_1 \neq \emptyset$. Denote by Φ the pair $(\varphi,\psi)\colon X \to \mathbf{R}^{p_1+p_2}$.

LEMMA 2.1. *The set $\Phi(X) + K$ has nonempty interior in $\mathbf{R}^{p_1+p_2}$ if and only if the interior of the set $M := \Phi(X) + ((intK_1) \times K_2)$ is nonempty.*

Proof. The if part is obvious since $M \subset \Phi(X) + K$. Suppose that $int(\Phi(X) + K) \neq \emptyset$. Choose $x \in X$, $k_i \in K_i\,(i = 1,2)$ such that $(\varphi(x) + k_1, \psi(x) + k_2) \in int(\Phi(X) + K)$. Then there exist U and V neighbourhoods of the origin in $\mathbf{R}^{p_1}$ and $\mathbf{R}^{p_2}$ respectively, such that

$$[\varphi(x) + k_1 + (intK_1) \cap U] \times [\psi(x) + k_2 + V] \subset [\varphi(x) + k_1 + U] \times [\psi(x) + k_2 + V] \subset \Phi(X) + K.$$

Therefore, for any $u \in intK_1 \cap U$ and $v \in V$, there exist $z \in X$, $l_1 \in K_1$ and $l_2 \in K_2$ such that

$$\varphi(x) + k_1 + u = \varphi(z) + l_1$$

and

$$\psi(x) + k_2 + v = \psi(z) + l_2.$$

Since $K_1 + intK_1 \subset intK_1$ we obtain $\varphi + k_1 + 2u = \varphi(z) + u + l_1 \in \varphi(z) + intK_1$ and $\psi(x) + k_2 + v \in \psi(z) + K_2$, or, in other words

$$[\varphi(x) + k_1 + 2((intK_1) \cap U)] \times [\psi(x) + k_2 + V] \subset \Phi(X) + ((intK_1) \times K_2).$$

Since in the left hand side of the inclusion we have an open set, then $intM$ is nonempty. ■

Next we prove the following "Farkas' lemma" for K-convexlike functions.

Let $\Phi = (\varphi,\psi): X \to \mathbb{R}^{p_1+p_2}$ and $K = \mathbb{R}_+^{p_1} \times \{0\} \subset \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$.

THEOREM 2.2. (i) *Assume that Φ is K-convexlike and the set $\Phi(X) + K$ has nonempty interior. If there is no $x \in X$ such that*

$$\begin{cases} \varphi(x) < 0 \\ \psi(x) = 0 \end{cases} \tag{4}$$

then there exist $y_1 \in \mathbb{R}_+^{p_1}$, $y_2 \in \mathbb{R}^{p_2}$ with $(y_1,y_2) \neq (0,0) \in \mathbb{R}^{p_1+p_2}$ such that

$$\langle y_1, \varphi(x)\rangle + \langle y_2, \psi(x)\rangle \geq 0, \ \forall x \in X. \tag{5}$$

(ii) *If there exist $y_1 \in \mathbb{R}_+^{p_1}\setminus\{0\}$ and $y_2 \in \mathbb{R}^{p_2}$ such that* (5) *holds, then there is no $x \in X$ such that* (4) *holds.*

Proof. (ii) is obvious, hence we have only to prove (i). Let $M := \Phi(X) + ((intK_1) \times K_2)$. Since Φ is K-convexlike, then M is nearly convex. By Aleman [1] and the hypothesis, the set $intM$ is nonempty and convex. On the other hand, by the assumption, (0,0) doesn't belong to M. Using a well-known separation theorem (see for instance Zeidler [16]) the sets $\{(0,0)\}$ and $intM$ can be separated by a hyperplane i.e., there exists $y = (y_1,y_2) \in \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$ with $y \neq (0,0)$ such that

$$\langle y, u\rangle > 0, \ \forall u \in intM. \tag{6}$$

We show that the relation

$$<y,u> \geq 0,\ \forall u \in M \tag{7}$$

holds, i.e. the same hyperplane separates the sets $\{(0,0)\}$ and M. Supposing the contrary, there exists $u_0 \in M$ such that $<y,u_0> < 0$. Fix $u_1 \in intM$. Since the function $f(t) := <y, tu_0 + (1-t)u_1>$ is continuous on $\mathbf{R}$, there exists $\lambda \in (0,1)$ such that $<y,u_\lambda> = 0$, where $u_\lambda := \lambda u_0 + (1-\lambda)u_1$. Let V be an open neighbourhood of u_1 such that $V \subset M$. Since M is nearly convex, there exists $\tau \in B\setminus\{1\}$ such that $u_2 := (1/(1-\tau))(u_\lambda - \tau u_0) \in V$. Then we have $u_\lambda = \tau u_0 + (1-\tau u_2) \in M$. We show that $u_\lambda \in intM$. Define the map $H: \mathbf{R}^{p_1+p_2} \to \mathbf{R}^{p_1+p_2}$ by $H(y) := (1/(1-\tau))(y - \tau u_0)$. Since this map is continuous, the set $U := H^{-1}(V)$ is an open neighbourhood of u_λ. It is obvious that $U \subset M$. Indeed, for each $u \in intM$ let $\bar{u} := H(u) \in V$. Then we have $u = \tau u_0 + (1-\tau\bar{u}) \in M$. Therefore $u_\lambda \in intM$ and taking into account (6) we obtain a contradiction with $<y,u_\lambda> = 0$. Hence, we have shown that (7) holds, i.e.

$$<y_1,v> + <y_2,w> \geq 0,\ \forall u=(v,w) \in M.$$

By this relation we clearly obtain $y_1 \in \mathbf{R}_+^{p_1}$, on one hand, and (5) on the other hand. ■

Theorem 2.2. is a lop-sided version of Farkas' lemma. The sufficiency needs the additional condition ($y_1 \neq 0$), which can be guaranted for instance by the assumption $0 \in int\psi(X)$. This is a natural restriction taking into account that in case of φ convex and ψ affine functions, the matrix corresponding to ψ is always supposed to have full row rank. In this case ψ is onto, therefore $0 \in int\psi(X)$. Under this additional constraint (given on system (4)) Theorem 2.2. can be stated out as a usual alternative theorem.

3. Contrained optimization. In this part we deal with problem (P) and its Lagrangian $L: X \times \mathbf{R}_+^m \times \mathbf{R}^k \to \mathbf{R}$ defined by

$$L(x,y_1,y_2) := f_0(x) + <y_1,f(x)> + <y_2,g(x)>.$$

Consider the function $h: \mathbf{R}_+^m \times \mathbf{R}^k \to \mathbf{R} \cup \{-\infty\}$ given by

$$h(y_1,y_2) := \inf_{x \in X} L(x,y_1,y_2).$$

The Lagrangian dual problem of (P) is

$$(D) \quad \begin{cases} \max h(y_1,y_2) \\ y_1 \in \mathbf{R}_+^m, y_2 \in \mathbf{R}^k \end{cases}$$

We introduce the primal and dual feasible solution sets

$$P := \{x \in X \mid f(x) \le 0,\ g(x) = 0\}$$

and

$$D := \{(y_1,y_2) \in \mathbf{R}_+^m \times \mathbf{R}^k \mid \inf_{x \in X} L(x,y_1,y_2) > -\infty\},$$

respectively. Consider $F = (f_0, f, g): X \to \mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^k$ and

$$K = \mathbf{R}_+ \times \mathbf{R}_+^m \times \{0\} \subset \mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^k$$

as in the introduction.

It is easy to see that $h(y_1,y_2) \le f_0(x)$ for all $x \in P$ and $(y_1,y_2) \in D$. We say that there is no *duality gap* between problems (P) and (D) if there exists $x^* \in P$ and $(y_1^*,y_2^*) \in D$ such that

$$f_0(x^*) = h(y_1^*,y_2^*).$$

In this case, problems (P) and (D) are in *perfect duality*.

THEOREM 3.1. *The following two assertions hold:*

(i) *Suppose that F is K-convexlike, and there exits* $\mu \in \mathbf{R}$ *such that* $(\mu,0,0) \in int(F(X) + K)$. *If* $x^* \in P$ *is an optimal solution of* (*P*), *then there exits* $(y_1^*,y_2^*) \in D$ *optimal solution of* (*D*) *such that*

$$f_0(x^*) = h(y_1^*,y_2^*) \tag{8}$$

and the complementarity condition

$$<y_1^*,f(x^*)> = 0$$

holds.

(ii) *If there exists* $x^* \in P$, $(y_1^*,y_2^*) \in D$ *such that* (8) *holds, then* x^* *and* (y_1^*,y_2^*) *are optimal solutions of* (*P*) *and* (*D*) *respectively.*

Proof. (i). It is obvious that $h(y_1,y_2) \le f_0(x)$ for all $x \in P$ and $(y_1,y_2) \in D$. Therefore, it is enough to prove that there exists $(y_1^*,y_2^*) \in D$ such that

$$f_0(x^*) \le h(y_1^*,y_2^*). \tag{9}$$

Define the functions

$$\varphi: X \to \mathbf{R} \times \mathbf{R}^m,\ \psi: X \to \mathbf{R}^k$$

by $\varphi(x) = (f_0(x) - f_0(x^*), f(x))$, $\psi(x) = g(x)$ and let $\Phi = (\varphi,\psi)$. Then Φ is K-convexlike (with K defined above) and by the regularity condition the set $\Phi(X) + K$ has nonempty interior. Appling Theorem 2.2. (i) it follows that there exists $(\bar\lambda_0,\bar y_1,\bar y_2) \neq (0,0,0)$ with $\bar\lambda_0 \in \mathbf{R}, \bar y_1 \in \mathbf{R}^m_+$ and $\bar y_2 \in \mathbf{R}^k$ such that

$$\bar\lambda_0 f_0(x^*) \le \bar\lambda_0 f_0(x) + <\bar y_1,f(x)> + <\bar y_2,g(x)>\ \forall x \in X. \tag{10}$$

Again by the regularity condition it follows that

$$(0,0) \in int\{f(x) + k, g(x) \mid x \in X, k \in \mathbf{R}_+^m\}.$$

Thus $\bar{\lambda}_0 \neq 0$. Dividing (10) by $\bar{\lambda}_0$ we obtain (9) with $y_1^* = \frac{1}{\bar{\lambda}_0}\bar{y}_1$ and $y_2^* = \frac{1}{\bar{\lambda}_0}\bar{y}_2$.

Part (ii) is obvious. ■

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$\mathcal{A}$-FIXED POINT THEOREMS FOR LOCCALY CONTRACTIVE MULTIVALUED OPERATORS AND APPLICATIONS TO FIXED POINT STABILITY

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REZUMAT. - Teoreme de $\mathcal{A}$-punct fix pentru operatori multivoci local contractivi şi aplicaţii la stabilitatea punctelor fixe. Folosindu-se noţiunile de $\mathcal{A}$-punct fix şi $\mathcal{A}$-stabilitate introduse recent de B.Ricceri şi O.Naselli Ricceri (vezi [11], [8]) se demonstrează o teoremă abstractă de existenţă pentru o incluziune funcţională de tipul $\varphi(s) \in F(s,\varphi(s))$ în ipoteza de local contractivitate a operatorului multivoc F. Ca şi consecinţă se obţine un rezultat asupra stabilităţii punctelor fixe. O aplicaţie a acestor rezultate la incluziuni diferenţiale depinzând de un parametru este de asemenea prezentată.

1. Introduction. The study of random fixed points was initiated by the Prague school of probabilities in the fifties. Recently the interest on this subject was revived especially after the survey article of Bharucha-Reid [2]. The theory of random fixed point has found important applications in random operator equations, random differential equations in Banach spaces and differential inclusions (see [2], [4], [5], [8]).

To be more precise, we recall that given a measurable space $(S,\mathcal{F})$ ($\mathcal{F}$ is a σ-algebra of subsets of S), a topological space X and a multifunction F from

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$S \times X$ into X, a measurable function $\varphi: S \to X$ is said to be a random fixed point of F if $\varphi(s) \in F(s,\varphi(s))$, for all $s \in S$.

In [9] O. Naselli-Ricceri considers the following problem:

Given a family $\mathcal{A}$ of singlevalued functions from S into X, find an $\mathcal{A}$-fixed point of F, i.e. $\varphi \in \mathcal{A}$ such that $\varphi(s) \in F(s,\varphi(s))$, for all $s \in S$. Under two basic assumptions: X is a complete metric space and for every $s \in S$, $F(s,\cdot)$ is a multivalued contraction with closed values, O. Naselli-Ricceri proved the existence of an $\mathcal{A}$-fixed point of F.

The purpose of this paper is to show that a similar result can be obtained for locally contractive multivalued operators. As consequences we obtain a result on fixed points stability. An application to a differential inclusion depending on a parameter is also considered.

We follow the technique given in [12] and [9].

2. Preliminary results. Let A, B be two nonempty sets. We will indicate by $\mathcal{P}(B)$ the family of all subsets of B and by $P(B)$ the family of all nonempty subsets of B. A multifunction (or multivalued operator) from A into B is a function from A into $\mathcal{P}(B)$.

Let $T: A \to P(A)$. A point $a \in A$ is said to be a fixed point of T if $a \in T(a)$. We will denote by Fix T the set of fixed points of T. Let $T: A \to P(B)$. We will denote by Graph T the set $\{(a,b) \in A \times B \mid b \in T(a)\}$. If C is a subset of B, we put $T^{-1}(C) = \{a \in A \mid T(x) \cap C \neq \emptyset\}$. If A, B are topological spaces, a multifunction $T: A \to P(B)$ is said to be lower semiconinuous at $a \in A$ if for every open set $C \subseteq B$ such that $a \in T^{-1}(C)$ one has $a \in \operatorname{int} T^{-1}(C)$. T is said to be lower semicontinuous (l.s.c.) in A if it is so at every point of A.

Let (X,d) be a generalized metric space (see [14]). If A, B are two non-empty subsets of X, $x \in X$ and $r > 0$, we will put:

$$B_d(x,r) := \{y \in X \mid d(x,y) < r\}$$

$$D_d(x,A) := \inf_{y \in A} d(x,y)$$

$$\rho_d(A,B) = \sup_{x \in A} D_d(x,B).$$

Let $P_{cl}(X)$ be the family of all non-empty closed subsets of X endowed with the generalized Hausdorff-Pompeiu metric defined by:

$$H_d(A,B) := \begin{cases} \max\{\rho_d(A,B), \rho_d(B,A)\}, & \text{if the supremum exists} \\ +\infty, & \text{otherwise} \end{cases}$$

$(P_{cl}(X),H_d)$ is a generalized metric space (see [14]).

Now, let S be a non-empty set, $\mathcal{G} \subseteq \mathcal{P}(S)$, X a topological space. We will say that $T: S \to P(X)$ is $\mathcal{G}$-measurable if $T^{-1}(C) \in \mathcal{G}$ for every open set $C \subseteq X$. Thus, if $\mathcal{G}$ is a topology then $\mathcal{G}$-measurability means lower semicontinuity. If $\mathcal{G}$ is a σ-algebra, we will say simply measurability instead of $\mathcal{G}$-measurability.

From now on, S will indicate a nonempty set, (X,d) a metric space and F a multifunction from $S \times X$ into X. Let us denote by $\mathcal{M}(S,X)$ the of all single-valued functions from S into X. We will always consider $\mathcal{M}(S,X)$ endowed with the generalized metric α_d defined by $\alpha_d(f,g) = \sup_{s \in S} d(f(s),g(s))$, for $f,g \in \mathcal{M}(S,X)$.

DEFINITION 2.1. Let $f \in \mathcal{A}$. We say that f is an $\mathcal{A}$-fixed point of F if

$f(s) \in F(s, f(s))$, for every $s \in S$.

DEFINITION 2.2. Let $T: S \to P(X)$ be a multivalued operator. We say that the multifunction T is $\mathcal{A}$-stable if the following two conditions are satisfied:

i) there exists $f \in \mathcal{A}$ such that $f(s) \in T(s)$ for every $s \in S$

ii) for every $\eta, r \in \mathbb{R}_+^* =]0,\infty[$ and every $g \in \mathcal{A}$ such that $T(s) \cap B_d(g(s),r) \neq \emptyset$ for all $s \in S$ there exists $h \in \mathcal{A}$ such that $h(s) \in T(s) \cap B_d(g(s), r+\eta)$ for all $s \in S$.

DEFINITION 2.3. Let $T: X \to P_{cl}(X)$ be a multifunction. Then T is said to be:

i) multivalued contraction if there exists a constant $k \in [0,1[$ such that:

$$H_d(T(x), T(y)) \leq kd(x,y), \text{ for every } x,y \in X.$$

ii) ε-locally contractive multivalued operator (where $\varepsilon > 0$) if there exists $k \in [0,1[$ such that:

$$H_d(T(x), T(y)) \leq kd(x,y), \text{ for every } x,y \in X \text{ with } d(x,y) < \varepsilon.$$

Finally, for every $s \in S$ we put $\Gamma_F(s) := \mathrm{Fix}(F(s,\cdot))$.

We will need auxiliary results on $\mathcal{G}$-measurability of multivalued operators.

PROPOSITION 2.4. ([9]) Let $\mathcal{G} \subseteq \mathcal{P}(S)$. *Assume that either* $\mathcal{G}$ *is closed under arbitrary union or* $\mathcal{G}$ *is closed under countable union and* X *is separable. Let* $\{T_n\}_{n\in\mathbb{N}}$ *be a sequence of* $\mathcal{G}$-*measurable multifunctions from* S *into* X *and let* $T: S \to P(X)$ *be such that*

$$\lim_{n\to\infty} \sup_{s\in S} H_d(T_n(s), T(s)) = 0.$$

Then T is $\mathcal{G}$-measurable.

PROPOSITION 2.5. *Let $\mathcal{G} \subseteq \mathcal{P}(S)$, (Y,d') be a metric space, $L \geq 0$ and $T: S\times X \to P(Y)$ be such that one has $H_{d'}(T(s,x), T(s,y)) \leq Ld(x,y)$ for every $s \in S$ and $x,y \in X$ with $d(x,y) < \varepsilon$. Moreover, suppose that:*

i) *there exists a dense subset D of X such that $F(\cdot,x)$ is $\mathcal{G}$-measurable for each $x \in D$.*

ii) *$\mathcal{G}$ is a topology or $\mathcal{G}$ is closed under finite intersection and countable union and D is countable.*

Then, for every $\mathcal{G}$-measurable function $\varphi: S \to X$, the multifunction $T(\cdot,\varphi(\cdot))$ is $\mathcal{G}$-measurable.

The following theorem is a consequence of a result given in [10]:

THEOREM 2.6. *Let (X,d) be a complete generalized metric space, $T: X \to P_{cl}(X)$ an ε-locally contractive multivalued operator. We suppose that there is an element $x_0 \in X$ such that $D_d(x_0, T(x_0)) < \varepsilon$. Then* Fix $T \neq \emptyset$.

Finally, the following proposition will be useful in the sequel:

PROPOSITION 2.7. [1] *Let X be a paracompact topological space, (Y,d') be a metric space, $G: X \to P(Y)$ be a* l.s.c. *multivalued operator and $g: X \to Y$ be a continuous single-valued function. Let $\varepsilon: X \to \mathbf{R}_+$ be* l.s.c. *Then the multivalued operator $T: X \to P(Y)$ given by $T(x) = G(x) \cap B_{d'}(g(x), \varepsilon(x))$ is* l.s.c. *for each $x \in X$ with $T(x) \neq \emptyset$.*

3. Basic results. The main result of this paper is the following:

THEOREM 3.1. *Let* (X,d) *be a complete metric space,* $\mathcal{A}$ *a nonempty closed subset of* $\mathcal{M}(S,X)$ *and* $F: S\times X \to P_{cl}(X)$ *a multivalued operator. Suppose that:*

i) *the multifunction* G_φ *(given by* $G_\varphi(s) = F(s,\varphi(s))$ *for every* $s \in S$*) is* $\mathcal{A}$*-stable for each* $\varphi \in \mathcal{A}$

ii) *there exists* $k \in [0,1[$ *such* $H_d(F(s,x),F(s,y)) \leq kd(x,y)$ *for every* $s \in S$ *and every* $x,y \in X$ *with* $d(x,y) < \varepsilon$

iii) *there exists* $\varphi_0 \in \mathcal{A}$ *such that* $\sup\limits_{s\in S} D_d(\varphi_0(s), F(s,\varphi_0(s)) < \varepsilon$.

Then F admits an $\mathcal{A}$*-fixed point* φ^* *such that* $\alpha_d(\varphi^*, \varphi_0) < \infty$.

Proof. For every $\varphi \in \mathcal{A}$ put

$T(\varphi) := \{\psi \in \mathcal{A} \mid \psi(s) \in F(s,\varphi(s))$, for every $s \in S\}$. Thanks to the $\mathcal{A}$-stability of G_φ one has $T(\varphi) \neq \emptyset$ for each $\varphi \in \mathcal{A}$. Let us prove that T is an ε-locally contractive multivalued operator from $\mathcal{A}$ into $P(\mathcal{A})$. Let $\varphi,\psi \in \mathcal{A}$ with $\alpha_d(\varphi,\psi) < \varepsilon$. Fix $f \in T(\varphi)$. Then, for every $s \in S$ we have:

$$D_d(f(s),F(s,\psi(s))) \leq H_d(F(s,\varphi(s)), F(s,\psi(s)).$$

Since $\alpha_d(\varphi,\psi) < \varepsilon$ it follows that $d(\varphi(s),\psi(s)) < \varepsilon$ for every $s \in S$. Then

$$H_d(F(s,\varphi(s)), F(s,\psi(s))) \leq kd(\varphi(s),\psi(s)) \leq k\alpha_d(\varphi,\psi)$$

for every $s \in S$. Hence $D_d(s,F(s,\psi(s))) \leq k\alpha_d(\varphi,\psi)$, for every $s \in S$. Therefore for every $\eta > 0$ we have

$$F(s,\psi(s)) \cap B_d\left(f(s), k\alpha_d(\varphi,\psi) + \frac{\eta}{2}\right) \neq \emptyset.$$

Thanks to the $\mathcal{A}$-stability of G_ψ there exists $g \in \mathcal{A}$ such that $g(s) \in F(s,\psi(s)) \cap$

$B_d(f(s))$, $k\alpha_d(\varphi,\psi) + \eta)$, for every $s \in S$. Hence, $g \in T(\psi)$ and $\alpha_d(f,g) \leq k$ $\alpha_d(\varphi,\psi) + \eta$. Since η is arbitrary, it follows that $D_{\alpha_d}(f, T(\psi)) \leq k\alpha_d(\varphi, \psi)$. This inequality holds for any $f \in T(\varphi)$ and so $\rho_{\alpha_d}(T(\varphi), T(\psi)) \leq k\alpha_d(\varphi, \psi)$. Changing the roles of φ and ψ, we obtain $\rho_{\alpha_d}(T(\psi), T(\varphi)) \leq k\alpha_d(\varphi, \psi)$.

Hence, for every $\varphi,\psi \in \mathcal{A}$ such that $\alpha_d(\varphi,\psi) < \varepsilon$ one has $H_{\alpha_d}(T(\varphi), T(\psi)) \leq k\alpha_d(\varphi, \psi)$. (where $k < 1$). Since X is a complete metric space and $\mathcal{A}$ is closed in $\mathcal{M}(S,X)$ it follows that $(\mathcal{A},\alpha_d)$ is a complete generalized metric space.

Thanks to the $\mathcal{A}$-stability of the multifunction G_{φ_0} the condition iii) from theorem implies that $\rho_d(\varphi_0, T(\varphi_0)) < \varepsilon$. Therefore, by Theorem 2.6 we obtain the conclusion. Moreover, by the proof of Theorem 2.1 of [10] we can derive that $\rho_d(\varphi^*, \varphi_0) < +\infty$. Q.E.D.

Now, we state some consequences of Theorem 3.1.

THEOREM 3.2. *Let S be a paracompact topological space, X a closed, convex subset of a Banach space $(E,\|\cdot\|)$, Z a subset of S with* $\dim_Z(Z) \leq 0$ (i.e.. $\dim(U) \leq 0$ *for every $U \subseteq Z$ which is closed in S, where* $\dim(U)$ *denotes the covering dimension of U; see also* [8]), *and F: $S\times X \to P_{cl}(X)$ a multivalued operator such that $F(s,x)$ is convex for every $(s,x) \in (S\setminus Z)\times X$. Suppose that:*

i) $F(\cdot,x)$ *is* l.s.c. *for every $x \in X$*

ii) *there exists a continuous function k: $S \to [0,1[$ such that $H_{\|\cdot\|}(F(s,x),F(s,y)) \leq k(s)\|x-y\|$ for each $s \in S$ and $x,y \in X$ with $\|x-y\| < \varepsilon$.*

Then for every closed subset V of S and every continuous function ψ: V

$\to X$ such that $\psi(s) \in F(s,\psi(s))$ for every $s \in V$ there exists a continuous function $\varphi: S \to X$ such that $\varphi(s) \in F(s,\varphi(s))$, for every $s \in S$ and that $\varphi|_V = \psi$. If, in addition $Z = S$, we can suppose that X is only closed.

Proof. Let V and ψ be as in statement. By Theorem 1.4 of [8], we can choose a continuous function $\tilde{\varphi}: S \to X$ such that $\tilde{\varphi}|_V = \psi$. Observe that the multifunction $F(\cdot,\tilde{\varphi}(\cdot))$ is closed-valued and by Proposition 2.5 it is l.s.c. Moreover, if $s \in S\setminus Z$, $F(s,\tilde{\varphi}(s))$ is convex. Consider the multifunction $G: S \to P(X)$ given by $G(s) = F(s,\tilde{\varphi}(s)) \cap B_{|\cdot|}(\tilde{\varphi}(s);\varepsilon)$, for every $s \in S$. By Proposition 2.7 G is l.s.c. and $G(s)$ is convex for each $s \in S\setminus Z$. Then, by Theorem 7.1 of [8] there exists a continuous function $\varphi_0: S \to X$ such that $\varphi_0(s) \in G(s)$, for every $s \in S$ and $\varphi_0|_V = \psi$. Therefore $\varphi_0(s) \in F(s,\tilde{\varphi}(s))$ and $\|\varphi_0(s) - \tilde{\varphi}(s)\| < \varepsilon$, for every $s \in S$.

Observe that, for every $s \in S$ one has:

$$D_{|\cdot|}(\varphi_0(s),F(s,\varphi_0(s))) \le H_{|\cdot|}(F(s,\tilde{\varphi}(s)),F(s,\varphi_0(s))) \le$$
$$\le k(s)\,\|\tilde{\varphi}(s) - \varphi_0(s)\|.$$

Let $\{a_n\}_{n\in\mathbf{N}}$ be an increasing and unbounded sequence of positive real numbers such that for every $n \in \mathbf{N}$ $a_n \ge \varepsilon$ and the set

$$S_n = \{ s\in S \mid \|\varphi_0(s)\| + \|\tilde{\varphi}(s)\| < a_n\} \cap \{ s\in S \mid k(s) < 1 - \frac{1}{a_n} \}$$

is nonempty. Then $\{S_n\}_{n\in\mathbf{N}}$ is an increasing sequence of open sets and $\bigcup_{n\in\mathbf{N}} S_n = S$. We prove that there exists a sequence $\{\varphi_n\}_{n\in\mathbf{N}^*}$ of continuous and bounded functions from S into X such that, for each $n \in \mathbf{N}^*$ one has:

$$\varphi_n(s) \in F(s,\varphi_n(s)),\ \text{for every } s \in \overline{S_n} \tag{1}$$

$$\varphi_n(s) \in F(s,\tilde{\varphi}(s)), \text{ for every } s \in \bar{S}_{n+1} \tag{2}$$

$$\varphi_n|_{\bar{S}_{n-1}} = \varphi_{n-1} \text{ (where } S_0 = \varnothing) \tag{3}$$

$$\varphi|_{V\cap\bar{S}_n} = \psi. \tag{4}$$

We construct such a sequence by induction. Let us construct φ_1. Denote $\mathcal{A}_1$ the family of all continuous functions from $\bar{S}_1$ into X and put $\mathcal{B}_1 = \{\varphi \in \mathcal{A}_1 | \varphi|_{V\cap\bar{S}_1} = \psi\}$. For each $\varphi \in \mathcal{B}_1$, the multifunction $F(\cdot,\varphi(\cdot))$ is $\mathcal{B}_1$-stable (see Example 1.3* of [7], Proposition 2.5 and Proposition 3.2 (α) by [12]). Now, observe that $\mathcal{B}_1$ is closed in $(\mathcal{M}(\bar{S}_1,X),\alpha_d)$, that $\varphi_0|_{\bar{S}_1} \in \mathcal{B}_1$, that $H_{\|\cdot\|}(F(s,x),F(s,y)) \le k(s)\|x-y\| \le \le \left(1-\frac{1}{a_1}\right)\|x-y\|$ for every $s \in S_1$ and $x,y \in X$ with $\|x-y\| < \varepsilon$ and that $\sup_{s\in\bar{S}_1} D_d(\varphi_0(s),F(s,\varphi_0(s))) < \varepsilon$. By Theorem 3.1 there exists a continuous function $\varphi_1^*:\bar{S}_1 \to X$ such that $\varphi_1^*|_{V\cap\bar{S}_1} = \psi$, $\varphi_1^*(s) \in F(s,\varphi_1^*(s))$, for each $s \in \bar{S}_1$ and $\sup_{s\in\bar{S}_1}\|\varphi_1^*(s)-\varphi_0(s)\|<\infty$. Therefore φ_1^* is bounded on $\bar{S}_1$.

Consider the singlevalued function $\varphi_1^{**}: \bar{S}_1 \cup \bar{S}_2 \to X$, defined by:

$$\varphi_1^{**}(s) = \begin{cases} \varphi_1^*(s), & s \in \bar{S}_1 \\ \varphi_0(s), & s \in \bar{S}_2 \end{cases}$$

φ_1^{**} is a continuous selection of the following multivalued operator

$$G_1: \bar{S}_1 \cup \bar{S}_2 \to P(X),\ G(s) = \begin{cases} F(s,\varphi_1^*(s)), & s \in \bar{S}_1 \\ F(s,\tilde{\varphi}(s)) \cap B(\tilde{\varphi}(s),\varepsilon), & s \in \bar{S}_2 \end{cases}$$

Then, by Theorem 1.4 of [8] it is possible to extend φ_1^{**} to a continuous and bounded function φ_1 on S which satisfies (1)-(4) for $n = 1$. Suppose now that bounded and continuous functions $\varphi_1, \ldots, \varphi_h$ from S into X satisfying (1)-(4) for $n = 1,2,\ldots,h$ have been constructed. Let us construct φ_{h+1}. To this end, denote by $\mathcal{A}_{h+1}$ the family of all continuous functions from $\bar{S}_{n+1}$ into X and put $\mathcal{B}_{h+1} = \{\varphi \in \mathcal{A}_{h+1} \mid \varphi|_{\bar{S}_h} = \varphi_h\}$. As in the case $n = 1$ $F(\cdot,\varphi(\cdot))$ is $\mathcal{B}_{h+1}$-stable. Observe that $\mathcal{B}_{h+1}$ is closed in $(\mathcal{M}(\bar{S}_{h+1},X), \alpha_d)$ and that $\varphi_h|_{\bar{S}_{h+1}} \in \mathcal{B}_{h+1}$. Moreover, for all $s \in \bar{S}_{h+1}$ one has:

$$D_d(\varphi_h(s), F(s,\varphi_h(s))) \leq H(F(s,\tilde{\varphi}(s)), F(s,\varphi_h(s))) \leq$$
$$\leq k(s)\,\|\tilde{\varphi}(s) - \varphi_h(s)\| < \varepsilon.$$

Hence $\sup\limits_{s\in\bar{S}_{h+1}} D_d(\varphi_h(s), F(s,\varphi_h(s))) < \varepsilon$.

Finally, one has $H_{\|\cdot\|}(F(s,x), F(s,y)) \leq k(s)\|x-y\| \leq \left(1 - \dfrac{1}{a_{h+1}}\right)\|x-y\|$, for every $s \in \bar{S}_{h+1}$ and $x,y \in X$ with $\|x-y\| < \varepsilon$. Therefore by Theorem 3.1 there exists a continuous function $\varphi^*_{h+1}: \bar{S}_{h+1} \to X$ such that $\varphi^*_{h+1}|_{\bar{S}_h} = \varphi_h$ and $\varphi^*_{h+1}(s) \in F(s,\varphi^*_{h+1}(s))$, for every $s \in \bar{S}_{h+1}$. Consider $\varphi^{**}_{h+1}: \bar{S}_{h+1} \cup \bar{S}_{h+2} \to X$ defined by

$$\varphi^{**}_{h+1}(s) = \begin{cases} \varphi^*_{h+1}(s), & \text{if } s \in \bar{S}_{h+1} \\ \varphi_0(s), & \text{if } s \in \bar{S}_{h+2} \end{cases}$$

φ^{**}_{h+1} is a continuous selection of the following multivalued operator

$$G_{h+1}: \bar{S}_{h+1} \cup \bar{S}_{h+2} \to P(X),\ G(s) = \begin{cases} F(s,\varphi^*_{h+1}(s)), & s \in \bar{S}_{h+1} \\ F(s,\tilde{\varphi}(s)) \cap B(\tilde{\varphi}(s),\varepsilon), & s \in \bar{S}_{h+2} \end{cases}$$

Then, by Theorem 1.4 of [8] it is possible to extend φ^{**}_{h+1} to a continuous and bounded function φ_{h+1} on S which satisfies (1)-(4) for $n = h+1$.

So, the sequence $\{\varphi_n\}_{n\in\mathbf{N}}$ has been constructed. Now, define $\varphi: S \to X$ by putting $\varphi(1) = \varphi_n(1)$ if $s \in S_n \setminus S_{n-1}$, $n \in \mathbf{N}$.

Then $\varphi(s) \in F(s,\varphi(s))$ for every $s \in A$ and $\varphi|_V = \psi$. The continuity of φ follows from (3).

Remark 3.3. Since every complete metric space can be isometrically embedded in a Banach space as a closed set, when $S = Z$ in theorem 3.2 we can suppose that (X,d) is a complete metric space.

DEFINITION 3.4. [6] Let (X,d) be a metric space and $\{A_n\}_{n\in\mathbf{N}}$ a sequence of non-empty subsets of X. The set:

$$\underline{\mathrm{Lim}}_{n\to\infty} A_n := \left\{x\in X \mid \lim_{n\to\infty} D_d(x, A_n) = 0\right\}$$

is called the topological lower limit of $\{A_n\}_{n\in\mathbf{N}}$.

The following result is a consequence of Theorem 3.2.

THEOREM 3.5. *Let (X,d) be a complete metric space and let U, $U_1, U_2, \ldots$ be a sequence of closed-valued multifunctions from X into itself which are ε-locally contractive multivalued operator with the same constant $k \in [0,1[$. Suppose that there exists a dense subset D of X, such that*

$$U(x) \subseteq \underline{\mathrm{Lim}}_{n\to\infty}\, U_n(x), \text{ for every } x \in D \tag{5}$$

Then $\mathrm{Fix}\, U \subseteq \underline{\mathrm{Lim}}_{n\to\infty} \mathrm{Fix}\,(U_n)$.

Proof. Let S be the one-point compactification of $\mathbf{N}$ with the usual topology. Define a multivalued operator F: $S\times X \to P(X)$ by putting

$$F(s,x) = \begin{cases} U_n(x), & \text{if } s = n,\ n\in\mathbf{N},\ x\in X \\ U(x), & \text{if } s = \infty,\ x\in X \end{cases}$$

F satisfies the hypotheses of Theorem 3.2. In particular, relation (6) is equivalent to the lower semicontinuity of $F(\cdot,x)$ at the point $s = \infty$ for $x \in D$. Then, choose $x_0 \in \mathrm{Fix}\, U$ and put $V = \{\infty\}$, $\psi(\infty) = x_0$. By Theorem 3.2 there exists a continuous function φ: $S \to X$ such that $\varphi(n) \in F(n,\varphi(n))$, for all $n \in \mathbf{N}$ and $\varphi(\infty) = x_0$. Thus, if we put $x_n = \varphi(n)$, $n \in \mathbf{N}$ we have $x_n \in \mathrm{Fix}\,(U_n)$.

Moreover, by the continuity of φ, we have $\lim_{n\to\infty} x_n = x_0$. Hence $x_0 \in \underline{\mathrm{Lim}}_{n\to\infty}\, \mathrm{Fix}\,(U_n)$. Q.E.D.

Now, we state the following result on the fixed points set stability.

THEOREM 3.6. *Let S be a first-countable topological space, (X,d) a compact metric space and $F: T\times X \to P_{cl}(X)$ a multivalued operator such that:*

i) *$F(\cdot,x)$ is* l.s.c. *for every $x \in D$ (where D is a dense subset of X)*

ii) *there exists an upper semicontinuous function $k: S \to [0,1[$ such that $H_d(F(s,x), F(s,y)) \le k(s)d(x,y)$, for every $s \in S$ and $x,y \in X$ with $d(x,y) < \varepsilon$.*

Then the multifunction Γ_F is lower semicontinuous.

Proof. Let $s_0 \in S$. By Proposition 2.1 of [4] to prove the l.s.c. of Γ_F at s_0 it suffices to show that

$$\Gamma_F(s_0) \subseteq \underline{\mathrm{Lim}}_{n\to\infty}\, \Gamma_F(s_n) \tag{6}$$

for every sequence $\{s_n\}_{n\in\mathbf{N}}$ of points of T convergent to t_0. To this purpose, consider the multifunction from X into itself defined as follows:

$U_n(x) = F(s_n,x)$, for every $x \in X$, $n \in \mathbf{N}$

$U(x) = F(s_0,x)$, for every $x \in X$.

Observe that by upper semicontinuity of k, all these multifunctions are ε-locally contractive with the same constant $k^* = \max\left\{k(s_0), \sup_{n\in\mathbf{N}} k(s_n)\right\} < 1$. Moreover, by (1) one has $U(x) \subseteq \underline{\mathrm{Lim}}_{n\to\infty}\, U_n(x)$, for every $x \in D$. Then, by Theorem 3.5 one has $\mathrm{Fix}(U) \subseteq \underline{\mathrm{Lim}}_{n\to\infty}\, \mathrm{Fix}(U_n)$. Q.E.D.

An application to a differential inclusions depending on a parameter, can be obtained from Theorem 3.6.

THEOREM 3.7. *Let* $(E,\|\cdot\|)$ *be a separable Banach space, S be a first countable topological space,* $[a,b]$ *is a compact real interval,* $p \in [1,+\infty[$ *and* $F: [a,b]\times E\times S \to P_{cl}(E)$ *is a multivalued operator such that the following conditions are satisfied:*

i) $F(\cdot,x,\lambda)$ *is measurable for all* $x \in E$, $\lambda \in S$

ii) *there exists a function* $k \in L^p[a,b])$ *such that*

$$D_d(F(t,x,\lambda), F(t,y,\lambda)) \leq k(t)\|x-y\|, \textit{ for } \text{a.e. } t \in [a,b] \textit{ and for all } \lambda \in S$$

and $x,y \in E$ *with* $\|x-y\| < \varepsilon$

iii) $F(t,x,\cdot)$ *is* l.s.c. *for* a.e. $t \in [a,b]$ *and all* $x \in E$.

Then the following assertions are equivalent:

(a) *For every convergent sequence* $\{\lambda_n\} \subset S$ *the set functions:*

$$A \to \int_A \left[D\left(O_E, F\left(t, O_E, \lambda_n\right)\right)\right]^p dt$$

are equai-absolutely continuous.

(b) *For every* $t_0 \in [a,b]$, $x_0 \in E$ *the multifunction* $\lambda \to \Gamma_{p,t_0,x_0}(\lambda)$ *is* l.s.c. *with respect to the norm-topology on* $AC_p([a,b]E)$ *and with non-empty, closed values (where* $\Gamma_{p,t_0,x_0}(\lambda)$ *denote the following set*: $\{\varphi \in AC_p([a,b],E) \mid \varphi'(t) \in F(t,\varphi(t),\lambda)$, a.e. *in* $[a,b]$, $\varphi(t_0) = x_0\}$.

Remark 3.8. For much more details on this subject see [11].

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COMMON FIXED POINT THEOREMS FOR A GENERALIZED SET-VALUED MAPPING

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REZUMAT. - Teoreme de punct fix pentru aplicaţii multivoce generalizate. Sunt demonstrate teoremele de punct fix comun pentru aplicaţii multivoce de tip contracţie generalizată. Rezultatele extind la cazul multivoc o teoremă stabilită în [9].

Abstract. In this paper we give common fixed point theorems for a generalized contractive type set-valued mapping. The results extend corresponding theorem due to F.Skoff [9] for point-valued mappings to set-valued mappings. Suitable examples are given.

1. Introduction. Let (X,d) be a complete metric space with a metric d. Let R^+ denotes the set of all non-negative real numbers and Φ be the family of mappings ϕ from R^+ into R^+ such that:

(i) ϕ is continuous and strictly increasing in R^+,

(ii) $\phi(t) = 0$ iff $t = 0$.

(iii) $\phi(t) \geq Mt^{\mu}$ for every $t > 0$, where $M > 0$, $\mu > 0$ are constant.

The following theorem was proved by F.Skoff in [9].

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THEOREM 1. *Let T be a self-mapping of (X,d) and $\phi \in \Phi$ such that for every $x,y \in X$,*

$$\text{(A)} \qquad \phi(d(Tx,Ty)) \leq a\phi(d(x,y)) + b\phi(d(x,Tx)) + c\phi(d(y,Ty)),$$

where $0 < a+b+c < 1$. *Then T has a unique fixed point.*

In [8], the author has considered functions $\phi \in \Phi$ such that $\phi(t) = t^n$, $n \in \mathbf{N}$, for every $t \geq 0$.

In the literature of fixed point theory many fixed point theorems of point-valued mappings can be extended to set-valued mappings under different contractive type conditions (See for example [1], [2], [3],[4], [6]).

The main purpose of this paper is to extend Theorem1 for point-valued mappings to set-valued mappings satisfying a generalized contractive condition.

Following Fisher [2], let $B(X)$ be the set of all nonempty, bounded subsets of X. We define the function $\delta(A,B)$ with A and B in $B(X)$ by:

$$\delta(A,B) = \sup\{d(a,b)]: a \in A \text{ and } b \in B\}.$$

If A is a single point $\{a\}$, we write

$$\delta(A,B) = \delta(a,B)$$

and if B consists also of a single point b, we write

$$\delta(A,B) = \delta(a,b).$$

It follows immediately from the definition that

$$\delta(A,B) = \delta(B,A) \geq 0,\ \delta(A,A) = \text{diam } A,$$

$$\delta(A,B) \leq \delta(A,C) + \delta(B,C),$$

for all A,B and C in $B(X)$ and if $\delta(A,B) = 0$, then $A = B = \{a\}$. If $\{A_n\}$ is a sequence in $B(X)$, we say that $\{A_n\}$ converges to $A \subseteq X$, and write $A_n \to A$, iff

(1) $a \in A$ implies that $a = \lim_{n\to\infty} a_n$ for some sequence $\{a_n\}$ with $a_n \in A_n$ for $n \in \mathbf{N}$, and

(2) for any $\varepsilon > 0$, $m \in \mathbf{N}$ such that

$$A_n \subseteq A_\varepsilon = \{x \in X: d(x,a) < \varepsilon \text{ for some } a \in A\}$$

for $n > m$.

The following lemma was proved in [2]:

LEMMA 1.1. *Suppose $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ and (X,d) is a complete metric space. If $A_n \to A \in B(X)$ and $B_n \to B \in B(X)$, then $\delta(A_n,B_n) \to \delta(A,B)$.*

The following definition is due to Jungck-Rhoades [5]:

DEFINITION 1.2. Let (X,d) be a metric space. Let $I:X \to X$ and $F:X \to B(X)$. F and I are δ-compatible iff $IFx \in B(x)$ for $x \in X$ and $\delta(IFx_n,FIx_n) \to 0$ whenever $\{x_n\}$ is a sequence in X such that $Ix_n \to t$ and $Fx_n \to \{t\}$ for some $t \in X$.

As stated in [5] F need not be single-valued even throung the conditions of the above definition are satisfied: consider, for example, $I:R \to R$ and $F:R \to B(R)$ defined by $Ix = \frac{x}{3}$ and $Fx = \left[0, \frac{x}{2}\right]$, where R denotes the reals with the usual topology.

Note that by definition 1.2, a function $F:X \to B(X)$ is continuous iff $x_n \to z$ in (X,d) implies $Fx_n \to Fz$ in $B(X)$.

Next, we give proposition for our main theorem as in [5]:

PROPOSITION 1.3. *Let* (X,d) *be a complete metric space. Suppose* $I:X \to X$ *and* $F:X \to B(X)$, *and* I *and* F *are* δ-*compatible*

(i) *Suppose the sequence* $\{Fx_n\}$ *converges to* $\{z\}$ *and* $\{Ix_n\}$ *converges to* z. *If* I *is continuous, then* $FIx_n \to \{Iz\}$.

(ii) *If* $\{Iu) = Fu$ *for some* $u \in X$, *then* $FIu = IFu$.

2. Main Result. We now state and prove the following theorem:

THEOREM 2.1. *Let* I *be mapping of a complete metric space* (X,d) *into itself and let* $F:X \to B(X)$. *Suppose there is an increasing, continuous function* $\phi:R^+ \to R^+$ *satisfying property* (ii) *such that for all* $x,y \in X$, $x \neq y$.

(I) $$\phi(\delta(Fx,Fy)) \leq a\,\phi(d(Ix,Iy)) + b\,[\phi(\delta(Ix,Fx)) + \phi(\delta(Iy,Fy))] + c\,\min\{\phi(\delta(Ix,Fy)),\ \phi(\delta(Iy,Fx))\},$$

where a, b, c *are constants satisfying* $a+2b+c < 1$.

If:

(a) $F(X) \subset I(X)$,

(b) I *is continuous,*

(c) (F,I) *is* δ-*compatible.*

Then F and I have a unique common fixed point u in X. Moreover $Fu = \{u\} = \{Iu\}$.

Proof. Let x_0 be an arbitrary point in X and define the sequence $\{x_n\}$ by

$$Ix_{n+1} \in Fx_n = X_n\,,\ n = 0,1,2,...$$

For simplicity, we put

$$\delta_n = \delta(X_n, X_{n+1}), \text{ for } n = 0,1,2,...$$

Using inequality (I) and property (ii), we have

$$\begin{aligned}\phi(\delta_{n+1}) &= \phi(\delta(Fx_{n+1}, Fx_{n+2})) \\ &\leq a\,\phi(d(Ix_{n+1}, Ix_{n+2})) + \\ &+ b\,[\phi(\delta(Ix_{n+1}, Fx_{n+1})) + \phi(\delta(Ix_{n+2}, Fx_{n+2}))] \\ &+ c\min\{\phi(Ix_{n+1}, Fx_{n+2})), \phi(\delta(Ix_{n+2}, Fx_{n+1}))\} \\ &\leq a\,\phi(\delta_n) + b[\phi(\delta_n) + \phi(\delta_{n+1})],\end{aligned}$$

So, we have

$$\phi(\delta_{n+1}) = \frac{a+b}{1-b}\,\phi(\delta_n) < \phi(\delta_n) \qquad (1)$$

Since ϕ is increasing, then $\{\delta_n\}$ is a decreasing sequence, which has a limit $\delta > 0$. Letting $n \to \infty$ in (1) and using property (i), then $\delta_n \geq \delta$ implies that

$$\phi(\delta) \leq \frac{a+b}{1-b}\,\phi(\delta) < \phi(\delta).$$

This is a contradiction. So $\phi(\delta) = 0$ implies $\delta = 0$.

Let z_n be an arbitrary point in X_n for $n = 0,1,2,...$ Now we show that $\{z_n\}$ is a Cauchy sequence in X. Since

$$\lim_{n\to\infty} d(z_n, z_{n+1}) \le \lim_{n\to\infty} \delta(X_n, X_{n+1}) = 0,$$

it suffices to prove that the sequence $\{z_{2n}\}$ is a Cauchy sequence. Suppose not, then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$, $\{n(k)\}$ such that for every $n \in \mathbf{N}\backslash\{0\}$, we find that $m(k) > n(k) > k$,

$$\varepsilon < d(z_{n(k)}, z_{m(k)}) \le \delta(X_{n(k)}, X_{m(k)}) \tag{2}$$

and

$$\delta(X_{n(k)-1}, X_{m(k)}) < \varepsilon. \tag{3}$$

For each $n \ge 0$, we put $S_n = \delta(X_{n(k)}, X_{m(k)})$. Then we have

$$\varepsilon \le S_n \le \delta(X_{n(k)-1}, X_{n(k)}) + \delta(X_{n(k)-1}, X_{m(k)}) < \delta_{n(k)-1} + \varepsilon.$$

Since $\{\delta_n\}$ converges to 0, $\{S_n\}$ converges to ε.

Furthermore, for each $n \ge 0$,

$$-\delta_{n(k)} - \delta_{m(k)} + S_n \le (X_{n(k)+1}, X_{m(k)+1}) \le \delta_{n(k)} + \delta_{m(k)} + S_n,$$

therefore the sequence $\{\delta(X_{n(k)+1}, X_{m(k)+1})\}$ converges also to ε.

Applying inequality (I), we get

$$\phi(\delta(X_{n(k)}, X_{m(k)})) \le \phi(\delta(Fx_{n(k)}, Fx_{m(k)})) \le a\phi(d(Ix_{n(k)}, Ix_{m(k)}))$$

$$+ b[\phi(\delta(Ix_{n(k)}, Fx_{n(k)}) + \phi(\delta(Ix_{m(k)}, Fx_{m(k)}))] +$$

$$+c\min\{\phi(\delta(Ix_{n(k)}, Fx_{m(k)})), \phi(\delta(Ix_{m(k)}, Fx_{n(k)}))\} \le a\phi(\delta(X_{n(k)-1}, X_{m(k)}) + \delta_{m(k)-1})$$

$$+ b[\phi(\delta_{n(k)-1}) + \phi(\delta_{m(k)-1})] +$$

$$+ c\min\{\phi(\delta(X_{n(k)-1}, X_{m(k)})), \phi(\delta(X_{n(k)-1}, X_{m(k)}) + \delta_{n(k)-1} + \phi(\delta_{m(k)-1}))\}$$

Using (3), we have

$$\begin{aligned}\phi(S_n) &= \phi(\delta(X_{n(k)}, X_{m(k)})) \\ &\leq a\phi(\varepsilon + \delta_{m(k)-1}) + b[\phi(\delta_{n(k)-1}) + \phi(\delta_{m(k)-1})] + \\ &+ c\min\{\phi(\varepsilon), \phi(\varepsilon + \delta_{n(k)-1} + \phi(\delta_{m(k)-1}))\}\end{aligned}$$

Letting $k \to \infty$ and using (i), (ii), we have

$$\begin{aligned}\phi(\varepsilon) &\leq a\phi(\varepsilon) + c\phi(\varepsilon) \\ &= (a+c)\,\phi(\varepsilon) < \phi(\varepsilon)\end{aligned}$$

a contradiction. Therefore the sequence $\{z_n\}$ is a Cauchy sequence in the complete space (X,d) and so it has a limit u in X. In particular, the sequence $\{Ix_n\}$ converges to u and further, the sequence of sets $\{Fx_n\}$ converges to $\{u\}$.

Since I is continuous, then $IIx_n \to Iu$. But I and F are δ-compatible, therefore $FIx_n \to \{Iu\}$ by proposition 1.3 (i). Consequently, since (I) holds, then

$$\begin{aligned}&\phi(\delta(\mathrm{FIx}_n, Fx_{n+1})) \leq a\phi\,(d(IIx_n, Ix_{n+1})) \\ &+ b\,[\phi(\delta(IIx_n, \mathrm{FIx}_n)) + \phi\,(\delta(Ix_{n+1}, Fx_{n+1}))] \\ &+ c\,\min\{\phi(\delta(IIx_n, Fx_{n+1})), \phi(\delta(Ix_{n+1}, \mathrm{FIx}_n))\}\end{aligned}$$

for $n \in \mathbb{N}$. As $n \to \infty$ we obtain

$$\begin{aligned}\phi(\delta(Iu,u)) &\leq a\phi(\delta(Iu, u)) + c\phi(\delta(Iu, u)) \\ &= (a+c)\,\phi\,(\delta(Iu, u)) \\ &< \phi\,(\delta(Iu, u))\end{aligned}$$

a contradiction by applying lema 1.1 and property (ii). Thus $Iu \quad u$.

Further

$$\phi(\delta(Fu, Fx_{n+1})) \le a\phi(d(Iu, Ix_{n+1}))$$
$$+ b[\phi(\delta(Iu, Fu)) + \phi(\delta(Ix_{n+1}, Fx_{n+1}))]$$
$$+ c\min\{\phi(\delta(Iu, Fx_{n+1})), \phi(\delta(Ix_{n+1}, Fu))\}$$

Letting $n \to \infty$, by lemma 1.1 and properties (i), (ii), it yields

$$\begin{aligned}\phi(\delta(Fu,u)) &\le a\phi(0) + b[\phi(\delta(u, Fu)) + \phi(0)] + c\phi(0)\\ &= b\phi(\delta(u, Fu))\\ &< \phi(\delta(u, Fu))\end{aligned}$$

a contradiction. Thus $\phi(\delta(u,Fu)) = 0$ implies $Fu = \{u\}$.

Let $v \in X$ be a common fixed point of F and I, $v \ne u$. Then

$$\begin{aligned}\phi(d(v,u)) &= \phi(\delta(Fv, Fu))\\ &\le a\phi(d(Iv,Iu)) + b[\phi(\delta(Iv,Fv)) + \phi(\delta(Iu, Fu))]\\ &\quad + c\min\{\phi(\delta(Iv, Fu)), \phi(\delta(Iu, Fv))\}\\ &= (a+c)\,\phi(d(v,u)) < \phi(d(v,u))\end{aligned}$$

a contradiction. So $v = u$. So u is the unique common fixed point of F and

Remark 1. In theorem 2.1 if we put $c = 0$ and $\phi(t) = t$ for every $t \ge 0$, obtain a result in Rus [8].

Remark 2. We note that if F is a single-valued mapping on X and I is identity mapping, theorem 2.1 reduces to the following result:

COROLLARY 1. *Let F be mapping of a complete metric space (X,d) itself. Suppose there is an increasing, continuous function ϕ: $R^+ \to R^+$ satisf* *property* (ii) *such that for all* $x,y \in X$, $x \neq Y$,

$$\text{(II)}\quad \phi(\delta(Fx,Fy)) \leq a\phi(d(x,y)) + b[\phi(\delta(x,Fx)) + \phi(\delta(y,Fy))] + c\min\{\phi(\delta(x,Fu)), \phi(\delta(y,Fx))\},$$

where a, b, c *are constants satisfying* $a+2b+c < 1$. *Then* F *has a unique fixed point in* X.

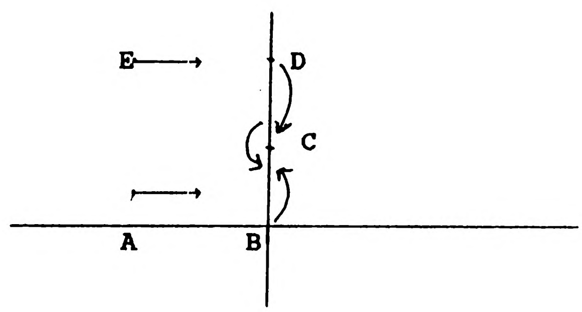
If we assume $c = 0$ in corollary 1, we obtain Theorem 1.

The following example shows that condition (II) is more general that condition (A).

Example 1. ([17]) Let X be the subsetsof R^2 defined by $X = \{A,B,C,D,E\}$, where $A = (-1,0)$, $B = (0,0)$, $C = \left(0, \frac{1}{2}\right)$, $D = (0,1)$, $E = (-1,1)$.

Let F: $X \to X$ be given by

$F(A) = B$, $F(B) = F(C) = F(D) = C$, $F(E) = D$.



Then F satisfies condition (II) by setting:

$a = \frac{3}{4}$, $b = 0$, $c = \frac{1}{5}$ and $\phi(t) = t^3$ for any $t \in R^+$.

However, F does not satisfy condition (A). For otherwise, choosing $x = A$ and

$y = E$ with $a+b+c < 1$, we would have

$$\phi(d(FA, FE)) = \phi(1) \le a\,\phi(1) + b\,\phi(1) + c\,\phi(1) < \phi(1),$$

which is a contradiction.

Motivated by the work of Fisher-Sessa [3], we give the following exan

which satisfy all requirements of Theorem 2.1.

Example 2. Let $X = [0,1]$ with Euclidean metric d and let $\phi(t) = t^4$. De

$$Fx = \left[0, \frac{x}{4+x}\right],\ Ix = \frac{x}{2},$$

for all $x \in X$.

Note that

$$F(X) = \left[0, \frac{1}{5}\right] \subset \left[0, \frac{1}{2}\right] = Ix,$$

and I is continuous mapping.

For any sequence $\{x_n\}$ in X, we have

$Ix_n \to 0$ as $x_n \to 0$,

$Fx_n \to 0$ as $x_n \to 0$

and

$$\delta(FIx_n, IFx_n) = \max\{\frac{x_n}{8+x_n}, \frac{x_n}{8+2x_n}\} \to 0 \text{ as } x_n \to 0,$$

Thus T, F are δ-compatible, $IFx_n \in B(X)$.

For any $x,y \in X$,

$$\phi(\delta(Fx, Fy)) = (\delta(Fx, Fy))^4$$
$$= \left(\max\left\{\frac{x}{4+x}, \frac{y}{4+y}\right\}\right)^4$$

$$= \max\left\{\left(\frac{x}{4+x}\right)^4, \left(\frac{y}{4+y}\right)^4\right\}$$

$$= \max\left\{\left(\frac{x}{4}\right)^4, \left(\frac{y}{4}\right)^4\right\}$$

$$= \left(\frac{1}{2}\right)^4 \max\left\{\left(\frac{x}{2}\right)^4, \left(\frac{y}{2}\right)^4\right\}$$

$$\leq \frac{1}{16} \max\left\{\left|\frac{x}{2} - \frac{y}{2}\right|^4, \left(\frac{x}{2}\right)^4, \left(\frac{y}{2}\right)^4\right\}$$

$$\leq \frac{1}{16}\left[\left|\frac{x}{2} - \frac{y}{2}\right|^4 + \left(\frac{x}{2}\right)^4 + \left(\frac{y}{2}\right)^4\right]$$

$$\leq \frac{1}{16}(d(Ix, Iy))^4 + \frac{1}{16}\left[(\delta(Ix, Fx))^4 + (\delta(Iy, Fy))^4\right] +$$

$$+ \frac{1}{16}\min\left\{(\delta(Ix, Fy))^4, (\delta(Iy, Fx))^4\right\}$$

$$= \frac{1}{16}\phi(d(Ix, Iy)) + \frac{1}{16}\left[\phi(\delta(Ix, Fx)) + \phi(\delta(Iy, Fy))\right] +$$

$$+ \frac{1}{16}\min\{\phi(\delta(Ix, Fy)), \phi(\delta(Iy, Fx))\}.$$

We see that (I) holds with $a = b = c = \frac{1}{16}$, so 0 is the unique common fixed point of F and I. Hence all hypotheses of Theorem 2.1 are satisfied.

In Theorem 2.1, if the function F is replaced by F_α, $\alpha \in \Lambda$ is an index set, we have the following:

THEOREM 2.2. *Let I be map of a complete metric space (X,d) into itself and let F_α: $X \to B(X)$. Suppose there is an increasing, continuous function ϕ:*

$R^+ \to R^+$ *satisfying property* (ii) *such that for all* $x,y \in X$, $x \neq y$.

$$\begin{aligned}\phi(\delta(F_\alpha x, F_\alpha y)) \quad & \leq a\,\phi(d(Ix, Iy)) \\ & + b\,[\phi(\delta(Ix, F_\alpha x)) + \phi(\delta(Iy, F_\alpha y))] \\ & + c\min\{\phi(\delta(Ix, F_\alpha y)), \phi(\delta(Iy, F_\alpha x))\},\end{aligned}$$

where a, b, c *are constants satisfying* $a+2b+c < 1$.

If:

(a) $\bigcup_{\alpha \in \Lambda} F_\alpha(X) \subset I(X)$,

(b) I *is continuous,*

(c) (F_α, I) *is* δ*-compatible for all* α *in* Λ.

Then F_α *and* I *have a unique common fixed point* u *in* X. *Moreover*

$F_\alpha u = \{u\}$ *for all* α *in* Λ.

If the continuity of I in Theorem 2.1 is replaced by the continuity of F, we have the following theorem:

THEOREM 2.3. *Let* (X,d), F *and* I *be defined as in* Theorem 2.1. *Suppose there is an increasing, continuous function* ϕ: $R^+ \to R^+$ *satisfying property* (ii) *such that the inequality* (I) *and the inequality*

(III) $\delta(Fx,Fx) \leq \delta(x,Fx)$ *holds for all* x,y *in* X.

If (a) *and* (c) *hold as in* Theorem 2.1 *and if* F *is continuous, then* F *and* I *have a unique common fixed point* u *in* X. *Further,*

$Fu = \{Iu\} = \{u\}$.

Proof. Define the sequence $\{x_n\}$ as in Theorem 2.1 so that $Ix_n \to u$, $\delta(Fx_n, u) \to 0$ as $n \to \infty$, and so $\delta(Fx_n, Ix_n) \to 0$ as $n \to \infty$. Since F is continuous, we have $\delta(FIx_n, Fu) \to 0$ as $n \to \infty$.

Since F and I are δ-compatible, we get

$$\delta(IFx_n, Fu) \leq \delta(IFx_n, FIx_n) + \delta(FIx_n, Fu)$$

$$\to 0 \text{ as } n \to \infty.$$

Since $Ix_{n+1} \in Fx_n$ and by inequality (I), we have

$$\phi(\delta(FIx_n, Fx_{n+1})) \leq a\,\phi(d(IIx_n, Ix_{n+1}))$$
$$+ b\,[\phi(\delta(IIx_n, FIx_n) + \phi(\delta(Ix_{n+1}, Fx_{n+1}))]$$
$$+ c\min\{\phi(\delta(IIx_n, Fx_{n+1})), \phi(\delta(Ix_{n+1}, FIx_n))\}$$
$$\leq a\,\phi(\delta(IFx_{n-1}, Ix_{n+1}))$$
$$+ b\,[\phi(\delta(IFx_{n-1}, FIx_n) + \phi(\delta(Ix_{n+1}, Fx_{n+1}))]$$
$$+ c\min\{\phi(\delta(IFx_{n-1}, Fx_{n+1})), \phi(\delta(Ix_{n+1}, FIx_n))\}.$$

As $n \to \infty$, by using condition (III) we have

$$\phi(\delta(Fu,u)) \leq a\,\phi(\delta(Fu, u))$$
$$+ b\,[\phi(\delta(Fu, Fu))) + \phi(0)]$$
$$+ c\min\{\phi(\delta(Fu, u))), \phi(\delta(Fu, u)))\}$$
$$\leq (a+b+c)\,\phi\,(\delta(Fu, u)))$$

$$< \phi(\delta(Fu, u)))$$

a contradiction. So $Fu = \{u\}$. Since the range of I contains the range of F, there exists a point u' such that $Iu' = u \in Fu$.

From the inequality (I), we obtain

$$\phi(\delta(FIx_{n+1}, Fu')) \leq a\,\phi(d(IIx_{n+1}, Iu'))$$

$$+ b\,[\phi(\delta(IIx_{n+1}, FIx_{n+1})) + \phi(\delta(Iu', Fu'))]$$

$$+ c\min\{\phi(\delta(IIx_{n+1}, Fu')), \phi(\delta(Iu', FIx_{n+1}))\}$$

$$\leq a\,\phi(\delta(IFx_n, Iu'))$$

$$+ b\,[\phi(\delta(IFx_n, FIx_n) + \phi(\delta(Iu', Fu'))]$$

$$+ c\min\{\phi(\delta(IFx_n, Fu')), \phi(\delta(Iu', FIx_{n+1}))\}.$$

As $n \to \infty$, we have

$$\phi(\delta(u, Fu')) \leq a\phi(0)$$

$$+ b\,[\phi(0) + \phi(\delta(u, Fu'))]$$

$$+ c\min\{\phi(\delta(u, Fu')), \phi(0)\}$$

$$= b\,\phi(\delta(u, Fu'))$$

$$< \phi(\delta(u, Fu'))$$

a contradiction and so $Fu' = \{u\}$. Since F and I are δ-compatible and $\delta(Fu', Iu') = \delta(u,u) = 0$, we have $\delta(IFu', FIu') = 0$ and hence

$$IFu' = \{Iu\} = FIu' = Fu = \{u\}.$$

So $Iu = u$. This proves that the point u is a common fixed point of I and F with $Fu = \{u\}$. The uniqueness of the common fixed point of I and F can be proved.

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ANIVERSĂRI

PROFESSOR JÓZSEF KOLUMBÁN AT HIS 60TH ANNIVERSARY

Wolfgang W. BRECKNER*

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József (Iosif) Kolumbán was born on 4th August 1935 in Gheorgheni, a small town that lies in the district Harghita, Romania, in a picturesque region, surrounded by the Carpathian Mountains. His parents were farmers. After he had studied at the primary and low secondary school in his native town, he attended the pedagogic high school from Miercurea-Ciuc between 1949 and 1953. He completed his education at the Faculty of Mathematics and Physics of the Bolyai University in Cluj (today's Cluj-Napoca). In 1957 he received his diploma in teaching mathematics and physics. After that he also attended the section of mathematical research of the Faculty of Mathematics and Physics of the Babeş University in Cluj and received a second diploma in 1958. In 1959 he became an assistant at the Faculty of Mathematics and Physics of the Babeş-Bolyai University, a new university which was founded in that year in Cluj by the union of the Babeş University with the Bolyai University. Since then József Kolumbán has been working at this university. The most important positions he held during his professional career were: 1969-1978 an assistant professor, 1978-1990 an associate professor, and since 1990 a full professor. His speciality is mathematical analysis,

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but besides this discipline he also taught measure theory, optimization theory, convex analysis and geometry of fractals.

In 1968 József Kolumbán received his doctoral degree in mathematics from the Babeş-Bolyai University with the thesis "The duality principle for a class of optimization problems". This thesis was written under the supervision of Tiberiu Popoviciu (1906-1975), a well-known master of numerical analysis.

The stays of József Kolumbán outside Romania had an impact on the quality of his teaching and scientific work. As a scholar of the Alexander von Humboldt Foundation he spent 15 months in 1972-1973 at Hamburg, working under the guidance of Lothar Collatz (1910-1990) at the Institute of Applied Mathematics. He was on further short period research stays at Hamburg, Augsburg, Bayreuth and Munich during 1991-1992, these stays being supported by the same above-mentioned foundation. In 1993 and 1995, respectively, he was a visiting professor for three months at the Eötvös Loránd University in Budapest.

Professor Kolumbán is a devoted researcher who has made significant contributions in various fields of mathematics such as approximation theory ([1], [3] - [7], [15]), optimization theory ([8] - [14], [17], [33], [34], [39], [43], [47]), mathematical analysis ([2], [16], [19], [23], [25] - [32], [36], [37], [40] - [42], [44] - [46], [48]) and the teaching of mathematics ([18], [20] - [22], [24], [35], [38], [49], [50]). The numbers in the brackets indicate the articles in the annexed list of publications by József Kolumbán where results belonging to the specified fields can be found.

On behalf of the members of the Department of Analysis and Optimization of the Faculty of Mathematics and Computer Science of the Babeş-Bolyai University, as well as, on behalf of the other colleagues and the students of this faculty, we congratulate Professor József Kolumbán with esteem, on his 60th birthday, wishing him good health, happiness and new satisfactions in his research work.

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