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MATRIX INEQUALITIES FOR COMMUTING MATRICES

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REZUMAT. - Inegalităţi matriciale pentru matrici care comută. O serie de inegalităţi clasice pentru numere pozitive sunt extinse la cazul matricilor care comută.

1. Introduction. It is well-known that for Hermitian positive definite commuting matrices A and B, the inequality between arithmetic and geometric is given by (see for example [1])

$$\sqrt{AB} \leq \frac{A+B}{2} \tag{1}$$

where $C \geq D$ means $C - D$ is positive semi-definite. (Similarly, $C > D$ means $C - D$ is positive definite.)

Of course, this inequality is a direct analogue of the corresponding inequality for positive numbers. In this paper, we show that similar analogues can also be given for many other classical inequalities.

1. Preliminaries. Let $A_i \in C^{v\times v}$ be pairwise commutative Hermitian matrices with eigenvalues $\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{iv}$. Then there exists a unitary matrix U such that for all i

$$A_i = U^*[\lambda_{i1}, \ldots, \lambda_{iv}]U \tag{2}$$

where $[\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{iv}]$ are diagonal matrices whose diagonal elements are the eigenvalues

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of A_i, each appearing as often as its multiplicity (see, for example, [2]). Further, by Theorem 4.14 from [3], for a real-valued function f, we have

$$f(A_i) = U^*[f(\lambda_{i1}), f(\lambda_{i2}), \ldots, f(\lambda_{i\nu})]U. \tag{3}$$

We denote by $S(J)$ the set of all Hermitian matrices with eigenvalues in an interval J.

3. Inequalities for convex functions.

THEOREM 1. *Let $f: J \to R$ be a continuous convex function. Then f is also a matrix-convex function on the set of commutable matrices from $S(J)$.*

Proof. We have for $a \in [0,1]$ and permutable matrices $A, B \in S(J)$ with eigenvalues $\lambda_1, \ldots, \lambda_\nu$ and $\mu_1, \ldots, \mu_\nu$ respectively

$$\begin{aligned} f(aA+(1-a)B) &= U^*[f(a\lambda_1+(1-a)\mu_1), \ldots, f(a\lambda_\nu+(1-a)\mu_\nu)]U \\ &\le U^*[af(\lambda_1)+(1-a)f(\mu_1), \ldots, af(\lambda_\nu)+(1-a)f(\mu_\nu)]U \\ &= af(A)+(1-a)f(B) \end{aligned}$$

i.e.

$$f(aA+(1-a)B) \le af(A)+(1-a)f(B) \tag{4}$$

which is the definition of a matrix-convex function.

We can now, by mathematical induction, obtain Jensen's inequality for convex functions. Moreover, we can prove directly a more general result with matrix weights instead of real weights as in (4).

THEOREM 2. (Jensen's inequality). *Let C_j, w_j, $j = 1, \ldots, n$ be commuting matrices such that $C_j \in S(J)$, $j = 1, \ldots, n$ and $w_j \in S(0,\infty)$ $j = 1, \ldots, n$. If $f: J \to R$ is a continuous convex function, then*

$$f\left(\frac{1}{W_n}\sum_{j=1}^{n} w_j C_j\right) \le \frac{1}{W_n}\sum_{j=1}^{n} w_j f(C_j) \tag{5}$$

where $W_n = \sum_{i=1}^{n} w_i$.

Proof. Let $\lambda_{j1}, \ldots, \lambda_{jN}$ and $w_{j1}, \ldots, w_{jN}$ $(j = 1, \ldots, n)$ be eigenvalues of A_j and w_j respectively. Then by (2) and (3),

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i A_i\right) = U^*\left[f\left(\frac{\sum_{i=1}^{n} w_{i1}\lambda_{i1}}{\sum_{i=1}^{n} w_{i1}}\right), \ldots, f\left(\frac{\sum_{i=1}^{n} w_{iN}\lambda_{iN}}{\sum_{i=1}^{n} w_{iN}}\right)\right]U$$

$$\le U^*\left[\frac{\sum_{i=1}^{n} w_{i1} f(\lambda_{i1})}{\sum_{i=1}^{n} w_{i1}}, \ldots, \frac{\sum_{i=1}^{n} w_{iN} f(\lambda_{iN})}{\sum_{i=1}^{n} w_{iN}}\right]U$$

$$= \frac{1}{W_n}\sum_{i=1}^{n} w_i f(A_i),$$

where we used the classical Jensen inequality for a convex function of real variables.

We can prove many other inequalities in a similar manner. We, therefore, give references to corresponding discrete inequalities but only a few actual proofs. (In fact, many of the results stated here can also be proved from Theorem 2.)

THEOREM 3. *Let* C_j, w_j, $j = 1, \ldots, n$ *be commuting matrices such that* $C_j \in S(J)$, $j = 1, \ldots, n$, $\bar{C} = \frac{1}{W_n}\sum_{i=1}^{n} w_i C_i \in S(J)$

$$w_1 > 0,\ w_i < 0,\ i = 2, \ldots, n,\ W_n > 0. \tag{6}$$

If $f : J \to R$ *is a convex function, then the reverse inequality in* (5) *holds.*

Now let us consider an index set function

$$F(J) = W_J f(A_J(C; w)) - \sum_{i \in J} w_i f(C_i) \tag{7}$$

where

$$W_J = \sum_{i \in J} w_i,\quad A_J(C; w) = \frac{1}{W_J}\sum_{i \in J} w_i C_i.$$

THEOREM 4. *Let* f *be a convex function on* J, T *and* K *are two finite nonempty* *subsets of N such that* $T \cap K = \varnothing$, $w = (w_i)_{i \in T \cup K}$ *and* $C = (C_i)_{i \in T \cup K}$ *are commuting matrices such that* $C_i \in S(J)$, $w_i \in S(R)(i \in T \cup K)$, $W_{T \cup K} > 0$, $A_t(C;w) \in S(J)$ $(t = T, K, T \cup K)$.

If $W_T > 0$ *and* $W_K > 0$, *then*

$$F(T \cup K) \leq F(T) + F(K). \tag{8}$$

If $W_T W_K < 0$, *we have the reverse inequality in* (8).

THEOREM 5. *If* $w_i > 0$, $i = 1, \ldots, n$, $I_k = \{1, \ldots, k\}$, *then*

$$F(I_n) \leq F(I_{n-1}) \leq \ldots \leq F(I_2) \leq 0, \tag{9}$$

but if (6) *is valid and* $A_{I_n}(C;w) \in S(J)$ *then the reverse inequalities in* (9) *hold.*

Theorems 3-5 in the real case are obtained in [4], [5], [6].

THEOREM 6. [7]. *Let the conditions of Theorem 2 be satisfied. Then*

$$\begin{aligned} f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i C_i\right) &= f_{n,n} \leq \ldots \leq f_{k+1,n} \leq f_{k,n} \leq \ldots \leq f_{1,n} \\ &= \frac{1}{W_n}\sum_{i=1}^{n} w_i f(C_i), \end{aligned} \tag{10}$$

where

$$f_{k,n} := \frac{1}{\binom{n-1}{k-1} W_n} \sum_{1 \leq i_1 < \ldots < i_k \leq n} (w_{i_1} + \ldots + w_{i_k}) f\left(\frac{w_{i_1} C_{i_1} + \ldots + w_{i_k} C_{i_k}}{w_{i_1} + \ldots + w_{i_k}}\right).$$

THEOREM 7. [8] *Let the conditions of Theorem 2 be fulfilled. Then*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i C_i\right) \leq \ldots \leq \bar{f}_{k+1,n} \leq \bar{f}_{k,n} \leq \ldots \leq \bar{f}_{1,n} = \frac{1}{W_n}\sum_{i=1}^{n} w_i f(C_i), \tag{11}$$

where

$$\bar{f}_{k,n} = \frac{1}{\binom{n+k-1}{k-1} W_n} \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq n} (w_{i_1} + \ldots + w_{i_k}) f\left(\frac{w_{i_1} C_{i_1} + \ldots + w_{i_k} C_{i_k}}{w_{i_1} + \ldots + w_{i_k}}\right).$$

THEOREM 8. [9]. *Let the conditions of Theorem 2 be fulfilled. Then*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i C_i\right) \le f_{k+1,n} \le f_{k,n} \le \dots \le f_{1,n} = \frac{1}{W_n}\sum_{i=1}^{n} w_i f(C_i) \tag{12}$$

where $1 \le k \le n-1$, *and*

$$f_{k,n} = \frac{1}{W_n^k} \sum_{i_1,\dots,i_k=1}^{n} w_{i_1}\dots w_{i_k} f\left(\frac{1}{k}(C_{i_1} + \dots + C_{i_k})\right).$$

THEOREM 9. [10,11]. *Let the conditions of Theorem 2 be fulfilled and let* q_i, $i=1,\dots,k$, *with* $Q_k := \sum_{i=1}^{n} q_i$, *be also strictly positive matrices commutable with* $\{C_j\}$ *and* $\{w_j\}$, *then*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i C_i\right) \le \frac{1}{W_n^k} \sum_{i_1,\dots,i_k=1}^{n} w_{i_1}\dots w_{i_k} f\left(\frac{1}{Q_k}\sum_{j=1}^{k} q_j C_{i_j}\right) \le \frac{1}{W_n}\sum_{i=1}^{n} w_i f(C_i). \tag{13}$$

THEOREM 10. [12,13]. *Let the conditions of Theorem 2 be fulfilled and let* $\bar{C} = \frac{1}{W_n}\sum_{i=1}^{n} w_i C_i$, $t_i \in [0,1]$, $i = 1,\dots,k-1$. *Then*

$$\begin{aligned} f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i C_i\right) &\le \tilde{f}_{n,1} \le \dots \le \tilde{f}_{n,k-1} \\ &\le \frac{1}{W_n^k} \sum_{i_1,\dots,i_k=1}^{n} w_{i_1}\dots w_{i_k} f(C_{i_1}(1-t_1) + \sum_{j=1}^{k-2} C_{i_{j+1}}(1-t_{j+1})t_1\dots t_j \\ &\quad + C_{i_k}t_1\dots t_{k-1}) \le \frac{1}{W_n}\sum_{i=1}^{n} w_i f(C_i), \end{aligned} \tag{14}$$

where

$$\tilde{f}_{n,k} = \frac{1}{W_n^k} \sum_{i_1,\dots,i_k=1}^{n} w_{i_1}\dots w_{i_k} f(C_{i_1}(1-t_1)$$

$$+ \sum_{j=1}^{k-1} C_{i_{j+1}}(1-t_{j+1})t_1\dots t_j + \bar{C}t_1\dots t_k).$$

THEOREM 11. [14]. *Let* q_i, $A_i \in S(R)$, $i = 1,\dots,n$ *be commuting matrices, and let the function g be defined by*

$$g(x) = \sum_{i=1}^{n} \frac{1}{q_i} f\left(q_i x A_i + (r-x)\sum_{k=1}^{n} A_k\right)$$

where $q_i > 0$, $i = 1,\dots,n$, *with* $\sum_{k=1}^{n}(1/q_k) = I$, $r \in R$, $q_i x A_i + (r-x)\sum_{k=1}^{n} A_k \in S(T)$, $i = 1,\dots,n$, *for all* x *from an interval* J *from* R. *If* $f : T \to R$ *is a convex function and* $|x| \leq |y| \, (xy > 0, y \in J)$, *then*

$$g(x) \leq g(y). \tag{15}$$

The function g *is also convex.*

Remark. Using the substitutions: $1/q_i \to w_i \left(\sum_{i=1}^{n} w_i = 1 \right)$, $q_i A_i \to X_i$, $r = 1$, we get that (15) is also valid if

$$g(x) = \sum_{i=1}^{n} w_i f\left(x X_i + (1 - x) \sum_{k=1}^{n} w_k X_k \right).$$

Remark. For some further generalizations of some of the previous results, see [15] and the references given there.

THEOREM 12. *Let the conditions of Theorem* 2 *be satisfied with* $J = [m,M]$. *Then*

$$\frac{1}{W_n} \sum_{i=1}^{n} w_i f(A_i) \leq \frac{MI - \tilde{A}}{M - m} f(m) + \frac{\tilde{A} - mI}{M - m} f(M) \tag{16}$$

where $\tilde{A} = \frac{1}{W_n} \sum_{i=1}^{n} w_i A_i$.

Proof. We use a converse of Jensen's inequality ([16])

$$\sum_{i=1}^{n} w_{ij} f(\lambda_{ij}) \leq \frac{M - \bar{\lambda}_j}{M - m} f(m) + \frac{\bar{\lambda}_j - m}{M - m} f(M) \quad (j = 1, \ldots, \nu)$$

$$\text{where } \bar{\lambda}_j = \frac{\sum_{i=1}^{n} w_{ij} \lambda_{ij}}{\sum_{i=1}^{n} w_{ij}} \quad (j = 1, \ldots, \nu),$$

and an appropriate representationn of commutative Hermitian matrices (see (2)).

THEOREM 13. [17]. *Let the conditions of Theorem* 12 *be satisfied and* f *also be positive strictly convex and twice differentiable. Then*

$$\frac{1}{W_n} \sum_{i=1}^{n} w_i f(A_i) \leq \lambda f\left(\frac{1}{W_n} \sum_{i=1}^{n} w_i A_i \right) \tag{17}$$

where, with $\phi = (f')^{-1}$, $\mu = \frac{f(M) - f(m)}{M - m}$, $\nu = \frac{Mf(m) - mf(M)}{M - m}$, λ *is the unique* *solution of the equation*

$$f \circ (\phi(\mu/\lambda) = (\mu/\lambda)\,\phi(\mu/\lambda) + (\nu/\lambda). \tag{18}$$

THEOREM 14. [17]. *Under the same conditions as Theorem* 13,

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i f(A_i) \le \lambda I + f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i A_i\right) \tag{19}$$

where

$$\lambda = \mu\phi(\mu) + \nu - f\circ\phi(\mu) \tag{20}$$

4. Inequalities for power means. There are, of course, many generalizations of (1). We shall give analogous results for power means.

Let $A = (A_1, \ldots, A_n)$, $w = (w_1, \ldots, w_n)$ be two n-tuples of positive definite matrices such that A_i, w_i are pairwise commutative.

The power mean of A with weight w of order r is given by

$$\begin{aligned} M_n^{[r]}(A;w) &= \left(\frac{1}{W_n}\sum_{i=1}^{n} w_i A_i^r\right)^{1/r}, \quad r \ne 0 \\ &= \exp\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i \log A_i\right), \quad r = 0 \end{aligned} \tag{21}$$

where $W_n = \sum_{i=1}^{n} w_i$. We write $M_n^{[0]}(A;w) = G_n(A;w)$.

Note that (2) and (3) give

$$M_n^{[r]}(A;w) = U^*[M_n^{[r]}(\{\lambda_{i1}\};\{w_{i1}\}), \ldots, M_n^{[r]}(\{\lambda_{i\nu}\};\{w_{i\nu}\})]U \tag{22}$$

where, again, λ_{ij}, ($j = 1, \ldots, \nu$) are eigenvalues of A_i, and w_{ij} ($j = 1, \ldots, \nu$) are eigenvalues of w_i.

Now using (22) and various results for means (see, for example, [4]) we can obtain many matrix inequalities for means (21). Here we give only some key results (again with appropriate reference to the real case).

THEOREM 15. ([4], p.159). *If $r \le s$, then*

$$M_n^{[r]}(A;w) \le M_n^{[s]}(A;w) \tag{23}$$

THEOREM 16. ([18]). *Let A_i, $i = 1, \ldots, n$ be positive commuting matrices such that*

$$0 \le mI \le A_i \le MI \ (i = 1, \ldots, n) \tag{24}$$

Let r, s, w_i, $i = 1, \ldots, n$ be real numbers such that $r < s$, $rs \ne 0$, $w_i > 0$, $i = 1, \ldots, n$.

Then

$$r[M_n^{[r]}(A;w)^r - aM_n^{[s]}(A;w)^s - bI] \ge 0 \tag{25}$$

where

$$a = (M^r - m^r)/(M^s - m^s), \quad b = (M^s m^r - M^r m^s)/(M^s - m^s). \tag{26}$$

THEOREM 17. [18] *Let the conditions of Theorem 16 be satisfied (with possibly rs = 0).*

Then

$$M_n^{[s]}(A;w) - M_n^{[r]}(A;w) \le \Delta I \tag{27}$$

where

$$\Delta = [\theta M^s + (1-\theta)m^s]^{1/s} - [\theta M^r + (1-\theta)m^r]^{1/r}.$$

θ is defined as follows. Let

$$h(t) = t^{1/s} - (at + b)^{1/r}$$

and a, b are defined by (26), if $r \ne 0$,

$$h(t) = t^{1/s} - m(M/m)^{(t-m^s)/(M^s-m^s)} \text{ if } r = 0$$

and

$$h(t) = -t^{1/r} + m(M/m)^{(t-m^r)/(M^r-m^r)}, \text{ if } s = 0.$$

Let J denote the open interval joining m^s to M^s if $s \ne 0$, and let $J = (M^r, m^r)$ if $s = 0$. Let $\bar{J}$ be the closure of J. Then there is a unique $t^* \in \bar{J}$ where $h(t)$ attains its maximum in $\bar{J}$.

(Observe that if $rs \neq 0$, then $at + b > 0$ at the end points of $\bar{J}$ and, therefore, throughout $\bar{J}$.) *Thus t^* lies in J. We set*

$$\theta = (t^* - m^s)/(M^s - m^s) \text{ if } s \neq 0,$$

and

$$\theta = (t^* - M^r)/(M^r - m^r) \text{ if } s = 0.$$

If $s \geq 1$, then t^* is the unique solution of $h'(t) = 0$ in J.

THEOREM 18. *Let the conditions of Theorem 16 be satisfied (with possibly $rs = 0$). Then if we set $\gamma = M/m$,*

$$M_n^{[s]}(A; w) \leq \tilde{\Delta} M_n^{[r]}(A; w) \tag{28}$$

where

$$\tilde{\Delta} = \left\{ \frac{r(\gamma^s - \gamma^r)}{(s-r)(\gamma^r - 1)} \right\}^{1/s} \left\{ \frac{s(\gamma^r - \gamma^s)}{(r-s)(\gamma^s - 1)} \right\}^{1/r} \quad \text{if } rs \neq 0$$

$$= \left(\frac{\gamma^{s/(\gamma - 1)}}{e \log \gamma^{s/(\gamma - 1)}} \right)^{1/s} \quad \text{if } r = 0$$

$$= \left(\frac{\gamma^{r/(\gamma - 1)}}{e \log \gamma^{r/(\gamma - 1)}} \right)^{-1/r} \quad \text{if } s = 0.$$

5. Hölder, Minkowski and related inequalities.

THEOREM 19. (Hölder's inequality) *Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be commuting matrices from $S(0, \infty)$, $p, q \in R$, $pq \neq 0$, $p^{-1} + q^{-1} = 1$. Then*

(a) *If $p > 1$*

$$\sum_{i=1}^{n} A_i B_i \leq \left(\sum_{i=1}^{n} A_i^p \right)^{1/p} \left(\sum_{i=1}^{n} B_i^q \right)^{1/q} \tag{29}$$

(b) *If either $p < 0$ or $q < 0$, then inequality (29) is reversed.*

Remark. It is worth noting that if w_i, $i = 1, \ldots, n$ are also commutable with A_i and B_i

and $w_i \in S(0,\infty)$, then (29) becomes

$$\sum_{i=1}^{n} w_i A_i B_i \le \left(\sum_{i=1}^{n} w_i A_i^p\right)^{1/p} \left(\sum_{i=1}^{n} w_i B_i^q\right)^{1/q} \tag{29'}$$

We need only put $A_i \to w_i^{1/p} A_i$ and $B_i \to w_i^{1/q} B_i$ in (29) to obtain (29′).

Another interesting form of (29) is the case when for $0 < s < 1$, (29′) is equivale

to

$$\sum_{i=1}^{n} w_i A_i^s B_i^{(1-s)} \le \left[\sum_{i=1}^{n} w_i A_i\right]^s \left[\sum_{i=1}^{n} w_i B_i\right]^{1-s} \tag{29''}$$

If $s > 1$ or $s < 0$, we have the reverse inequality.

A further interesting form of the Hölder inequality (29′) is

$$\left(\sum_{i=1}^{n} w_i A_i^p\right)^{1/p} \left(\sum_{i=1}^{n} w_i B_i^q\right)^{1/q} \left(\sum_{i=1}^{n} w_i C_i^r\right)^{1/r} \ge 1 \tag{29'''}$$

where $A_i B_i C_i = I$ and $p^{-1} + q^{-1} + r^{-1} = 0$, and all but one of p, q, r are positive. If all but on

negative, (29‴) is reversed.

For the corresponding real cases, see [4, pp. 136-139].

THEOREM 20. *Let* w_i, $A_i \in S(0, \infty)$, $i = 1, \ldots, n$ *be commuting matrices,* $0 < t <$ $< r$; $t, s, r \in R$. *Then*

$$\left[\sum_{i=1}^{n} w_i A_i^s\right]^{r-t} \le \left[\sum_{i=1}^{n} w_i A_i^t\right]^{r-s} \left[\sum_{i=1}^{n} w_i A_i^r\right]^{s-t} \tag{30}$$

Proof. Substituting $p = \dfrac{r-t}{r-s}$, $q = \dfrac{r-t}{s-t}$, $A_i^p \to A_i^t$, $B_i^q \to A_i^r$ in (29′) leads (30).

Remark. Theorem 20 gives a matrix version of the well-known Liapunov inequalit

Let us consider the expression

$$G(p) = \sum_{i=1}^{n} w_i A_i^p$$

Setting in (29"), $A_i \to A_i^u$, $B_i \to A_i^v$, gives

$$G(su + (1-s)v) \le G(u)^s G(v)^{1-s}, \tag{31}$$

Thus, $G(p)$ is an operator log-convex function, i.e., the function $G : R_+ \to S(0,\infty)$ satisfies (31).

Another interesting function is $\tilde{H}(s) = \left(\sum_{i=1}^{n} w_i A_i^{1/s}\right)^s$ for $s > 0$. Setting in Liapunov's inequality (30), $r \to x^{-1}$, $t \to y^{-1}$, $r(s-t)/[s(r-t)] = \lambda$, so that $1-\lambda = t(r-s)/[s(r-t)]$ and $s = [\lambda x + (1-\lambda)y]^{-1}$, gives

$$\tilde{H}(\lambda x + (1-\lambda)y) \le \tilde{H}(x)^\lambda \tilde{H}(y)^{1-\lambda} \tag{32}$$

for $0 < x < y$, $0 < \lambda < 1$.

THEOREM 21. ([4, p.143]) *Let w_i, $A_i \in S(0,\infty)$, $i = 1, \ldots, n$ be commuting matrices, $0 < r < s$. If $w_i \ge I$, $i = 1, \ldots, n$ then*

$$\left[\sum_{i=1}^{n} w_i A_i^s\right]^{1/s} \le \left[\sum_{i=1}^{n} w_i A_i^r\right]^{1/r}. \tag{33}$$

THEOREM 22. ([19, 20]) *Let w_j, $A_{ij} \in S(0,\infty)$ ($i = 1, \ldots, n, j = 1, \ldots, m$) be commuting matrices and let λ_i ($i = 1, \ldots, n$) be positive numbers.*

(a) *If $\lambda_1 + \ldots + \lambda_n = 1$, then*

$$\sum_{j=1}^{m} w_j A_{1j}^{\lambda_1} \ldots A_{nj}^{\lambda_n} \le \left(\sum_{j=1}^{m} w_j A_{1j}\right)^{\lambda_1} \cdots \left(\sum_{j=1}^{m} w_j A_{nj}\right)^{\lambda_n} \tag{34}$$

(b) *If $w_j \ge I$, $j = 1, \ldots, m$ and $\lambda_1 + \ldots + \lambda_n \ge 1$, then* (34) *also holds.*

Remark. Further extensions of Theorems 21 and 22 can be obtained analogous to the corresponding real cases [19], [20] and [21].

Let us now consider the function ([4, p.154])

$$g(x) = \prod_{i=1}^{n} \left(\sum_{t=1}^{m} w_t A_{it}^{q_i x} \prod_{k=1}^{n} A_{kt}^{r-x}\right)^{1/q_i} \tag{35}$$

where $q_i > 0$ ($i = 1, \ldots, n$) with $\sum_{k=1}^{n} q_k^{-1} = 1$, $r \in R$, $x \in R$, $A_{it} \in S(0,\infty)$, $i = 1, \ldots, n$; $t = 1, \ldots, m$. If $|x| < |y|$ ($xy > 0$), then

$$g(x) \le g(y) \tag{36}$$

As a special case, we have the following matrix analogue of the well-known Callebaut inequalities:

$$\begin{aligned} \left(\sum_{i=1}^{n} w_i A_i B_i\right)^2 &\le \sum_{i=1}^{n} w_i A_i^{\alpha} B_i^{2-\alpha} \sum_{i=1}^{n} w_i A_i^{2-\alpha} B_i^{\alpha} \\ &\le \sum_{i=1}^{n} w_i A_i^{\beta} B_i^{2-\beta} \sum_{i=1}^{n} w_i A_i^{2-\beta} B_i^{\beta} \le \sum_{i=1}^{n} w_i A_i^2 \sum_{i=1}^{n} w_i B_i^2 \end{aligned} \tag{37}$$

provided either $1 \le \alpha \le \beta \le 2$ or $0 \le \beta \le \alpha \le 1$.

Moreover another improvement of the Cauchy inequality can be given as follows ([4, p.155]):

If $1 \le x \le 1$, then

$$\left(\sum_{i=1}^{n} A_i B_i + x \sum_{\substack{i,j=1\\ i \ne j}}^{n} A_i B_j\right)^2 \le \left(\sum_{i=1}^{n} A_i^2 + 2x \sum_{\substack{i,j=1\\ i<j}}^{n} A_i A_j\right)\left(\sum_{i=1}^{n} B_i^2 + 2x \sum_{\substack{i,j=1\\ i<j}} B_i B_j\right). \tag{38}$$

THEOREM 23. (Minkowski's Inequality [22, pp. 25-27]). *Let* $A_1, \ldots, A_n, B_1, \ldots, B_n \in S(0,\infty)$ *be commuting matrices and let* $p \in R$. *If* $p > 1$, *then*

$$\left[\sum_{i=1}^{n} (A_i + B_i)^p\right]^{1/p} \le \left[\sum_{i=1}^{n} A_i^p\right]^{1/p} + \left[\sum_{i=1}^{n} B_i^p\right]^{1/p}. \tag{39}$$

If $p < 1$, $p \ne 0$, *then inequality* (39) *is reversed.*

Another Minkowski inequality is given by the following:

THEOREM 24 [22, p.26]. *Let* $A_1, \ldots, A_n; B_1, \ldots, B_n; w_1, \ldots, w_n \in S(0,\infty)$ *be commuting matrices. Then*

$$G_n(A+B;w) \geq G_n(A;w) + G_n(B;w)$$

THEOREM 24′ ([4, p.170]). *Let* $A_{ij} \in S(0,\infty)$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ *and let* $u = (u_1, \ldots, u_m)$, $v = (v_1, \ldots, v_n)$ *be sequences of strictly positive definite matrices. Set* $A^{(j)} = (A_{1j}, \ldots, A_{mj})$, $1 \leq j \leq n$ *and* $A_{(i)} = (A_{i1}, \ldots, A_{in})$, $1 \leq i \leq m$. *If* $-\infty < r \leq s < \infty$, *then*

$$M_n^{[s]}(M_m^{[r]}(A^{(j)};u);v) \leq M_m^{[r]}(M_n^{[s]}(A_{(i)};v);u) \tag{41}$$

THEOREM 25 ([23, p.57], [20]). *Let A and B be two n-tuples of permutable operators from S(R) such that*

$$B_1^2 - B_2^2 - \ldots - B_n^2 > 0 \quad or \quad A_1^2 - A_2^2 - \ldots - A_n^2 > 0. \tag{42}$$

Then

$$(A_1^2 - A_2^2 - \ldots - A_n^2)(B_1^2 - B_2^2 - \ldots - B_n^2) \leq (A_1B_1 - A_2B_2 - \ldots - A_nB_n). \tag{43}$$

THEOREM 26 ([20)]. *Let A and B be two n-tuples of commuting matrices from* $S(0,\infty)$ *such that*

$$A_1^p - A_2^p - \ldots - A_n^p > 0 \;\; and \;\; B_1^q - B_2^q - \ldots - B_n^q > 0 \tag{44}$$

then, if $p > 1$, $p^{-1} + q^{-1} = 1$,

$$(A_1^p - A_2^p - \ldots - A_n^p)^{1/p}(B_1^q - B_2^q - \ldots - B_n^q)^{1/q} \leq A_1B_1 - A_2B_2 - \ldots - A_nB_n. \tag{45}$$

In case $p < 1$ $(p \neq 0,\ p^{-1} + q^{-1} = 1)$, *inequality* (45) *is reversed.*

THEOREM 27 ([20]). *Let A and B be two n-tuples of permutable operators from* $S(0,\infty)$ *such that*

$$A_1^p - A_2^p - \ldots - A_n^p > 0 \;\; and \;\; B_1^p - B_2^p - \ldots - B_n^p > 0. \tag{46}$$

Then for $p > 1$

$$\begin{aligned} &[(A_1^p - A_2^p - \ldots - A_n^p)^{1/p} + (B_1^p - B_2^p - \ldots - B_n^p)^{1/p}]^p \\ &\leq (A_1 + B_1)^p - (A_2 + B_2)^p - \ldots - (A_n + B_n)^p. \end{aligned} \tag{47}$$

Inequality (47) *is reversed if* $p < 0$ $(p \neq 0)$.

Converses of Hölder's and Minkowski's inequalities. Now, we give some convers[e]

of (29) and (39), i.e. of Hölder's and Minkowski's matrix inequalities.

THEOREM 28. *Let* $A_1, \dots, A_n, B_1, \dots, B_n \in S(0, \infty)$ *be commuting matrices such th[at]*

$$mI \le A_k^{1/q}/B_k^{1/p} \le MI \quad (k = 1, 2, \dots, n) \tag{48}$$

where $0 < m < M$, $p^{-1} + q^{-1} = 1$, $p > 0$. *If*

$$a = (M^{-q} - m^{-q})/(M^p - m^p), \quad b = (M^p m^{-q} - M^{-q} m^p)/(M^p - m^p),$$

then

$$q\left(\sum_{k=1}^n B_k^q - a\sum_{k=1}^n A_k^p - b\sum_{k=1}^n A_k B_k\right) \le 0. \tag{49}$$

THEOREM 29. *Let the conditions of Theorem* 28 *be satisfied. If* $p > 1$, *then*

$$\left[\sum_{k=1}^n A_k^p\right]^{1/p} \left[\sum_{k=1}^n B_k^q\right]^{1/q} \le K \sum_{k=1}^n A_k B_k \tag{50}$$

where

$$K = \left[\frac{q}{p+q}\,\frac{\gamma^p - \gamma^{-q}}{1 - \gamma^{-q}}\right]^{1/p} \left[\frac{p}{p+q}\,\frac{\gamma^p - \gamma^{-q}}{\gamma^p - 1}\right]^{1/q} \tag{51}$$

and $\gamma = M/m$. *If* $0 < p < 1$, *the reverse inequality of* (50) *holds.*

THEOREM 30. *Let the conditions of Theorem* 28 *be satisfied. Then for* $p > 1$,

$$\left\{\frac{\sum_{k=1}^n A_k^p}{\sum_{k=1}^n A_k B_k}\right\}^{1/p} - \left\{\frac{\sum_{k=1}^n A_k B_k}{\sum_{k=1}^n B_k^q}\right\}^{1/q} \le \tilde{\Delta} I \tag{52}$$

where $\tilde{\Delta} = \max_{m^p \le t \le M^p} h(t) = h(t^*)$, $h(t) = t^{1/p} - (at+b)^{-1/q}$, t^* *being the unique solution of* $h'(t) = 0$ *in* (m^p, M^p).

THEOREM 31. *Assume the hypotheses of Theorem* 28, *except that instead of* (48),

assume

$$mI \le [A_k(A_k+B_k)^{-1}]^{1/q} \le MI,\ mI \le [B_k(A_k+B_k)^{-1}]^{1/q} \le MI \tag{53}$$

for every $k = 1, \ldots, n$. *If* $p > 1$, *then*

$$\left(\sum_{k=1}^{n} A_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} B_k^p\right)^{1/p} \le K\left(\sum_{k=1}^{n} (A_k+B_k)^p\right)^{1/p} \tag{54}$$

where K *is given by* (51). *If* $p < 1$, *we have the reverse inequality in* (54).

Following an idea of W.J. Everitt (see [4, pp. 151-152]), if $J \subset N$, $J \neq \varnothing$, let us define the following set functins

$$\chi(J) = \left[\sum_{i\in J} A_i^p\right]^{1/p}\left[\sum_{i\in J} B_i^q\right]^{1/q} - \sum_{i\in J} A_i B_i$$

and

$$\mu(J) = \left\{\left[\sum_{i\in J} A_j^p\right]^{1/p} + \left[\sum_{i\in J} B_j^p\right]^{1/p}\right\}^p - \sum_{i\in J} (A_i+B_i)^p$$

where A_i, $B_i \in S(0,\infty)$, $i \in J$.

THEOREM 32. (a) *If* $p,q > 0$, $p^{-1}+q^{-1} = 1$; $J,K \subset N$, $J \cap K = \varnothing$, $J \neq \varnothing$, $K \neq \varnothing$, $A_i, B_i \in S(0,\infty)$, $i \in J \cup K$. *Then*

$$\chi(J) + \chi(K) \le \chi(J \cup K) \tag{55}$$

(b) *If* $p > 1$ *or* $p < 0$, *then with* J *and* K *as in* (a),

$$\mu(J) + \mu(K) \le \mu(J \cup K) \tag{56}$$

The following result is a generalization of the Dresher inequality (for this and related results see [24]):

THEOREM 33. *Let* u_j, v_i, w_j, A_{ij}, $B_{ij} \in S(0,\infty)$, $i = 1, \ldots, n$, $j = 1, \ldots, m$ *be commuting matrices and let* $p,r,\alpha,\beta \in R$ *be such that* $\alpha p - \beta r = 1$.

(a) *If* $p \ge 1$, $0 < r \le 1$, $\alpha > 0$, $\beta > 0$, *then*

$$\frac{\left[\sum_{j=1}^{m} u_j \left[\sum_{i=1}^{n} v_i A_{ij}\right]^p\right]^\alpha}{\left[\sum_{j=1}^{m} w_j \left[\sum_{i=1}^{n} v_i B_{ij}\right]^r\right]^\beta} \le \sum_{i=1}^{n} v_i \frac{\left[\sum_{j=1}^{m} u_j A_{ij}^p\right]^\alpha}{\left[\sum_{j=1}^{m} w_j B_{ij}^r\right]^\beta} \tag{57}$$

(b) *If* $0 \le p \le 1$, $r < 0$, $\alpha > 0$, $\beta > 0$, *the reverse inequality in* (57) *holds.*

COROLLARY 1. ([24]) *Let* u_j, v_i, w_j, A_{ij}, B_{ij} *be defined as in Theorem* 33. *If* $p \ge 1 > r > 0$, $p,r \in R$ *then*

$$\left\{\frac{\sum_{j=1}^{m} u_j \left(\sum_{i=1}^{n} v_i A_{ij}\right)^p}{\sum_{j=1}^{m} w_j \left(\sum_{i=1}^{n} v_i B_{ij}\right)^r}\right\}^{\frac{1}{p-r}} \le \sum_{i=1}^{n} v_i \left\{\frac{\sum_{j=1}^{m} u_j A_{ij}^p}{\sum_{j=1}^{m} w_j B_{ij}^r}\right\}^{\frac{1}{p-r}} \tag{58}$$

If $1 \ge p > 0 > r$, *then the reverse inequality in* (58) *holds.*

This is the case $\alpha = \beta$ of Theorem 33.

6. Generalizations of power means. Let $J \subseteq R$ and suppose $M : J \to R$ is continuous and strictly monotone. Let $A = (A_1, \ldots, A_n)$ be an n-tuple with elements from $S(J)$, $w = (w_1, \ldots, w_n)$ an n-tuple with elements from $S(0,\infty)$ such that $W_n = \sum_{i=1}^{n} w_i > 0$, and let all matrices A_i, w_i be commutable. Then the quasi-arithmetic M-mean of A with weight w is

$$M_n(A; w) = M^{-1}\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i M(A_i)\right) \tag{59}$$

THEOREM 34. ([4], p.226]) *Let* F *and* G *be two strictly monotone functions defined on* J, G *increasing (decreasing). Then for all n-tuples of commuting matrices A, w such that* $A_i \in S(J)$, $w_i \in S(0,\infty)$, $i = 1, \ldots, n$

$$F_n(A; w) \le G_n(A; w) \tag{60}$$

if G *if convex (concave) with respect to* F *(i.e. if* $G \circ F^{-1}$ *is convex). If* G *is decreasing* *(increasing) and G is convex (concave) with respect to F, inequality (60) is reversed.*

Further, let $K : J_1 \to R$, $L : J_2 \to R$, $M : J_3 \to R$, $f : J_1 \times J_2 \to J_3$ be continuous functions, M increasing. Consider special cases of the inequality

$$f(K_n(A;w), L_n(B;w)) \geq M_n(f(A,B);w) \tag{61}$$

where $f(A,B) = (f(A_1,B_1), \dots, f(A_n,B_n))$ and where $A = (A_1, \dots, A_n)$, $B = (B_1, \dots, B_n)$, $w = (w_1, \dots, w_n)$, n-tuples with elements $A_i \in S(J_1)$, $B_i \in S(J_2)$, $w_i \in S(0,\infty)$, $i = 1, \dots, n$, are commutable operators.

THEOREM 35. ([4, p.251]) *If $f(x,y) = x + y$, so that*

$$\tilde{H}(s,t) = M(K^{-1}(s) + L^{-1}(t))$$

and if $E = K'/K''$, $F = L'/L''$, $G = M'/M''$ and all of $K', L', M', K'', L'', M''$ are positive, then

$$K_n(A;w) + L_n(B;w) \leq M_n(A + B;w)$$

if

$$G(x+y) \geq E(x) + F(y).$$

THEOREM 36 ([4, p.252]). *If $f(x,y) = xy$, so that*

$$\tilde{H}(s,t) = M(K^{-1}(s)L^{-1}(t))$$

and if $A(x) = K'(x)/(K'(x) + xK''(x))$, $B(x) = L'(x)/(L'(x) + xL''(x))$, $C(x) = M'(x)/(M'(x) + xM''(x))$, and K', L', M', A, B and C are positive, then

$$K_n(A;w)L_n(B;w) \leq M_n(AB;w),$$

(here AB means $(A_1B_1, \dots, A_nB_n)$) if

$$C(x,y) \geq A(x) + B(y)$$

THEOREM 37. ([16]). *Let F and G be two strictly monotone continuous functions defined on $J = [m,M]$, G increasing (decreasing) and G convex (concave) with respect to F.*

Let $A = (A_1, \ldots, A_n)$ *and* $w = (w_1, \ldots, w_n)$ *be two n-tuples of matrices such that* $A_i \in S($ $w_i \in S(0,\infty)$ *with all matrices commutable. Then*

$$(F(M) - F(m))G_n(A;w) - (G(M) - G(m))F_n(A;w) \le (F(M)G(m) - G(M)F(m))I (6$$

If G *is decreasing (increasing) and* G *is convex (concave) with respect to* F*, inequality* (6 *is reversed.*

Another generalization of power means can be given by

$$M_{n,a}(A;w)_p = \left\{ \frac{\sum_{i=1}^{n} w_i A_i^{a+p}}{\sum_{i=1}^{n} w_i A_i^{p}} \right\}^{1/a}, \quad a \ne 0$$

$$= \exp\left\{ \frac{\sum_{i=1}^{n} w_i A_i^{p} \log A_i}{\sum_{i=1}^{n} w_i A_i^{p}} \right\}, \quad a = 0$$

where $A = (A_1, \ldots, A_n)$, $w = (w_1, \ldots, w_n)$ are n-tuples of strictly positive matrices, and a matrices are commutable.

THEOREM 38 [25]. *Let* $a,b,p,q \in R$ *be such that*

$$||a| - |b|| + a + 2p \le b + 2q. \qquad (63)$$

Then

$$M_{n,a}(A;w)_p \le M_{n,b}(A;w)_q. \qquad (64)$$

THEOREM 39. [26] *Let* $a, b_1, \ldots, b_k, p, q_1, \ldots, q_k \in R$, $k \ge 2$. *Further, let*

$$Q_0 = a^- - p,\ Q_i = b_i^+ + q_i,\ i = 1, \ldots, k$$

$$Q_0^* = a^+ + p,\ Q_i^* = b_i - q_i,\ i = 1, \ldots, k,$$

where $a^+ = (|a| + a)/2$ *and* $a^- = (|a| - a)/2$ *and for* $i = 0, 1, \ldots, k$, *let*

$$H_i = \begin{cases} \left(\sum_{\substack{j=0 \\ j \neq i}}^{k} Q_j^{-1} \right) & \text{when } \prod_{\substack{j=0 \\ j \neq i}}^{k} Q_j \neq 0 \\ 0, & \text{when } \prod_{\substack{j=0 \\ j \neq i}}^{k} Q_j = 0. \end{cases}$$

Then the inequality

$$M_{n,a}(A_1, \dots, A_k; w)_p \leq M_{n,b_1}(A_1; w)_{q_1} \dots M_{n,b_k}(A_k; w)_{q_k} \tag{65}$$

holds for all n-tuples $A_1, \dots, A_k$, w *of strictly positive commutable matrices*

$((A_1 \dots A_k) = (A_{11} \dots A_{k1}, \dots, A_{1n} \dots A_{kn}))$ *if*

$$Q_i \geq 0, \text{ and } H_i \geq Q_i^*(i = 0, \dots, k) \tag{66}$$

THEOREM 40. [27]. *Let* $a, b_1, \dots, b_k, p, q_1, \dots, q_k \in R$, $k \geq 2$. *Then the inequality*

$$M_{n,a}(A_1 + \dots + A_k; w)_p \leq M_{n,b_1}(A_1; w)_{q_1} + \dots + M_{n,b_k}(A_k; w)_{q_k} \tag{67}$$

holds for all n-tuples $A_1, \dots, A_k, w$ *of strictly positive commutable matrices* $(A_1 + \dots + A_k = (A_{11} + \dots + A_{k1}, \dots, A_{1n} + \dots + A_{kn}))$ *if*

$$\max\{p + a^+, 1\} \leq q_i + b_i^+ \tag{68}$$

and

$$\max\{p - a^-, 0\} \leq \min\{q_i - b_i^-, 1\} \tag{69}$$

hold for every $i = 1, \dots, k$. *The reverse inequality holds in* (67) *if*

$$\min\{p + a^+, 1\} \geq \max\{q_i + b_i^+, 0\} \tag{70}$$

and

$$\min\{p - a^-, 0\} \geq q_i - b_i^- \tag{71}$$

hold for $i = 1, \dots, k$.

THEOREM 41. [28]. *Let* $0 < m < M < \infty$ *and let* a, b, p, q *be fixed numbers such that* (31) *holds. Further let* $\gamma = M/m$ *and* t_0 *be the unique positive root of the equation*

$$\lambda_{a,p}(\gamma)(\gamma^q + t)(\gamma^{b+q} + t) = \lambda_{b,q}(\gamma)(\gamma^p + t)(\gamma^{a+p} + t)$$

where, for $t > 0$.

$$\lambda_{a,p}(t) = \begin{cases} t^p \dfrac{t^{a-1}}{a}, & a \neq 0 \\ t^p \log t, & a = 0, \end{cases}$$

and

$$\Gamma_{a,p}(t,\gamma) = \begin{cases} ((\gamma^{a+p} + t)/(\gamma^p + t))^{1/a}, & a \neq 0 \\ \exp((\gamma^p \log\gamma)/(\gamma^p + t)), & a = 0. \end{cases}$$

Then

$$M_{n,b}(A;w)_q \leq C(m,M)\, M_{n,a}(A;w)_p \tag{72}$$

where

$$C(m,M) = \Gamma_{b,q}(t_0,\gamma)/\Gamma_{a,p}(t_0,\gamma). \tag{73}$$

The quasi-arithmetic mean (59) can be generalized as follows:

Let $\phi : J \to R_+$ be a strictly positive function, $F : J \to R$ a strictly monotone function $A \in S(J)^n$, $w \in S(0,\infty)$. Then define, for commutable A_i, w_i,

$$F_n(A;\phi) = F^{-1}\left(\frac{\sum_{i=1}^{n} w_i \phi(A_i) F(A_i)}{\sum_{i=1}^{n} w_i \phi(A_i)}\right) \tag{74}$$

THEOREM 42 [19]. *Let K,L,M be three differentiable strictly monotone functions from the closed interval J to R; let* ϕ,ψ,χ *be three functions from J to* R_+, *and let* $A,B \in S(J)^n$. *Then*

$$K_n(A;\phi) + L_n(B;\psi) \geq M_n(A+B;\chi), \tag{75}$$

holds if for all $u,v,s,t \in J$ *the following inequality holds.*

$$\frac{M(u+v) - M(t+s)}{M'(t+s)}\frac{\chi(u+v)}{\chi(t+s)} \leq \frac{K(u)-K(t)}{K'(t)}\frac{\phi(u)}{\phi(t)} + \frac{L(v)-L(s)}{L'(s)}\frac{\chi(v)}{\chi(s)}. \tag{76}$$

THEOREM 43.([29]) *With the notation of Theorem 42*

$$M_n(A;\chi) \leq K_n(A;\phi) \tag{77}$$

if, for all $u,t \in J$,

$$\left(\frac{M(u)-M(t)}{M'(t)}\right)\frac{\chi(u)}{\chi(t)} \le \left(\frac{K(u)-K(t)}{K'(t)}\right)\frac{\phi(u)}{\phi(t)}.$$

7. Symmetric means. Let $\mathcal{A} = (A_1, \ldots, A_n)$ be an n-tuple of positive commutable matrices. The normalized k-th elementary symmetric of $\mathcal{A}$ can be given by

$$p_n^{[0]}(\mathcal{A}) = I \quad \text{and} \quad p_n^{[k]}(\mathcal{A}) = \binom{n}{k}^{-1} \sum_{1 \le i_1 < \ldots i_k \le n} \prod_{j=1}^{k} A_{i_j} \tag{79}$$

Then we have a matrix version of the well-known Newton inequality

$$\left(p_n^{[r]}(\mathcal{A})\right)^2 \ge p_n^{[r-1]}(\mathcal{A})\, p_n^{[r+1]}(\mathcal{A}) \tag{80}$$

where r is an integer, $1 \le r \le n$-1.

We also have the matrix version of MacLaurin's inequalities

$$p_n^{[1]}(\mathcal{A}) \ge \left(p_n^{[2]}(\mathcal{A})\right)^{1/2} \ge \left(p_n^{[3]}(\mathcal{A})\right)^{1/3} \ge \ldots \ge \left(p_n^{[n]}(\mathcal{A})\right)^{1/n} \tag{81}$$

Further let $e_n^{[r]}(A) = \binom{n}{r} p_n^{[n]}(A)$. Then the following operator version of the generalized Marcus-Lopes inequality ([4, pp. 306-309]) is valid:

Let A and B be n-tuples of strictly positive matrices, r and s integers, $1 \le r \le s \le n$. Then

$$\left\{\frac{e_n^{[s]}(A+B)}{e_n^{[s-r]}(A+B)}\right\}^{1/r} \ge \left\{\frac{e_n^{[s]}(A)}{e_n^{[s-r]}(A)}\right\}^{1/r} + \left\{\frac{e_n^{[s]}(B)}{e_n^{[s-r]}(B)}\right\}^{1/r} \tag{82}$$

The case $r = s$ of (82) is

$$\left\{e_n^{[s]}(A+B)\right\}^{1/s} \ge \left\{e_n^{[s]}(A)\right\}^{1/s} + \left\{e_n^{[s]}(B)\right\}^{1/s} \tag{83}$$

Similarly, we also have the following: [3, p.106] (for possible generalizations see [4, pp. 301-303]):

Let $A = (A_1, \ldots, A_n)$ and let $\underset{\sim}{A} = (A_1, \ldots, A_n, A_{n+1})$ where $A_1, \ldots, A_{n+1}$ are positive commutable matrices. If $1 \le r \le k \le n$, then

$$\frac{\left(p_n^{[r]}(A)\right)^k}{\left(p_n^{[k]}(A)\right)^r} \le \frac{\left(p_{n+1}^{[r]}(A)\right)^{k+1}}{\left(p_{n+1}^{[k+1]}(A)\right)^r} \qquad (8$$

Of course, we can also consider the r th complete symmetric function of an n-tup

$A = (A_1, \ldots, A_n)$ of positive commutable matrices:

$$q_n^{[r]}(A) = \binom{n+r-1}{r}^{-1} \sum \left(\prod_{i=1}^{n} A_i^{i_i}\right)$$

where the sum is over all $\binom{n+r-1}{r}$ n-tuples of nonnegative integers such that $\sum_{j=1}^{n} i_j = r$

in addition $C_n^{[0]}(A)$ is defined as I.

If r is a positive integer, then we have ([4, p.315]):

$$\left(q_n^{[r]}(A)\right)^2 \le q_n^{[r-1]}(A)\, q_n^{[r+1]}(A) \qquad (85)$$

and ([4, p. 314]), if $1 \le r \le s$,

$$\left(q_n^{[r]}(A)\right)^{1/r} \le \left(q_n^{[s]}(A)\right)^{1/s}. \qquad (86)$$

Moreover we can consider the following generalizations of elementary and comple

symetric functions of an n-tuple $A = (A_1, \ldots, A_n)$ of positive commutable matrices:

$$w_n^{[k,s]}(A) = \binom{ns}{k}^{-1} \sum \left\{\prod_{j=1}^{n} \lambda_{i_j} A_j^{i_j}\right\} \quad (s > 0)$$

$$= (-1)^k \binom{ns}{k}^{-1} \sum \left\{\prod_{j=1}^{n} \lambda_{i_j} A_j^{i_j}\right\} \quad (s < 0)$$

where s is a non-zero real number, k is a natural number,

$$\lambda_i = \binom{s}{i} \qquad \text{if } s > 0$$

$$= (-1)^i \binom{s}{i} \quad \text{if } s < 0,$$

and the summation is over all non-negative n-tuples $(i_1, \ldots, i_n)$ such that $\sum_{j=1}^{n} i_j = k$.

We have ([4, pp. 317-323])

(i) If $s > 0$, k an integer and $1 \leq k \leq s$ when s is not an integer or $1 \leq k \leq ns$ if s is an integer, the

$$\left(w_n^{[k,s]}(A)\right)^2 \geq w_n^{[k-1,s]}(A)\, w_n^{[k+1,s]}(A). \tag{87}$$

(ii) If $s < 0$, inequality (87) is reversed.

(iii) If $s > 0$, k and ℓ integers, $1 \leq k < \ell \leq s+1$, when s is not an integer or $1 \leq k < \ell \leq ns$ when s is an integer, then

$$\left(w_n^{[\ell,s]}(A)\right)^{1/\ell} \leq \left(w_n^{[k,s]}(A)\right)^{1/k} \tag{88}$$

(iv) If $s < 0$ inequality (88) is reversed.

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SUPERADDITIVITY AND HERMITE-HADAMARD'S INEQUALITIES

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REZUMAT. - Superaditivitate şi inegalităţile lui Hermite-Hadamard. Îmbunătăţim în anumite sensuri inegalităţile lui Hermite-Hadamard, valabile pentru funcţii convexe pe $[a,b]$

$$f(A(a,b)) \leq A(f;a,b) \leq A(f(a), f(b))$$

unde $A(f;a,b)$ reprezintă media aritmetică integrală a funcţiei f pe $[a,b]$, iar $A(a,b)$ media aritmetică a numerelor a şi b. De exemplu $A(f;a,b)$ se înlocuieşte cu o funcţională liniară izotonă, simetrică într-un anume sens. De asemenea, inegalităţile se demonstrează pentru clase mai largi de funcţii, care le includ pe cele convexe.

1. An inequality for superadditive functions. Let us consider the sets of continuous, convex, starahaped respectively superadditive functions on $[a,b]$, given by:

$$C[a,b] = \{f : [a,b] \to R, f \text{ continuous}\}$$

$$K[a,b] = \{f \in C[a,b]; f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$
$$\forall x,y \in [a,b], \forall t \in [0,1]\}$$

$$St[a,b] = \{f \in C[a,b]; (f(x) - f(a))/(x-a) \leq$$
$$(f(y) - f(a))/(y-a), a < x < y \leq b\}$$

respectively

$$S[a,b] = \{f \in C[a,b]; f(x) + f(y) \leq f(x+y-a) + f(a),$$
$$\forall x, y, x+y-a \in [a,b]\}.$$

For $a = 0$ we denote by $C(b)$, $K(b)$, $St(b)$ respectively $S(b)$ the corresponding sets of functions, submitted also to the condition $f(0) = 0$.

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A.M. Bruckner and E. Ostrow have proved in [1] the strict inclusions:

$$K(b) \subset St(b) \subset S(b).$$

Simple proofs and generalizations of the results of [1] may be found in [5].

Starting from some properties of superadditive sequences (see [6]) at the 3 International Symposium on Functional Equations (August 22-28, 1993, Debrecen, Hung we have proposed the following problem: find some positive functions p of $C[0,b]$, diffe from the identity function, with the property that the inequality:

$$\int_0^x p(t)\left[\frac{f(x)}{x} - \frac{f(t)}{t}\right]dt \geq 0, \ \forall\ x \in [0,b]$$

hold for every $f \in S(b)$.

Of course, for $f \in St(b)$ the inequality (1) is valid for all positive p. On the other s foi the identity function, $p(x) = x$, (1) is valid for all $f \in S(b)$. Indeed we have:

LEMMA 1. *For every $f \in S(b)$ holds the inequality*:

$$\int_0^x f(t)\,dt \leq \frac{x f(x)}{2},\ \forall\ x \in (0,b]$$

Proof. We have:

$$f(t) + f(x-t) \leq f(x),\ \forall\ t \in [0,x].$$

Integrating on $[0,x]$ we get (2).

Remark 1. We can write (2) as:

$$\frac{1}{x}\int_0^x f(t)\,dt \leq \frac{f(x) + f(0)}{2}$$

which is one of Hermite-Hadamard's inequalities, as we see at once.

2. Hermite-Hadamard's inequalities. Let us denote by $A(f;a,b)$ and $A(a,b)$ integral arithmetic mean of f on $[a,b]$ respectively the arithmetic mean of a and b given

$$A(f;a,b) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

and

$$A(a,b) = \frac{a+b}{2}.$$

The inequalities of Hermite-Hadamard, valid for every function f from $K[a,b]$ are:

$$f(A(a,b)) \le A(f;a,b) \le A(f(a),f(b)). \tag{4}$$

In (3) we see that the second inequality of (4) holds for all f in $S(b)$. In fact it is valid for all superadditive functions, even of a weak kind.

DEFINITION 1. The function f is called weakly superadditive on $[a,b]$ if it verifies:

$$f(a+t) + f(b-t) \le f(a) + f(b),\ \forall\ t \in [0,(b-a)/2]. \tag{5}$$

Let us denote by $wS[a,b]$ the set of all these functions.

THEOREM 1. *The inequality*

$$A(f;a.b) \le A(f(a),f(b)) \tag{6}$$

is valid for every f of $wS[a,b]$.

Proof. Integrating (5) on $[0,b-a]$, where it is valid in fact, we get (6).

Similarly we can extend the set of functions for which the first inequality of (4) is valid.

DEFINITION 2. The function f is weakly Jensen convex on $[a,b]$ if:

$$\frac{f(a+t) + f(b-t)}{2} \ge f\left(\frac{a+b}{2}\right),\ \forall\ t \in \left[0,\frac{b-a}{2}\right] \tag{7}$$

We denote by $wJ[a,b]$ the set of all such functions.

THEOREM 2. *If $f \in wJ[a,b]$ then:*

$$A(f;a,b) \ge f(A(a,b)). \tag{8}$$

Proof. In fact (7) is valid for $t \in [0,b-a]$ and integrating on this interval, we get (8).

We can characterize the functions from $wS[a,b]$ and those from $wJ[a,b]$. For this we begin with the following:

LEMMA 2. *For every function $f \in C[a,b]$ we can determine two functions f_1, f_2 :* $[0, (b-a)/2] \to R$ such that:

$$f(x) = \begin{cases} f_1(x-a) & , for\ x \in \left[a, \frac{a+b}{2}\right] \\ f_1\left(\frac{b-a}{2}\right) + f_2\left(\frac{b-a}{2}\right) - f_1(b-x), & for\ x \in \left(\frac{a+b}{2}, b\right] \end{cases} \tag{9}$$

Proof. Of course:

$$f_1(t) = f(a+t) \quad \text{for} \quad t \in [0,(b-a)/2]$$

and

$$f_2(t) = f((b-a)/2) + c - f(b-t) \text{ for } t \in [0,(b-a)/2]$$

where c is an arbitrary real number.

Using it we can obtain the desired characterizations, which permit also the construction of such functions.

THEOREM 3. *The function f belongs to:*

a) $wS[a,b]$ *if and only if*

$$f_1(t) - f_1(0) \leq f_2(t) - f_2(0);$$

b) $wJ[a,b]$ *if and only if*

$$f_1(t) - f_1((b-a)/2) \geq f_2(t) - f_2((b-a)/2).$$

Remark 2. If we take in (9) $f_1 = f_2$ arbitrary, we get a function f with the property:

$$f(a+t) + f(b-t) = f(a) + f(b) = 2f((a+b)/2), \ \forall\ t \in [0,(b-a)/2]$$

thus it is contained in $wS[a,b] \cap wJ[a,b]$, as are also all the convex functions.

3. Symmetric linear functionals. The inequalities (4) were generalized in [3] replacing the integral arithmetic mean $A(f; a,b)$ by an arbitrary isotonic linear functional but also with the modification of the first and of the last terms. In what follows we want to do the same change of $A(f;\ a,b)$ but with the preservation of the inequalities (4). And this will be done, as in the previous paragraph, not only for convex functions.

Let $L(\cdot;\ a,b)$: $C[a,b] \to R$ be an isotonic linear functional, that is, for $t,s \in R$, $f,g \in C[a,b]$:

$$L(f;\ a,b) \geq 0 \text{ if } f \geq 0$$

$$L(tf + sg;\ a,b) = tL(f;\ a,b) + sL(g;\ a,b).$$

Analysing the proofs of Theorem 1 (or Lemma 2) and Theorem 2 we see that for our intention we can use a special type of functionals. If $f \in C[a,b]$ we denote by $f\_$ the function defined by:

$$f\_(x) = f(a+b-x) \text{ for } x \in [a,b].$$

DEFINITION 3. The functional $L(\cdot;\ a,b)$ is symmetric if:

$$L(f\_;\ a,b) = L(f;\ a,b),\ \forall f \in C[a,b].$$

THEOREM 4. *If* $L(\cdot;\ a,b)$ *is a symmetric isotonic linear functional, with* $L(1;\ a,b) = 1$, *then:*

$$L(f;\ a,b) \leq A(f(a), f(b)),\ \forall f \in wS[a,b]$$

and

$$L(f;\ a,b) \geq f(A(a,b)),\ \forall f \in wJ[a,b].$$

Proof. Indeed (5) is equivalent with:

$$f(x) + f\_(x) \leq f(a) + f(b) \text{ for } x \in [a,b]$$

and (9) with:

$$f(x) + f\_(x) \geq 2f(A(a,b)) \text{ for } x \in [a,b]$$

and we have only to apply the functional $L(\cdot;\ a,b)$.

Remark 3. If $g \in C[a,b]$ is symmetric with respect to $A(a,b)$, the functional defined by:

$$L(f;a,b) = \int_a^b f(x)\,g(x)\,dx \Big/ \int_a^b g(x)\,dx$$

satisfies all the hypothesis of Theorem 4. So we get a generalization of Hermite-Hadamard's inequalities which include the result of L. Fejér from [2] (established also only for convex functions).

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INEQUALITIES RELATED TO A CERTAIN INTEGRAL INEQUALITY ARISING IN THE THEORY OF DIFFERENTIAL EQUATIONS

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REZUMAT. - Inegalităţi în legătură cu o integrală din teoria ecuaţiilor diferenţiale. În lucrare sunt puse în evidenţă un număr de inegalităţi integrale şi discrete care pot fi folosite în studiul unor clase de ecuaţii diferenţiale sau cu diferenţe.

Abstract. In this paper we establish a number of integral and discrete inequalities related to a certain nonlinear integral inequality used in the theory of differential equations. The inequalities that we propose here can be used as tools in the study of certain new classes of differential and finite difference equations.

1. Introduction. In a recent paper [5, p.257], H. Engler proved the following inequality.

LEMMA. *Let* $c > 0$, $a \in L^1(0, T; R^+)$ *and assume that the function* $w : [0,T] \to [1,\infty)$ *satisfies*

$$w(t) \leq c\left[1 + \int_0^t a(s)\, w(s) \operatorname{Log} w(s)\, ds\right],\ 0 \leq t \leq T. \tag{1}$$

Then

$$w(t) \leq c^{\exp\left[c\int_0^t a(s)\,ds\right]},\ 0 \leq t \leq T. \tag{2}$$

A slight variant of this inequality is established by A. Haraux in his lecture notes [7, p.139] published in 1981. An important feature of this inequality is that it contains an extra

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logarithmic factor in the integrand on the right side in (1) and it is useful in the situations for which the other available inequalities in the literature do not apply directly. Due to the successful utilization of the above inequality and its variants in applications given in [5, 7], it is natural to expect that some new generalizations and extensions of the inequality given in the above Lemma would be equally important in certain new applications.

Our main objective here is to establish a number of new integral and discrete inequalities related to the inequality given in the above Lemma, which can be used as tools in the study of certain new classes of differential and finite difference equations in one and two independent variables. Altough, a great many papers have been written on various types of integral and discrete inequalities [1-4, 6, 8-15], we believe that the inequalities established in this paper are new to the literature and will prove their importance to achieve a diversity of desired goals in various applications.

2 Integral inequalities. Let $I = [0, T]$, $T > 0$ is finite but can be arbitrarily large. Let R denote the set of real numbers and $R_+ = [0, \infty)$, $R_+^0 = (0, \infty)$ and $R_1 = [1, \infty)$. We write $p \in L^1(I, R)$ whenever p is measurable from $I \to R$ and $\int_0^T |p(t)|\, dt < \infty$. Let $I_\alpha = [0, \alpha]$, $I_\beta = [0, \beta]$, $E = I_\alpha \times I_\beta$ for $\alpha, \beta \in R$, $\alpha > 0$, $\beta > 0$ and $E_1 = \{(x,y) : 0 \leq x \leq x_1, 0 \leq y \leq y_1\} \subset E$. We define the differential operators D_1 and D_2 by $D_1 r(x,y) = \frac{\partial}{\partial x} r(x,y)$ and $D_2 r(x,y) = \frac{\partial}{\partial y} r(x,y)$, where $r(x, y)$ is some function defined for $(x, y) \in E$. We write $r \in L^1(E, R)$ whenever r is measurable from $E \to R$ and $\int_0^\alpha \int_0^\beta |r(x,y)|\, dy\, dx < \infty$.

A fairly general version of Lemma is given in the following theorem.

THEOREM 2.1 *Let $a, b \in L^1(I, R_+)$, and assume that the function $u : I \to R_1$ satisfies*

$$u(t) \leq u_0 + \int_0^t a(s)\, u(s) \left(\mathrm{Log}\, u(s) + \int_0^s b(\tau)\, \mathrm{Log}\, u(\tau)\, d\tau \right) ds, \tag{2.1}$$

for $t \in I$, where $u_0 \geq 1$ is a constant. Then

$$u(t) \leq u_0^{\left[1 + \int_0^t a(s) \exp\left(\int_0^s |a(\tau) + b(\tau)| d\tau\right) ds\right]}, \tag{2.2}$$

for $t \in I$.

Proof. Define a function $v(t)$ by the right side of (2.1), then we have

$$v'(t) = a(t)\, u(t) \left(\mathrm{Log}\, u(t) + \int_0^t b(\tau)\, \mathrm{Log}\, u(\tau)\, d\tau\right). \tag{2.3}$$

Using the fact that $u(t) \leq v(t)$ in (2.3) we obtain

$$\frac{v'(t)}{v(t)} \leq a(t) \left(\mathrm{Log}\, v(t) + \int_0^t b(\tau)\, \mathrm{Log}\, v(\tau)\, d\tau\right). \tag{2.4}$$

Integrating both sides of (2.4) from 0 to t we get

$$\mathrm{Log}\, v(t) \leq \mathrm{Log}\, u_0 + \int_0^t a(s) \left(\mathrm{Log}\, v(s) + \int_0^s b(\tau)\, \mathrm{Log}\, v(\tau)\, d\tau\right) ds. \tag{2.5}$$

Now an application of Theorem 1 given in [8, p.758] we have

$$\begin{aligned} \mathrm{Log}\, v(t) &\leq \left[1 + \int_0^t a(s) \exp\left(\int_0^s [a(\tau) + b(\tau)]\, d\tau\right) ds\right] \mathrm{Log}\, u_0 \\ &= \mathrm{Log} u_0^{\left[1 + \int_0^t a(s) \exp\left(\int_0^s |a(\tau) + b(\tau)| d\tau\right) ds\right]}. \end{aligned} \tag{2.6}$$

From (2.6) we observe that

$$v(t) \leq u_0^{\left[1 + \int_0^t a(s) \exp\left(\int_0^s |a(\tau) + b(\tau)| d\tau\right) ds\right]}. \tag{2.7}$$

Inserting this into (2.1) implies (2.2) and the proof of the theorem is complete.

Remark 1. It is easy to observe that, by setting $b(t) = 0$ and $a(t) = u_0\, a^*(t)$ in Theorem 2.1 we arrive at the inequality given in Lemma with c u_0.

A slight variant of Theorem 2.1 is embodied in the following theorem.

THEOREM 2.2. *Let a, b, u, u_0 be as defined in Theorem 2.1 and*

$$u(t) \leq u_0 + \int_0^t a(s)\, u(s)\, \mathrm{Log}\, u(s) \left(\mathrm{Log}\, u(s) + \int_0^s b(\tau)\, \mathrm{Log}\, u(\tau)\, d\tau\right) ds, \tag{2.8}$$

for $t \in I$. *If* $\operatorname{Log} u_0 \int_0^t a(\tau) \exp\left(\int_0^\tau b(k)\,dk\right) d\tau < 1$ *for all* $t \in I$, *then*

$$u(t) \leq u_0^{\exp\left[\int_0^t a(s)\,Q(s)\,ds\right]}, \tag{2.9}$$

for $t \in I$, *where*

$$Q(t) = \frac{(\operatorname{Log} u_0) \exp\left(\int_0^t b(\tau)\,d\tau\right)}{1 - (\operatorname{Log} u_0) \int_0^t a(\tau) \exp\left(\int_0^\tau b(k)\,dk\right) d\tau}. \tag{2.10}$$

Proof. Defining a function $v(t)$ by the righ side of (2.8), we obtain as the proof of Theorem 2.1

$$\operatorname{Log} v(t) \leq \operatorname{Log} u_0 + \int_0^t a(s) \operatorname{Log} v(s) \left(\operatorname{Log} v(s) + \int_0^s b(\tau) \operatorname{Log} v(\tau)\,d\tau\right) ds.$$

Now by applying Theorem 1 given in [12, p.21] and following the last arguments as the proof of Theorem 2.1 given above, we get the desired inequality in (2.9).

Another interesting generalization of Lemma is established in the following theorem.

THEOREM 2.3. *Let* a, u, u_0 *be as defined in Theorem* 2.1. *Let* f *be a continuous nondecreasing function defined on* R, *and* $f > 0$ *on* R_+^0 *and* $f(0) = 0$. *If*

$$u(t) \leq u_0 + \int_0^t a(s)\,u(s)\,f(\operatorname{Log} u(s))\,ds, \tag{2.11}$$

for $t \in I$, *then for* $0 \leq t \leq t_1$, t, $t_1 \in I$,

$$u(t) \leq \exp\left[F^{-1}\left[F(\operatorname{Log} u_0) + \int_0^t a(s)\,ds\right]\right], \tag{2.12}$$

where

$$F(r) = \int_{r_0}^r \frac{ds}{f(s)}, \quad r \geq 0,\ r_0 > 0, \tag{2.13}$$

F^{-1} *is the inverse of* F *and* t_1 *is chosen so that*

$$F(\operatorname{Log} u_0) + \int_0^t a(s)\,ds \in \operatorname{Dom}(F^{-1}),$$

for all t *lying in the subinterval* $0 \leq t \leq t_1$ *of* I.

Proof. By setting $v(t)$ is equal to the right side of (2.11) and following the arguments as in the proof of Theorem 2.1 upto the inequality (2.5) we obtain

$$\operatorname{Log} v(t) \le \operatorname{Log} u_0 + \int_0^t a(s) f(\operatorname{Log} v(s))\, ds. \tag{2.14}$$

Now by applying Bihari's inequality given in [4, p.3] we have

$$\operatorname{Log} v(t) \le F^{-1}\left[F(\operatorname{Log} u_0) + \int_0^t a(s)\, ds\right]. \tag{2.15}$$

From (2.15) we observe that

$$v(t) \le \exp\left[F^{-1}\left[F(\operatorname{Log} u_0) + \int_0^t a(s)\, ds\right]\right]. \tag{2.16}$$

The desired inequality in (2.12) now follows by substituting (2.16) in (2.11). The subinterval for t is obvious.

Remark 2. We note that in Theorem 2.3, if we take $f(r) = r$, then (2.12) reduces to (2) with $c = u_0$ and if we take $f(r) = r^{\alpha}$, $0 < \alpha < 1$, then (2.12) reduces to

$$u(t) \le \exp\left[\left[(\operatorname{Log} u_0)^{1-\alpha} + (1-\alpha)\int_0^t a(s)\, ds\right]^{1/(1-\alpha)}\right]. \tag{2.17}$$

We next establish the following inequality which can be used in certain situations.

THEOREM 2.4. *Let a, u, u_0, f be as defined in Theorem 2.3. If*

$$u(t) \le u_0 + \int_0^t a(s)\, u(s)\left(\operatorname{Log} u(s) + \int_0^s a(\tau) f(\operatorname{Log} u(\tau))\, d\tau\right) ds, \tag{2.18}$$

for $t \in I$, then for $0 \le t \le t_1$, $t, t_1 \in I$,

$$u(t) \le u_0 \exp\left[\int_0^t a(s)\, \Omega^{-1}\left[\Omega(\operatorname{Log} u_0) + \int_0^s a(\tau)\, d\tau\right] ds\right], \tag{2.19}$$

where

$$\Omega(r) = \int_{r_0}^{r} \frac{ds}{s + f(s)}, \quad r \ge 0,\ r_0 > 0, \tag{2.20}$$

Ω^{-1} is the inverse of Ω and t_1 is chosen so that

$$\Omega(\operatorname{Log} u_0) + \int_0^t a(\tau)\, d\tau \in \operatorname{Dom}(\Omega^{-1}),$$

for all t lying in the subinterval $0 \le t \le t_1$ of I.

The proof of this theorem follows by the same arguments as in the proof of Theorem 2.1 and applying Theorem 2 given in [11, p.9] with suitable modifications. We omit the details.

Our next result deals with the two independent variable generalization of the inequality given in Lemma.

THEOREM 2.5. *Let* $p \in L^1(E, R_+)$ *and assume that the function* $z : E \to R_+$ *satisfies*

$$z(x,y) \leq z_0 + \int_0^x \int_0^y p(s,t)\, z(s,t) \operatorname{Log} z(s,t)\, dt\, ds, \tag{2.21}$$

for $(x,y) \in E$, *where* $z_0 \geq 1$ *is a constant. Then*

$$z(x,y) \leq z_0^{\exp\left[\int_0^x \int_0^y p(s,t)\, dt\, ds\right]}, \tag{2.22}$$

for $(x,y) \in E$.

Proof. Define a function $v(x,y)$ by the right side of (2.21). Then

$$D_2 D_1 v(x,y) = p(x,y)\, z(x,y) \operatorname{Log} z(x,y). \tag{2.23}$$

Using the fact that $z(x,y) \leq v(x,y)$ in (2.23) we obtain

$$D_2 D_1 v(x,y) \leq p(x,y)\, v(x,y) \operatorname{Log} v(x,y). \tag{2.24}$$

From (2.24) we observe that (see [14, p.492])

$$D_2\left(\frac{D_1 v(x,y)}{v(x,y)}\right) \leq p(x,y) \operatorname{Log} v(x,y). \tag{2.25}$$

Now keeping x fixed in (2.25) and setting $y = t$ and integrating both sides from 0 to y and using the fact that $D_1 v(x,0) = 0$, we get

$$\frac{D_1 v(x,y)}{v(x,y)} \leq \int_0^y p(x,t) \operatorname{Log} v(x,t)\, dt. \tag{2.26}$$

Keeping y fixed in (2.26) and setting $x = s$ and integrating both sides of (2.26) from 0 to x and using the fact that $v(0,y) = z_0$, we obtain

$$\operatorname{Log} v(x,y) \leq \operatorname{Log} z_0 + \int_0^x \int_0^y p(s,t) \operatorname{Log} v(s,t)\, dt\, ds. \tag{2.27}$$

Now an application of the inequality given in [14, p.492] we have

$$\operatorname{Log} v(x,y) \le \left[\exp\left[\int_0^x\int_0^y p(s,t)\,dt\,ds\right]\operatorname{Log} z_0\right] = \operatorname{Log} z_0^{\exp\left[\int_0^x\int_0^y p(s,t)\,dt\,ds\right]}. \tag{2.28}$$

From (2.28) we observe that

$$v(x,y) \le z_0^{\exp\left[\int_0^x\int_0^y p(s,t)\,dt\,ds\right]}. \tag{2.29}$$

Now using (2.29) in (2.21) we get the desired inequality in (2.22) and the proof of the theorem is complete.

In the following theorem we establish the two independent variable generalizations of the results established above in Theorem 2.1, 2.3 and 2.4 which can be used in some applications.

THEOREM 2.6. *Let $p, q \in L^1(E, R_+)$ and assume that the function $z : E \to R_1$ satisfies*

$$z(x,y) \le z_0 + \int_0^x\int_0^y p(s,t)\,z(s,t)\Big(\operatorname{Log} z(s,t) + \int_0^s\int_0^t q(\xi,\eta)\operatorname{Log} z(\xi,\eta)\,d\xi\,d\eta\Big)\,dt\,ds, \tag{2.30}$$

for $(x,y) \in E$, where $z_0 \ge 1$ is a constant. Then

$$z(x,y) \le z_0^{\left[1+\int_0^x\int_0^y p(s,t)\exp\left[\int_0^s\int_0^t [p(\xi,\eta)+q(\xi,\eta)]\,d\xi\,d\eta\right]dt\,ds\right]}, \tag{2.31}$$

for $(x,y) \in E$.

THEOREM 2.7. *Let p, z, z_0 be as defined in Theorem 2.5. Let g be a continuously differentiable function defined on R_+ and $g > 0$ on R_+^0 and $g(0) = 0$, $g' \ge 0$ on R_+^0. If*

$$z(x,y) \le z_0 + \int_0^x\int_0^y p(s,t)\,z(s,t)\,g(\operatorname{Log} z(s,t))\,dt\,ds, \tag{2.32}$$

for $(x,y) \in E$, then for $(x,y) \in E_1 \subset E$,

$$z(x,y) \le \exp\left[G^{-1}\left[G(\operatorname{Log} z_0) + \int_0^x\int_0^y p(s,t)\,dt\,ds\right]\right], \tag{2.33}$$

where

$$G(r) = \int_{r_0}^{r} \frac{ds}{g(s)}, \; r \geq 0, \; r_0 > 0, \tag{2.34}$$

G^{-1} *is the inverse of* G *and* $(x,y) \in E_1 \subset E$ *is chosen so that*

$$G(\mathrm{Log}\, z_0) + \int_0^x \int_0^y p(s,t)\, dt\, ds \in \mathrm{Dom}\,(G^{-1}),$$

for $(x,y) \in E_1 \subset E$.

THEOREM 2.8. *Let* p, z, z_0, g *be as defined in Theorem* 2.7. *If*

$$z(x,y) \leq z_0 + \int_0^x \int_0^y p(s,t)\, z(s,t) \Big(\mathrm{Log}\, z(s,t) + \int_0^s \int_0^t p(\xi,\eta)\, g(\mathrm{Log}\, z(\xi,\eta))\, d\xi\, d\eta \Big) dt\, ds, \tag{2.35}$$

for $(x,y) \in E$, *then for* $(x,y) \in E_1 \subset E$,

$$z(x,y) \leq z_0 \exp\left[\int_0^x \int_0^y p(s,t)\, B^{-1}\Big[B(\mathrm{Log}\, z_0) + \int_0^s \int_0^t p(\xi,\eta)\, d\xi\, d\eta \Big] dt\, ds \right], \tag{2.36}$$

where

$$B(r) = \int_{r_0}^{r} \frac{ds}{s + g(s)}, \; r \geq 0, \; r_0 > 0, \tag{2.37}$$

B^{-1} *is the inverse of* B *and* $(x,y) \in E_1 \subset E$ *is chosen so that*

$$B(\mathrm{Log}\, z_0) + \int_0^x \int_0^y p(\xi,\eta)\, d\xi\, d\eta \in \mathrm{Dom}\,(B^{-1}),$$

for $(x,y) \in E_1 \subset E$.

The proofs of Theorems 2.6, 2.7 and 2.8 follows by the same arguments as in the proof of Theorem 2.5 and using Theorem 3 in [14, p.496], Theorem 1 in [3, p.162] and Theorem 2 in [3, p.164] respectively with suitable modifications. The details are omitted.

3. Discrete inequalities. In this we establish some discrete inequalities which can be used in the study of certain finite difference equations. In what follows we let $N_0 = \{0, 1, 2, \ldots\}$. For any function $u(t)$, $t \in N_0$, we define the operator Δ by $\Delta u(t) = u(t+1) - u(t)$. For $t_1 > t_2$, $t_1, t_2 \in N_0$, we use the usual conventions $\sum_{s=t_1}^{t_2} u(s) = 0$ and $\prod_{s=t_1}^{t_2} u(s) = 1$. We write $u \in M^1(N_0, R)$ whenever $u : N_0 \to R$ and $\sum_{s=0}^{\infty} |u(s)| < \infty$. For any function $z(x,y)$, $x,y \in N_0$ we define the operators $\Delta_1 z(x,y) = z(x+1,y) - z(x,y)$, $\Delta_2 z(x,y) = z(x,y+1) - z(x,y)$ and $\Delta_2\Delta_1 z(x,y) = \Delta_2[\Delta_1 z(x,y)]$. We write $z \in M^1(N_0 \times N_0, R)$ whenever $z : N_0 \times N_0 \to R$ and $\sum_{s=0}^{\infty}\sum_{t=0}^{\infty} |z(s,t)| < \infty$.

An interesting and useful discrete analogue of Lemma is embodied in the following theorem.

THEOREM 3.1. *Let $a \in M^1(N_0, R_+)$ and assume that the function $u : N_0 \to R_1$ satisfies*

$$u(t) \le u_0 + \sum_{s=0}^{t-1} a(s)\, u(s) \operatorname{Log} u(s), \tag{3.1}$$

for $t \in N_0$, where $u_0 \ge 1$ is a constant. Then

$$u(t) \le u_0^{\prod_{s=0}^{t-1}[1+a(s)]}, \tag{3.2}$$

for $t \in N_0$.

Proof. Define a function $v(t)$ by the right side of (3.1), then

$$v(t+1) - v(t) = a(t)\, u(t) \operatorname{Log} u(t). \tag{3.3}$$

Using the fact that $u(t) \le v(t)$ in (3.3) we get

$$v(t+1) \le [1 + a(t) \operatorname{Log} v(t)]\, v(t). \tag{3.4}$$

Multiplying both sides of (3.4) by $\prod_{s=0}^{t} [1 + a(s) \operatorname{Log} v(s)]^{-1}$ and summing up both sides of the resulting inequality from 0 to t-1 it follows that (see [9, p.149])

$$v(t) \prod_{s=0}^{t-1} [1 + a(s) \operatorname{Log} v(s)]^{-1} \le u_0,$$

which implies

$$v(t) \leq u_0 \prod_{s=0}^{t-1} [1 + a(s) \operatorname{Log} v(s)] \leq$$
$$\leq u_0 \exp\left(\sum_{s=0}^{t-1} a(s) \operatorname{Log} v(s)\right). \tag{3.5}$$

From (3.5) we observe that

$$\operatorname{Log} v(t) \leq \operatorname{Log} u_0 + \sum_{s=0}^{t-1} a(s) \operatorname{Log} v(s). \tag{3.6}$$

Now by applying the inequality given in Lemma 1 in [10, p.348] we get

$$\operatorname{Log} v(t) \leq \left(\prod_{s=0}^{t-1} [1 + a(s)]\right) \operatorname{Log} u_0 =$$
$$= \operatorname{Log} u_0^{\prod_{s=0}^{t-1} [1 + a(s)]}. \tag{3.7}$$

From (3.7) we have

$$v(t) \leq u_0^{\prod_{s=0}^{t-1} [1 + a(s)]}. \tag{3.8}$$

Now using (3.8) in (3.1) we get the required inquality in (3.2). The proof of the theorem is complete.

In the following two theorems we establish the discrete analogues of the inequalities established in our Theorems 2.1 and 2.2.

THEOREM 3.2. *Let* $a, b \in M^1(N_0, R_+)$ *and assume that the function* $u : N_0 \to R_1$ *satisfies*

$$u(t) \leq u_0 + \sum_{s=0}^{t-1} a(s) u(s) \left(\operatorname{Log} u(s) + \sum_{\tau=0}^{s-1} b(\tau) \operatorname{Log} u(\tau)\right), \tag{3.9}$$

for $t \in N_0$, *where* $u_0 \geq 1$ *is a constant. Then*

$$u(t) \leq u_0^{\left[1 + \sum_{s=0}^{t-1} a(s) \prod_{\tau=0}^{s-1} [1 + a(\tau) + b(\tau)]\right]}, \tag{3.10}$$

for $t \in N_0$.

Proof. By setting $v(t)$ is equal to the right side of (3.9) and following the arguments as in the proof of Theorem 3.1 upto the inequality (3.6) with suitable modifications we obtain

$$\operatorname{Log} v(t) \leq \operatorname{Log} u_0 + \sum_{s=0}^{t-1} a(s)\left(\operatorname{Log} v(s) + \sum_{\tau=0}^{s-1} b(\tau) \operatorname{Log} v(\tau)\right). \tag{3.11}$$

Now by applying Theorem 1 given in [10, p.149] and following the last arguments as in the proof of Theorem 3.1 given above, we get the required inequality in (3.10).

THEOREM 3.3. *Let a, b, u, u_0 be as defined in Theorem 3.2 and*

$$u(t) \leq u_0 + \sum_{s=0}^{t-1} a(s)\, u(s) \operatorname{Log} u(s)\left(\operatorname{Log} u(s) + \sum_{\tau=0}^{s-1} b(\tau) \operatorname{Log} u(\tau)\right), \tag{3.12}$$

for $t \in N_0$. If $(\operatorname{Log} u_0) \sum_{\tau=0}^{t-1} a(\tau) \prod_{k=0}^{\tau} [1 + b(k)] < 1$ *for $t \in N_0$, then*

$$u(t) \leq u_0^{\prod_{s=0}^{t-1} [1 + a(s) Q_0(s)]}, \tag{3.13}$$

for $t \in N_0$, where

$$Q_0(t) = \frac{(\operatorname{Log} u_0) \prod_{\tau=0}^{t-1} [1 + b(\tau)]}{1 - (\operatorname{Log} u_0) \sum_{\tau=0}^{t-1} a(\tau) \prod_{k=0}^{\tau} [1 + b(k)]}. \tag{3.14}$$

The proof of this theorem follows by an argument as in the proof of Theorem 3.2 and applying Theorem 3 given in [13, p.318] with suitable modifications and hence we omit it here.

We next establish the following discrete analogue of our results given in Theorems 2.3 and 2.4.

THEOREM 3.4. *Let a, u, u_0 be as defined in Theorem 3.1 and f be a function as defined in Theorem 2.3. If*

$$u(t) \leq u_0 + \sum_{s=0}^{t-1} a(s)\, u(s)\, f(\operatorname{Log} u(s)), \tag{3.15}$$

for $t \in N_0$, then for $0 \leq t \leq t_1$, $t, t_1 \in N_0$,

$$u(t) \le \exp\left[F^{-1}\left[F(\operatorname{Log} u_0) + \sum_{s=0}^{t-1} a(s)\right]\right], \tag{3.16}$$

where F, F^{1} are as defined in Theorem 2.3 and $t_1 \in N_0$ is chosen so that

$$f(\operatorname{Log} u_0) + \sum_{s=0}^{t-1} a(s) \in \operatorname{Dom}(F^{-1}),$$

for all $t \in N_0$ lying in the subset $0 \le t \le t_1$ of N_0.

Proof. Defining a function $v(t)$ by the right side of (3.15) and following the same arguments as in the proof of Theorem 3.1 upto the inequality (3.6) we get

$$\operatorname{Log} v(t) \le \operatorname{Log} u_0 + \sum_{s=0}^{t-1} a(s) f(\operatorname{Log} v(s)). \tag{3.17}$$

Define a function $m(t)$ by the right side of (3.17), then we obtain

$$\Delta m(t) = a(t) f(\operatorname{Log} v(t)). \tag{3.18}$$

Using the fact that Log $v(t) \le m(t)$ in (3.18) we have

$$\frac{\Delta m(t)}{f(m(t))} \le a(t). \tag{3.19}$$

Now from (2.13) and (3.19) we observe that

$$F(m(t+1)) - F(m(t)) = \int_{m(t)}^{m(t+1)} \frac{ds}{f(s)} \le \frac{\Delta m(t)}{f(m(t))} \le a(t). \tag{3.20}$$

Setting $t = s$ in (3.20) and substituting $s = 0, 1, 2, \ldots, t\text{-}1$ and using the fact that $m(0)$=Log u_0, we obtain

$$F(m(t)) \le F(\operatorname{Log} u_0) + \sum_{s=0}^{t-1} a(s). \tag{3.21}$$

Now using the bound on $m(t)$ from (3.21) in (3.17) we have

$$\operatorname{Log} v(t) \le F^{-1}\left[F(\operatorname{Log} u_0) + \sum_{s=0}^{t-1} a(s)\right]. \tag{3.22}$$

From (3.22) we observe that

$$v(t) \le \exp\left[F^{-1}\left[F(\operatorname{Log} u_0) + \sum_{s=0}^{t-1} a(s)\right]\right]. \tag{3.23}$$

Now using (3.23) in (3.15) we get the required inequality in (3.16). The subdomain $0 \le t \le t_1$ of N_0 is obvious.

THEOREM 3.5. *Let a, u, u_0, f be as defined in Theorem 3.4. If*

$$u(t) \le u_0 + \sum_{s=0}^{t-1} a(s)\, u(s) \left(\operatorname{Log} u(s) + \sum_{\tau=0}^{s-1} a(\tau) f(\operatorname{Log} u(\tau)) \right), \tag{3.24}$$

for $t \in N_0$, *then for* $0 \le t \le t_1$, $t, t_1 \in N_0$,

$$u(t) \le u_0 \exp\left[\sum_{s=0}^{t-1} a(s)\, \Omega^{-1} \left[\Omega(\operatorname{Log} u_0) + \sum_{\tau=0}^{s-1} a(\tau) \right] \right], \tag{3.25}$$

where Ω, Ω^{-1} *are as defined in Theorem* 2.4 *and* $t_1 \in N_0$ *is chosen so that*

$$\Omega(\operatorname{Log} u_0) + \sum_{\tau=0}^{t-1} a(\tau) \in \operatorname{Dom}(\Omega^{-1}),$$

for all $t \in N_0$ *lying in the subset* $0 \le t \le t_1$ *of* N_0.

The proof of this theorem follows by the same arguments as in the proof of Theorem 3.4 given above in view of the proof of Theorem 1 given in [10, p.349] with suitable modifications. We omit it here.

Our next results are the discrete extensions of Theorems 3.1 and 3.2 given above to two independent variables.

THEOREM 3.6. *Let* $p \in M^1(N_0 \times N_0, R_+)$ *and assume that the function* $z : N_0 \times N_0 \to R_1$ *satisfies*

$$z(x,y) \le z_0 + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} p(s,t)\, z(s,t) \operatorname{Log} z(s,t), \tag{3.26}$$

for $x, y \in N_0$, *where* $z_0 \ge 1$ *is a constant. Then*

$$z(x,y) \le z_0^{\prod_{s=0}^{x-1} \left[1 + \sum_{t=0}^{y-1} p(s,t) \right]}, \tag{3.27}$$

for $x, y \in N_0$.

Proof. Define a function $v(x,y)$ by

$$v(x,y) = z_0 + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} p(s,t)\, z(s,t) \operatorname{Log} z(s,t). \tag{3.28}$$

From (3.28) and using the fact that $z(x,y) \le v(x,y)$ we observe that

$$\Delta_1 v(x,y+1) - \Delta_1 v(x,y) \le p(x,y)\, v(x,y) \operatorname{Log} v(x,y). \tag{3.29}$$

From the definition of $v(x,y)$ we observe that $v(x,y) \le v(x,y+1)$, for $x,y \in N_0$. Using this and the fact that $\Delta_1 v(x,y) \ge 0$ in (3.29) we observe that

$$\frac{\Delta_1 v(x,y+1)}{v(x,y+1)} - \frac{\Delta_1 v(x,y)}{v(x,y)} \le p(x,y)\,\mathrm{Log}\, v(x,y). \tag{3.30}$$

Now keeping x fixed in (3.30), set $y = t$ and sum over $t = 0, 1, 2, \ldots, y-1$ to obtain the estimate

$$\frac{\Delta_1 v(x,y)}{v(x,y)} \le \sum_{t=0}^{y-1} p(x,t)\,\mathrm{Log}\, v(x,t). \tag{3.31}$$

Here we have used the fact that $\Delta_1 v(x,0) = 0$. From (3.31) we observe that

$$v(x+1,y) \le v(x,y)\left[1 + \sum_{t=0}^{y-1} p(x,t)\,\mathrm{Log}\, v(x,t)\right]. \tag{3.32}$$

Now keeping y fixed in (3.32) set $x = s$ and substituting $s = 0, 1, 2, \ldots, x-1$ successively we obtain the estimate

$$\begin{aligned} v(x,y) &\le z_0 \prod_{s=0}^{x-1}\left[1 + \sum_{t=0}^{y-1} p(s,t)\,\mathrm{Log}\, v(s,t)\right] \\ &\le z_0 \exp\left[\sum_{s=0}^{x-1}\sum_{t=0}^{y-1} p(s,t)\,\mathrm{Log}\, v(s,t)\right]. \end{aligned} \tag{3.33}$$

From (3.33) we observe that

$$\mathrm{Log}\, v(x,y) \le \mathrm{Log}\, z_0 + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} p(s,t)\,\mathrm{Log}\, v(s,t). \tag{3.34}$$

From (3.34) and by following exactly the same arguments as above upto the inequality (3.33) we get

$$\begin{aligned} \mathrm{Log}\, v(x,y) &\le [\prod_{s=0}^{x-1}\left[1 + \sum_{t=0}^{y-1} p(s,t)\right]\mathrm{Log}\, z_0 \\ &= \mathrm{Log}\, z_0^{\prod_{s=0}^{x-1}\left[1 + \sum_{t=0}^{y-1} p(s,t)\right]}. \end{aligned} \tag{3.35}$$

From (3.35) we observe that

$$v(x,y) \le z_0^{\prod_{s=0}^{x-1}\left[1 + \sum_{t=0}^{y-1} p(s,t)\right]}. \tag{3.36}$$

Using (3.36) in (3.26) we get the required inequality in (3.27) and the proof of the theorem is complete.

THEOREM 3.7. *Let* $p,q \in M^4(N_0 \times N_0, R_+)$ *and assume that the function* $z : N_0 \times N_0 \to R_1$ *satisfies*

$$z(x,y) \le z_0 + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} p(s,t) z(s,t) \Big[\operatorname{Log} z(s,t) + \sum_{m=0}^{s-1} \sum_{n=0}^{t-1} q(m,n) \operatorname{Log} z(m,n) \Big], \tag{3.37}$$

for $x,y \in N_0$, *where* $z_0 \ge 1$ *is a constant. Then*

$$z(x,y) \le z_0^{\left[1 + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} p(s,t) Q^*(s,t)\right]}, \tag{3.38}$$

for $x,y \in N_0$, *where*

$$Q^*(x,y) = \prod_{m=0}^{x-1} \left[1 + \sum_{n=0}^{y-1} [p(m,n) + q(m,n)] \right], \tag{3.39}$$

for $x,y \in N_0$.

The proof of this theorem follows by closely looking at the proof of Theorem 3.6 given above and the proof of Theorem 1 given in [10, p.349] and hence we omit it here.

To this end we establish the following results which are further extensions of our Theorems 3.4 and 3.5 given above to two independent variables.

THEOREM 3.8. *Let* p, z, z_0 *be as defined in Theorem* 3.6 *and let* f *be as defined in Theorem* 2.3. *If*

$$z(x,y) \le z_0 + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} p(s,t) z(s,t) f(\operatorname{Log} z(s,t)), \tag{3.40}$$

for $x,y \in N_0$, *then for* $(x,y) \in E_2 \subset N_0 \times N_0$,

$$z(x,y) \le \exp\left[F^{-1} \left[F(\operatorname{Log} z_0) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} p(s,t) \right] \right], \tag{3.41}$$

where F, F^{-1} *are as defined in Theorem* 2.3 *and* $(x,y) \in E_2$ *is chosen so that*

$$F(\operatorname{Log} z_0) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} p(s,t) \in \operatorname{Dom}(F^{-1}),$$

for all $(x,y) \in E_2 \subset N_0 \times N_0$.

THEOREM 3.9. *Let p, z, z_0, f be as defined in Theorem* 3.8. *If*

$$z(x,y) \le z_0 + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} p(s,t)\, z(s,t) \Big[\operatorname{Log} z(s,t) + \sum_{m=0}^{s-1} \sum_{n=0}^{t-1} p(m,n)\, f(\operatorname{Log} z(m,n)) \Big], \tag{3.42}$$

for $x, y \in N_0$, *then for* $(x,y) \in E_2 \subset N_0 \times N_0$,

$$z(x,y) \le z_0 \exp\Big[\sum_{s=0}^{x-1} \sum_{t=0}^{y-1} p(s,t)\, \Omega^{-1} \Big[\Omega(\operatorname{Log} z_0) + \sum_{m=0}^{s-1} \sum_{n=0}^{t-1} p(m,n) \Big] \Big], \tag{3.43}$$

where Ω, Ω^{-1} *are as defined in Theorem* 2.4 *and* $(x,y) \in E_2$ *is chosen so that*

$$\Omega(\operatorname{Log} z_0) + \sum_{m=0}^{x-1} \sum_{n=0}^{y-1} p(m,n) \in \operatorname{Dom}(\Omega^{-1}),$$

for all $(x,y) \in E_2 \subset N_0 \times N_0$.

The details of the proofs of Theorems 3.8 and 3.9 proceed much as in the proof of Theorem 3.6 and observing the proofs of other theorems given above and can safely be omitted.

Remark. We note that the inequalities established in Theorems 2.5-2.8 and Theorems 3.6-3.9 can be extended very easily to many independent variables. The precise formulation of these results is very close to that of the results mentioned above with suitable modifications. We leave it for the reader to fill in where needed.

4. An application. As already mentioned above, it is easy to observe that the inequalities established in this paper can be used as handy tools in the qualitative analysis of certain classes of differential and finite difference equations for which the inequalities available in the literature do not apply directly. For example, consider the following finite difference equation

$$\Delta x(t) = p(t)x(t) \operatorname{Log} |x(t)|, \quad x(0) = x_0, \tag{4.1}$$

for $t \in N_0$, where $x : N_0 \to R$, $p \in M^1(N_0,R)$. The equation (4.1) is equivalent to the equation

$$x(t) = x_0 + \sum_{s=0}^{t-1} p(s)x(s) \operatorname{Log} |x(s)|. \tag{4.2}$$

From (4.2) we observe that

$$\begin{aligned} 1 + |x(t)| &\le 1 + |x_0| + \sum_{s=0}^{t-1} |p(s)|\,|x(s)|\,|\operatorname{Log} |x(s)|| \\ &\le 1 + |x_0| + \sum_{s=0}^{t-1} |p(s)|\,(1 + |x(s)|) \operatorname{Log} (1 + |x(s)|). \end{aligned} \tag{4.3}$$

Now an application of Theorem 3.1 with $u(t) = 1 + |x(t)|$ to the inequality (4.3) yields

$$|x(t)| \le \left[(1 + |x_0|)^{\prod_{s=0}^{t-1} [1 + |p(s)|]} - 1 \right]. \tag{4.4}$$

The inequality (4.4) obtains the bound on the solution of (4.1) in terms of the known functions. There are many possible applications of the inequalities established in this paper, but the one presented here is sufficient to convey the importance of our results to the literature. Various applications of these inequalities will appear elsewhere.

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CLASSICAL SOLUTIONS OF THE DARBOUX PROBLEM FOR PARTIAL FUNCTIONAL-DIFFERENTIAL INCLUSIONS IN BANACH SPACE

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REZUMAT. - Soluţii clasice ale problemei lui Darboux pentru incluziuni funcţional-diferenţiale cu derivate parţiale. În lucrare considerăm problema lui Darboux pentru incluziuni funcţional-diferenţiale cu derivate parţiale de forma (*P*), unde $F:Q \times X^{4+\nu} \to 2^X$, f: $[0,a] \to X$, g: $[0,b] \to X$, α_i: $Q \to [0,a]$, β_i: $Q \to [0,b]$, $i = \overline{1,\nu}$, $Q = [0,a] \times [0,b]$ şi X este un spaţiu Banach. Demonstrăm că mulţimea soluţiilor clasice ale acestei probleme este un retract într-un spaţiu de funcţii adecvat.

Introduction. Let a, b be two positive real numbers; Q the rectangle $[0,a] \times [0,b]$; X a Banach space; F a multifunction F: $Q \times X^{4+\nu} \to 2^X$.

Given $f \in C^1([0,a],X)$, $g \in C^1([0,b],X)$, with $f(0) = g(0)$ and α_i: $Q \to [0,a]$, β_i: $Q \to [0,b]$continuous, $i = \overline{1,\nu}$, we consider the Darboux problem (*P*):

$$
(P)\begin{cases}
\dfrac{\partial^2 u}{\partial x \partial y}(x,y) \in F(x,y,u(\cdot,\cdot),\dfrac{\partial u}{\partial x}(\cdot,\cdot),\dfrac{\partial u}{\partial y}(\cdot,\cdot),\dfrac{\partial^2 u}{\partial x \partial y}(\cdot,\cdot),\dfrac{\partial^2 u}{\partial x \partial y}(\alpha_1(x,y),\beta_1(x,y)),\dots, \\
\qquad\qquad \dfrac{\partial^2 u}{\partial x \partial y}(\alpha_\nu(x,y),\beta_\nu(x,y))) \\
\qquad\qquad u(x,0) = f(x) \quad \text{for} \quad 0 \le x \le a \\
\qquad\qquad u(0,y) = g(y) \quad \text{for} \quad 0 \le y \le b
\end{cases}
$$

Besides problem (*P*), given $z_0, z_{i0} \in F(0,0,f(0),f'(0),g'(0),z_0,z_{10},z_{20},\dots,z_{\nu 0})$, $i = \overline{1,\nu}$, we consider also the problem (P_0):

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$$(P_0)\begin{cases}\dfrac{\partial^2 u}{\partial x\partial y}(x,y)\in F\left(x,y,u(\cdot,\cdot),\dfrac{\partial u}{\partial x}(\cdot,\cdot),\dfrac{\partial u}{\partial y}(\cdot,\cdot),\dfrac{\partial^2 u}{\partial x\partial y}(\cdot,\cdot),\dfrac{\partial^2 u}{\partial x\partial y}(\alpha_1(x,y),\beta_1(x,y)),\dots,\right.\\ \qquad\left.\dfrac{\partial^2 u}{\partial x\partial y}(\alpha_\nu(x,y),\beta_\nu(x,y))\right)\\ u(x,0)=f(x)\quad\text{for}\quad 0\le x\le a\\ u(0,y)=g(y)\quad\text{for}\quad 0\le y\le b\\ \dfrac{\partial^2 u}{\partial x\partial y}(0,0)=z_0\\ \dfrac{\partial^2 u}{\partial x\partial y}(\alpha_i(0,0),\beta_i(0,0))=z_{i0},\quad i=\overline{1,\nu}\end{cases}$$

A function $u: Q\to X$ is said to be classical solution of (P) if

$u,\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y},\dfrac{\partial^2 u}{\partial x\partial y}(\alpha_i(x,y),\beta_i(x,y))\in C^0(Q,X)$ and, for every $(x,y)\in Q$ one has

$$\frac{\partial^2 u}{\partial x\partial y}(x,y)\in F\left(x,y,u(x,y),\frac{\partial u}{\partial x}(x,y),\frac{\partial u}{\partial y}(x,y),\frac{\partial^2 u}{\partial x\partial y}(x,y),\frac{\partial^2 u}{\partial x\partial y}(\alpha_1(x,y),\beta_1(x,y)),\dots,\frac{\partial^2 u}{\partial x\partial y}(\alpha_\nu(x,y),\beta_\nu(x,y))\right)$$

$u(x,0)=f(x)$, $u(0,y)=g(y)$.

A classical solution of (P_0) is any classical solution of (P) whose second mixed derivative assumes at (0,0) the prescribed values z_0, z_{i0}, $i=\overline{1,\nu}$.

Denote by $\Gamma(f,g,F)$ (respectively $\Gamma(f,g,z_0,z_{10},z_{20},\dots,z_{\nu 0},F)$) the set of all classical solutions of (P) (resp. (P_0)). In general, $\Gamma(f,g,F)\neq\varnothing$ does not imply $\Gamma(f,g,z_0,z_{10},z_{20},\dots,z_{\nu 0},F)\neq\varnothing$, [5].

Because of the multi-valuedness of F, the sets $\Gamma(f,g,F)$, $\Gamma(f,g,z_0,z_{10},z_{20},\dots,z_{\nu 0},F)$ in several cases, contain many elements and so it makes sense to perform a qualitative study of them, for instance, from a topological point of view.

In this direction, we prove that, under suitable assumptions, each of the sets $\Gamma(f,g,F)$, $\Gamma(f,g,z_0,z_{10},z_{20},\dots,z_{\nu 0},F)$ is a retract of an appropiate function space. Under the same assumptions, we prove also that these sets depend - in a Lipschitzian way - on $f, g, z_0, z_{10}, z_{20}, \ldots, z_{\nu 0}, F$. Furthermore, given 4+ν sequences $\{f_n\}, \{g_n\}, \{z_n\}, \{z_{in}\}$, $i = \overline{1,\nu}$, $\{F_n\}$, we state some sufficient conditions under which each $u \in \Gamma(f,g,F)$ (resp. $u \in \Gamma(f, g, z_0, z_{10}, z_{20}, \ldots, z_{\nu 0}, F)$) is the limit, with respect to an appropiate metric, of a sequence $\{u_n\}$ such that $u_n \in \Gamma(f_n, g_n, F_n)$ (resp. $\Gamma(f_n, g_n, z_n, z_{1n}, z_{2n}, \ldots, z_{\nu n}, F_n)$) for every $n \in N$.

The problem (P) is more general than a similar problem (P) of [5], because F also depends on ν **additional** arguments $u(\alpha_i(x,y), \beta_i(x,y))$, $i = \overline{1,\nu}$. Anyway our paper is essentially based on [5] and, therefore the reasonements and the results are similar to the ones thereof, with slight modifications.

For the case when F is single-valued, a thorough study of the equation in Problem (P) is performed by M. Kwapisz and J. Turo [2]. The functions α_i and β_i occur in the evaluations imposed on F in stating the sufficient conditions for the existence of solutions. The differential inclusion may have various particular forms, depending upon the form of the multifunction F, namely:

a) $$\frac{\partial^2 u}{\partial x \partial y}(x,y) \in F(x,y,u(\gamma_1(x,y),\delta_1(x,y)),\frac{\partial u}{\partial x}(\gamma_2(x,y),\delta_2(x,y)),\frac{\partial u}{\partial y}(\gamma_3(x,y),\delta_3(x,y)),\frac{\partial^2 u}{\partial x \partial y}(\alpha(x,y),\beta(x,y))).$$

b) $$\frac{\partial^2 u}{\partial x \partial y} \in F(x, y, u(x,y), \frac{\partial u}{\partial x}(x,y), \frac{\partial u}{\partial y}(x,y)),$$

studied by W. Sosulski [12], [13], and by A.N. Vityuk [26].

c) $$\frac{\partial^2 u}{\partial x \partial y} \in F(x, y, u(x,y))$$

was studied by G. Teodoru in a series of papers [16]-[25], by V. Staicu [14], [15], and by I. Kubiaczyk [3].

d) The case when F depends on x,y only, the inclusion $\frac{\partial^2 u}{\partial x \partial y} \in F(x,y)$ was studied by F.S. de Blasi and J. Myjak [1].

e) $F = F(x, y, u(x,y), \frac{\partial^2 u}{\partial x \partial y}(x,y))$

1. Preliminaries (Notations and definitions) [5]. Let X,Y be two non-empty sets. A multifunction Φ from into Y (that is, $\Phi: X \to 2^Y$) is a function from X into the family of all non-empty subsets of Y. If $\Omega \subseteq Y$, we put $\Phi^-(\Omega) = \{x \in X: \Phi(x) \cap \Omega \neq \emptyset\}$. If X, Y are two topological spaces we say that Φ is **lower semicontinuous** if, for every open set $\Omega \subseteq Y$, the set $\Phi^-(\Omega)$ is open in X. A (single-valued) function φ: $X \to Y$ is said to be a **selection** of Φ if $\varphi(x) \in \Phi(x)$ for all $x \in X$. If (Σ, δ) is a metric space, for every $x \in \Sigma$ and every non-empty sets $A,B \subseteq \Sigma$, we put:

$$\delta(x, A) = \inf_{z \in A} \delta(z, B), \quad \delta^*(A, B) = \sup_{z \in A} \delta(z, B)$$

and consider a Hausdorff-Pompeiu metric [11]

$$\delta_H(A, B) = \max \{\delta^*(A, B), \delta^*(B, A)\}.$$

Moreover, if $\{A_n\}_{n \in N}$ is a sequence of non-empty subsets of Σ, we put (as in [5])

$$\underset{n \to \infty}{Li}\, A_n = \left\{x \in \Sigma : \lim_{n \to \infty} \delta(x, A_n) = 0\right\}.$$

Let (X,d), (Y,ρ) be two metric spaces. A multifunction $\Phi: X \to 2^Y$ is said to be **Lipschitzian** if there exists a real number $L > 0$ (Lipschitz constant) such that

$$\rho_H(\Phi(x), \Phi(z)) \le L d(x, z)$$

for all $x,z \in X$. If $L < 1$, we say that Φ is a **multivalued contraction.**

Notice that any Lipschitzian multifunction is lower semicontinuous.

Let $Q \subseteq \mathbf{R}^2$ be a compact rectangle and $(X, \|\cdot\|)$ be a Banach space. We denote by $C^0(Q,X)$ the space of all continuous functions from Q into X endowed with the norm

$$\|u\|_{C^0(Q,X)} = \max_{(x,y)\in Q} \|u(x,y)\|.$$

Let $E(Q,X)$ denote the space of all functions $u:Q \to X$ such that

$u, \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial^2 u}{\partial x \partial y}, \dfrac{\partial^2 u}{\partial x \partial y}(\alpha_i, \beta_i) \in C^0(Q,X)$, $i = \overline{1,\nu}$, endowed with the norm

$$\|u\|_{E(Q,X)} = \|u\|_{C^0(Q,X)} + \left\|\frac{\partial u}{\partial x}\right\|_{C^0(Q,X)} + \left\|\frac{\partial u}{\partial y}\right\|_{C^0(Q,X)} + \left\|\frac{\partial^2 u}{\partial x \partial y}\right\|_{C^0(Q,X)} + \sum_{i=1}^{\nu} \left\|\frac{\partial^2 u}{\partial x \partial y}(\alpha_i,\beta_i)\right\|_{C^0(Q,X)}.$$

If $I \subseteq R$ is a compact interval, we denote by $C^1(I,X)$ the space of all continuously differentiable function from I into X, endowed with the norm

$$\|\varphi\|_{C^1(I,X)} = \max_{t\in I} \|\varphi(t)\| + \max_{t\in I} \|\varphi'(t)\|.$$

A subset Γ of a topological space Σ is said to be a **retract** of Σ if $\Gamma \neq \Phi$ and there exists a continuous function $\psi:\Sigma \to \Gamma$ such that $\psi(x) = x$, for every $x \in \Gamma$, i.e. $\psi_{|\Gamma} = 1_\Gamma$, [5], [8], [11].

2. Results. Everywhere in what follows a,b are two positive real numbers; Q is a rectangle $[0,a] \times [0,b]$; $(X, \|\cdot\|)$ is a real Banach space; d is the metric induced by $\|\cdot\|$; D is the metric induced by $\|\cdot\|_{E(Q,X)}$.

Given the non-negative real numbers, $L, \mu, \bar{\mu}$, $i = \overline{1,\nu}$, we denote by $\mathcal{F}_{L,\mu,\bar{\mu},X}$ the family of all lower semicontinuous multifunctions $F: Q \times X^{4+\nu} \to 2^X$, with closed and convex values, such that for every $(x,y,z_1',z_2',z_3',\dots,z_{4+\nu}')$, $(x,y,z_1'',z_2'',z_3'',\dots,z_{4+\nu}'') \in Q \times X^{4+\nu}$ the following inequalities hold:

$$d_H(F(x,y,z_1',z_2',z_3',\dots,z_{4+\nu}'), F(x,y,z_1'',z_2'',z_3'',\dots,z_{4+\nu}'')) \le$$
$$\le L\sum_{i=1}^{3} \|z_i' - z_i''\| + \mu\|z_4' - z_4''\| + \bar{\mu}\sum_{i=1}^{\nu} \|z_{4+i}' - z_{4+i}''\|.$$

Let G be the set of all $(f,g) \in C^1([0,a],X) \times C^1([0,b],X)$ such that $f(0) = g(0)$. Given

$(f,g) \in G$ and $F: Q \times X^{4+\nu} \to 2^X$ we put

$$\Gamma(f,g,F) = \Big\{ u \in E(Q,X) : \frac{\partial^2 u}{\partial x \partial y}(x,y) \in F(x,y,u(x,y), \frac{\partial u}{\partial x}(x,y), \frac{\partial u}{\partial y}(x,y), \frac{\partial^2 u}{\partial x \partial y}(x,y),$$

$$\frac{\partial^2 u}{\partial x \partial y}(\alpha_1(x,y), \beta_1(x,y)), \ldots, \frac{\partial^2 u}{\partial x \partial y}(\alpha_\nu(x,y), \beta_\nu(x,y))), \; u(x,0) = f(x), \; u(0,y) = g(y) \text{ for all}$$

$(x,y) \in Q\Big\}$.

Further, if $z_0, z_{i0} \in F(0,0,f(0),f'(0),g'(0),z_0,z_{10},\ldots,z_{\nu 0}\}$ we put

$$\Gamma(f,g,z_0,z_{10},z_{20},\ldots,z_{\nu 0},F) = \{u \in \Gamma(f,g,F) : \frac{\partial^2 u}{\partial x \partial y}(0,0) =$$
$$= z_0, \frac{\partial^2 u}{\partial x \partial y}(\alpha_1(0,0), \beta_1(0,0)) = z_{10}, \ldots, \frac{\partial^2 u}{\partial x \partial y}(\alpha_\nu(0,0), \beta_\nu(0,0)) = z_{\nu 0}\}$$

We obviously have, from these definitions, $\Gamma(f,g,z_0,z_{10},z_{20},\ldots,z_{\nu 0},F) \subset \Gamma(f,g,F)$

In the proof of two theorems in paper in the sequel, we will often apply the following lemm [5].

LEMMA 2.1. *Let Σ be a paracompact topological space, $(Y, \|\cdot\|_Y)$ a Banach spa $\Phi: \Sigma \to 2^Y$ a lower semicontinuous multifunction, with closed and convex values,* $\varphi: \Sigma \to$ *a continuous function; $\beta: X \to [0,+\infty[$ a lower semicontinuous function such th $\rho(\varphi(x), \Phi(x)) \le \beta(x)$ for all $x \in X$, where ρ is the metric induced by $\|\cdot\|_Y$. Then, for ev $\varepsilon > 0$, there exists a continuous selection ψ of Φ such that $\|\varphi(x) - \psi(x)\|_Y \le \beta(x) + \varepsilon$ for $x \in \Sigma$.*

LEMMA 2.2. *Let (Σ,δ) be a metric space, $(Y, \|\cdot\|_Y)$ a normed space, and $T: \Sigma \to$ a Lipschitzian function with Lipschitz constant L. Then, for every $y,v \in Y$ and every no empty sets $A,B \subseteq \Sigma$,*

$$\rho_H(y + T(A), v + T(B)) \le \|y - v\|_Y + L\delta_H(A,B),$$

ρ being the metric induced by $\|\cdot\|_Y$.

Our first result is the following:

THEOREM 2.1. *Let be* $L \in]0, \infty[$, $\mu \in [0, 1[$, $\bar{\mu} \in [0, 1]$ *such that* $1 - \mu - \bar{\mu}\nu \neq 0$ *and* $F, G \in \mathcal{F}_{L,\mu,\bar{\mu},X}$. *We put*

$$c = (ab + a + b + \nu)\left\{\min_{\lambda \in \left]\frac{3}{1-\mu-\bar{\mu}\nu}, +\infty\right[} \frac{e^{\lambda L(ab+a+b)}}{1 - (3\lambda^{-1} + \mu + \bar{\mu}\nu)}\right\}$$

Then, for every $(f,g) \in G$, $\{z_0, z_{10}, z_{20}, \ldots, z_{\nu 0}\} \subset F(0,0,f(0),f'(0),g'(0),z_0,z_{10},z_{20},\ldots,z_{\nu 0})$, *the following assertions hold:*

(i) *each of the sets* $\Gamma(f,g,F)$, $\Gamma(f,g,z_0,z_{10},z_{20},\ldots,z_{\nu 0},F)$ *is a retract of the space* $E(Q,X)$;

(ii) *if* F *is a single valued function, then the set* $\Gamma(f,g,F)$ *is a singleton*;

(iii) *for every* $(h,l) \in G$, *such that* $\{z_0,z_{10},z_{20},\ldots,z_{\nu 0}\} \subset G(0,0,h(0),h'(0),l'(0),z_0,z_{10},\ldots,z_{\nu 0})$ *we have*

$$D_H(\Gamma(f,g,z_0,z_{10},z_{20},\ldots,z_{\nu 0},F),\ \Gamma(h,l,z_0,z_{10},z_{20},\ldots,z_{\nu 0},G)) \le$$
$$\le (cL+1)\left(\|f-h\|_{C^1([0,a],X)} + \|g-l\|_{C^1([0,b],X)} + \|f(0)-h(0)\|\right) +$$
$$+ c \sup_{\xi \in Q \times X^{4+\nu}} d_H(F(\xi), G(\xi))$$

(iv) *if* $\mu = 0$, $\bar{\mu} = 0$, *then for every* $(h,l) \in G$ *and every* $\{\omega_0, \omega_{10}, \ldots, \omega_{\nu 0}\} \subset G(0,0,h(0),h'(0),l'(0),w_0,\omega_{10},\omega_{20},\ldots,\omega_{\nu 0})$, *we have*

$$D_H(\Gamma(f,g,z_0,z_{10},z_{20},\ldots,z_{\nu 0},F), \Gamma(h,l,z_0,z_{10},z_{20},\ldots,z_{\nu 0},G)) \le$$
$$\le (cL+1)\left(\|f-h\|_{C^1([0,a],X)} + \|g-l\|_{C^1([0,b],X)} + \|f(0)-h(0)\|\right) +$$
$$+ c\left(\|z_0-\omega_0\| + \sum_{i=1}^{\nu}\|z_{i0}-\omega_{i0}\| + \sup_{\xi \in Q \times X^{4+\nu}} d_H(F(\xi),G(\xi))\right)$$

(v) *for every* $(h,l) \in G$

$$D_H(\Gamma(f,g,F), \Gamma(h,l,G)) \le (cL+1)(\|f-h\|_{C^1([0,a],X)} + \|g-l\|_{C^1([0,b],X)} +$$
$$629 + \|f(0)-h(0)\|) + c \sup_{\xi \in Q \times X^{4+\nu}} d_H(F(\xi),G(\xi))$$

Proof. Fix $(f,g) \in G$, $z_0, z_{i0} \in F(0,0,f(0),f'(0),g'(0),z_0,z_{10},z_{20},\dots,z_{\nu 0})$, $i = \overline{1,\nu}$.
Fix also $\lambda \in \left]\frac{3}{1-\mu-\overline{\mu}\nu}, +\infty\right[$.

Renorm the space $C^0(Q,X)$, by the Bielecki norm [5] equivalent with $\|\cdot\|_{C^0(Q,X)}$. For each $\varphi \in C^0(Q,X)$, put:

$$\|\varphi\|_0 = \max_{(x,y)\in Q} e^{-\lambda L(xy+x+y)} \|\varphi(x,y)\|.$$

For each $\psi \in C^0(Q,X)$, put:

$$\Phi(f,g,\psi,F) = \{\varphi \in C^0(Q,X)\colon \varphi(x,y) \in F(x,y,\eta(f,g,\psi)(x,y)) \text{ for all } (x,y)\in Q\},$$

where

$$\eta(f,g,\psi)(x,y)) = \Bigg(f(x)+g(y)-f(0)+\int_0^x\int_0^y \psi(s,t)\,ds\,dt,\ f'(x)+\int_0^y \psi(x,t)\,dt,$$
$$g'(y)+\int_0^x \psi(s,y)\,ds,\ \psi(x,y),\ \psi(\alpha_1(x,y),\beta_1(x,y)),\dots,\psi(\alpha_\nu(x,y),\beta_\nu(x,y))\Bigg)$$

the integrals being understood in the sense of Riemann (as usually, $\int_0^x\int_0^y \psi(s,t)\,ds\,dt$ stands for $\int_0^x \left(\int_0^y \psi(s,t)\,dt\right) ds$).

Further, put

$C_{z_0,z_{10},\dots,z_{\nu 0}} =$

$\{\varphi \in C^0(Q,X)\colon \varphi(0,0) = 0, \varphi(\alpha_1(0,0),\beta_1(0,0)) = z_{10},\dots,\varphi(\alpha_\nu(0,0),\beta_\nu(0,0)) = z_{\nu 0}\}$

as well as

$$\Phi(f,g,z_0,z_{10},z_{20},\dots,z_{\nu 0},\psi,F) = \Phi(f,g,\psi,F) \cap C_{z_0,z_{10},\dots,z_{\nu 0}},$$

for every $\psi \in C_{z_0,z_{10},\dots,z_{\nu 0}}$. By Theorem 3.2" and Example 1.3* of [6], $\Phi(f,g,\psi,F) \neq \varnothing$ for all $\psi \in C^0(Q,X)$ and $\Phi(f,g,z_0,z_{10},z_{20},\dots,z_{\nu 0},\psi,F) \neq \varnothing$ [5], for all $\psi \in C_{z_0,z_{10},\dots,z_{\nu 0}}$. We now prove that the multifunctions $\Phi(f,g,\cdot,F)$ and $\Phi(f,g,z_0,z_{10},z_{20},\dots,z_{\nu 0},\cdot,F)$ are multivalued contractions with respect to the metric, say σ, induced by $\|\cdot\|_0$, with Lipschitz constant $L_1 = 3\lambda^{-1} + \mu + \overline{\mu}\nu$. Since $\lambda > \frac{3}{1-\mu-\overline{\mu}\nu}$ it results that $L_1 < 1$.

We prove this only for $\Phi(f,g,z_0,z_{10},z_{20},\dots,z_{\nu 0},\cdot,F)$, since the proof for

$\Phi(f, g, \cdot, F)$, is quite similar. Thus, fix $\gamma, \psi \in C_{z_0, z_{10}, \ldots, z_{\nu 0}}$, $\varphi \in \Phi(f, g, z_0, z_{10}, z_{20}, \ldots, z_{\nu 0}, \gamma, F)$ and $\varepsilon > 0$. Then, for every $(x,y) \in Q$, we have

$$d(\varphi(x,y), F(x,y,\eta(f,g,\psi)(x,y))) \leq$$

$$\leq L\left(\left\| \int_0^x \int_0^y (\gamma(s,t) - \psi(s,t)) ds\,dt \right\| + \left\| \int_0^y (\gamma(x,t) - \psi(x,t)) dt \right\| + \left\| \int_0^y (\gamma(s,y) - \psi(s,y)) ds \right\| \right) +$$

$$+ \mu \|\gamma(x,y) - \psi(x,y)\| + \bar{\mu} \sum_{i=1}^{\nu} \|\gamma(\alpha_i(x,y), \beta_i(x,y)) - \psi(\alpha_i(x,y), \beta_i(x,y))\|.$$

By Example 1.3* of [6], the multifunction $\Psi : Q \to 2^X$ defined by putting

$$\Psi(x,y) = \begin{cases} F(x,y,\eta(f,g,\psi)(x,y)) & \text{if } (x,y) \in Q \setminus \{(0,0)\} \\ \{z_0, z_{10}, \ldots, z_{\nu 0}\} & \text{if } (x,y) = (0,0) \end{cases}$$

is lower semicontinuous. Hence, by Lemma 2.1, [5], ψ admits a continuous selection β such that

$$\|\beta(x,y) - \varphi(x,y)\| \leq$$

$$\leq L\left(\left\| \int_0^x \int_0^y (\gamma(s,t) - \psi(s,t)) ds\,dt \right\| + \left\| \int_0^t (\gamma(x,t) - \psi(x,t)) dt \right\| + \left\| \int_0^x (\gamma(s,y) - \psi(s,y)) ds \right\| \right) + \tag{1}$$

$$+ \mu \|\gamma(x,y) - \psi(x,y)\| + \bar{\mu} \sum_{i=1}^{\nu} \|\gamma(\alpha_i(x,y), \beta_i(x,y)) - \psi(\alpha_i(x,y), \beta_i(x,y))\| + \varepsilon$$

for every $(x,y) \in Q$. In particular, remark that $\beta \in \Phi(f, g, z_0, z_{10}, \ldots, z_{\nu 0}, \psi, F)$.

Let us now evaluate $\|\beta - \varphi\|_0$. For every $(x,y) \in Q$, we have

$$\left\| \int_0^x \int_0^y (\gamma(s,t) - \psi(s,t)) ds\,dt \right\| + \left\| \int_0^y (\gamma(x,t) - \psi(x,t)) dt \right\| + \left\| \int_0^x (\gamma(s,y) - \psi(s,y)) ds \right\| \leq$$

$$\leq \|\gamma - \psi\|_0 \left(\int_0^x \int_0^y e^{\lambda L(st+s+t)} ds\,dt + \int_0^y e^{\lambda L(xt+x+t)} dt + \int_0^x e^{\lambda L(sy+s+y)} ds \right) \leq \tag{2}$$

$$\leq \frac{\|\gamma-\psi\|_0}{\lambda L}\left(\int_0^x\int_0^y \frac{\partial^2 e^{\lambda L(st+s+t)}}{\partial s\,\partial t}ds\,dt + \int_0^y \frac{\partial e^{\lambda L(xt+x+t)}}{\partial t}dt + \int_0^x \frac{\partial e^{\lambda L(sy+s+y)}}{\partial s}\right) \leq$$

$$\leq \frac{3\,\|\gamma-\psi\|_0}{\lambda L}\, e^{\lambda L(xy+x+y)}$$

From (1) and (2), we readily obtain

$$\|\beta-\varphi\|_0 \leq (3\lambda^{-1}+\mu+\bar{\mu}\nu)\,\|\gamma-\psi\|_0 + \varepsilon.$$

This shows that

$$\sigma^*(\Phi(f,g,z_0,z_{10},z_{20},\dots,z_{\nu 0},\gamma,F),\ \Phi(f,g,z_0,z_{10},z_{20},\dots,z_{\nu 0},\psi,F)) \leq$$
$$\leq (3\lambda^{-1}+\mu+\bar{\mu}\nu)\,\|\gamma-\psi\|_0.$$

Interchanging the roles of γ and ψ, we get

$$\sigma^*(\Phi(f,g,z_0,z_{10},z_{20},\dots,z_{\nu 0},\psi,F),\ \Phi(f,g,z_0,z_{10},z_{20},\dots,z_{\nu 0},\gamma,F)) \leq$$
$$\leq (3\lambda^{-1}+\mu+\bar{\mu}\nu)\,\|\gamma-\psi\|_0.$$

Hence, our assertion is proved, since we deduce from the preceding inequalities that

$$\sigma_H(\Phi(f,g,z_0,z_{10},z_{20},\dots,z_{\nu 0},\gamma,F),\ \Phi(f,g,z_0,z_{10},z_{20},\dots,z_{\nu 0},\psi,F)) \leq$$
$$\leq (3\lambda^{-1}+\mu+\bar{\mu}\nu)\,\|\gamma-\psi\|_0.$$

Let us now put

$$P(f,g,F) = \{\varphi \in C^0(Q,X) : \varphi \in \Phi(f,g,\varphi,F)\}$$

as well as

$$P(f,g,z_0,z_{10},z_{20},\dots,z_{\nu 0},F) =$$
$$= \{\varphi \in C_{z_0,z_{10},\dots,z_{\nu 0}} : \varphi \in \Phi(f,g,z_0,z_{10},z_{20},\dots,z_{\nu 0},\varphi,F)\}.$$

Then, taking into account that all sets $\Phi(f,g,\psi,F)$, $\Phi(f,g,z_0,z_{10},z_{20},\dots,z_{\nu 0},\psi,F)$ are convex and closed, by Theorem 1 of [9] the set $P(f,g,F)$ is a retract of $C^0(Q,X)$ and the set $P(f,g,z_0,z_{10},z_{20},\dots,z_{\nu 0},F)$ is a retract of $C_{z_0,z_{10},\dots,z_{\nu 0}}$.

Since $C_{z_0,z_{10},\dots,z_{\nu 0}}$ is convex and closed, it is a retract of $C^0(Q,X)$. Hence

$P(f,g,z_0,z_{10},\ldots,z_{\nu 0},F)$ is, in its turn, a retract of $C^0(Q,X)$. Now, consider the operator $T\colon C^0(Q,X) \to E(Q,X)$ defined by

$$T(\psi)(x,y) = \int_0^x \int_0^y \psi(s,t)\,ds\,dt$$

for every $\psi \in C^0(Q,X)$, $(x,y) \in Q$. Let us remark that

$$\Gamma(f,g,F) = \varphi_{f,g} + T(P(f,g,F)) \tag{3}$$

$$\Gamma(f,g,z_0,z_{10},z_{20},\ldots,z_{\nu 0},F) = \varphi_{f,g} + T(P(f,g,z_0,z_{10},z_{20},\ldots,z_{\nu 0},F)) \tag{4}$$

where $\varphi_{f,g}$ denotes the function $(x,y) \to f(x) + g(y) - f(0)$, $(x,y) \in Q$.

Let θ_X be the zero element of X. Put

$$V_0 = \{u \in E(Q,X)\colon u(x,0) = u(0,y) = \theta_X \text{ for all } (x,y) \in Q\}.$$

The operator T is a linear homeomorfism from $C^0(Q,X)$ onto V_0. Therefore, each of the sets $T(P(f,g,F))$, $T(P(f,g,z_0,z_{10},z_{20},\ldots,z_{\nu 0},F))$ is a retract of V_0. But V_0 being closed and convex, it is a retract of $E(Q,X)$, and so each of the sets $T(P(f,g,F))$, $T(P(f,g,z_0,z_{10},z_{20},\ldots,z_{\nu 0},F))$ is a retract of $E(Q,X)$. Now, assertion (i) immediately follows from (3) and (4).

On the basis of what was seen above, assertion (ii) is an immediate consequence of the classical contraction mapping principle of Banach - Caccioppoli.

Let us prove (iii). Put

$$M = \sup_{\xi \in Q \times X^{4+\nu}} d_H(F(\xi), G(\xi)).$$

Naturally, we suppose that $M < +\infty$. Let $(h,l) \in G$ be such that

$$\{z_0, z_{10}, z_{20}, \ldots, z_{\nu 0}\} \subset G(0,0,h(0),h'(0),l'(0),z_0,z_{10},z_{20},\ldots,z_{\nu 0}).$$

From the proof of (i), we know that each of the multifunctions

$\Phi(f,g,z_0,z_{10},z_{20},\ldots,z_{\nu 0},\cdot,F)$, $\Phi(h,l,z_0,z_{10},z_{20},\ldots,z_{\nu 0},\cdot,G)$ is a multivalued contraction from $(C_{z_0,z_{10},\ldots,z_{\nu 0}},\sigma)$ into itself, with the Lipschitz constant $L_1 = 3\lambda^{-1} + \mu + \bar{\mu}\nu$. Then, by

Lemma 1 of [4], we get

$$\sigma_H(P(f,g,z_0,z_{10},z_{20},\dots,z_{v0},F),\ P(h,l,z_0,z_{10},z_{20},\dots,z_{v0},G)) \le \frac{1}{1-(3\lambda^{-1}+\mu+\bar{\mu}v)}\cdot$$

$$\cdot \sup_{\psi\in C_{z_0,z_{10},\dots,z_{v0}}} \sigma_H(\Phi(f,g,z_0,z_{10},\dots,z_{v0},\psi,F),\Phi(h,l,z_0,z_{10},\dots,z_{v0},\psi,G)). \qquad (5)$$

Fix $\psi \in C_{z_0,z_{10},\dots,z_{v0}}$ and $\varphi \in \Phi(f,g,z_0,z_{10},z_{20},\dots,z_{v0},F)$. Then, for every $(x,y)\in Q$, we have

$$d(\varphi(x,y),G(x,y,\eta(h,l,\psi)(x,y))) \le$$

$$\le d_H(F(x,y,\eta(f,g,\psi)(x,y)),F(x,y,\eta(h,l,\psi)(x,y))) +$$

$$+ d_H(F(x,y,\eta(h,l,\psi)(x,y)),\ G(x,y,\eta(h,l,\psi)(x,y))) \le$$

$$\le L\big(\|f(x)+g(y)-f(0)-h(x)-l(y)+h(0)\| + \|f'(x)-h'(x)\| + \|g'(y)-l'(y)\|\big) + M \le$$

$$\le L\big(\|f-h\|_{C^1([0,a],X)} + \|g-l\|_{C^1([0,b],X)} + \|f(0)-h(0)\|\big) + M.$$

Then, applying Lemma 2.1 [5] as in the proof of (i), for every $\varepsilon > 0$ we get $u\in\Phi(h,l,z_0,z_{10},z_{20},\dots,z_{v0},\psi,G)$ such that

$$\|u-\varphi\|_{C^0(Q,X)} \le L\big(\|f-h\|_{C^1([0,a],X)} + \|g-l\|_{C^1([0,b],X)} + \|f(0)-h(0)\|\big) + M + \varepsilon.$$

From this, by means of a usual reasoning, it follows that

$$\sigma_H(\Phi(f,g,z_0,z_{10},z_{20},\dots,z_{v0},\psi,F),\ \Phi(h,l,z_0,z_{10},z_{20},\dots,z_{v0},\psi,G)) \le$$

$$\le L\big(\|f-h\|_{C^1([0,a],X)} + \|g-l\|_{C^1([0,b],X)} + \|f(0)-h(0)\|\big) + M + \varepsilon.$$

Therefore, by (5), we have

$$\sigma_H(P(f,g,z_0,z_{10},z_{20},\dots,z_{v0},F),\ P(h,l,z_0,z_{10},z_{20},\dots,z_{v0},G)) \le$$

$$\le \frac{1}{1-(3\lambda^{-1}+\mu+\bar{\mu}v)}\Big\{L\big(\|f-h\|_{C^1([0,a],X)} + \|g-l\|_{C^1([0,b],X)} + \|f(0)-h(0)\|\big) + M + \varepsilon\Big\}. \qquad (6)$$

Now, remark that, for every $\varphi\in C^0(Q,X)$, one has

$$\|T(\varphi)\|_{E(Q,X)} \le (ab+a+b+v)\|\varphi\|_{C^0(Q,X)} \le (ab+a+b+v)\,e^{\lambda L(ab+a+b)}\|\varphi\|_0. \qquad (7)$$

Then, applying Lemma 2.2 [5] and taking into account (4) and (7), we obtain

$$D_H(\Gamma(f,g,z_0,z_{10},z_{20},\dots,z_{v0},F),\ \Gamma(f,g,z_0,z_{10},z_{20},\dots,z_{v0},G)) \le$$

$$\leq \|\varphi_{f,g}-\varphi_{h,l}\|_{E(Q,X)}+(ab+a+b+\nu)e^{\lambda L(ab+a+b)}\sigma_H(P(f,g,z_0,z_{10},\dots,z_{\nu 0},F),P(h,l,z_0,z_{10},\dots,z_{\nu 0},G)).$$

But

$$\|\varphi_{f,g}-\varphi_{h,l}\| \leq \|f-h\|_{C^1([0,a],X)}+\|g-l\|_{C^1([0,b],X)}+\|f(0)-h(0)\|.$$

Using this inequality and (6) we have

$$D_H(\Gamma(f,g,z_0,z_{10},z_{20},\dots,z_{\nu 0},F),\Gamma(h,l,z_0,z_{10},z_{20},\dots,z_{\nu 0},G)) \leq$$

$$\leq \left[1+\frac{(ab+a+b+\nu)e^{\lambda L(ab+a+b)}}{1-(3\lambda^{-1}+\mu+\bar{\mu}\nu)}L\right]\left(\|f-h\|_{C^1([0,a],X)}+\|g-l\|_{C^1([0,b],X)}+\|f(0)-h(0)\|\right)+$$

$$+\frac{(ab+a+b+\nu)e^{\lambda L(ab+a+b)}}{1-(3\lambda^{-1}+\mu+\bar{\mu}\nu)}M.$$

From this, since λ is an arbitrary number greater than $\frac{3}{1-\mu-\bar{\mu}\nu}$ and using the notation for c and M, it result that

$$D_H(\Gamma(f,g,z_0,z_{10},z_{20},\dots,z_{\nu 0},F),\Gamma(h,l,z_0,z_{10},z_{20},\dots,z_{\nu 0},G)) \leq$$

$$\leq (cL+1)\left(\|f-h\|_{C^1([0,a],X)}+\|g-l\|_{C^1([0,b],X)}+\|f(0)-h(0)\|\right)+c\sup_{\xi\in Q\times X^{4+\nu}} d_H(F(\xi),G(\xi))$$

i.e. (iii) follows.

To prove (iv), we notice that, when $\mu=0$ and $\bar{\mu}=0$, we can define the multifunctions $\psi\to\Phi(f,g,z_0,z_{10},z_{20},\dots,z_{\nu 0},\psi,F)$, $\psi\to\Phi(h,l,\omega_0,\omega_{10},\omega_{20},\dots,\omega_{\nu 0},\psi,G)$ in the whole space $C^0(Q,X)$. They are again multivalued contractions, with respect to σ, with Lipschitz constant $L_1 = 3\lambda^{-1}+\mu+\bar{\mu}\nu$. Hence, by Lemma 1 of [4], we have

$$\sigma_H(P(f,g,z_0,z_{10},z_{20},\dots,z_{\nu 0},F),\ P(h,l,\omega_0,\omega_{10},\omega_{20},\dots,\omega_{\nu 0},G) \leq \frac{1}{1-(3\lambda^{-1}+\mu+\bar{\mu}\nu)}\cdot$$

$$\cdot\sup_{\psi\in C^0(Q,X)}\sigma_H(\Phi(f,g,z_0,z_{10},z_{20},\dots,z_{\nu 0},\psi,F),\ \Phi(h,l,\omega_0,\omega_{10},\omega_{20},\dots,\omega_{\nu 0},\psi,G)).$$

On the other hand, proceeding as in the proof of (iii), it is possible to obtain

$$\sup_{\psi\in C^0(Q,X)}\sigma_H(\Phi(f,g,z_0,z_{10},z_{20},\dots,z_{\nu 0},\psi,F),\ \Phi(h,l,\omega_0,\omega_{10},\omega_{20},\dots,\omega_{\nu 0},\psi,G)) \leq$$

$$\leq L\left(\|f-h\|_{C^1([0,a],X)}+\|g-l\|_{C^1([0,b],X)}+\|f(0)-h(0)\|\right)+\|z_0-\omega_0\|+\sum_{i=1}^{\nu}\|z_{i0}-\omega_{i0}\|+M.$$

Now, the remainder of proof goes on exactly as before.

Finally, the proof of (v) is entirely similar to that of (iii), and hence we omit it.

Now, we prove the following approximation result.

THEOREM 2.2. *Let* $L \in]0, +\infty[$, $\mu \in [0, 1[$, $\bar{\mu} \in [0, 1[$ *and* $F, F_1, F_2, \ldots,$ *be a sequence in* $\mathcal{F}_{L,\mu,\bar{\mu},X}$. *Assume that, for every continuous function* $\beta: Q \to X^{4+\nu}$ *and every continuous selection* γ *of the multifunction* $(x,y) \to F(x,y,\beta(x,y))$,

$$\lim_{n\to\infty} \sup_{(x,y)\in Q} d(\gamma(x,y), F_n(x,y,\beta(x,y))) = 0$$

Then, for every sequence $(f,g), (f_1,g_1), (f_2,g_2), \ldots,$ *in G such that*

$$\lim_{n\to\infty} \max\left\{\|f_n - f\|_{C^1([0,a],X)}, \|g_n - g\|_{C^1([0,b],X)}\right\} = 0,$$

the following assertion hold:

(i_1) $\Gamma(f,g,F) \subseteq \underset{n\to\infty}{Li}\ \Gamma(f_n,g_n,F_n)$;

(i_2) *if* $\{z_0, z_{10}, \ldots, z_{\nu 0}\} \subset \bigcap_{n\in N} F_n(0,0,f_n(0),f_n'(0),g_n'(0),z_0,z_{10},\ldots,z_{\nu 0}) \cap$

$$\cap F(0,0,f(0),f'(0),g'(0),z_0,z_{10},\ldots,z_{\nu 0})$$

then $\Gamma(f,g,z_0,z_{10},z_{20},\ldots,z_{\nu 0},F) \subseteq \underset{n\to\infty}{Li}\ \Gamma(f_n,g_n,z_0,z_{10},z_{20},\ldots,z_{\nu 0},F_n)$;

(i_3) *if* $\mu = 0$, $\bar{\mu} = 0$ *and* $\{z_0,z_{10},z_{20},\ldots,z_{\nu 0}\} \subset F(0,0,f(0),f'(0),g'(0),z_0,z_{10},\ldots,z_{\nu 0})$, *then for every sequence* $\{z_n,z_{1n},z_{2n},\ldots,z_{\nu n}\}$ *in* $X^{1+\nu}$ *converging to* $\{z_0,z_{10},z_{20},\ldots,z_{\nu 0}\} \in X^{1+\nu}$ *for* $n \to \infty$, *such that* $\{z_n,z_{1n},z_{2n},\ldots,z_{\nu n}\} \subset F_n(0,0,f_n(0),f_n'(0),g_n'(0),z_0,z_{10},\ldots,z_{\nu 0})$ *for all* $n \in N$, *one has*

$$\Gamma(f,g,z_0,z_{10},z_{20},\ldots,z_{\nu 0},F) \subseteq \underset{n\to\infty}{Li}\ \Gamma(f_n,g_n,z_n,z_{1n},z_{2n},\ldots,z_{\nu n},F_n).$$

Proof. The proofs of (i_1), (i_2), (i_3) are similar. So, we limit ourselves to present only that of (i_3). Let $\lambda > 3$. Keeping the same notations from the proof of Theorem 2.1, we know that the multifunctions $\Phi(f,g,z_0,z_{10},z_{20},\ldots,z_{\nu 0},\cdot,F)$, $\Phi(f_n,g_n,z_n,z_{1n},z_{2n},\ldots,z_{\nu n},\cdot,F_n)$ $(n \in N)$ are closed-valued contractions from $(C^0(Q,X),\sigma)$ into itself, with the same

Lipschitz constant $3\lambda^{-1}$. Let $\psi \in C^0(Q, X)$ be fixed. We claim that

$$\Phi(f, g, z_0, z_{10}, z_{20}, \dots, z_{\nu 0}, \psi, F) \subseteq \lim_{n\to\infty} \Phi(f_n, g_n, z_n, z_{1n}, z_{2n}, \dots, z_{\nu n}, \psi, F_n). \tag{9}$$

Indeed, let $\varphi \in \Phi(f, g, z_0, z_{10}, z_{20}, \dots, z_{\nu 0}, \psi, F)$. Then, for every $(x,y) \in Q$, $n \in N$, we have

$$d(\varphi(x,y), F_n(x,y,\eta(f_n, g_n, \psi)(x,y))) \le$$

$$\le d(\varphi(x,y), F_n(x,y,\eta(f,g,\psi)(x,y))) + d_H(F_n(x,y,\eta(f,g,\psi)(x,y)), F_n(x,y,\eta(f_n,g_n,\psi)(x,y))).$$

Because F_n is Lipschitzian multifunction with $\mu = 0$ and $\overline{\mu} = 0$ it result

$$d_H(F_n(x,y,\eta(f,g,\psi)(x,y)), F_n(x,y,\eta(f_n,g_n,\psi)(x,y))) \le$$

$$\le L\left(2\|f_n - f\|_{C^1([0,a];X)} + \|g_n - g\|_{C^1([0,b];X)}\right).$$

Hence

$$d(\varphi(x,y), F_n(x,y,\eta(f_n,g_n,\psi)(x,y))) \le d(\varphi(x,y), F(x,y,\eta(f,g,\psi)(x,y))) +$$

$$+ L\left(2\|f_n - f\|_{C^1([0,a];X)} + \|g_n - g\|_{C^1([0,b];X)}\right).$$

Applying Lemma 2.1 [5] as in the proof of Theorem 2.1, we get a sequence $\{\varphi_n\}$ in $C^0(Q,X)$ such that, for every $(x,y) \in Q$, $n \in N$, we have

$$\varphi_n(x,y) \in F_n(x,y,\eta(f_n,g_n,\psi)(x,y))$$

$$\varphi_n(0,0) = z_n,\ \varphi_n(\alpha_i(0,0), \beta_i(0,0)) = z_{i0},\ i = \overline{1,\nu}, \text{ and}$$

$$\|\varphi_n(x,y) - \varphi(x,y)\| \le d(\varphi(x,y), F_n(x,y,\eta(f,g,\psi)(x,y))) +$$

$$+ L\left(2\|f_n - f\|_{C^1([0,a];X)} + \|g_n - g\|_{C^1([0,b];X)}\right) + \|z_n - z_0\| + \sum_{i=1}^{\nu} \|z_{\in} - z_{i0}\| + \frac{1}{n}.$$

Hence $\varphi_n(x,y) \in F_n(x,y,\eta(f_n,g_n,\psi)(x,y))$ i.e. $\varphi_n \in \Phi(f_n, g_n, z_n, z_{1n}, z_{2n}, \dots, z_{\nu n}, \psi, F)$ and, by (8)

$$\lim_{n\to\infty} \|\varphi_n - \varphi\|_{C^0(Q,X)} = 0.$$

This proves (9). Now, we can apply Theorem 3.4 of [7]. By that result, we have

$$P(f, g, z_0, z_{10}, z_{20}, \dots, z_{\nu 0}, F) \subseteq \mathop{Li}_{n\to\infty} P(f_n, g_n, z_n, z_{1n}, z_{2n}, \dots, z_{\nu n}, F_n). \tag{10}$$

For $n \in N$, we have

$$\Gamma(f_n, g_n, z_n, z_{1n}, z_{2n}, \dots, z_{\nu n}, F) = \varphi_{f_n, g_n} + T(P(f_n, g_n, z_n, z_{1n}, z_{2n}, \dots, z_{\nu n}, F)) \qquad (4_n)$$

From (10) and (4_n) (i_3) easily follows.

We point out that some sufficient conditions in order that (8) holds are given in [8] (Proposition 2.1 and 2.2). We also stress that Theorems 2.1 and 2.2 are similar to the Theorems 2.1 and 2.2 of [5].

Finally, we consider the Darboux problem $(\bar{P})$ for single-valued function, which results as a particular case of the Darboux problem (P).

In the proof of (i) in Theorem 2.1, more precisely of the fact that $\Phi(f, g, z_0, z_{10}, z_{20}, \dots, z_{\nu 0}, \cdot, F)$ is a multivalued contraction, it results a continuous selection β of the multifunction $\Psi: Q \to 2^X$, such that (1) holds. We remark that the proof for the multifunction $\Phi(f, g, \cdot, F)$ is similar. In this case $\Psi(x,y) = F(x,y,\eta(f,g,\psi)(x,y)), (x,y) \in Q$. We denote by $\bar{f}$ a continuous selection of Ψ which results by Lemma 2.1 [5]. We have

$$\|\bar{f}(x,y) - \varphi(x,y)\| \le$$
$$\le L\left(\left\|\int_0^x\int_0^y(\gamma(s,t) - \psi(s,t))ds\,dt\right\| + \left\|\int_0^y(\gamma(x,t) - \psi(x,t))dt\right\| + \left\|\int_0^x(\gamma(x,y) - \psi(s,y))ds\right\|\right) + \qquad (\bar{1})$$
$$+ \mu\|\gamma(x,y) - \psi(x,y)\| + \bar{\mu}\sum_{i=1}^{\nu}\|\gamma(\alpha_i(x,y), \beta_i(x,y)) - \psi(\alpha_i(x,y), \beta_i(x,y))\| + \varepsilon, \ (x,y) \in Q.$$

In particular, remark that $\bar{f} \in \Phi(f, g, \psi, F)$.

We denote by $(\bar{P})$ the Darboux problem obtained from (P) substituting F by the continuous selection $\bar{f}$. This way we obtain the Darboux problem studied by Kwapisz and Turo [2].

$$(\bar{P})\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y}(x,y) = \\ = \bar{f}(x,y,u(\cdot,\cdot),u_x(\cdot,\cdot),u_y(\cdot,\cdot),u_{xy}(\cdot,\cdot),u_{xy}(\alpha_1(x,y)),\beta_1(x,y)),\dots,u_{xy}(\alpha_\nu(x,y),\beta_\nu(x,y))) \\ u(x,0) = f(x), \quad 0 \le x \le a \\ u(0,y) = g(y), \quad 0 \le y \le b. \end{cases}$$

The Darboux problem $(\bar{P})$ is equivalent to the problem of solving the equation

$$z(x,y) = \bar{f}(x,y,(f(\xi) + g(\eta) - f(0) + \int_0^\xi\int_0^\eta z(s,t)ds\,dt)_Q, (f'(\xi) + \int_0^\eta z(\xi,t)dt)_Q,$$

$$(g'(\eta) + \int_0^\xi z(s,\eta)ds)_Q, (z(\xi,\eta))_Q, z(\alpha_1(x,y), \beta_1(x,y)), \dots, z(\alpha_\nu(x,y), \beta_\nu(x,y))), \tag{11}$$

where

$$u(x,y) = f(x) + g(y) - f(0) + \int_0^x\int_0^y z(s,t)ds\,dt.$$

Putting in the equation (11)

$$\tilde{f}(x,y,(z(\xi,\eta))_Q, (p(\xi,\eta))_Q, (q(\xi,\eta))_Q, (s(\xi,\eta))_Q, r_1, r_2, \dots, r_\nu) =$$

$$= \bar{f}(x,y,(f(\xi+g(\eta)-f(0)+z(\xi,\eta))_Q, (f'(\xi)+p(\xi,\eta))_Q, (g'(\eta)+q(\xi,\eta))_Q, (s(\xi,\eta))_Q, r_1, r_2, \dots, r_\nu)$$

we get an equation of the form

$$z(x,y) = \tilde{f}(x,y, \left(\int_0^\xi\int_0^\eta z(s,t)ds\,dt\right)_Q, \left(\int_0^\eta z(\xi,t)dt\right)_Q, \left(\int_0^\xi z(s,\eta)ds\right)_Q,$$

$$(z(\xi,\eta))_Q, z(\alpha_1(x,y), \beta_1(x,y)), \dots, z(\alpha_\nu(x,y), \beta_\nu(x,y))). \tag{12}$$

which is the analogue of equation (4) in [2], where that problem is studied in full detail; those results may be readily transferred to our problem $(\bar{P})$ by simply replacing σ, r, f, Δ, E (= a Banach space) with f, g, $\tilde{f}$, Q, X, respectively.

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METHOD OF AVERAGING FOR INTEGRAL-DIFFERENTIAL INCLUSIONS

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REZUMAT. - Metoda medierii pentru incluziuni integro-diferenţiale. În lucrare se prezintă o teoremă de tip Bogolubov pentru incluziuni integro-diferenţiale de forma (*).

1. Introduction and notations. The aim of this paper is to present the Bogolubov's type theorem (see [1]) for integral-differential inclusions of the form

$$\dot{x}(t) \in F(t, x_t, \dot{x}_t) + \int_0^t \phi(t, s, x_s, \dot{x}_s)\, ds \qquad (*)$$

where F is a multifunction with values that are nonempty compact convex subsets of n-dimensional Euclidean space $\mathbf{R}^n$ and ϕ is a measurable single-valued mapping with values in $\mathbf{R}^n$.

Section 2 of this paper contains the proof of the existence theorem for (*). The results obtain in this section generalize the results of A.F. Filippov ([3]). Further on, in Section 3 we prove the theorem of middling for inclusions (*). The results of this section generalize the results of W.A. Płotnikov and O.G. Rubyk ([5]) where the system $\dot{x} \in F(t,x) + \int_0^t \phi(t,s,x(s))ds$ was intestigated.

Let C_0 and L_0 denote the Banach spaces of all continuous and Lebesque integrable functions, respectively, of $[-r,0]$ into $\mathbf{R}^n$ with the norms

$$\|x\|_0 = \sup_{-r \le t \le 0} |x(t)| \text{ and } |y|_0 = \int_{-r}^{0} |y(t)|\, dt$$

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for $x \in C_0$ and $y \in L_0$, respectively, where $|\cdot|$ denotes the Euclidean norm. For a given function u: $[-r,T] \to \mathbf{R}^n$ and fixed $t \in [0,T]$ we denote $u_t(s) = u(t+s)$ for $s \in [-r,0]$, $r \geq 0$, $T > 0$. Finally, let us denote by (Comp $\mathbf{R}^n$,H) and (Conv $\mathbf{R}^n$,H) the metric space of all nonempty compact and nonempty compact convex, respectively, subsets of n-dimensional Euclidean space $\mathbf{R}^n$ with the Hausdorff metric H.

Assume that the multivalued mapping F: $[0,T] \times C_0 \times L_0 \to$ Comp $\mathbf{R}^n$ and the mapping ϕ: $[0,T] \times [0,T] \times C_0 \times L_0 \to \mathbf{R}^n$ satisfy the following conditions:

(a) $F(\cdot, u, v)$: $[0,T] \to$ Comp $\mathbf{R}^n$ is measurable for fixed $(u,v) \in C_0 \times L_0$,

(b) there exists a $M > 0$ such that $H(F(t,u,v),\{0\}) \leq M$ for $(t,u,v) \in [0,T] \times C_0 \times L_0$,

(c) $F(t, \cdot, \cdot)$: $C_0 \times L_0 \to$ Comp $\mathbf{R}^n$ is Lipschitzean with respect to (u,v), i.e. there exists a Lebesque integrable function k: $[0,T] \to \mathbf{R}^+$ such that $H(F(t,u,v), F(t,\bar{u},\bar{v})) \leq k(t)(\|u-\bar{u}\|_0 + |v-\bar{v}|_0)$ for (u,v), $(\bar{u},\bar{v}) \in C_0 \times L_0$ and almost every $t \in [0,T]$,

(d) ϕ: $[0,T] \times [0,T] \times C_0 \times L_0 \to \mathbf{R}^n$ is continuous,

(e) $\phi(t,s,\cdot,\cdot)$: $C_0 \times L_0 \to \mathbf{R}^n$ satisfies for fixed $(t,s) \in [0,T] \times [0,T]$ the Lipschitz condition of the form $|\phi(t,s,u,v) - \phi(t,s,\bar{u},\bar{v})| \leq \mu(t,s)(\|u-\bar{u}\|_0 + |v-\bar{v}|_0)$ where μ: $[0,T] \times [0,T] \to \mathbf{R}^+$ is a Lebesque integrable function,

(f) there exists a Lebesque integrable function λ: $[0,T] \times [0,T] \to \mathbf{R}^+$ such that $|\phi(t,s,u,v)| \leq \lambda(t,s)$ for $(t,s) \in [0,T] \times [0,T]$ and $(u,v) \in C_0 \times L_0$, where $\int_{t_1}^{t_2} dt \int_0^t \lambda(t,s)\,ds \leq k_1(t_2 - t_1)$, $\forall\, t_1, t_2,\ t_2 \geq t_1 \geq 0$.

THEOREM 1. *Let* δ: $[0,T] \to \mathbf{R}$ *be a nonnegative Lebesque integrable function and* $\varphi \in C_0$ *be absolutely continuous. Suppose* $F: [0,T] \times C_0 \times L_0 \to \mathrm{Comp}\,\mathbf{R}^n$ *and* $\phi: [0,T] \times [0,T] \times C_0 \times L_0 \to \mathbf{R}^n$ *satisfies* (a)-(f) *and furthermore let* y: $[-r,T] \to \mathbf{R}^n$ *be an absolutely continuous mapping such that:*

(g) $y(t) = \varphi(t)$ *for* $t \in [-r,0]$,

(h) $\rho(\dot{y}(t), F(t,y_t,\dot{y}_t) + \int_0^t \phi(t,s,y_s,\dot{y}_s)\,ds) \leq \delta(t)$ for a.e. $t \in [0,T]$.

Then there is a solution $x(\cdot)$ *of an initial value problem:*

$$\begin{cases} x(t) = \varphi(t) & \text{for } t \in [-r,0], \\ \dot{x}(t) \in F(t,x_t,\dot{x}_t) + \int_0^t \phi(t,s,x_s,\dot{x}_s)\,ds & \text{for a.e. } t \in [0,T] \end{cases} \tag{1}$$

such that

$$|x(t) - y(t)| \leq \xi(t) \quad \text{for } t \in [0,T] \tag{2}$$

where $\xi(t) = \int_0^t \delta(s)\,e^{2(m(t)-m(s))}\,ds$ *and* $m(t) = \int_0^t \left(k(r) + \int_0^r \mu(r,s)\,ds\right) dr.$

Proof. Let y: $[-r,T] \to \mathbf{R}^n$ be a given absolutely continuous function satisfying (g) and (h). Since (in general) $\dot{y}(t) \notin F(t,y_t,\dot{y}_t) + \int_0^t \phi(t,s,y_s,\dot{y}_s)\,ds$ for a.e. $t \in [0,T]$, then by the measurable selection Theorem there exists a measurable function v^0 such that $v^0(t) \in F(t,y_t,\dot{y}_t) + \int_0^t \phi(t,s,y_s,\dot{y}_s)\,ds$ and $|v^0(t) - \dot{y}(t)| = \rho(\dot{y}(t), F(t,y_t,\dot{y}_t) + \int_0^t \phi(t,s,y_s,\dot{y}_s)\,ds \leq \delta(t)$ for a.e. $t \in [0,T]$.

Let us call x^1 the absolutely continuous function defined by

$$\begin{cases} x^1(t) = \varphi(t) & \text{for } t \in [-r,0], \\ x^1(t) = \varphi(0) + \int_0^t v^0(s)\,ds & \text{for } t \in [0,T] \end{cases}$$

We have $|x^1(t) - y(t)| \leq |y(0) - \varphi(0)| + \int_0^t |v^0(s) - \dot{y}(s)|\,ds \leq \int_0^t \delta(s)\,ds$ for $t \in [0,T]$.

We shall define now a sequence of absolutely continuous functions (x^i) in the following way

$$\begin{cases} x^i(t) = \varphi(t) \quad \text{for} \quad t \in [-r, 0], \\ x^i(t) = \varphi(0) + \int_0^t v^{i-1}(s)\,ds \quad \text{for a.e. } t \in [0, T]. \end{cases}$$

where v^{i-1} is a measurable function such that $v^{i-1} \in F\left(t, x_t^{i-1}, \dot{x}_t^{i-1}\right) + \int_0^t \phi\left(t, s, x_s^{i-1}, \dot{x}_t^{i}\right.$

and $|v^{i-1}(t) - \dot{x}^{i-1}(t)| = \rho(\dot{x}^{i-1}(t), (F(t, x_t^{i-1}, \dot{x}_t^{i-1}) + \int_0^t \phi(t, s, x_s^{i-1}, \dot{x}_s^{i-1})\,ds)$ for a.e. $t \in$

Hence and by conditions (c) and (e) for a.e. $t \in [0,T]$ we obtain

$$|\dot{x}^i(t) - \dot{x}^{i-1}(t)| = |v^{i-1}(t) - \dot{x}^{i-1}(t)| \le$$

$$\le H(F(t, x_t^{i-2}, \dot{x}_t^{i-2}) + \int_0^t \phi(t, s, x_s^{i-2}, \dot{x}_s^{i-2})\,ds, \quad F(t, x_t^{i-1}, \dot{x}_t^{i-1}) +$$

$$+ \int_0^t \phi(t, s, x_s^{i-1}, \dot{x}_s^{i-1})\,ds) \le H(F(t, x_t^{i-2}, \dot{x}_t^{i-2}), \ F(t, x_t^{i-1}, \dot{x}_t^{i-1})) +$$

$$+ \int_0^t |\phi(t, s, x_s^{i-2}, \dot{x}_s^{i-2}) - \phi(t, s, x_s^{i-1}, \dot{x}_s^{i-1})|\,ds \le k(t)(\|x_t^{i-2} - x_t^{i-1}\|_0 +$$

$$+ |\dot{x}_t^{i-2} - \dot{x}_t^{i-1}|_0) + \int_0^t \mu(t,s)(\|x_s^{i-2} - x_s^{i-1}\|_0 + |\dot{x}_s^{i-2} - \dot{x}_s^{i-1}|_0)\,ds \le$$

$$\le k(t)(\|x^{i-2} - x^{i-1}\|_{[0,t]} + |\dot{x}^{i-2} - \dot{x}^{i-1}|_{[0,t]}) + \int_0^t \mu(t,s)(\|x^{i-2} - x^{i-1}\|_{[0,s]} +$$

$$+ |\dot{x}^{i-2} - \dot{x}^{i-1}|_{[0,s]})\,ds,$$

where $\|x\|_{[0,t]} = \sup_{0 \le \tau \le t} |x(\tau)|$ and $|y|_{[0,t]} = \int_0^t |y(\tau)|\,d\tau$.

By the definition of x^2 we have

$$\begin{cases} x^2(t) = \varphi(t) \quad \text{for } t \in [-r, 0], \\ x^2(t) = \varphi(0) + \int_0^t v^1(s)\,ds \quad \text{for a.e. } t \in [0, T]. \end{cases}$$

Therefore $\dot{x}^2(t) = v^1(t) \in F(t, x_t^1, \dot{x}_t^1) + \int_0^t \phi(t, s, x_s^1, \dot{x}_s^1)\,ds$ and

$$|\dot{x}^2(t) - \dot{x}^1(t)| = \rho(\dot{x}^1(t), F(t, x_t^1, \dot{x}_t^1) + \int_0^t \phi(t, s, x_s^1, \dot{x}_s^1)\, ds \le$$

$$\le H(F(t, y_t, \dot{y}_t) + \int_0^t \phi(t, s, y_s, \dot{y}_s)\, ds,\ F(t, x_t^1, \dot{x}_t^1) + \int_0^t \phi(t, s, x_s^1, \dot{x}_s^1)\, ds) \le$$

$$\le k(t)(\|y_t - x_t^1\|_0 + |\dot{y}_t - \dot{x}_t^1|_0) + \int_0^t \mu(t, s)(\|y_s - x_s^1\|_0 + |\dot{y}_s - \dot{x}_s^1|_0)\, ds \le$$

$$\le k(t)(\sup_{0 \le \tau \le t} |y(\tau) - x^1(\tau)| + \int_0^t |\dot{y}(\tau) - \dot{x}^1(\tau)|\, d\tau) +$$

$$+ \int_0^t \mu(t, s) \sup_{0 \le \tau \le s} |y(\tau) - x^1(\tau)|\, ds + \int_0^t \mu(t, s)(\int_0^s |\dot{y}(\tau) - \dot{x}^1(\tau)|\, d\tau)\, ds \le$$

$$\le 2k(t) \int_0^t \delta(\tau)\, d\tau + 2 \int_0^t \mu(t, s)(\int_0^s \delta(\tau)\, d\tau)\, ds \le$$

$$\le 2k(t) \int_0^t \delta(\tau)\, d\tau + 2 \int_0^t \delta(\tau)(\int_\tau^t \mu(t, s)\, ds)\, d\tau \le$$

$$\le 2\, (\int_0^t k(t)\delta(\tau)\, d\tau + \int_0^t \delta(\tau)\, [\int_0^t \mu(t,s)\, ds]\, d\tau) = 2 \int_0^t \delta(\tau)\, (k(t) + \int_0^t \mu(t,s)\, ds)\, d\tau$$

for almost every $t \in [0,T]$.

Furthermore for $t \in [0,T]$ we have

$$|x^2(t) - x^1(t)| \le \int_0^t |v^1(r) - v^0(r)|\, dr \le 2 \int_0^t (\int_0^r \delta(\tau)(k(r) + \int_0^r \mu(r,s)\, ds)\, d\tau)\, dr \le$$

$$\le 2 \int_0^t \delta(\tau)\, (\int_\tau^t [\, k(r) + \int_0^r \mu(r,s)\, ds\,]\, dr\,)\, d\tau = 2 \int_0^t \delta(\tau)\, [\, m(t) - m(\tau)\,]\, d\tau$$

Using the induction we can show that for every $i \ge 2$

$$|\dot{x}^i(t) - \dot{x}^{i-1}(t)| \le k(t)\, 2^{i-1} \int_0^t \delta(\tau)\, \frac{[\, m(t) - m(\tau)\,]^{i-2}}{(i-2)!}\, d\tau \tag{3}$$

for a.e. $t \in [0,T]$ and

$$|x^i(t) - x^{i-1}(t)| \le \frac{2^{i-1}}{(i-1)!} \int_0^t \delta(\tau)\, [\, m(t) - m(\tau)]^{i-1}\, d\tau \text{ for } t \in [0,T]. \tag{4}$$

Assume we have defined our functions x^i up to i n. Let us consider $x^n(\cdot)$. By measurability of a multivalues mapping $F(\cdot,u,v)$ and by assumption (d) there exists a measurable function v^n such that

$$v^n(t) \in F(t, x_t^n, \dot{x}_t^n) + \int_0^t \phi(t, s, x_s^n, \dot{x}_s^n)\, ds \text{ and } |v^n(t) - \dot{x}^n(t)| =$$
$$= \rho(\dot{x}^n(t), F(t, x_t^n, \dot{x}_t^n) + \int_0^t \phi(t, s, x_s^n, \dot{x}_s^n)\, ds) \text{ for a.e. } t \in [0,T].$$

Define now x^{n+1} by setting

$$\begin{cases} x^{n+1}(t) = \varphi(t) & \text{for } t \in [-r, 0], \\ x^{n+1}(t) = \varphi(0) + \int_0^t v^n(s)\, ds & \text{for a.e. } t \in [0, T]. \end{cases} \tag{5}$$

We have

$$|\dot{x}^{n+1}(t) - \dot{x}^n(t)| \le k(t)\, 2^n \int_0^t \delta(\tau) \frac{[m(t) - m(\tau)]^{n-1}}{(n-1)!} d\tau \text{ for a.e. } t \in [0, T].$$

We obtain

$$|x^{n+1}(t) - y(t)| \le |x^{n+1}(t) - x^n(t)| + |x^n(t) - x^{n-1}(t)| + \ldots + |x^1(t) - y(t)| \le$$
$$\le \int_0^t \delta(\tau) e^{2[m(t) - m(\tau)]} d\tau \le \xi(t) \text{ for } t \in [0, T].$$

The inequality (4) imply that (x^n) is a Cauchy sequence of $C([0, T], \mathbb{R}^n)$. Let $x = \lim_{n\to\infty} x^n$. Similarly from (3) it follows that (v^n) converges pointwise almost everywhere to a measurable function v. Hence, passing to the limit as $n \to \infty$ in (5) we get

$$\begin{cases} x(t) = \varphi(t) \text{ for } t \in [-r, 0], \\ x(t) = \varphi(0) + \int_0^t v(s)\, ds \text{ for a.e. } t \in [0, T]. \end{cases}$$

But for almost every $t \in [0, T]$, $v^n(t) \in F(t, x_t^n, \dot{x}_t^n) + \int_0^t \phi(t, s, x_s^n, \dot{x}_s^n)\, ds$ and

$$|v^n(t) - \dot{x}^n(t)| = \rho(\dot{x}^n(t), F(t, x_t^n, \dot{x}_t^n) + \int_0^t \phi(t, s, x_s^n, \dot{x}_s^n)\, ds).$$

Therefore for a.e. $t \in [0,T]$ we have

$$|v^n(t) - \dot{x}^n(t)| \le \rho(\dot{x}^n(t), F(t, x_t, \dot{x}_t) + \int_0^t \phi(t, s, x_s, \dot{x}_s)\, ds) +$$
$$+ H(F(t, x_t, \dot{x}_t) + \int_0^t \phi(t, s, x_s, \dot{x}_s)\, ds,\ F(t, x_t^n, \dot{x}_t^n) + \int_0^t \phi(t, s, x_s^n, \dot{x}_s^n)\, ds).$$

Hence we obtain

$$\begin{cases} x(t) = \varphi(t) \text{ for } t \in [-r, 0], \\ \dot{x}(t) \in F(t, x_t, \dot{x}_t) + \int_0^t \phi(t, s, x_s, \dot{x}_s)\, ds \text{ for a.e. } t \in [0, T], \end{cases}$$

which completes the proof.

3. The Bogolubov's type theorem. In this part we will study integral-differential inclusions of the form

$$\begin{cases} x(t) = \varphi(t) \text{ for } t \in [-r, 0], \\ \dot{x}(t) \in \varepsilon F(t, x_t, \dot{x}_t) + \varepsilon \int_0^t \phi(t, s, x_s, \dot{x}_s)\, ds \text{ for a.e. } t \geq 0, \end{cases} \tag{6}$$

where F: $[0, \infty) \times C_0 \times L_0 \to \text{Conv } \mathbf{R}^n$, φ: $[-r, 0] \to \mathbf{R}^n$ is a given absolutely continuous function, ϕ: $[0, \infty) \times [0, \infty) \times C_0 \times L_0 \to \mathbf{R}^n$ and $\varepsilon > 0$ is a small parameter.

Assume that together with the conditions (a) - (b), (d) - (f) mappings F: $[0, \infty) \times C_0 \times L_0 \to \text{Conv } \mathbf{R}^n$ and ϕ: $[0, \infty) \times [0, \infty) \times C_0 \times L_0 \to \mathbf{R}^r$ satisfy the following conditions:

(i) there exists a limit

$$\lim_{L \to \infty} H\left(\frac{1}{L}\int_0^L F(s, u, v)\, ds,\ F_0(u, v)\right) = 0$$

uniformly with respect to $(u, v) \in C_0 \times L_0$, where the integral is mean in Aumann's -Hukuhara's sense.

(j) there exists a function ϕ_0: $[0, \infty) \times C_0 \times L_0 \to \mathbf{R}^n$ such that

$$\lim_{L \to \infty} \frac{1}{L} \int_0^L dt \int_0^t [\phi(t, s, u, v) - \phi_0(t, u, v)]\, ds = 0,$$

(k) $\forall\, t_1, t_2,\ t_2 \geq t_1 \geq 0$ we have $\int_{t_1}^{t_2} dt \int_0^t \mu(t, s) ds \leq k_2(t_2 - t_1)$ and

$$\int_{t_1}^{t_2} dt \int_0^t \mu(t,s)(t-s)\,ds \le k_3(t_2-t_1) \text{ where } \mu \text{ is a function of (e),}$$

(l) $\phi_0(t,\cdot,\cdot): C_0 \times L_0 \to \mathbf{R}^n$ satisfies for fixed $t \in [0,\infty)$ the Lipschitz conditio

of the form

$$|\phi_0(t,u,v) - \phi_0(t,\bar{u},\bar{v})| \le \mu_0(t)(\|u-\bar{u}\|_0 + |v-\bar{v}|_0)$$

where $\mu_0: [0,\infty) \to \mathbf{R}^+$ is a Lebesque integrable function and moreov

$\forall\ t_1, t_2,\ t_2 \ge t_1 \ge 0$ we have

$$\int_{t_1}^{t_2} t\mu_0(t)\,dt \le \mu_1(t_2-t_1) \text{ and } \int_{t_1}^{t_2} t^2\mu_0(t)\,dt \le \mu(t_2-t_1),$$

(m) there exists a function $\lambda_1: [0,\infty) \to \mathbf{R}^+$ such that $|\phi_0(t,u,v)| \le \lambda_1(t)$ an

$\forall\ t_1, t_2,\ t_2 \ge t_1 \ge 0$ we have $\int_{t_1}^{t_2} t\lambda_1(t)\,dt \le \lambda_2(t_2-t_1)$.

Furthermore the mapping F satisfies the Lipschitz conditions (c) with a number $k>0$.

We shall consider (6) together with the middling inclusions

$$\begin{cases} y(t) = \varphi(t) \quad \text{for } t \in [-r,0], \\ \dot{y}(t) \in \varepsilon F_0(y_t,\dot{y}_t) + \varepsilon \int_0^t \phi_0(t,y_s,\dot{y}_s)\,ds \text{ for a.e. } t \ge 0. \end{cases} \quad (7)$$

The main result of this paper is contained in a following theorem:

THEOREM 2. *Let* $F: [0,\infty) \times C_0 \times L_0 \to \text{Conv}\,\mathbf{R}^n$, $\phi: [0,\infty) \times [0,\infty] \times C_0 \times L_0 \to \mathbf{R}$ *and* $\phi_0: [0,\infty) \times C_0 \times L_0 \to \mathbf{R}^n$ *satisfy the conditions* (a)-(f) *and* (i)-(m). *Suppose that given a problems* (6) *and* (7) *together with the initial conditions* $x(t) = y(t) = \varphi(t) = const$ *for* $t \in [-r,0]$. *Then, for each* $\eta > 0$ *and* $T > 0$ *there exists a* $\varepsilon^0(\eta,T)$ *such that for every* $\varepsilon \in (0,\varepsilon^0$ *the following conditions are satisfied:*

(1°) *for each solution* $x(\cdot)$ *of* (6) *there exists a solution* $y(\cdot)$ *of* (7) *such that*

$$|x(t) - y(t)| \le \eta \quad \text{for } t \in [-r, \frac{T}{\varepsilon}], \quad (8$$

(2°) *for each solution* $y(\cdot)$ *of* (7) *there exists a solution* $x(\cdot)$ *of* (6) *such that* (8 *holds.*

Proof. In a similar as in ([4]) it can be proved that $F_0: C_0 \times L_0 \to \operatorname{Conv}\mathbf{R}^n$ is a boundary mapping and satisfies the Lipschitz condition with a number $k > 0$.

Let $x(\cdot)$ be a solution of (6). Then

$$\begin{cases} x(t) = \text{const} \quad \text{for } t \in [-r, 0], \\ x(t) = \varphi(0) + \varepsilon \int_0^t v(\tau)\, d\tau \quad \text{for } t \in [0, T/\varepsilon] \end{cases}$$

where $v(t) \in F(t, x_t, \dot{x}_t) + \int_0^t \phi(t, s, x_s, \dot{x}_s)\, ds$. To prove this theorem we shall consider the solution $y(\cdot)$ of the inclusion (7) in such a way that, for $t \in [-r,0]$, $x(t) = y(t) = \text{const}$ hence $|x(t) - y(t)| = 0 \le \eta$. We will prove inequality (8) on the interval $[0, T/\varepsilon]$. To do this let us divide the interval $[0, T/\varepsilon]$ on m-subintervals $[t_i, t_{i+1}]$, where $t_i = \dfrac{iT}{\varepsilon m}$, $i = 0, 1, \dots, m-1$, and write a solution $x(\cdot)$ in the form

$$\begin{cases} x(t) = \text{const} \quad \text{for } t \in [-r, 0], \\ x(t) = x(t_1) + \varepsilon \int_{t_1}^t v(\tau)\, d\tau \quad \text{for } t \in [t_1, t_{i+1}]. \end{cases} \tag{9}$$

Let us consider a function $x^1(\cdot)$ defined by

$$\begin{cases} x^1(t) = \text{const} \quad \text{for } t \in [-r, 0], \\ x^1(t) = x^1(t_i) + \varepsilon \int_{t_i}^t v^1(\tau)\, d\tau \quad \text{for } t \in [t_i, t_{i+1}] \end{cases} \tag{10}$$

where $v^1(\cdot)$ are measurable functions such that $v^1(t) \in F(t, x^1_{t_i}, \dot{x}^1_{t_i}) + \int_0^t \phi(t, s, x^1_{t_i}, \dot{x}^1_{t_i})\, ds$ and

$$|v^1(t) - v(t)| = \rho\Big(v(t), F(t, x^1_{t_i}, \dot{x}^1_{t_i}) + \int_0^t \phi(t, s, x^1_{t_i}, \dot{x}^1_{t_i})\, ds\Big) =$$

$$= \min_{z(t) \in F(t, x^1_{t_i}, \dot{x}^1_{t_i}) + \int_0^t \phi(t, s, x^1_{t_i}, \dot{x}^1_{t_i})\, ds} |z(t) - v(t)|.$$

Mapping v^1 exists because set-valued function $F(t, x_{t_i}^1, \dot{x}_{t_i}) + \int_0^t \phi(t, s, x_{t_i}^1, \dot{x}_{t_i}^1)\, ds$ is measurable and has compact and convex values ([3]).

By virtue of (9) and (10) for every $t \in [t_i, t_{i+1}]$, $s \le t$, we have

$$|x(t) - x^1(t_i)| \le |x(t_i) - x^1(t_i)| + \varepsilon \int_{t_i}^t |v(\tau)|\, d\tau \le \le \delta_i + \varepsilon(M + k_1)(t - t_i), \text{ where } \delta_i = |x(t_i) - x^1(t_i)|,\ i = 0, \dots, m-1, \tag{11}$$

and

$$|x(t) - x(s)| \le \varepsilon(M + k_1)(t - s), \tag{12}$$

$$|x(t) - x(t_i)| \le \varepsilon \int_{t_i}^t |v(\tau)|\, d\tau \le \varepsilon(M + k_1)(t - t_i) \le (M + k_1)\, T/m, \tag{13}$$

$$|x^1(t) - x^1(t_i)| \le (M + k_1)\, T/m. \tag{14}$$

By the properties of mappings F and ϕ we have

$$|v(t) - v^1(t)| \le$$

$$\le H(F(t, x_t, \dot{x}_t) + \int_0^t \phi(t, s, x_s, \dot{x}_s)\, ds,\ F(t, x_{t_i}, \dot{x}_{t_i}^1) + \int_0^t \phi(t, s, x_{t_i}^1, \dot{x}_{t_i}^1)\, ds) \le$$

$$\le H(F(t, x_t, \dot{x}_t), F(t, x_{t_i}^1, \dot{x}_{t_i}^1)) + \int_0^t |\phi(t, s, x_s, \dot{x}_s) - \phi(t, s, x_{t_i}^1, \dot{x}_{t_i}^1)|\, ds \le$$

$$\le k[\|x_t - x_{t_i}^1\|_0 + |\dot{x}_t - \dot{x}_{t_i}^1|_0] + \int_0^t \mu(t, s)(\|x_{t_i}^1 - x_s\|_0 + |\dot{x}_{t_i}^1 - \dot{x}_s|_0)\, ds.$$

By the definition of $x(\cdot)$ and virtue of (11), (12) it follows that

$$\|x_t - x_{t_i}^1\|_0 \le \|x_t - x_{t_i}\|_0 + \|x_{t_i} - x_{t_i}^1\|_0 =$$

$$= \sup_{-r \le s \le 0} |x(t+s) - x(t+s)| + \sup_{-r \le s \le 0} |x(t+s) - x^1(t+s)| \le$$

$$\le \sup_{-r \le s \le 0} \varepsilon \int_{t_i+s}^{t+s} |v(\tau)|\, d\tau + \sup_{t_i - r \le \tau \le t_i} (|x(t_i) - x^1(t_i)| + \varepsilon \int_{\tau}^{t_i} |v^1(\tau) - v(\tau)|\, d\tau) \le$$

$$\le \varepsilon(M + k_1)|t - t_i| + \delta_i + 2\varepsilon(M + k_1)|t_i - \tau| \le [(M + k_1)T]/m + \delta_i + 2\varepsilon(M + k_1)r, \tag{15}$$

where $t \in [t_i, t_{i+1}]$, Furthermore

$$|\dot{x}_t - \dot{x}^1_{t_i}|_0 = \int_{-r}^{0} |\dot{x}(t+s) - \dot{x}^1(t_i+s)|\, ds =$$

$$= \begin{cases} 0, \text{ for every } (t+s), (t_i+s) \in [-r, 0], \\ \varepsilon \int_{-r}^{0} |v(t+s)|\, ds \le \varepsilon (M+k_1) r, \text{ for } (t_i+s) \in [-r, 0], (t+s) \in [0, T/\varepsilon], \\ \varepsilon \int_{-r}^{0} |v(t+s) - v^1(t_i+s)|\, ds \le 2\varepsilon (M+k_1) r, \text{ for e. } (t+s), (t_i+s) \in [0, T/\varepsilon]. \end{cases} \tag{16}$$

Now, by virtue of (15) and (16), respectively, we have for $t_i \le s \le t$

$$\begin{aligned} \|x^1_{t_i} - x_s\|_0 &\le \|x^1_{t_i} - x_t\|_0 + \|x_t - x_s\|_0 \le \\ &\le [(M+k_1)T]/m + \delta_i + 2\varepsilon (M+k_1) r + \varepsilon (M+k_1)(t-s), \end{aligned} \tag{17}$$

and
$$|\dot{x}^1_{t_i} - \dot{x}_s|_0 = \int_{-r}^{0} |\dot{x}^1(t_i+\tau) - \dot{x}(s+\tau)|\, d\tau \le$$

$$= \begin{cases} 0, \text{ for every } (t_i+\tau), (s+\tau) \in [-r, 0], \\ \varepsilon (M+k_1) r, \text{ for } (t_i+\tau) \in [-r, 0], (s+\tau) \in [0, T/\varepsilon], \\ 2\varepsilon (M+k_1) r, \text{ for every } (t_i+\tau), (s+\tau) \in [0, T/\varepsilon]. \end{cases} \tag{18}$$

Hence, in virtue of (15), (16), (17) and (18) we have

$$\begin{aligned} |v(t) - v^1(t)| &\le k(\delta_i + [(M+k_1)T]/m + 4\varepsilon (M+k_1) r) + \\ &+ \int_0^t \mu(t,s)(\delta_i + [(M+k_1)T]/m + 4\varepsilon (M+k_1) r + \varepsilon (M+k_1)(t-s))\, ds \le \\ &[\delta_i + [(M+k_1)T]/m + 4\varepsilon (M+k_1) r][k + \int_0^t \mu(t,s)\, ds] + \varepsilon (M+k_1) \int_0^t \mu(t,s)(t-s)\, ds. \end{aligned} \tag{19}$$

Now, profit from (9), (10) and (19) we obtain

$$\begin{aligned} \delta_{i+1} &= |x(t_{i+1}) - x^1(t_{i+1})| \le |x(t_i) - x^1(t_i)| + \varepsilon \int_{t_i}^{t_{i+1}} |v(\tau) - v^1(\tau)| \le \\ &\le \delta_i + \varepsilon (t_{i+1} - t_i)([\delta_i + [(M+k_1)T]/m + 4\varepsilon (M+k_1) r][k + \int_0^t \mu(t,s)\, ds] + \\ &+ \varepsilon (M+k_1) \int_0^t \mu(t,s)(t-s)\, ds) \le \\ &\le \delta_i + \frac{T}{m}([\delta_i + [(M+k_1)T]/m + 4\varepsilon (M+k_1) r][k+k_2] + \varepsilon (M+k_1) k_3) = \\ &= \delta_i (1 + [(k+k_2)T]/m) + \frac{T}{m}(M+k_1)[(k+k_2)(\frac{T}{m} + 4\varepsilon r) + \varepsilon k_3] \le \delta_i (1 + \frac{a}{m} t) + \frac{b}{m}, \end{aligned}$$

where $(k+k_2)T = a$ and $b = [(M+k_1)T]/m\,[(k+k_2)(T+4\varepsilon mr) + \varepsilon mk_3]$.

Then

$$\delta_{i+1} \le \delta_i\left(1+\frac{a}{m}\right) + \frac{b}{m} \le \left(1+\frac{a}{m}\right)\left[\delta_{i-1}\left(1+\frac{a}{m}\right)+\frac{b}{m}\right]+\frac{b}{m} \le \ldots \le$$

$$\le \left(1+\frac{a}{m}\right)^{i+1}\delta_0 + \left(1+\frac{a}{m}\right)^{i}\frac{b}{m} + \ldots + \frac{b}{m} \le$$

$$\le \frac{b}{m}\left(1+\left(1+\frac{a}{m}\right)+\left(1+\frac{a}{m}\right)^2+\ldots+\left(1+\frac{a}{m}\right)^i\right) \le \frac{b}{m}\left[\left(1+\frac{a}{m}\right)^{i+1}-1\right] \le$$

$$\le \frac{b}{a}\left(e^{a}-1\right) = g(m,\varepsilon)\left[e^{(k+k_2)T}-1\right],$$

where $g(m,\varepsilon) = \dfrac{M+k_1}{m(k+k_2)}\left[(k+k_2)(T+4\varepsilon mr)+\varepsilon mk_3\right]$. (20)

Hence, for $t \in [t_i, t_{i+1}]$, by virtue of the inequality (13), (14) and (20) we have

$$|x(t)-x^1(t)| \le |x(t)-x(t_i)| + |x(t_i)-x^1(t_i)| + |x^1(t_i)-x^1(t)| \le$$
$$\le \frac{2(M+k_1)T}{m} + g(m,\varepsilon)\left(e^{(k+k_2)T}-1\right). \qquad (21)$$

Now we shall consider the function

$$\begin{cases} \xi(t) = \text{const} \quad \text{for } t \in [-r,0], \\ \xi(t) = \xi(t_i) + \varepsilon\displaystyle\int_{t_i}^{t} w(\tau)d\tau \ \text{ for } t \in [t_i, t_{i+1}], \end{cases}$$

where $t_i = \dfrac{iT}{\varepsilon m}$, $i = 0, 1, \ldots, m-1$, $w(\cdot)$ are measurable functions such that

$$|v^1(t) - w(t)| = \min_{z(t)\in F_0(x^1_{t_i},\dot{x}^1_{t_i}) + t\phi_0(t,x^1_{t_i},\dot{x}^1_{t_i})} |v^1(t) - z(t)|.$$

Measurable mappings w exists because set-valued $F_0(x^1_{t_i},\dot{x}^1_{t_i}) + t\phi_0(t,x^1_{t_i},\dot{x}^1_{t_i})$ has compact and convex values ([3]).

Let us notice that by virtue of conditions (i) and (j) for each $\eta > 0$ there exists a $L_0(\eta)$ such that for every $L > L_0$ we have inequalities

$$H\Big(\frac{1}{L}\int_0^L [F(\tau, x_{t_i}^1, \dot{x}_{t_i}^1) + \int_0^\tau \phi(\tau, s, x_{t_i}^1, \dot{x}_{t_i}^1)\, ds]\, d\tau,$$
$$\frac{1}{L}\int_0^L [F_0(x_{t_i}^1, \dot{x}_{t_i}^1) + \tau\phi_0(\tau, x_{t_i}^1, \dot{x}_{t_i}^1)]\, d\tau\Big) \le \eta .$$

Therefore

$$\frac{1}{L}\Big(H\Big(\int_0^L [F(\tau, x_{t_i}^1, \dot{x}_{t_i}^1) + \int_0^\tau \phi(\tau, s, x_{t_i}^1, \dot{x}_{t_i}^1)\, ds]\, d\tau,$$
$$\int_0^L [F_0(x_{t_i}^1, \dot{x}_{t_i}^1) + \tau\phi(\tau, x_{t_i}^1, \dot{x}_{t_i}^1)]\, d\tau\Big) \le \eta.$$

In particular, for $\frac{T}{\varepsilon m} > L_0$ and for every $i \in \{0, 1, ..., m\text{-}1\}$

(·)
$$H\Big(\int_0^{\frac{iT}{\varepsilon m}} [F(\tau, x_{t_i}^1, \dot{x}_{t_i}^1) + \int_0^\tau \phi(\tau, s, x_{t_i}^1, \dot{x}_{t_i}^1)\, ds]\, d\tau,$$
$$\int_0^{\frac{iT}{\varepsilon m}} \Big[F_0(x_{t_i}^1, \dot{x}_{t_i}^1) + \tau\phi_0(\tau, x_{t_i}^1, \dot{x}_{t_i}^1)\Big]\, d\tau\Big) \le \eta\frac{iT}{\varepsilon m},$$

and

(:)
$$H\Big(\int_0^{\frac{(i+1)T}{\varepsilon m}} \Big[F(\tau, x_{t_i}^1, \dot{x}_{t_i}^1) + \int_0^\tau \phi(\tau, s, x_{t_i}^1, \dot{x}_{t_i}^1)\, ds\Big]\, d\tau,$$
$$\int_0^{\frac{(i+1)T}{\varepsilon m}} \Big[F_0(x_{t_i}^1, \dot{x}_{t_i}^1) + \tau\phi_0(\tau, x_{t_i}^1, \dot{x}_{t_i}^1)\Big]\, d\tau\Big) \le \eta\frac{(i+1)T}{\varepsilon m}.$$

Let us observe that $\frac{(i+1)T}{\varepsilon m} = t_{i+1}$ and $\frac{iT}{\varepsilon m} = t_i$. By virtue of (.), (:) and the Hausdorff metric condition ([2]) we have

$$H\Big(\int_{t_i}^{t_{i+1}} [F(\tau, x_{t_i}^1, \dot{x}_{t_i}^1) + \int_0^\tau \phi(\tau, s, x_{t_i}^1, \dot{x}_{t_i}^1)\, ds]\, d\tau,$$
$$\int_{t_i}^{t_{i+1}}\Big[F_0(x_{t_i}^1, \dot{x}_{t_i}^1) + \tau\phi_0(\tau, x_{t_i}^1, \dot{x}_{t_i}^1)\Big]\, d\tau\Big) \le \frac{iT}{\varepsilon m}\eta + \frac{(i+1)T}{\varepsilon m}\eta \le \frac{T}{\varepsilon m}\eta(2m+1).$$

Hence

$$H\Big(\frac{\varepsilon m}{T}\int_{t_i}^{t_{i+1}} [F(\tau, x_{t_i}^1, \dot{x}_{t_i}^1) + \int_0^\tau \phi(\tau, s, x_{t_i}^1, \dot{x}_{t_i}^1)\, ds]\, d\tau,$$

$$\frac{\varepsilon m}{T}\int_{t_i}^{t_{i+1}}\left[F_0(x_{t_i}^1,\dot{x}_{t_i}^1)+\tau\phi_0(\tau,x_{t_i}^1,\dot{x}_{t_i}^1)\right]d\tau)\le(2m+1)\eta \text{ for } \frac{T}{\varepsilon m}>L_0(\eta).$$

Therefore

$$H(\frac{\varepsilon m}{T}\int_{t_i}^{t_{i+1}}[F(\tau,x_{t_i}^1,\dot{x}_{t_i}^1)+\int_0^{\tau}\phi(\tau,s,x_{t_i}^1,\dot{x}_{t_i}^1)\,ds]\,d\tau,$$

$$F_0(x_{t_i}^1,\dot{x}_{t_i}^1)+\frac{\varepsilon m}{T}\int_{t_i}^{t_{i+1}}\tau\phi(\tau,x_{t_i}^1,\dot{x}_{t_i}^1)\,d\tau)\le\eta_1=(2m+1)\eta,$$

$$\text{for } \frac{T}{\varepsilon m}>L_0\left(\frac{\eta_1}{2m+1}\right), \text{ then for } \varepsilon<\varepsilon^0(\eta_1,m)=\frac{T}{mL_0(\eta_1/2m+1)}.$$

Hence, it follows that

$$\frac{\varepsilon m}{T}\int_{t_i}^{t_{i+1}}|v^1(\tau)-w(\tau)|\,d\tau\le\eta_1$$

and

$$|x^1(t_{i+1})-\xi(t_{i+1})|\le|x^1(t_i)-\xi(t_i)|+\varepsilon\int_{t_i}^{t_{i+1}}|v^1(\tau)-w(\tau)|\,d\tau\le$$

$$\le\ldots\le m\varepsilon\frac{T}{\varepsilon m}\eta_1+\eta_1 T, \text{ where } i=0,1,\ldots,m-1. \tag{22}$$

Using the inequality (22) and the fact that for $t\in[t_i,t_{i+1}]$, $|\xi(t)-\xi(t_i)|\le(M+k_1)\frac{T}{m}$ and $|x^1(t)-x^1(t_i)|\le(M+k_1)\frac{T}{m}$,

$$|x^1(t)-\xi(t)|\le|x^1(t)-x^1(t_i)|+|x^1(t_i)-\xi(t_i)|+|\xi(t_i)-\xi(t)|\le$$

$$\le 2(M+k_1)\frac{T}{m}+\eta_1 T. \tag{23}$$

Moreover using the definition of the mapping ξ, similary, as in the proof of the inequality (12) and (13) we obtain

$$|\xi(t_i)-\xi(s)|\le|\xi(t_i)-\xi(t)|+|\xi(t)-\xi(s)|\le(M+k_1)\left[\frac{T}{m}+\varepsilon(t-s)\right] \tag{24}$$

Because the mapping F_0 satisfies the Lipschitz condition with a number $k>0$ then by assumption (1) it follows that

$$H(F_0(\xi_t,\dot{\xi}_t)+\int_0^t\phi_0(t,\xi_s,\dot{\xi}_s)\,ds,\ F_0(x_{t_i}^1,\dot{x}_{t_i}^1)+t\phi_0(t,x_{t_i}^1,\dot{x}_{t_i}^1))\le$$

$$\leq k\left(\| \xi_t - x^1_{t_i} \|_0 + | \dot{\xi}_t - \dot{x}^1_{t_i} |_0\right) + \int_0^t \mu_0(t)\left(\| \xi_s - x^1_{t_i} \|_0 + | \dot{\xi}_s - \dot{x}^1_{t_i} |_0\right) ds.$$

Similarly, as in the proof of the inequalities (15), (16), (17), (18) using the definitions of the norm $\|\cdot\|_0$ and $|\cdot|_0$ and the definition of the mappings $\xi(\cdot)$ and $x^1(\cdot)$ and making use of the inequality (22) and (24) we obtain

$$\| \xi_t - x^1_{t_i} \|_0 \leq \| \xi_t - \xi_{t_i} \|_0 + \| \xi_{t_i} - x^1_{t_i} \|_0 \leq$$

$$\leq \sup_{-r \leq s \leq 0} \varepsilon \int_{t_i+s}^{t+s} | w(\tau) | \, d\tau + \sup_{t_i - r \leq \tau \leq t_i} \left(| \xi(t_i) - x^1(t_i) | + \varepsilon \int_{t_i}^{\tau} | w(\tau) - v^1(\tau) | \, d\tau \right) \leq$$

$$\leq \frac{(M + \lambda_2)T}{m} + \eta_1 T + \varepsilon(2M + \lambda_2 + k_1)r.$$

and

$$| \dot{\xi}_t - \dot{x}^1_{t_i} |_0 = \int_{-r}^{0} | \dot{\xi}(t+s) - \dot{x}^1(t_i+s) | \, ds =$$

$$= \begin{cases} 0 \quad \text{for } (t+s), (t_i+s) \in [-r, 0], \\ \varepsilon \int_{-r}^{0} |w(t+s)| \, ds \leq \varepsilon (M + \lambda_2) r \quad \text{for } (t_i+s) \in [-r, 0], (t+s) \in [0, T/\varepsilon], \\ \varepsilon \int_{-r}^{0} |w(t+s) - v^1(t_i+s)| \, ds \leq \varepsilon (2M + \lambda_2 + k_1) r \text{ for } (t+s), (t_i+s) \in [0, T/\varepsilon]. \end{cases}$$

Furthermore, for $t_i \leq s \leq t$,

$$\| \xi_s - x^1_{t_i} \|_0 \leq \| \xi_s - \xi_{t_i} \|_0 + \| \xi_{t_i} - x^1_{t_i} \|_0 \leq$$

$$\leq (M + k_1)\left[\frac{T}{m} + \varepsilon(t - s)\right] + \eta_1 T + \varepsilon(2M + \lambda_2 + k_1) r,$$

and

$$| \dot{\xi}_s - \dot{x}^1_{t_i} |_0 = \int_{-r}^{0} | \dot{\xi}(s+\tau) - \dot{x}^1(t_i+\tau) | \, d\tau =$$

$$\leq \begin{cases} 0 \quad \text{for } (t_i+\tau), (\tau+s) \in [-r, 0], \\ \varepsilon (M + \lambda_2) r \quad \text{for } (t_i+\tau) \in [-r, 0], (s+\tau) \in [0, T/\varepsilon], \\ \varepsilon (2M + \lambda_2 + k_1) r \quad \text{for } (t_i+\tau), (s+\tau) \in [0, T/\varepsilon]. \end{cases}$$

Hence

$$H(F_0(\xi_t,\dot\xi_t) + \int_0^t \phi_0(t,\xi_s,\dot\xi_s)\,ds,\ F_0(x_t^1,\dot x_t^1) + t\phi_0(t,x_t^1,\dot x_t^1)) \le$$

$$\le k\,[\frac{(M+\lambda_2)T}{m} + \eta_1 T + 2\varepsilon(2M+\lambda_2+k_1)r] +$$

$$+ \int_0^t \mu_0(t)\,[\frac{(M+k_1)T}{m} + \eta_1 T + \varepsilon(t-s)(M+k_1) + 2\varepsilon(2M+\lambda_2+k_1)r]\,ds \le$$

$$\le 2\varepsilon r(k+t\mu_0(t))(2M+\lambda_2+k_1) + kT(\frac{M+\lambda_2}{m}+\eta_1) + t\mu_0(t)T(\frac{M+k_1}{m}+\eta_1) +$$

$$+ \frac{1}{2}\varepsilon t^2\mu_0(t)(M+k_1) = b(m,\varepsilon,\eta_1,t). \qquad (25)$$

Now, in virtue of inequality (25) and using the definition $\xi(\cdot)$ we have

$$\rho\left(\dot\xi(t),\ \varepsilon F_0(\xi_t,\dot\xi_t) + \varepsilon_0\int_0^t \phi_0(t,\xi_s,\dot\xi_s)\,ds\right) \le$$

$$\le \rho\left(\dot\xi(t),\ \varepsilon F_0(x_t^1,\dot x_t^1) + \varepsilon\int_0^t \phi_0(t,x_t^1,\dot x_t^1)\,ds\right) +$$

$$+ H\left(\varepsilon F_0(x_t^1,\dot x_t^1) + \varepsilon\int_0^t \phi_0(t,x_t^1,\dot x_t^1)\,ds,\ \varepsilon F_0(\xi_t,\dot\xi_t) + \varepsilon\int_0^t \phi_0(t,\xi_s,\dot\xi_s)\,ds\right) \le$$
$$\le \varepsilon\, b(m,\varepsilon,\eta_1,t).$$

On the ground of Filippov's type theorem (see Theorem 1) there exists the solution $y(\cdot)$ of (7) that for $t \in [0, T/\varepsilon]$

$$|\xi(t) - y(t)| \le \varepsilon\int_0^t c(m,\varepsilon,\eta_1)\,e^{2\varepsilon[m(t)-m(\tau)]}\,d\tau$$

where $m(t)$ is the function from Theorem 1 and $c(m,\varepsilon,\eta_1)$ is a function which we get with $b(m,\varepsilon,\eta_1,t)$ using the assumption (1). In this case

$$m(t) = \int_0^t (k + \int_0^r \mu_0(r)\,ds)\,dr \quad \text{and} \quad c(m,\varepsilon,\eta_1) =$$

$$= 2\varepsilon r(k + \mu_1)(2M + \lambda_2 + k_1) + kT\left(\frac{M+\lambda_2}{m} + \eta_1\right) + \mu_1 T\left(\frac{M+k_1}{m} + \eta_1\right) + \frac{1}{2}\varepsilon\mu_2(M+k_1).$$

Hence

$$|\xi(t) - y(t)| \le \varepsilon\, c(m,\varepsilon,\eta_1)\int_0^t e^{2\varepsilon\int_\tau^t (k + \int_0^t \mu_0(r)\,ds)\,dr}\,d\tau \le$$

$$\le \varepsilon\, c(m,\varepsilon,\eta_1)\int_0^t e^{2\varepsilon(k+\mu_1)(t-\tau)}\,d\tau \le T\,c(m,\varepsilon,\eta_1)\,e^{2T(k+\mu_1)}. \tag{26}$$

In virtue of the inequalities (21), (23) and (26) it follows

$$|x(t) - y(t)| \le |x(t) - x^1(t)| + |x^1(t) - \xi(t)| + |\xi(t) - y(t)| \le$$

$$\le \frac{4T(M+k_1)}{m} + \eta_1 T + g(m,\varepsilon)\left[e^{(k+k_1)T} - 1\right] + T\,c(m,\varepsilon,\eta_1)\,e^{2T(k+\mu_1)}. \tag{27}$$

Therefore, there exists a m, ε and η_1 such that we get the inequality $|x(t) - y(t)| \le \eta$ for $t \in [0, T/\varepsilon]$ because $g(m,\varepsilon) \to 0$ and $c(m,\varepsilon,\eta_1) \to 0$ if $m \to \infty$ and $\varepsilon \to 0$.

Adopting now the procedure presented above we get condition (2°). It this way the proof is completed for $t \in [-r, \tau/\varepsilon]$.

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ON THE ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF SOME DIFFERENTIAL EQUATIONS OF THE FOURTH ORDER

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REZUMAT. - Asupra comportării asimptotice a soluţiilor unor ecuaţii diferenţiale de ordinul patru. În lucrare sunt date condiţii suficiente pentru ca toate soluţiile ecuaţiei (1.1) să fie uniform mărginite şi să tindă la zero când $t \to \infty$.

Abstract. The main purpose of this lecture is to give sufficient conditions, which ensure that all solutions of (1.1) are uniformly bounded and tend to zero as $t \to \infty$.

1. Introduction and statement of the result. This work deals with the asymptotic behaviour of the solutions of non-autonomous differential equations of the form

$$x^{(4)} + a(t)\varphi(\dot{x}, \ddot{x}, \dot{x})\dot{x} + b(t)f(x, \dot{x}, \ddot{x}) + c(t)g(\dot{x}) + d(t)h(x) = p(t, x, \dot{x}, \ddot{x}, \dot{x}) \quad (1.1)$$

in which the functions a, b, c, d, φ, f, g, h and p are continuous and depend only on the arguments displayed explicitly. The dots as usual indicate differentiation with respect to t. All functions and solutions are supposed to be real.

Abou-El-Ela [1] presented sufficient conditions for the uniform global asymptotic stability of the zero solution of the equation

$$x^{(4)} + f_1(\dot{x}, \ddot{x})\dot{x} + f_2(\ddot{x}) + f_3(\dot{x}) + \alpha_4 x = 0.$$

Hara [3] and Abou-El-Ela [2] investigated the asymptotic behaviour of the solutions of the equations

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$$x^{(4)} + a(t)f(\ddot{x})\dot{x} + b(t)\phi(\dot{x},\ddot{x}) + c(t)g(\dot{x}) + d(t)h(x) = p(t,x,\dot{x},\ddot{x},\dot{x})$$

and

$$x^{(4)} + a(t)f_1(\dot{x},\ddot{x})\dot{x} + b(t)f_2(\dot{x},\ddot{x}) + c(t)f_3(\dot{x}) + d(t)f_4(x) = p(t,x,\dot{x},\ddot{x},\dot{x})$$

respectively.

We shall henceforth assume that the functions a, b, c, d are positive and differentiable in $R^+=[0,\infty)$ and that the derivatives $\frac{\partial}{\partial y}\varphi(y,z,u), \frac{\partial}{\partial u}\varphi(y,z,u), \frac{\partial}{\partial x}f(x,y,z), \frac{\partial}{\partial y}(x,y,z), g'(y)$ and $h'(x)$ exist and continuous for all x, y, z and u.

The main objective of this paper is to prove the following:

THEOREM. *In addition to the basic assumption on a, b, c, d, φ, f, g, h and p suppose that:*

(i) $A \geq a(t) \geq \alpha_0 > 0, B \geq b(t) \geq b_0 > 0, C \geq c(t) \geq c_0 > 0, D \geq d(t) \geq d_0 > 0,$ for $t \in R^+$

(ii) $\varphi(y,z,u) \geq \alpha_1 > 0,$ *for all y, z and u;* $\alpha_2 > 0$, $\alpha_4 > 0$.

(iii) $g(0) = 0$ *and* $g'(y) \geq \alpha_3 > 0$ *for all y.*

(iv) *There is a finite constant* $\delta_0 > 0$ *such that*

$$a_0 b_0 c_0 \alpha_1 \alpha_2 \alpha_3 - C^2\alpha_3 g'(y) - A^2 D\alpha_1 \alpha_4 \varphi(y,z,0) \geq \delta_0 \text{ for all } y \text{ and } z.$$

(v) $$0 \leq g'(y) - \frac{g(y)}{y} \leq \delta_1 < \frac{2D\alpha_4\delta_0}{Ca_0c_0^2\alpha_1\alpha_3^2},\ y \neq 0.$$

(vi) $$\left(\frac{1}{z}\right)\int_0^z \varphi(y,\zeta,0)\,d\zeta - \varphi(y,z,0) \leq \delta_2 < \frac{2\delta_0}{Aa_0^2c_0\alpha_1^2\alpha_3} \text{ for all } y \text{ and } z \neq 0.$$

(vii) $$y\frac{\partial}{\partial y}\varphi(y,z,0) \leq 0 \text{ and } z\frac{\partial}{\partial y}\varphi(y,z,0) \leq 0 \text{ for all } y \text{ and } z.$$

(viii) $$f(x,y,0) = 0,\ \frac{\partial}{\partial y}f(x,y,z) \leq 0,\ y\int_0^z \frac{\partial}{\partial x}f(x,y,\zeta)\,d\zeta \leq 0 \text{ for all } x, y \text{ and } z \text{ and}$$

$$0 \leq \frac{f(x,y,z)}{z} - \alpha_2 \leq \frac{\varepsilon_0 c_0^3 \alpha_3^3}{BD^2\alpha_4^2}$$ *for all x,y and z ≠ 0, where* ε_0 *is a positive constant such that*

$$\varepsilon_0 < \varepsilon = \min\left[\frac{1}{a_0\alpha_1}, \frac{D\alpha_4}{c_0\alpha_3}, \frac{\delta_0}{4a_0c_0\alpha_1\alpha_3\Delta_0}, \frac{Cc_0\alpha_3}{4D\alpha_4\Delta_0}\left(\frac{2\alpha_4 D\delta_0}{Ca_0\alpha_1c_0^2\alpha_3^2} - \delta_1\right), \frac{Aa_0\alpha_1}{4\Delta_0}\left(\frac{2\delta_0}{Aa_0^2c_0\alpha_1^2\alpha_3} - \delta_2\right)\right] \quad (1.2)$$

with $\Delta_0 = \left(\dfrac{a_0b_0c_0\alpha_1\alpha_2}{C} + \dfrac{a_0b_0c_0\alpha_2\alpha_3}{AD\alpha_4}\right)$.

(ix) $h(0) = 0,\ h(x)\operatorname{sgn} x > 0\ (x \neq 0),\ H(x) \equiv \int_0^x h(\zeta)\,d\zeta \to \infty$ *as* $|x| \to \infty$ *and* $0 \le \alpha_4 - h'(x) \le \dfrac{\varepsilon\Delta_0 a_0^2\alpha_1^2}{D}$ *for all* x.

(x) $z\dfrac{\partial}{\partial u}\varphi(y,z,u) + \Delta_2 y\dfrac{\partial}{\partial u}\varphi(y,z,u) \ge 0$ *for all y,z and u, where*

$$\Delta_2 = \frac{\alpha_4 D}{c_0\alpha_3} + \varepsilon. \quad (1.3)$$

(xi) $\int_0^\infty \gamma_0(t)\,dt < \infty,\ d'(t) \to \infty$ *as* $t \to \infty$, *where*

$$y_0(t) = |a'(t)| + b'_+(t) + |c'(t)| + |d'(t)|,\ b'_+(t) = \max\{b'(t), 0\}.$$

(xii) $|p(t,x,y,z,u)| \le p_1(t) + p_2(t)\left[H(x) + y^2 + z^2 + u^2\right]^{\frac{\delta}{2}} + \Delta\left(y^2 + z^2 + u^2\right)^{\frac{1}{2}}$, *where* Δ,δ *are constants such that* $0 \le \delta \le 1$, $\Delta \ge 0$ *and* $p_1(t)$, $p_2(t)$ *are non-negative continuous functions satisfying*

$$\int_0^\infty p_i(t)\,dt < \infty \quad (i = 1,2). \quad (1.4)$$

If Δ *is sufficiently small, then every solution* $x(t)$ *of* (1.1) *is uniformly bounded and satisfies*

$$x(t) \to 0,\ \dot{x}(t) \to 0,\ \ddot{x}(t) \to 0,\ \dot{x}(t) \to 0 \text{ as } t \to \infty \quad (1.5)$$

Remark 1. When we take $a(t) = b(t) = c(t) = d(t) = 1$, and $\varphi(\dot{x},\ddot{x},\dot{x}) = f_1(\dot{x},\ddot{x})$ and $f(x,\dot{x},\ddot{x}) = f_2(\ddot{x})$, and $h(x) = \alpha_4 x$ and finally $p(t,x,\dot{x},\ddot{x},\dot{x}) = 0$, conditions (i)-(xii) of the theorem reduced to those of Abou-El-Ela [1].

Remark 2. When $\varphi(\dot{x},\ddot{x},\dot{x}) = f(\ddot{x})$, and $f(x,\dot{x},\ddot{x}) = \phi(\dot{x},\ddot{x})$, then the conditions (i)-(xii) become similar to those of Hara [3].

Remark 3. When $\varphi(\dot{x},\ddot{x},\dot{x}) = f_1(\dot{x},\ddot{x})$, and $f(x,\dot{x},\ddot{x}) = f_2(\dot{x},\ddot{x})$, then the conditions (i)-(xii) of the theorem are reduced to those of Abou-El-Ela [2].

2. The Function V_0 (t,x,y,z,u). Consider, instead of (1.1), the equivalent system

$$\begin{aligned}&\dot{x} = y,\ \dot{y} = z,\ \dot{z} = u\\ &\dot{u} = -a(t)\varphi(y,z,u)u - b(t)f(x,y,z) - c(t)g(y) - d(t)h(x) + p(t,x,y,z,u)\end{aligned} \tag{2.1}$$

derived from it by setting $y = \dot{x}$, $z = \ddot{x}$ and $u = \dddot{x}$.

The main tool, in the proof of the theorem, is the function $V_0 = V_0(t,x,y,z,u)$ defined by

$$\begin{aligned}2V_0 = {} & 2\Delta_2 d(t)\int_0^x h(\zeta)d\zeta + 2c(t)\int_0^y g(\eta)d\eta + 2\Delta_1 b(t)\int_0^z f(x,y,\zeta)d\zeta + 2a(t)\int_0^z \zeta\varphi(y,\zeta,0)d\zeta\\ & + 2\Delta_2 a(t)y\int_0^z \varphi(y,\zeta,0)d\zeta + [\Delta_2\alpha_2 b(t) - \Delta_1\alpha_4 d(t)]y^2 - \Delta_2 z^2 + \Delta_1 u^2 + 2d(t)yh(x)\\ & + 2\Delta_1 d(t)zh(x) + 2\Delta_1 c(t)zg(y) + 2\Delta_2 yu + 2zu + k\end{aligned} \tag{2.2}$$

where

$$\Delta_1 = \frac{1}{a_0\alpha_1} + \varepsilon, \tag{2.3}$$

Δ_2 being the constant by (1.3) and k is a positive constant to be determined later in the proof.

Let me first discuss some important inequalities.

Using (2.3), (i) and (ii) we have

$$\Delta_1 - \frac{1}{a(t)\varphi(y,z,0)} \geq \varepsilon \text{ for all } y,\ z \text{ and all } t \in R^+. \tag{2.4}$$

Following a similar procedure used in [2], it is also possible to show that

$$\alpha_2 b(t) - \Delta_1 c(t)g'(y) - \Delta_2 a(t)\varphi(y,z,0) \geq \frac{\delta_0}{a_0 c_0\alpha_1\alpha_3} - \varepsilon\Delta_0 \tag{2.5}$$

for all y, z and $t \in R^+$.

Let ϕ_1 be the function defined by

$$\phi_1(y,z,0) = \begin{cases}\left(\frac{1}{z}\right)\int_0^z \varphi(y,\zeta,0)\,d\zeta, & z \neq 0,\\ \varphi(y,0,0), & z = 0,\end{cases} \tag{2.6}$$

Then from (ii) and (vi) we get

$$\phi_1(y,z,0) \geq \alpha_1 > 0 \text{ for all } y \text{ and } z, \tag{2.7}$$

$$\phi_1(y,z,0) - \varphi(y,z,0) \leq \delta_2 \text{ for all } y \text{ and } z. \tag{2.8}$$

From (2.3), (2.7) and (i) we obtain

$$\Delta_1 - \frac{1}{a(t)\phi_1(y,z,0)} \geq \varepsilon \text{ for all } y,z \text{ and all } t \in R^1. \tag{2.9}$$

Since $\phi_1(y,z,0) = \varphi(y,\tilde{z},0)$, $\tilde{z} = \theta z$, $0 \leq \theta \leq 1$, then

$$\alpha_2 b(t) - \Delta_1 c(t) g'(y) - \Delta_2 a(t)\phi_1(y,z,0) \geq \frac{\delta_0}{a_0 c_0 \alpha_1 \alpha_3} - \varepsilon\Delta_0 \text{ for all } y,z \text{ and all } t \in R^1. \tag{2.10}$$

The properties of the function $V_0 = V_0(t,x,y,z,u)$ are summarized in Lemma 1 and Lemma 2.

LEMMA 1. *Suppose that the conditions* (i)-(ix) *of the theorem hold. Then there are positive constants* D_1 *and* D_2 *such that*

$$D_1[H(x) + y^2 + z^2 + u^2 + k] \leq V_0 \leq D_2[H(x) + y^2 + z^2 + u^2 + k] \tag{2.11}$$

for all x,y,z, and u.

Proof. Rewrite the function $2V_0(t,x,y,z,u)$ as follows:

$$2V_0 = \left[2\Delta_2 d(t)\int_0^x h(\zeta)d\zeta - \frac{d^2(t)h^2(x)}{c(t)\phi_3(y)}\right] + \left[\Delta_2\alpha_2 b(t) - \Delta_1\alpha_4 d(t) - \Delta_2^2 a(t)\phi_1(y,z,0)\right]y^2$$
$$+2c(t)\int_0^y g(\eta)d\eta - c(t)y^2\phi_3(y) + 2\Delta_1 b(t)\int_0^z f(x,y,\zeta)d\zeta - \left[\Delta_2 + \Delta_1^2 c(t)\phi_3(y)\right]z^2 + 2a(t)\int_0^z \zeta\varphi(y,\zeta,0)d\zeta$$
$$-a(t)z^2\phi_1(y,z,0) + \left[\Delta_1 - \frac{1}{a(t)\phi_1(y,z,0)}\right]u^2 + \frac{a(t)}{\phi_1(y,z,0)}\left[\frac{u}{a(t)} + z\phi_1(y,z,0) + \Delta_2 y\phi_1(y,z,0)\right]^2$$
$$+\frac{c(t)}{\phi_3(y)}\left[\frac{d(t)}{c(t)}h(x) + y\phi_3(y) + \Delta_1 z\phi_3(y)\right]^2 + k,$$

where the function $\phi_3(y)$ is the same as in [2].

Since $f(x,y,0) = 0$ and $\dfrac{f(x,y,z)}{z} \geq \alpha_2$ $(z \neq 0)$, it is clear that

$$2\Delta_1\, b(t)\int_0^z f(x,y,\zeta)\,d\zeta \geq \Delta_1\,\alpha_2\, b(t)z^2.$$

By using (2.9) we obtain

$$\left[\Delta_1 - \frac{1}{a(t)\phi_1(y,z,0)}\right]u^2 \geq \varepsilon u^2.$$

Hence

$$2V_0 \geq V_1 + V_2 + V_3 + \varepsilon u^2 + k,$$

where

$$V_1 = 2\Delta_2 d(t)\int_0^{x} h(\zeta)\,d\zeta - \frac{d^2(t)h^2(x)}{c(t)\phi_3(y)},$$

$$V_2 = \left[\Delta_2\alpha_2 b(t) - \Delta_1\alpha_4 d(t) - \Delta_2^2 a(t)\phi_1(y,z,0)\right]y^2 + 2c(t)\int_0^{y} g(\zeta)d\zeta - c(t)y^2\phi_3(y),$$

$$V_3 = \left[\Delta_1\alpha_2 b(t) - \Delta_2 - \Delta_1^2 c(t)\phi_3(y)\right]z^2 + 2a(t)\int_0^{z}\zeta\varphi(y,\zeta,0)d\zeta - a(t)z^2\phi_1(y,z,0).$$

The functions V_1, V_2 and V_3 can be estimated as in [2]. These estimates show that

$$V_1 \geq 2\varepsilon d_0\int_0^{x} h(\zeta)\,d\zeta, \tag{2.13}$$

$$V_2 \geq \frac{C}{4}\left(\frac{2\alpha_4 D\delta_0}{Ca_0\alpha_1 c_0^2\alpha_3^2} - \delta_1\right)y^2, \tag{2.14}$$

$$V_3 \geq \frac{A}{4}\left(\frac{2\delta_0}{Aa_0^2 c_0\alpha_1^2\alpha_3} - \delta_2\right)z^2. \tag{2.15}$$

Then, combining these results, we obtain

$$2V_0 \geq 2\varepsilon d_0 H(x) + \frac{C}{4}\left(\frac{2\alpha_4 D\delta_0}{Ca_0\alpha_1 c_0^2\alpha_3^2} - \delta_1\right)y^2 + \frac{A}{4}\left(\frac{2\delta_0}{Aaz_0^2 c_0\alpha_1^2\alpha_3} - \delta_2\right)z^2 + \varepsilon u^2 + k.$$

In this case it is clear that there exists a positive constant D_1 such that

$$V_0 \geq D_1[\,H(x) + y^2 + z^2 + u^2 + k\,].$$

Under the assumptions of the theorem that we can say

$$\varphi(y,z,0) < \frac{a_0 b_0 c_0\alpha_2\alpha_3}{A^2 D\alpha_4},\ g(y) \leq \frac{a_0 b_0 c_0\alpha_1\alpha_2}{C^2}y,\ f(x,y,z) \leq (\alpha_2 + \frac{\varepsilon c_0^3\alpha_3^3}{BD^2\alpha_4^2})z \ \text{ and } \ h^2(x) \leq 2\alpha_4 H(x).$$

Therefore we can see that there exists a positive constant D_2 which satisfies

$$V_0 \le D_2[H(x) + y^2 + z^2 + u^2 + k].$$

Now the proof of Lemma 1 is complete.

LEMMA 2. *Subject to the hypotheses* (i)-(xii) *of the theorem, there exist positive constants* D_4, D_5 *and* D_6 *such that*

$$\dot{V}_0 \le -D_5(y^2 + z^2 + u^2) + \sqrt{3}\,D_6(y^2 + z^2 + u^2)^{\frac{1}{2}}\{p_1(t) + p_2(t)\} + $$
$$+ \sqrt{3}\,D_6 p_2(t)[H(x) + y^2 + z^2 + u^2] + D_4\gamma_0 V_0. \tag{2.16}$$

Proof. A straightforward calculation using the identity

$$\frac{d}{dt}V_0 = \frac{\partial V_0}{\partial u}\dot{u} + \frac{\partial V_0}{\partial z}u + \frac{\partial V_0}{\partial y}z + \frac{\partial V_0}{\partial x}y + \frac{\partial V_0}{\partial t}$$

yields

$$\frac{d}{dt}V_0 = -\Delta_1 a(t)u^2\varphi(y,z,u) - \Delta_2 b(t)yf(x,y,z) - \Delta_2 c(t)yg(y) - b(t)zf(x,y,z) + u^2 +$$
$$+ \Delta_1 b(t)z\int_0^z \frac{\partial}{\partial y}f(x,y,\zeta)d\zeta + a(t)z\int_0^z \zeta\frac{\partial}{\partial y}\varphi(y,\zeta,0)d\zeta + \Delta_2 a(t)yz\int_0^z \frac{\partial}{\partial y}\varphi(y,\zeta,0)d\zeta +$$
$$+ [\Delta_2\alpha_2 b(t) - \Delta_1\alpha_4 d(t)]yz + \Delta_2 a(t)z\int_0^z \varphi(y,\zeta,0)d\zeta + \Delta_1 c(t)z^2 g'(y) + d(t)y^2h'(x) +$$
$$+ \Delta_1 d(t)yzh'(x) - a(t)[\varphi(y,z,u) - \varphi(y,z,0)]zu - \Delta_2 a(t)[\varphi(y,z,u) - \varphi(y,z,0)]yu +$$
$$+ \Delta_1 b(t)y\int_0^z \frac{\partial}{\partial x}f(x,y,\zeta)d\zeta + (\Delta_2 y + z + \Delta_1 u)p(t,x,y,z,u) + \frac{\partial V_0}{\partial t}.$$

From (vii) and (viii) we have

$$z\int_0^z \zeta\frac{\partial}{\partial y}\varphi(y,\zeta,0)d\zeta \le 0,\ z\int_0^z y\frac{\partial}{\partial y}\varphi(y,\zeta,0)d\zeta \le 0 \text{ and } z\int_0^z \frac{\partial}{\partial y}f(x,y,\zeta)d\zeta \le 0.$$

Thus it follows that

$$\frac{d}{dt}V_0 \le -(V_4 + V_5 + V_6 + V_7 + V_8 + V_9) + (\Delta_2 y + z + \Delta_1 u)p(t,x,z,u) + \frac{\partial V_0}{\partial t} \tag{2.17}$$

where

$$V_4 = \Delta_2 c(t)yg(y) - \alpha_4 d(t)y^2,$$

$$V_5 = [\alpha_2 b(t) - \Delta_1 c(t) g'(y)]z^2 - \Delta_2 a(t) z \int_0^z \varphi(y,\zeta,0)\, d\zeta,$$

$$V_6 = [\Delta_1 a(t)\varphi(y,z,u) - 1]u^2,$$

$$V_7 = z b(t) f(x,y,z) - \alpha_2 b(t) z^2 + \Delta_2 b(t) y f(x,y,z) - \alpha_2 \Delta_2 b(t) yz,$$

$$V_8 = \alpha_4 d(t) y^2 - d(t) h'(x) y^2 + \Delta_1 \alpha_4 d(t) yz - \Delta_1 d(t) h'(x) yz,$$

$$V_9 = a(t)[\varphi(y,z,u) - \varphi(y,z,0)]zu + \Delta_2 a(t)[\varphi(y,z,u) - \varphi(y,z,0)]yu.$$

The functions V_4 and V_8 can be estimated as in [2]. In fact the estimates there show that

$$V_4 \geq \varepsilon c_0 \alpha_3 y^2, \quad V_8 \geq -\varepsilon \Delta_0 z^2 \tag{2.18}$$

Now

$$\begin{aligned} V_5 &= [\alpha_2 b(t) - \Delta_1 c(t) g'(y) - \Delta_2 a(t)\phi_1(y,z,0)]z^2 \\ &\geq \left(\frac{\delta_0}{a_0 c_0 \alpha_1 \alpha_3} - \varepsilon \Delta_0\right) z^2 \end{aligned} \tag{2.19}$$

by (2.10).

We find from (i), (ii) and (2.3)

$$\begin{aligned} V_6 &= [\Delta_1 a(t)\varphi(y,z,u) - 1]u^2 \\ &\geq \varepsilon a_0 \alpha_1 u^2 \end{aligned} \tag{2.20}$$

$$\begin{aligned} V_7 &= b(t)\left[\frac{f(x,y,z)}{z} - \alpha_2\right](z^2 + \Delta_2 yz), \text{ for } z \neq 0 \\ &\geq -(1/4)\Delta_2^2 b(t)\left[\frac{f(x,y,z)}{z} - \alpha_2\right] y^2 < \text{by (viii)}. \end{aligned}$$

In case we use (i), (viii) and (1.3) for $z \neq 0$, then we find

$$(1/4)\Delta_2^2 b(t)\left[\frac{f(x,y,z)}{z} - \alpha_2\right] \leq \frac{b(t)}{4}\left(\frac{D\alpha_4}{c_0\alpha_3} + \varepsilon\right)^2 \frac{\varepsilon_0 c_0^3 \alpha_3^3}{BD^2\alpha_4^2} \leq \varepsilon_0 c_0 \alpha_3,$$

since $\varepsilon \leq \dfrac{D\alpha_4}{c_0\alpha_3}$ by (1.2). Thus it follovs, $V_7 \geq -(\varepsilon_0 c_0 \alpha_3)y^2$ for all y and $z \neq 0$, but $V_7 =$ when $z = 0$, hence we have

$$V_7 \geq -(\varepsilon_0 c_0 \alpha_3) y^2 \text{ for all } y \text{ and } z. \quad (2.2$$

From (x) for $u \neq 0$ we obtain

$$V_9 = a(t)[z\varphi_u(y,z,\theta u) + \Delta_2 y\varphi_u(y,z,\theta u)]u^2 \geq 0,\ 0 \leq \theta \leq 1$$

but $V_9 = 0$ when $u = 0$. Hence

$$V_9 \geq 0 \text{ for all } y,z \text{ and } u. \tag{2.22}$$

Combining the estimates for V_4, V_5, V_6, V_7, V_8 and V_9 with (2.17) we have

$$\dot{V}_0 \leq -(\varepsilon-\varepsilon_0)c_0\alpha_3 y^2 - \left(\frac{\delta_0}{a_0c_0\alpha_1\alpha_3} - 2\varepsilon\Delta_0\right)z^2 - \varepsilon a_0\alpha_1 u^2 + (\Delta_2 y + z + \Delta_1 u)p(t,x,y,z,u) + \frac{\partial V_0}{\partial t}$$

$$\leq -(\varepsilon-\varepsilon_0)c_0\alpha_3 y^2 - \left(\frac{\delta_0}{2a_0c_0\alpha_1\alpha_3}\right)z^2 - \varepsilon a_0\alpha_1 u^2 + (\Delta_2 y + z + \Delta_1 u)p(t,x,y,z,u) + \frac{\partial V_0}{\partial t}.$$

since $\varepsilon < \dfrac{\delta_0}{4a_0c_0\alpha_1\alpha_3\Delta_0}$ by (1.2)

An easy calculation from (2.2) shows that

$$\frac{\partial V_0}{\partial t} = a'(t)\left[\int_0^z \zeta\varphi(y,\zeta,0)d\zeta + \Delta_2 y\int_0^z \varphi(y,\zeta,0)d\zeta\right] + b'(t)\left[\Delta_1\int_0^z f(x,y,\zeta)d\zeta + (\Delta_2/2)\alpha_2 y^2\right]$$

$$+ c'(t)\left[\int_0^y g(\eta)d\eta + \Delta_1 z\, g(y)\right] + d'(t)\left[\Delta_2\int_0^x h(\xi)d\xi - (\Delta_1/2)\alpha_4 y^2 + h(x)y + \Delta_1 h(x)z\right].$$

On the basis of the proof of [2, Lemma 2] it can be shown

$$\frac{\partial V_0}{\partial t} \leq D_3[|a'(t)| + b'(t) + |c'(t)| + |d'(t)|][H(x) + y^2 + z^2 + u^2] \leq D_4\gamma_0 V_0, \text{ where } D_4 = \frac{D_3}{D_1}.$$

The rest of the proof is similar to the proof of [2; Lemma 2] and hence is omitted.

3. Completion of the Proof.

Proof. It follows from the proof of the theorem in [2] that all the solutions ($x(t)$, $y(t)$, $z(t)$, $u(t)$) of (2.1) are uniformly bounded and hence is omitted.

Also the remainder of the claim can be proved by using the techniques similar to those used by Abou-El-Ela [2].

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ON THE MONOTONICITY PROPERTIES OF A SEQUENCE OF OPERATORS OF MEYER-KÖNIG AND ZELLER TYPE

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REZUMAT. - Asupra proprietăţilor de monotonie ale unui şir de operatori de tip Meyer-König şi Zeller. Într-o lucrare anterioară [14] autorul a introdus şi studiat o clasă de operatori liniari pozitivi $M_m^{\langle\alpha\rangle}$, depinzând de un parametru α ne-negativ, de tip Meyer-König şi Zeller. În lucrarea de faţă se deduc reprezentările (7) şi (12), în funcţie de diferenţele divizate de ordinul al doilea, pentru diferenţa a doi termeni consecutivi ai şirului $(M_m^{\langle\alpha\rangle}f)$. Aceste reprezentări au permis stabilirea unor proprietăţi de monotonie, în raport cu m, ale acestui şir, în cazul când se presupune că funcţia f este convexă (respectiv concavă) pe intervalul [0,1]. Rezultate similare pentru operatorii $S_m^{\langle\alpha\rangle}$, definiţi la (3), au fost găsite în lucrarea [10] a autorului.

Summary. In a previous paper [14], the author has introduced and investigated a class of a parameter dependent linear positive operators $M_m^{\langle\alpha\rangle}$ ($\alpha \geq 0$), of Meyer-König and Zeller type. In the present paper one deduces the representations (7) and (12), in terms of second-order divided differences, for the difference of two consecutive terms of the sequence $(M_m^{\langle\alpha\rangle}f)$. These representations permit to establish some monotonicity properties, with respect to m, of this sequence, in the case when we assume that the function f is convex (respectively concave) on the interval [0,1]. Similar results for the operators $S_m^{\langle\alpha\rangle}$, defined at (3), were obtained in the paper [10] of the author.

1. In 1970 we have introduced and investigated in the paper [14] a class of a parameter-dependent linear positive operators $M_m^{\langle\alpha\rangle}$, of Meyer-König and Zeller type

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Denoting by $u^{(n,h)}$ the factorial power of non-negative order n and increment h of u, that is

$$u^{(n,h)} = u(u-h)\dots(u-(n-1)h),\ u^{(0,h)} = 1,$$

the operator $M_m^{<\alpha>}$, associated to a real-valued function f, defined and bounded on the interval $[0,1]$, is defined by

$$\left(M_m^{<\alpha>}f\right)(x) = \sum_{k=0}^{\infty} w_{m,k}^{<\alpha>}(x) f\left(\frac{k}{m+k}\right), \tag{1}$$

where α is a non-negative parameter, $x \in [0,1]$, while

$$\begin{aligned} w_{m,k}^{<\alpha>}(x) &= \binom{m+k}{k} \frac{x^{(k,-\alpha)}(1-x)^{(m+1,-\alpha)}}{1^{(m+1+k,-\alpha)}} = \\ &= \binom{m+k}{k} \frac{x(x+\alpha)\dots(x+(k-1)\alpha)(1-x)(1-x+\alpha)\dots(1-x+m\alpha)}{(1+\alpha)(1+2\alpha)\dots(1+(m+k)\alpha)}. \end{aligned} \tag{2}$$

One observes that if $x = 0$ then $\left(M_m^{<\alpha>}f\right)(0) = f(0)$, while if $x = 1$ it is convenient to take

$$\left(M_m^{<\alpha>}f\right)(1) = \lim_{x \nearrow 1} \left(M_m^{<\alpha>}f\right)(x) = f(1).$$

It is obvious that $M_m^{<\alpha>}$ includes as a special case ($\alpha = 0$) the well known operator of Meyer-König and Zeller [7], defined by

$$\left(M_m f\right)(x) = \sum_{k=0}^{\infty} w_{m,k}(x) f\left(\frac{k}{m+k}\right),$$

where

$$w_{m,k}(x) = \binom{m+k}{k} x^k (1-x)^{m+1},$$

obtained by these authors using the negative binomial (Pascal) probability distribution.

2. In our paper [14] we gave an integral representation of $M_m^{<\alpha>}$, by using a beta transform of the operator M_m, which is valid for $\alpha > 0$ and $0 < x < 1$, namely

$$\left(M_m^{<\alpha>}f\right)(x) = \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \int_0^1 t^{\frac{x}{\alpha}-1} (1-t)^{\frac{1-x}{\alpha}-1} \left(M_m f\right)(t)\, dt,$$

where by $B(a,b)$ we denote the beta function

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt \quad (a,b>0).$$

As we have mentioned above, we should take into consideration that for $x = 0$ and $x = 1$ we take

$$\left(M_m^{<\alpha>}f\right)(0) = f(0), \quad \left(M_m^{<\alpha>}f\right)(1) = f(1).$$

A similar representation was given in 1968 in our paper [10] for a Bernstein type operator $S_m^{<\alpha>}$ (see also [11], [15], [16], [5], [6] and [1]), defined by

$$\left(S_m^{<\alpha>}f\right)(x) = \sum_{k=0}^{m} \binom{m}{k} \frac{x^{(k,-\alpha)}(1-x)^{(m-k,-\alpha)}}{1^{(m,-\alpha)}} f\left(\frac{k}{m}\right), \tag{3}$$

which for $\alpha = 0$ reduces to the classical Bernstein operator B_m.

In [10] we have shown that $S_m^{<\alpha>}f$ can be obtained from $B_m f$ by using a beta transform, namely for any $\alpha > 0$ and $x \in (0,1)$ we have

$$\left(S_m^{<\alpha>}f\right)(x) = \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} \left(B_m f\right)(t)\,dt,$$

while for $x = 0$ and $x = 1$ we have to consider that $\left(S_m^{<\alpha>}f\right)(0) = f(0)$, $\left(S_m^{<\alpha>}f\right)(1) = f(1)$.

In [11] we have presented the connection of this operator with the probability distribution of Markov-Polya (introduced in 1917 by A.A. Markov [4] and encountered in 1923 by F. Eggenberger-G. Polya [2] and used in 1930 by G. Polya [8] for the study of contaggeous diseases). The numerical characteristics of this distribution were studied in detail in our papers [12] and [13].

Concerning our operator $M_m^{<\alpha>}$, defined at (1) and (2), we want to mention that in the paper [20] J. Swetits and B. Wood, referring to a probabilistic method, in connection with the Markov-Polya urn scheme, used by us in [11] for constructing the operator $S_m^{<\alpha>}$, have presented a variation of the Pascal urn scheme, in the sense in which was generalized in [4] and [2] the Bernoulli urn scheme. It permits to give a probabilistic interpretation to $M_m^{<\alpha>}$

In our paper [14] we have proved that this operator reproduces the linear functions and that $M_m^{<\alpha>} e_2 \to e_2$, uniformly on [0,1] if we assume that $\alpha = \alpha(m) \to 0$ when $m \to \infty$.

3. In the paper [17] we investigated the remainder in the approximation formula

$$f(x) = \left(M_m^{<\alpha>} f\right)(x) + \left(R_m^{<\alpha>} f\right)(x). \tag{4}$$

Now we can formulate that result in a new form enunciating the following

THEOREM 1. *If we use the notation*

$$w_{m,k}^{<\alpha>}(u, v) = \binom{m+k}{k} \frac{u^{(k,-\alpha)} v^{(m+1,-\alpha)}}{(u+v)^{(m+1+k,-\alpha)}} \tag{5}$$

and if we assume that all the second-order divided differences of the function f are bounded on [0,1], *then the remainder of formula* (4) *can be expressed under the following form*

$$\left(R_m^{<\alpha>} f\right)(x) = -x(1-x) \sum_{k=0}^{\infty} (m+1+k)^{-1} w_{m-1,k}^{<\alpha>}(x+\alpha, 1-x+\alpha) \left[x, \frac{k}{m+k}, \frac{k+1}{m+1+k}; f\right], \tag{6}$$

where the brackets represent the symbol for divided differences.

In the special case $\alpha = 0$ this formula was established in [19].

It is easily seen that from this theorem there follows

COROLLARY 1. *If on the interval* [0,1] *the function f is convex* (*concave*), *then for any* $x \in [0,1]$ *and any* $m \in \mathbb{N}$ *we have*

$$f(x) \le \left(M_m^{<\alpha>} f\right)(x) \ (\textit{respectively}\ f(x) \ge \left(M_m^{<\alpha>} f\right)(x)).$$

If f is strictly convex (*concave*) *then these inequalities are strict on the interval* (0,1).

4. Now we shall investigate a monotonicity property of the sequence $\left(M_m^{<\alpha>} f\right)$, when f is a convex function. The key to do this lies in proving

THEOREM 2. *If all the second-order divided differences of the function f are bounded* *on* [0,1] *and* $\alpha \geq 0$, $m \in \mathbb{N}$, *then we have*

$$D_m(f;x,\alpha) = \left(M_{m+1}^{<\alpha>}f\right)(x) - \left(M_m^{<\alpha>}f\right)(x) = \\ = -\frac{x(1-x)}{m(1+\alpha)}\sum_{k=0}^{\infty}\frac{m+k}{(m+1+k)(m+2+k)}\cdot w_{m-1,k}^{<\alpha>}(x+\alpha,\, 1-x+\alpha)\cdot D^2(f;m,k), \tag{7}$$

where

$$D^2(f;m,k) = \left[\frac{k}{m+1+k}, \frac{k+1}{m+2+k}, \frac{k+1}{m+1+k}; f\right]. \tag{8}$$

Proof. If we use the folloving identities

$$(1-x)^{(m+2,-\alpha)} = [1-x+(m+1)\alpha]\,(1-x)^{(m+1,-\alpha)} = \\ = \{[1+(m+1+k)\alpha] - (x+k\alpha)\}\cdot(1-x)^{(m+1,-\alpha)}, \\ 1^{(m+2+k,-\alpha)} = [1+(m+1+k)\alpha]\cdot 1^{(m+1+k,-\alpha)},$$

then we decompose the sum

$$\left(M_{m+1}^{<\alpha>}f\right)(x) = \sum_{k=0}^{\infty}\binom{m+1+k}{k}\frac{x^{(k,-\alpha)}(1-x)^{(m+2,-\alpha)}}{1^{(m+2+k,-\alpha)}}f\left(\frac{k}{m+1+k}\right)$$

into two parts, namely

$$\left(M_{m+1}^{<\alpha>}f\right)(x) = A_m(f;x,\alpha) - C_m(f;x,\alpha),$$

where

$$A_m(f;x,\alpha) = \sum_{k=0}^{\infty}\binom{m+1+k}{k}\frac{x^{(k,-\alpha)}(1-x)^{(m+1,-\alpha)}}{1^{(m+1+k,-\alpha)}}f\left(\frac{k}{m+1+k}\right),$$

$$C_m(f;x,\alpha) = \sum_{k=0}^{\infty}\binom{m+1+k}{k}\frac{x^{(k+1,-\alpha)}(1-x)^{(m+1,-\alpha)}}{1^{(m+2+k,-\alpha)}}f\left(\frac{k}{m+1+k}\right).$$

Now let us detach from the first sum the term corresponding to $k = 0$ and set k-1 $= j$ in the remaining sum; if we denote further the index of summation by k, we can write

$$A_m(f;x,\alpha) = \frac{(1-x)^{(m+1,-\alpha)}}{1^{(m+1,-\alpha)}}f(0) + \\ + \sum_{k=0}^{\infty}\binom{m+2+k}{k+1}\frac{x^{(k+1,-\alpha)}(1-x)^{(m+1,-\alpha)}}{1^{(m+2+k,-\alpha)}}f\left(\frac{k+1}{m+2+k}\right).$$

Therefore we obtain

$$\left(M_{m+1}^{<\alpha>}f\right)(x) = \frac{(1-x)^{(m+1,-\alpha)}}{1^{(m+1,-\alpha)}}f(0) +$$

$$+\sum_{k=0}^{\infty}\binom{m+2+k}{k+1}\frac{x^{(k+1,-\alpha)}(1-x)^{(m+1,-\alpha)}}{1^{(m+2+k,-\alpha)}}f\left(\frac{k+1}{m+2+k}\right) -$$

$$-\sum_{k=0}^{\infty}\binom{m+1+k}{k}\frac{x^{(k+1,-\alpha)}(1-x)^{(m+1,-\alpha)}}{1^{(m+2+k,-\alpha)}}f\left(\frac{k}{m+1+k}\right).$$

Now in the expression of $\left(M_m^{<\alpha>}f\right)(x)$ we detach the first term and in the remaining sum we set k-1 $= j$; denoting again the index of summation by k, we get

$$\left(M_m^{<\alpha>}f\right)(x) = \frac{(1-x)^{(m+1,-\alpha)}}{1^{(m+1,-\alpha)}}f(0) +$$

$$+\sum_{k=0}^{\infty}\binom{m+1+k}{k+1}\frac{x^{(k+1,-\alpha)}(1-x)^{(m+1,-\alpha)}}{1^{(m+2+k,-\alpha)}}f\left(\frac{k+1}{m+1+k}\right).$$

If we take into account (9) and (10), the difference from the left member of the equality (7) can be expressed under the following form

$$C_m(f;x,\alpha) = \sum_{k=0}^{\infty}\binom{m+2+k}{k+1}\frac{x^{(k+1,-\alpha)}(1-x)^{(m+1,-\alpha)}}{1^{(m+2+k,-\alpha)}}f\left(\frac{k+1}{m+2+k}\right) -$$

$$-\sum_{k=0}^{\infty}\binom{m+1+k}{k}\frac{x^{(k+1,-\alpha)}(1-x)^{(m+1,-\alpha)}}{1^{(m+2+k,-\alpha)}}f\left(\frac{k}{m+1+k}\right) -$$

$$-\sum_{k=0}^{\infty}\binom{m+1+k}{k+1}\frac{x^{(k+1,-\alpha)}(1-x)^{(m+1,-\alpha)}}{1^{(m+2+k,-\alpha)}}f\left(\frac{k+1}{m+1+k}\right).$$

Consequently we are able to write

$$C_m(f;x,\alpha) = -\sum_{k=0}^{\infty}\frac{x^{(k+1,-\alpha)}(1-x)^{(m+1,-\alpha)}}{1^{(m+2+k,-\alpha)}}\left\{\binom{m+1+k}{k}f\left(\frac{k}{m+1+k}\right) - \right.$$

$$\left. - \binom{m+2+k}{k+1}f\left(\frac{k+1}{m+2+k}\right) + \binom{m+1+k}{k+1}f\left(\frac{k+1}{m+1+k}\right)\right\}.$$

Since

$$\binom{m+1+k}{k} = \frac{m+1+k}{m+1}\binom{m+k}{k},$$

$$\binom{m+2+k}{k+1} = \frac{(m+1+k)(m+2+k)}{(k+1)(m+1)}\binom{m+k}{k},$$

$$\binom{m+1+k}{k+1} = \frac{m+1+k}{k+1}\binom{m+k}{k}$$

and

$$x^{(k+1,-\alpha)} = x(x+\alpha)^{(k,-\alpha)}$$

$$(1-x)^{(m+1,-\alpha)} = (1-x)(1-x+\alpha)^{(m,-\alpha)},$$

we obtain at once

$$D_m(f;x,\alpha) = -x(1-x)\sum_{k=0}^{\infty}\binom{m+k}{k}\frac{(x+\alpha)^{(k,-\alpha)}(1-x+\alpha)^{(m,-\alpha)}}{1^{(m+k+2,-\alpha)}}(m+1+k)\cdot$$

$$\cdot\left\{\frac{1}{m+1}f\left(\frac{k}{m+1+k}\right) - \frac{(m+2+k)}{(m+1)(k+1)}f\left(\frac{k+1}{m+2+k}\right) + \frac{1}{k+1}f\left(\frac{k+1}{m+1+k}\right)\right\}.$$

Now it is helpful to consider the divided difference (8), for which, by a straightforward calculation, we obtain the following explicit expression

$$D^2(f;m,k) = \frac{(m+1+k)^2(m+2+k)}{m+1}f\left(\frac{k}{m+1+k}\right) -$$

$$-\frac{(m+1+k)^2(m+2+k)^2}{(m+1)(k+1)}f\left(\frac{k+1}{m+2+k}\right) + \frac{(m+1+k)^2(m+2+k)}{k+1}f\left(\frac{k+1}{m+1+k}\right) =$$

$$= (m+1+k)^2(m+2+k)\left\{\frac{1}{m+1}f\left(\frac{k}{m+1+k}\right) - \frac{m+2+k}{(m+1)(k+1)}f\left(\frac{k+1}{m+2+k}\right) + \frac{1}{k+1}f\left(\frac{k+1}{m+1+k}\right)\right\}.$$

Consequently we have

$$(m+1+k)\left\{\frac{1}{m+1}f\left(\frac{k}{m+1+k}\right) - \frac{m+2+k}{(m+1)(k+1)}f\left(\frac{k+1}{m+2+k}\right) + \frac{1}{k+1}f\left(\frac{k+1}{m+1+k}\right)\right\} =$$

$$= \frac{1}{(m+1+k)(m+2+k)}D^2(f;m,k)$$

and we finally obtain

$$D_m(f;x,\alpha) = -x(1-x)\sum_{k=0}^{\infty}\binom{m+k}{k}\frac{(x+\alpha)^{(k,-\alpha)}(1-x+\alpha)^{(m,-\alpha)}}{(m+1+k)(m+2+k)\,1^{(m+2+k,-\alpha)}}D^2(f;m,k). \tag{11}$$

If now we insert in (5): $u = x + \alpha$, $v = 1 - x + \alpha$ and replace m by m-1, then we get

$$w_{m-1,k}^{\langle\alpha\rangle}(x+\alpha, 1-x+\alpha) = \binom{m-1+k}{k}\frac{(x+\alpha)^{(k,-\alpha)}(1-x+\alpha)^{(m,-\alpha)}}{(1+2\alpha)^{(m+k,-\alpha)}}.$$

Taking this into account and the relation

$$\binom{m+k}{k} = \frac{m+k}{k}\binom{m-1+k}{k},$$

we are led to the representation (7), ehich we desired to establish.

Remark. Since we can write

$$x(x+\alpha)^{(k,-\alpha)} = x^{(k,-\alpha)}(x+k\alpha),\ (1-x)(1-x+\alpha)^{(m,-\alpha)} = (1-x)^{(m+1,-\alpha)},$$

formula (11) permits to give also the following representation for $D_m(f;x,\alpha)$:

$$D_m(f;x,\alpha) = -\sum_{k=0}^{\infty}\frac{(x+k\alpha)\,w_{m,k}^{\langle\alpha\rangle}(x)}{(m+1+k)(m+2+k)(1+(m+1+k)\alpha)}\cdot D^2(f;m,k). \tag{12}$$

For the classical operators M_m of Meyer-König and Zeller we have to replace $\alpha = 0$ and we obtain

$$(M_{m+1}f)(x) - (M_m f)(x) = -x\sum_{k=0}^{\infty}\frac{w_{m,k}(x)}{(m+1+k)(m+2+k)}\cdot D^2(f;m,k). \tag{13}$$

Remark. A similar representation was given, without proof, in the paper [3], but there was omitted the factor x in the second member.

Invoking Corollary 1 and the representation (7) or (12), we can state

THEOREM 3. *If on the interval* [0,1] *the function f is convex (concave), then on this interval we have*

$$f \le M_{m+1}^{\langle\alpha\rangle}f \le M_m^{\langle\alpha\rangle}f \ (respectively\ f \ge M_{m+1}^{\langle\alpha\rangle}f \ge M_m^{\langle\alpha\rangle}f).$$

If f is not linear and is strictly convex (concave) on [0,1], *then these inequalities are strict on the interval* (0,1).

Referring to these results, we can say that the approximation of continuous convex (concave) functions f by means of $M_m^{\langle\alpha\rangle}f$ occurs monotonically from above (below).

In the case of the Bernstein polynomials, a representation similar with that given at (7) was deduced in our paper [9], where we have investigated also the monotonicity of the sequence of the first order derivatives of these polynomials. Extensions to higher derivatives were investigated in our paper [18]. As we have mentioned in that paper, in the case when we use operators which can be expressed by means of factorial powers, of increment $h = -\alpha$, then if we want to investigate the monotonicity of the prederivatives of the sequences of such operators, then the differentiation operator should be replaced by the Nörlund difference operator $D_{-\alpha}$.

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EXISTENCE OF PERIODIC SOLUTIONS FOR SOME INTEGRAL EQUATIONS ARISING IN INFECTIOUS DISEASES

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REZUMAT. - Existenţa soluţiilor periodice pentru o clasă de ecuaţii integrale care modelează propagarea unor epidemii. În lucrare se stabilesc două rezultate de existenţă relativ la ecuaţia (1), extinzându-se unele rezultate din [2].

Introduction. The nonlinear integral equation:

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))\,ds \tag{1}$$

can be interpreted as a model for the spread of certain infectious diseases with periodic contact rate that varies seasonally. In Eq. (1), $x(t)$ represents the proportion of the infectives in the population at time t, τ is the length of time an individual remains infectious and $f(t,x(t))$ is the proportion of new infectives per unit time which satisfies:

(H_1) $f(t,x)$ is nonnegative and continuous for $-\infty < t < \infty$ and $x \geq 0$;

(H_2) $f(t,0) = 0$ for $-\infty < t < \infty$ and there exists $w > 0$ such that $f(t+w, x) = f(t, x)$ for all $-\infty<t<\infty$ and $x \geq 0$.

Obviously, $x(t) \equiv 0$ is the trivial solution of Eq. (1). In [1,5] were given sufficient conditions for the existence of nontrivial, nonnegative, periodic and continuous solutions of Eq. (1). The aim of this paper is to prove two new existence theorems. The first theorem gives us sufficient conditions for the existence of at least one, positive, periodic and continuous solution of Eq. (1). Our assumptions are different from those in [1,5] and there

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are cases where our result applies while those in [1,5] do not.

The second theorem, which is related to the first one, gives us sufficient conditions for the existence of at least $n+2$ ($n \in \mathbb{N}$), nontrivial, nonnegative, periodic and continuous solutions of Eq. (1). This result extends in a simple way Theorem 1 from [2].

The proofs are based on the fixed point index theory.

2. Main results.

THEOREM 1. *Assume* (H_1), (H_2) *hold and that the following two conditions are satisfied:*

(H_3) *there exist the real numbers* $a > 0$ *and* $\alpha > 1$ *such that:*

$$\sup_{t \in [0,w]} \frac{f(t,a)}{a} < \frac{1}{\alpha\tau} \tag{2}$$

and

$$\frac{\alpha}{\alpha-1} a \leq \inf\left\{x > a \mid \text{ there exists } t \in [0,w] \text{ such that } f(t,x) = \frac{x}{\alpha\tau}\right\} \tag{3}$$

(*where, if the set is empty we use convention* $\inf \varnothing = +\infty$);

(H_4) *there exist* b, *with* $0 < b < a$, *and a nonnegative, continuous function* $g(t)$ *with period* w, *such that:*

$$f(t,x) \geq g(t) \text{ for } t \in [0,w] \text{ and } x \in \left[b, \frac{\alpha}{\alpha-1} a\right[\tag{4}$$

and

$$\int_{t-\tau}^{t} g(s)ds \geq b \text{ for all } t \in [0,w]. \tag{5}$$

Then Eq. (1) has at least one, positive, w-periodic and continuous solution $x_1(t)$ such that:

$$b \leq \min_{t \in [0,w]} x_1(t) \leq \max_{t \in [0,w]} x_1(t) < \frac{\alpha}{\alpha-1} a \tag{6}$$

Proof. Let E be the Banach space of all continuous and w-periodic functions $x(t)$ on

R with norm:

$$\|x\| = \sup_{-\infty<t<+\infty} |x(t)| = \max_{0\le t\le w} |x(t)|,$$

let $K = \{x \in E \,|\, x(t) \geq b \text{ for } t \in \mathbf{R}\}$, let $U = \left\{x \in K \,|\, \|x\| < \frac{\alpha}{\alpha-1} a\right\}$.

Clearly, K is a normal cone of E and U is a nonempty, open, bounded subset of K.

For each $\lambda \in [0,1]$ we define the nonlinear integral operator;

$$T_\lambda x(t) = (1-\lambda)a + \lambda \int_{t-\tau}^{t} f(s, x(s))ds.$$

It is easy to show, using the conditions (H_1)-(H_4) and the Theorem of Ascoli-Arzelá, that the operators T_λ are completely continuous from $\bar{U}$ into K. Now we shall prove that, for every $\lambda \in [0,1]$, the operator T_λ has no fixed points on the boundary ∂U of U with respect to K.

Suppose contrary, there exist $\lambda_0 \in [0,1]$ and $x_0 \in K$ such that $\|x_0\| = \frac{\alpha}{\alpha-1} a$ and $T_{\lambda_0} x_0 = x_0$. From $\|x_0\| = \frac{\alpha}{\alpha-1} a$, we deduce that there exists at least one $t_0 \in \mathbf{R}$ such that $x_0(t_0) = \frac{\alpha}{\alpha-1} a$, and by $T_{\lambda_0} x_0 = x_0$ we obtain:

$$\frac{\alpha}{\alpha-1} a = x_0(t_0) = T_{\lambda_0} x_0(t_0) = (1-\lambda_0)a + \lambda_0 \int_{t_0-\tau}^{t_0} f(s, x_0(s))ds. \tag{7}$$

If $\lambda_0 = 0$, we find from (7) that $\frac{\alpha}{\alpha-1} a = a$, or equivalently $\alpha = 0$, which contradicts (H_3). Therefore we have:

$$\lambda_0 > 0. \tag{8}$$

Now (H_3) and the continuity of $f(t,x)$ imply that there exists $\delta > 0$ such that:

$$f(t,x) \leq \frac{x}{\alpha\tau} \leq \frac{1}{\alpha\tau} \cdot \frac{\alpha}{\alpha-1} a, \quad \text{for } t \in \mathbf{R},\ x \in \left]a-\delta, \frac{\alpha}{\alpha-1} a\right]. \tag{9}$$

If $x_0(s) > a-\delta$, for all $s \in [t_0-\tau, t_0]$, then from $\|x_0\| = \frac{\alpha}{\alpha-1} a$ we get $a-\delta < x_0(s) \leq \frac{\alpha}{\alpha-1} a$, $s \in [t_0-\tau, t_0]$ and, in this case, using (9), (7) implies:

$$\frac{\alpha}{\alpha-1} a \leq (1-\lambda_0)a + \lambda_0 \frac{1}{\alpha\tau} \cdot \frac{\alpha}{\alpha-1} a \int_{t_0-\tau}^{t_0} ds < (1-\lambda_0)a + \lambda_0 \frac{\alpha}{\alpha-1} a,$$

and, consequently $\lambda_0 > 1$, which contradicts $\lambda_0 \in [0,1]$.

If there exists $s_0 \in [t_0-\tau,\ t_0]$ such that $x_0(s_0) \leq a-\delta$, then, observing that $x_0(s_0) < a < x_0(t_0) = \frac{\alpha}{\alpha-1}a$, the continuity of x_0 implies that there exists at least one $t_1 \in]s_0,t_0] \subset [t_0-\tau,\ t_0]$ such that $x_0(t_1) = a$. Choosing now

$$t_1 = \sup\{t \in [t_0 - \tau, t_0] \mid x_0(t) = a\}$$

it is easy to show that:

$$t_0 - \tau < t_1 < t_0,$$

$$x_0(t_1) = a,$$

and, using $\|x_0\| = \frac{\alpha}{\alpha-1}a$,

$$a < x_0(s) \leq \frac{\alpha}{\alpha-1}a, \text{ for all } s \in]t_1, t_0].$$

By $x_0(t_1) = a$ and $T_{\lambda_0}x_0 = x_0$ we get:

$$a = x_0(t_1) = T_{\lambda_0}x_0(t_1) = (1-\lambda_0)a + \lambda_0\int_{t_1-\tau}^{t_1} f(s,x_0(s))ds$$

From the last equality and (7), using $t_0-\tau < t_1 < t_0$ and the nonnegativity of function $f(t,x)$ we have:

$$\frac{\alpha}{\alpha-1}a - a = \lambda_0\left[\int_{t_0-\tau}^{t_0} f(s,x_0(s))ds - \int_{t_1-\tau}^{t_1} f(s,x_0(s))ds\right] =$$

$$= \lambda_0\left[\int_{t_1}^{t_0} f(s,x_0(s))ds - \int_{t_1-\tau}^{t_0-\tau} f(s,x_0(s))ds\right] \leq \tag{10}$$

$$\leq \lambda_0\int_{t_1}^{t_0} f(s,x_0(s))ds$$

By $a < x_0(s) \leq \frac{\alpha}{\alpha-1}a,\ s \in]t_1, t_0]$, using (9) we find

$$\int_{t_1}^{t_0} f(s,x_0(s))ds \leq \frac{1}{\alpha\tau}\cdot\frac{\alpha}{\alpha-1}a\int_{t_1}^{t_0} ds < \frac{a}{\alpha-1} \tag{11}$$

Since $\lambda_0 \in]0,1]$ (see (8)), (10) and (11) imply:

$$\frac{\alpha}{\alpha-1}a - a < \frac{a}{\alpha-1}, \text{ a contradiction.}$$

Nevertheless, the assumption that there exist $\lambda_0 \in [0,1]$ such that T_{λ_0} has fixed points on the boundary ∂U of U with respect to K, is false. Therefore, the fixed point index

$i(T_\lambda, U, K)$ does not depend on λ. In particular we have

$$i(T_1, U, K) = i(T_0, U, K) = 1,$$

consequently, T_1 has at least one fixed point $x_1 \in U$ which, obviously, satisfies (6). So our theorem is completely proved. ■

Here is an elementary example which shows that Theorem 1 applies while the theorems from [1,5] do not:

Example 1. Let us consider Eq. (1) with $\tau = 1$ and

$$f(t,x) = \begin{cases} 11\,(\sin t{+}2)\sqrt{x}\,, & \text{for } t \in \mathbf{R},\ x \in [0,1] \\ (\sin t{+}2)\left[\dfrac{2}{15}(x-4)^4 + \dfrac{1}{5}\right], & \text{for } t \in \mathbf{R},\ x > 1 \end{cases}$$

Choosing $a = 4$, $\alpha = 5$, $b = \frac{1}{5}$ and $g(t) \equiv \frac{1}{5}$ we can apply our theorem obtaining the existence of at least one, positive, continuous, 2π-periodic solution $x_1(t)$ such that:

$$\frac{1}{5} \le \min_{t\in[0,2\pi]} x_1(t) \le \max_{t\in[0,2\pi]} x_1(t) < 5.$$

On the other hand, one can verify that theorems from [1,5] do not apply in this case.

Let us see the behaviour of a function $f(t,x)$ which satisfies (H_1)-(H_4). For an arbitrary t_0, the dependence on x of the function $f(t_0,x)$ is as follows from Fig. 1 (see at the end of this paper). Shortly, (H_3) is equivalent with the fact that, for every $t_0 \in \mathbf{R}$, the graphic of $f(t_0,x)$, for $a \le x < \frac{\alpha}{\alpha-1}a$, is under the right line $y = \frac{x}{\alpha\tau}$; (H_4) is weaker than the fact that, for every $t_0 \in \mathbf{R}$, the graphic of $f(t_0,x)$, for $b \le x < \frac{\alpha}{\alpha-1}a$, is above the right line $y = \frac{b}{\tau}$; by (H_3) and (H_4) the behaviour of $f(t,x)$, for $0 \le x < b$ and $x \ge \frac{\alpha}{\alpha-1}a$, is unconditioned.

Let $n \in \mathbf{N}$ be an arbitrary natural number and let us list two more conditions for convenience:

(H_3') There exist the real numbers $a_0, a_1, \ldots, a_{n+1}$, $0 \le a_0 < a_1 < \ldots < a_{n+1} \le +\infty$ such that, for every $j \in \overline{0, n+1}$ there are satisfied:

$$\sup_{t \in [0,w]} \frac{f(t, a_j)}{a_j} < \frac{1}{\alpha_j \tau}, \text{ where } \alpha_j > 1 \quad (2$$

(if $a_j = 0$ or $a_j = \infty$ we assume that the limit $\overline{\lim_{x \to a_j}} \frac{f(t, a_j)}{a_j}$ exists uniformly with respect $t \in [0,w]$ and we denote this limit by $\frac{f(t, a_j)}{a_j}$),

and

$$\frac{\alpha_j}{\alpha_j - 1} a_j \leq \inf \left\{ x > a_j \mid \text{there exists } t \in [0,w] \text{ such that } f(t,x) = \frac{x}{\alpha_j \tau} \right\} \quad (3$$

(where, if the set is empty we use the convention $\inf \varnothing = +\infty$);

(H_4') There exist the numbers $b_1, b_2, \ldots, b_{n+1}$ and the nonnegative, w-periodic, continuo functions $g_1(t), g_2(t), \ldots, g_{n+1}(t)$ such that:

$$0 \leq \frac{\alpha_0}{\alpha_0 - 1} a_0 < b_1 < a_1 < \frac{\alpha_1}{\alpha_1 - 1} a_1 < b_2 < a_2 < \ldots < \frac{\alpha_n}{\alpha_n - 1} a_n < b_{n+1} < a_{n+1} \leq +\infty$$

and, for every $j \in \overline{1, n+1}$, there are satisfied:

$$f(t,x) \geq g_j(t), \text{ for } t \in [0,w],\ x \in \left[b_j, \frac{\alpha_j}{\alpha_j - 1} a_j \right[\quad (4$$

and

$$\int_{t-\tau}^{t} g_j(s) ds > b_j, \text{ for all } t \in [0,w]. \quad (5$$

Now, we are ready to state our second theorem:

THEOREM 2. *If* (H_1), (H_2), (H_3') *and* (H_4') *are satisfied, then* Eq. (1) *has at least n+ distinct, nontrivial, nonnegative, w-periodic and continuous solutions* $x_0(t)$, $x_1(t)$, $x_2(t)$, ... $x_{n+1}(t)$ *such that:*

$$b_j < \min_{t \in [0,w]} x_j(t) \leq \max_{t \in [0,w]} x_j(t) < \frac{\alpha_j}{\alpha_j - 1} a_j \text{ for every } j \in \overline{1, n+1} \quad (6'$$

and

$$\max_{t \in [0,w]} x_0(t) < \frac{\alpha_{n+1}}{\alpha_{n+1} - 1} a_{n+1}. \quad (7'$$

Proof. Let E be the same Banach space as well in Theorem 1, let

$$K = \{x \in E \mid x(t) \geq 0 \text{ for all } -\infty < t < +\infty\}$$

and, for every $j \in \overline{1,n}$ let

$$U_j = \left\{x \in K \mid b_j < \min_{t \in [0,w]} x(t) \leq \max_{t \in [0,w]} x(t) < \frac{\alpha_j}{\alpha_j - 1} a_j \right\}.$$

Clearly, K is a normal cone of E, and, for every $j \in \overline{1,n}$, U_j is a nonempty, open and bounded subset of K.

Let T be the nonlinear integral operator:

$$(Tx)(t) = \int_{t-\tau}^{t} f(s, x(s)) ds.$$

It is easy to show, using (H_1), (H_2) and Theorem of Ascoli-Arzela, that T is completely continuous from K into K.

Now we shall prove that, for every $j \in \overline{1,n}$, the fixed point index:

$$i(T, U_j, K) = 1.$$

For an arbitrary fixed $j \in \overline{1,n}$ we can consider the homotopy

$$\mathcal{H}_j : [0,1] \times \bar{U}_j \to K$$

$$\mathcal{H}_j(\lambda, x)(t) = (1 - \lambda) a_j + \lambda \int_{t-\tau}^{t} f(s, x(s))\, ds,$$

which is completely continuous from $[0, 1] \times \bar{U}_j$ into K. If we suppose that $\mathcal{H}_j(\lambda, \cdot)$ has a fixed point $\tilde{x}$ on the boundary ∂U_j of U_j with respect to K, then $\|\tilde{x}\| = \frac{\alpha_j}{\alpha_j - 1} a_j$ or $\min_{t \in [0,w]} \tilde{x}(t) = b_j$. By a similar proof as in Theorem 1, we obtain that $\|\tilde{x}\| = \frac{\alpha_j}{\alpha_j - 1} a_j$ is impossible. The second case implies that there exists $t_0 \in [0,w]$ such that:

$$b_j = \tilde{x}(t) = \mathcal{H}_j(\lambda, \tilde{x})(t) = (1 - \lambda) a_j + \lambda \int_{t_0-\tau}^{t_0} f(s, \tilde{x}(s))\, ds \tag{12}$$

Therefore $\lambda > 0$, otherwise we have $b_j = a_j$, which contradicts $b_j < a_j$. Using $\tilde{x} \in U_j$, by (4') and (5'), we deduce:

$$\int_{t_0-\tau}^{t_0} f(s, \tilde{x}(s)) ds \geq \int_{t_0-\tau}^{t_0} g_j(s) ds > b_j \tag{13}$$

and from (12) and (13), since $\lambda > 0$, we obtain

$$b_j > (1-\lambda)a_j + \lambda b_j$$

which contradicts $\lambda \in [0,1]$ and $b_j < a_j$.

So $\mathcal{H}_j(\lambda,\cdot)$ has no fixed points on the boundary of U_j with respect to K and consequently:

$$i(T, U_j, K) = i(\mathcal{H}_j(1,\cdot), U_j, K) = i(\mathcal{H}_j(0,\cdot), U_j, K) = 1.$$

Hence

$$i(T, U_j, K) = 1, \text{ for all } j \in \overline{1,n}.$$

Now, we shall construct three more nonempty, open and bounded subsets of K: U_0, U_{n+1}, U_{n+2} such that:

$$i(T, U_0, K) = i(T, U_{n+1}, K) = i(T, U_{n+2}, K) = 1.$$

If $a_0 \neq 0$ then $U_0 = \left\{ x \in K \mid \|x\| < \dfrac{\alpha_0}{\alpha_0 - 1} a_0 \right\}$. For the calculus of the fixed point index $i(T, U_0, K)$ we define the homotopy:

$$\mathcal{H}_0 : [0,1] \times \bar{U}_0 \to K$$

$$\mathcal{H}_0(\lambda, x)(t) = \lambda \int_{t-\tau}^{t} f(s, x(s))ds,$$

which is completely continuous from $[0,1] \times \bar{U}_0$ into K.

If we suppose that $\mathcal{H}_0(\lambda,\cdot)$ has a fixed point on the boundary of U_0 with respect to K, then $\|\tilde{x}\| = \dfrac{\alpha_0}{\alpha_0 - 1} a_0$ and, by a similar proof as in Theorem 1, we obtain a contradiction. Therefore:

$$i(T, U_0, K) = i(\mathcal{H}_0(1,\cdot), U_j, K) = i(\mathcal{H}_0(0,\cdot), U_j, K) = 1.$$

If $a_0 = 0$, then we shall construct U_0 as in Theorem 1 from [2]. Shortly, from:

$$\sup_{t\in[0,w]} \frac{f(t,0)}{0} < \frac{1}{\alpha_0 \tau}, \ \alpha_0 > 1$$

where

$\frac{f(t,0)}{0} = \overline{\lim_{x \to +0}} \frac{f(t,x)}{x}$ uniformly with respect to $t \in [0,w]$, we deduce that there exists a real number r such that:

$$0 < r < b_1$$

and

$$f(t,x) \le \frac{x}{\alpha_0 \tau} < \frac{r}{\tau} \text{ for all } t \in [0,w],\ x \in]0,r],$$

which implies, using (H_1) and (H_2):

$$0 \le f(t,x) < \frac{r}{\tau} \text{ for all } t \in [0,w],\ x \in [0,r].$$

Let $U_0 = \{x \in K |\ \|x\| < r\}$. Now, it is easy to prove that $T(\bar{U}_0) \subset U_0$ and consequently (see [6])

$$i(T, U_0, K) = 1.$$

To construct U_{n+1} and U_{n+2} we have to distinguish following cases:

If $a_{n+1} < \infty$ then

$$U_{n+1} = \left\{ x \in K \mid b_{n+1} < \min_{t \in [0,w]} x(t) \le \max_{t \in [0,w]} x(t) < \frac{\alpha_{n+1}}{\alpha_{n+1} - 1} a_{n+1} \right\}$$

$$U_{n+2} = \left\{ x \in K \mid \|x\| < \frac{\alpha_{n+1}}{\alpha_{n+1} - 1} a_{n+1} \right\}$$

The calculus of $i(T, U_{n+1}, K)$ is, now, similar to the calculus of $i(T, U_j, K)$, for $j \in \overline{1,n}$, and we obtain the same result:

$$i(T, U_{n+1}, K) = 1.$$

Moreover, the calculus of $i(T, U_{n+2}, K)$ is similar to the calculus of $i(T, U_0, K)$, for $a_0 \ne 0$ and the result is:

$$i(T, U_{n+2}, K) = 1.$$

If $a_{n+1} = \infty$, then from:

$$\sup_{t\in[0,w]} \frac{f(t,\infty)}{\infty} < \frac{1}{\alpha_{n+1}\tau},\ \alpha_{n+1} > 1,$$

where $\frac{f(t,\infty)}{\infty} = \overline{\lim_{x\to\infty}} \frac{f(t,x)}{x}$ uniformly with respect to $t \in [0,w]$, we deduce that there exists $\ell > b_{n+1}$ such that

$$0 \le f(t,x) < \frac{x}{\alpha_{n+1}\tau} \text{ for all } t \in [0,w],\ x \ge \ell.$$

Hence

$$0 \le f(t,x) < \frac{x}{\alpha_{n+1}\tau} + \beta, \text{ for all } t \in [0,w],\ x \ge 0, \tag{14}$$

where $\beta = \max_{x\in[0,\ell]} \max_{t\in[0,w]} f(t,x)$. Now, choosing $R > b_{n+1}$ such that $\frac{R}{\alpha_{n+1}\tau} + \beta < \frac{R}{\tau}$, we deduce, from (14) and (4'), that:

$$0 \le f(t,x) < \frac{R}{\tau} \text{ for all } t \in [0,w],\ x \in [0,R],$$

and

$$g_{n+1}(t) \le f(t,x) < \frac{R}{\tau} \text{ for all } t \in [0,w],\ x \in [b_{n+1}, R].$$

Let

$$U_{n+1} = \left\{x \in K \mid b_{n+1} < \min_{t\in[0,w]} x(t) \le \max_{t\in[0,w]} x(t) < R\right\}$$

$$U_{n+2} = \{x \in K \mid \|x\| < R\}.$$

Now it is easy to prove that $T(\bar{U}_{n+1}) \subset U_{n+1}$ and $T(\bar{U}_{n+2}) \subset U_{n+2}$ and consequently (see [6]):

$$i(T, U_{n+1}, K) = i(T, U_{n+2}, K) = 1$$

Nevertheless, we constructed the family of nonempty, open, and bounded subsets of K: $\{U_j \mid j \in \overline{0, n+2}\}$, with the properties:

$$i(T, U_j, K) = 1, \text{ for all } j \in \overline{0, n+2}, \tag{15}$$

$$U_j \subset U_{n+2}, \text{ for all } j \in \overline{0, n+1}, \tag{16}$$

and

$$U_j \cap U_k = \varnothing \text{ for all } j,k \in \overline{0, n+1} \text{ provided that } j \ne k. \tag{17}$$

Using the additivity propertie of the fixed point index, by (15)-(17) we deduce:

$$i(T, U_{n+2}\setminus(\bigcup_{j=0}^{n+1}\overline{U}_j), K) = i(T, U_{n+2}, K) - \sum_{j=0}^{n+1} i(T, U_j, K) = -(n+1) \neq 0, \qquad (18)$$

since $n \in \mathbf{N}$

Now, (15) and (18) imply that there exist $x_j \in U_j$, for $j = \overline{1, n+1}$, and $x_0 \in U_{n+2}\setminus\left(\bigcup_{j=0}^{n+1}\overline{U}_j\right)$ such that $Tx_j = x_j$, for $j = \overline{0, n+1}$. By definitions of U_j, $j = \overline{0, n+2}$, and by (17), it is obvious that $x_0, x_1, \ldots, x_{n+1}$ are distinct, nontrivial, nonnegative, w-periodic and continuous of Eq. (1) with the properties (6') and (7').

So, our theorem is completely proved. ■

Remarks. 1° In particular, for $n = 0$, $a_0 = 0$, $a_1 = \infty$ condition (3') become superfluous and we obtain Theorem 1 from [2]. But, clearly, our second Theorem is an extension of Theorem 1 from [2] even $n = 0$. Here is an example where Theorem 2 applies and Theorem 1 from [2] does not:

Example 2. Let us consider Eq. (1) with $\tau = 1$ and

$$f(t,x) = \begin{cases} \left(\frac{4}{5}\right)^4 \frac{1}{6}(\sin t + 2)\sqrt{x}, & \text{for } t \in \mathbf{R},\ x \in \left[0, \frac{1}{100}\right[\\ \frac{20^3}{3}(\sin t + 2)\left(x - \frac{1}{20}\right)^4, & \text{for } t \in \mathbf{R},\ x \in \left[\frac{1}{100}, x_0\right[\\ (\sin t + 2)\left[\frac{2}{15}(x-4)^4 + \frac{1}{5}\right], & \text{for } t \in \mathbf{R},\ x \in [x_0, +\infty[\end{cases}$$

where $x_0 \in \left]\frac{1}{5}, 1\right[$ such that $f(t,x)$ is continuous in x_0.

For $n = 0$, $a_0 = \frac{1}{20}$, $\alpha_0 = 2$, $a_1 = 4$, $\alpha_1 = 5$, $b_1 = \frac{1}{5}$ and $g_1(t) = \frac{1}{5}(\sin t + 2)$ we can apply our second theorem, on the other hand:

$$\overline{\lim_{x \to +0}} \frac{f(t,x)}{x} = +\infty = \overline{\lim_{x \to +\infty}} \frac{f(t,x)}{x}$$

and, obviously, Theorem 1 from [2] do not apply.

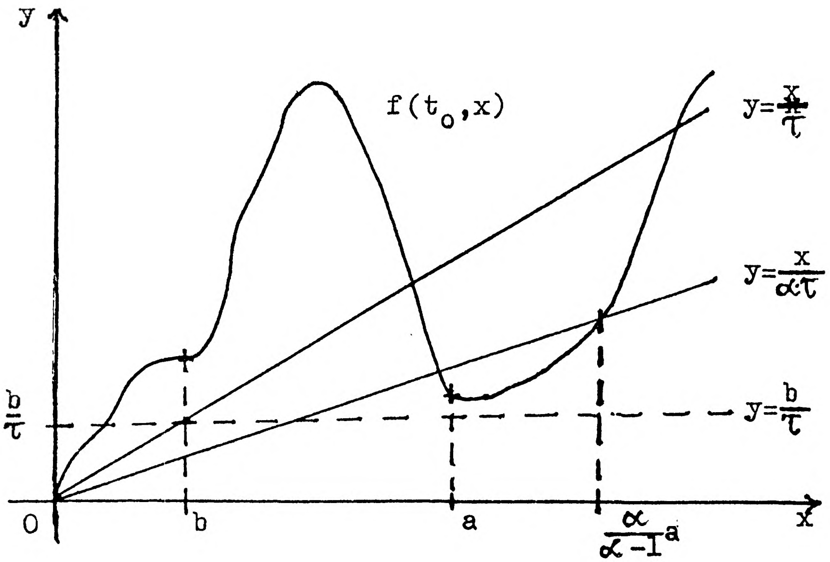
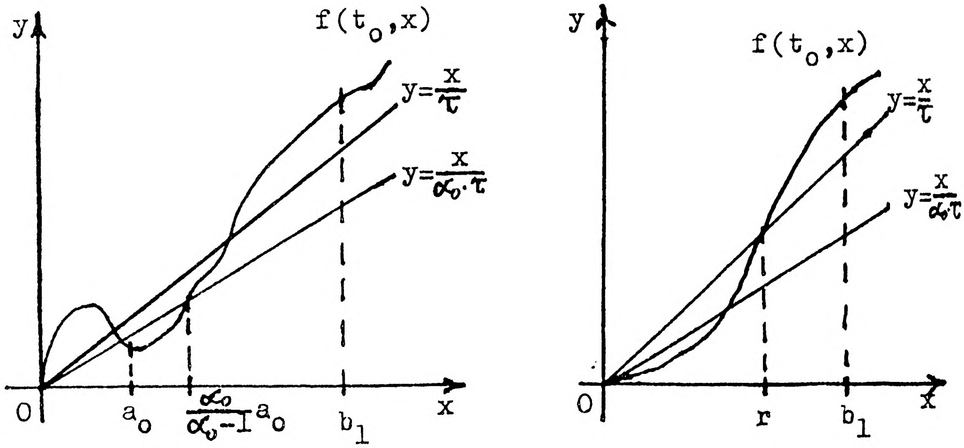


Fig.1



a) $a_o \neq 0$ b) $a_o = 0$

2° (H_3') and (H_4') can be related to (H_3) and (h_4) respectively. The essence is that the function $f(t,x)$, for an arbitrary fixed $t = t_0$, has a similar behaviour on the intervals $x \in \left[b_j, \frac{\alpha_j}{\alpha_j - 1}\right[$, $j = \overline{1, n+1}$, as follows from Fig. 1. The existence of the solution x_0 is caused by the behaviour of the function $f(t = t_0, x)$, for an arbitrary fixed $t = t_0$, on the interval $x \in \left[0, \frac{\alpha_0}{\alpha_0 - 1} a_0\right[$ (if $a_0 \neq 0$, see Fig. 2a) or in a neighborhood of $x = 0$ (if $a_0 = 0$, see Fig. 2b). The conclusion of the second theorem implies that the solutions $x_1, x_2, \ldots, x_{n+1}$ do not intersect, but it is possible that x_0 intersects one or more from the other solutions.

3_0 Theorem 1 holds even $a = \infty$. In this case, we assume that:

$\frac{f(t,a)}{a} = \overline{\lim_{x \to +\infty}} \frac{f(t,x)}{x}$ uniformly with respect to $t \in [0,w]$. With clear modifications, the proof is similar to the calculus of $i(T, U_{n+1}, K)$, when $a_{n+1} = \infty$, from Theorem 2.

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