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ON CERTAIN SUBCLASSES OF STARLIKE FUNCTIONS

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Dedicated to Professor M. Balázs on his 65th anniversary

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REZUMAT. - Asupra unor clase de funcţii stelate. În lucrare sunt date condiţii suficiente pentru ca pentru o funcţie $f\in A_n$ să aibă loc inegalitatea $|zf'(z)/f(z)-1|<1$ în discul unitate. Astfel de funcţii sunt stelate.

Abstract. For n a positive integer, let A_n denote the class of functions

$$f(z) = z + a_{n+1}z^{n+1} + \dots$$

that are analytic in the unit disc $U = \{z\in\mathbf{C} : |z|<1\}$. In this paper, certain sufficient conditions on a function $f\in A_n$ are found such that $|zf'(z)/f(z)-1|<1$ in U. It is well-known that in this case f is a starlike function.

1. Introduction. Let $H = H(U)$ denote the class of functions analytic in U if $f,F\in H$ and F is univalent, then the function f is subordinate to F, written $f\prec F$, or $f(z)\prec F(z)$, if $f(0) = F(0)$ and $f(U)\subset F(U)$.

For n a positive integer and $a\in\mathbf{C}$, let

$$H[a,n] = \left\{f\in H : f(z) = a + a_nz^n + a_{n+1}z^{n+1} + \dots, z\in U\right\}$$

and

$$A_n = \left\{f\in H : f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots, z\in U\right\},$$

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with $A_1 = A$.

The main result of this paper is given by

THEOREM 1. *Let n be a positive integer and let* $\alpha > 0$. *If* $g \in A_n$ *and*

$$\frac{zg'(z)}{g(z)} \prec 1 + z + \frac{n\alpha z}{1+z} \tag{1}$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + z,$$

where

$$f(z) = \left(\frac{1}{\alpha}\int_0^z g^{1/\alpha}(w)\, w^{-1}\, dw\right)^{\alpha}. \tag{2}$$

If we set

$$J(\alpha, f; z) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf''(z)}{f'(z)} + 1\right), \tag{3}$$

then Theorem 1 can be restated in the following equivalent form

THEOREM 2. *Let n be a positive integer and let* $\alpha > 0$. *If* $f \in A_n$ *and*

$$J(\alpha, f; z) \prec 1 + z + \frac{n\alpha z}{1+z},$$

then $zf'(z)/f(z) \prec 1 + z$, *i.e.*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1, \text{ for } z \in U.$$

If we take $k(z) = ze^z$, then for $f \in A_n$ and $\alpha > 0$ the above result can be rewritten in the following symmetric form

$$J(\alpha, f; z) \prec J(\alpha, k; z) \Rightarrow J(0, f; z) \prec J(0, k; z).$$

2. Preliminaries. We will need the following lemmas to prove Theorem 1.

LEMMA 1. *Let q be analytic and univalent on* $\overline{U}$ *except for at most one pole on* ∂U, *and* $q'(\zeta) \neq 0$ *at other points on* ∂U. *Let* $q(0) = a$ *and* $p \in H[a,n]$. *If p is not subordinate to q, then there exist points* $z_0 \in U$, $\zeta_0 \in \partial U$ *and an* $m \geq n$, *for which* $p(|z| < |z_0|) \subset q(U)$,

(i) $p(z_0) = q(\zeta_0)$ *and*

(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$.

More general versions of this lemma are given in [1,Lemma 1] and [3,Lemma 2.3].

LEMMA 2. *Let n be a positive integer and let $\alpha > 0$. Let $P \in H[1,n]$ and suppose that*

$$P(z) \prec 1 + z + \frac{n\alpha z}{1+z} = h(z). \tag{4}$$

If $p \in H[1,n]$ satisfies the differential equation

$$\alpha z p'(z) + P(z)p(z) = 1 \tag{5}$$

then $p(z) \prec 1/(1+z)$.

Proof. If we put $q(z) = 1/(1+z)$, then

$$h(z) = \frac{1}{q(z)} - n\alpha\frac{zq'(z)}{q(z)}.$$

Since

$$\mathrm{Re}\,[(1+z)h'(z)] = \mathrm{Re}\left[1 + z + \frac{n\alpha}{1+z}\right] > 0, \text{ for } z \in U,$$

we deduce that h is close-to-convex, hence univalent in U.

The domain $h(U)$ is symmetric with respect to the real axis and we have

$$w = h(e^{i\theta}) = re^{i\theta/2} = u + iv, \quad \theta \in (-\pi, \pi),$$

where

$$r = r(\theta) = 2\cos(\theta/2) + \frac{n\alpha}{2\cos(\theta/2)} \tag{6}$$

and

$$\begin{cases} u = 2\cos^2(\theta/2) + \dfrac{n\alpha}{2} \\ v = u\tan(\theta/2). \end{cases} \tag{7}$$

From (4) and (5) we deduce

$$P(z) = \frac{1}{p(z)} - \alpha\frac{zp'(z)}{p(z)} \prec h(z). \tag{8}$$

If we suppose that p is not subordinate to q, then, according to Lemma 1, there exist points $z_0 \in U$ and $\zeta_0 \in \partial U$ and an $m \geq n$, such that $p(z_0) = q(\zeta_0)$ and $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$. Hence from (8) we deduce

$$P(z_0) = \frac{1}{q(\zeta_0)} - \alpha \frac{m\zeta_0 q'(\zeta_0)}{q(\zeta_0)} = 1 + \zeta_0 + \frac{m\alpha\zeta_0}{1+\zeta_0}.$$

If we set $\zeta_0 = e^{i\theta}$, then $P(z_0) = \mathrm{R}e^{i\theta/2}$, with

$$R = R(\theta) = 2\cos(\theta/2) + \frac{m\alpha}{2\cos(\theta/2)} \geq r,$$

where r is given by (6). This shows that $P(z_0) \notin h(U)$, which contradicts (8). Hence we must have $p \prec q$. □

3. Proof of Theorem 1. Let $g \in A_n$ satisfy (1) and let define

$$p(z) = \frac{1}{\alpha g^{1/\alpha}(z)} \int_0^z g^{1/\alpha}(w) w^{-1} dw = \frac{1}{\alpha} \int_0^1 \left[\frac{g(tz)}{g(z)}\right]^{1/\alpha} t^{-1} dt.$$

It is easy to show that $p \in H[1,n]$ and it satisfies (5), with $P(z) = zg'(z)/g(z)$. Hence, by Lemma 2, we deduce that $p(z) \prec 1/(1+z)$. Since $p(z) \neq 0$, for $z \in U$, we can define the analytic function $f \in A_n$ by

$$f(z) = g(z)\,[p(z)]^{\alpha}, \tag{9}$$

which is given by (2). Using (5), from (9) we obtain

$$\frac{zf'(z)}{f(z)} = P(z) + \alpha\frac{zp'(z)}{p(z)} = \frac{1}{p(z)},$$

hence $zf'(z)/f(z) \prec 1 + z$. □

From (2) and (3) we easily obtain

$$J(\alpha, f; z) = \frac{zg'(z)}{g(z)}$$

and we deduce that Theorem 2 is equivalent to Theorem 1.

4. Particular cases. From (6) we deduce

$$M(\alpha, n) = \min r(\theta) = \begin{cases} 2\sqrt{n\alpha}, & \text{for } 0 < \alpha n \le 4 \\ 2 + \dfrac{n\alpha}{2}, & \text{for } n\alpha > 4. \end{cases} \tag{10}$$

Similarly, from (7) we have

$$\rho^2 = (u-1)^2 + v^2 = \frac{4u^2}{2u - n\alpha} - 2u + 1, \text{ for } \frac{n\alpha}{2} \le u \le 2 + \frac{n\alpha}{2}$$

and we deduce

$$\min \rho = 1 + \frac{n\alpha}{2}. \tag{11}$$

Using (10) and (11), from Theorem 2 we deduce the following particular results.

COROLLARY 1. *If* $f \in A_n$, $\alpha > 0$ *and*

$$|J(\alpha, f; z)| < M(\alpha, n), \quad z \in U,$$

where $J(\alpha, f; z)$ *is given by* (3) *and* $M(\alpha, n)$ *is given by* (10), *then*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1, \quad z \in U.$$

Example 1. If we take $n\alpha = 1$ in Corollary 1 we obtain

$$f \in A_n \text{ and } \left|(n-1)\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} + 1\right| < 2n \Rightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| < 1.$$

For $n = 1$ we get

$$f \in A \text{ and } \left|\frac{zf''(z)}{f'(z)} + 1\right| < 2 \Rightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| < 1.$$

This last result was obtained in [2,Theorem 5].

Example 2. If we take $\alpha = 1$ and $n = 2$ in Corollary 1 we obtain

$$f \in A_2 \text{ and } \left|\frac{zf''(z)}{f'(z)} + 1\right| < 2\sqrt{2} \Rightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| < 1.$$

COROLLARY 2. *If* $f \in A_n$, $\alpha > 0$ *and*

$$|J(\alpha, f; z) - 1| < 1 + \frac{n\alpha}{2}, \; z \in U,$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1, \; z \in U$$

For $n = 1$ this result was obtained in [4,Theorem 3].

Example 3. If we take $n\alpha = 1$ in Corollary 2 we obtain

$$f \in A_n \text{ and } \left|(n-1)\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} + 1 - n\right| < \frac{3n}{2} \Rightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| < 1.$$

For $n = 1$ we get

$$f \in A \text{ and } \left|\frac{f''(z)}{f'(z)}\right| \leq \frac{3}{2} \Rightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| < 1.$$

Example 4. If we take $\alpha = 1$ and $n = 2$ in Corollary 2 we obtain

$$f \in A_2 \text{ and } \left|\frac{f''(z)}{f'(z)}\right| \leq 2 \Rightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| < 1.$$

In particular, if we consider the error function

$$f(z) = \operatorname{erf}(z) = \int_0^z e^{-w^2} dw,$$

then $f \in A_2$ and $|f''(z)/f'(z)| \leq 2$. Hence $f = \operatorname{erf}$ satisfies $|zf'(z)/f(z) - 1| < 1$, which is equivalent to

$$\operatorname{Re} \int_0^1 e^{(1-t^2)z^2} dt > \frac{1}{2}, \text{ for } z \in U.$$

5. Other equivalent form of Theorem 1. If we set

$$f(z) = \left[\frac{g(z)}{z}\right]^{1/\alpha},$$

then (1) becomes

$$\alpha \frac{zf'(z)}{f(z)} \prec z + \frac{n\alpha z}{1+z}$$

and if we put $\beta = 1/\alpha$, we deduce the following equivalent form of Theorem 1.

THEOREM 3. *Let n be a positive integer and let $\beta > 0$. If $f \in H[1,n]$ satisfies*

$$\frac{zf'(z)}{f(z)} \prec \beta z + \frac{nz}{1+z},$$

then

$$\operatorname{Re} \frac{F(z)}{f(z)} > \frac{1}{2}, \text{ for } z \in U,$$

where

$$F(z) = \frac{\beta}{z^{\beta}} \int_0^z f(w) w^{\beta-1} dw = \beta \int_0^1 f(tz) t^{\beta-1} dt. \tag{12}$$

Using (11), with $\alpha = 1/\beta$, from Theorem 3 we obtain the following result

COROLLARY 3. *If $f \in H[1,n]$, $\beta > 0$ and*

$$\left|\frac{f'(z)}{f(z)}\right| \leq \beta + \frac{n}{2}, \quad z \in U,$$

then

$$\mathrm{Re}\, \frac{F(z)}{f(z)} > \frac{1}{2}, \quad \textit{for } z \in U,$$

where F is given by (12).

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ELIPTIC SYSTEMS WITH DISCONTINUOUS NONLINEARITY

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REZUMAT. - Sisteme eliptice cu neliniaritate discontinuă. În lucrare este studiată problema la limită (1) în cazul când f nu depinde de x şi prezintă discontinuităţi de speţa întâia în raport cu variabilele $u_1,\dots,u_N$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary $\partial\Omega$. We consider the boundary value problem

$$L_i u = f_i(\cdot, u) \text{ in } \Omega,\ u|_{\partial\Omega} = 0 \quad i = 1, \dots, N, \tag{1}$$

$u = (u_1, \dots, u_N)$, where L_i are second order linear differential operators with real coefficients

$$L_i u = -\sum_{j=1}^{N} \sum_{k,l=1}^{n} \frac{\partial}{\partial x_k}\left[a_{kl}^{ij} \frac{\partial u_j}{\partial x_k \partial x_l}\right] + \sum_{j=1}^{N} \sum_{k=1}^{n} a_k^{ij} \frac{\partial u_j}{\partial x_k} + \sum_{j=1}^{N} a_0^{ij} u_j \quad i = 1, \dots, N \tag{2}$$

and $f: \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a given function.

If the system (1) is strongly elliptic [14] or weakly closed [3], the coefficients of L_i are bounded, f does not depend on u and is square integrable, then problem (1) is of Fredholm type; particulary if the unicity holds for problem (1), then there exists a unique weak (variational) solution u of (1) and an estimate of the form

$$\|u\|_{H_0^1(\Omega, \mathbb{R}^N)} \le C \|f\|_{L^2(\Omega, \mathbb{R}^N)} \tag{3}$$

is true.

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If the functions f_i depend on u and satisfy Caratheodory conditions, then many existence results were obtained using methods like: variational-, continuation-, monoton operators method etc.

Ourdays in many tehnical problems appear systems where the functions f_i are discontinuous in the variables $u_1, \ldots, u_N$ [1, 2, 4, 5, 6, 11].

Some authors study such problems, substituting f_i by a multivalued mapping, considering instead of $f(x, u_0)$ the jump of f in u_0. This way appear boundary value problems as

$$Lu \in \mathscr{F}(\cdot, u) \text{ in } u|_{\partial\Omega} = 0 \tag{4}$$

[7, 9, 12, 13]. The solutions of (4) are not always solutions of the initial considered problem (1), but among the solutions of (4) there are often functions which are solutions of (1) too.

For simplicity we shall study the solvability of the boundary value problem (1) in the case when f does not depend on x and has discontinuity of first kind with respect to the variables $u_1, \ldots, u_N$.

The used spaces are:

$$L^2(\Omega, \mathbb{R}^N) = \left\{ u = (u_1, \ldots, u_N) \mid u_i \in L^2(\Omega),\ i = 1, \ldots, N \right\}$$

with the scalar product resp. norm

$$(u, v)_{L^2(\Omega, \mathbb{R}^N)} = \int_\Omega \sum_{i=1}^N u_i v_i \, dx,\ \|u\|^2_{L^2(\Omega, \mathbb{R}^N)} = \int_\Omega \sum_{i=1}^N u_i^2 \, dx; \tag{5}$$

$$H_0^1(\Omega, \mathbb{R}^N) = \left\{ u \in L^2(\Omega, \mathbb{R}^N) \mid \frac{\partial u_i}{\partial x_k} \in L^2(\Omega),\ u_i|_{\partial\Omega} = 0,\ i = 1, \ldots, N,\ k = 1, \ldots, n \right\},$$

with the scalar product resp. norm

$$(u, v)_{H_0^1(\Omega, \mathbb{R}^N)} = \int_\Omega \sum_{i=1}^N \sum_{k=1}^n \frac{\partial u_i}{\partial x_k} \frac{\partial v_i}{\partial x_k} \, dx,\ \|u\|^2_{H_0^1(\Omega, \mathbb{R}^N)} = \int_\Omega \sum_{i=1}^N \sum_{k=1}^n \left| \frac{\partial u_i}{\partial x_k} \right|^2 dx; \tag{6}$$

$$L_+^2(\Omega,\mathbb{R}^N) = \left\{ u \in L^2(\Omega,\mathbb{R}^N) \mid u_i(x) \geq 0 \text{ a.e. on } \Omega \right\}.$$

We shall assume that $L_2(\Omega,\mathbb{R}^N)$ and hence also $H_0^1(\Omega,\mathbb{R}^N)$ is partially ordered by $u \leq v$ if and only if $v-u \in L_+^2(\Omega,\mathbb{R}^N)$. If $\underline{u},\bar{u} \in L^2(\Omega,\mathbb{R}^N)$ and $\underline{u} \leq \bar{u}$, we denote $[\underline{u},\bar{u}] = \left\{ u \in L^2(\Omega,\mathbb{R}^N) \mid \underline{u} \leq u \leq \bar{u} \right\}$.

Assume that the coefficients of L_i are from $L^\infty(\Omega)$ and build the bilinear forms

$$a_i(u,v_i) = \int_\Omega \sum_{j=1}^N \sum_{k,l=1}^n a_{kl}^{ij} \frac{\partial u_j}{\partial x_k}\frac{\partial v_i}{\partial x_l}dx + \int_\Omega \sum_{j=1}^N \sum_{k=1}^n a_k^{ij} \frac{\partial u_j}{\partial x_k} v_i\,dx + \int_\Omega \sum_{j=1}^N a_0^{ij} u_j v_i\,dx \quad i=1,\dots,N \tag{7}$$

and

$$a(u,v) = \sum_{i=1}^N a_i(u,v_i) \quad u,v \in H_0^1(\Omega,\mathbb{R}^N). \tag{8}$$

We say that $u \in H_0^1(\Omega,\mathbb{R}^N)$ is a weak solution of (1) if $f_i(u) \in L^2(\Omega)$ and

$$a_i(u,v_i) = \int_\Omega f_i(u)v_i\,dx \quad \text{for all } v_i \in H_0^1(\Omega) \quad i=1,\dots,N. \tag{9}$$

u is called weak upper solution if $f_i(u) \in L^2(\Omega)$ and

$$a_i(u,v_i) \geq \int_\Omega f_i(u)v_i\,dx \quad \text{for all } v_i \in H_0^1(\Omega) \cap L_+^2(\Omega). \tag{10}$$

Similary, u is weak lower solution if $f_i(u) \in L^2(\Omega)$ and

$$a_i(u,v_i) \leq \int_\Omega f_i(u)v_i\,dx \quad \text{for every } v_i \in H_0^1(\Omega) \cap L_+^2(\Omega). \tag{11}$$

We shall impose the following hypotheses on the operators L_i and on the functions f_i:

α_1) $a_{kl}^{ij}, a_k^{ij}, a_0^{ij} \in L^\infty(\Omega)$

α_2) The system (1) is strongly elliptic or weakly closed

α_3) There exists a positive constant M_1 such that for the elliptic operator $L+M_1I$ the weak maximum and minimum principle is true in the sense that

$$a_i(u,v_i) + M_1(u_i,v_i)_{L^2(\Omega)} \geq 0 \;\forall\, v_i \in H_0^1(\Omega) \cap L_+^2(\Omega) \tag{12}$$

implies that $u(x) \leq 0$ a.e. on Ω, and from

$$a_i(u,v_i) + M_1(u_i,v_i)_{L^2(\Omega)} \leq 0 \;\forall\, v_i \in H_0^1(\Omega) \cap L_+^2(\Omega) \tag{13}$$

results that $u(x) \geq 0$ a.e. on Ω.

Conditions α_2, α_3 are obviously fulfilled if $L_i u$ contains only the function u_i,

$$L_i u = \sum_{k,l=1}^{n} \frac{\partial}{\partial x_k}\left[a_{kl}^i \frac{\partial u_i}{\partial x_l}\right] + \sum_{k=1}^{n} a_k^i \frac{\partial u_i}{\partial x_k} + a_0^i u_i \quad i = 1, \ldots, N \tag{14}$$

and there exists $\mu > 0$ such that

$$\sum_{k,l=1}^{n} a_{kl}^i(x)\xi_k\xi_l \geq \mu \sum_{k=1}^{n} \xi_k^2 \quad \forall\, x \in \Omega, \quad \forall\, \xi \in \mathbb{R}^N,\ i = 1, \ldots, N. \tag{15}$$

For the maximum and minimum principle for elliptic systems see [8, 10].

β_1) There exists a positive constant M_1 such that the functions

$$F_i(t) = f_i(t) + Mt_i \quad t \in \mathbb{R}^N \quad i = 1, \ldots, N$$

are monotone increasing for every $M \geq M_1$ e.g.

$$F_i(t^1) \leq F_i(t^2) \quad \text{if} \quad t_i^1 \leq t_i^2 \quad i = 1, \ldots, N.$$

β_2) There exist a finite or countable number of surfaces $S_k \subset \mathbb{R}^N$ for which we have a representation

$$S_k = \left\{(t_1, \ldots, t_N) \in \mathbb{R}^N \mid t_N = \varphi_{N,k}(t'),\ t' = (t_1, \ldots, t_{N-1}) \in \mathbb{R}^{N-1}\right\},\ \text{where } \varphi_{N,k} \in C^1(\mathbb{R}^{N-1})$$

and

$$\varphi_{N,k}(t') > \varphi_{N,k-1}(t') \quad \forall\, t' \in \mathbb{R}^{N-1}, \quad \forall\, k.$$

The functions $f_i : \mathbb{R}^N \to \mathbb{R}^N$ are continuous on $\mathbb{R}^N \setminus \bigcup_k S_k$, f has one-side limits on S_k, e.g.

$$f^-(t) = \lim_{\substack{\tau \to t \\ \tau_N < t_N}} f(\tau_1, \ldots, \tau_N), \quad f^+(t) = \lim_{\substack{\tau \to t \\ \tau_N > t_N}} f(\tau_1, \ldots, \tau_N)$$

exist and are finite.

We assume also

γ_1) the boundary value problem (1) has a lower and an upper solution $\underline{u}$ and $\bar{u}$ such that $\underline{u} \leq \bar{u}$.

LEMMA 1. *If conditions* β_1), β_2) *and* γ_1) *are fulfilled and* $M \geq \max\{M_1, M_2\}$, then

1° *For every* $u \in [\underline{u}, \bar{u}]$ *we have* $F(u) = f(u) + Mu \in L^2(\Omega, \mathbb{R}^N)$.

2° *If* $u, v \in [\underline{u}, \bar{u}]$ *and* $u \leq v$ *then* $F(u) \leq F(v)$

3⁰ *There exists a constant* $C_1 > 0$ *such that*

$$\| F(u)\|_{L^2(\Omega, \mathbb{R}^N)} \le C_1 \text{ for every } u \in [\underline{u}, \bar{u}]$$

(C_1 *depends on* M).

Proof. Sentence 2⁰ results immediately from condition β_1.

From condition β_1 and β_2 we obtain that $F(u): \Omega \to \mathbb{R}^N$ is measurable and

$$F_i(\underline{u}) \le F_i(u) \le F_i(\bar{u}) \quad \forall u \in [\underline{u}, \bar{u}].$$

Thus

$$-|F_i(\underline{u})| \le F_i(\underline{u}) \le F_i(u) \le F_i(\bar{u}) \le |F_i(\bar{u})| \quad i = 1, \dots, N,$$

$$|F_i(\underline{u})| \le |F_i(\underline{u})| + |F_i(\bar{u})| \quad \text{and} \quad |F_i(u)|^2 \le 2(|F_i(\underline{u})|^2 + |F_i(\bar{u})|^2).$$

But $F_i(\underline{u}), F_i(\bar{u}) \in L^2(\Omega)$, so $F_i(u) \in L^2(\Omega)$ and

$$\| F_i(u)\|^2_{L^2(\Omega)} \le 2\| F_i(\underline{u})\|^2_{L^2(\Omega)} + 2\| F_i(\bar{u})\|^2_{L^2(\Omega)} = K_i^2 \quad i = 1, \dots, N,$$

so 1⁰ and 3⁰ are proved with the constant $C_1^2 = \sum_{i=1}^{N} K_i^2$.

LEMMA 2. *Assume that conditions* α_1, α_2, α_3, β_1, β_2 *and* γ_1 *are fulfilled. Let* $w \in [\underline{u}, \bar{u}]$ *a fixed function,* $M_0 = \max\{M_1, M_2\}$. *Then the boundary value problem*

$$Lu + Mu = f(w) + Mw \text{ in } \Omega, \; u|_{\partial\Omega} = 0 \; M \ge M_0 \tag{16}$$

has a unique weak solution $u \in [\underline{u}, \bar{u}]$.

Proof. From lemma 1 results that $f(w) + Mw \in L^2(\Omega, \mathbb{R}^N)$. In this case the unique solvabilty of the problem (16) results from conditions α_1, α_2, α_3. u is a weak solution of (16) and $\bar{u}$ a weak upper solution of (1), thus

$$a_i(u, v_i) + M(u_i, v_i)_{L^2(\Omega)} = (F_i(w), v_i)_{L^2(\Omega)} + M(w_i, v_i)$$

and

$$a_i(\bar{u}, v_i) + M(\bar{u}_i, v_i)_{L^2(\Omega)} \ge (F_i(\bar{u}), v_i)_{L^2(\Omega)} + M(w_i, v_i), \quad \forall v_i \in H_0^1(\Omega) \cap L_+^2(\Omega).$$

From these two we obtain

$$a_i(\bar{u} - u, v_i) + M(\bar{u}_i - u_i, v_i) \geq (F_i(\bar{u}) - F_i(w), v_i)_{L^2(\Omega)} \geq 0 \ \forall\ v_i \in H_0^1(\Omega) \cap L_+^2(\Omega),$$

which according to the maximum principle gives $\bar{u} - u \geq 0$ a.e. on Ω. In a simillar way we obtain the inequality $\underline{u} - u \leq 0$, thus $\underline{u} \leq u \leq \bar{u}$.

Let M_3 a constant greater then $M_0 = \max\ (M_1, M_2)$. We consider the family of the boundary value problems (16) when w describes the interval $[\underline{u}, \bar{u}]$ and $M \in [M_0, M_3]$. We denote by u_{wM} the solution of problem (16).

LEMMA 3. *There exists a positive constant C_2 depending on $\underline{u}, \bar{u}$ and M_3 such that*

$$\|u_{wM}\|_{H_0^1(\Omega,\mathbb{R}^N)} \leq C_2 \quad \forall\ w \in [\underline{u}, \bar{u}], \quad \forall\ M \in [M_0, M_3].$$

Proof. The coefficients of the operator $L + M\ I$ are from the space $L^\infty(\Omega)$, $f(w) + Mw \in L^2(\Omega, \mathbb{R}^N)$, then from the conditions α_1, α_2, α_3 results that there exists a constant $C > 0$ such that for the solution u_{wM} of problem (16) we have

$$\|u_{wM}\|_{H_0^1(\Omega,\mathbb{R}^N)} \leq C\|f(w) + Mw\|_{L^2(\Omega,\mathbb{R}^N)}.$$

But from lemma 1 results that

$$\|f(w) + Mw\|_{L^2(\Omega,\mathbb{R}^N)} \leq C_1(M)\ \forall\ w \in [\underline{u}, \bar{u}],$$

so

$$\|u_{wM}\|_{H_0^1(\Omega,\mathbb{R}^N)} \leq C_2 \quad \forall\ w \in [\underline{u}, \bar{u}], \quad \forall\ M \in [M_0, M_3]$$

with a constant C_2 conveniently choosen.

LEMMA 4. *Let $u^1, u^2, \ldots, u^k, \ldots$ a monotone sequence (increasing or decreasing) from $H_0^1(\Omega, \mathbb{R}^N)$ for which there exists a constant C_3 such that*

$$\|u^k\|_{H_0^1(\Omega,\mathbb{R}^N)} \leq C_3 \quad k = 1, 2, \ldots,$$

then the sequence $(u^k)_{k\geq 1}$ is strongly convergent in $H_0^1(\Omega, \mathbb{R}^N)$.

For the proof see [4].

THEOREM 1. *Let $\underline{u}, \bar{u} \in H_0^1(\Omega, \mathbb{R}^N)$ be one lower resp. upper solution of the* *boundary value problem* (1). *Assume that the hypotheses* α_1, α_2, α_3, β_1, β_2 *and* γ_1 *are fulfilled and* $f^+(t) = f(t)$ (*or* $f^-(t) = f(t)$) *for every* $t \in \bigcup_k S_k$, *then there exists at least one weak solution* $u \in [\underline{u}, \bar{u}]$ *of problem* (1).

Proof. Following the method presented by S. Carl in [4] a constructive iterative method may be given, solving an infinite sequence of variational boundary value problems (with Dirichlet date).

We chose an $M \in [M_0, M_1]$. Let $u^0 = \bar{u}$ and $u^1 \in H_0^1(\Omega, \mathbb{R}^N)$ the unique weak solution of the problem

$$Lu^1 + Mu^1 = f(u^0) + Mu^0 \text{ in } \Omega,\ u^1|_{\partial\Omega} = 0. \tag{17}$$

We have then

$$a_i(u^0, v) + M(u_i^0, v)_{L^2(\Omega)} \geq (f_i(u^0), v)_{L^2(\Omega)} + M(u_i^0, v)_{L^2(\Omega)}$$

and

$$a_i(u^1, v) + M(u_i^1, v)_{L^2(\Omega)} = (f_i(u^0), v)_{L^2(\Omega)} + M(u_i^0, v)_{L^2(\Omega)}$$

for every $v \in H_0^1(\Omega) \cap L_+^2(\Omega)$. From these two we have

$$a_i(u^0 - u^1, v) + M(u_i^0 - u_i^1, v)_{L^2(\Omega)} \geq 0 \ \forall\, v \in H_0^1(\Omega) \cap L_+^2(\Omega).$$

Similarly

$$a_i(\underline{u} - u^1, v) + M(\underline{u}_i - u_i^1, v)_{L^2(\Omega)} \leq 0 \ \forall\, v \in H_0^1(\Omega) \cap L_+^2(\Omega).$$

Using the maximum resp. minimum principle we obtain

$$\underline{u} \leq u^1 \leq u^0 = \bar{u}.$$

In the same manner the sequence $u^1, u^2, \ldots, u^k, \ldots$ is built solving the boundary value problems

$$Lu^{k+1} + Mu^{k+1} = f(u^k) + Mu^k \text{ in } \Omega,\ u^{k+1}|_{\partial\Omega} = 0 \quad k = 1, 2, \ldots \tag{18}$$

It is obvious that

$$\underline{u} \le u^{k+1} \le u^k \le \dots \le u^1 \le u^0 = \bar{u}. \tag{19}$$

According to lemma 3 the sequence $(u^k)_{k\ge 1}$ is bounded

$$\|u^k\|_{H_0^1(\Omega,\mathbb{R}^N)} \le C_2 \quad \forall\ k \in \mathbf{N},$$

and then from lemma 4 results that the sequence $(u^k)_{k\in\mathbf{N}}$ is strongly convergent in $L^2(\Omega,\mathbb{R}^N)$ and weakly convergent in $H_0^1(\Omega,\mathbb{R}^N)$. From the strong convergence of $(u^k)_{k\in\mathbf{N}}$ results that there exists a subsequence of $(u^k)_{k\in\mathbf{N}}$ convergent almost every-where. Let $u(x)$ the limit of the convergent subsequence. The sequence $(u^k)_{k\in\mathbf{N}}$ being monotone decreasing, results that the whole sequence $(u^k)_{k\in\mathbf{N}}$ is convergent a.e. to $u \cdot u^{k+1}$ is the weak solution of problem (18), thus

$$\begin{aligned} a(u^{k+1}, v) + M(u^{k+1}, v)_{L^2(\Omega,\mathbb{R}^N)} = \\ = \int_\Omega \sum_{i=1}^N f_i(u^k)\, v_i\, dx + M(u^k, v)_{L^2(\Omega,\mathbb{R}^N)} \quad \forall\ v \in H_0^1(\Omega,\mathbb{R}^N). \end{aligned} \tag{20}$$

$u^k \to u$ in $L^2(\Omega,\mathbb{R}^N)$, $u^k \to u$ (weakly) in $H_0^1(\Omega,\mathbb{R}^N)$, consequently

$$\lim\ a(u^{k+1}, v) + M(u^{k+1}, v)_{L^2(\Omega,\mathbb{R}^N)} = a(u, v) + M(u, v)_{L^2(\Omega,\mathbb{R}^N)}.$$

We show that the limit of the right side of (20) exists and is equal to

$$\int_\Omega \sum_{i=1}^N f_i(u)\, v_i\, dx + M(u, v)_{L^2(\Omega,\mathbb{R}^N)}.$$

$u^k(x)$ converges decreasing to $u(x)$ a.e. on Ω, f_i is continuous on $\mathbb{R}^N \setminus \bigcup_j S_j$, $f_i(t) = f_i^*(t)$ on S_j, thus $f_i(u^k(x)) \to f_i(u(x))$ a.e. on Ω, and from lemma 3 results that

$$\left| \int_\Omega f_i(u^k(x))\, v_i(x)\, dx \right| \le \|f_i(u^k)\|_{L^2(\Omega)} \cdot \|v_i\|_{L^2(\Omega)} \le C \quad k = 1, 2, \dots$$

where C is a conveniently chosen constant. Thus we can pass to limit in the right side of (20) too, and we obtain

$$a(u, v) = \int_\Omega \sum_{i=1}^N f_i(u)\, v_i\, dx \quad \forall\ v \in H_0^1(\Omega,\mathbb{R}^N),$$

which means that u is a weak solution of problem (1).

If $f_i(t) = f_i^-(t)$ on S_p then a similar construction of the sequence $(u^k)_{k\in\mathbf{N}}$ is made, starting with the element $u^0 = \underline{u}$ (lower solution) and continuing by

$$Lu^1 + Mu^1 = f(\underline{u}) + M\underline{u} \text{ in } \Omega,\ u^1|_{\partial\Omega} = 0,$$

$$Lu^{k+1} + Mu^{k+1} = f(u^k) + Mu^k \text{ in } \Omega,\ u^{k+1}|_{\partial\Omega} = 0 \tag{21}$$

□

It is obvious, that for both cases $f_i(t) = f_i^+(t)$ and $f_i(t) = f_i^-(t)$ we may start the interative method with any $u^0 \in [\underline{u}, \bar{u}]$. The sequences built by the algorithm (18) resp. (21) may converge to an element different from that obtained in the proof of theorem 1. The following result holds:

THEOREM 2. a) *If in the theorem* 1 $f_i(t) = f_i^+(t)$ $\forall\ t \in \bigcup_j S_j$, *then the solution of the boundary value problem* (1) *obtained in the proof of theorem* 1 *is maximal in the sense that for all solutions* $w \in [\underline{u}, \bar{u}]$ *of problem* (1) *we have* $w \le u$.

b) *If* $f_i(t) = f_i^-(t)$ $\forall\ t \in \bigcup_j S_j$, *then the solution* u *obtained by algorithm* (21) *is minimal, that is* $w \ge u$ *for any solution* $w \in [\underline{u}, \bar{u}]$ *of the problem* (1).

Proof. a) Let $w \in [\underline{u}, \bar{u}]$ a solution of (1), then w is a lower solution too. Repeating the construction from theorem 1 on the interval $[w, \bar{u}]$ with the starting element $u^0 = \bar{u}$ we obtain again the same solution u obtained in the proof of theorem 1. This u, according to the proof of theorem 1 belongs to interval $[w, \bar{u}]$ thus $w \le u$. The proof of b) is similary.

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ON THE MONOTONICITY OF THE SEQUENCE FORMED BY THE FIRST ORDER DERIVATIVES OF A FAVARD-SZASZ TYPE OPERATOR

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REZUMAT. - Asupra monotoniei şirului format de derivatele de ordinul unu ale unui operator de tip Favard-Szasz. În această lucrare se studiază proprietatea de monotonie a şirului format cu prima derivată a unui operator de tip Favard-Szasz, obţinut cu ajutorul polinoamelor lui Appell. În felul acesta se extinde un rezultat al lui D.D.Stancu [6] relativ la operatorul lui Bernstein.

Abstract. In this paper one studies the monotonicity property of the sequence formed by the first order derivatives of an operator of type Favard-Szasz. The result obtained by the author represents an extension of an earlier result established by D.D. Stancu for the Bernstein operator.

I. Introduction. The study of the monotonicity of a sequence of positive linear operators and of their derivatives was the aim of a lot of papers O. Aramă [1], D.D. Stancu [6], [7] have studied the monotonicity of the sequence of Bernstein polynomials and of the sequence formed by their derivatives. The study has been extended to other linear positive operators, for instance to Favard-Szasz and Baskakov operators: [2], [3], [5], [8]. The aim of this paper is to study the monotonicity of the sequence formed by the first derivatives of a

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Favard-Szasz type operator obtained by A. Jakimovski and D. Leviatan [4] by means of Appell polynomials.

One considers $g(z) = \sum_{n=0}^{\infty} a_n z^n$ an analytic function in the disk $|z| < R$, $R > 1$ and suppose $g(1) \neq 0$, $a_n \in R$ for $n = 0,1,\ldots$. Define the Appell polynomials $p_k(x)$ ($k \geq 0$) by

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k. \tag{1}$$

To each function f defined in $[0,\infty)$ we associate the operators P_n, defined by

$$(P_n f)(x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \quad n = 1, 2, \ldots \tag{2}$$

The case $g(z) = 1$ yields the Favard-Szasz operators

$$(S_n f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right)$$

B. Wood [9] has proved that the operator P_n is positive in $[0,\infty)$ if and only if $\frac{a_n}{g(1)} \geq 0$, $n = 0, 1, \ldots$. B. Wood also studied the monotonicity of the sequence (P_n). He proved:

THEOREM A. *Let P_n be positive in $[0,\infty)$, $n = 1,2,\ldots$. Assume $\sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right)$ converges uniformly and absolutely in $[0,b]$, for any $b > 0$ and suppose f is convex and monotone nondecreasing in $[0,\infty)$. Then*

$$(P_n f)(x) - (P_{n+1} f)(x) \geq 0, \quad n = 0, 1, \ldots \text{ for } 0 \leq x < \infty.$$

For the Favard-Szasz operators, $\{S_n\}$, E.W. Cheney and A. Sharma have proved in their paper [2].

THEOREM B. *If the function f is defined and non-concave of the first order on the interval $[0,\infty)$, then the sequence of the Szasz-Mirakyan operators $\{(S_n f)(x)\}$, $n = 1, 2, \ldots$ is non-increasing on this interval.*

$$(S_n f)(x) \geq (S_{n+1} f)(x), \quad x \in [0, \infty), \quad n = 1, 2, \ldots$$

II. The aim of this paper is to study the monotonicity of the sequence formed by the first order derivatives of the operators P_n.

We will use the following

DEFINITION. A real-valued function f is called non-concave of the order k on the interval I, if $[x_1, x_2, \ldots, x_{k+2}; f] \geq 0$ for any system of $k+2$ points of the interval I.

LEMMA [3]. *Let f be non-concave of the first and the second orders on the interval $[0,\infty)$. Then for any three points $x_1, x_2, x_3 \in [0,\infty)$, $x_1 < x_2 < x_3$ and an arbitrary positive number α there holds*

$$[x_1 + a, x_2 + a, x_3 + a; f] \geq [x_1, x_2, x_3; f] \geq 0.$$

We will prove the following theorem

THEOREM. *Let P_n be positive in $[0,\infty)$, $n = 0,1,\ldots$ and assume that the sequence $\{(P_n f)(x)\}$ converges uniformly and absolutely in $[0,a]$ for any $a > 0$. If the function f is non-concave of the first and second order on the interval $[0,\infty)$, then the sequence formed by the first order derivatives of the sequence $\{(P_n f)(x)\}$ is non-increasing on this interval, i.e.*

$$(P_n' f)(x) \geq (P_{n+1}' f)(x), \ x \in [0,\infty), \ n = 1, 2, \ldots$$

Proof. First we note that $p_k'(x) = p_{k-1}(x)$, $(p_{-1}(x) = 0)$. By derivation of $(P_n f)(x)$, we obtain

$$(P_n f)'(x) = \frac{e^{-nx}}{g(1)} \sum_{\nu=0}^{\infty} p_\nu(nx) \left[\frac{\nu}{n}, \frac{\nu+1}{n}; f\right]$$

It follows that

$$\Delta(P_n f)'(x) = (P_n f)'(x) - (P_{n+1} f)'(x) =$$

$$= \frac{e^{-(n+1)x}}{g(1)} \sum_{\nu=0}^{\infty} e^{x} p_\nu(nx) \left[\frac{\nu}{n}, \frac{\nu+1}{n}; f\right] - \frac{e^{-(n+1)x}}{g(1)} \sum_{\nu=0}^{\infty} p_\nu[(n+1)x] \left[\frac{\nu}{n+1}, \frac{\nu+1}{n+1}; f\right] \tag{3}$$

First we compute the first term of this difference, by taking account of the assumed convergence and of the expression of Appell polynomial $p_\nu(nx) = \sum_{s=0}^{\nu} a_{\nu-s} \frac{(nx)^s}{s!}$

We have

$$\sum_{\nu=0}^{\infty} e^{x} p_\nu(nx)\left[\frac{\nu}{n},\frac{\nu+1}{n};f\right]=\sum_{k=0}^{\infty}\frac{x^k}{k!}\sum_{\nu=0}^{\infty}p_\nu(nx)\left[\frac{\nu}{n},\frac{\nu+1}{n};f\right]=$$

$$=\sum_{\nu=0}^{\infty}\frac{x^\nu}{\nu!}\sum_{k=0}^{\infty}p_k(nx)\left[\frac{k}{n},\frac{k+1}{n};f\right]=\sum_{\nu=0}^{\infty}\frac{x^\nu}{\nu!}\sum_{k=0}^{\infty}\left(\sum_{s=0}^{k}a_{k-s}\frac{(nx)^s}{s!}\right)\left[\frac{k}{n},\frac{k+1}{n};f\right]$$

In this product, the coefficient of x^ν is:

$$\frac{1}{\nu!}\sum_{s=0}^{\infty}a_s\left[\frac{s}{n},\frac{s+1}{n};f\right]+\frac{n}{1!(\nu-1)!}\sum_{s=1}^{\infty}a_{s-1}\left[\frac{s}{n},\frac{s+1}{n};f\right]+\ldots+$$

$$+\frac{n^k}{k!(\nu-k)!}\sum_{s=k}^{\infty}a_{k-s}\left[\frac{s}{n},\frac{s+1}{n};f\right]+\ldots+\frac{n^\nu}{\nu!}\sum_{s=\nu}^{\infty}a_{s-\nu}\left[\frac{s}{n},\frac{s+1}{n};f\right]=$$

$$=\sum_{k=0}^{\nu}\frac{n^k}{k!(\nu-k)!}\sum_{s=k}^{\infty}a_{s-k}\left[\frac{s}{n},\frac{s+1}{n};f\right]$$

Thus, we obtain

$$\sum_{\nu=0}^{\infty}e^{x}p_\nu(nx)\left[\frac{\nu}{n},\frac{\nu+1}{n};f\right]=\sum_{\nu=0}^{\infty}x^\nu\left\{\sum_{k=0}^{\nu}\frac{n^k}{k!(\nu-k)!}\sum_{s=k}^{\infty}a_{s-k}\left[\frac{s}{n},\frac{s+1}{n};f\right]\right\}$$

In the same way we compute the second term of (3) and we obtain:

$$\Delta(P_nf)'(x)=\frac{e^{-(n+1)x}}{g(1)}\sum_{\nu=0}^{\infty}x^\nu\left\{\sum_{k=0}^{\nu}\frac{n^k}{k!(\nu-k)!}\sum_{s=k}^{\infty}a_{s-k}\left[\frac{s}{n},\frac{s+1}{n};f\right]\right\}-$$

$$-\frac{e^{-(n+1)x}}{g(1)}\sum_{\nu=0}^{\infty}x^\nu\frac{(n+1)^\nu}{\nu!}\sum_{s=\nu}^{\infty}a_{s-\nu}\left[\frac{s}{n+1},\frac{s+1}{n+1};f\right]=$$

$$=\frac{e^{-(n+1)x}}{g(1)}\sum_{\nu=0}^{\infty}x^\nu\left\{\sum_{k=0}^{\nu}\frac{n^k}{k!(\nu-k)!}\sum_{s=k}^{\infty}a_{s-k}\left[\frac{s}{n},\frac{s+1}{n};f\right]-\right.$$

$$\left.-\frac{(n+1)^\nu}{\nu!}\sum_{s=\nu}^{\infty}a_{s-\nu}\left[\frac{s}{n+1},\frac{s+1}{n+1};f\right]\right\}$$

Next, we will prove that

$$\sum_{k=0}^{\nu} \frac{n^k}{k!(\nu-k)!} \sum_{s=k}^{\infty} \frac{a_{s-k}}{g(1)} \left[\frac{s}{n}, \frac{s+1}{n}; f\right] - \frac{(n+1)^{\nu}}{\nu!} \sum_{s=\nu}^{\infty} \frac{a_{s-\nu}}{g(1)} \left[\frac{s}{n+1}, \frac{s+1}{n+1}; f\right] \geq 0 \qquad (4)$$

We consider the function $F_n(x) = f\left(x + \frac{1}{n}\right) - f(x)$, $n \in N$ fixed, $x \in [0,\infty)$.

From the preceding Lemma, for $a = \frac{1}{n}$ we obtain $\left[x_1+\frac{1}{n}, x_2+\frac{1}{n}, x_3+\frac{1}{n}; f\right] \geq [x_1, x_2, x_3; f] \geq 0$

By using the recurrence formula for divided differences we obtain

$$\frac{f\left(x_3+\frac{1}{n}\right) - f\left(x_2+\frac{1}{n}\right)}{x_3 - x_2} - \frac{f\left(x_2+\frac{1}{n}\right) - f\left(x_1+\frac{1}{n}\right)}{x_2 - x_1} \geq \frac{f(x_3) - f(x_2)}{x_3 - x_2} - \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

and thus it results that

$$\frac{F_n(x_3) - F_n(x_2)}{x_3 - x_2} - \frac{F_n(x_2) - F_n(x_1)}{x_2 - x_1} \geq 0, \text{ i.e. } [x_1, x_2, x_3; F] \geq 0,$$

for any points $x_1, x_2, x_3 \in [0, \infty)$, $x_1 < x_2 < x_3$.

It results that the function F_n is non-concave of the first order.

Let $a_k = \frac{1}{(n+1)^{\nu}} \frac{n^k}{k!(\nu-k)!} \nu!$ be, $k = \overline{0,\nu}$ and

$x_{k,i} = \frac{k+i}{n}$, $k = \overline{0,\nu}$, $i = 0, 1 \ldots$ We have

$\sum_{k=0}^{\nu} a_k = 1$ and $\bar{x} = \sum_{k=0}^{\nu} a_k x_{k,i} = \frac{\nu}{n+1} + \frac{i}{n} = \frac{\nu+i}{n+1} + \left(\frac{i}{n} - \frac{i}{n+1}\right)$

Because $\frac{a_n}{g(1)} \geq 0$, $n = 0, 1, \ldots$ and F_n is non-concave of the first order, we have:

$$\frac{a_i}{g(1)} F_n\left(\frac{\nu+i}{n+1}\right) \leq \frac{a_i}{g(1)} F_n(\bar{x}) \leq \frac{a_i}{g(1)} \sum_{k=0}^{\nu} a_k F_n(x_{k,i})$$

and

$$\sum_{i=0}^{\infty} \frac{a_i}{g(1)} F_n\left(\frac{\nu+i}{n+1}\right) \leq \sum_{i=0}^{\infty} \frac{a_i}{g(1)} \sum_{k=0}^{\nu} a_k F_n(x_{k,i}).$$

It result that

$$\frac{1}{g(1)} \sum_{i=0}^{\infty} a_i \left[f\left(\frac{\nu+i}{n+1} + \frac{1}{n}\right) - f\left(\frac{\nu+i}{n+1}\right)\right] \leq$$

$$\leq \frac{1}{g(1)} \sum_{i=0}^{\infty} a_i \sum_{k=0}^{\nu} \frac{1}{(n+1)^{\nu}} \frac{n^k \nu!}{k!(\nu-k)!} \left[f\left(\frac{k+i}{n} + \frac{1}{n}\right) - f\left(\frac{k+i}{n}\right)\right]$$

Now we can introduce the divided differences and we obtain:

$$\frac{1}{g(1)}\sum_{i=0}^{\infty} a_i\left[\frac{\nu+i}{n+1},\frac{\nu+i}{n+1}+\frac{1}{n};f\right]\le$$

$$\le\frac{1}{g(1)}\sum_{i=0}^{\infty} a_i\sum_{k=0}^{\nu}\frac{1}{(n+1)^{\nu}}\frac{n^{k}\nu!}{k!(\nu-k)!}\left[\frac{k+i}{n},\frac{k+i}{n}+\frac{1}{n};f\right].$$

It follows that

$$\frac{1}{g(1)}\frac{\nu!}{(n+1)^{\nu}}\sum_{i=0}^{\infty} a_i\sum_{k=0}^{\nu}\frac{n^{k}}{k!(\nu-k)!}\left[\frac{k+i}{n},\frac{k+i}{n}+\frac{1}{n};f\right]\ge$$

$$\ge\frac{1}{g(1)}\sum_{i=0}^{\infty} a_i\left[\frac{\nu+i}{n+1},\frac{\nu+i}{n+1}+\frac{1}{n};f\right]\ge\frac{1}{g(1)}\sum_{i=0}^{\infty} a_i\left[\frac{\nu+i}{n+1},\frac{\nu+i}{n+1}+\frac{1}{n+1};f\right]$$

Now, in the left side we use notation $i = s\text{-}k$ and in the right side we note $i = s\text{-}\nu$ and it result:

$$\frac{1}{g(1)}\frac{\nu!}{(n+1)^{\nu}}\sum_{s=k}^{\infty} a_{s-k}\sum_{k=0}^{\nu}\frac{n^{k}}{k!(\nu-k)!}\left[\frac{s}{n},\frac{s+1}{n};f\right]\ge$$

$$\ge\frac{1}{g(1)}\sum_{s=\nu}^{\infty} a_{s-\nu}\left[\frac{s}{n+1},\frac{s+1}{n+1};f\right]$$

and so we obtain the desired inequality (4):

$$\frac{1}{g(1)}\sum_{k=0}^{\nu}\frac{n^{k}}{k!(\nu-k)!}\sum_{s=k}^{\infty} a_{s-k}\left[\frac{s}{n},\frac{s+1}{n};f\right]-$$

$$-\frac{1}{g(1)}\frac{(n+1)^{\nu}}{\nu!}\sum_{s=\nu}^{\infty} a_{s-\nu}\left[\frac{s}{n+1},\frac{s+1}{n+1};f\right]\ge 0.$$

Remark. Ivana Horova [3] has studied the monotonicity of the sequence $(S'_n f)(x)$, which can be obtained from the operators P_n in the case $g(z) = 1$. Her result is

THEOREM C. *Let f be non-concave of the first and the second orders on the interval $[0,\infty)$. Then the sequence formed by the first order derivatives of the Szasz-Mirakyan operators is non-increasing on this interval, i.e.*

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$$(S_n f)'(x) \geq (S_{n+1} f)'(x),\ x \in [0, \infty),\ n = 1, 2, \ldots$$

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IMPLICIT FUNCTION THEOREMS AND VARIATIONAL INEQUALITIES

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REZUMAT. - **Teoreme de funcţii implicite şi inegalităţi variaţionale.** Articolul prezintă teoreme de funcţii implicite,care se utilizează în rezolvarea unor ecuaţii generalizate şi la studiul senzitivităţii inegalităţilor variaţionale.Totodată se realizează o legătură între teoremele care utilizează condiţii de monotonie şi cele care utilizează generalizări ale conceptului de tare aproximare.

1. Introduction. In this paper we present an implicit function theorem for set valued maps, based on a generalization of the concept of strong aproximation,introduced by S.M.Robinson [8] and used also by A.L. Dontcov and W.W.Hager [4].
We will use this implicit function theorem to discuss senzitivity of perturbed variational inequalities with monotonicity conditions. Monotonicity conditions for implicit function theorems and variational inequalities were used by S.Dafermos [2],W.Alt and I.Kolumbán [1] and by G.Kassay and I.Kolumbán [5],[6].

In [1] W.Alt and I.Kolumbán showed that,in Hilbert spaces,their theorems can be used to prove the theorems of [7] and [8].

2. Definitions and preliminary results. Let us recall that the distance from a point x to

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a set A in a metric space (X,ρ) is defined as:

$$d(x,A)=\inf\{\rho(x,y) : y \in A\}$$

and the excess e from the set A to the set C is given by:

$$e(C,A) = \sup\{d(x,A) : x \in C\}.$$

We denote by $B(a,r)$ the closed ball centered at a with radius r and by B_X the closed unit ball.

DEFINITION 2.1. Let X,Y be linear normed spaces.

a) We say that the set-valued map $\Sigma : Y \to X$ is pseudo-Lipschitz around $(y_0,x_0) \in \mathrm{graf}\Sigma$ with modulus λ if there exist neighborhoods V of y_0 and U of x_0 such that

$$\Sigma(y_1) \cap U \subset \Sigma(y_2) + \lambda\|y_1 - y_2\|B_X,$$

for all $y_1, y_2 \in V$.

b) We say that the set-valued map $\Sigma : Y \to X$ is quasi-Lipschitz around $(y_0,x_0) \in \mathrm{graf}\Sigma$ with modulus λ if for every neighborhood U of x_0 there exists a neighborhood V of y_0 such that

$$[\Sigma(y_1) \cap U] \cap [\Sigma(y_2) + \lambda\|y_1 - y_2\|B_X] \neq \emptyset$$

for all $y_1, y_2 \in V$.

DEFINITION 2.2. Let X,Y,Z be linear normed spaces.

a) We say that the map $f : X \times Y \to Z$ is lipschitz in y uniformly in x around (x_0,y_0) if there exist neighborhoods U of x_0 and V of y_0 and $L \geq 0$ such that

$$\|f(x,y_1) - f(x,y_2)\| \leq L\|y_1 - y_2\|$$

for all $x \in U$ and $y_1, y_2 \in V$.

b) We say that the map $g : X \to Z$ strongly approximates the map $f : X \times Y \to Z$ in x at (x_0,y_0) if for every $\varepsilon > 0$, there exist neighborhoods U of x_0 and V of y_0 such that

$$\|f(x_1,y) - g(x_1) - [f(x_2,y) - g(x_2)]\| \leq \varepsilon\|x_1 - x_2\|,$$

for all $x_1,x_2 \in U$ and $y \in V$.

We will use the following lemma [3]:

LEMMA 2.3. *Let (X,ρ) be a complete metric space, let $\Phi : X \to X$ be a closed-valued map,let $\xi_0 \in X,r$ and λ be such that $0 \leq \lambda < 1$, $d(\xi_0,\Phi(\xi_0)) < r(1-\lambda)$ and $e(\Phi(x_1) \cap B(\xi_0,r),\Phi(x_2)) \leq \lambda\rho(x_1,x_2)$,for all $x_1,x_2 \in B(\xi_0,r)$.*

Then Φ has a fixed point in $B(\xi_0,r)$. If Φ is single-valued,then the fixed point is unique.

A.L.Dontchev and W.W.Hager proved in [4] the following implicit function theorem:

THEOREM 2.4. *Let X be a Banach space, let Y and Z be normed linear spaces. Consider a* map *$f:X\times Y \to Z$, a set-valued* map *$F : X \to Z$ and the (possibly set-valued)* map *$\Sigma : Y \to X$ defined by $\Sigma(y) = \{x \in X : o \in f(x,y) + F(x)\}$. Let $x_0 \in X, y_0 \in Y, x_0 \in \Sigma(y_0)$, let U_0 be a neighborhood of x_0, let $g : U_0 \to Z$ be a map,let Z_0 be a neighborhood of $z_0 = g(x_0) - f(x_0,y_0)$. Suppose that:*

i) *f is Lipschitz in y uniformly in x around (x_0,y_0) with modulus l.*

ii) *g strongly approximates f in x at (x_0,y_0)*

iii) *there exists a (possibly set-valued) map $\Psi : Z_0 \to X$, with the following properties:*

-$x_0 \in \Psi(z_0)$

-*$\Psi(z)$ is a closed set included in $(g + F)^{-1}(z)$, for all $z \in Z_0$*

-*Ψ is pseudo-Lipschitz around $(z_0,x0)$ with modulus γ.*

Then, for each $\gamma_1 > \gamma$, the following properties hold:

1) Σ is quazi-Lipschitz around (y_0,x_0) with modulus $\gamma_1 l$.

2) If $\Psi(z) = (g + F)^{-1}(z)$, for all $z \in Z_0$,then Σ is pseudo-Lipschitz around (y_0,x_0) with modulus $\gamma_1 l$.

3) If $\Psi(z) = (g + F)^{-1}(z)$, for all $z \in Z_0$ and, in addition, the map $z \mapsto \Psi(z) \cap U_0$ is single-valued

in Z_0, then there exist neighborhoods U of x_0 and V of y_0 such that the map $y \mapsto \Sigma(y) \cap U$ is single-valued and Lipschitz in V with modulus $\gamma_1 L$.

3. Implicit function theorems.

THEOREM 3.1. *Let X be a Banach space, Y, Z be linear normed spaces, let $f : X \times Y \to Z$, $F : X \times Y \to Z$, $\Sigma : Y \to X$ defined by $\Sigma(y) = \{x \in X : 0 \in f(x,y) + F(x,y)\}$, let $x_0 \in \Sigma(y_0)$ Suppose that:*

i) *there exist neighborhoods U of x_0, V of y_0 and $L \geq 0$ such that*

$$\|f(x,y) - f(x,y_0)\| \leq L\|y - y_0\|, \text{ for all } x \in U,\ y \in V.$$

For an $\varepsilon > 0$

ii) *there exists a map $g_\varepsilon : X \to Z$, and $r_\varepsilon > 0$ such that*

$$\|f(x_1,y) - g_\varepsilon(x_1) - f(x_2) + g_\varepsilon(x_2)\| \leq \varepsilon \|x_1 - x_2\|,$$

for all $x_1, x_2 \in B(x_0, r_\varepsilon)$ and $y \in B(x_0, r_\varepsilon)$.

iii) *there exists a (possibly set-valued) map $\Psi_\varepsilon : Z \times Y \to X$ with the following properties:*

- $x_0 \in \Psi_\varepsilon(z_0, y_0)$, *where* $z_0 = g_\varepsilon(x_0) - f(x_0, y_0)$
- $\Psi_\varepsilon(z,y)$ *is a closed set included in* $[g_\varepsilon + F(\cdot, y)]^{-1}(z)$
- *there exist neighborhoods Z_0 of z_0, Y_0 of y_0, X_0 of x_0 and $\gamma_\varepsilon > 0$, $\beta_\varepsilon > 0$ such that:*

$$e(\Psi_\varepsilon(z_1,y) \cap X_0, \Psi_\varepsilon(z_2,y)) \leq \gamma_\varepsilon \|z_1 - z_2\|,$$

$$e(\Psi_\varepsilon(z_0,y_0) \cap X_0,\ \Psi_\varepsilon(z,y)) \leq \gamma_\varepsilon(\|z - z_0\| + \beta_\varepsilon \|y - y_0\|) \tag{1}$$

for all $z, z_1, z_2 \in Z_0$ and $y_0 \in Y_0$.

- $\gamma_\varepsilon \cdot \varepsilon < 1$.

Then there exist $\gamma_1 > 0$, $a > 0$, $b > 0$ such that

$$[\Sigma(y) \cap B(x_0, a)] \cap [\{x_0\} + \gamma_1 \|y - y_0\| B_X] \neq \varnothing,$$

for all $y \in B(y_0,b)$.

Proof. Let $\varepsilon > 0$, r_ε, and g_ε given by the assumptions. We can choose a number $r_{\varepsilon 1} > 0$ such that: $r_{\varepsilon 1} \leq r_\varepsilon$, $B(x_0,r_{\varepsilon 1}) \subset U \cap X_0$, $B(y_0,r_{\varepsilon 1}) \subset V \cap Y_0$.

If $x \in B(x_0,r_{\varepsilon 1})$ and $y \in B(y_0, r_{\varepsilon 1})$ then

$$\|g_\varepsilon(x) - f(x,y) - z_0\| = \|g_\varepsilon(x) - f(x,y) - g_\varepsilon(x_0) + f(x_0,y_0)\| \leq$$

$$\leq \|g_\varepsilon(x) - f(x,y) - g_\varepsilon(x_0) - f(x_0,y)\| + \|f(x_0,y) - f(x_0,y_0)\| \leq$$

$$\leq \varepsilon\|x - x_0\| + L\|y - y_0\| \leq (\varepsilon + L)r_{\varepsilon 1}.$$

We can choose a number $r > 0$ such that $r \leq r_{\varepsilon 1}$ and $g_\varepsilon(x) - f(x,y) \in Z_0$, for all $x \in B(x_0,r)$ and $y \in B(y_0,r)$.

Let $\gamma > 0$ such that $\gamma_\varepsilon/(1-\gamma_\varepsilon \cdot \varepsilon) < \gamma$.

Let $a = r$, $0 < b < \min\{r, r/[2\gamma(L + \beta_\varepsilon)]\}$.

Let $\Phi : X \times Y \to X$ defined by $\Phi(x,y) = \Psi_\varepsilon(g_\varepsilon(x) - f(x,y),y)$. If $x \in \Phi(x,y)$ then $x \in \Sigma(y)$.

Let $y \in B(y_0,b)$. Then

$$d(x_0,\Phi(x_0,y)) \leq e(\Phi(x_0,y_0) \cap B(x_0,a),\Phi(x_0,y)) \leq e(\Phi(x_0,y_0) \cap X_0,\Phi(x_0,y)) \leq$$

$$\leq \gamma_\varepsilon(\|f(x_0,y) - f(x_0,y_0)\| + \beta_\varepsilon\|y - y_0\|) \leq \gamma_\varepsilon(L + \beta_\varepsilon)\|y - y_0\| <$$

$$< \gamma(L + \beta_\varepsilon)\|y - y_0\|(1 - \gamma_\varepsilon \cdot \varepsilon).$$

Let $x_1,x_2 \in B(x_0,\gamma(L + \beta_\varepsilon)\|y - y_0\|)$. From $\gamma(L + \beta_\varepsilon)\|y - y_0\| \leq \gamma(L + \beta_\varepsilon)b < a/2$ we have

$$e(\Phi(x_1,y) \cap B(x_0,\gamma(L + \beta_\varepsilon)\|y - y_0\|),\Phi(x_2,y)) \leq e(\Phi(x_1,y) \cap X_0,\Phi(x_2,y)) \leq$$

$$\leq \gamma_\varepsilon\|f(x_1,y) - g(x_1) - f(x_2,y) + g(x_2)\| \leq \gamma_\varepsilon \cdot \varepsilon\,\|x_1 - x_2\|.$$

We apply Lemma 2.3 with $\xi_0 = x_0$, $\lambda = \gamma_\varepsilon \cdot \varepsilon$, $r = \gamma(L + \beta_\varepsilon)\|y - y_0\|$ and we obtain a fixed point for $\Phi(\cdot,y)$ in $B(x_0,r)$. Hence, for all $y \in B(y_0,b)$, there exists an $x \in B(x_0,\gamma(L + \beta_\varepsilon)\|y-y_0\|)$ such that $x \in \Phi(x,y)$. This means that $x \in \Sigma(y)$ and $[\Sigma(y) \cap B(x_0,a)] \cap [\{x_0\} + \gamma(L + \beta_\varepsilon)\|y - y_0\|B_X] \neq \varnothing$.

COROLLARY 3.2. *If we replace property* (1) *from Theorem 3.1 with*

$\max \{e(\Psi(z,y_0)\cap X_0, \Psi(z,y)), e(\Psi(z,y)\cap X_0, \Psi(z,y_0))\} \le \gamma_\varepsilon(\|z - z_0\| + \beta_\varepsilon\|y - y_0\|)$, for all

$z \in Z_0$, $y \in Y_0$, *and we take* $\Psi_\varepsilon(z,y) = [g_\varepsilon + F(\cdot,y)]^{-1}(z)$, *then*

$$[\Sigma(y)\cap B(x_0,a/2)] \subset \Sigma(y_0) + \gamma(L + \beta_\varepsilon)\|y - y_0\|B_X$$

for all $y \in B(y_0,b)$.

Proof. For all $x \in B(x_0,a)$ we have $x \in \Phi(x,y)$ if and only if $x \in \Sigma(y)$.

We have shown in Theorem 3.1 that $\Sigma(y)\cap B(x_0,a/2) \ne \emptyset$, for all $y \in B(y_0,b)$. Let $y \in B(y_0,b)$ and $x \in \Sigma(y)\cap B(x_0,a/2)$. Then $x \in \Phi(x,y)$ and

$$d(x,\Phi(x,y_0)) \le e(\Phi(x,y)\cap X_0,\Phi(x,y_0)) \le \gamma_\varepsilon(L + \beta_\varepsilon)\|y - y_0\| \le$$

$$\le \gamma(L + \beta_\varepsilon)\|y - y_0\|(1 - \gamma_\varepsilon\varepsilon)$$

Because of $r = \gamma(L + \beta_\varepsilon)\|y - y_0\| < a/2$, $x_1,x_2 \in B(x_0,a)$ for all $x_1,x_2 \in B(x,r)$ and hence

$$e(\Phi(x_1,y_0)\cap B(x,r), \Phi(x_2,y_0)) \le e(\Phi(x_1,y_0)\cap X_0, \Phi(x_2,y_0)) \le$$

$$\le \gamma_\varepsilon\varepsilon\|x_1 - x_2\|$$

We apply Lemma 2.3 with $\xi_0 = x$, $\lambda = \gamma_\varepsilon\varepsilon$, $r = \gamma(L + \beta_\varepsilon)\|y - y_0\|$ and we obtain a fixed point for $\Phi(\cdot,y_0)$ in $B(x,\gamma(L + \beta_\varepsilon)\|y - y_0\|)$. This means that for $y \in B(y_0,b)$, there exists $x \in \Sigma(y_0)\cap B(x,\gamma(L + \beta_\varepsilon)\|y - y_0\|)$ and this completes the proof.

COROLLARY 3.3. *If in Theorem 3.1* $\Psi_\varepsilon = [g_\varepsilon + F(\cdot,y)]^{-1}$ *is single-valued, then* $\Sigma(y)\cap B(x_0,a) = \Sigma_1(y)$ *is single-valued and*

$$\|\Sigma_1(y) - \Sigma_1(y_0)\| \le \gamma(L + \beta_\varepsilon)\|y - y_0\|, \text{ for all } y \in B(y_0,b)$$

Proof. In the proof of Theorem 3.1 we can apply Lemma 2.3 with $\xi_0 = x_0$, $\lambda = \gamma_\varepsilon\varepsilon$, $r = a$ and we obtain for all $y \in B(y_0,b)$ a unique fixed point for $\Phi(\cdot,y)$ in $B(x_0,a)$. In this case $\Sigma_1(y)$ is Lipschitz at y_0.

4. Application to variational inequalities. Let X be a Hilbert space, let Y be a normed linear space, let K be a closed and convex subset of X and let $h : X \times Y \to X$ be a mapping.

Let us consider the following problems:

$\mathrm{VI}_K(y)$ find $x \in K$, such that

$$\langle h(x,y), u - x \rangle \geq 0, \text{ for all } u \in K,$$

and the equivalent generalized equation.

$\mathrm{GE}_K(y)$ find $x \in K$, such that

$$0 \in h(x,y) + N_K(x),$$

where

$$N_K(x) = \begin{cases} \{x^* \in X : \langle x^*, u - x \rangle \leq 0, \quad \text{for all} \quad u \in K\}, & \text{if } x \in K \\ \varnothing, \quad \text{if} \quad x \notin K \end{cases}$$

THEOREM 4.1. *Suppose that:*

i) x_0 *is a solution for* $\mathrm{GE}_K(y_0)$ *(or for* $\mathrm{VI}_K(y_0)$*)*

ii) *there exist neighborhoods* U *of* x_0, V *of* y_0, *and* $L > 0$ *such that*

$$\|h(x,y) - h(x,y_0)\| \leq L\|y - y_0\|, \text{ for all } x \in U, \ y \in V$$

iii) *there exists a neighborhood* Y_0 *of* y_0 *and* $\beta > 0$ *such that*

$$\langle h(x_1,y) - h(x_2,y), x_1 - x_2 \rangle \geq \beta\|x_1 - x_2\|^2, \text{ for all } x_1, x_2 \in X, \ y \in Y_0$$

iv) $h(\cdot, y)$ *is hemicontinuous for all* $y \in Y_0$

Then, for all $\varepsilon > 0$ *there exists a neighborhood* U_ε *of* x_0, V_ε *of* y_0 *and* $\gamma > 0$ *such that: for all* $y \in V_\varepsilon$ *there exists a unique solution* $x(y) \in U_\varepsilon$ *of* $\mathrm{GE}_K(y)$ *(or* $\mathrm{VI}_K(y)$*)* and $\|x(y) - x_0\| \leq \gamma\|y - y_0\|$.

Proof. Let $F(x,y) = h(x,y) + N_K(x)$.

From assumption iii) and from the maximal monotonicity of $N_K(\cdot)$ follows that $F(\cdot,y)$ is maximal monoton for all $y \in Y_0$. Let $\varepsilon > 0$ and $g_\varepsilon : X \to X$ defined by $g_\varepsilon(x) = \varepsilon x$. In this case $\Psi_\varepsilon(\cdot,y) = (g_\varepsilon + F(\cdot,y))^{-1}$ is well defined and single-valued.

Let $x_1 = \Psi_\varepsilon(z_1,y)$, $x_2 = \Psi_\varepsilon(z_2,y)$. Then $z_1 \in x_1 + h(x_1,y) + N_K(x_1)$ and $z_2 \in \varepsilon x_2 + h(x_2,y) + N_K(x_2)$. The monotonicity of $N_K(\cdot)$ implies:

$$0 \le \langle z_1 - \varepsilon x_1 - h(x_1,y) - z_2 + \varepsilon x_2 + h(x_2,y), x_1 - x_2 \rangle = \langle z_1 - z_2, x_1 - x_2 \rangle -$$

$$< \varepsilon \langle x_1 - x_2, x_1 - x_2 \rangle - \langle h(x_1,y) - h(x_2,y), x_1 - x_2 \rangle \le \|z_1 - z_2\| \|x_1 - x_2\| -$$

$$- \varepsilon \|x_1 - x_2\|^2 - \beta \|x_1 - x_2\|^2, \text{ and hence}$$

$$\|x_1 - x_2\| \le (1/(\varepsilon + \beta)) \|z_1 - z_2\|, \text{ for } x_1 \ne x_2.$$

Let $x_0 = \Psi_\varepsilon(z_0,y_0)$, $x = \Psi_\varepsilon(z,y)$, $z_0 = \varepsilon x_0$. Then $z_0 - \varepsilon x_0 - h(x_0,y_0) \in N_K(x_0)$, $z - \varepsilon x - h(x,y) \in N_K(x)$ and hence

$$0 \le \langle z - \varepsilon x - h(x,y) - z_0 + \varepsilon x_0 + h(x_0,y_0), x - x_0 \rangle = \langle z - z_0, x - x_0 \rangle - \varepsilon \langle x - x_0, x - x_0 \rangle$$

$$-\langle h(x,y) - h(x,y_0), x - x_0 \rangle - \langle h(x,y_0) - h(x_0,y_0), x - x_0 \rangle \le \|z - z_0\| \|x - x_0\| -$$

$$-\varepsilon \|x - x_0\|^2 + L\|y - y_0\| \|x - x_0\| - \beta \|x - x_0\|^2. \text{ Then}$$

$$\|x - x_0\| \le (1/(\varepsilon + \beta))(\|z - z_0\| + L\|y - y_0\|), \text{when } x \ne x_0.$$

We can use Theorem 3.1 and Corollary 3.3 with $f = 0$, $F(x,y) = h(x,y) + N_K(x)$, $g_\varepsilon(x) = \varepsilon x$, $\Psi_\varepsilon(z,y) = [g_\varepsilon + F(\cdot,y)]^{-1}(z)$, $\gamma_\varepsilon = 1/(\varepsilon+\beta)$ to completes the proof.

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ON THE GENERALIZED MINTY'S INEQUALITY

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Dedicated to Professor M. Balázs on his 65th anniversary

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REZUMAT. - Asupra inegalității generalizate a lui Minty. Se dau condiții necesare și suficiente pentru inegalitatea generalizată a lui Minty din 1970. Se obțin astfel extinderi ale teoremelor lui Kirszbraun, Grünbaum și Minty.

Abstract. Some necessary and sufficient conditions for the generalized Minty's inequality are given. One obtains in this way, extensions for the theorems of Kirszbraun, Grünbaum and Minty.

1. Introduction. The well-known theorem of Kirszbraun [8] states that a non-expansive function from $\mathbb{R}^n$ to itself, with domain a finite pointset, can be extended to a larger domain including any arbitrarily chosen point so as to be nonexpansive. More precisely, we have:

THEOREM A (Kirszbraun [8]). *Let* $(x_1, y_1), \ldots, (x_m, y_m) \in \mathbb{R}^n \times \mathbb{R}^n$ *be such that*

$$\|x_i - x_j\| \leq \|y_i - y_j\|, \quad 1 \leq i, j \leq m \tag{1}$$

Then for each $y \in \mathbb{R}^n$ *there exists* $x \in \mathbb{R}^n$ *such that*

$$\|x_i - x\| \leq \|y_i - y\|, \quad 1 \leq i \leq m \tag{2}$$

This theorem was rediscovered by Valentine [13] using different methods. In 1962

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Minty [9] proved the same fact for a monotone function:

THEOREM B (Minty [9]). *Let* $(x_1, y_1), \ldots, (x_m, y_m) \in \mathbb{R}^n \times \mathbb{R}^n$ *such that*

$$<x_i - x_j, y_i - y_j> \geq 0, \quad 1 \leq i, j \leq m. \tag{3}$$

Then for each $y \in \mathbb{R}^n$ *there exists a point* $x \in \mathbb{R}^n$ *such that*

$$<x_i - x, y_i - y> \geq 0, \quad 1 \leq i \leq m. \tag{4}$$

In the same year, Grünbaum [5] gave a common generalization of these two theorems, namely:

THEOREM C (Grünbaum [5]). *Let* $(x_1, y_1), \ldots, (x_m, y_m) \in \mathbb{R}^n \times \mathbb{R}^n$ *satisfying* (3). *Let reals* α, β *be given,* $\alpha^2 + \beta^2 > 0$. *Then there exists a point* $z \in \mathbb{R}^n$ *such that*

$$<x_i + \alpha z, y_i + \beta z> \geq 0, \quad 1 \leq i \leq m. \tag{5}$$

In 1964 Debrunner and Flor [3] improved Minty's theorem by showing that the desired value $x \in \mathbb{R}^n$ in formula (2) could always be chosen in the convex hull of the given $x_1, \ldots, x_m$; several different proofs of this fact have been given (see [10],[2]). It is easy to see that an immediate consequence of Kirszbraun's theorem is that any Lipschitz function defined on a subset of a Hilbert space can be extended to the whole space so as to satisfy the same Lipschitz-inequality.

In a paper from 1970, Minty [11] gave a unified method for proving all the above results. This result leads to a generalization of Banach's theorem [1] concerning the extensibility of Lipschitz-Hölder continuous functions. His proof is based on the classical minimax theorem of J. von Neumann [12]. In their recent paper [7] the authors gave a general saddle point theorem which convers a lot of special cases known in the literature. The proof of this result is a simple application of the well-known separation theorem of convex sets in finite dimensional spaces.

In this note we give two extensions of Minty's theorem [11] by using different proofs. First we show that this result is an easy consequence of our Theorem 2.2 (section 2). Then by Theorem 2.2 we deduce a necessary and sufficient condition for the generalized Minty's inequality [11] (Theorem 2.3 in section 2) and as well, a generalization of Minty's theorem [11] by relaxing the convexity assumption (Theorem 2.4). Another extension of Kirszbraun's theorem has been given by Karamardian [6] whose proof is based on the fixed point theorem of Kakutani. We show (Theorem 2.5) that by the well-known Ky Fan's minimax inequality [4] which has been published in the same volume as Karamardian's paper [5], one can obtain a similar generalization of Minty's theorem [11].

2. The Kirszbraun's function. First we reproduce the theorem of Minty [11]. Let X be a vector space over the reals and Y be a nonempty set.

Let

$$\Delta_m := \left\{ \lambda \in \mathbb{R}^m \mid \lambda = (\lambda_1, \ldots, \lambda_m),\ \lambda_i \geq 0,\ \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

A function $\phi : X \times Y \times Y \to \mathbb{R}$ is called Kirszbraun function (K-function) if

(a) for each $y_1, y_2 \in Y$ the function $\phi(\cdot, y_1, y_2)$ is convex on X;

(b) for any sequence $(x_1, y_1), \ldots, (x_m, y_m)$ in $X \times Y$, any $y \in Y$ and any $\lambda \in \Delta_m$ we have

$$\sum_{i,j=1}^{m} \lambda_i \lambda_j \phi(x_i - x_j, y_i, y_j) \geq k \sum_{i=1}^{m} \lambda_i \phi(x_i - x, y_i - y)$$

where $x := \sum_{i=1}^{m} \lambda_j x_j$ and k is a positive constant which may depend on the sequence $(x_1, y_1), \ldots, (x_m, y_m)$.

If X is a finite-dimensional space, then ϕ is called a finite-dimensional K-function if it satisfies the above definition with m replaced by $1 + \dim X$.

THEOREM 2.1 (Minty [11]). (α) *Let $\phi : X \times Y \times Y \to \mathbb{R}$ be a K-function and*

$(x_1, y_1), \ldots, (x_m, y_m) \in X \times Y$ *such that*

$$\phi(x_i - x_j, y_i, y_j) \leq 0 \text{ for all } 1 \leq i, j \leq m \tag{6}$$

and let $y \in Y$. *Then there exists* $x \in co\{x_1, \ldots, x_m\}$ *such that*

$$\phi(x_i - x, y_i, y) \leq 0 \text{ for all } 1 \leq i \leq m, \tag{7}$$

where $co\, M$ *denotes the convex hull of* $M \subseteq X$.

(β) *The same statement holds if* X *is finite-dimensional, and* ϕ *is a corresponding finite-dimensional K-function.*

In the following we shall prove Theorem 2.1 using another method based on a recent result of the authors ([7], Theorem 1). Let A and B be nonempty sets and $\varphi: A \times B \to \mathbb{R}$ be a given function. For each $b \in B$ and $\delta > 0$ define the sets

$$U(b, \delta) := \{a \in A \mid \varphi(a, b) + \delta < 0\}.$$

The function φ is said to be *weakly convex-like in its first variable*, if for every finite sets $\{a_1, \ldots, a_m\} \subseteq A$ and $\{b_1, \ldots, b_n\} \subseteq B$ the inequality

$$\max_{1 \leq j \leq n} \sum_{i=1}^{m} \lambda_i \varphi(a_i, b_j) \geq \inf_{a \in A} \max_{1 \leq j \leq n} \varphi(a, b_j)$$

holds for all $\lambda \in \Delta_m$. The function φ is said to be *weakly concave-like in its first variable* if $-\varphi$ is weakly convex-like in its first variable.

THEOREM 2.2 ([5], Theorem 1). *Suppose that* φ *is weakly concave-like in its first variable and the following condition is satisfied:*

if the system $\{U(b, \delta) \mid b \in B, \delta > 0\}$ *covers* A, *then it contains a finite subcover.* (8)

Then the following two assertions are equivalent: there exists $a \in A$ *such that*

$$\varphi(a, b) \geq 0 \text{ for all } b \in B; \tag{9}$$

$$\sup_{a \in A} \sum_{j=1}^{n} \mu_j \varphi(a, b_j) \geq 0 \tag{10}$$

for each finite set $\{b_1, \ldots, b_n\} \subseteq B$ *and* $\mu \in \Delta_n$.

Proof of Theorem 2.1. Let $A = co\{x_1, \ldots, x_m\}$, $B = \{(x_1, y_1), \ldots, (x_m, y_m)\} \subseteq X \times Y$ and $\psi(a, b_i) := -\phi(x_i - a, y_i, y)$, where $b_i := (x_i, y_i)$, $1 \leq i \leq m$. The function ψ being concave in its first variable it is also weakly concave-like in its first variable. Since ψ is concave in its first variable and the affine hull of A is a finite-dimensional space, then ψ is continuous in its first variable on A. Now, since A is cmpact, then condition (8) is satisfied. On the other hand the inequality

$$\sup_{a \in A} \sum_{j=1}^{m} \mu_j \psi(a, b_j) \geq 0$$

follows easily by (b) and (6). Then we have the assertion by Theorem 2.2. ∎

Remark 2.1. Condition (6) is not necessary for (7). To see this, consider the following simple example. Let $X = \mathbb{R}$ and $\phi: \mathbb{R} \times Y \times Y \to \mathbb{R}$ the K-function given by $\phi(x, y_1, y_2) = x$. Then for arbitrary $(x_1, y_1), \ldots, (x_m, y_m) \in \mathbb{R} \times Y$ and $y \in Y$ the inequality $\phi(x_i - x, y_i, y) = x_i - x \leq 0$, $1 \leq i \leq m$ holds for $x = \max\{x_1, \ldots, x_m\}$, while the inequality $\phi(x_i - x_j, y_i, y_j) = x_i - x_j \leq 0$, $1 \leq i, j \leq m$ fails provided the set $\{x_1, \ldots, x_m\}$ contains more than one element. On the other hand, it can be shown that under some additional hypothesis on ϕ, conditions (6) and (7) are in fact equivalent. Suppose Y is also a vector space. Let $\psi: X \times Y \to \mathbb{R}$ be convex in its first variable and suppose

$$\sum_{i,j=1}^{m} \lambda_i \lambda_j \psi(x_i - x_j, y_i - y_j) \geq 2 \sum_{i=1}^{m} \lambda_i \psi(x_i - x, y_i)$$

for each $(x_1, y_1), \ldots, (x_m, y_m) \in X \times Y$ and $\lambda \in \Delta_m$, where x stands for $\sum_{j=1}^{m} \lambda_j x_j$. Then $\phi(x, y_1, y_2) = \psi(x, y_1 - y_2)$ is a K-function. Suppose further that for each $(x, y) \in X \times Y$ we have

$$\psi(x, y) \leq 0 \Rightarrow \psi(-x, -y) \leq 0 \tag{11}$$

and

$$\psi(x, y) > 0 \Rightarrow \psi\left(\frac{x}{2}, \frac{y}{2}\right) > 0 \tag{12}$$

Then conditions (6) and (7) are equivalent. In fact, by Theorem 2.1 we have only to prove that (7) implies (6). Supposing the contrary, there exist $i_0, j_0 \in \{1, \ldots, m\}$ such that $\psi(x_{i_0} - x_{j_0}, y_{i_0} - y_{j_0}) > 0$. Take $y := \frac{y_{i_0} + y_{j_0}}{2}$. By (7) there exists $x_0 \in X$ such that $\psi\left(x_i - x_0, y_i - \frac{y_{i_0} + y_{j_0}}{2}\right) \le 0$ for each $1 \le i \le m$. Then for $i = i_0$, and $i = j_0$ respectively, we obtain

$$\psi\left(x_{i_0} - x_0, \frac{y_{i_0} - y_{j_0}}{2}\right) \le 0 \tag{13}$$

and

$$\psi\left(x_{j_0} - x_0, \frac{y_{j_0} - y_{i_0}}{2}\right) \le 0. \tag{14}$$

Taking into account (11), the last inequality yields

$$\psi\left(x_0 - x_{j_0}, \frac{y_{i_0} - y_{j_0}}{2}\right) \le 0.$$

Thus, by (12) and the convexity of $\psi(\cdot, y)$ we obtain the following contradiction

$$0 < \psi\left(\frac{x_{i_0} - x_{j_0}}{2}, \frac{y_{i_0} - y_{j_0}}{2}\right) \le \frac{1}{2}\psi\left(x_{i_0} - x_0, \frac{y_{i_0} - y_{j_0}}{2}\right) + \frac{1}{2}\psi\left(x_0 - x_{j_0}, \frac{y_{i_0} - y_{j_0}}{2}\right) \le 0.$$

It is easy to see that Kirszbraun's theorem (Theorem A) follows from the case $\psi(x, y) = \|x\|^2 - \|y\|^2$, Minty's theorem (Theorem B) is the case where $\psi(x,y)$ is a bilinear form and Grünbaum's theorem (Theorem C) is contained in the case $\psi(x, y) = k_1(\|x\|^2 - \|y\|^2) + k_2\langle x, y\rangle$ with nonnegative k_1, k_2. Since in all of these particular cases the function ψ satisfies (11) and (12), then these theorems give necessary and sufficient conditions for the existence of x satisfying the desired properties. As we could see, in case of the more general Theorem 2.1, this is not true. However, Theorem 2.2 allows us to give the following characterization of (7) which is also a generalization of Theorem 2.1.

THEOREM 2.3. *Let $\phi: X \times Y \to \mathbb{R}$ be convex in its first variable and $(x_1, y_1), \ldots, (x_m, y_m) \in X \times Y$. Then the following two assertions are equivalent:*

there exists $x \in X$ such that

$$\phi(x_i - x, y_i) \leq 0 \tag{15}$$

for each $\lambda \in \Delta_m$

$$\inf_{x \in co\{x_1,\dots,x_m\}} \sum_{i=1}^{m} \lambda_i \phi(x_i - x, y_i) \leq 0 \tag{16}$$

In the following we state a generalized form of Theorem 2.3, in which the convexity of ϕ in its first variable is relaxed.

THEOREM 2.4. *Let* ϕ: $X \times Y \to \mathbb{R}$ *and* $(x_1, y_1), \dots, (x_m, y_m) \in X \times Y$. *Suppose that the following assertions hold:*

for each finite set $\{a_1, \dots, a_k\} \subseteq A$ *and each subset* $\{(x_{i_1}, y_{i_1}), \dots, (x_{i_p}, y_{i_p})\}$ *of* $(x_1, y_1), \dots, (x_m, y_m)$ *we have*

$$\max_{1 \leq j \leq p} \sum_{s=1}^{k} \lambda_s \phi(x_{i_j} - a_s, y_{i_j}) \geq \inf_{a \in A} \max_{1 \leq j \leq p} \phi(x_{i_j} - a, y_{i_j})$$

for each $\lambda \in \Delta_k$. (17)

for each $y \in Y$, *the restriction of* $\phi(\cdot, y)$ *to each finite dimensional subspace of* X *is lower semicontinuous;* (18)

Then assertions (15) and (16) are equivalent.

Proof. Take φ: $A \times \{(x_1, y_1), \dots, (x_m, y_m)\} \to \mathbb{R}$, $\varphi(a, b_i) := -\phi(x_i - a, y_i)$, where $b_i := (x_i, y_i)$, $1 \leq i \leq m$. Then φ is weakly concave-like in its first variable. Since by (18) φ is lower semicontinuous in its first variable on A and A is compact, then (8) also holds. Now the conclusion follows by Theorem 2.2.

Next we give an example for a function ϕ satisfying (17). A function f: $\mathbb{R} \to \mathbb{R}$ is said to be *Ky Fan convex like* if for each $\lambda \in [0,1]$, each $x_1, \dots, x_n \in \mathbb{R}$ and each $a_1, a_2 \in co\{x_1, \dots, x_n\}$ there exists $a_3 \in co\{x_1, \dots, x_n\}$ such that $f(x_i - a_3) \leq \lambda f(x_i - a_1) + (1 - \lambda) f(x_i - a_2)$, $1 \leq i \leq n$. It is easy to see that each monotone function is Ky Fan convex like, where a_3 can be chosen as max $\{a_1, a_2\}$ if f is increasing and min $\{a_1, a_2\}$ if f is decreasing. Now consider the function ϕ defined on $\mathbb{R}\times Y$ with real values given by $\phi(x,y) := f(x) + g(y)$, where $f: \mathbb{R} \to \mathbb{R}$ is a Ky Fan convex like function and $g: Y \to \mathbb{R}$ is an arbitrary function. Then ϕ satisfies (17) for each $\{(x_1, y_1), \dots, (x_m, y_m)\} \in X \times Y$. Indeed, let $\{a_1, a_2, \dots, a_k\} \subseteq A = co\{x_1, \dots, x_m\}$, $\lambda \in \Delta_k$ and $\{(x_{i_1}, y_{i_1}), \dots, (x_{i_p}, y_{i_p})\} \subseteq \{(x_1, y_1), \dots, (x_m, y_m)\}$. Since f is Ky Fan convex like, then there exists $\bar{a} \in A$ such that

$$f(x_j - \bar{a}) \le \sum_{i=1}^{k} \lambda_i f(x_j - a_i), \ 1 \le j \le n.$$

Therefore,

$$\max_{1\le j\le p} \sum_{s=1}^{k} \lambda_s [f(x_{i_j} - a_s) + g(y_{i_j})] \ge \max_{1\le j\le p} [f(x_{i_j} - \bar{a}) + g(y_{i_j})] \ge$$
$$\ge \inf_{a\in A} \max_{1\le j\le p} [f(x_{i_j} - a) + g(y_{i_j})],$$

so ϕ satisfies (17).

Finally we give another extension of Theorem 2.1 (Minty's theorem) which proof is based on Ky Fan's minimax inequality [4].

Let X be a vector space over the reals, Y be a nonempty set, $\phi: X \times Y \times Y \to \mathbb{R}$ and $(x_1, y_1), \dots, (x_m, y_m) \in X \times Y$ be given.

THEOREM 2.5. *Suppose that for each $y_1, y_2 \in Y$, the restriction of $\phi(\cdot, y_1, y_2)$ to each finite dimensional subspace of X is lower semicontinuous and the inequalities*

$$0 \ge \sum_{i,j=1}^{m} \lambda_i \lambda_j \phi(x_i - x_j, y_i, y_j) \ge k \sum_{i=1}^{m} \lambda_i \phi(x_i - x, y_i, y),$$

hold for each $y \in Y$ and $\lambda \in \Delta_m$, where $x := \sum_{j=1}^{m} \lambda_j x_j$ and k is a positive constant which may depend on $(x_1, y_1), \dots, (x_m, y_m)$. Then for each $y \in Y$ there exists $x \in co\{x_1, \dots, x_m\}$ such that

$$\phi(x_i - x, y_i, y) \le 0 \text{ for each } 1 \le i \le m.$$

Proof. Let $y \in Y$ and $h: \Delta_m \times \Delta_m \to \mathbb{R}$ be such that

$$h(\lambda, \mu) = \sum_{i=1}^{m} \lambda_i \phi\left(x_i - \sum_{j=1}^{m} \mu_j x_j, y_i, y\right).$$

It is easy to verify that $h(\lambda, \lambda) \le 0$ for each $\lambda \in \Delta_m$, $h(\cdot, \mu)$ is quasiconcave for each $\mu \in \Delta_m$

(since it is linear) and $h(\mu, \cdot)$ is lower semicontinuous on Δ_m for each $\lambda \in \Delta_m$. Then by Ky Fan's minimax inequality [4] it follows the existence of an element $\mu_0 \in \Delta_m$ such that

$$h(\lambda, \mu_0) \leq 0 \text{ for each } \lambda \in \Delta_m.$$

Now the assertion follows if we put instead of λ the vectors

$$(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 1) \in \Delta_m.$$

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SUR LES PROBLEMES DE BIRECCURRENCE GENERALISE POUR DES CONNEXIONS SEMI-SYMETRIQUE METRIQUES

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REZUMAT. - Asupra unor probleme de birecurență generalizată pentru conexiuni metrice semisimetrice. În lucrare sunt stabilite mai multe proprietăți ale conexiunilor semisimetrice.

Soit L_n une variété differentiable à n dimensions, de classe C^∞ et g une métrique riemanienne sur L_n de composantes g_{ij} dans une carte locale (u,φ). Nous allons noter par ∇ la connexion Levi-Civita, correspondant à g de coéfficients $\left\{ {i \atop jk} \right\}$ dans la carte locale (u,φ).

Soit dans L_n une connexion D semi-symétrique [1], [6] de coéfficients:

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + \omega_k\delta^i_j - \omega_j\delta^i_k \tag{1}$$

Nous avons:

$$T^i_{jk} = \omega_k\delta^i_j - \omega_j\delta^i_k \tag{2}$$

$$g_{ij,k} = 0 \tag{3}$$

où $T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$ et par la virgule on a noté la dérivée covariante par rapport à D.

Nous allons noter par R^i_{jkh} les composantes du tenseur de courbure pour la connexion D, par $R_{jk} = R^i_{jki}$ le tenseur de Ricci, par $R = g^{ij}R_{ij}$ sa courbure scalaire, par T^i_{jkh} le tenseur de courbure D-concirculaire [4], par Z^i_{jkh} le tenseur de courbure D-coharmonique [4], par W^i_{jkh}

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le tenseur de courbure D-projective [5] et par C^{i}_{jkh} le tenseur de courbure D-conforme.

DEFINITION 1. S'il existe deux tenseurs covariants φ_{rs} et a_{rs} et un tenseur A^{i}_{jkh} tel que:

$$R^{i}_{jkh,rs} = \varphi_{rs} R^{i}_{jkh} + a_{rs} A^{i}_{jkh} \tag{4}$$

on dira que L_n est D-biréccurrente généralisé.

Si dans (4) nous appliquons une contraction par i et k nous obtenons:

$$R_{jh,rs} = \varphi_{rs} R_{jh} + a_{rs} A_{jh} \tag{5}$$

ou

$$A_{jh} = A^{i}_{jih} \tag{6}$$

DEFINITION 2. Une variété L_n douée d'une D-connexion pour laquelle on a (5) est nomée Ricci-D-biréccurrente généralisée.

De (4) et (5) on a:

PROPOSITION 1. *Une variété L_n douée d'une D-connexion semi-symétrique métrique D-biréccurrente généralisée est aussi Ricci-D-biréccurrente généralisée avec les mêmes tenseurs φ_{rs} et a_{rs} et A_{jh} donée de* (6). *La réciproque n'est pas en général vraie.*

Si dans (5) nous multipliqons contracté par g^{jh} nous obtenons:

$$R_{,rs} = \varphi_{rs} R + a_{rs} A \tag{7}$$

où:

$$A = A_{jh} g^{jh} \tag{8}$$

DEFINITION 3. Une variété L_n douée d'une D-connexion pour laquelle on a (7) est nommée de courbure scalaire D-birécurrente généralisée.

De (5) et (7) on a:

PROPOSITION 2. *Une variété L_n douée d'une D-connexion Ricci-D-biréccurrente* *généralisée est aussi de courbure scalaire D-biréccurrente généralisée avec A donné de* (8).

Soit maintenant T^i_{jkh} le tenseur de courbure D-concirculaire.

DEFINITION 4. S'il existe deux tenseurs covariants φ_{rs} et a_{rs} et un tenseur B^i_{jkh} tel que:

$$T^i_{jkh,rs} = \varphi_{rs} T^i_{jkh} + a_{rs} B^i_{jkh} \tag{9}$$

on dira que L_n est concirculaire D-biréccurrente généralisée.

De:

$$T^i_{jkh} = R^i_{jkh} - \frac{R}{n(n-1)} (g_{jk} \delta^i_h - g_{jh} \delta^i_k) \tag{10}$$

par dérivation covariante deux fois et tenant compte de (5) et (7) il en résulte:

PROPOSITION 3. *Les variétés L_n D-biréccurrentes généralisées sont aussi concirculaires D-biréccurrentes généralisées avec les mêmes tenseurs φ_{rs} et a_{rs} et B^i_{jkh} donné de*:

$$B^i_{jkh} = A^i_{jkh} - \frac{A}{n(n-1)} (g_{jk} \delta^i_h - g_{jh} \delta^i_k) \tag{11}$$

Pour la réciproque, de (10) par dérivation covariante deux fois et tenant compte de (7) et (9) il résulte (4) avec:

$$A^i_{jkh} = B^i_{jkh} + \frac{A}{n(n-1)} \tag{12}$$

Donc:

PROPOSITION 4. *Une variété L_n concirculaire D-biréccurrente généralisée est D-biréccurrente généralisée si elle est avec courbure scalaire D-biréccurrente généralisée et A^i_{jkh} donné de* (12).

De (9) par contraction en i et k on obtient:

$$T_{jh,rs} = \varphi_{rs} T_{jh} + a_{rs} B_{jh} \tag{13}$$

où $T_{jh} = T^i_{jih}$, $B_{jh} = B^i_{jih}$. Donc on a:

PROPOSITION 5. *Dans une variété L_n concirculaire D-biréccurrente généralisée, le tenseur contracté du tenseur concirculaire de courbure est aussi D-biréccurrent généralisé avec les mêmes tenseurs φ_{rs} et a_{rs} et $B_{jh} = B^{i}_{jih}$.*

Le tenseur:

$$E_{ij} = R_{ij} - \frac{R}{n} g_{ij} \tag{14}$$

sera nommé tenseur *D*-Einstein.

DEFINITION 5. S'il existe deux tenseurs φ_{rs}, a_{rs} et un tenseur N_{ij} tel que:

$$E_{ij,rs} = \varphi_{rs} E_{ij} + a_{rs} N_{ij} \tag{15}$$

on ira que L_n est *D*-Einstein biréccurrente généralisée.

De (5), (7), (14), et (15) il résulte:

PROPOSITION 6. *Une variété L_n Ricci D-biréccurrente généralisée est aussi D-Einstein biréccurrente généralisée avec $N_{ij} = A_{ij} - \frac{A}{n} g_{ij}$ et une variété L_n D-Einstein biréccurrente généralisée est Ricci-D-biréccurrente généralisée si elle est de courbure scalaire D-biréccurrente généralisée et $A_{ij} = N_{ij} + \frac{A}{n} g_{ij}$.*

De (10) par contraction en i et k et de (14) il résulte:

$$T_{jh} = T^{i}_{jih} = E_{jh} \tag{16}$$

et de (13) et (16) il résulte (15) avec $N_{ij} = B_{ij}$.

Donc:

PROPOSITION 7. *Une variété L_n concirculaire D-biréccurrente généralisée est aussi D-Einstein biréccurrente généralisée.*

Soit:

$$Z^{i}_{jkh} = R^{i}_{jkh} - \frac{1}{n-2} \left(R_{jh}\delta^{i}_{k} - R_{jk}\delta^{i}_{h} + g_{jh}R^{i}_{k} - g_{jk}R^{i}_{h} \right) \tag{17}$$

le tenseur D-coharmonique de courbure [4].

DEFINITION 6. S'il existe deux tenseurs φ_{rs} et a_{rs} et un tenseur H^i_{jkh} tel que:

$$Z_{jkh,rs} = \varphi_{rs} Z^i_{jkh} + a_{rs} H^i_{jkh} \tag{18}$$

on dira que L_n est coharmonique D-biréccurrente généralisée.

De (4), (5) et (17) il résulte:

PROPOSITION 8. *Les variétés L_n D-biréccurrentes généralisées sont aussi coharmonique D-biréccurrentes généralisées avec les mêmes tenseurs φ_{rs} et a_{rs} et H^i_{jkh} est donné de*:

$$H^i_{jkh} = A^i_{jkh} + \frac{1}{n-2}\left(A_{jh}\delta^i_k - A_{jk}\delta^i_h + g_{jh}A^i_k - g_{jk}A^i_h \right) \tag{19}$$

Pour la réciproque de (17) par dérivation covariante deux fois et tenant compte de (5) et (18) il résulte (4) avec:

$$A^i_{jkh} = H^i_{jkh} - \frac{1}{n-2}\left(A_{jh}\delta^i_k - A_{jk}\delta^i_h + g_{jh}A^i_k - g_{jk}A^i_h \right) \tag{20}$$

Donc:

PROPOSITION 9. *Une variété L_n coharmonique D-biréccurrente généralisée est D-biréccurrente généralisée si elle est Ricci-D-biréccurrente généralisée*.

De (17) par contraction en i et k il résulte:

$$Z^i_{jih} = Z_{jh} = -\frac{R}{n-2} g_{jh} \tag{21}$$

et de (18) par la même contraction il résulte:

$$Z_{jh,rs} = \varphi_{rs} Z_{jh} + a_{rs} H_{jh}$$

où $H_{jh} = H^i_{jih}$ et de (21)

$$R_{,rs} = \varphi_{rs} R - a_{rs}\frac{n-2}{n} H_{jh}$$

Donc on a:

PROPOSITION 10. *Dans une variété L_n coharmonique D-biréccurrente généralisée, Z_{jh} et R sont D-biréccurrentes généralisées*.

Des propositions 4 et 10 il résulte:

PROPOSITION 11. *Si une variété L_n est simultanement concirculaire D-biréccurrente généralisée et coharmonique D-biréccurrente généralisée, elle est D-biréccurrente généralisée.*

Soit:

$$W^i_{jkh} = R^i_{jkh} - \frac{1}{n-1}(R_{jk}\delta^i_h - R_{jh}\delta^i_k) \tag{22}$$

le tenseur D-projectif de courbure [5].

DEFINITION 7. S'il existe deux tenseurs φ_{rs} et a_{rs} et un tenseur P^i_{jkh} tel que:

$$W^i_{jkh} = \varphi_{rs} W^i_{jkh} + a_{rs} P^i_{jkh} \tag{23}$$

on dira que L_n est projective D-biréccurrente.

De (4), (5) et (22) il résulte:

PROPOSITION 12. *Les variétés L_n D-biréccurrentes généralisées sont aussi projectives D-biréccurrentes généralisées avec les mêmes tenseurs φ_{rs} et a_{rs} et:*

$$P^i_{jkh} = A^i_{jkh} - \frac{1}{n-1}(A_{jk}\delta^i_h - A_{jh}\delta^i_k) \tag{24}$$

Pour la réciproque de (22) par dérivation covariante deux fois et tenant compte de (5) et (23) il résulte (4) avec:

$$A^i_{jkh} = P^i_{jkh} + \frac{1}{n-1}(A_{jk}\delta^i_h - A_{jh}\delta^i_k) \tag{25}$$

Donc:

PROPOSITION 13. *Une variété L_n projective D-biréccurrente généralisée et Ricci D-biréccurrente généralisée est D-biréccurrente généralisée avec A^i_{jkh} donné de (25).*

Si on multiplie contracté (22) par g_{jk} on obtient:

$$W_h^{\ i} = \frac{n}{n-1}\left(R_h^{\ i} - \frac{R}{n}\delta^i_h\right)$$

ou

$$W_{ik} = \frac{n}{n-1} (R_{ik} - \frac{R}{n} g_{ik}) = \frac{n}{n-1} E_{ik} \tag{26}$$

De (23) et (26) il résulte:

PROPOSITION 14. *Dans une variété L_n projective D-biréccurrente généralisée le tenseur W_{ik} est aussi D-biréccurrente généralisé et L_n est D-Einstein biréccurrente généralisée.*

En tenant compte de relations [2]:

$$W^{i}_{jkh} = T^{i}_{jkh} - \frac{1}{n-1} (T_{jh}\delta^{i}_{k} - T_{jk}\delta^{i}_{h})$$
$$T^{i}_{jkh} = W^{i}_{jkh} + \frac{1}{n} (W_{jh}\delta^{i}_{k} - W_{jk}\delta^{i}_{h})$$

il résulte:

PROPOSITION 15. *Une variété L_n projective D-biréccurrente généralisée est aussi concirculaire D-biréccurrente généralisée et réciproquement, une variété L_n concirculaire D-biréccurrente généralisée est aussi projective D-biréccurrente généralisée.*

Soit:

$$C^{i}_{jkh} = R^{i}_{jkh} - \frac{1}{n-2} (R_{jh}\delta^{i}_{k} - R_{jk}\delta^{i}_{h} + g_{jh}R^{i}_{k} - g_{jk}R^{i}_{h}) + \\ + \frac{R}{(n-1)(n-2)} (g_{jh}\delta^{i}_{k} - g_{jk}\delta^{i}_{h}) \tag{27}$$

le tenseur D-conforme de courbure qui coïncide avec le tenseur de courbure conforme de V.

DEFINITION 8. S'il existe deux tenseurs φ_{rs} et a_{rs} et un tenseur K^{i}_{jkh} tel que:

$$C^{i}_{jkh,rs} = \varphi_{rs} C^{i}_{jkh} + a_{rs} K^{i}_{jkh} \tag{28}$$

on dira que L_n est conforme D-biréccurrente généralisée.

De (4), (5), (7) et (27) il résulte:

PROPOSITION 16. *Les variétés D-biréccurrentes généralisées sont aussi conformes D-biréccurrentes généralisées avec les mêmes tenseurs φ_{rs} et a_{rs} et:*

$$K^{i}_{jkh} = A^{i}_{jkh} - \frac{1}{n-2} (A_{jh}\delta^{i}_{k} - A_{jk}\delta^{i}_{h} + g_{jh}A^{i}_{k} - g_{jk}A^{i}_{h}) +$$

$$+\frac{A}{(n-1)(n-2)}(g_{jh}\delta^i_k - g_{jk}\delta^i_h) \tag{29}$$

En tenant compte [2] des relations:

$$C^i_{jkh} = Z^i_{jkh} - \frac{1}{n-1}(Z_{jh}\delta^i_k - Z_{jk}\delta^i_h)$$

$$C^i_{jkh} = T^i_{jkh} - \frac{1}{n-2}(T_{jh}\delta^i_k - T_{jk}\delta^i_h + g_{jh}T^i_k - g_{jk}T^i_h)$$

$$C^i_{jkh} = W^i_{jkh} - \frac{1}{n(n-2)}(W_{jh}\delta^i_k - W_{jk}\delta^i_h) -$$

$$- \frac{n-1}{n(n-2)}(g_{jh}W^i_k - g_{jk}W^i_h)$$

il résulte:

PROPOSITION 17. *Une variété L_n coharmonique D-biréccurrente généralisée, ou concirculaire D-biréccurrente généralisée, ou projective D-biréccurrente généralisée, est aussi conforme D-biréccurrente généralisée, entre les tenseurs K^i_{jkh}, H^i_{jkh}, B^i_{jkh} et P^i_{jkh} il y a les mêmes relations qu'entre C^i_{jkh}, Z^i_{jkh}, T^i_{jkh} et W^i_{jkh}.*

Observation: Les tenseurs B^i_{jkh}, H^i_{jkh}, P^i_{jkh}, K^i_{jkh} s'expriment avec le tenseur A^i_{jkh} et son contracté de la même manière que T^i_{jkh}, Z^i_{jkh}, W^i_{jkh} et C^i_{jkh} s'expriment avec R^i_{jkh} et son contracté.

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SEPARATION PROPERTIES IN CATEGORIES OF LIMIT AND PRETOPOLOGICAL SPACES

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REZUMAT. - Proprietăţi de separare în spaţii cu limită şi spaţii pretopologice. Sunt date caracterizări explicite ale proprietăţilor de separare în spaţii cu limită, pseudotopologige şi pretopologice.

Abstract. In this paper, an explicit characterizations of the separation properties for the topological categories of limit spaces, pseudotopological spaces, and pretopological spaces are given. Furthermore, the relationships among various forms of the separation properties as well as some invariance properties of them are investigated.

Introduction. There are (see [1]) various ways of generalizations of the separation axioms of topology to an arbitrary topological category. For example, there are four various notions of T_0 and each equivalent to the usual T_0, one notion of T_1 which is equivalent to the usual T_1, and eight various notions of T_2 (Hausdorff) and each is equivalent to the usual T_2.

In this paper, we have shown the followings:

1. To characterize separation properties in topological categories of limit spaces, pseudotopological spaces, and pretopological spaces.

2. To investigate the relationships among the various forms of the separation properties in above mentioned categories.

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3. To determine the invariance properties (i.e. closed under formation of products, subspaces, and quotient spaces) of each of the separation properties in above categories.

4. Nel's [7] result has been extented.

Let E and B be any categories. The functor $U: E \to B$ is said to be topological if it is concrete (i.e faithful and amnestic (i.e. if $U(f)=id$ and f is an isomorphism, then $f=id$)), has small (i.e sets) fibers, and for which every U-source has an initial lift or, equivalently, for which each U-sink has a final lift [5] p. 125 or [6] p. 279.

Recall, [6] p. 279, that an object X of E called indiscrete (resp. discrete) iff every map from an E-object to X (resp. every map from X to an E-object) is an E-morphism.

Let X be a nonempty set and $X^2 = X \times X$ be the cartesian product of X with itself, and $X^2 \vee_{\Delta} X^2$ be two distinct copies of X^2 identified along the diagonal. A point (x,y) in $X^2 \vee_{\Delta} X^2$ will be denoted by $(x,y)_1$ ($(x,y)_2$) if (x,y) is in the first (resp. the second) component of $X^2 \vee_{\Delta} X^2$. Clearly $(x,y)_1 = (x,y)_2$ iff $x = y$.

1.1 DEFINITIONS. The principal axis map $A : X^2 \vee_{\Delta} X^2 \to X^3$ is given by $A(x,y)_1 = (x,y,x)$ and $A(x,y)_2 = (x,x,y)$. The skewed axis map $S : X^2 \vee_{\Delta} X^2 \to X^3$ is given by $S(x,y)_1 = (x,y,y)$ and $S(x,y)_2 = (x,x,y)$, and the fold map, $\nabla : X^2 \vee_{\Delta} X^2 \to X^2$ is given by $\nabla(x,y)_i = (x,y)$ for $i = 1,2$. Note that $\pi_1 S = \pi_{11}$, $\pi_2 S = \pi_{21}$, and $\pi_3 S = \pi_{22}$, where $\pi_k : X^3 \to X$ is the k-th projection $k = 1,2,3$ and $\pi_{ij} = \pi_i + \pi_j : X^2 \vee_{\Delta} X^2 \to X$.

Let $U: E \to$ Sets, the category of sets, be topological and X an object in E with $UX=B$.

1.2 DEFINITIONS

1. X is T_0 iff X does not contain an indiscrete subspace with (at least) two points [9] p. 316.

2. X is T_0' iff the initial lift of the U-source $\{id : B^2 \vee_{\Delta} B^2 \to U(B^2 \vee_{\Delta} B^2)' = B^2 \vee_{\Delta} B^2$

and $\nabla : B^2 V_\Delta B^2 \to UD(B^2) = B^2\}$ is discrete, where $(B^2 V_\Delta B^2)'$ is the final lift of the U- sink $\{i_1, i_2 : U(X^2) = B^2 \to B^2 V_\Delta B^2\}$ and $D(B^2)$ is the discrete structure on B^2 [1] p. 338.

3. X is $\overline{T}_0$ iff the initial lift of the U- source $\{A : B^2 V_\Delta B^2 \to U(X^3) = B^3$ and $\nabla : B^2 V_\Delta B^2 \to UD(B^2) = B^2\}$ is discrete, [1] p. 338.

4. X is T_1 iff the initial lift of the U- source $\{S : B^2 V_\Delta B^2 \to U(X^3) = B^3$ and $\nabla : B^2 V_\Delta B^2 \to UD(B^2) = B^2\}$ is discrete, [1] p. 338.

5. X is Pre T_2' iff the initial lift of the U- source $\{S : B^2 V_\Delta B^2 \to U(X^3) = B^3\}$ and the final of the U- sink $\{i_1, i_2 : U(X^2) = B^2 \to B^2 V_\Delta B^2\}$ agree [1] p. 338.

6. X is Pre $\overline{T}_2$ iff the initial lift of the U- sources $\{S : B^2 V_\Delta B^2 \to U(X^3) = B^3\}$ and $\{A : B^2 V_\Delta B^2 \to U(X^3) = B^3\}$ agree [1] p. 338

7. X is T_2' iff X is T_0' and Pre T_2' [1] p. 338.

8. X is $\overline{T}_2$ iff X is $\overline{T}_0$ and Pre $\overline{T}_2$ [1] p. 338.

9. X is KT_2 iff X is T_0' and Pre $\overline{T}_2$ [4].

10. X is LT_2 iff X is $\overline{T}_0$ and Pre T_2' [4].

11. X is MT_2 iff X is T_0 and Pre T_2' [4].

12. X is NT_2 iff X is T_0 and Pre $\overline{T}_2$ [4].

1.3 ***Remark***. For the category of topological spaces, TOP, we have:

1. All of the T_0's in 1.2 are equivalent and they reduce to the usual T_0 (see [1] p. 338 and [9] p. 316).

2. All of the T_2's in 1.2 are equivalent and they reduce to T_2, the Hausdorff condition. This follows from part (1) and [1] p. 338.

Let A be a set and K be a function which assigns to each point x of A a set of proper filters (i.e a filter δ is proper iff δ does not contain the empty set, $\varnothing$ i.e $\delta \neq [\varnothing]$) (the "filters converging to x") is called a convergence structure on A ((A,K) a filter convergence space) iff it satisfies the following two conditions.

1. $[x] \in K(x)$ for each $x \in A$ (where $[x] = \{ B \subset A \mid x \in B \}$.

2. $\beta \supset \alpha \in K(x)$ implies $\beta \in K(x)$ for any proper filter β on A. A map $f : (A,K) \to (B,L)$ between filter convergence spaces is called continous iff $\alpha \in K(x)$ implies $f(\alpha) \in L(f(x))$ (where $f(\alpha)$ denotes the filter generated by $\{ f(D) \mid D \in \alpha \}$. The category of filter convergence spaces and continous maps is denoted by FCO (see [8] p. 354). A filter convergence space (A,K) is said to be a local filter convergence space if $\alpha \cap [x] \in K(x)$ whenever $\alpha \in K(x)$ [7] p. 1374, a limit space if $\alpha \cap \beta \in K(x)$ whenever $\alpha, \beta \in K(x)$ [7] p. 1374, a pseudotopological space if a filter $\alpha \in K(x)$ whenever all the ultrafilters containing α belongs to $K(x)$ [7] p. 1374, a pretopological space if the intersections, N_x, of all filters in $K(x)$ belongs to $K(x)$ [7] p. 1374. These spaces are the objects of the full subcategories, LFCO, Lim, PsT, and PrT of FCO.

1.4 A source $\{ f_i : (A,K) \to (A_i,K_i)\ i \in I \}$ is an initial lift in Lim, PsT, and PrT iff $\alpha \in K(x)$ precisely when $f_i\alpha \in K_i(f_i(x))$ for all $x \in K(x)$ and $i \in I$ [7] p. 1374.

1.5 An epi sink $\{ i_1, i_2 : (B^2, K) \to (B^2 V_\Delta B^2, L) \}$, where i_1, i_2 are the canonical injections, in Lim, PsT, and PrT is a final lift iff for any filter α on the wedge and any point z in wedge, $\alpha \supset i_k\alpha_1$ for some $\alpha_1 \in K(u)$, $u \in B^2-\Delta$, and $i_k u = z$ for $k = 1$ or 2 $\alpha \supset i_1\alpha_1 \cap i_2\alpha_2$ for some $\alpha_1, \alpha_2 \in K(u)$, u is on the diagonal. This is a special case of [7] p. 1374.

Separation properties. We now give the characterizations of each of the separation properties defined in 1.2 for Lim, PsT, and PrT and investigate some of their invariance properties.

2.1 THEOREM.

1. (A,K) *in* Lim, PsT *or* PrT *is* $\overline{T}_0$ *or* T_0 *iff for each distinct pair of points x and y in* A, $[x] \notin K(y)$ *or* $[y] \notin K(x)$ [3] *and* [9] p. 318.

2. *All objects of* Lim, PsT, *or* PrT *are* T_0'. The proof is similar to [3].

3. (A,K) *in* Lim, PsT *or* PrT *is* Pre $\overline{T}_2$ *iff for any pair x and y in A if* $K(x) \cap K(y) \neq \{[\varnothing]\}$, *where* $\varnothing$ *is the empty set, then* $K(x) = K(y)$. It follows from [3] since the initial lifts in FCO, Lim, PsT, and PrT are the same.

4. (A,K) *in* Lim, PsT *or* PrT *is* $\overline{T}_2$ *or* NT_2 *iff for any* $x \neq y$ *in* A, $K(x) \cap K(y) = \{[\varnothing]\}$. It follows from (1), (3), and Definition 1.2.

5. (A,K) *in* Lim, PsT, *or* PrT *is* KT_2 *iff* (A,K) *is* Pre $\overline{T}_2$. It follows from parts (2) *and* (3) and Definition 1.2.

2.2 THEOREM. *Let* α_{11}, α_{21}, *and* α_{22} *be proper filters on A*.

1. *If* $\alpha = \pi_{11}^{-1}\alpha_{11} \cup \pi_{21}^{-1}\alpha_{21} \cup \pi_{22}^{-1}\alpha_{22}$, *then* α *is proper iff either* (a) $\alpha_{22} \cup \alpha_{21}$ *or* (b) $\alpha_{11} \cup \alpha_{21}$ *is proper*.

2. *There exists a proper filter* β *on* $A^2 \vee_\Delta A^2$ *such that* $\pi_{11}\beta = \alpha_{11}$, $\pi_{21}\beta = \alpha_{21}$, *and* $\pi_{22}\beta = \alpha_{22}$ *iff* (i) *if* (a) *fails, then* $\alpha_{11} = \alpha_{21}$, (ii) *if* (b) *fails, then* $\alpha_{22} = \alpha_{21}$, (iii) *if neither* (a) *nor* (b) *fails then* $\alpha_{11} \cap \alpha_{22} \subset \alpha_{21}$.

For the proof see [2] p. 103.

2.3 THEOREM. $X = (A,K)$ *in* Lim, PsT *or* PrT *is* Pre T_2' *iff for any* $x \neq y$ *in* A, $K(x) \cap K(y) = \{[\varnothing]\}$.

Proof. Suppose X is Pre T_2'. We show that if $x \neq y$, then $K(x) \cap K(y) = \{[\varnothing]\}$. If $K(x) \cap K(y) \neq \{[\varnothing]\}$, then there exists a proper filter $\alpha \in K(x) \cap K(y)$. Let $\alpha_{11} = [x]$,

$\alpha_{21} = \alpha = \alpha_{22}$, and $z = (x,y)_2$. Note that $\alpha_{21} \cup \alpha_{22} = \alpha$ is proper. If further $\alpha_{11} \cup \alpha_{21} = [x] \cup \alpha$ is improper, then by 2.2 there exists a proper filter β such that $\pi_{11}\beta = [x]$ $\pi_{21}\beta = \alpha = \pi_{22}\beta$. Note also that since case (ii) of 2.2 holds, there exists an element W of β in the form $W = (\{x\} \times U)_1$, $\{x\} \in [x]$, $U \in \alpha$ i.e W lies in the first component of the wedge. Since X is Pre T_2', it follows that $\beta \supset i_2\beta_1$ for some $\beta_1 \in K'(x,y)$ (i.e $\pi_1\beta_1 \in K(x)$ and $\pi_2\beta_1 \in K(y)$), and $i_2(x,y) = (x,y)_2$. Hence if $N \in i_2\beta_1$, then $N \supset i_2 M$ for some $M \in \beta_1$ and consequently $N \supset i_2 M \supset i_2((G \times H) = (G \times H)_2$ for some $G \in \pi_1\beta_1$ and $H \in \pi_2\beta_1$. Note that $W \cap (G \times H) = \{(x,x)\}$ or $\varnothing$ is in β since β is a filter. Since β is proper, $\beta = [(x,x)]$ and consequently $[x] = \pi_{21}\beta = \alpha$, a contradiction since $\alpha \cup [x]$ is improper. Hence $\alpha \cup [x]$ must be proper and thus $\alpha \subset [x]$. Since $\alpha \in K(y)$, it follows that $[x] \in K(y)$. Let $\beta = [(x,y)_2]$ and $z = (x,y)_1$, and note that $\pi_{11}\beta = [x] \in K(x)$, $\pi_{21}\beta = [x] \in K(y)$, and $\pi_{22}\beta = [y] \in K(y)$. However, $\beta \not\supset i_1\delta$ for any $\delta \in K'(x,y)$ since $x \neq y$. Therefore, $K(x) \cap K(y) = \{[\varnothing]\}$ for all $x \neq y$ in A.

Conversely, we show that if the condition holds, then X is Pre T_2' i.e for any filter β on the wedge and any point z in wedge, (I) $\beta \supset i_k\alpha_1$ for some $\alpha_1 \in K'(u)$, $u \in B^2\text{-}\Delta$, and $i_k u = z$ for $k = 1$ or 2, or $\beta \supset i_1\alpha_1 \cap i_2\alpha_2$ for some $\alpha_1, \alpha_2 \in K'(u)$, u is not on the diagonal, $i_1 u = z$ $i_2 u$ iff (II) $\pi_{11}\beta \in K(\pi_{11}z)$, $\pi_{21}\beta \in K(\pi_{21}z)$, and $\pi_{22}\beta \in K(\pi_{22}z)$, by 1.1, 1.2, 1.4, and 1.5. By [4], (I) implies (II). To show (II) implies (I) note that if β is improper, then clearly $\beta \supset i_1\pi_{11}\beta$ since $i_1\pi_{11}\beta$ is improper. Assume β is proper. If $z = (x,y)_1$, $x \neq y$, and $\pi_{11}\beta \in K(x)$, $\pi_{21}\beta \in K(y)$, and $\pi_{22}\beta \in K(y)$, then we apply 2.2 with $\alpha_{11} = \pi_{11}\beta$, $\alpha_{21} = \pi_{21}\beta$, and $\alpha_{22} = \pi_{22}\beta$. If (a) of 2.2 fails, then by (i) $\pi_{11}\beta = \pi_{21}\beta$ and consequently $\pi_{11}\beta \in K(x) \cap K(y)$, a contradiction. If (b) of 2.2 fails, then by (ii) $\pi_{22}\beta = \pi_{21}\beta$. Note that

(b) of 2.2 fails means that there exists an element W of β in the form $W = (U_{11} \times U_{22})_1$ i.e W lies on the first component of the wedge. Let $\beta_1 = \pi_1^{-1}\pi_{11}\beta \cup \pi_2^{-1}\pi_{22}\beta$ and note that $\pi_1\beta = \pi_{11}\beta \in K(x)$ and $\pi_2\beta = \pi_{22}\beta \in K(y)$ and consequently $\beta_1 \in K'(x,y)$. We claim that $i_1\beta_1 = \pi_{11}^{-1}\pi_{11}\beta \cup \pi_{21}^{-1}\pi_{21}\beta \cup \pi_{22}^{-1}\pi_{22}\beta = \beta_0$. We first show that $i_1\beta_1 \subset \beta_0$. If $W \in i_1\beta_1$ then $W \supset (V_1 \times V_2)_1$ for some $V_1 \in \pi_{11}\beta$ and $V_2 \in \pi_{22}\beta = \pi_{21}\beta$. Since (b) of 2.2 fails, $\pi_{11}\beta \cup \pi_{21}\beta$ is improper and consequently $V \cap U = \emptyset$ for $V \in \pi_{11}\beta$ and $U \in \pi_{21}\beta$. Clearly, $V \cap V_1 \in \pi_{11}\beta$ and $U \cap U_1 \in \pi_{22}\beta = \pi_{21}\beta$ (since they are filters), and consequently $\pi_{11}^{-1}(V \cap V_1) \cap \pi_{21}^{-1}(U \cap V_2) \cap \pi_{22}^{-1}(U \cap V_2) = ((V \cap V_1) \times (U \cap V_2))_1 \in \beta_0$ since $(U \cap V_2) \cap (V \cap V_1) = \emptyset$. But $W \supset (V_1 \times V_2)_1 \supset ((V \cap V_1) \times (U \cap V_2))_1 \in \beta_0$ and consequently $W \in \beta_0$ i.e $i_1\beta_1 \subset \beta_0$. We now show that $\beta_0 \subset i_1\beta_1$. If $W \in \beta_0$, then $W \supset (U \times V)_1 \vee ((U \cap V) \times V)_2$ for some $U \in \pi_{11}\beta$, $V \in \pi_{22}\beta = \pi_{21}\beta$. We further may assume that $U \cap V = \emptyset$ since $\pi_{11}\beta \cup \pi_{21}\beta$ is improper. Thus $W \supset (U \times V)_1$ and consequently $(U \times V)_1 \in i_1\beta_1$. Therefore $W \in i_1\beta_1$. This shows that $i_1\beta_1 = \beta_0$. But by 3.3 of [2] p.99, $\beta_0 \subset \beta$ i.e $i_1\beta_1 \subset \beta$.

If neither (a) nor (b) of 2.2 fails, then by (iii) $\pi_{11}\beta \cap \pi_{22}\beta \subset \pi_{21}\beta$. Since $\pi_{11}\beta \cup \pi_{21}\beta$ is proper ((b) of 2.2 holds) and is in $K(x) \cap K(y)$, we get a contradiction. Hence only case (ii) of 2.2 holds and in this case $\beta \supset i_1\beta_1$. If $z = (x,y)_2$, then $\pi_{11}\beta \in K(x)$, $\pi_{21}\beta \in K(x)$, and $\pi_{22}\beta \in K(y)$. We apply 2.2. If (a) of 2.2 fails, then by (i) $\pi_{11}\beta = \pi_{21}\beta$. Let $\beta_1 = \pi_1^{-1}\pi_{11}\beta \cup \pi_2^{-1}\pi_{22}\beta$ and clearly $\beta_1 \in K'(x,y)$. By an argument similar to the one used above, one can easily show that $i_2\beta_1 \subset \beta$. If case (ii) or (iii) of 2.2 holds, then $\pi_{22}\beta \cup \pi_{21}\beta$ is proper and is in $K(x) \cap K(y)$, a contradiction. Finally, if $x = y$ i.e $z = (x,x)_1 = (x,x)_2$, then $\pi_{11}\beta$, $\pi_{21}\beta$, and $\pi_{22}\beta$ are in $K(x)$. If case (i) or (ii) of 2.2 holds, then we have $\beta \supset i_1\beta_1$ or $\beta \supset i_2\beta_1$ for some $\beta_1 \in K'(x,x)$, respectively (see above cases for the proof). If case (iii) of 2.2

holds, then we have $\beta \supset i_1\beta_1$ or $\beta \supset i_2\beta_1$ for some $\beta_1 \in K'(x,x)$, respectively (see above cases for the proof). If case (iii) of 2.2 holds, then we have $\pi_{11}\beta \cap \pi_{22}\beta \subset \pi_{21}\beta$. Note that $\pi_{11}\beta \cap \pi_{22}\beta \in K(x)$ since X is in Lim, PsT, or PrT. If we let $\beta_0 = \pi_{11}^{-1}\pi_{11}\beta \cup \pi_{21}^{-1}(\pi_{11}\beta \cap \pi_{22}\beta) \cup \pi_{22}^{-1}\pi_{22}\beta$, then by [2] p.99, $\beta_0 \subset \beta$ and $\pi_{11}\beta_0 = \pi_{11}\beta \in K(x)$, $\pi_{21}\beta_0 = \pi_{11}\beta \cap \pi_{22}\beta \in K(x)$, and $\pi_{22}\beta_0 = \pi_{22}\beta \in K(x)$. Let $\beta_1 = \pi_1^{-1}\pi_{11}\beta \cup \pi_2^{-1}\pi_{22}\beta$ and note that $\beta_1 \in K'(x,x)$. We will show that $\beta_0 = i_1\beta_1 \cap i_2\beta_1$. If $W \in \beta_0$, then $W \supset (U \times (N \cap V))_1 \vee ((U \cap V) \times N)_2$ for some $U \in \pi_{11}\beta$, $V \in \pi_{11}\beta \cap \pi_{22}\beta$, and $N \in \pi_{22}\beta$. Since all of these are filters, $U \cap V \in \pi_{11}\beta$ and $V \cap N \in \pi_{22}\beta$, and consequently $W \supset ((U \cap V) \times (N \cap V))_1 \vee ((U \cap V) \times (V \cap N))_2$. Since $((U \cap V) \times (N \cap V)) \in \beta_1$, $W \in i_1\beta_1 \cap i_2\beta_1$ (since $((U \cap V) \times (N \cap V))_1 \in i_1\beta_1$, $((U \cap V) \times (N \cap V))_1 \vee ((U \cap V) \times (V \cap N))_2 \in i_1\beta_1 \cap i_2\beta_1$, and $((U \cap V) \times (V \cap N))_2 \in i_2\beta_1$). This shows $\beta_0 \subset i_1\beta_1 \cap i_2\beta_1$. On the other hand, if $W \in i_1\beta_1 \cap i_2\beta_1$, then $W \supset (U \times V)_1 \vee (U \times V)_2$ for some $U \in \pi_{11}\beta, V \in \pi_{22}\beta$. Note that $N = U \cup V \in \pi_{11}\beta \cap \pi_{22}\beta$ and $\pi_{11}^{-1}(U) \cap \pi_{21}^{-1}(N) \cap \pi_{22}^{-1}(V) = (U \times (N \cap V))_1 \vee ((U \cap N) \times V)_2 = (U \times V)_1 \vee (U \times V)_2 \in \beta_0$ and consequently $W \in \beta_0$. This shows (II) implies (I) which completes the proof.

2.4 THEOREM. $X = (B,K)$ *in* Lim, PsT, *or* PrT *is* T_2', LT_2, *or* MT_2 *iff for any distinct pair of points x and y in B,* $K(x) \cap K(y) = \{[\phi]\}$.

Proof. Combine 2.1, 2.3, and Definition 1.2.

2.5 *Remark* (1) For the categories of Lim, PsT or PrT, we have $\overline{T}_0 = T_0$ implies T_0' and $T_2' = LT_2 = MT_2 = \overline{T}_2 = NT_2$ implies KT_2 but the converse of each implication is not true, in general.

(2) We also have some relationships among our notions T_2 's and Nel 's T_2 [7]. In Lim, PsT, and PrT, his T_2 is equivalent to our NT_2, T_2', LT_2, MT_2, and $\overline{T}_2$ and his T_2 implies

our KT_2.

(3) If (A,K) is an indiscrete (with at lest two points) i.e $\forall\ x \in A$, $K(x) = F(X)$, the set of all filters on A, then (A,K) is clearly KT_2 but not any of other T_2, T_1, T_0, and $\overline{T}_0$. However, in TOP we know that every T_2 topological space is both T_0 and T_1 space.

Let (A,L) and (X,K) be objects of Lim, PsT, or PrT.

2.6 THEOREM. *Let $f : (A, L) \rightarrow (X, K)$ be the initial lift* (see 1.4). *If (X,K) is LT_2, $\overline{T}_2$, T_2', MT_2, KT_2, and NT_2, then (A,L) is LT_2, $\overline{T}_2$, T_2', MT_2, KT_2, and NT_2, respectively.*

Proof. It follows easily from 2.1, 2.4, and f is being initial.

2.7 THEOREM. *Let $f : (A, L) \rightarrow (X, K)$ be the initial lift and (X,K) be T_0, $\overline{T}_0$, or T_1, then (A,L) is T_0, $\overline{T}_0$, or T_1, respectively iff f is mono.*

Proof. If f is mono, then the result follows easily. Suppose (A,L) is T_0, $\overline{T}_0$, or T_1 but f is not mono i.e $\exists\ x \neq y$ in A such that $f(x) = f(y)$. Note that $f([x]) = [f(x)] = [f(y)] \in K(f(y)) \cap K(f(x))$ implies $[x]$ and $[y]$ are in $L(x) \cap L(y)$, which is a contradiction since (A,L) is T_0, $\overline{T}_0$, or T_1.

2.8 COROLLARY. *Suppose (X,K) is "T_i" space for $i = 0,1,2$. Then every subspace of (X,K) (i.e $f : (A,L) \rightarrow (X,K)$ is mono and initial) is "T_i" space for $i = 0,1,2$.*

Proof. It follows from 2.6 and 2.7.

2.9 THEOREM. *The cartesian product $(X = \{\prod X_i \mid i \in I\}, K)$ is "T_k", for $k = 0,1,2$, iff each (X_i,K_i) is "T_k" for $k = 0,1,2$.*

Proof. Suppose (X,K) is "T_k", $k = 0,1,2$. Then it is easy to see that each (X_i,K_i) is isomorphic to some slice in (X,K) and consequently by 2.8, (X_i,K_i) is "T_k" $(k = 0,1,2)$ for all i.

Conversely, suppose each (X_i,K_i) is T_1 but (X,K) is not T_1 i.e by 2.1, $\exists\, x \neq y$ in X such $[x] \in K(y)$. It follows that $\exists\, j \in I$ such that $x_j \neq y_j$ in X_j and $\pi_j([x]) = [x_j] \in K_j(\pi_j(y) = y_j)$, a contradiction. Hence (X,K) must be T_1. The proof for T_0 and $\bar{T}_0$ is similar. Suppose each (X_i,K_i) is T_2'. For any $x \neq y$ in X, let $\alpha \in K(x) \cap K(y)$. Note that $\pi_i\alpha \in K_i(\pi_i x = x_i) \cap K_i(\pi_i y = y_i)$. Since $x \neq y$, $\exists\, j \in I$ such that $x_j \neq y_j$ in X_j. In particular, by 2.4 and 2.1, $\pi_j\alpha = [\phi]$ which implies $\alpha = [\phi]$. Hence by 2.4 and 2.1, (X,K) is T_2'. Note that T_2' is equivalent to $\bar{T}_2$, NT_2, MT_2, LT_2.

Suppose each (X_i,K_i) is KT_2 and for any $x \neq y$ in X, $K(X) \cap K(y) \neq \{[\phi]\}$. Then there exists a proper filter $\beta \in K(x) \cap K(y)$ and conse quently $\pi_i\beta \in K_i(x_i) \cap K_i(y_i)\ \forall\, i$. Since each (X_i,K_i) is KT_2, $K_i(x_i) = K_i(y_i)$ for all i. Let α be in $K(x)$. Then $\pi_i\alpha \in K_i(x_i) = K_i(y_i)$ for all i and consequently $\alpha \in K(y)$. By interchanging the role of x and y, we get $K(y) \subset K(x)$ if $K(x) \cap K(y) \neq \{[\phi]\}$. Hence $K(x) = K(y)$ and by 2.1, (X,K) is KT_2.

We now generalize Nel's, [7], result for full subcategories KT_2E, $T_0'E$, and T_1E, where E = Lim, PsT, or PrT (objects of these subcategories are KT_2, T_0', and T_1 limit spaces, pseudopological spaces, and pretopological spaces). Nel has showed that a full subcategory, HLim (objects of HLim are Hausdorff limit convergence spaces) of Lim is a cartesian closed initially structured category.

2.10 LEMMA. *Each of subcategories defined above are cartesian closed initially structured categories.*

Proof. It follows easily from 2.1 and the results 1.13, 1.14, and 2.5 of [7].

Let (X,K) be in Lim, PsT, or PrT and a F be nonempty subset of X. Let $q : (X,K) \rightarrow (X/F,L)$ be the quotient map that identifying F to a point, *

2.11 LEMMA. *Let α and β be proper filters on X. Then $q\alpha \cup q\beta$ is proper iff either*

$\alpha \cup \beta$ is proper or $\alpha \cup [F]$ and $\beta \cup [F]$ are proper.

Proof. Suppose $q\alpha \cup q\beta$ is proper. We show that if $\alpha \cup [F]$ is improper, then $\alpha \cup \beta$ is proper. Suppose it is not proper i.e there exist U in α and V in β such that $U \cap V = \phi$. Since $\alpha \cup [F]$ is improper, $\exists\, W \in \alpha$ such that $W \cap F = \phi$. Note that $U \cap W \in \alpha$ and $U \cap W = q(U \cap W)$ in $q\alpha$. It follows easily that $\phi = q(U \cap W) \cap q(V) \in q\alpha \cup q\beta$, a contradiction. Similarly, if $\beta \cup [F]$ is improper, then $\alpha \cup \beta$ is proper.

Conversely, suppose $\alpha \cup \beta$ is proper, but $q\alpha \cup q\beta$ is improper. Then $\exists\, U \in q\alpha$ and $\exists\, V \in q\beta$ such that $\phi = U \cap V \supset q(U_1) \cap q(V_1)$ for some $U_1 \in \alpha$ and $V_1 \in \beta$. It follows that $\phi = U_1 \cap V_1 \in \alpha \cup \beta$, a contradiction. Suppose $\alpha \cup [F]$ and $\beta \cup [F]$ are proper. Suppose also that $q\alpha \cup q\beta = [\phi]$. Then $\exists\, U \in q\alpha$ and $\exists\, V \in q\beta$ such that $\phi = U \cap V \supset q(U_1) \cap q(V_1)$ for some $U_1 \in \alpha$ and $V_1 \in \beta$. But this is a contradiction since $U_1 \cap F \neq \phi \neq V_1 \cap F$ and $q(U_1 \cap F) = {}^* = q(V_1 \cap F)$.

2.12 THEOREM. *If (X,K) in* Lim *or* PsT *is "T_2", then $(X/F,L)$ is "T_2".*

Proof. Let a and b be any distinct pair of points in X/F and α be in $L(a) \cap L(b)$. If α is improper, then we are done. Soppose α is proper q is the quotient map, [7], implies $\exists$ $\beta_i \in K(x_i)$ and $\exists\, \delta_j \in K(y_j)$ such that $\alpha \supset q(\bigcap_{i=1}^{n} \beta_i)$, $\alpha \supset q(\bigcap_{j=1}^{k} \delta_j)$, and $qx_i = a$, $qy_j = b$, where $i = 1, \ldots, n$ and $j = 1, \ldots, k$. Note that $q(\bigcap_{j=1}^{n} \beta_i) \cup q(\bigcap_{j=1}^{k} \delta_j)$ is proper and by 2.11, either $(\bigcap_{i=1}^{n} \beta_i) \cup (\bigcap_{j=1}^{k} \delta_j)$ is proper or $(\bigcap_{i=1}^{n} \beta_i) \cup [F]$ and $(\bigcap_{j=1}^{k} \delta_j) \cup [F]$ are proper. If the first case holds, then there exist s and t such that $\beta_s \cup \delta_t$ is proper and consequently is in $K(x_s) \cap K(y_t)$, a contradiction since (X,K) is T_2', LT_2, MT_2, T_2, and NT_2 $(x_s \neq y_t)$. If the second case holds, then it is easy to see that there exist s and t, and points d, c in F such that $\beta_s \cup [d]$ and $\delta_t \cup [c]$ are proper. It follows that $\beta_s \subset [d]$ and $\delta_t \subset [c]$, and consequently $[d] \in K(x_s)$ and $[c] \in K(y_t)$. If $d \neq x_s$ or $c \neq y_t$, then we get a contradiction since (X,K) is "T_2". If $d = x_s$ and $c = y_t$, then $a = qx = qd = * = qc = b$ a contradiction since $a \neq b$. Hence the second case can not occur, either. Thus α must be improper. By 2.1 and 2.2,, we get the result.

Suppose (X,K) is KT_2 and for any $a \neq b$ in X/F, let $\alpha \in L(a) \cap L(b)$ with $\phi \notin \alpha$. q is the quotient map implies $\exists\ \beta_s \in K(x_s)$ and $\exists\ \delta_t \in K(y_t)$ such that $\beta_s \cup \delta_t$ is proper or $\beta_s \subset [d]$ and $\delta_t \subset [c]$ for some $c,d \in F$ (see above argument). If the first case holds, then $\beta_s \cup \delta_t$ is in $K(x_s) \cap K(y_t)$ and by 2.1, $K(x_s) = K(y_t)$. Hence $L(a) = L(b)$. If the second case holds and $x_s \neq d$ or $y_t \neq c$, then $[d] \in K(x_s) \cap K(d)$ or $[c] \in K(y_t) \cap K(c)$, respectively. Since (X,K) is KT_2, $K(x_s) = K(d)$ or $K(y_t) = K(c)$, and consequently $L(a) = L(b)$ since $qd = * = qc$. If $x_s = d$ and $y_t = c$, then $a = qd = * = qc = b$, a contradiction to $a \neq b$.

Note that 2.12 holds for (X,K) in PrT where the finite intersection is replaced by the intersections, N_x, of all filters in $K(x)$.

2.13 THEOREM. *If (X,K) is T_1 or T_0, then $(X/F,L)$ is T_1 or T_0.*

We only prove for T_1 since the proof for T_0 is similar. By 2.1, we need to show that for any distinct pair of points a and b in X/F, $[a] \notin L(b)$. Suppose $[a] \in L(b)$ for some $a \neq b$ and $a \neq * \neq b$. Since q is quotient ([7] p. 1375), $\exists\ \alpha \in K(x)$ such that $[a] \supset q\alpha$ and $qx = b$. By Lemma 3.16 of [2], we get $[a] \supset \alpha$ and consequently $[a] \in K(x)$, a contradiction (X is T_1). Suppose $a = * \neq b$. Then $[*] \supset q\alpha$ for some α in $K(x)$ and $x = qx = b$. By Lemma 3.16 of [2], $\alpha \cup [F]$ is proper and consequently $\exists\ d \in F$ such that $\alpha \subset [d]$. Hence $[d] \in K(x)$, a contradiction. Suppose $a \neq * = b$. Then there exist $x_i \in F$ and $\beta_i \in K(x_i)$, $i = 1,2,\ldots,n$ such that $[a] \supset q(\bigcap_{i=1}^{n} \beta_i)$ and $qx_i = * = b$. It follows easily that $\exists\ k \in \{1,2,\ldots,n\}$ such that $q\beta_k \subset [a]$. By Lemma 3.16 of [2] p. 106, we get $\beta_k \subset [a]$ and consequently $[a] \in K(x_k)$, a contradiction.

2.14 *Remark*. Let (X,K) be in Lim, PsT or PrT and R be an equivalence relation on X. Then Theorems 2.12 and 2.13 hold for a quotient space $(X/R,L)$. However, in TOP these results do not hold, in general.

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CONTINUOUS APPROXIMATE SOLUTIONS TO THE NEUTRAL DELAY DIFFERENTIAL EQUATIONS BY A SIMPLIFIED PICARD'S METHOD

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REZUMAT. - Soluţii aproximative continue pentru ecuaţii diferenţiale cu argument întârziat de tip neutral prin metoda lui Picard modificată. Rezolvarea numerică a ecuaţiilor diferenţiale cu argument întârziat prezintă o deosebită importanţă. Procedeele cunoscute până acum, cum ar fi metoda colocaţiei, metoda seriilor de puteri sau metoda paşilor nu sunt suficient de eficiente. În această lucrare se prezintă o metodă de tip Picard simplificată pentru rezolvarea numerică a următoarei probleme:

$$y'(t) = f(t, y(t), y(g(t)), y'(g(t)), t \in [a, b]$$
$$y(t) = \varphi(t), y'(t) = \varphi'(t), t \in [\alpha, a], \alpha \le a < b$$

unde $\alpha \le g(t) < t$, iar $\varphi \in C^{m+1}([\alpha,a], \mathbb{R})$, $m > 1$ fiind un număr natural dat. Metoda prezentată este o variantă simplificată a metodei aproximaţiilor succesive a lui Picard, aplicată problemei de mai sus.

Abstract. Numerical solutions of neutral delay differential equations has become increasingly important. Collocation procedure with polynomial spline, power series and method of steps in the independent variable makes it seems a slow and memory consuming process. A simplified version of Picard's Method, it is not necessary to store and manipulate all the coefficients of the k th approximation in order to the $(k+1)$ th approximation. We consider the following first order initial value (Cauchy) problem for neutral delay differential

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equations (NSSEs).

$$y'(t) = f(t, y(t), y(g(t)), y'(g(t))), \; t \in [a, b]$$

$$y(t) = \phi(t), \; y'(t) = \phi'(t), \; t \in [\alpha, a], \; \alpha \le a < b$$

The function g is called the delay argument and it is assumed to be continuous on $[a,b]$ and satisfies the inequality $\alpha \le g(t) < t$. Also initial function ϕ has property $\phi \in C^{m-1}([\alpha, a], R)$, where $m > 1$ is given natural number. We pressent an efficient procedure for computing continuous approximate solution to initial value problem for neutral delay differential equations. The method is a simplification of Picard's method of succesive substitutions.

1. Introduction. In recent years, there has been a growing interest in the numerical treatment of NDDEs. This type of equation is a differential equation in which the highest order derivative of the unknow function appears in the equation evaluated both at the present time t as well as at some past time $t-\tau$. Besides its theorical interest, the study of NDDEs has some importance in applications. For example, NDDEs appear in problems concerning network containing lossless transmission lines. Such networks arise in high speed computers where lossless transmission lines are used to interconnect switcing circuits. See for more detail [4] and the reference cited there in. Also such type of equations appear in many fields of application such as; physics, engineering, biology, medicine, economics, etc.

The purpose of the present study is to extend and generalize some results from the ordinary case to the neutral one. Consider the following first order initial value problem for NDDEs.

$$y'(t) = f(t, y(t), y(g(t)), y'(g(t))), \; t \in [a, b]$$

$$y(t) = \phi(t), \; y'(t) = \phi'(t), \; t \in [\alpha, a], \; \alpha \le a < b \qquad (1.1)$$

Where $f: [a, b] \times C^1[a, b] \times C^1[\alpha, b] \times C[\alpha, b] \rightarrow R$

The function g, called the delay function and is assumed to be continuous on the interval $[a,b]$ to satisfy the inequality $\alpha \leq g(t) \leq t$ and $\phi \in C^{m-1}([\alpha,a],R)$, where $m > 1$ is a given natural number. For the qualitative behaviour of the solution y, in particular the presence of jump discontinuites, which can occur in various derivatives of the solution, even if f, g and ϕ are analytic in their arguments, the reader is referred for example to Bellen [1], Driver [3].

Denote the jump - discontinuites by $\{\xi_j\}$ which are the roots of the equations $g(\xi_j) = \xi_{j-1}$, $\xi_0 = a$ is the jump - discontinuity of ϕ. Since in this paper g does not depend on y (no state - depend delay), we can consider the jump - discontinuities to be known for sufficiently high order derivatives and they are disposed in the form

$$\xi_0 < \xi_1 < \dots < \xi_{k-1} < \xi_k < \dots < \xi_m$$

The notation used in this paper is taken from [5] and [2]. Assume the following conditions (A) satisfied:

Condition A:

A_1: For any $x \in C^1(\alpha,b]$ the maping $t \rightarrow f(t, x(t), x(\cdot), x'(\cdot))$ is continuous on $[a,b]$.

A_2: The Lipschitz condition holds:

$$\|f(t, x_1(t), y_1(t), z_1(t)) - f(t, x_2(t), y_2(t), z_2(t))\| \leq$$

$$L_1(\|x_1 - x_2\|_{[a,t]} + \|y_1 - y_2\|_{[\alpha,t-\delta]} + \|z_1 - z_2\|_{[\alpha,t-\delta]}) + L_2\|z_1 - z_2\|_{[\alpha,t]} \qquad (1.2)$$

with $L_1 \geq 0$, $0 \leq L_2 < 1$, $\delta > 1$ for any $t \in [a,b]$, for all $x_1, x_2, y_1, y_2 \in C[\alpha, b]$ $z_1, z_2 \in C^1[\alpha, b]$. Here

$C^i[a, b]$, $i = 0, 1, 2$ denotes the space of all function of class C^i from $[a,b]$ in to R and

$$\|x\|_{[\alpha,t]} := \operatorname{Sup}\{\|x(s)\| : s \in [\alpha, t]\} \tag{1.3}$$

Function $f(t, y(t), y(\cdot), y'(\cdot))$ has continous partial derivatives with respect to its arguments of all orders up to the $(n+1)$ th inclusive in the neighbourhood of a point $[\xi_j \le t_i \le \xi_{j+1}]$. Then Taylor formula neighbourhood of t_i will held

$$\Delta f(t, y(t), y(\cdot), y'(\cdot)) = \frac{1}{1!}df + \frac{1}{2!}d^2 f + \ldots + \frac{1}{n!}d^n f + \frac{1}{(n+1)!}d^{n+1} f + \ldots \tag{1.4}$$

Since the all orders of partial derivatives are assumed to exist, each one of then must have a maximum value $M_1 > 0$ on the consedering interval. The following procedure yields to an approximating function $u \to R, m \in C^{m-1}([a,b], R)$ which is defined on each subinterval

$[\xi_i, \xi_{i+j}]$ $(j = 0, 1, \ldots, M-1)$

2. Description of the metod. Consider the first interval $[\xi_0, \xi_1]$ **The proposed method for the approximate solution of (1.1) can recursively defined as follows:**

$$u_0 = \phi(t), \quad t \in [\alpha, b] \quad \alpha \le a < b \tag{2.1}$$

$$u_0' = f(t, \phi(t), \phi(\cdot), \phi'(\cdot)) \tag{2.2}$$

$$u_j = \int_t u_{j-1}' ds, \ \xi_0 = t_0 < t < \xi_1, \ j = 1, 2, \tag{2.3}$$

where u_{j-1}' is the term of degree $j-1$ in the expansion of $f(t, u^{(j-1)}(t), u^{(j-1)}(\cdot), u'^{(j-1)}(\cdot))$ as a power series in t. And

$$u^{(j)} = u_0 + u_1 + \ldots + u_j \tag{2.4}$$

is a polynomial in t

THEOREM 2.1. ***Let condition (A) hold, then there exist an unique approximating solution of the problem*** **(1.1)** ***given by*** **(2.4)** ***which is identical to first n+1 terms of the series representation of the solution of (1.1) by Picard's method on each of the interval*** $[\xi_j, \xi_{j+1}]$

The proof follows by induction. First let us show that defined initial function ϕ and linear terms in (2.4) are equal to the correspondinng terms computed by Picard's method.

$$y_0(t) = \phi(t), \quad t \in [\alpha, a] \tag{2.5}$$

$$y_m(t) = \phi(t) + \int_{t_0}^{t} f(s, y_{m-1}(s), y_{m-1}(\cdot), y'_{m-1}(\cdot))\, ds \tag{2.6}$$

$$\xi_j \le t_0 < t < \xi_{j+1} \quad \text{and} \quad T = \xi_{j+1} - \xi_j$$

convergence of the Picard sequence $\{y_n\}$ to a solution $y(t)$ of (1.1) is guaranteed with Condition (A). Thus the solution of (1.1) computed by Picard's method can be written in the form

$$y(t) = p_0 + p_1 t + p_2 t^2 + \quad + p_n t^n + \ldots \tag{2.7}$$

on the interval $[\xi_0, \xi_1]$. Here $p_i(i \le n)$ are depends on the particular form of f and its partial derivatives with respect to all arguments of all orders up to $(n+1)$ th. The first approximation to the solution of (1.1) by the Picard's iteration

$$y_1(t) = \phi(t) + \int_{t_0}^{t} f(s, \phi(s), \phi(\cdot), \phi'(\cdot))\, ds \tag{2.8}$$

integration of power series expansion of $f(s, \phi(s), \phi(\cdot), \phi'(\cdot))$ in s we can obtain

$$y_1(t) = \phi(t) + c_1 t + c_2 t^2 + \tag{2.9}$$

The term $\phi(t)$ remains invariant in all successive approximation $y_m(t)$, $m = 2, 3,$ The equality

$$\phi(t) = y_0 = p_0 \tag{2.10}$$

follows from (2.5) and (2.7).

Evaluation of equation (2.3) with $j = 1$ it follows,

$$u_1 = c_1(t) \tag{2.11}$$

which is a result of comparison of (2.8) with (2.9)

LEMMA 2.1. *Assume that the function $f(t, z(t), z(\cdot), z'(\cdot))$ is analytic in all arguments* *and has continuous partial derivatives of any orders. Then it can be expanded is a power serie of t. The term of degree k of the argument z(t):*

$$z(t) = z_0 + z_1 t + z_2 t^2 + \ldots + z_k t^k + \tag{2.12}$$

of the function f contributes only to the values of term of equal or higher degree in the resulting expansion of the function as a power series in t. The proof of the Lemma is simply a modification of Lemma 1 of [5].

From Lemma 2.1 and invariance of the form $\phi(t)$ it follows that

$$c_1 = p_1 \tag{2.13}$$

Thus it is easily seen that the equality

$$u_j = p_j t^j \tag{2.14}$$

is satisfied for $j = 0, 1$. Assume that (2.14) holds for $j = 0, 1, 2, \ldots, n-1$. Substitution of the solution $y(t)$ in equation (2.6) followed by the expansion of $f(s, y(s), y(\cdot), y'(\cdot))$ as a power series in s results in

$$y(t) = \phi(t) + \int_{t_0}^{t} (d_0 + d_1 s + d_2 s^2 + \ldots + d_{n-1} s^{n-1} + d_n s^n + \ldots)\, ds \tag{2.15}$$

After integration we obtain

$$y(t) = p_0 + p_1 t + p_2 t^2 + \ldots + p_n t^n \tag{2.16}$$

where

$$p_0 = \phi(t), \quad p_k = \frac{d_{k-1}}{k}, \quad k = 1, 2, \ldots \tag{2.17}$$

The constants d_{n-1} $(n = 1, 2, \ldots)$ in (2.15) depend only on the coefficients p_i $i \leq n-1$ of $y(t)$ and the particular form of $f(t, y(t), y(\cdot), y'(\cdot))$. Then $(n-1)$ iterations carried out using equations (2.1)-(2.3) result in the expression

$$u^{(n-1)}(t) = u_0 + u_1 + \ldots + u_{n-1} \tag{2.18}$$

where the terms u_i satisfy (2.14) by the inductive assumption. The variable u'_{n-1} is by definition, the term of $f(t, y^{[n-1]}(t), y^{[n-1]}(\cdot), y'^{[n-1]}(\cdot))$ as a power series in t Lemma 1 and the inductive assumption (2.14) guarantee

$$u'_{n-1} = d_{n-1}\, t^{n-1} \tag{2.19}$$

Substitution of (2.19) in (2.3) yields

$$u_n = \frac{d_{n-1}}{n}\, t^n \tag{2.20}$$

and the desired result

$$u_n = p_n\, t^n \tag{2.21}$$

follows by direct comparison of (2.20) and (2.17). This completes the proof of the theorem. Above processes can be repete on each consecutive interval $[\xi_k, \xi_{k+1}]$.

3. Accuracy and Error analysis of the method. Using equations (2.1)-(2.3), computed polynomial

$$u^{(n)}(t) = \sum_{i=0}^{N} p_i\, t^i \tag{3.1}$$

is a truncation of the infinite series

$$y(t) = \sum_{i=0}^{\infty} p_i\, t^i \tag{3.2}$$

which represents the exact solution of equation (1.1) in the interval $\xi_j \le t \le \xi_{j+1}$. Picard sequence $\{y_m\}$ generated with (2.5)-(2.6) can be written in the form

$$y_m(t) = \sum_{i=0}^{m} p_i\, t^i + \sum_{i=m+1}^{\infty} q_i\, t^i \tag{3.3}$$

where the q_i depend on the number of iterations m, the particular form of the function $f(t, y(t), y(\cdot), y'(\cdot))$ and the initial function $\phi(t)$. Following recall [6] the estimation of the error is:

$$\| y(t) - y_m(t) \| \le \left(e^{3LT} - \sum_{k=0}^{m-1} \frac{(3Lt)^k}{k!} \right) \| y_1(t) - \phi(t) \| \tag{3.4}$$

where $k := \max\{k_1, k_2\}$ and the particular form (3.3) of $y_m^{(t)}$ can be written in the following form:

$$\|u(t) - u^{(m)}(t)\| \le \left(e^{3LT} - \sum_{k=0}^{m-1} \frac{(3LT)^k}{k!}\right)\|y_1(t) - \phi(t)\| + \left\|\sum_{i=m+1}^{\infty} q_i t^i\right\| \tag{3.5}$$

This is a bound on the error of the truncated series (3.1) which is computed using equations (2.1)-(2.3). Here the symbol $\|\ \|$ denotes the norm $\|v(t)\| = \max_{t\in[0,T]} |v(t)|$. The inequality (3.5) leads to reducing the magnitude of the error and there are two techniques;

a) Computation of a polynomial of degree higher than m and

b) Reduction of the length of the interval. Since the limited capacity of machine, coefficients of higher powers of t tend to cause underflow or overflow in the computer. Also efficiency becomes the limiting factor when dealing with nonlinear equations. The number of machine operations required to compute the term grows with square of the degree of the resulting polynomial. Clearly the error term (3.5) in the approximations $u^{(m)}(t)$ vanishes as $T \to 0$. If the computed polynomial $u^{(m)}(t)$ represent the solution within the interval $\xi_j \le t \le h \le \xi_{j+1}$ where $h = T/N$, (N is sufficiently large positive integer) then the error in the approximate solution $u^{(m)}(t)$ can be made arbitrarily small within the interval $\xi_j \le t \le \xi_{j+1}$.

The procedure can be repeated on the consecutive intervals to times to obtain the solution on the complete interval. The procedure described is usually caled analytic continuation. Thus the approximate solution $u(t)$ on the complete interval is given as a set of polynomials of degree m.

The error bound given by inequality (3.5) may be extended to the case that Lipschitz condition (1.2) is satisfied for $y_1(t)$ and $y_2(t)$. In this case the Lipschitz constant depends on r and j where r is the radius of the ball: $u(0, r) = \{x : \|x - x_0\| \le r\}$ in which the polynomial represent a solution of the equation. Thus the inequality

$$r \ge e^{3LT}\|y_1(t) - \phi\|$$

must be satisfied. The convergence of the iteration (2.1)-(2.3) to the solution of (1.1) is guaranteed and the error bound (3.5) holds.

4. Conclusions. The most important advantage of the method lies in the fact that it is an efficient procedure to compute continuous approximate to initial value problems in ordinary differential equations, delay differential equations and neutral delay differential equations. The analytic condition scheme described in the procedure represents the solution in the form of piecewise polynomial functions of desired degree. This allows the possibility of a stable differentiation algoritm.

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ON THE USE OF INTERPOLATORY POSITIVE OPERATORS FOR COMPUTING THE MOMENTS OF THE RELATED PROBABILITY DISTRIBUTIONS

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Dedicated to Professor M. Balázs on his 65[th] anniversary

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REZUMAT. - Asupra utilizării operatorilor liniari pozitivi de tip interpolator pentru calculul momentelor distribuţiilor probabilistice asociate. În partea întâia a lucrării am introdus unele notaţii şi definiţii ale momentelor de ordinul r relative la un punct dat, cele ordinare, centrale şi factoriale. Cu ajutorul diferenţelor lui zero şi numerelor lui Stirling de speţa a doua am dat formule explicite pentru momentul ordinar şi cel factorial de un ordin dat. Prin folosirea relaţiilor de recurenţă pentru diferenţele lui zero sau pentru numerele lui Stirling, se pot calcula în mod succesiv aceste momente.

Abstract. It is known that the theory of approximation by means of linear positive operators is connected with the theory of probability, as there has been illustrated in a concrete case already by S.N. Bernstein [1] in 1912. In a paper by J.P. King [6] there has been proved that if we have a sequence of linear positive operators (L_m), where $L_m: C[a,b] \to C[a,b]$, such that $(L_m e_0)(x) = L_m(1;x) = 1$ for $x \in [a,b]$, then there exists a sequence of independent random variables (Y_m) such that $(L_m f)(x) = E[f(Y_m)]$. Moreover $L_m f \to f$, uniformly on $[a,b]$, for each $f \in C[a,b]$, if and only if $E[Y_m] \to x$ and $\mathrm{Var}[Y_m] \to 0$ as $m \to \infty$. In the paper [14] D.D. Stancu has used probabilistic methods for constructing and investingating linear positive operators useful in constructive approximation theory of functions. In the same paper he started from the Newton interpolation formula for representing

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a linear interpolatory positive operator by means of the factorial moments of the related probability distribution and the finite differences of the function involved on the equally spaced nodes used by the operator.

By means of such representations we deduce explicit formulas for the ordinary moments of the corresponding discrete probability distributions.

In the first part of the paper we introduce some notations and definitions connected with the moments of an order r with respect to a given point, the ordinary, the central and the factorial moments. By means of differences of zero and of Stirling numbers of the second kind we give explicit formulas for an ordinary and a factorial moment. If one uses the recurrence relations for the differences of zero and for the Stirling numbers, one can calculate successively these moments.

1. It is known that with the distribution of a random variable there are associated certain numbers called the parameters of the distribution, which have an important role in many problems from mathematical statistics. These parameters are represented by various types of moments of the considered random variable and by some functions of them, as well as by the order parameters, formed by quantiles of a certain given order $p \in (0,1)$, and some functions of them.

Let $(\Omega, \mathcal{B}, P)$ be a probability space. Suppose Ω is a fixed sample space, $\mathcal{B}$ is a Borel field of subsets of Ω, while P is a probability measure on Ω. Let X be a random variable on Ω with cumulative distribution F_X.

We shall consider a single-valued function $g : \mathbb{R} \to \mathbb{R}$, integrable with respect to F_X. The following Riemann-Stieltjes integral

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x)\, dF_X(x) \tag{1}$$

represents the expected value of the random variable $g(X)$, if we require that the inequality

$$\int_{-\infty}^{+\infty} |g(x)|\, dF_X(x) < \infty$$

is satisfied.

One can see that $E[g(X)]$ is just a weighted average with weights determined by F_X.

If X is a discrete random variable then the corresponding distribution $F_X(x) = P(X \le x) = p_X(x)$ is a step function with a finite or countable infinite number of jump points $\{x_k\}$ on the real axis, at which jumps $\{p_X(x_k)\}$ occur. The jump of $F_X(x)$ at x_k is given by

$$p_X(x_k) = F_X(x_k) - F_X(x_k - 0).$$

The function $p_X(x)$ represents the discrete probability function, which is characterized by the fact that it is positive if x coincides with a point of the sequence $\{x_k\}$, is zero when we have no jumps at x and in addition we have

$$\sum_k p_X(x_k) = 1.$$

In this discrete case we can write

$$F_X(x) = \sum_{(k)} p(x_k),$$

the sum being extended to all those k for which we have $x_k \le x$.

Consequently, the expected value (1) can be expressed by the formula

$$E[g(X)] = \sum_k g(x_k) p_X(x_k).$$

If the random variable X has a continuous probability density function ρ, then the expected value of $g(X)$ is given by the following formula

$$E[g(X)] = \int_{-\infty}^{+\infty} \rho(x)\, g(x)\, dx,$$

under the assumption that this integral is absolutely convergent.

2. Several special choices of $g(X)$ are of great interest in probability theory and mathematical statistics.

Let us first consider that

$$g(X) = (X - a)^r,$$

where r is a natural number and a is any real number.

The moment of order r, with respect to the point a, is defined - if it exists - as the mean value of the random variable $g(X) = (X-a)^r$, namely

$$\nu_r(a) = E\left[(X-a)^r\right].$$

If $a = 0$ then $\nu_r(0) = \nu_r$ represents the r^{th} moment of X about the origin, or the r^{th} ordinary moment of X.

The r^{th} absolute moment of the random variable X is defined by

$$\nu_r' = E\left[|X|^r\right].$$

It is obvious that the absolute moment of an even order equals the ordinary moment of the same order.

It is known that the moments with respect to the expected value $m = \nu_1 = E[X]$ are of great importance in probability theory and mathematical statistics. We denote them by μ_r.

We then have

$$\mu_r = E\left[(X - \nu_1)^r\right].$$

This number is called the central moment of order r of X.

It is easily seen that μ_r can be expressed by means of the ordinary moments, namely

$$\mu_r = \sum_{j=0}^{r} (-1)^j \binom{r}{j} \nu_1^j \cdot \nu_{r-j}$$

For $r = 1, 2, 3$ we obtain

$$\mu_1 = 0,\ \mu_2 = \nu_2 - \nu_1^2,\ \mu_3 = \nu_3 - 3\nu_1\nu_2 + 2\nu_1^3.$$

The central moment of second order, that is the mean quadratic deviation of the random variable X from its expected value ν_1, represents the variance of X. We have

$$\mathrm{Var}[X] = \mu_2 = \sigma_X^2 = \nu_2 - \nu_1^2,$$

σ_X being the standard deviation of X.

3. Considering now that we choose the factorial power of X:

$$g(X) = X^{[s]} = X(X-1)\dots(X-s+1),$$

we arrive at the factorial moment of X, namely

$$\nu_{[s]} = E\left[X^{[s]}\right] = \int_{-\infty}^{+\infty} x^{[s]}\, dF_X(x).$$

In certain kind of problems, involving discrete random variables, it is often convenient to determine the moments ν_r by first evaluating the factorial moments

$$\nu_{[s]} = \sum_k x_k^{[s]} \cdot p_X(x_k).$$

By using the Newton interpolation formula, corresponding to the function $f(x) = e_r(x) = x^r = (r \in \mathbb{N}_0)$ and to the nodes $0, 1, 2, \dots, r$, we obtain the standard monomial e_r in terms of the factorial powers $x^{[j]}$ and the finite differences of order j, with the step $h = 1$ and the starting point $a=0$, namely

$$x^r = \sum_{j=1}^{r} x^{[j]} \cdot \frac{(\Delta^j e_r)(0)}{j!}$$

Consequently, if we introduce the Stirling numbers of the second kind

$$S_j^r = \frac{\Delta^j 0^r}{j!},$$

by means of the differences of zero $\Delta^j 0^r = (\Delta^j e_r)(0)$, we can obtain relations between the ordinary moment ν_r and the factorial moments of the random variable X:

$$\nu_r = \sum_{j=1}^{r} S_j^r \cdot \nu_{[j]} = \sum_{j=1}^{r} \nu_{[j]} \cdot \frac{\Delta^j 0^r}{j!}$$

For the calculation of the differences of zero one may use a recurrence relation due to E.T.Whittaker and G.Robinson [16]:

$$\Delta^j 0^r = j(\Delta^j 0^{r-1} + \Delta^{j-1} 0^{r-1}).$$

By dividing this relation by $j!$ we obtain the following recurence relation for the Stirling numbers of the first kind:

$$S_j^r = jS_j^{r-1} + S_{j-1}^{r-1},$$

which is suitable for constructing a table of their values. In [4] can be found such a table for $r = 2(1)25,\ j = 2(1)r$

4. By using the Maclaurin expansion we can represent the factorial powers in terms of the ordinary powers and of the Stirling numbers of the first kind:

$$s_p^i = \frac{1}{i!} \cdot D^i 0^{[p]} = \frac{1}{i!}\left(x^{[p]}\right)^{(i)}_{x=0},$$

and we obtain

$$x^{[p]} = \sum_{i=1}^{p} s_p^i \cdot x^i$$

Consequently, the factorial moments can be expressed by means of the ordinary moments and the Stirling numbers of the first kind according to the following formula

$$\nu_{[p]} = \sum_{i=1}^{p} s_p^i \cdot \nu_i$$

5. The factorial moments can be computed by using the corresponding factorial moment generating function, defined, if the expected value of the random variable $g(X) = t^X$ exists, by the following formula

$$g(t) = E\left[t^X\right] = \int_{-\infty}^{+\infty} t^x \, dF_X(x).$$

It is easily seen that the factorial moment of order j is given by

$$\nu_{[j]} = g_X^{(j)}(1).$$

In the discrete case we get

$$g_X(t) = \sum_k t^{x_k} \cdot p_X(x_k).$$

6. For the computation of the moments of some important discrete random variables we can use certain representations of interpolatory linear positive operators.

In [14] and later in [7], by starting from an idea of W. Feller [2], there were presented probabilistic methods for constructing linear positive operators useful in the theory of approximation of functions.

Let (Y_m) be a sequence of one-dimensional random variables and let $F_{m,x}$ be the distribution of Y_m. For any real-valued function f, bounded on the real-axis, such that the mean value $E\left[|f(y_m)|\right]$ exists, one defines a linear positive operator L_m^f by means of the formula

$$E\left[f(Y_m)\right] = L_m^f(x) = \int_{-\infty}^{\infty} f(y)\, dF_{m,x}(y).$$

We shall use the notations

$$L_m^f(x) = (L_m f)(x) = L_m(f(t); x).$$

It is evidently that we have $L_m e_0 = 1$.

If we further assume that $(L_m e_1)(x) = x$ and that

$$\mathrm{Var}\,[Y_m] = L_m((t-x)^2; x) = \sigma_m^2(x) \to 0$$

as $m \to \infty$, then we say that L_m is an operator of Feller type.

In the paper [14] D.D. Stancu has proved that, if f is a bounded uniformly continuous real-valued function on the real axis, then the order of approximation of f by means of L_m^f can be evaluated by using the modulus of continuity of the function f, according to the following inequality

$$\left|f(x) - L_m^f(x)\right| \leq \left(1 + \beta_m \sqrt{m}\right) \omega\left(f; \frac{1}{\sqrt{m}}\right).$$

where

$$\beta_m = \sup \sigma_m(x) \text{ for } x \in I.$$

7. An important method for constructing concrete linear positive operators useful in constructive function theory is given by $L_m^f = E[Y_m]$, where

$$Y_m = \frac{1}{m}\left[X_1 + X_2 + \ldots + X_m\right],$$

under the hypothesis that (X_n) is a sequence of independent and identically distributed random variables.

Now let us present several illustrative examples.

(i) If X_j has a zero-one distribution, then Y_m has the binomial distribution

$$b(k; m, x) = \binom{m}{k} x^k (1-x)^{m-k}$$

and we obtain the Bernstein operator B_m, defined by

$$(B_m f)(x) = \sum_{k=0}^{m} \binom{m}{k} x^k (1-x)^{m-k} f\left(\frac{k}{m}\right), \quad x \in [0, 1].$$

(ii) Let us assume that X_j has a Poisson distribution with the parameter x. In this case Y_m has also a Poisson distribution but with the new parameter mx and we obtain the Mirakian-Favard-Szasz operator L_m defined by

$$(L_m f)(x) = \sum_{k=0}^{\infty} \frac{(mx)^k}{k!} \cdot e^{-mx} f\left(\frac{k}{m}\right), \quad x \in [0, \infty).$$

(iii) By using the Pascal probability distribution one can arrive at the Meyer-König and Zeller operator, defined by the formula

$$(M_m f)(x) = \sum_{k=0}^{\infty} \binom{m+k}{k} x^k (1-x)^{m+1} f\left(\frac{k}{m+k}\right), \quad x \in [0, 1].$$

(iv) In the paper [12] D.D. Stancu has introduced and investigated a parameter-dependent linear positive operator $S_m^{\langle\alpha\rangle}$, defined for any function $f: [0, 1] \to \mathbb{R}$, by the formula

$$\left(S_m^{<\alpha>}f\right)(x) = \sum_{k=0}^{m} w_{m,k}^{<\alpha>}(x) f\left(\frac{k}{m}\right), \quad (\alpha \geq 0) \tag{2}$$

where, by using the factorial powers, we have

$$w_{m,k}^{<\alpha>}(x) = \binom{m}{k} \frac{x^{[k,-\alpha]}(1-x)^{[m-k,-\alpha]}}{1^{[m,-\alpha]}}, \tag{3}$$

with the notation for factorial powers

$$x^{[n,h]} = x(x-h)\dots(x-(n-1)h), \quad x^{[0,h]} = 1.$$

The same author has deduced this operator in [14], by starting from the Markov-Polya probability distribution.

(v) We mention also the Baskakov operator, defined by

$$(L_m f)(x) = \sum_{k=0}^{\infty} \binom{m+k-1}{k} \frac{x^k}{(1+x)^{m+k}} f\left(\frac{k}{m}\right), \quad x \geq 0,$$

which corresponds to a variant of the negative binomial distribution [4].

3. By starting from the Newton interpolation formula, D.D. Stancu has obtained representations in terms of finite differences of the function f of the linear positive operators of interpolatory type.

This representations has the following form

$$L_m^f(x) = \sum_{k=0}^{m} q_{m,k}(x) f\left(\frac{k}{m}\right) = f(0) + \sum_{j=1}^{m} \frac{1}{j!} \cdot \nu_{(j)}(x) \left(\Delta^j_{\frac{1}{m}} f\right)(0),$$

where

$$\nu_{(j)}(x) = \sum_{k=0}^{m} k^{[j]} q_{m,k}(x).$$

In the case (i) of the binomial distribution, the factorial moment generating function is

$$g(t) = \sum_{k=0}^{m} t^k b(k; m, x) = (1 - x + tx)^m,$$

so that we get immediately

$$g^{(j)}(t) = m^{[j]} x^j (1 - x + tx)^{m-j}.$$

Consequently, the factorial moment of the binomial distribution will be

$$\nu_{[j]} = m^{[j]} \cdot x^j.$$

Therefore we obtain the following representation of the Bernstein polynomial

$$B_m^j(x) = \sum_{j=0}^{m} \binom{m}{j} x^j \cdot \left(\Delta^j_{\frac{1}{m}} f\right)(0).$$

Now if we take $f(x) = m^r x^r$, where $r \in \mathbf{N}_0$, we have

$$\left(\Delta^j_{\frac{1}{m}} f\right)(0) = (\Delta_1^j e_r)(0) = \Delta^j 0^r$$

It follows that for the ordinary moments of the binomial distribution we obtain the following explicit expressions

$$\nu_r(m;p) = \sum_{j=1}^{r} \binom{m}{j} p^j \cdot \Delta^j 0^r \tag{4}$$

or

$$\nu_r(m;p) = \sum_{j=1}^{m} m^{[j]} p^j S_j^r, \tag{5}$$

by using the differences of zero and respectively the Stirling numbers of the second kind.

In the case of the Markov-Polya probability distribution

$$P(X=k) = P_m(k;a,b,c) = \binom{m}{k} \frac{a^{[k,-c]} b^{[m-k,-c]}}{(a+b)^{[m,-c]}} \tag{6}$$

we can use the representation in terms of the differences of the function f on the nodes k/m, given in [14] for the Bernstein type polynomial $S_m^{\langle\alpha\rangle} f$ of D.D. Stancu.

$$\left(S_m^{\langle\alpha\rangle} f\right)(x) = \sum_{j=0}^{m} \binom{m}{j} \frac{x^{[j,-\alpha]}}{1^{[j,-\alpha]}} \left(\Delta^j_{\frac{1}{m}} f\right)(0),$$

where

$$x^{[j,-\alpha]} = x(x+\alpha)\ldots(x+(j-1)\alpha)$$

$$1^{[j,-\alpha]} = (1+\alpha)(1+2\alpha)\ldots(1+(j-1)\alpha),$$

while $\alpha = c/(a+b)$ and $x = a/(a+b)$.

By taking $f(x) = m^r x^r$ we obtain the following explicit expression for the ordinary moment of the Markov-Polya probability distribution

$$\nu_r(m;a,b,c) = \sum_{j=1}^{r} \binom{m}{j} \frac{a(a+c)\dots(a+(j-1)c)}{(a+b)(a+b+c)\dots(a+b+(j-1)c)} \Delta^j 0^r \tag{7}$$

If we use the factorial power notation and the Stirling numbers of the second kind, we can give the following more compact expression for moment

$$\nu_r(m;a,b,c) = \sum_{j=1}^{m} m^{[j]} \frac{a^{[j,-c]}}{(a+b)^{[j,-c]}} \cdot S_j^r. \tag{8}$$

In the paper [10] T. Sródka established a recurrence relation for the ordinary moments of the Markov-Polyai distribution, while in [15] D.D. Stancu has obtained a recurrence relation for the central moments of this distribution.

If we take $c = -1$ we obtain the hypergeometric distribution, defined by

$$P(X=k) \frac{\binom{Np}{k}\binom{Nq}{m-k}}{\binom{N}{m}},$$

where $N = a+b$, $p = a/N$, $q = b/N$, $\max\{0, m-N_q\} \le k \le \min\{m, Np\}$

The proceding formulas permit us to see that in this case the ordinary moments are given by

$$\nu_r = \sum_{j=1}^{r} \binom{m}{j} \frac{(Np)^{[j]}}{N^{[j]}} \Delta^j 0^r = \sum_{j=1}^{r} \frac{m^{[j]}(Np)^{[j]}}{N^{[j]}} S_j^r,$$

formulas which are due to J. Riordan [9]

If we use the notations

$$\frac{a}{a+b} = p,\ \frac{b}{a+b} = q,\ \frac{c}{a+b} = s,$$

then the Markov-Polya distribution (4) can be written under the form

$$P(X=k) = \binom{m}{k} \frac{p^{[k,-s]} q^{[m-k,-s]}}{1^{[m,-s]}}$$

We assume now that p and q depend on m in such a way that for $m \to \infty$ we have $p \to 0$, $s \to 0$ and $mp \to \frac{Q}{P}$. According to a result from W. Feller [2], we obtain as a limiting case the negative binomial distribution

$$P(X = k) = \binom{m+k-1}{k} P^{m} Q^{k},$$

where $m = \lambda P\backslash Q$ and $k \in \mathbf{N}_0$.

If we take into account this limit case, the formulas (7) and (8) permit us to deduce the following expressions for the ordinary moments of the preceding negative binomial distribution

$$\nu_r = \sum_{j=1}^{r} \binom{m+j-1}{j} \cdot \left(\frac{Q}{P}\right)^j \Delta^j 0^r = \sum_{j=1}^{r} (m+j-1)^{[j]} \left(\frac{Q}{P}\right)^j \cdot S_j^r.$$

We conclude this paper by making the remark that if we take the parameter c to be zero in the formulas (7) and (8) then we arrive respectively at the formulas (4) and (5) for the binomial distribution.

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RELATIVISTIC ORBITAL PERTURBATIONS IN A SPHERICAL POST-NEWTONIAN GRAVITATIONAL FIELD

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Dedicated to Professor M. Balázs on his 65th anniversary

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REZUMAT. - Perturbaţii orbitale relativiste într-un câmp gravitaţional sferic post-newtonian. Se studiază mişcarea de tip eliptic a unui corp de probă într-un câmp gravitaţional post-newtonian sferic, pe baza teoriei clasice a perturbaţiilor. Se determină perturbaţiile relativiste de ordinul întâi în cinci elemente orbitale independente, pe durata unei perioade nodale.

1. Introduction. Let M be a mass generating a spherical gravitational field, and let $\mu = GM$ (G = gravitational constant) be its gravitational parameter. Consider a test orbiting M in this field. Using standard post-Newtonian coordinates (t, x), its relative motion is described by the equation (e.g. [6]; see also [7])

$$dV/dt = -\mu x/r^3 + (\mu/c^2)\{2(\beta+\gamma)(\mu/r^4)x - \gamma(V^2/r^3)x + 2(\gamma+1)[(x\cdot V)/r^3]V\}, \quad (1)$$

where dV/dt = acceleration of the test body, r = radial coordinate, c = speed of light, β = post-Newtonian parameter describing the amount of nonlinearity of the gravitational field, γ = post-Newtonian parameter describing the space curvature (β and γ are the so-called Eddington-Robertson parameters [8]).

Introducing the new radial coordinate

$$r' = r + \alpha\mu/c^2, \quad (2)$$

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where α = gauge parameter [2], equation (1) becomes

$$dV/dt = -\mu x/r^3 + (\mu/c^2)\{2(\beta + \gamma - 2\alpha)(\mu/r^4)x$$

$$- (\gamma + \alpha)(V^2/r^3)x + 3\alpha[x\cdot V)^2/r^5]x +$$

$$+ 2(\gamma + 1 - \alpha)[(x\cdot V)/r^3]V\}, \tag{3}$$

where the primes for r, x, V were dropped.

After all, the equation of motion can be written as [6]

$$dV/dt = -\mu x/r^3 + a_{PN}, \tag{4}$$

and the motion of the test body may be treated perturbatively. In other words, although the motion in the post-Newtonian field is unperturbed, we shall consider a_{PN} to be a perturbing acceleration undergone by the test body moving in the Newtonian field.

Let us describe the relative motion of the test body by means of the Keplerian orbital elements $\{y \in Y;\ u\}$, all time-dependent, where

$$Y = \{p, q = e\cos\omega, k = e\sin\omega, \Omega, i\}, \tag{5}$$

and p = semilatus rectum, e = eccentricity, ω = argument of pericentre, Ω = longitude of ascending node, i = inclination, u = argument of latitude. Since the motion will be treated perturbatively, we shall estimate analytically the first order deviations of the elements (5) from their initial values, over one nodal period, regarding these differences as relativistic perturbations.

2. Basic equations. Since the nodal period was chosen as basic time interval, let us describe the motion by means of Newton-Euler equations written with respect to u (e.g. [1, 5])

$$dp/du = 2(Z/\mu)r^3 T,$$

$$dq/du = (Z/\mu)\{r^3kBCW/(pD) + r^2T[r(q+A)/p + A] + r^2BS\},$$

$$dk/du = (Z/\mu)\{-r^3qBCW/(pD) + r^2T[r(k+B)/p + B] - r^2AS\},$$

$$d\Omega/du = (Z/\mu)r^3BW/(pD),$$

$$di/du = (Z/\mu)r^3AW/p,$$

$$dt/du = Zr^2(\mu p)^{-1/2}, \tag{6}$$

where $Z = [1 - r^2C\dot{\Omega}/(\mu p)^{1/2}]^{-1}$, $A = \cos u$, $B = \sin u$, $C = \cos i$, $D = \sin i$, S, T, W = radial, transverse, and binormal components of the perturbing acceleration, respectively.

Since the perturbing acceleration is a relativistic one, the elements (5) have small variations over one revolution of the test body. Consequently, they may be considered constant and equal to their initial values $y_0 = y(u_0) = y(u(t_0))$, $y \in Y$, in the right-hand side of equations (6), and these ones can be separately integrated. So, we can write $y = y_0 + \Delta y$, where the first order variations are found from

$$\Delta y = \int_0^{2\pi} (dy/du)\, du, \; y \in Y, \tag{7}$$

with the integrands provided by (6). The integrals are therefore estimated by successive approximations, limiting the process to the first order approximation.

3. Perturbing acceleration. The components S, T, W of a_{PN} have respectively the expressions [6]

$$S = (\mu/c^2)[\mu/(a^3(1-e^2)^3](1 + e\cos v)^2[(2\beta + \gamma - 3\alpha) +$$

$$+ (\gamma + 2)e^2 + 2(\beta - 2\alpha)e\cos v - (2\gamma + 2 - \alpha)e^2\cos^2 v],$$

$$T = 2(\mu/c^2)[\mu/(a^3(1-e^2)^3](1 + e\cos v)^3(\gamma + 1 - \alpha)e\sin v, \tag{8}$$

$$W = 0,$$

where a = semimajor axis, v = true anomaly.

Using the well-known formulae

$$p = a(1 - e^2), \tag{9}$$

$$v = u - \omega, \tag{10}$$

the definition of q and k, and the orbit equation in polar coordinates

$$r = p/(1 + e\cos v), \tag{11}$$

expressions (8) acquire the form

$$\begin{aligned} S &= (\mu^2/(c^2pr^2))[L_1 + L_2Aq + L_2Bk + (L_3 + L_4A^2)q^2 + 2L_4ABqk + \\ &\quad + (L_3 + L_4B^2)k^2], \\ T &= (\mu^2/(c^2r^3))L_5(Bq - Ak), \\ W &= 0, \end{aligned} \tag{12}$$

where we abbreviated

$$\begin{aligned} L_1 &= 2\beta + \gamma - 3\alpha, \\ L_2 &= 2\beta - 4\alpha, \\ L_3 &= \gamma + 2, \\ L_4 &= -2\gamma - 2 - \alpha, \\ L_5 &= 2\gamma + 2 - \alpha. \end{aligned} \tag{13}$$

4. Variations of orbital elements. Let us introduce (12) in (6), then use (11) in the equivalent form

$$r = p/(1 + Aq + Bk). \tag{14}$$

Also observe, by the fourth and the sixth equations (6) that $Z = 1$ (because $W = 0$). After performing all calculations, the equations of motion become

$$dp/du = 2(\mu/c^2)L_5(Bq - Ak),$$

$$
\begin{aligned}
dq/du &= (\mu/(c^2p))\{L_1B + (L_2 + 2L_5)ABq + \\
&+ [L_2 - (L_2 + 2L_5)A^2]k + \\
&+ [(L_3 + L_5)B + (L_4 + L_5)A^2B]q^2 + \\
&+ [2L_4A - 2(L_4 + L_5)A^3]qk + \\
&+ [(L_3 - L_5)B + (L_4 + L_5)B^3]k^2\} \qquad (15)\\
dk/du &= -(\mu/(c^2p))\{L_1A + [L_2 - (L_2 + 2L_5)B^2]q + \\
&+ (L_2 + 2L_5)ABk + \\
&+ [(L_3 - L_5)A + (L_4 + L_5)A^3]q^2 + \\
&+ [2L_4B - 2(L_4 + L_5)B^3]qk + \\
&+ [(L_3 + L_5)A + (L_4 + L_5)AB^2]k^2\}, \\
d\Omega/du &= 0, \\
di/du &= 0, \\
dt/du &= p^{3/2}\mu^{-1/2}(1 + Aq + Bk)^{-2}
\end{aligned}
$$

So, the expressions in the right-hand side of equations (15) contain only explicit functions of u (through A and B) and quantities considered constant over one revolution of the test body (according to the considerations made in Section 2)

Performing now the integrals (7) with the integrands provided by the first five equations (15) considered separately, we obtain the first order relativistic variations of the orbital elements (5) over one nodal period

$$
\begin{aligned}
\Delta p &= 0, \\
\Delta q &= \pi(\mu/(c^2p_0))(L_2 - 2L_5)k_0, \\
\Delta k &= \pi(\mu/(c^2p_0))(2L_5 - L_2)q_0, \qquad (16)\\
\Delta\Omega &= 0,
\end{aligned}
$$

$$\Delta_i = 0.$$

5. Comments. Observe that, in a first order approximation, the shape and dimensions of the orbit are not affected over one nodal period. Indeed, taking into account the definition of q and k, and the second and third formulae (16), we easily get

$$\Delta e = \Delta q \cos\omega_0 + \Delta k \sin\omega_0 = 0. \tag{17}$$

By (9), (17), and the first formula (16), follows immediately that $\Delta a = 0$, too.

Taking again into consideration the definition of q and k, and the second and third formulae (16), we obtain

$$\begin{aligned}\Delta\omega &= (\Delta k \cos\omega_0 - \Delta q \sin\omega_0)/e_0 = \\ &= \pi(\mu/(c^2 p_0))(2L_3 - L_2),\end{aligned} \tag{18}$$

or, by (13)

$$\Delta\omega = 2\pi(\mu/(c^2 p_0))(2\gamma - \beta + 2), \tag{19}$$

which means apsidal motion.

Considering the spherical Einstein post-Newtonian gravitational field ($\beta = \gamma = 1$), formula (19) becomes

$$\Delta\omega = 6\pi\mu/(c^2 p_0), \tag{20}$$

that is, the well-known pericentre shift.

The last two expressions (16) show that the position of the orbit plane remains unchanged. Indeed, due to the post-Newtonian conservation of the angular momentum, the motion is planar.

Another remark to be made is that an initially circular orbit will come back after one nodal period to the same circular shape and the same radius.

As a final remark, if the integrals (7) are performed between the initial (u_0) and current (u) positions, the variations of the orbital elements (5) can further be used to determine the relativistic perturbations in the nodal period (e.g. [3, 4]).

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ANIVERSĂRI

PROFESSOR MARTIN BALAZS AT HIS 65th BIRTHDAY

Professor Martin Balázs was born on 17th June 1929 in LUETA - Harghita, Romania. He attended the primary school betwen 1936-1942 in his native village, the secondary school betwen 1942-1949 in Odorheiu Secuiesc and betwen 1949-1952 the Faculty of Mathematics and Physics of the "Bolyai" University in Cluj. In 1968 he obtained a Ph. D. degree in Mathematics (under the supervision of the great Romanian mathematician G. Călugăreanu.) He has been working at the "Babeş-Bolyai" University of Cluj-Napoca since 1951, first as assistant professor betwen 1951-1954, as a lecturer betwen 1954-1972, as associate professor betwen 1972-1990 and as full professor since 1990. He has tought Mathematical Analysis, Theory of Complex Functions, Theory of Real Functions, Differential Equations, General Mathematics a.s.o.

Professor M. Balázs obtained remarkable scientific results in the following domains (see "List of Publications").

General Topology ([2], [3], [4]).

Divided Differences in Various Spaces ([20], [21], [29], [30], [31], [81], [82]).

Approximate Solving the Operator Equations ([6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [22], [23], [27], [34], [35], [36], [37], [40], [44], [48], [50], [60], [62], [64], [69], [85], [89]).

Applications of the Iterative Method in Solving the Integral and Differential Equations ([28], [32], [33], [56], [57], [63], [67], [68], [71], [83], [85], [89]).

Teaching of Mathematics in Secondary and in Faculty ([1], [41], [45], [46], [49], [51], [54], [55], [58], [59], [61], [66], [70], [72], [73], [77], [78], [80], [84], [88], [90], [94]).

Professor M. Balázs has taken part in about 100 conferences presenting lectures in Bucharest, Cluj, Iaşi, Timişoara, Oradea, Baia Mare, Tg. Mureş, Budapest, Debrecen, Veszprém a.s.o.

Since 1984 Professor M. Balázs has been the Chairman of the Seminar of Teaching Mathematics, where he was also Editos of 10 volumes of this seminar. He was member of the editorial board of the following publications: Studia Univ. "Babeş-Bolyai" and Matematikai Lapok.

Professor M. Balázs was appointed scientific secretary of the Senate of the "Babeş-Bolyai" University betwen 1977-1986 and vicedean of the Faculty of Mathematics betwen 1990-1992.

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