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ON PRIMITIVE π-SOLVABLE GROUPS

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REZUMAT. - Asupra grupurilor primitive π-rezolubile. Lucrarea conţine unele proprietăţi ale grupurilor finite primitive, în particular ale grupurilor primitive π-resolubile. Rezultate similare cu unele rezultate ale lui O. ORE date in [4] pentru grupuri primitive resolubile sunt obţinute pentru grupuri primitive π-resolubile şi subgrupurile lor normale minimale.

Abstract. The paper contains some properties of finite primitive groups, particularly of primitive groups, particularly of primitive π-solvable groups. Similar results to some of O. ORE given in [4] for primitive solvable groups are obtained for primitive π-solvable groups and their minimal normal subgroups.

1. Preliminaries. All groups considered are finite. We shall denote by π an arbitrary set of primes and by π' the complement to π in the set of all primes. A group G is said to be *π-solvable* if every chief factor of G is either a solvable π-group or a π' -group. Particularly, for π the set of all primes one obtain the notion of solvable group.

Of special interest in the *formation theory* are the so called "primitive groups", which we define below.

DEFINITION 1.1. a) Let G be a group and W be a subgroup of G. We define:

$\mathrm{core}_G W = \cap \{W^g / g \in G\}$,

where $W^g = g^{-1}Wg$.

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b) A maximal subgroup W of G is caled a *stabilizer* of G if $core_G = 1$.

c) The group G is said to be a *primitive group* if there is a subgroup W of G which is a stabilizer of G.

The following BAER's theorems ([1]) will be useful in our considerations:

THEOREM 1.2. *A solvable minimal normal subgroup of a finite group is abelian.*

THEOREM 1.3. *If S is a stabilizer of a finite group G and $N \neq 1$ is a normal subgroup of G, then:* a) $C_G(N) \cap S = 1$; b) $C_G(N) = 1$ *or $C_G(N)$ is a minimal normal subgroup of G, where*

$C_G(N) = \{ g \in G \mid \forall n \in N, gn = ng \}$.

THEOREM 1.4. *If the group G has a maximal subgroup with core 1, then the following two properties of G are equivalent:*

(1) *There is one and only one minimal normal subgroup of G; there is a prime common divisor of the indexes in G of all maximal subgroups with core 1;*

(2) *There is a $\neq 1$ normal solvable subgroup of G.*

2. Primitive groups. We first give some properties of finite primitive groups.

THEOREM 2.1. *Let G be a primitive group and W a stabilizer of G. Then:*

i) *for any normal subgroup $K \neq 1$ of G, we have $KW = G$;*

ii) *for any minimal normal subgroup M of G, we have $MW = G$;*

iii) *there is not a normal subgroup $K \neq 1$ of G such that $K \leq W$.*

Proof. i) Let $K \neq 1$ a normal subgroup of G. Since W is maximal in G we have $KW = W$ or $KW = G$. Suppose that $KW = W$. It follows that $K \leq W$ and so $K^g \leq W^g$ for any $g \in G$. This implies $K \leq core_G W = 1$, hence $K = 1$, contrary to hypothesis.

ii) Follows immediately from i).

iii) Suppose that there is a normal subgroup $K \neq 1$ of G such that $K \leq W$. By i), $KW = G$. But from $K \leq W$ follows $KW = W$. So $W = G$, in contradiction to the fact that W is a maximal subgroup of G. ■

THEOREM 2.2. *If G is a primitive group, then any abelian normal subgroup $A \neq 1$ of G is a minimal normal subgroup of G.*

Proof. Let W be a stabilizer of G. By 2.1.i), $G = AW = WA$. We have that $A \cap W$ is a normal subgroup of G. Indeed, if $x \in G$ and $a \in A \cap W$, let us prove that $x^{-1}ax \in A \cap W$. If $x = wb$, with $w \in W$ and $b \in A$, then

$$x^{-1}ax = (wb)^{-1} a(wb) = (b^{-1}w^{-1})a(wb) = b^{-1}(w^{-1}aw)b.$$

Since $A \cap W$ is normal in W, we have $w^{-1}aw \in A \cap W$ and so $w^{-1} aw \in A$. Using that A is abelian, we obtain

$$x^{-1}ax = b^{-1}b\,(w^{-1}aw) = w^{-1}aw \in A \cap W.$$

Further, $A \cap W = 1$, since if we suppose that $A \cap W \neq 1$, then, $A \cap W$ being a normal subgroup of G with $A \cap W \leq W$, it follows a contradiction with 2.1., iii).

Since $A \neq 1$ is a normal subgroup of G, there is a minimal normal subgroup M of G such that $M \leq A$. We shall prove that $M = A$. Let $a \in A$. By 2.1. ii), $G = MW$. So $a = mw$, where $m \in M$ and $w \in W$. It follows that $w = m^{-1} a \in A \cap W = 1$ and so $w = 1$. Hence $a = m \in M$. ■

3. Minimal normal subgroups of primitive π-solvable groups. The main purpose of this paper is to give some results on minimal normal subgroups in primitive π-solvable groups. These results are a generalization of some ORE's theorems from [4] given for primitive solvable groups.

THEOREM 3.1. *Let G be a primitive π-solvable group. If G has a minimal normal subgroup which is a solvable π-group, then G has one and only one minimal normal subgroup.*

Proof. Let N be a minimal normal subgroup of G such that N is a solvable π-group. By 1.4., G has one and only one minimal normal subgroup. ■

Theorem 3.1. has the following two important corollaries:

COROLLARY 3.2. *If G is a primitive π-solvable group, then G has at most one minimal normal subgroup which is a solvable π-group.*

COROLLARY 3.3. *If a primitive π-solvable group G has a minimal normal subgroup which is a solvable π-group, then G has no minimal normal subgroups which are π' -groups.*

THEOREM 3.4. *If G is a primitive π-solvable group and N is a minimal normal subgroup of G which is a solvable π-group, then* $C_G(N) = N$.

Proof. By 1.2., N is abelian. Hence $N \leq C_G(N)$.

Let W be a stabilizer of G. Using 1.3. b), we obtain that $C_G(N) = 1$ or $C_G(N)$ is a minimal normal subgroup of G. If $C_G(N) = 1$, it follows that $N = 1$, contrary to our hypothesis. Hence $C_G(N)$ is a minimal normal subgroup of G. From this and from $N \neq 1$, $N \leq C_G(N)$, we conclude that $N = C_G(N)$. ■

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ON CERTAIN IDENTITIES FOR MEANS

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REZUMAT. - Despre câteva identităţi pentru medii. Demonstrând identităţi valabile pentru mediile logaritmice, identrice, aritmetice, geometrice, exponenţiale, etc., în această lucrare obţinem metode comune pentru deducerea unor inegalităţi speciale relative la aceste medii.

1. Introduction. Let $I = I(a,b) = \frac{1}{e}(b^b/a^a)^{1/(b-a)}$ for $a \neq b; I(a,a) = a\ (a,b > 0)$ denote the identric mean of the positive real numbers a and b. Similarly, consider the logarithmic mean $L = L(a,b) = (b-a)/\log(b/a)$ for $a \neq b$; $L(a,a) = a$. Usually, the arithmetic and geometric means are denoted by $A = A(a,b) = \frac{a+b}{2}$, and $G = G(a,b) = \sqrt{ab}$, respectively. We shall consider also the exponential mean $E = E(a,b) = (ae^a - be^b)/(e^a - e^b) - 1$ for $a \neq b$; $E(a,a) = a$.

These means are connected to each others by many relations, especially inequalities which are valid for them. For a survey of results, as well as an extended bibliography, see e.q. H.Alzer [1], J.Sándor [6], J.Sándor and Gh.Toader [8]. The aim of this paper is to prove certain identities for these means and to connect these identities with some known results. As it will be shown, exact identities give a powerful tool in proving inequalities. Such a method appears in [6] (Section 4 (page 265) and Section 6 (pp.268-269)), where it is proved that

$$\log \frac{I^2(\sqrt{a},\sqrt{b})}{I(a,b)} = \frac{G-L}{L} \tag{1}$$

where $G = G(a,b)$ etc. This identity enabled the author to prove that (see [6], p.265)

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$$L^2 \leq I \cdot \left(\frac{A+G}{2}\right) \cdot e^{(G-L)/L} \tag{2}$$

and

$$L \leq I \cdot e^{(G-L)/L} \quad \left(\text{i.e. } \log\frac{I}{L} \geq 1 - \frac{G}{L}\right) \tag{3}$$

In a recent paper [9] it is shown how this inequality improves certain known results.

In [10] appears without proof the identity

$$\log\frac{I}{G} = \frac{A-L}{L} \tag{4}$$

We will prove that relations of type (1) and (4) have interesting consequences, giving sometimes short proofs for known results or refinements of these results.

2. Identities and inequalities. Identity (4) can be proved by a simple verification, it is more interesting the way of discovering it. By

$$\log I(a,b) = \frac{b\log b - a\log a}{b-a} - 1 = \frac{b(\log b - \log a)}{b-a} + \log a - 1 \tag{*}$$

it follows that $\log I(a,b) = \dfrac{b}{L(a,b)} + \log a - 1$, and by symmetry, $\log I(a,b) = \dfrac{a}{L(a,b)} + \log b - 1$ $(a \neq b)$ i.e.

$$\log\frac{I}{a} = \frac{b}{L} - 1 \quad \text{and} \quad \log\frac{I}{b} = \frac{a}{L} - 1 \tag{5}$$

Now, by addition of the two identities from (5) we get relation (4). From (5), by multiplication it results:

$$\log\frac{I}{a} \cdot \log\frac{I}{b} = \frac{G^2}{L^2} - 2\frac{A}{L} + 1; \tag{6}$$

and similarly:

$$\log\frac{I}{a} / \log\frac{I}{b} = \frac{b-L}{a-L} \tag{7}$$

An analogous identity to (1) can be proved by considering the logarithm of identric mean. Indeed, apply the formula (*) to $a \to \sqrt[3]{a}$, $b \to \sqrt[3]{b}$. After some elementary transformations, we arrive at:

$$\log \frac{I^3\left(\sqrt[3]{a}, \sqrt[3]{b}\right)}{I(a,b)} = \frac{6\sqrt[3]{G^2 \cdot M}}{L} - 2, \tag{8}$$

where $M = A_{1/3}(a,b) = \left(\frac{\sqrt[3]{a} + \sqrt[3]{b}}{2}\right)^3$ denotes the power mean of order 1/3. (More generally, one defines $A_k = A_k(a,b) = \left(\frac{a^k + b^k}{2}\right)^{1/k}$). Now, Lin's inequality states that $L(u,v) \le M(u,v)$ (see [5]), and Stolarsky's inequality ([11]) that $I(u,v) \ge A_{2/3}(u,v)$. Thus one has

$$I^3(\sqrt[3]{a}, \sqrt[3]{b}) \ge \left\{\left(\frac{(a^{2/3})^{1/3} + (b^{2/3})^{1/3}}{2}\right)^3\right\}^{3/2} \ge L^{3/2}(a^{2/3}, b^{2/3})$$

by the above inequalities applied to $u = a^{1/3}$, $v = b^{1/3}$ and $u = a^{2/3}$, $v = b^{2/3}$, respectively. Thus

$$I^3(\sqrt[3]{a}, \sqrt[3]{b}) \ge L^{3/2}(a^{2/3}, b^{2/3}) \tag{9}$$

This inequality, via (8) gives:

$$L^{3/2}(a^{2/3}, b^{2/3}) \le I(a,b) \cdot e^{6\sqrt[3]{G^2 M}/L - 2} \tag{10}$$

or

$$\log \frac{I}{L^{3/2}(a^{2/3}, b^{2/3})} \ge 2 - \frac{6\sqrt[3]{G^2 M}}{L} \tag{11}$$

This is somewhat similar (but more complicated) to (3).

Finally, we will prove certain less known series representations of $\log \frac{A}{G}$ and $\log \frac{I}{G}$, with applications.

First, let us remark that

$$\log \frac{A(a,b)}{G(a,b)} = \log \frac{a+b}{2\sqrt{ab}} = \log \frac{1}{2}\left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}\right).$$

Put $z = \frac{b-a}{b+a}$ (with $b > a$), i.e. $t = \frac{1+z}{1-z}$, where $t = \frac{a}{b} \in (0,1)$. Since

$$\frac{1}{2}\left(\sqrt{\frac{1+z}{1-z}} + \sqrt{\frac{1-z}{1+z}}\right) = \frac{1}{\sqrt{1-z^2}} \quad \text{and} \quad \log 1/\left(\sqrt{1-z^2}\right) = -\frac{1}{2}\log(1-z^2) =$$

$= \frac{1}{2}z^2 + \frac{1}{4}z^4 + \ldots$ (by $\log(1-u) = -u - \frac{u^2}{2} - \frac{u^3}{3} - \ldots$), we have obtained:

$$\log\frac{A(a,b)}{G(a,b)} = \sum_{k=1}^{+\infty}\frac{1}{2k}\left(\frac{b-a}{b+a}\right)^{2k} \tag{12}$$

In a similar way, we have

$\frac{A}{L} - 1 = \frac{a+b}{2}\left(\frac{\log b - \log a}{b-a}\right) - 1 = \frac{1}{2z}\log\frac{1+z}{1-z} - 1 = \frac{1}{z}\operatorname{arc\,tanh} z - 1 = \frac{z^3}{3} + \frac{z^4}{5} + \ldots,$

implying, in view of (4),

$$\log\frac{I(a,b)}{G(a,b)} = \sum_{k=1}^{+\infty}\frac{1}{2k+1}\cdot\left(\frac{b-a}{b+a}\right)^{2k} \tag{13}$$

The identities (12) and (13) have been transmitted (without proof) to the author by H.J.Seiffert (particular letter). We note that parts of these relations have appeared in other equivalent forms in a number of places. For (13) see e.g. [4] (Nevertheless, (4) is not used, and the form is slightly different).

Clearly, (12) and (13) imply, in a simple manner, certain inequalities.

By $\frac{z^2}{3} < \frac{z^2}{3} + \frac{z^4}{5} + \ldots < \frac{z^2}{3}(1 + z^2 + z^4 + \ldots) = \frac{z^2}{3}\cdot\frac{1}{1-z^2}$

we get

$$1 + (b-a)^2/3(b+a)^2 < \log\frac{I}{G} < 1 + (b-a)^2/12ab \tag{14}$$

improving the inequality $I > G$.

On the same lines, since $\frac{z^2}{2} < \frac{z^2}{2} + \frac{z^4}{4} + \ldots < \frac{1}{2}(z^2 + z^4 + \ldots) = \frac{z^2}{2}\cdot\frac{1}{1-z^2}$ one obtains

$$\frac{1}{2}(b-a)^2/(b+a)^2 < \log\frac{A}{G} < (b-a)^2/8ab \tag{15}$$

3. Applications. We now consider some new applications of the found identities.

a) Since it is well-known that $\log x < x-1$ for all $x > 0$, by (4) we get

$$A\cdot G < L\cdot I \tag{16}$$

discovered by H.Alzer [1]. By considering the similar inequality $\log x > 1 - \frac{1}{x}$ $(x > 0)$, via (4) one obtains

$$\frac{A}{L} + \frac{G}{I} > 2 \qquad (17)$$

due to H.J.Seiffert (Particular letter).

b) The double-inequality $G(x,1) < L(x,1) < A(x,1)$ for $x > 1$ (see the References from [5]) can be written as

$$2\cdot\frac{x-1}{x+1} < \log x < \frac{x-1}{\sqrt{x}} \quad (x > 1) \qquad (18)$$

Let $x = \frac{I}{G} > 1$ in (18). By using (4) one obtains:

$$2\cdot\frac{I-G}{I+G} < \frac{A-L}{L} < \frac{I-G}{\sqrt{IG}}. \qquad (19)$$

These improve (16) and (17), since $2\cdot(I-G)/(I+G) > 1 - G/I$ and $(I-G)/\sqrt{IG} < I/G - 1$. Let us remark also that, since it is known that ([1]) $I - G < A - L$, the right side of (19) implies

$$\sqrt{IG} < \frac{I-G}{A-L}\cdot L < L, \qquad (20)$$

improving $\sqrt{IG} < L$ (see [2]).

c) For an other improvement of (16), remark that the following elementary inequality is known:

$$e^{x} > 1 + x + \frac{x^2}{2} \quad (x > 0) \qquad (21)$$

This can be proved e.g. by the classical Taylor expansion of the exponential function. Now, let $x=A/L-1$ in (21). By (4) one has $I = Ge^{A/L-1} > G\left[\frac{A}{L} + \frac{1}{2}\left(\frac{A}{L}-1\right)^2\right] = \frac{1}{2}G\left(1+\frac{A^2}{L^2}\right)$. Thus we have:

$$G\cdot A < \frac{1}{2}\cdot\frac{G}{L}(L^2 + A^2) < L\cdot I, \qquad (22)$$

since the left side is equivalent with $2LA < L^2 + A^2$. This result has been obtained in cooperation with H.J.Seiffert.

d) Let us remark that one has always $\log\frac{I}{a}\cdot\log\frac{I}{b} < 0$, since, when $a \neq b$, I lies between a and b. So, from (6) one gets

$$G^2 + L^2 < 2A \cdot L, \tag{23}$$

complementing the inequality $2A \cdot L < A^2 + L^2$.

e) By identities (1) and (4) one has

$$\frac{2G + A}{L} = 3 + \log \frac{I^4(\sqrt{a}, \sqrt{b})}{I(a,b) \cdot G(a,b)} \tag{24}$$

In what follows we shall prove that

$$\frac{I^4(\sqrt{a}, \sqrt{b})}{I(a,b) \cdot G(a,b)} \geq 1, \tag{25}$$

thus (by (24)), obtaining the inequality

$$L \leq \frac{2G + A}{3} \tag{26}$$

due to B.C.Carlson [3]. In fact, as we will see, a refinement will be deduced.

Let us define a new mean, namely

$$S = S(a,b) = (a^a \cdot b^b)^{1/2A} = (a^a \cdot b^b)^{1/(a+b)} \tag{27}$$

which is indeed a mean, since if $a < b$, then $a < S < b$. First remark that in [6] (inequality (30)) it is proved that

$$A^2 \leq I(a^2, b^2) \leq S^2(a,b) \tag{28}$$

(However the mean S is not used there). In order to improve (28), let us apply Simpson's quadrature formula (as in [7])

$\int_a^b f(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}(\xi)$, $\xi \in (a,b)$, to the function $f(x) = x \log x$. Since $f^{(4)}(x) > 0$ and $\int_a^b x \log x\, dx = \frac{b^2 - a^2}{4} \log I(a^2, b^2)$ (see [6], relation (3.1)), we can deduce that

$$I^3(a^2, b^2) \leq S^2 \cdot A^4 \tag{29}$$

Now, we note that for the mean S the following representation is valid:

$$S(a,b) = \frac{I(a^2, b^2)}{I(a,b)} \tag{30}$$

This can be discovered by the method presented in part 2 of this paper (see also [6]).

By (29) and (30) one has

$$I(a^2, b^2) \le A^4/I^2(a,b) \tag{31}$$

which is stronger than relation (25). Indeed, we have $I(a^2,b^2) \le A^4/I^2(a,b) \le$ $\le I^4(a,b)/G^2(a,b)$, since this last inequality is

$$I^3 \ge A^2 \cdot G \tag{32}$$

due to the author [7]. Thus we have (by putting $a \to \sqrt{a}$, $b \to \sqrt{b}$ in (31))

$$\frac{I^4(\sqrt{a},\sqrt{b})}{I(a,b)\,G(a,b)} \ge \frac{A^4(\sqrt{a},\sqrt{b})}{I^2(\sqrt{a},\sqrt{b})\,I(a;b)} \ge 1 \tag{33}$$

giving (by (1)):

$$\frac{2G+A}{L} \ge 3 + \log\frac{A^4(\sqrt{a},\sqrt{b})}{I^2(\sqrt{a},\sqrt{b})\cdot I(a,b)} \ge 3, \tag{34}$$

improving (26).

f) If $a < b$, then $a < I < b$ and the left side of (5), by taking into account of (18), implies

$$1 - \frac{a}{I} < 2\cdot\left(\frac{I-a}{I+a}\right) < \frac{b-L}{L} < \frac{I-a}{\sqrt{aI}} < \frac{I}{a} - 1 \tag{35}$$

Remark that the weaker inequalities of (35) yields

$$\frac{b}{L} + \frac{a}{I} > 2 \tag{36}$$

similarly, from (5) (right side) one obtains:

$$\frac{a}{L} + \frac{b}{I} > 2 \tag{37}$$

g) For the exponential mean E a simple observation gives $\log I(e^a, e^b) = E(a,b)$, so via (4) we have

$$E - A = \frac{A(e^a,e^b)}{L(e^a,e^b)} - 1 \tag{38}$$

Since $A > L$, this gives the inequality

$$E > A \tag{39}$$

due to Gh.Toader [11]. This simple proof explains in fact the meaning of (39). Since

$I^3 > A^2 G$ (see [6]), the following refinement is valid:

$$E > \frac{A + 2\log A(e^a, e^b)}{3} > A, \qquad (40)$$

where the last inequality holds by $(e^a + e^b)/2 > e^{(a+b)/2}$, i.e. the Jensen-convexity of e^x.

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A GENERAL MATRIX INEQUALITY FOR COMMUTING MATRICES

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REZUMAT. - O inegalitate matricială generală pentru matrici care comută. În lucrare se arată că o serie de inegalităţi cunoscute pentru numere reale au un analog matricial.

Introduction. In this paper, we show that many known inequalities for real numbers have analogous matrix inequalities in the case of commuting matrices.

We make use of the following notation: $S(J)$ denotes the totality of all Hermitian matrices whose spectra are contained in the interval J. $A \geq B$ means $A - B$ is positive semi-definite, while $A > B$ means $A - B$ is positive definite.

$\bar{J} = J_1 \times J_2 \times \ldots \times J_m$ where $J_1, \ldots, J_m$ are intervals in R. If $\bar{x} = (x_1, \ldots, x_n)$, $x_i \in J_i$, $i = 1, \ldots, m$ we write $\bar{x} \in \bar{J}$. Similarly, if $\bar{A} = (A_1, \ldots, A_m)$, $A_i \in S(J_i)$, $i = 1, \ldots, m$, we write $\bar{A} \in S(\bar{J})$.

If $A_i \in S(J_i)$, $i = 1, \ldots, m$ are pairwise commuting Hermitian matrices, then there exists a Hermitian matrix H and m polynomials $p_i(t)$ $(i = 1, \ldots, m)$ with real coefficients, such that

$$A_i = p_i(H) \, (i = 1, \ldots, m)$$

(see [1, p.77]). Therefore, for $f : \bar{J} \to R$,

$$f(\bar{A}) = f(p_1(H), \ldots, p_m(H)) = F(H) \tag{1}$$

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Let λ_j, $j = 1, \dots, n$ be eigenvalues of H and let μ_{ij} $(j = 1, \dots, n)$ be eigenvalues of $A_i (i = 1, \dots, m)$. Then

$$\mu_{ij} = p_i(\lambda_j) \in J_i,\ j = 1, \dots, n;\ i = 1, \dots, m \tag{2}$$

On the other hand, there exists a unitary matrix U such that ([1, p.71])

$$\begin{aligned} f(\bar{A}) &= F(H) = U \operatorname{diag}[F(\lambda_1), \dots, F(\lambda_n)]\, U^* \\ &= U \operatorname{diag}[f(\mu_{11}, \dots, \mu_{m1}), \dots, f(\mu_{1n}, \dots, \mu_{mn})]\, U^* \end{aligned} \tag{3}$$

This leads to the following

THEOREM 1. *Let $f : \bar{J} \to R$ be a continuous function such that*

$$f(\bar{x}) \geq 0,\ \forall \bar{x} \in \bar{J}. \tag{4}$$

If $\bar{A} \in S(\bar{J})$ and all A_i are pairwise commuting Hermitian matrices, then

$$f(\bar{A}) \geq 0. \tag{5}$$

Proof. From (4), it follows that every diagonal element in the diagonal matrix in (3) is non-negative. This gives (5).

Applications.

THEOREM 2. *Let $f : \bar{J} \to R$ be a continuous convex function. Then f is also a matrix convex function of several variables on the set of commuting matrices $\bar{A} \in S(\bar{J})$.*

Proof. Let $\lambda \in (0, 1)$, $\bar{x}, \bar{y} \in \bar{J}$. Further, let $g(\bar{x}, \bar{y})$ be defined by

$$g(\bar{x}, \bar{y}) = \lambda f(\bar{x}) + (1 - \lambda) f(\bar{y}) - f(\lambda \bar{x} + (1 - \lambda)\bar{y})$$

Since the function f is convex, we have that

$$g(\bar{x}, \bar{y}) \geq 0 \tag{6}$$

which, by Theorem 1, for $\bar{x} \to (\bar{x}, \bar{y})$, implies

$$g(\bar{A}, \bar{B}) \geq 0 \tag{7}$$

where $\bar{A}, \bar{B} \in S(\bar{J})$, i.e.,

$$f(\lambda\bar{A} + (1-\lambda)\bar{B}) \leq \lambda f(\bar{A}) + (1-\lambda) f(\bar{B}) \tag{8}$$

which is the definition of a matrix convex function of several variables (see, for example, [2]).

We can now use known results for matrix-convex functions obtained from (8), and give corresponding results for the commuting case. Moreover, we can use directly, known real inequalities to obtain more general results.

THEOREM 3. (Jensen's inequality). *Let* $C_{ji}, w_j, j = 1, \ldots, n,\ i = 1, \ldots, m$ *be commuting matrices such that* $\bar{C}_j = (C_{j1}, \ldots, C_{jm}) \in S(\bar{J}),\ j = 1, \ldots, n$ *and* $w_j \in S(0, \infty),\ j = 1, \ldots, n$. *If* $f: \bar{J} \to R$ *is a continuous convex function, then*

$$f\left(\frac{1}{W_n}\sum_{j=1}^{n} w_j \bar{C}_j\right) \leq \frac{1}{W_n}\sum_{j=1}^{n} w_j f(\bar{C}_j) \tag{9}$$

where $W_n = \sum_{i=1}^{n} w_i$.

Proof. Let us consider the function

$$g(\bar{x}_1, \ldots, \bar{x}_n, w_1, \ldots, w_n) = \frac{1}{W_n}\sum_{j=1}^{n} w_j f(x_j) - f\left(\frac{1}{W_n}\sum_{j=1}^{n} w_j \bar{x}_j\right).$$

We have

$$g(\bar{x}_1, \ldots, \bar{x}_n, w_1, \ldots, w_n) \geq 0$$

and (5) gives (9).

Remark. The above result for $m = 1$ was obtained in [3]. Similarly some of our subsequent results will reduce to those found in [3] for $m = 1$. However, here we only give results that cannot be obtained by the method of [3].

We now list operator inequalities but only give references to the corresponding discrete inequalities.

THEOREM 4. *Let* $C_{ji}, w_j, j = 1, \ldots, n,\ i = 1, \ldots, m$ *be commutable matrices such that* $\bar{C}_j \in S(\bar{J}), j = 1, \ldots, n$; $\frac{1}{W_n}\sum_{i=1}^{n} w_i \bar{C}_i \in S(\bar{J})$

$$w_1 > 0, \ w_i < 0, \ i = 2, \dots, n, \ W_n > 0. \tag{10}$$

If $f : \bar{J} \to \mathbf{R}$ is a convex function, then the reverse inequality in (9) *holds.*

Now let us consider an index set function

$$F(J) = W_J f(A_J(C; w)) - \sum_{i \in J} w_i f(\bar{C}_i)$$

where

$$W_J = \sum_{i \in J} w_i, \ A_J(C; w) = \frac{1}{W_J} \sum_{i \in J} w_i \bar{C}_i.$$

THEOREM 5. *Let f be a matrix convex function on $\bar{J}$, T and K are two finite nonempty subsets of* $\mathbf{N}$ *such that $T \cap K = \phi$, $w = (w_i)_{i \in T \cup K}$ and $C = (C_{ij})_{i \in T \cup K, j \in \{1, \dots, m\}}$ is a set of commutable matrices such that $\bar{C}_i \in S(\bar{J})$, $w_i \in S(R) (i \in T \cup K)$, $W_{T \cup K} > 0$, $A_t(C; w) \in S(\bar{J})$ $(t = T, K, T \cup K)$.*

If $W_T > 0$ and $W_K > 0$, then

$$F(T \cup K) \le F(T) + F(K). \tag{11}$$

If $W_T W_K < 0$, we have the reverse inequality in (11).

THEOREM 6. *If $w_i > 0$, $i = 1, \dots, n$, $I_k = \{1, \dots, k\}$, then*

$$F(I_n) \le F(I_{n-1}) \le \dots \le F(I_2) \le 0, \tag{12}$$

but if (10) *is valid and $A_{I_n}(C; w) \in S(J)$ then the reverse inequalities in* (12) *are valid.*

Theorems 4-6 in the real case are obtained in [5], [6], [4].

THEOREM 7. [7]. *Let the conditions of Theorem 3 be satisfied. Then*

$$\begin{aligned} f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i \bar{C}_i\right) &= f_{n,n} \le \dots \le f_{k+1,n} \le f_{k,n} \le \dots \le f_{1,n} \\ &= \frac{1}{W_n}\sum_{i=1}^{n} w_i f(\bar{C}_i), \end{aligned} \tag{13}$$

where

$$f_{k,n} := \frac{1}{\binom{n-1}{k-1} W_n} \sum_{1 \le i_1 < \ldots < i_k \le n} (w_{i_1} + \ldots + w_{i_k}) f\left(\frac{w_{i_1}\bar{C}_{i_1} + \ldots + w_{i_k}\bar{C}_{i_k}}{w_{i_1} + \ldots + w_{i_k}} \right).$$

THEOREM 8. [8]. *Let the conditions of Theorem 3 be fulfilled. Then*

$$f\left(\frac{1}{W_n} \sum_{i=1}^{n} w_i \bar{C}_i \right) \le \ldots \le \bar{f}_{k+1,n} \le \bar{f}_{k,n} \le \ldots \le \bar{f}_{1,n} = \frac{1}{W_n} \sum_{i=1}^{n} w_i f(\bar{C}_i), \tag{14}$$

where

$$\bar{f}_{k,n} = \frac{1}{\binom{n+k-1}{k-1} W_n} \sum_{1 \le i_1 \le \ldots \le i_k \le n} (w_{i_1} + \ldots + w_{i_k}) f\left(\frac{w_{i_1}\bar{C}_{i_1} + \ldots + w_{i_k}\bar{C}_{i_k}}{w_{i_1} + \ldots + w_{i_k}} \right).$$

THEOREM 9. [9]. *Let the conditions of Theorem 3 be fulfilled. Then*

$$f\left(\frac{1}{W_n} \sum_{i=1}^{n} w_i \bar{C}_i \right) \le \underset{-k+1,n}{f} \le \underset{-k,n}{f} \le \ldots \le \underset{-1,n}{f} = \frac{1}{W_n} \sum_{i=1}^{n} w_i f(\bar{C}_i) \tag{15}$$

where $1 \le k \le n-1$, *and*

$$\underset{-k,n}{f} = \frac{1}{W_n^k} \sum_{i_1,\ldots,i_k=1} w_{i_1} \ldots w_{i_k} f\left(\frac{1}{k} (\bar{C}_{i_1} + \ldots + \bar{C}_{i_k}) \right).$$

THEOREM 10. [10,11]. *Let the conditions of Theorem 3 be fulfilled and let* q_i, $i = 1, \ldots, k$, *with* $Q_k := \sum_{i=1}^{k} q_i$, *be also strictly positive operators permutable with* $\{C_{ji}\}$ *and* $\{w_j\}$, *then*

$$f\left(\frac{1}{W_n} \sum_{i=1}^{n} w_i \bar{C}_i \right) \le \frac{1}{W_n^k} \sum_{i_1,\ldots,i_k=1}^{n} w_{i_1} \ldots w_{i_k} f\left(\frac{1}{Q_k} \sum_{j=1}^{k} q_j \bar{C}_{i_j} \right)$$

$$\le \frac{1}{W_n} \sum_{i=1}^{n} w_i f(\bar{C}_i). \tag{16}$$

THEOREM 11. [12,13]. *Let the conditions of Theorem 3 be fulfilled and let* $\tilde{C} = \frac{1}{W_n} \sum_{i=1}^{n} w_i \bar{C}_i$, $t_i \in [0,1]$, $i = 1, \ldots, k-1$. *Then*

$$f\left(\frac{1}{W_n} \sum_{i=1}^{n} w_i \bar{C}_i \right) \le \tilde{f}_{n,1} \le \quad \le$$

$$\le \frac{1}{W_n^k} \sum_{i_1,\ldots,i_k=1} w_{i_1} \ldots w_{i_k} f((\bar{C}_{i_1}(1 - t_1) + \quad \tag{17}$$

$$+ \bar{C}_{i_k} t_i \dots t_{k-1}) \le \frac{1}{W_n} \sum_{i=1}^{n} w_i f(\bar{C}_i).$$

where

$$\tilde{f}_{n,k} = \frac{1}{W_n^k} \sum_{i_1, \dots, i_k = 1}^{n} w_{i_1} \dots w_{i_k} f(\bar{C}_{i_1} (1 - t_1)$$

$$+ \sum_{j=1}^{k-1} \bar{C}_{i_{j+1}} (1 - t_{j+1}) t_1 \dots t_j + \tilde{C} t_1 \dots t_k).$$

THEOREM 12. [14]. *Let q_i, $A_i \in S(R^m)$, $i = 1, \dots, n$ be commuable matrices, and let the function g be defined by*

$$g(x) = \sum_{i=1}^{n} \frac{1}{q_i} f\left(q_i x A_i + (r - x) \sum_{k=1}^{n} A_k \right)$$

where $q_i > 0$, $i = 1, \dots, n$, *with* $\sum_{k=1}^{n} (1/q_k) = I$, $r \in R$, $q_i x A_i + (r - x) \sum_{k=1}^{n} A_k \in S(\bar{T})$, $i = 1, \dots, n$, *for all x from an interval J from R. If $f : \bar{T} \to R$ is a convex function and if* $|x| \le |y| (xy > 0, y \in J)$, *then*

$$g(x) \le g(y). \tag{18}$$

The function g is also convex.

Remark. Using the substitutions: $1/q_i \to w_i \left(\sum_{i=1}^{n} w_i = I \right)$, $q_i A_i \to X_i$, $r = 1$, we get that (18) is also valid if

$$g(x) = \sum_{i=1}^{n} w_i f\left(x X_i + (1 - x) \sum_{k=1}^{n} w_k X_k \right).$$

Remark. For some further generalizations of some of the previous results, see [15] and the references given there.

THEOREM 13. [16]. *Let $A_j, w_j, j = 1, \dots, n$ be commuting matrices such that $A_j \in S(J)$, $w_j \in (0, \infty)$, $j = 1, \dots, n$, $J = [m, M]$. Let $f: J \to R$ be a continuous convex function and let $F: \tilde{J}^2 \to R$ be continuous and increasing in its first variable, $\tilde{J}$ an interval in the range of f. Then*

$$F\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i f(A_i), f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i A_i\right)\right) \le \left\{\max_{m\le x\le M}\left[F\left(\frac{M-x}{M-m}f(m)+\frac{x-m}{M-m}f(M), f(x)\right)\right]\right\} I. \tag{19}$$

Remark. Inequality (19) is a generalization of corresponding results from [3].

Let $J \subseteq R$ and suppose $M: J \to R$ is continuous and strictly monotone. Let $A = (A_1, \ldots, A_n)$ be an n-tuple with elements from $S(J)$, $w = (w_1, \ldots, w_n)$ an n-tuple with elements from $S(0,\infty)$ such that $W_n = \sum_{i=1}^{n} w_i > 0$, and let all matrices A_i, w_i be commutable. Then the quasi-arithmetic M-mean of A with weight w is

$$M_n(A; w) = M^{-1}\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i M(A_i)\right). \tag{20}$$

We now give generalizations of some results for three quasi-arithmetic means obtained in [3]. (We use analogous real case from [4, pp. 248-253]).

Let $K: J_1 \to R$, $L: J_2 \to R$, $M: J_3 \to R$, $f: J_1 \times J_2 \to J_3$ be continuous functions, M increasing. Consider the inequality

$$f(K_n(A; w), L_n(B; w)) \ge M_n(f(A,B); w), \tag{21}$$

(here $f(A,B) = (f(A_1,B_1), \ldots, f(A_n,B_n))$) where $A = (A_1, \ldots, A_n)$, $B = (B_1, \ldots, B_n)$, $w = (w_1, \ldots, w_n)$, are n-tuples whose elements $A_i \in S(J_1)$, $B_i \in S(J_2)$, $w_i \in S(0,\infty)$, $i = 1, \ldots, n$, are commutable metrices.

THEOREM 14. *If the function*

$$\tilde{H}(s,t) = M(f(K^{-1}(s), L^{-1}(t)))$$

is concave, then (21) *holds. If* $\tilde{H}$ *is convex, then inequality* (21) *is reversed.*

THEOREM 15. [16] *Let* $\phi(u,v)$ *be a continuous real-valued function defined on* $J^2 (J = [m,M])$, *nondecreasing in u, G increasing and convex with respect to F. Let A and w be n-tuples of operators,* $A_i \in S(J)$, $w_i \in S(0,\infty)$, *all permutable. Then*

$$\phi(G_n(A;w), F_n(A;w)) \le$$
$$\left\{\max_{\theta\in[0,1]} \phi\left[G^{-1}(\theta G(m) + (1-\theta)G(M)),\ F^{-1}(\theta F(m) + (1-\theta)F(M))\right]\right\} I. \quad (22)$$

The quasi-arithmetic mean (20) can be generalized as follows:

Let $\phi : J \to R_+$ be a strictly positive function, $F : J \to R$ a strictly monotone function $A \in S(J)^n$, $w \in S(0,\infty)$. Then define, for permutable A_i, w_i,

$$F_n(A;\phi) = F^{-1}\left(\frac{\sum_{i=1}^{n} w_i \phi(A_i) F(A_i)}{\sum_{i=1}^{n} w_i \phi(A_i)}\right) \quad (23)$$

THEOREM 16 [17]. *Let K,L,M be three differentiable strictly monotone functions from the closed interval J to R; let ϕ,ψ,χ be three functions from J to R_+, $f: J^2 \to J$ and let $A, B \in S(J)^n$. Then*

$$f(K_n(A;\phi), L_n(B;\psi)) \ge M_n(f(A,B);\chi), \quad (24)$$

where $f(A,B) = (f(A_1,B_1), \ldots, f(A_n,B_n))$, holds if for all $u,v,s,t \in J$ the following inequality holds.

$$\left(\frac{M\circ f(u,v) - M\circ f(t,s)}{M'\circ f(t,s)}\right)\frac{\chi\circ f(u,v)}{\chi\circ f(t,s)} \le$$

$$\left(\frac{K(u)-K(t)}{K'(t)}\right)\frac{\phi(u)}{\phi(t)} f_1'(t,s) + \left(\frac{L(v)-L(s)}{L'(s)}\right)\frac{\psi(v)}{\psi(s)} f_2'(t,s).$$

Remark. For some special cases of theorem 16 see [3].

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A NOTE ON CERTAIN NEW INTEGRODIFFERENTIAL INEQUALITIES

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REZUMAT. - Notă asupra unor noi inegalităţi integrodiferenţiale. Scopul acestei note este de a stabili două noi inegalităţi integrodiferenţiale.

Abstract. The object of this note is to establish two new integrodifferential inequalities involving a function and its derivatives by using a fairly elementary analysis.

1. Introduction. In a paper published in 1932, G.H.Hardy and J.E.Littlewood [3] proved the following integrodifferential inequality:

$$\left(\int_0^\infty f'^{\,2} ds\right)^2 \le 4\left(\int_0^\infty f^2 ds\right)\left(\int_0^\infty f''^{\,2} ds\right), \tag{1}$$

where f is a real-valued, twice continuously differentiable function on $[0, \infty)$ and such that f and f'' are in the space $L^2(0,\infty)$. Due to its importance in various applications, this result has attracted a great many authors over the years and a number of papers related to this inequality have appeared in the literature. An excellent survey of the work on such inequalities together with many references are contained in the recent paper by W.D.Evans and W.N.Everitt [1]. The main purpose of the present note is to establish two new integral inequalities involving a real-valued function and its first and second derivative which claim their origin to the well known Hardy-Littlewood inequality given in (1). The analysis used in the proof is elementary and our results provide new estimates on this type of inequality.

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2. Main results. We first introduce some notations which will be used to formulate our results.

$$P_\alpha[t, f'(t)] = t^{\alpha+1} f'^2(t),$$

$$Q_\alpha[t, f(t)] = t^\alpha f^2(t),$$

$$R_\alpha[t, f'(t), f''(t)] = t^\alpha[(\alpha+1)^2 f'^2(t) + t^2 f''^2(t)],$$

$$S_\alpha[t, f(t), f'(t), f''(t)] = t^{\alpha+1}(t+\alpha+2) f^2(t) + t^{\alpha+1}(2t+\alpha+2) f'^2(t) + t^{\alpha+2} f''^2(t),$$

$$A_\alpha[t, f(t), f'(t)] = t^\alpha[(\alpha+1) f^2(t) + t f'^2(t)],$$

$$B_\alpha[t, f'(t), f''(t)] = t^\alpha[(2t+\alpha+1)^2 f'^2(t) + t^2 f''^2(t)],$$

$$C_\alpha[t, f(t), f'(t), f''(t)] = \frac{3}{2} t^{\alpha+1}(t+\alpha+2) f^2(t) + \frac{1}{2} t^{\alpha+1}(4t+\alpha+2) f'^2(t) + \frac{1}{2} t^{\alpha+2} f''^2(t),$$

where $\alpha \geq 0$ is a real constant and $f(t)$ is a real-valued function defined on $I = [0, \infty)$ and $f'(t)$, $f''(t)$ are the first and second derivatives of the function $f(t)$ for $t \in I$.

Our main result is given in the following theorem.

THEOREM 1. *Let $\alpha \geq 0$ be a real constant and f be a real-valued continuous function defined on I such that f' and f'' exist and continuous on I such that* $\int_0^\infty Q_\alpha[t, f(t)]\,dt < \infty$, $\int_0^\infty R_\alpha[t, f'(t), f''(t)]\,dt < \infty$, $\int_0^\infty S_\alpha[t, f(t), f'(t), f''(t)]\,dt < \infty$.

Then

$$\left(\int_0^\infty P_\alpha[t, f'(t)]\,dt\right)^2 \leq 4\left(\int_0^\infty Q_\alpha[t, f(t)]\,dt\right)\left(\int_0^\infty R_\alpha[t, f'(t), f''(t)]\,dt\right). \tag{2}$$

Proof. Let a be a real constant such that $0 < a < \infty$. Integrating by parts we have the following identity

$$\int_0^a \left[t^{\alpha+1} - \frac{1}{a} t^{\alpha+2}\right] f'^2(t)\,dt = -\int_0^a f(t) \frac{d}{dt}\left(\left[t^{\alpha+1} - \frac{1}{a} t^{\alpha+2}\right] f'(t)\right) dt. \tag{3}$$

From (3) we observe that

$$\int_0^a P_\alpha[t,f'(t)]\,dt = \frac{1}{a}\int_0^a [t^{\alpha+2}f'^2(t) + t^{\alpha+2}f(t)f''(t) + (\alpha+2)t^{\alpha+1}f(t)f'(t)]\,dt$$

$$- \int_0^a \left(t^{\frac{\alpha}{2}}f(t)\right)\left(t^{\frac{\alpha}{2}}[tf''(t) + (\alpha+1)f'(t)]\right)dt. \tag{4}$$

Squaring both sides of (4) and using the elementary inequalities

$(c_1+c_2)^2 \le 2\left(c_1^2+c_2^2\right)$, $c_1c_2 \le \frac{1}{2}\left(c_1^2+c_2^2\right)$, ($c_1, c_2$ reals) and Schwarz inequality, we observe that

$$\left(\int_0^a P_\alpha[t,f'(t)]\,dt\right)^2$$

$$\le 2\left[\frac{1}{a^2}\left(\int_0^a [t^{\alpha+2}f'^2(t) + t^{\alpha+2}f(t)f''(t) + (\alpha+2)t^{\alpha+1}f(t)f'(t)]\,dt\right)^2\right.$$

$$\left.+\left(\int_0^a \left(t^{\frac{\alpha}{2}}f(t)\right)\left(t^{\frac{\alpha}{2}}[tf''(t) + (\alpha+1)f'(t)]\right)dt\right)^2\right]$$

$$\le 2\left[\frac{1}{a^2}\left(\int_0^a \left[t^{\alpha+2}f'^2(t) + t^{\alpha+2}\frac{1}{2}\left(f^2(t) + f''^2(t)\right)\right.\right.\right.$$

$$\left.\left.+ (\alpha+2)t^{\alpha+1}\frac{1}{2}\left(f^2(t) + f'^2(t)\right)\right]dt\right)^2$$

$$\left.+\left(\int_0^a \left(t^{\frac{\alpha}{2}}f(t)\right)\left(t^{\frac{\alpha}{2}}[tf''(t) + (\alpha+1)f'(t)]\right)dt\right)^2\right]$$

$$\le \frac{2}{a}\int_0^a \left[\frac{1}{2}t^{\alpha+1}(t+\alpha+2)f^2(t) + \frac{1}{2}t^{\alpha+1}(2t+\alpha+2)f'^2(t) + \frac{1}{2}t^{\alpha+2}f''^2(t)\right]^2 dt$$

$$+ 2\left(\int_0^a Q_\alpha[t,f(t)]\,dt\right)\left(\int_0^a t^\alpha[tf''(t) + (\alpha+1)f'(t)]^2\,dt\right)$$

$$\le \frac{1}{2a}\int_0^a \left(S_\alpha[t,f(t),f'(t)f''(t)]\right)^2 dt$$

$$+ 4\left(\int_0^a Q_\alpha[t,f(t)]\,dt\right)\left(\int_0^a R_\alpha[t,f'(t),f''(t)]\,dt\right). \tag{5}$$

Now by letting $a \to \infty$ on both sides of (5), we get the desired inequality in (2). This completes the proof.

A slightly different version of Theorem 1 is established in the following theorem.

THEOREM 2. *Let* α, f, f', f'' *be as defined in Theorem* 1 *and such that* $\int_0^\infty Q_\alpha[t, f(t)]\,dt < \infty$, $\int_0^\infty B_\alpha[t, f'(t), f''(t)]\,dt < \infty$,

$\int_0^\infty C_\alpha[t, f(t), f'(t), f''(t)]\,dt < \infty$. *Then*

$$\left(\int_0^\infty A_\alpha[t, f(t), f'(t)]\,dt\right)^2 \le 4\left(\int_0^\infty Q_\alpha[t, f(t)]\,dt\right)\left(\int_0^\infty B_\alpha[t, f'(t), f''(t)]\,dt\right). \tag{6}$$

Proof. Let a be a real constant such that $0 < a < \infty$. In view of the hypotheses we have the identity (4) obtained in the proof of Theorem 1. Integrating by parts we have the following identity:

$$\int_0^a \frac{d}{dt}\left[t^{\alpha+1} - \frac{1}{a}t^{\alpha+2}\right]f^2(t)\,dt = -2\int_0^a \left[t^{\alpha+1} - \frac{1}{a}t^{\alpha+2}\right]f(t)f'(t)\,dt. \tag{7}$$

From (7) we observe that

$$\begin{aligned}\int_0^a (\alpha+1)t^\alpha f^2(t)\,dt = {} & \frac{1}{a}\int_0^a \left[(\alpha+2)t^{\alpha+1}f^2(t) + 2t^{\alpha+2}f(t)f'(t)\right]dt \\ & - 2\int_0^a t^{\alpha+1}f(t)f'(t)\,dt.\end{aligned} \tag{8}$$

From (4) and (8) we observe that

$$\begin{aligned}&\int_0^a A_\alpha[t, f(t), f'(t)]\,dt \\ &\quad = \frac{1}{\alpha}\int_0^a \big[t^{\alpha+2}f'^2(t) + t^{\alpha+2}f(t)f''(t) + (\alpha+2)t^{\alpha+1}f(t)f'(t) \\ &\quad + (\alpha+2)t^{\alpha+1}f^2(t) + 2t^{\alpha+2}f(t)f'(t)\big]dt \\ &\quad - \int_0^a \left(t^{\frac{\alpha}{2}}f(t)\right)\left(t^{\frac{\alpha}{2}}\left[tf''(t) + (2t+\alpha+1)f'(t)\right]\right)dt.\end{aligned} \tag{9}$$

Now by following exactly the same arguments as in the proof of Theorem 1 given below the identity (4) with suitable changes, we get the desired inequality in (6). The proof is complete.

Finally we note that, in the special case when $\alpha = 0$, the inequalities in (2) and (6) reduce to the inequalities which we believe are new to the literature and are of independent interest.

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ON SOME LINEAR INEQUALITIES

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REZUMAT. - Asupra unor inegalităţi liniare. Lucrarea conţine o nouă demonstraţie a unei teoreme a lui Tiberiu Popoviciu, precum şi aplicaţii la studiul unor inegalităţi.

1. Let X be a real linear space and S a connected topological space. Consider a linear functional b on X and also a family $A = \{a_s : s \in S\}$ of linear functionals on X. For $x \in X$ let us denote by v_x the function $v_x(s) = a_s(x)$, $s \in S$.

Suppose that

(1) There exists $x_0 \in X$ such that $a_s(x_0) > 0$ for all $s \in S$.

(2) If $x \in X$ and $a_s(x) > 0$ for every $s \in S$, then $b(x) > 0$.

(3) For each $x \in X$, v_x is continuous on S.

Under the above assumptions we have

THEOREM 1 ([6], [7]). *For every $x \in X$ there exists $s \in S$ such that $b(x)/b(x_0) = a_s(x)/a_s(x_0)$.*

Related results and many applications are given in [6]-[9]. In this note we present some more applications.

2. Let $n \geq 2$, $X = C[0,1]$ and

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$$S = \left\{ (t_1, \ldots, t_n) \in R^n \quad 0 \le t_1 < \ldots < t_n \le 1 \,,\ t_2 - t_1 = \ldots = t_n - t_{n-1} \right\}.$$

For $s = (t_1, \ldots, t_n) \in S$ let $a_s(x) = [t_1, \ldots, t_n ; x]$, $x \in C[0,1]$.

Here $[t_1, \ldots, t_n ; x]$ denotes the divided difference of the function x on $t_1, \ldots, t_n$.

Let $0 \le u_1 < \ldots < u_n \le 1$ be fixed and consider the functional $b(x) = [u_1, \ldots, u_n ; x]$, $x \in C[0,1]$. Denote $x_0(t) = t^{n-1}$, $t \in [0,1]$. Then $b(x_0) = a_s(x_0) = 1$, $s \in S$. Condition (2) is satisfied: see, for example, [2,pp.244-246]. Clearly condition (3) is also satisfied.

By applying Theorem 1 we obtain an alternative proof of the following result of T.Popoviciu (see also [3]).

THEOREM 2 ([5]). *Given* $0 \le u_1 < \ldots < u_n \le 1$, *for every* $x \in C[0,1]$ *there exist* *n equidistant points* $t_1 < \ldots < t_n$ *in* [0,1] *such that* $[u_1, \ldots, u_n ; x] = [t_1, \ldots, t_n ; x]$.

3. Let μ be a probability measure on [0,1] whose support does not reduce to a single point. Let

$$b(x) = \int_0^1 x(t)\,d\mu(t) - x\left(\int_0^1 t\,d\mu(t)\right), \quad x \in C[0,1].$$

Let $S = \left\{ (t_1, t_2, t_3) \in R^3 : 0 \le t_1 < t_2 < t_3 \le 1 \,,\ t_3 - t_2 = t_2 - t_1 \right\}$.

Denote $a_s(x) = [t_1, t_2, t_3 ; x]$, $x \in C[0,1]$, $s = (t_1, t_2, t_3) \in S$.

Let $x_0 \in C[0,1]$ be a strictly convex function. From Theorem 1 we deduce

THEOREM 3. *For every* $x \in C[0,1]$ *there exists* $s = (t_1, t_2, t_3) \in S$ *such that*

$$\left(\int_0^1 x(t)\,d\mu(t) - x\left(\int_0^1 t\,d\mu(t)\right)\right) \Big/ \left(\int_0^1 x_0(t)\,d\mu(t) - x_0\left(\int_0^1 t\,d\mu(t)\right)\right) = [t_1, t_2, t_3 ; x] / [t_1, t_2, t_3 ; x_0].$$

This is an improvement of a result of H.Alzer [1]. Related results involving probability Radon measures on compact convex subsets of locally convex spaces are to be found in [6] and [8].

4. In the above setting, instead of the given functional b let us consider

$$b(x) = \int_0^1 x(t)\left(\frac{1}{6} - t + t^2\right)dt, \; x \in C[0,1].$$

Condition (2) is satisfied by virtue of an inequality of Levin and Stečkin [4]. Theorem 1 shows that for every $x \in C[0,1]$ there exists $s = (t_1, t_2, t_3) \in S$ such that

$$\left(\int_0^1 x(t)\left(\frac{1}{6} - t + t^2\right)dt\right) \Big/ \left(\int_0^1 x_0(t)\left(\frac{1}{6} - t + t^2\right)dt\right) = [t_1, t_2, t_3; x] / [t_1, t_2, t_3; x_0].$$

Similar results can be stated for many known functionals b which are strictly positive on strictly convex functions.

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ON THE HALLENBECK AND RUSCHEWEYH TYPE OF DIFFERENTIAL SUBORDINATION

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REZUMAT. - Asupra subordonării diferenţiale de tip Hallenbeck-Ruscheweyh. În lucrare sunt date condiţii suficiente asupra lui q şi γ, pentru ca implicaţia (1) să aibă loc.

1. Introduction. Let $H(U)$ be the space of functions which are analytic in the unit disc U. If $f, g \in H(U)$, we say that f is subordinate to g, written $f(z) \prec g(z)$, if g is univalent in U, $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

We denote by $S^c(\alpha)$, $\alpha < 1$, the class of functions $f(z) = z + a_2 z^2 + \ldots$ which are analytic in U and satisfy

$$\operatorname{Re}(1 + zf''(z)/f'(z)) > \alpha, \quad z \in U$$

and $S^C_{(\alpha)}$ is called the class of convex functions of order α; furthmore, $S^C = S^c(0)$, represents the class of convex functions in U.

In [1], Hallenbeck and Ruscheweyh proved that if $h(z) = q(z) + \frac{1}{\gamma} z q'(z)$ is convex and univalent in U, where $q \in H(U)$, $\operatorname{Re}\gamma \geq 0$, $\gamma \neq 0$ and $p \in H(U)$, then

$$p(z) + \frac{1}{\gamma} z p'(z) \prec h(z) \text{ implies } p(z) \prec q(z). \tag{1}$$

A natural problem is to find conditions on h, or on q for which the implication (1) is true for some negative values of $\operatorname{Re}\gamma$. In our paper we determine sufficient conditions on q and γ, where $\operatorname{Re}\gamma$ may be negative, so that the implication (1) holds.

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2. Preliminaries. We shall need the following definitions and lemmas to prove our main results.

We denote by $s^*(\alpha)$, $\alpha < 1$, the class of functions $f(z) = z + a_2 z^2 + \dots$ which are analytic in U and satisfy

$$\operatorname{Re} zf'(z)/f(z) > \alpha, \quad z \in U$$

and $S^*(\alpha)$ represents the class of starlike functions of order α in U.

LEMMA A [5]. *If* $f \in S^*(\alpha)$ *where* $\alpha \in [-1/2, 1)$ *and* $F(z) = \frac{2}{z}\int_0^z f(t)\,dt$, *then* $F \in S^*(\delta(\alpha))$ *where*

$$\delta(\alpha) = \begin{cases} \dfrac{\alpha(2\alpha - 1)}{2(2^{-2\alpha} + \alpha - 1)} - 1, & \text{for } \alpha \neq 1/2 \text{ and } \alpha \neq 0 \\ \dfrac{1}{2(1 - \ln 2)} - 1, & \text{for } \alpha = 1/2 \\ \dfrac{1}{2(2\ln 2 - 1)} - 1, & \text{for } \alpha \neq 0 \end{cases} \tag{2}$$

and this result is sharp.

LEMMA B [6]. *The function* $\delta(\alpha)$ *is continuous, positive, increasing and convex on* $[-1/2, 1)$.

LEMMA C. *Let g be a convex (univalent) function in U; if* $f \in H(U)$ *and*

$$\operatorname{Re} f'(z)/g'(z) > 0, \quad z \in U$$

then f is univalent in U.

Note that this lemma represents the criterion of univalence (close-to-convexity) of Ozaki and Kaplan [7], [2].

We say that a function $L(z;t)$, $z \in U$ and $t \geq 0$ is a subordination (or a Loewner) chain if $L(\cdot;t)$ is analytic and univalent in U for all $t \geq 0$, $L(z;\cdot)$ is continuously differentiable on $[0, +\infty)$ for all $z \in U$ and $L(z;s) \prec L(z;t)$ when $0 \leq s \leq t$.

LEMMA D [8, p.159]. *The function* $L(z;t) = a_1(t)z + \ldots$ *with* $a_1(t) \neq 0$ *for all* $t \geq 0$ *is a subordination chain if and only if*

$$\mathrm{Re}\, z\frac{\partial L/\partial z}{\partial L/\partial t} > 0$$

for all $z \in U$ *and* $t \geq 0$.

LEMMA E [3]. *Let* $f \in H(U)$ *and let* $g \in H(\bar{U})$ *be univalent in* $\bar{U}$ *with* $f(0) = g(0)$. *If* $f(z)$ *is not subordinate to* $g(z)$ *there exists points* $z_0 \in U$ *and* $\xi_0 \in \partial U$, *and* $m \geq 1$ *for which* $f(|z| < |z_0|) \subset g(U)$, $f(z_0) = g(\xi_0)$ *and* $z_0 f'(z_0) = m\xi_0 g'(\xi_0)$.

3. Main results.

LEMMA 1. *Let* $h \in S^c(\alpha)$ *with* $\alpha \in [-1/2, 1)$ *and* $q(z) = \frac{2}{z}\int_0^z h(t)\,dt$. *Then* $q \in S^c(\delta(\alpha))$, *where* $\delta(\alpha)$ *is given by* (2) *and this result is sharp.*

Proof. If we let $P(z) = 1 + \frac{zq''(z)}{q'(z)}$, then $P(0) = 1$ and a simple calculus shows that

$$P(z) + \frac{zP'(z)}{P(z)+1} = 1 + \frac{zh''(z)}{h'(z)} \equiv H(z). \tag{3}$$

From $h \in S^c(\alpha)$, using the above relation, we obtain

$$P(z) + \frac{zP'(z)}{P(z)+1} = H(z) \prec \frac{1+(1-2\alpha)z}{1+z}.$$

Since $\mathrm{Re}\,(H(z)+1) > \alpha + 1 \geq 0$, $z \in U$, by using Theorem 1 from [4] we deduce that the differential equation (3) has a solution $P \in H(U)$ with $P(0) = 1$ and according to Lemma A we have

$$\mathrm{Re}\, P(z) > \delta(\alpha), \quad z \in U$$

i.e. $q \in S^c(\delta(\alpha))$, and this result is sharp.

THEOREM 1. *Let* $h \in S^c(\alpha)$ *with* $\alpha \in [-1/2, 1)$ *and* $q(z) = \frac{2}{z}\int_0^z h(t)\,dt$. *Suppose that* $\mathrm{Re}\,\gamma \geq -\delta(\alpha)$ *and* $\gamma \neq 0$, *where* $\delta(\alpha)$ *is given by* (2).

If $p \in H(U)$ *with* $p(0) = q(0)$, *then* $p(z) + \frac{1}{\gamma} zp'(z) \prec q(z) + \frac{1}{\gamma} zq'(z)$ *implies* $p(z) \prec q(z)$.

Proof. Since $q(z) = \frac{2}{z}\int_0^z h(t)\,dt$ satisfies the conditions of Lemma 1 we obtain $q \in S^c(\delta(\alpha))$ and by Lemma B we have $\delta(\alpha) \geq \delta(-1/2) = 0$, hence $q \in S^c$.

If we denote $L(z;t) = \gamma q(z) + (1+t) zq'(z)$, then $L(z;t)$ is analytic in U for all $t \geq 0$ and is continuously differentiable on $[0, +\infty)$ for all $z \in U$.

He have also

$$\frac{\partial L}{\partial z}(0;t) = q'(0)(1 + t + \gamma), \quad \text{for} \quad t \geq 0. \tag{4}$$

Since $q'(0) = 1$ and $\operatorname{Re}\gamma \geq -\delta(\alpha) > -1$ for $\alpha \in [-1/2, 1)$, from (4) we deduce that

$$\frac{\partial L}{\partial z}(0;t) \neq 0 \text{ for all } t \geq 0. \tag{5}$$

A simple calculation yields that

$$\operatorname{Re} z \frac{\partial L/\partial z}{\partial L/\partial t} = \operatorname{Re}\left\{\gamma + (1+t)\left(1 + \frac{zq''(z)}{q'(z)}\right)\right\}. \tag{6}$$

From $q \in S^c(\delta(\alpha))$ and $\operatorname{Re}\gamma \geq -\delta(\alpha)$ we have

$$\operatorname{Re}\left\{\gamma + (1+t)\left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > \operatorname{Re}\gamma + \delta(\alpha) \geq 0$$ for all $z \in U$ and $t \in 0$, and combining this with (6) we conclude that

$$\operatorname{Re} z \frac{\partial L/\partial z}{\partial L/\partial t} > 0, \quad z \in U, \quad t \geq 0. \tag{7}$$

From (5) and (7), according to Lemma D we deduce that $L(z;t)$ is a subordination chain.

If we let $G(z) \equiv L(z;0) = \gamma q(z) + zq'(z)$, hence

$$\operatorname{Re}\frac{G'(z)}{q'(z)} = \operatorname{Re}\left\{\gamma + 1 + \frac{zq''(z)}{q'(z)}\right\}. \tag{8}$$

From $q \in S^c(\delta(\alpha))$ and $\operatorname{Re}\gamma \geq -\delta(\alpha)$, by using (8) we obtain

$$\mathrm{Re}\,\frac{G'(z)}{q'(z)} > 0, \quad z \in U$$

and because $q \in S^c$, according to Lemma C we conclude that G is close-to-convex, hence univalent, in U.

Without loss of generality we can assume that the functions $p(z)$ and $q(z)$ satisfy the conditions of the theorem on the closed disc $\bar{U}$. If not, then we can replace the functions $p(z)$ and $q(z)$ by $p_r(z) = p(rz)$ and $q_r(z) = q(rz)$, where $0 < r < 1$. These new functions satisfy the conditions of the theorem on $\bar{U}$. We would then prove that $p_r(z) \prec q_r(z)$ for all $0 < r < 1$ and letting $r \to 1^-$ we obtain $p(z) \prec q(z)$.

Suppose that p is not subordinate to q. According to Lemma E, there exists points $z_0 \in U$ and $\xi_0 \in \partial U$ and $t \geq 0$ such that $p(z_0) = q(\xi_0)$ and $z_0 p'(z_0) = (1+t)\xi_0 q'(\xi_0)$. Then

$$\gamma p(z_0) + z_0 p'(z_0) = \gamma q(\xi_0) + (1+t)\xi_0 q'(\xi_0) = L(\xi_0; t)$$

and because $L(z;t)$ is a subordination chain we get

$$L(\xi_0; t) \notin G(U) \quad \text{i.e.}$$

$\gamma p(z_0) + z_0 p'(z_0) \notin G(U)$ which contradicts the assumption; hence $p(z) \prec q(z)$ and the proof of the theorem is complete.

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ABOUT A CONJECTURE

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REZUMAT. - Asupra unei conjecturi. În lucrare se îmbunătăţeşte un rezultat obţinut recent de H.H. Gonska şi C. Cottin.

1. Introduction. Let $\mathrm{Lip}_M(\alpha,[0,1])$ be the set of real-valued continuous functions satisfying inequality:

$$|f(x)-f(y)| \le M|x-y|^{\alpha}$$

for all $x,y \in [0,1]$, with $M>0$ and $\alpha \in (0,1]$ independent of x and y.

Let $\mathrm{Lip}^*_M(\alpha,[0,1])$ be the set:

$$\mathrm{Lip}^*_M(\alpha,[0,1]) =: \left\{ f \in \mathbf{C}[0,1],\ \omega_2(f,t) \le Mt^{\alpha} \right\}$$

$\alpha \in (0,2]$. Here ω_2 is the well-known modulus of smoothness of order two.

Lindvall [4] and Brown/Elliot/Paget [2] proved the following result:

Let $f \in \mathrm{Lip}_M(\alpha,[0,1])$. *Then* $B_n f \in \mathrm{Lip}_M(\alpha,[0,1])$.

This result was generalized in 1991 by G.A. Anastassiou, C. Cottin and H.H. Gonska [1]. They proved:

For Bernstein operators B_n *one has, for all* $f \in \mathbf{C}[0,1]$ *and* $\delta \ge 0$,

$$\omega_1(B_n f;\delta) \le 1 \cdot \tilde{\omega}_1(f;\delta) \le 2 \cdot \omega_1(f;\delta)$$

Here $\tilde{\omega}_1(f;\cdot)$ denotes the least concave majorant of $\omega_1(f;\cdot)$. The constants 1 and 2 are best

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possible.

In [3] C. Cottin and H.H. Gonska prove that if $f \in \mathrm{Lip}^*_M(\alpha,[0,1])$ then $B_n f \in$ $\in \mathrm{Lip}^*_{4.5M}(\alpha,[0,1])$. They posed the following problem (Problem 3.6):

Is it true for the classical Bernstein operators B_n that:

$$f \in \mathrm{Lip}^*_M(\alpha,[0,1]) \textit{ implies } B_n f \in \mathrm{Lip}^*_M(\alpha,[0,1]),\ 0<\alpha\le 2\ ?$$

In this paper we show that for $\alpha \in [0,1]$ and $f \in \mathrm{Lip}^*_M(\alpha,[0,1])$ implies $B_n f \in \mathrm{Lip}^*_{2M}(\alpha,[0,1])$. If $f \in \mathrm{Lip}^*_M(\alpha,[0,1])$ and f is convex or concave then $B_n f \in \mathrm{Lip}^*_M(\alpha,[0,1])$.

2. Results.

THEOREM 1. *Let* $f \in \mathrm{Lip}^*_M(\alpha,[0,1])$, $\alpha \in (0,1]$. *Then* $B_n f \in \mathrm{Lip}^*_{2M}(\alpha,[0,1])$.

Proof. Let $E = \{(0,0,0),(1,0,0),(1,1,0),(1,1,1)\} \subset \mathbb{R}^3$ and $V = \operatorname{conv} E$. We observe that we have:

$$V = \{(x,y,z) \in [0,1]^3 \mid x \ge y \ge z\}.$$

Let $B_n^V(g)$ be the Bernstein polynomial which is attached to the function g, $g \in C(V)$ and the simplex V.

For each $g \in C(V)$ we have:

$$B_n^V(g;x,y,z) = \sum_{i+j+k+p=n} \frac{n!}{i!j!k!p!}(x-y)^i (y-z)^j z^k (1-x)^p g\left(\frac{i+j+k}{n},\frac{j+k}{n},\frac{k}{n}\right). \tag{1}$$

If $g(x,y,z) = g(x)$, $\forall\ (x,y,z) \in V$ we know that

$$B_n^V(g;x,y,z) = B_n(g;x) = \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i} g\left(\frac{i}{n}\right).$$

Using this observation we have:

$$B_n(f;u) - 2B_n(f;v) + B_n(f;w) = B_n^V(g;u,v,w) \tag{2}$$

where $g: V \to \mathbb{R}$, $g(u,v,w) = f(u) - 2f(v) + f(w)$.

For $h \in (0,½)$ and $x \in [h,1-h]$ let $u=x+h$, $v=x$, $w=x-h$. From (1) and (2) we obtain

$$B_n(f;x+h) - 2B_n(f;x) + B_n(f;x-h) =$$

$$= \sum_{i+j+k+p=n} \frac{n!}{i!j!k!p!} h^{i+j}(x-h)^k(1-x-h)^p \cdot \left(f\left(\frac{i+j+k}{n}\right) - 2f\left(\frac{j+k}{n}\right) + f\left(\frac{k}{n}\right)\right) = \quad (3)$$

$$= \sum_{i+j+k+p=n} \frac{n!}{i!j!k!p!} h^{i+j}(x-h)^k(1-x-h)^p \left(f\left(\frac{i+j+k}{n}\right) - f\left(\frac{j+k}{n}\right) - f\left(\frac{i+k}{n}\right) + f\left(\frac{k}{n}\right)\right).$$

From (3) we have

$$B_n(f;x+h) - 2B_n(f;x) + B_n(f;x-h) = B_n^V(u;x+h,x,x-h) \quad (4)$$

where $u(x,y,z) = f(x) - f(x-y+z) - f(y) + f(z)$.

If $f \in \mathrm{Lip}_M^*(\alpha,[0,1])$, $\alpha \in (0,1]$, then

$$|u(x,y,z)| = \left|\left(f(x) - 2f\left(\frac{x+y}{2}\right) + f(z)\right) - \left(f(x-y+z) - 2f\left(\frac{x+y}{2}\right) + f(y)\right)\right| \le$$

$$\le \left|f(x) - 2f\left(\frac{x+z}{2}\right) + f(z)\right| + \left|f(x-y+z) - 2f\left(\frac{x+z}{2}\right) + f(y)\right| \le$$

$$\le M\left(\frac{x-z}{2}\right)^\alpha + M\left(\frac{|x+z-2y|}{2}\right)^\alpha$$

Because $\left|\frac{x+z-2y}{2}\right|^\alpha \le \left(\frac{x-z}{2}\right)^\alpha$ it follows that:

$$|u(x,y,z)| \le 2M\left(\frac{x-z}{2}\right)^\alpha \quad (5)$$

From (4) and (5) we obtain:

$$|B_n(f;x+h)-2B_n(f;x)+B_n(f;x-h)| = |B_n^V(u;x+h,x,x-h)| \le B_n^V(|u|;x+h,x,x-h) \le$$

$$\le 2M\left(\frac{B_n(e_1;x+h) - B_n(e_1;x-h)}{2}\right)^\alpha = 2Mh^\alpha$$

Using the fact that the function $h:[0,1] \to [0,1]$, $h(t) = t^\alpha$ is concave on the interval $[0,1]$.

With this the theorem is proved.

THEOREM 2. *Let $f \in Lip_M^*(\alpha,[0,1])$, $\alpha \in (0,1]$. If f is a convex function or a concave function then $B_n f \in Lip_M^*(\alpha,[0,1])$.*

Proof. If $f \in Lip_M^*(\alpha,[0,1])$ and f is a convex function we have:

$$u(x,y,z) \geq 2f\left(\frac{x+z}{2}\right) - f(x-y+z) - f(y) \geq -M\left(\frac{x-z}{2}\right)^{\alpha}$$

and

$$u(x,y,z) \leq f(x) - 2f\left(\frac{x+z}{2}\right) + f(z) \leq M\left(\frac{x-z}{2}\right)^{\alpha}$$

so

$$|u(x,y,z)| \leq M\left(\frac{x-z}{2}\right)^{\alpha} \tag{6}$$

In the same way it shows that if $f \in Lip_M^*(\alpha, [0,1])$ and f is a concave function u satisfies inequality (6).

From the inequality (6) we obtain the inequality:

$$|B_n(f;x+h) - 2B_n(f;x) + B_n(f;x-h)| \leq B_n^V(|u|; x+h, x, x-h) \leq Mh^{\alpha}.$$

With this the theorem is proved.

THEOREM 3. *If $f \in Lip_M^*(2,[0,1])$ then $B_n f \in Lip_M^*(2,[0,1])$.*

Proof. The Theorem 3 is obtained from the equality:

$$B_n''(f;x) = n(n-1)\sum_{k=0}^{n}\left(f\left(\frac{k+2}{n}\right) - 2f\left(\frac{k+1}{n}\right) + f\left(\frac{k}{n}\right)\right)\binom{n-2}{k}x^k(1-x)^{n-k-2}.$$

and from

$$|B_n(f;x+h) - 2B_n(f;x) + B_n(f;x-h)| \leq h^2\|B_n'' f\| \leq h^2 M\frac{n-1}{n}.$$

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ON THE CONSTRUCTION OF APPROXIMATING LINEAR POSITIVE OPERATORS BY PROBABILISTIC METHODS

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REZUMAT. - Construcţia unor operatori liniari de aproximare prin metode probabilistice. Lucrarea prezintă o metodă probabilistică de construire a unui şir de operatori liniari pozitivi utilizaţi în teoria aproximării uniforme a funcţilor continue de două variabile. Se studiază convergenţa şirului şi se evaluează ordinul de aproximare. În final sunt prezentate exemple care extind în plan rezultatele obţinute în [2].

1. Introduction. Connections between probability and positive linear operators are discussed in many papers. Also a lot of generalizations and investigations of the classical operators of discret type (Bernstein, Szasz, Mirakyan, Meyer-König and Zeller and others) were studied [1], [3], [4], [5], [6], [7], [8], [9], [10] and the literature cited there.

This paper develops a general probabilistic method for constructing positive linear operators useful in the theory of uniform approximation of continuous functions of two variables. We study this sequence of operators by applying the well known theorem of Bohman-Korovkin. Then we evaluate the orders of approximation in terms of the modulus of continuity; in the last section of this paper we present some examples which extend the results of G.C.Jain and S.Pethe [2].

2. Construction of the operators. Let (Ω, A, P) be a probability space in the sence of Kolmogorov, i.e. Ω is an arbitrary abstract space, A is a σ-algebra of subsets of Ω and P

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a probability measure in Ω and on A. Let $(X_n)_{n\geq 1}$, $(Y_m)_{m\geq 1}$ be two real sequences of random variables having the following distributions:

$P(X_n = x_{ni}) = p_{ni}(x,y)$, $(i \in A \subseteq \mathbf{N})$, $P(Y_m = y_{mj}) = q_{mj}(x,y)$, $(j \in B \subseteq \mathbf{N})$ where $(x,y) \in I \times J \subset \mathbf{R} \times \mathbf{R}$; I and J are not necessarily bounded intervals. Let $\left(p_{mj}^{ni}\right)$ be defined by:

$$p_{mj}^{ni}(x,y) = P\left(X_n = x_{ni} \text{ and } Y_m = y_{mj}\right)$$

By the definition of this distribution we have:

$$\sum_{i\in A} p_{mj}^{ni} = q_{mj}, \quad \sum_{j\in B} p_{mj}^{ni} = p_{ni} \quad \text{and consequently} \quad \sum_{i\in A}\sum_{j\in B} p_{mj}^{ni} = 1 \tag{1}$$

Now we consider the operators defined by:

$$\left(L_{mn} f\right)(x,y) = \sum_{i\in A}\sum_{j\in B} p_{mj}^{ni}(x,y) f\left(x_{ni}, y_{mj}\right) \tag{2}$$

where f is a continuous function on $I \times J$.

It is clear that the operators defined in (2) are linear positive. Therefore they are monotone. Let us calculate the values of our operator for the test functions $e_{kl} : D \to \mathbf{R}$, where $e_{kl}(x,y) = x^k y^l$; $k,l \in \{0,1,2\}$, $k+l \leq 2$. Using (1) we have:

$$\left(L_{mn} e_{00}\right)(x,y) = \sum_{i\in A}\sum_{j\in B} p_{mj}^{ni} = 1 = e_{00}(x,y)$$

Let us agree to denote the expectation of the random variable Z by $E(Z)$ and the moments about the origin by $\nu_k(Z)$, the subscript indicating the order of the moment. In these notations:

$$\left(L_{mn} e_{10}\right)(x,y) = \sum_{i\in A}\sum_{j\in B} p_{mj}^{ni} x_{ni} = \sum_{i\in A} p_{ni} x_{ni} = E\left(X_n\right)$$

Analogous:

$$\left(L_{mn} e_{01}\right)(x,y) = E\left(Y_m\right)$$

Also:

$$\left(L_{mn} e_{20}\right)(x,y) = \sum_{i\in A}\sum_{j\in B} p_{mj}^{ni} x_{ni}^2 = \sum_{i\in A}\left(\sum_{j\in B} p_{mj}^{ni}\right) x_{ni}^2 = \sum_{i\in A} p_{ni} x_{ni}^2 = \nu_2\left(X_n\right)$$

Analogous: $(L_{mn}e_{02})(x,y) = \nu_2(Y_m)$

At last we compute:

$$(L_{mn}e_{11})(x,y) = \sum_{j\in B}\left(\sum_{i\in A} p_{mj}^{ni} x_{ni}\right) y_{mj} = E(X_n Y_m) \le \left(\nu_2(X_n)\nu_2(Y_m)\right)^{1/2},$$

according to the Schwartz inequality. By making use of the results established above, in concordance with the well-known Korovkin's theorem in probabilistic form, we can state:

THEOREM. *Let* $(L_{mn})_{(m,n)\in \mathbf{N}\times\mathbf{N}}$ *be introduced in* (2) *and* $f \in C(I\times J)$

If

i) $\lim_n \nu_k(X_n) = x^k,\ k\in\{1,2\}$

ii) $\lim_m \nu_k(Y_m) = y^k,\ k\in\{1,2\}$

iii) $\underset{(n,m)}{Lim}\, E(X_n Y_m) \ge xy$

then we have: $\lim_{(n,m)} (L_{mn}f) = f$ *uniformly on* $I\times J$.

3. Order of approximation. For evaluating the corresponding orders of approximation, it is convenient to make use of the modulus of continuity of f, which is defined by:

$\omega_f(\delta_1,\delta_2) = \max|f(x',y') - f(x'',y'')|$, for $|x'-x''|\le\delta_1$, $|y'-y''|\le\delta_2$, δ_1,δ_2 being positive numbers.

We will need the following known property of the modulus of continuity

$$\omega_f(\lambda_1\delta_1,\lambda_2\delta_2) \le (1+\lambda_1+\lambda_2)\,\omega_f(\delta_1,\delta_2),\quad (\lambda_1>0,\ \lambda_2>0) \tag{3}$$

We can write succesively:

$$\left|(L_{mn})(x,y) - f(x,y)\right| = \left|\sum_{j\in B}\sum_{i\in A} p_{mj}^{ni} f(x_{ni},y_{mj}) - f(x,y)\right| \le$$
$$\le \sum_{j\in B}\sum_{i\in A} p_{mj}^{ni}\left|f(x_{ni},y_{mj}) - f(x,y)\right| \le \sum_{j\in B}\sum_{i\in A} p_{mj}^{ni}\,\omega\left(|x_{ni}-x|,|y_{mj}-y|\right).$$

But by using the inequality (3) we have:

$$\omega\left(|x_{ni}-x|,\,|y_{mj}-y|\right) = \omega\left(\frac{1}{\delta_1}|x_{ni}-x|\delta_1,\ \frac{1}{\delta_2}|y_{mj}-y|\delta_2\right) \le$$
$$\le \left(1+\delta_1^{-1}|x_{ni}-x|+\delta_2^{-1}|y_{mj}-y|\right)\omega(\delta_1,\delta_2).$$

Thus we obtain:

$$\sum_{j\in B}\sum_{i\in A}\left(1+\delta_1^{-1}|x_{ni}-x|+\delta_2^{-1}|y_{mj}-y|\right)p_{mj}^{ni}(x,y) =$$
$$= 1+\delta_1^{-1}\sum_{i\in A}p_{ni}|x_{ni}-x| + \delta_2^{-1}\sum_{j\in B}q_{mj}|y_{mj}-y|.$$

Now we shall apply the inequality of Cauchy: $\sum_{l\in C} a_l b_l \le \left(\sum_{l\in C} a_l^2 \sum_{l\in C} b_l^2\right)^{1/2}$

By choosing first $C = A$, $a_l = \sqrt{p_{nl}}$, $b_l = \sqrt{p_{nl}}\ |x_{nl}-x|$, we obtain:

$$\sum_{i\in A}p_{ni}|x_{ni}-x| \le \left(\left(\sum_{i\in A}p_{ni}\right)\left(\sum_{i\in A}p_{ni}|x_{ni}-x|^2\right)\right)^{1/2} = \left(\left(\sum_{i\in A}p_{ni}\right)x^2 -\right.$$

$$\left.-2\left(\sum_{i\in A}x_{ni}p_{ni}\right)x+\sum_{i\in A}p_{ni}x_{ni}^2\right)^{1/2} = \left(x^2-2E(X_n)x+\nu_2(X_n)\right)^{1/2} \le \sqrt{D(X_n)}$$

$D(X_n)$ being variance of the random variable X_n.

It is easy to see that the equality holds for $x = E(X_n)$.

At the second step we choose : $C = B$, $a_l = \sqrt{q_{ml}}$, $b_l = \sqrt{q_{ml}}\ |y_{ml}-y|$ and we get an analogous result:

$$\sum_{j\in B}q_{mj}|y_{mj}-y| \le \sqrt{D(Y_m)}$$

Now we can state the following proposition:

THEOREM. *If $f\in C(I\times J)$ and $L_{mn}f$ is defined at* (2), *then the following inequality* $\left|(L_{mn}f)(x,y)-f(x,y)\right| \le \left(1+\delta_1^{-1}D^{1/2}(X_n)+\delta_2^{-1}D^{1/2}(Y_m)\right)\omega_f(\delta_1,\delta_2)$ *holds.*

Examples.

A. Let us consider that $I = J = [0,1]$; $x_{ni} = \frac{i}{n}$, $(i=\overline{0,n})$; $y_{mj} = \frac{j}{m}$, $(j=\overline{0,m})$.

We assume that $(X_n)_{n\ge 1}$ follows a generalization of the binomial distribution with

$$p_{ni}(x,\beta) = \frac{1}{i!}\prod_{k=0}^{i-1}\left(\frac{x}{\beta}+k\right)\sum_{r=0}^{i}(-1)^r\binom{i}{r}(1-x-\beta r)^n \tag{4}$$

We assume that the product $\prod_{k=0}^{i-1}\left(\frac{x}{\beta}+k\right)$ is defined as (1) when $i = 0$ and we consider that β may depend on the number n. The limiting conditions on β are $\beta < \frac{1-x}{n}$ if $\beta \geq 0$ and $|\beta| < \frac{x}{n}$ if $\beta < 0$.

$(Y_m)_{m\geq 1}$ are identically distributed and $q_{mj}(y,\alpha) = p_{mj}(y,\alpha)$. It is easy to verify that: $E(X_n) = x(1+\beta_1(n))$ and $\nu_2(X_n) = x\left(\frac{1}{n}+\frac{\beta_1(n)}{n}+(n-1)x(x+\beta)(1+\beta_2(n))\right)$, where

$$\beta_1(n) = \frac{(1+\beta)^n-1}{n\beta}-1, \quad \beta_2(n) = \frac{1+(1+2\beta)^n-2(1+\beta)^n}{n(n-1)\beta^2}-1$$

The assumptions $\lim_n n\beta(n) = 0 = \lim_m m\alpha(m)$ assure that the conditions (i) and (ii) of the theorem are satisfied.

In order to the third condition, we take into account that

$$E(X_nY_m) = \sum_{i=1}^{n}\sum_{j=1}^{m}\frac{i}{n}\frac{j}{m}p_{ni}q_{mj} = \frac{1}{nm}\left(\sum_{i=1}^{n} ip_{ni}\right)\left(\sum_{j=1}^{m} jq_{mj}\right) =$$

$= xy(1+\beta_1(n))(1+\alpha_1(m)) \geq xy$ where $\alpha_1(m) = \frac{(1+\alpha)^m-1}{m\alpha}-1$

We notice that the random variables were considered independent.

B. Let $n \to \infty$, $x \to 0$, $\beta \to 0$ in (4) such that $nx = \alpha$, $n\beta = \gamma$ then the limiting probability $p_{ni}(\alpha,\gamma)$ of obtaining i successes is given by

$p_{ni}(\alpha,\gamma) = \prod_{k=0}^{i-1}(\alpha+k\gamma)\frac{e^{-\alpha}}{i!}\left(\frac{1-e^{-\gamma}}{\gamma}\right)^i$, $\alpha > 0$, $\gamma > 0$, $i \geq 1$ and $p_{n0} = e^{-nx}$. This case represents a generalization of the Szasz-Mirakyan operator. Also we define:

$$X_n\begin{pmatrix} i/n \\ p_{ni}(nx,\gamma)\end{pmatrix}_{i\geq 0} \qquad Y_m\begin{pmatrix} j/m \\ p_{mj}(my,\gamma)\end{pmatrix}_{j\geq 0}$$

If $f \in C([0,\infty) \times [0,\infty))$ and if $(\gamma,\gamma') \to (0,0)$ as $(n,m) \to (\infty,\infty)$ then the sequence $\left(L_{mn}^{<\gamma,\gamma'>}f\right)_{(m,n)\in \mathbb{N}\times\mathbb{N}}$ converges uniformly to f on $[a,b] \times [c,d]$ where $0 \leq a < b < \infty$, $0 \leq c < d < \infty$. Indeed there holds:

$$E(X_n) = \alpha n^{-1}(e^{\gamma}-1)\gamma^{-1} \to x,\ n \to \infty,\ \gamma \to 0$$

$$\nu_2(X_n) = \frac{x}{n}(nx+\gamma)\left(\frac{e^{\gamma}-1}{\gamma}\right)^2 + \frac{x}{n}\frac{e^{\gamma}-1}{\gamma} \to x^2,\ n \to \infty,\ \gamma \to 0$$

and similar results for $E(Y_m)$ and $\nu_2(Y_m)$.

$$E(X_n Y_m) = \sum_{i\geq 1}\sum_{j\geq 1} ij(nm)^{-1} p_{ni}(nx,\gamma) p_{mj}(my,\gamma') \to xy,\ (n,m) \to (\infty,\infty)$$

This completes the proof.

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ϕ-MONOTONE AND ϕ-CONTRACTIVE OPERATORS IN HILBERT SPACES

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REZUMAT. - Operatori ϕ-monotoni şi ϕ-contractivi în spaţii Hilbert. Lucrarea introduce noţiunea de ϕ-monotonie, care generalizează conceptul de tare-monotonie, şi arată că familia operatorilor ϕ-monotoni este echivalentă, într-un anume sens, cu cea a operatorilor ϕ-contractivi.

Introduction. The aim of this paper is to establish a relation between a class of monotone operators, by a hand, and a class of contractive mappings, on the other hand, using a generalization of the contraction mapping principle due to KRASNOSELSKI and STECENKO [6]. As shown by CEA [3], which applies this results in optimization theory, if the operator G satisfies certain monotonic conditions, then $T_\gamma = I - \gamma G$, for a certain $\gamma > 0$, is a contraction. In [5] DINCĂ argued that CEA's conditions are only sufficient and, consequently furnishes necessary and sufficient conditions, obtaining the following generalization of CEA's result: G is strongly monotone and Lipschitz operator if and only if there exists $\gamma > 0$ such that T_γ is a contraction.

Theorem 3.1 extends these results, by means of some new concepts, and states that G is ϕ-monotone operator if and only if there is $\gamma > 0$ such that T_γ is ϕ-contraction.

1. Comparison functions. Various concepts of comparison functions was defined and

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intensively studied in connection with the generalized contraction mapping principle, see, for example RUS,A.I. [7], [8], BERINDE,V. [1], [2]. In the present paper we need comparison functions defined without the monotone increasing condition.

DEFINITION 1.1. A mapping $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *comparison function* if

(i) φ is continuous;

(ii) $0 < \varphi(t) < t$, for $t > 0$.

Let's denote by ϕ the set of all comparison functions. Obviously, ϕ is nonempty and contains both linear and nonlinear functions as shown by

Example 1.1. If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t) = at$, $0 < a < 1$, $t \in \mathbb{R}_+$, then $\varphi \in \phi$.

Exampme 1.2. If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t) = t(1-t)$, for $0 \le t < 1$ and $\varphi(t) = t-1$, for $t \ge 1$, then $\varphi \in \phi$, but φ is nonlinear.

LEMMA 1.1. *Let* $\varphi \in \phi$ *be a comparison function. Then*

a) $\varphi(0) = 0$;

b) $0 < 2t\varphi(t) - \varphi^2(t) < t^2$, *for* $t > 0$.

Proof.

a) From (ii) we obtain $\lim_{t \to 0} \varphi(t) = 0$, hence, by (i), $\varphi(0) = 0$.

b) Since $\varphi \in \phi$, we have $\varphi(t) \ge 0$ and $t - \varphi(t) \ge 0$, $t \in \mathbb{R}_+$. Then, for every $t \in \mathbb{R}_+$,

$$2t\varphi(t) - \varphi^2(t) = \varphi(t)[t + (t - \varphi(t))] \ge 0$$

Finally, for $t > 0$, $t - \varphi(t) > 0$, then $2t\varphi(t) - \varphi^2(t) > 0$ and $(t - \varphi(t))^2 > 0$, that is, $2t\varphi(t) - \varphi^2(t) < t^2$, which completes the proof..

We are now able to give the following

DEFINITION 1.2. Let $\varphi \in \phi$ be a comparison function. A function $r_\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ given by $r_\varphi(t) = \sqrt{2t\varphi(t) - \varphi^2(t)}$ is called the *transformate* of φ.

LEMMA 1.2. *Let $\varphi \in \phi$ be a comparison function and r_φ its transformate. Then*

a) $r_\varphi \in \phi$;

b) *The mapping $r: \phi \to \phi$, $r(\varphi) = r_\varphi$ is bijective;*

c) $\varphi(t) < r_\varphi(t)$, *for each $t > 0$.*

Proof.

a) follows from Lemma 1.1.

b) It suffices to show that for any $\psi \in \phi$ there exists a unique $\varphi \in \phi$ there a unique $\varphi \in \phi$ such that

$$2t\varphi(t) - \varphi^2(t) = \psi^2(t), \quad t \in \mathbf{R}_+. \tag{1}$$

First, we observe that for $t = 0$, it follows $\varphi(0) = 0$. Then, let $t \neq 0$ be arbitrary but fixed. Denote $a = \dfrac{\psi(t)}{t}$, $x = \dfrac{\varphi(t)}{t}$. Since $\psi \in \phi$, $a \in (0,1)$. From (1) we obtain the equation

$$x^2 - 2x + a^2 = 0$$

which has a unique solution $x \in (0,1)$, $x = 1 - \sqrt{1 - a^2}$.

Hence

$$\varphi(t) = t - \sqrt{t^2 - \psi^2(t)}, \quad t \in \mathbf{R}_+,$$

is the unique solution of (1), that is, r is bijective.

c) It is obvious.

DEFINITION 1.3. Let (X,d) be a metric space. A mapping $f: X \to X$ is called *ϕ-contraction* if there exists a comparison function $\varphi \in \phi$ such that

$$d(f(x),f(y)) \leq d(x,y) - \varphi(d(x,y)), \quad \forall\, x,y \in X. \tag{2}$$

We need the following generalization of the contraction mapping principle

THEOREM 1.1. (KRASNOSELSKI and STECENKO [6], RUS,A.I. [7]).

Let (X,d) be a complete metric space and $f: X \to X$ a ϕ-contraction. Then f has a *unique fixed point that can be found by using functional iteration starting at an arbitrary point x_0 in X.*

2. ϕ-monotone operators. Let H be a real Hilbert space whose norm and inner product are denoted as usually by $\|\cdot\|$ and $<\cdot,\cdot>$, respectively. An operator $G : H \to H$ is called *monotone operator* if

$$<Gu - Gv, u - v> \geq 0, \quad \forall u,v \in H. \tag{3}$$

G is called *strictly monotone operator* if equality in (3) implies $u = v$.

G is said to be *strongly monotone operator* if there exists $m > 0$ such that

$$<Gu - Gv, u - v> \geq m \cdot \| u - v \|^2, \quad \forall u,v \in H. \tag{4}$$

The operator G is called *Lipschitz operator* if there exists $M > 0$ such that

$$\| Gu - Gv \| \leq M \cdot \| u - v \|, \quad \forall u,v \in H. \tag{5}$$

LEMMA 2.1. *If G: $H \to H$ is an operator which satisfies* (4) *and* (5) *then $m \leq M$.*

Proof. Since G is strongly monotone, hence monotone, the Cauchy-Schwarz inequality yields.

$$<Gu - Gv, u - v> \leq \| Gu - Gv \| \cdot \| u - v \|, \quad \forall u,v \in H$$

which together with (4) and (5) gives $m \| u - v \|^2 \leq M \cdot \| u - v \|^2, \ \forall u,v \in H$, that is $m \leq M$.

Remark. If G: $H \to H$ is strongly monotone, then G is injective. For an injective operator G and a given comparison function ψ, let denote by $t_1 = t_1(G,\psi)$, $t_2 = t_2(G,\psi)$ $t_1 < t_2$, the (assumed) real roots of the quadratic equation

$$\|Gu - Gv\|^2 t^2 - 2 \cdot <Gu - Gv, u - v> \cdot t + \psi^2 (\| u - v \|) = 0, \quad u,v \in H, \ u \neq v. \tag{6}$$

DEFINITION 2.1. We say that an injective operator G: $H \to H$ is a *ϕ-monotone* *operator* if there exists $\psi \in \phi$ such that

(m$_1$) $\quad <Gu - Gv, u - v> \geq \|Gu - Gv\| \cdot \psi(\|u - v\|), \ \forall\, u,v \in H;$

(m$_2$) $\quad \bigcap_{\substack{u,v\in H\\ u\neq v}} [t_1(G,\psi), t_2(G,\psi)] \neq \phi.$

LEMMA 2.2. *Any strongly monotone and Lipschitz operator is a ϕ-monotone operator.*

Proof. Assume G satisfies (4) and (5). To prove (m$_1$) it suffices to show that there exists $\psi \in \phi$ such that

$$m \cdot \|u - v\|^2 \geq M \cdot \|u - v\| \cdot \psi(\|u - v\|), \ \forall\, u,v \in H$$

or equivalently,

$$m\,\|u - v\| \geq M \cdot \psi(\|u - v\|), \ \forall\, u,v \in H. \qquad (*)$$

If $m = M$, (*) holds for any $\psi(t) = at$, $0 < a < 1$, $t \in \mathbf{R}_+$ and if $m < M$, (*) holds for $\psi(t) = \frac{m}{M}t$. For the second part of the proof, let us observe that $[t_1,t_2]$ is the solution of the inequation obtained from (6) replacing "=" by "≤", hence (m$_2$) is equivalent to the following condition: there exists $\gamma > 0$ such that

$$\|Gu - Gv\|^2\gamma^2 - 2 \cdot <Gu - Gv, u - v> \cdot \gamma + \varphi^2(\|u - v\|) \leq 0, \ \forall\, u,v \in H.$$

Using (4) and (5) we have

$$\begin{aligned} \|Gu - Gv\|^2\gamma^2 - 2 \cdot <Gu - Gv, u - v> \cdot \gamma + \varphi^2(\|u - v\|) \leq \\ \leq (M^2\gamma^2 - 2\gamma m) \cdot \|u - v\|^2 + \varphi^2(\|u - v\|), \end{aligned} \qquad (7)$$

hence, to prove (7) it suffices to show that for certain $\gamma > 0$ and $\varphi = r_\psi$ the inequality

$$(M^2\gamma^2 - 2\gamma m) \cdot \|u - v\|^2 + \varphi^2(\|u - v\|) \leq 0, \ \forall\, u,v \in H \qquad (8)$$

holds.

If $m = M$, then $\psi(t) = at$, $0 < a < 1$ and (8) holds for $\gamma = \frac{1}{M}$, since

$$(m^2\gamma^2 - 2\gamma M)\|u - v\|^2 + a^2\|u - v\|^2 < (M^2\gamma^2 - 2\gamma M + 1)\|u - v\|^2 = 0.$$

If $m < M$, when $\varphi(t) = \frac{m}{M}t$, (8) holds for $\gamma = \frac{m}{M^2}$.

Indeed, in this case we have

$$(M^2 \cdot \gamma^2 - 2\gamma m) \| u - v\|^2 + \varphi^2(\| u - v \|) = \left(M^2 \cdot \frac{m^2}{M^4} - 2\frac{m^2}{M^2} + \frac{m^2}{M^2}\right) \cdot \| u - v \|^2 = 0 .$$

Remark. The class of ϕ-monotone operators is larger than the class of strongly monotone and Lipschitz operators as shown by theorem 3.1 together with theorem 1.1 and theorem 3.2.

3. Fixed points. Let H be, as in the previous section, a real Hilbert space and let G: $H \to H$ be a given operator. For every $\gamma > 0$, let us define the operator T_γ: $H \to H$, given by

$$T_\gamma = I - \gamma G, \tag{9}$$

where I is the identity operator.

Such a procedure plays an important role in many practical problems, when the problem of solving the operatorial equation

$$G(x) = 0$$

is reduced (if possibly) to the fixed point problem:

$$T_\gamma(x) = x \tag{10}$$

Thus we are interested to convert the monotonic hypothesis on G in adequate conditions of contractive type on T_γ, in order to obtain an iterative method to solve (10).

The main result of this paper is given by the following

THEOREM 3.1. *Let H be a real Hilbert space, G: $H \to H$ a given operator and let T_γ be the operator defined by* (9).

Then, G is a ϕ-monotone operator if and only if there exists $\gamma > 0$ *such that T_γ is a ϕ-contraction.*

Proof. Assume T_γ is a φ-contraction. We shall prove that there exists $\psi \in \phi$ such that (m_1) and (m_2) holds. We have from (9)

$$\begin{aligned}\|T_\gamma u - T_\gamma v\|^2 &= \|u - v - \gamma(Gu - Gv)\|^2 = \\ &= \|u - v\|^2 - 2\gamma \cdot \langle Gu - Gv, u - v\rangle + \gamma^2 \cdot \|u - v\|^2, \ \forall\, u, v \in H\end{aligned} \tag{11}$$

But T_γ is φ-contraction if and only if there exists $\varphi \in \phi$ such that

$$\| T_\gamma u - T_\gamma v \| \le \| u - v \| - \varphi(\| u - v \|), \ \forall\, u, v \in H \tag{12}$$

Thus from (11) and (12) we deduce that there exists $\varphi \in \phi$ and $\gamma > 0$ such that

$$\|Gu - v\|^2 \cdot \gamma^2 - 2 \cdot \langle Gu - Gv, u - v\rangle \gamma + r_\varphi^2(\| u - v \|) \le 0, \ \forall\, u, v \in H, \tag{13}$$

that is, the inequation

$$\|Gu - Gv\|^2 \cdot t^2 - 2 \cdot \langle Gu - Gv, u - v\rangle \cdot t + r_\varphi^2(\| u - v \|) \le 0, \ u, v \in H \tag{14}$$

has a positive solutions $t = \gamma$ and γ does not depend on $u,v \in H$.

This implies, by a hand, that

$$\langle Gu - Gv, u - v\rangle \ge 0, \ \forall\, u, v \in H$$

(otherwise (13) is impossible for $u \ne v$), and on the other hand that

$$\langle Gu - Gv, u - v\rangle^2 - \| Gu - Gv \|^2 \cdot r_\varphi^2(\| u - v \|) \ge 0, \ \forall\, u, v \in H \tag{15}$$

From (14) and (15) we obtain (m_1) with $\psi = r_\varphi$.

Let $t_1(G, \psi)$, $t_2(G, \psi)$, $t_1(G, \psi) < t_2(G, \psi)$ the real roots of the equation associate to (14). Then $\gamma \in [t_1(G, \psi), t_2(G, \psi)]$, for each $u,v \in H$, $u \ne v$, that is (m_2) holds. Hence G is φ-monotone operator. The converse is obvious. The proof is now complete.

Remark. From the proof of Lemma 2.2 it results that if G is strongly monotone and Lipschitz operator then G is ϕ_1-monotone, where $\phi_1 = \{\varphi \in \phi \,/\, \varphi(t) = at, \ 0 < a < 1\}$ is the class of linear comparison functions. Thus we obtain from theorem 3.1 the results of DINCĂ ([5], theorem 12.32, p.520; see also DINCĂ, BLEBEA [4]).

THEOREM 3.2. *Let H be a real Hilbert space, $G: H \to H$ a given operator and T_γ:*

$H \to H$ defined by (9). *Then, G is strongly monotone and Lipschitz operator if and only if there exists $\gamma > 0$ such that T_γ is a contraction.*

Remark. There exist nonlinear comparison functions, see example 1.2, hence the class of ϕ-monotone operators is larger than the class of strongly monotone and Lipschitz operators. From theorem 3.1 we obtain following.

CORROLARY 3.1. *Let H be a real Hilbert space and $G: H \to H$ a ϕ-monotone operator. Then the equation*

$$G(x) = 0$$

has a unique solution $x^* \in H$ and there exists $\gamma > 0$ such that the sequence of successive approximations $(x_n)_{n \in \mathbb{N}}$ defined by

$$x_{n+1} = x_n - \gamma \cdot G(x_n), \quad n \geq 0$$

converges to x^*, for each $x_0 \in H$.

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ON SOME APPLICATIONS OF WHITNEY'S THEOREM

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REZUMAT. - Asupra unor aplicaţii ale teoremei lui Whitney. În lucrare sunt prezentate două aplicaţii ale teoremei lui Whitney. Prima referitoare la problema aproximării subvarietăţilor din $\mathbf{R}^n$ prin hipersuprafeţe netede compacte. A doua în legătură cu o problemă de separare în $\mathbf{R}^n$.

In this paper we shall present two applications of the well known Whitney's theorem. The first one is refering to the problem concerning the approximation of the compact topological submanifolds from $\mathbf{R}^n$ by compact smooth hypersuperfaces. The second one is related to the following separation problem in the space $\mathbf{R}^n$: if G is an open subset of $\mathbf{R}^n$ such that $\text{ext}(G) \neq \varnothing$ then $\mathbf{R}^n \setminus (\partial G \cap \partial(\text{ext}G))$ is not connected.

Let us begin with some notations and with the so called Whitney's theorem.

Consider $K(\mathbf{R}^n)$ the family of all compact subsets of $\mathbf{R}^n$ endowed with the following metric

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\right\}.$$

Here $d(x,M)$ denotes the distance from the element x to the subset M. It is known that $(K(\mathbf{R}^n),H)$ is a complete metric space ([2] p. 100) and that the function $d(\cdot,M)$: $\mathbf{R}^n \to \mathbf{R}$ is continuous.

THEOREM (Whitney). *If C is a closed subset of $\mathbf{R}^n$ then there exist a positive and smooth function f:$\mathbf{R}^n \to \mathbf{R}$ such that the preimage of zero by f is even the subset C.*

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Our first result is the following

THEOREM 1. *Let G an open subset of* $\mathbf{R}^n$ *whose border is a compact subset. Then there exists a sequence* $\{V_n\}_{n\in N}$ *of compact smooth hypersuperfaces such that* $\lim_{m\to\infty} H(V_m, \partial G) = 0$, *i.e.* $V_m \xrightarrow{H} \partial G$.

For the proof we need some lemmas.

LEMMA 1. *Let A be a subset of* $\mathbf{R}^n$, *x and y be two elements in A and* $\mathbf{R}^n \setminus A$, *respectively. If c*: $[0,1] \to \mathbf{R}^n$ *is a continuous path joining x with y* ($c(0) = x$, $c(1) = y$) *then there exists* $\xi \in [0,1]$ *such that* $c(\xi) \in \partial A$.

Proof. If x or y belongs to ∂A we can take $\xi = 0$ or $\xi = 1$. Otherwise $y \in \mathbf{R}^n \setminus \bar{A}$ and so there exist an open neigborhood U of y such that $U \cap A = \emptyset$, hence $U \subseteq \mathbf{R}^n \setminus A$. But since c is continuous, there exists $\varepsilon > 0$ such that $c([\varepsilon, 1]) \subseteq U \subseteq \mathbf{R}^n \setminus A$. If $\xi = \inf\{\varepsilon \in [0,1] : c([\varepsilon, 1]) \subseteq \mathbf{R}^n \setminus A\}$ we will show that $c(\xi) \in \partial A$. Let ε_n be a sequence of real numbers such that $\xi < \varepsilon_m < 1, c([\varepsilon_m, 1]) \subseteq \mathbf{R}^n \setminus A$ for all $m \geq 0$ and $\lim_{m\to\infty} \varepsilon_m = \xi$. Therefore, $c(\xi) \in \overline{\mathbf{R}^n \setminus A}$. If we suppose $c(\xi) \notin \bar{A}$, there exists an open neighborhood U' of $c(\xi)$ such that $U' \cap A = \emptyset$, namely $U' \subseteq \mathbf{R}^n \setminus A$. Since c is continuous in ξ, there exists $\delta > 0$ such that $c((\xi - \delta, \xi + \delta)) \subseteq U'$ and therefore, $c([\xi - \delta\backslash 2, 1]) = c([\xi - \delta\backslash 2, \varepsilon_m] \cup [\varepsilon_m, 1]) = c([\xi - \delta\backslash 2, \varepsilon_m]) \cup c([\varepsilon_m, 1]) \subseteq U' \cup (\mathbf{R}^n \setminus A) = \mathbf{R}^n \setminus A$. Thus $c([\xi - \delta\backslash 2, 1]) \subseteq \mathbf{R}^n \setminus A$ (m was choosen sufficiently large such that $\xi < \varepsilon_m < \xi + \delta$). But this is a contradiction with the choosing of ξ. Hence $c(\xi) \in \bar{A}$ which together with $c(\xi) \in \overline{\mathbf{R}^n \setminus A}$ implies that $c(\xi) \in \partial A$.

LEMMA 2. *If A is a subset of* $\mathbf{R}^n$ *such that A and* $\mathbf{R}^n \setminus A$ *are unbounded, then the border* ∂A *is unbounded too.*

Proof. Let $r > 0$ be an arbitrary strictly positive number and r_1, r_2 two numbers such that $r < r_1 < r_2$ and $S_{r_1}(0) \cap A \neq \emptyset$, $S_{r_2}(0) \cap (\mathbf{R}^n \setminus A) \neq \emptyset$ (here $S_r(0)$ denotes the sfere of radius

r having the center in the origin).

If $S_{r_1}(0) \subseteq A$ and $S_{r_2}(0) \subseteq \mathbf{R}^n \setminus A$ we choose a point $x_0 \in S_{r_1}(0)$ and consider the path $c\colon [0,1] \to \mathbf{R}^n$ given by $c(t) = [(1-t)r_1 + tr_2]\frac{x_0}{r_1}$. Notice that $c(0) = x_0 \in S_{r_1}(0)$ and $c(1) = \frac{r_2}{r_1}x_0 \in S_{r_2}(0)$ and therefore by the previous lemma there exists $\xi \in [0,1]$ such that $c(\xi) \in \partial A$. But since $\| c(\xi) \| = (1-\xi)r_1 + \xi r_2$ this implies that $\| c(\xi) \| > r$. Further on, we suppose that at least one of the two spheres $S_{r_1}(0)$ or $S_{r_2}(0)$, for example $S_{r_1}(0)$, contains points of A and also points of $\mathbf{R}^n \setminus A$.

Let x_0, y_0 be two points of the sphere $S_{r_1}(0)$ such $x_0 \in A$ and $y_0 \in \mathbf{R}^n \setminus A$. Because the sphere $S_{r_1}(0)$ is pathwise connected, there exists a path $c\colon [0,1] \to S_{r_1}(0)$ joining the points x_0 and y_0 ($c(0) = x_0$, $c(1) = y_0$) and therefore by the previous Lemma there exists $\xi' \in [0,1]$ such that $c(\xi') \in \partial A$. But since $c(\xi) \in S_{r_1}(0)$, this implies $\| c(\xi') \| = r_1 > r$.

Proof of Theorem 1. We shall consider two cases acording to the boundness of G.

a) Suppose that G is unbounded. Using the hypotheses of the theorem and the previous Lemma we can conclude that $\mathbf{R}^n \setminus G$ is compact and therefore there exists an open ball B such that $\mathbf{R}^n \setminus G \subseteq B$. But since $\partial B \subseteq \mathbf{R}^n \setminus B$ and $\mathbf{R}^n \setminus B \subseteq G$ it follows that $\partial B \subseteq G$.

Let $f\colon \mathbf{R}^n \to \mathbf{R}$ be a smooth and positive function such that $f^{-1}(0) = \mathbf{R}^n \setminus G$. We consider the function $g = f|_B\colon B \to \mathbf{R}$ and the number $m = \inf\{f(x) : x \in \partial B\}$. Obviously $m > 0$ and from a weak formulation of Sard's theorem ([1] p.221 or [4] p.21) we can choose a strictly decreasing sequence $(y_n)_{n \geq 0}$ of positive regular values of g such that $\lim y_n = 0$, $g^{-1}(y_n) \neq \emptyset$ and $0 < y_n < m$ for all $n \in \mathbf{N}$. From the theorem of pre-image ([1] p.204) we get that $V_n = g^{-1}(y_n)$ are smooth submanifolds of B having dimension n-1 (i.e. V_n are smooth hypersurfaces of $\mathbf{R}^n$). Because B is a bounded subset of $\mathbf{R}^n$ and $V_n \subseteq B$ for all $n \in \mathbf{N}$, V_n are bounded too. We are going to show that V_n are also closed subsets of $\mathbf{R}^n$, proving that

$g^{-1}(y_n) = f^{-1}(y_n) \cap \bar{B}$. The inclusion $g^{-1}(y_n) \subseteq f^{-1}(y_n)$ is obvious for all $n \geq 0$. For the other one, if $x \in f^{-1}(y_n) \cap \bar{B}$, we have $f(x) = y_n$ and $x \in \bar{B} = B \cup \partial B$. But since $y_n < m$ we get $x \notin \partial B$ and therefore $x \in B$, i.e. $x \in g^{-1}(y_n)$. Hence $V_n = f^{-1}(y_n)$ are closed and bounded hypersuperfaces of $\mathbf{R}^n$.

Further we intend to show that $V_n \xrightarrow{H} \partial G$ or, equivalently, $\lim \sup\{d(x, \partial G) \mid x \in V_n\} = 0$ and $\lim \sup\{d(x, V_n) \mid x \in \partial G\} = 0$. Suppose that there exists $\varepsilon_0 > 0$ such that for each $k \in \mathbf{N}$ one can find $n_k \in \mathbf{N}$, $n_k \geq k$ with the property that $\sup\{d(x, \partial G) \mid x \in V_{n_k}\} > \varepsilon_0$. Therefore there exist $x_{n_k} \in V_{n_k}$ such that $d(x_{n_k}, \partial G) \geq \varepsilon_0$. Since $x_{n_k} \in V_{n_k}$ and $V_{n_k} \subseteq B$ for all $k \in \mathbf{N}$, we obtain that $x_{n_k} \in B$ for all $k \in \mathbf{N}$. But since B is bounded there exists a convergent subsequence of $(x_{n_k})_{k \geq 0}$, denoted in the same way.

Let $x_0 = \lim x_{n_k}$. Since $d(x_{n_k}, \partial G) \geq \varepsilon_0$ for all $k \in \mathbf{N}$ and because the function $d(\cdot, \partial G)$ is continous, we get that $d(x_0, \partial G) \geq \varepsilon_0$. On the other hand $\lim x_{n_k} = x_0$ implies $\lim g(x_{n_k}) = g(x_0)$. But $g(x_{n_k}) = y_{n_k}$ and so $g(x_0) = 0$ i.e. $x_0 \in \mathbf{R}^n \setminus G \subseteq \overline{\mathbf{R}^n \setminus G}$. Because $x_{n_k} \in G$ and $x_0 = \lim x_{n_k}$, it follows that $x_0 \in \bar{G}$. Therefore $x_0 \in \bar{G} \cap \overline{\mathbf{R}^n \setminus G} = \partial G$, whence $d(x_0, \partial G) = 0$, a contradiction with $d(x_0, \partial G) \geq \varepsilon_0$. Therefore $\lim \sup\{d(x, \partial G) \mid x \in V_n\} = 0$.

Further, to complete the proof of this case we must show that $\lim \sup\{d(x, V_n) \mid x \in \partial G\} = 0$. For this purpose we shall show that for each x belonging to ∂G, there exists $n = n(\varepsilon, x) \in \mathbf{N}$ and W_x an open neighborhood of x such that $d(y, V_p) < \varepsilon$ for all $y \in W_x$ and all $p \geq n(\varepsilon, x)$.

Let x be an element of ∂G; this means that $B_\varepsilon(x) \cap G \cap B \neq \emptyset$, namely if $z \in B_\varepsilon(x) \cap G \cap B$ then $\|z - x\| < \varepsilon$ and $f(z) > 0$. Because $\lim y_n = 0$, there exists $n(\varepsilon, x) \in \mathbf{N}$ such that $y_n < f(z)$ for all $n \geq n(\varepsilon, x)$. We consider the function $h_z : [0,1] \to \mathbf{R}$ $h_z(t) = f[x + t(z - x)]$. Obviously h_z is a continuous function with $h_z(0) = f(x) = 0$ and $h_z(1) = f(z)$. But since $0 < y_n < f(z)$ for all

$n \geq n(\varepsilon,x)$ we can conclude that there exists $t_n \in (0,1)$ such that $h_x(t_n) = y_n$ for all $n \geq n(\varepsilon,x)$, namely $f[t_n z + (1-t)x] = y_n$ $n \geq n(\varepsilon,x)$ and so $t_n z + (1-t)x \in g^{-1}(y_n) = V_n$ $n \geq n(\varepsilon,x)$. We therefore get that $d(x,V_n) \leq \| x - t_n z - (1-t_n)x \| \leq t_n \| z - x \| < \varepsilon$ for all $n \geq n(\varepsilon,x)$. Because $d(x,V_{n(\varepsilon,x)}) < \varepsilon$ and $d(\cdot,V_{n(\varepsilon,x)})$ is a continuous function these implies that there exists W_x an open neighborhood of x such that $d(y,V_{n(\varepsilon,x)}) < \varepsilon$ for all $y \in W_x$. Further we are going to show that $d(y,V_n) < \varepsilon$ for all $y \in W_x \cap \partial G = U_x$ and all $n \geq n(\varepsilon,x)$. Indeed, because $V_{n(\varepsilon,x)}$ is compact there exists $v \in V_{n(\varepsilon,x)}$ such that $d(y,V_{n(\varepsilon,x)}) = \| y - v \|$ (where y is a fixed element of W_x). Considering the function $\alpha : [0,1] \to \mathbf{R}$ $\alpha(t) = f[ty + (1-t)v]$ we can remark that $\alpha(0) = y_{n(\varepsilon,x)}$, $\alpha(1) = 0$ and $0 < y_n < y_{n(\varepsilon,x)}$ for all $n > n(\varepsilon,x)$. Hence there exists $t_n \in (0,1)$ such that $\alpha(t_n) = y_n$, that is $f[t_n y + (1-t_n)v] = y_n$ if $n > n(\varepsilon,x)$. This implies that $t_n y + (1-t_n)v \in V_n$ if $n > n(\varepsilon,x)$. The last relation proves that

$$d(y,V_n) \leq \| y - t_n y - (1-t_n)v \| = (1-t_n)\| y - v \| \leq \| y - v \| = d(y,V_{n(\varepsilon,x)})$$

for all $n > n(\varepsilon,x)$, which finishes the proof of the above assertion. Hence we constructed an open covering $\{U_x\}_{x \in \partial G}$ of the compact set ∂G. Therefore there exist $x_1, x_2, \dots, x_n \in \partial G$ such that $\partial G = U_{x_1} \cup U_{x_2} \cup \dots \cup U_{x_n}$. It is easy to see that $d(x,V_n) < \varepsilon$ for all $x \in \partial G$ and $n \geq n_\varepsilon = \max\{ n(\varepsilon,x_1), \dots, n(\varepsilon,x_n) \}$, namely $\sup\{ d(x,V_n) \mid x \in \partial G \} < \varepsilon$ for all $n \geq n_\varepsilon$.

b) In the case G is bounded we can apply an analogous reasoning for a smooth function $f : \mathbf{R}^n \to \mathbf{R}$ with property that $f^{-1}(0) = \mathbf{R}^n \setminus G$.

COROLLARY. *The set of compact and smooth hypersuperfaces of* $\mathbf{R}^n$ *is dense in the set of all compact topological submanifolds of* $\mathbf{R}^n$ *(with respect to the metric* H*).*

Proof. It is sufficient to prove that each compact topological submanifold M of $\mathbf{R}^n$ can be aproximated by a sequence of compact smooth hypersuperfaces of $\mathbf{R}^n$. Because M is a compact topological submanifold this implies that $\mathbf{R}^n \setminus M$ is open and $M = \partial(\mathbf{R}^n \setminus M)$. So using the previous theorem we can conclude that there exists a sequence of compact smooth hypersuperfaces V_n such that $\lim H(V_n, M) = 0$, namely $V_n \xrightarrow{H} M$.

Remark. From Lemma 1 we get that, if G is an open subset of $\mathbb{R}^n$ such that $\operatorname{ext} G \neq \emptyset$ then $\mathbb{R}^n \setminus \partial G$ is not connected. In what follows we shall prove the following

THEOREM 2. *If G is an open subset of $\mathbb{R}^n$ such that $\operatorname{ext} G \neq \emptyset$, then $\mathbb{R}^n \setminus (\partial G \cap \partial \operatorname{ext} G)$ is not connected.*

Proof. Let $F_1, F_2 : \mathbb{R}^n \to \mathbb{R}$ be two positive functions such that $F_1^{-1}(0) = \bar{G}$ and $F_2^{-1}(0) = \overline{\operatorname{ext} G}$. We are going to show that $K^{-1}(0) \subseteq \partial G \cap \partial(\operatorname{ext} G)$ where $K = F_1 - F_2$. It is easy to see that $F_1^{-1}(0) \cap F_2^{-1}(0) = \partial G \cap \partial(\operatorname{ext} G)$. So the above relation is equivalent with $(\mathbb{R}^n \setminus F_1^{-1}(0)) \cup (\mathbb{R}^n \setminus F_2^{-1}(0)) \subseteq \mathbb{R}^n \setminus K^{-1}(0)$. If x belongs to $\mathbb{R}^n \setminus F_1^{-1}(0)$ then $F_1(x) > 0$, that is $x \in \mathbb{R}^n \setminus \bar{G}$. But because $\bar{G}$ is closed it follows that $\mathbb{R}^n \setminus \bar{G}$ is open and obviously $\mathbb{R}^n \setminus \bar{G} \subseteq \mathbb{R}^n \setminus G$. So we get that $x \in \operatorname{ext} G$, namely $F_2(x) = 0$ and therefore $x \in \mathbb{R}^n \setminus K^{-1}(0)$. Hence $\mathbb{R}^n \setminus F_1^{-1}(0) \subseteq \mathbb{R}^n \setminus K^{-1}(0)$. In a similar way one can show that $\mathbb{R}^n \setminus F_2^{-1}(0) \subseteq \mathbb{R}^n \setminus K^{-1}(0)$. Therefore the relation $K^{-1}(0) \subseteq \partial G \cap \partial(\operatorname{ext} G)$ is valid.

It is easy to see that $G, \operatorname{ext} G \subseteq \mathbb{R}^n \setminus [\partial G \cap \partial(\operatorname{ext} G)]$. Further we suppose that $\mathbb{R}^n \setminus [\partial G \cap \partial(\operatorname{ext} G)]$ is connected, so it is pathwise connected and therefore if $x \in G$ and $y \in \operatorname{ext} G$, there exists $c : [0,1] \to \mathbb{R}^n$ such that $c(0) = x$, $c(1) = y$ and $\operatorname{Im} c \subseteq \mathbb{R}^n \setminus [\partial G \cap \partial \operatorname{ext} G]$. On the other hand we have the following relations:

$$K(c(0)) = K(x) = F_1(x) - F_2(x) = -F_2(x) \leq 0 \text{ and}$$

$$K(c(1)) = K(y) = F_1(y) - F_2(y) = F_1(y) \geq 0.$$

So, there exists $\eta \in [0,1]$ such that $K(c(\eta)) = 0$ or $c(\eta) \in K^{-1}(0)$, namely $c(\eta) \in \partial G \cap \partial \operatorname{ext} G$. But this contradicts the initial assumption and therefore the proof is complete.

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ON THE CHORD METHOD IN FRECHET SPACES

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REZUMAT. - Asupra metodei coardei în spaţii Fréchet. În lucrare se reia, şi se îmbunătăţeşte un rezultat demonstrat in [1], privind rezolvarea aproximativă a unei ecuaţii $P(X) = 0$, folosind algoritmul

$$x_{n+1} = x_n - \Lambda_n P(x_n)$$

cu $\Lambda_n = [x_n, x_{n-1}; P]^{-1}$, $P: X \to Y$, X şi Y fiind spaţii Fréchet.

1. It is known [3] that in the Banach space the unmodified and the modified Newton-Kantorovici methods for solving the operator equation generate stable solutions which means that the methods still converge even if the initial approximate x_0 is replaced by another $\tilde{x}_0$ but close enough to x_0.

This property is still valable even in the case of the unmodified and modified chord methods, which means that the convergence holds even if the approximates x_0, x_{-1} are replaced by $\tilde{x}_0, \tilde{x}_{-1}$, close enough.

We prove this results in what follows.

2. Let be the equation

$$P(x) = \theta \tag{1}$$

where P: $X \to Y$ is a continuous nonlinear mapping, X and Y Fréchet spaces [4], $\theta \in Y$ the

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null element of the space.

Let be any $x_0, x_{-1} \in D \subset X$ and $\Lambda_n = [x_n, x_{n-1}; P]^{-1}$ the generalized quotient [2] of P.

Starting from the initial approximations x_0, x_{-1} and using the algorithm

$$x_{n+1} = x_n - \Lambda_n P(x_n) \tag{2}$$

known as "the Chord method", the sequence (x_n) is generated, each term of it being an approximate of the solution of (1).

To apply the iterative method (2), at each step the mapping $[x_k, x_{k-1}; P]^{-1}$ is needed. To avoid this inconvenient, a "modified" method may be applied

$$x_{n+1} = x_n - [x_0, x_{-1}; P]^{-1} P(x_n) \tag{2'}$$

which, to generate the (x_n) approximations, uses only the mapping

$$\Lambda_0 = [x_0, x_{-1}; P]^{-1}.$$

Concerning the convergence of Chord method (2), in [1] the following theorem is proved.

THEOREM A. *Supposing the existence of any continuous linear mapping* $\Lambda \in (X, Y)^{\#}$ *which has an inverse, and the following conditions fulfilled for initial approximates* $x_0, x_{-1} \in S \subset X$:

1^0 $\| \Lambda P(x_i) \| < \bar{\eta}_i$, $i = 0, -1$ *and* $\bar{\eta}_0 \le 1/4\, \bar{\eta}_{-1}$;

2^0 $\| \Lambda [x_0, x_{-1}: P] - I \| \le a < 1$, *I beeing the identical mapping;*

3^0 $\| \Lambda [u, v, w; P] \| \le \bar{K}$, $\forall\, u, v, w \in S(x_0, 5/4\, \bar{\eta}_{-1})$;

4^0 $\bar{h}_0 = \dfrac{\bar{K}\, \bar{\eta}_{-1}}{(1-a)^2} \le 1/4$

then the equation (1) *has a solution* $x^* \in S$, *which is the limit of sequence* (x_n) *generated by* (2), *the order of convergence being*

$$\| x^* - x_n \| \le \frac{1}{2^{s_n - 1}} q^{s_{n-1}} (4\bar{h}_0)^{s_n} \bar{\eta}_0 \tag{3}$$

where $0 < q < 1$, *and* s_n *is the general term of the sequence of partial summs of Fibonacci sequence, with* $u_1 = u_2 = 1$.

We prove the following

THEOREM. *In the conditions of Theorem A, having the constants* η_0, η_{-1}, k *and* $h < 1/4$, *with* $\| \tilde{x}_i - x_i \| \le b/2$ *for* $i = 0, -1$, *with*

$$b = \frac{1 - 4k\eta_{-1}}{4k} .$$

the (2) *and* (2') *methods are convergente, for any* $\tilde{x}_0$ *and* $\tilde{x}_{-1}$ *initial approximates.*

Proof. Taking in theorem 1 as $\Lambda \in (Y,X)^{\#}$ the mapping

$$\Lambda_0 = [x_0, x_{-1}; P]^{-1}$$

and replacing (x_0, x_{-1}) by $(\tilde{x}_0, \tilde{x}_{-1})$, we have

$$\begin{aligned}
\Lambda_0 P(\tilde{x}_0) &= \Lambda_0 P(x_0) + \Lambda_0[\tilde{x}_0, x_0; P](\tilde{x}_0 - x_0) = \\
&= \Lambda_0 P(x_0) + \big(I + \Lambda_0[\tilde{x}_0, x_0; P] - I\big)(\tilde{x}_0 - x_0) = \\
&= \Lambda_0 P(x_0) + \Big(I + \Lambda_0\big([\tilde{x}_0, x_0; P] - [x_0, x_{-1}; P]\big)\Big)(\tilde{x}_0 - x_0) = \\
&= \Lambda_0 P(x_0) + (\tilde{x}_0 - x_0) + \Lambda_0[\tilde{x}_0, x_0, x_{-1}; P](\tilde{x}_0 - x_{-1})(\tilde{x}_0 - x_0)
\end{aligned}$$

and based on the hypothesis

$$\| \Lambda_0 P(\tilde{x}_0) \| \le \eta_0 + b/2 + K(b^2/4) = \eta_0 .$$

Analogously, we may prove that

$$\| \Lambda_0 P(\tilde{x}_{-1}) \| \le \eta_{-1} + b/2 + K(b^2/4) = \bar{\eta}_{-1} .$$

so the condition 1^0 is true.

Also, we have

$$
\begin{aligned}
&) \mid \Lambda_0[\tilde{x}_0, \tilde{x}_{-1}; P] - I \mid (= \\
&=) \mid \Lambda_0\big([\tilde{x}_0, \tilde{x}_{-1}; P] - [x_0, x_{-1}; P] + [\tilde{x}_0, x_{-1}; P] - [\tilde{x}_0, x_{-1}; P]\big) = \\
&=) \mid \Lambda_0\big([\tilde{x}_0, \tilde{x}_{-1}, x_{-1}; P](\tilde{x}_{-1} - x_{-1}) + [\tilde{x}_0, x_0, x_{-1}; P](\tilde{x}_0 - x_0)\big) \mid (
\end{aligned}
$$

where from, based on the hypothesis, it results

$$) \mid \Lambda_0[\tilde{x}_0, \tilde{x}_{-1}; P] - I \mid (\le kb \le 1$$

so $kb = a$, and condition 2^0 is fulfilled.

Obviously, for $k = \bar{k}$ condition 3^0 is true.

The constant $\bar{h}_0$ is given by

$$\bar{h}_0 = \frac{\bar{k}\bar{\eta}_{-1}}{(1-d)^2} = \frac{1}{4}\,\frac{4k\eta_{-1} + 2kb + k^2b^2}{(1-kb)^2} \le \frac{1}{4}.$$

So, the theorem is proved.

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ANALYSE DE L'OPERATEUR DE NEWMARK

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REZUMAT. - Analiza operatorului lui Newmark. Faţă de analiza clasică a operatorului Newmark, lucrarea prezintă elemente noi constând în:
- o nouă deducere a formulelor operatorului şi o detaliere a coeficienţilor operatorului pentru unul şi mai multe grade de libertate;
- metoda Newton şi metoda punctului fix împreună cu o generalizare a limitei de convergenţă prezentată în [5] pentru un grad de libertate;
- evaluarea alegerii pasului de timp pe criteriul satisfacerii ipotezelor metodei.

1. Introduction. L'opérateur de Newmark [1,2,3,4] est l'un opérateurs les plus utilisés en Dynamique des structures pour l'intégration directe des équations du mouvement. L'étude qui suit présente un nouveau point de vue sur l'opérateur de Newmark, par: (1) une voie de déduction des formules de l'opérateur; (2) detaillement de la structure des coefficients β,γ qui contrôlent l'opérateur; (3) nouvelle définition de la limite de convergence; (4) conditions pour le choix du pas de yemps Δt.

En supposant que l'accélération Ü est dérivable sur l'intervalle $I = [t_0, t_0+\Delta t]$, on montre que la méthode de Newmark se déduit de l'hypothése $\dddot{U}(t) = \text{constant}$ pour $t \in I$.

Les considérations portent sur l'équation differentielle du mouvement d'un système non-linéaire

$$M\ddot{U} + g(\dot{U}) + f(U) = P(t) \tag{1}$$

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où: $U(t) = [u_1(t),...,u_s(t)]^T$ est le vecteur des degrés de liberté dynamique, M matrice d'inertie, g et f sont les fonctions non-linéaires d'amortissement et de rigidité respectivement, et P est le vecteur des excitations. En particulier, pour un système linèaire, $g(\dot{U}) = C\dot{U}$ et $f(U) = KU$.

2. Déduction des formules de l'opérateur. Soit t_0 un moment arbitraire et $t_1 = t_0 + \Delta t$ où Δt = petit. On suppose $\ddot{U}$ dérivable sur I et on dveloppe u_i et $\dot{u}_i$ en série de Taylor jusqu'aux termes en $\ddot{u}_i$ et on prend le reste sous la forme de Schlömilcn-Roche. Supprimons pour convenence l'indice "i" et désignons par les indices "0" et "1" les valeurs des fonctions $u, \dot{u}, \ddot{u}$ en t_0 et t_1 respectivement, il suit:

$$u_1 = u_0 + \dot{u}_0\Delta t + \ddot{u}_0(\Delta t)^2 + \frac{\dddot{u}(t_0+\Theta\Delta t)}{2p}(1-\Theta)^{3-p}(\Delta t)^3 \tag{2}$$

$$\dot{u}_1 = \dot{u}_0 + \ddot{U}_0\Delta t + \frac{\dddot{u}(t_0+\Theta'\Delta t)}{p'}(1-\Theta')^{2-p'}(\Delta t)^2 \tag{3}$$

où p et p' sont des nombres naturels arbitrairement choisis et $\Theta \in (0,1)$, $\Theta' \in (0,1)$ sont associés *à* p et p', respectivement.

Supposons Δt suffisament petit pour qu'on ait

$$\dddot{u}(t) = constant, \; t \in [t_0, t_1] \tag{4}$$

il résulte

$$\dddot{u}(t) = \frac{\ddot{u}(t_1) - \ddot{u}(t_0)}{\Delta t} = \frac{\Delta\ddot{u}_1}{\Delta t} \tag{5}$$

Désignons dans les relations (3), (4) par $\bar{u}_1$ et $\bar{\dot{u}}_1$ les séries de Taylor tronquées des fonctions u et $\dot{u}$ à savoir:

$$\bar{u}_1 = u_0 + \dot{u}_0\Delta t + \ddot{u}_0(\Delta t)^2/2, \; \bar{\dot{u}}_1 = \dot{u}_0 + \ddot{u}_0\Delta t \tag{6}$$

et posons

$$\beta = \frac{(1-\Theta)^{3-p}}{2p}, \gamma = \frac{(1-\Theta')^{2-p'}}{p'} \tag{7}$$

Les relations (7) deviennent

$$u_1 = \bar{u}_1 + \beta(\Delta\ddot{u}_1)(\Delta t)^2, \; \dot{u}_1 = \bar{\dot{u}}_1 + \gamma(\Delta\ddot{u}_1)(\Delta t) \tag{8}$$

où, selon (6)

$$\ddot{u}_1 = \ddot{u}_0 + \Delta\ddot{u}_1 \tag{9}$$

Les relations (8,9) sont les formules qui définissent l'opérateur de Newmark pour un système à un seul degré de liberté. Les coefficients β et γ sont définis par (7).

Remarques:

1) En choisisant en (2,3) $p = 3$ et $p' = 2$, on élimine la necéssité d'estimer θ et θ'; il résulte $\beta = 1/6$ et $\gamma = 1/2$.

2) Selon la déduction précédente, toute méthode définie par les relations (8,9) peut être nommée **méthode de l'accélération linéaire**, puisque l'hypothèse (4) revient à la condition $\ddot{u}$ = fonction linéaire sur I.

3) Conformément à (1) dans laquelle les fonctions f et g sont supposées dérivables sur I, l'hypothèse $\ddot{U}$ = dérivable sur I revient à la condition que $P(t)$ soit dérivable sur I; en particulier, si $P(t)$ est dérivable, à l'exception d'un nombre fini de points, sur l'intervalle de réponse $[t_0,TT]$, la condition imposéee à $\ddot{U}$ se réalisera en choisissant ces points comme extremités des intervalles $[t_0,t_0+\Delta t]$ - v# 4.

En écrivant les relations (8,9) pour $i = \overline{1,s}$, on obtient les formules de l'opérateur pour un système à plusieurs degrés de liberté sous la forme suivante:

$$U_1 = \bar{U}_1 + diag\,[\beta_i]\,\Delta\ddot{U}_1(\Delta t)^2 \tag{10}$$

$$\dot{U}_1 = \bar{\dot{U}}_1 + diag\,[\gamma_i]\,\Delta\ddot{U}_1(\Delta t) \tag{11}$$

$$\tilde{U} = \tilde{U}_0 + \Delta\tilde{U}_1 \tag{12}$$

où

$$\beta_i = \frac{(1-\Theta_i)^{3-p_i}}{2p_i},\ \gamma = \frac{(1-\Theta_i')^{2-p_i'}}{p_i'} \tag{13}$$

Les séries de Taylor tronqées sont, suivant (6)

$$\overline{U}_1 = U_0 + \dot{U}_0\Delta t + \ddot{U}_0(\Delta t)^2/2,\ \bar{\dot{U}}_1 = \dot{U}_0 + \ddot{U}_0\Delta t \tag{14}$$

L'accélération initiale $\ddot{U}_0$ se détermine de l'équation (1) écrite pour l'instant $t = t_0$.

Si dans les relations (10,11,13) **on choisit**

$$p_i = p,\ p_i' = p';\ i = \overline{1,s} \tag{15}$$

et **on estime** $\Theta_i = \Theta$, $\Theta_i' = \Theta'$; $i = \overline{1,s}$, il en résulte les formules habituelles de l'opérateur de Newmark [1,2,3], sous la forme donnée en [4]:

$$U_1 = \overline{U}_1 + \beta(\Delta t)^2\Delta\ddot{U}_1 \tag{16}$$

$$\dot{U}_1 = \bar{\dot{U}}_1 + \gamma(\Delta t)\Delta\ddot{U}_1 \tag{17}$$

$$\ddot{U}_1 = \ddot{U}_0 + \Delta\ddot{U}_1 \tag{18}$$

Remarque: En particulier, les relations (16,17) sont vraies quelles que soient $\theta \in (0,1)$, $\theta' \in (0,1)$, si l'on choisit en (13) $p_i = 3$ et $p_i' = 2$, $i = \overline{1,s}$, choix qui conduit aux coefficients $\beta_i = 1/6$, $\gamma_i = 1/2$, $i = \overline{1,s}$ dans les équations (10,11).

3. Intégration de l'équation (1). L'équation (1) écrite pour $t = t_1$ est:

$$M\ddot{U}_1 + g(\dot{U}_1) + f(U_1) = P(t_1) \tag{19}$$

En substituant U_1 et $\dot{U}_1$ des relations (16,17) dans l'équation (1), cette dernière devient

$$M(\ddot{U}_0 + \Delta\ddot{U}_1) + g(\bar{\dot{U}}_1 + \gamma\Delta t\ddot{U}_1) + f(\overline{U}_1 + \beta(\Delta t)^2\Delta\ddot{U}_1) = P(t_1) \tag{20}$$

3.1 Methode de Newton. L'équation (20) se résout par la méthode de Newton, selon le schéma d'intération suivant, où W désigne la variation $\Delta\ddot{U}_1$ de l'accélération:

$$J(W_n)\delta_{n+1} = -F(W_n),\ W_{n+1} = W_n + \delta_{n+1};\ W_0 = 0 \tag{21}$$

où:

$$F(W) = MW + g(\bar{\dot{U}}_1 + \gamma(\Delta t)W) + f(\bar{U}_1 + \beta(\Delta t)^2 W) + M\ddot{U}_0 - P(t_1). \tag{22}$$

$J(W)$ est le jacobien de F, à savoir:

$$J(W) = M + B(\bar{\dot{U}}_1 + \gamma(\Delta t)W)\gamma\Delta t + A(\bar{U}_1 + \beta(\Delta t)^2 W)\beta(\Delta t)^2 \tag{23}$$

où A et B sont les jacobiens de g et f, respectivement.

L'itération (21) se poursuit jusqu'à ce que l'un des suivants tests est satisfait:

$$\|\delta_{n+1}\| \le \varepsilon,\ \|F(W_n)\| \le \varepsilon_1 \tag{24}$$

où ε et ε_1 sont choisis d'avance.

3.2 Méthode du point fixe. Limite de convergence. L'équation (20) peut être mise sous la forme suivante, qui permet la résolution par la méthode du point fixe:

$$W = G(W) \tag{25}$$

où W désigne toujours $\Delta\ddot{U}_1$ et

$$G(W) = M^{-1}[-g(\gamma(\Delta t)W + \bar{\dot{U}}_1) - f(\beta(\Delta t)^2 W + \bar{U}_1) + P(t_1)] - \ddot{U}_0 \tag{26}$$

L'équation (26) se résout selon le schéma:

$$W_{n+1} = G(W_n);\ W_0 = 0 \tag{27}$$

Une condition suffisante de convergence des itérées (27) est d'avoir

$$\|J_G(W)\|_\infty < 1 \tag{28}$$

sur un voisinage de la solution de (20) -v.[6]; on a désigné par J_G le jacobien de G, défini par

(29).

En (29) les dérivées partielles sont calculées aux points $(\gamma\Delta t W + \bar{U}_1)$ et $(\beta(\Delta t)^2 W + \bar{U}_1)$ respectivement.

$$J_G(W) = M^{-1}\left[-\left(\frac{\partial g_i}{\partial u_j}\gamma\Delta t + \frac{\partial f_i}{\partial u_j}\beta(\Delta t)^2\right)\right]_{i,j=1,s} \tag{29}$$

Soit dans un voisinage des points $\bar{U}_1$ et $\bar{\dot{U}}_1$ respectivement

$$\left|\frac{\partial g_i}{\partial u_j}\right| < b, \left|\frac{\partial f_i}{\partial u_j}\right| < a \tag{30}$$

et

$$\frac{1}{m} = \|M^{-1}\|_\infty \tag{31}$$

En particulier, si M est diagonale, $M = \mathrm{diag}[m_i]$, l'on a $m = \max_{i=1,s}|m_i|$. De (30,31) il résulte

$$\|J_G\|_\infty \leq \frac{1}{m}s(b\gamma\Delta t + a\beta(\Delta t)^2) \tag{32}$$

et, une condition suffisante pour qu'on ait (28) est:

$$a\beta(\Delta t)^2 + b\gamma(\Delta t) - \frac{m}{s} < 0 \tag{33}$$

laquelle équivaut à

$$0 < \Delta t < \Delta t_c \tag{34}$$

où

$$\Delta t_c = \frac{-b\gamma \pm \sqrt{(b^2\gamma^2 + 4(m/s)a\beta)}}{2a\beta} \tag{35}$$

Application: pour un système linéaire, $W = \Delta\ddot{U}_1$ se détermine directement de (20). En appliquant la condition (30) à ce cas où $g(\dot{U}) = C\dot{U}$, $f(U) = KU$, il vient:

$$b = \max_{i,j}|c_{ij}|,\ a = \max_{i,j}|k_{ij}| \tag{36}$$

En particulier, pour un système à un degré de liberté, où $b = c = 2\zeta\omega m$ et $a = k = \omega^2 m$, la condition (36) devient

$$\Delta t_c = \frac{-\zeta\gamma + \sqrt{(\zeta^2\gamma^2 + \beta^2)}}{\beta} \qquad (37)$$

qui définit la limite de convergence. Pour $\zeta = 0$ retrouve la valeur déduite en [5], à savoir $\Delta t = 1/(\omega\beta)$.

4. Choisir le pas de temps Δt. La dimension du pas Δt est dictée par trois catégories de conditions, visant à assurer:

I) les hypothèses de l'opérateur

II) la convergence de la méthode de résolution de l'équation (20)

III) la stabilité de l'opérateur

Généralement on choisira un pas de temps constant, en le modifiant seulement localement en vue de satisfaire la condition I-a.

I-a) $\ddot{U}$ **soit dérivable sur** $I = [t_0, t_0+\Delta t]$

Tenant compte de l'hypothèse que f et g sont supposées dérivables sur I, il rèsulte que $P(t)$ doit elle-aussi être dérivable sur I. Si P est continue sur $[T_0,TT]$ et elle est dérivable sur cet intervalle à l'exception d'un nombre fini de points t', alors ces points doivent être des points de division de l'intervalle et Δt devra être modifié en conséquence.

Si une des composantes p_i de P est discontinue en t', par exemple $p_i(t'-0)=p_i(t')=p_i^{(1)}$ et $p_i(t'+0)=p_i^{(2)}$, alors on prendra t' comme point de division et on considerera une variation linéaire de p_i sur $[t', t'+\Delta t']$ entre les valeurs $p_i^{(1)}$ et $p_i^{(2)}$, où $\Delta t' \leq \Delta t$.

I-b) $\dddot{U}(t)$ **constant sur** I

C'est la condition la plus critique et bienqu'elle ne puisse générer une borne explicite pour

Δt, il faut avoir en attention le fait que, en supposant $\dddot{U}$ continue sur un voisinage de t_0, $0 < \Delta t < \delta$ implique $\|\dddot{U}(t_1) - \dddot{U}(t_0)\| < \varepsilon(\delta)$, de sorte que I-a est réalisée d'autant mieux que Δt est plus petit.

Une indication sur une borne pour Δt peut s'obtenir comme suit: considérons un système à un degré de liberté et supposons que $\dddot{U}(t)$ est dérivable et la dérivée $u^{(4)}$ est bornée sur l'intervalle $[t_0,TT]$; on peut supposer Δt suffisamment petit pour que $\dddot{u}$ soit monotone sur I; alors, la condition que la variation de $\dddot{u}$ sur I soit petite peut s'écrire

$$|\dddot{U}(t_1) - \dddot{U}(t_0)| \leq \varepsilon V_3 \tag{38}$$

où V_3 est la variation de $\dddot{u}$ sur $[T_0,TT]$ et $\in$ limite la variation de $\dddot{u}$ sur I à une fraction de V_3. L'équation (38) peut s'écrire

$$|u^{(4)}(\xi)|\Delta t \leq \varepsilon V_3 \tag{39}$$

où $t_0 < \xi < t_0+\Delta t$. Il suffit de prendre

$$\Delta t \leq \varepsilon\frac{V_3}{M_4} \tag{40}$$

où

$$|u^{(4)}(t)| \leq M_4,\ t\in[T_0,TT] \tag{41}$$

En particulier pour un système en vibration libre, $u(t) = a\cos(\omega t+\varnothing)$, il rèsulte $V_3 = 2a\omega^3$, $M_4 = a\,\omega^4$ et la condition (39) devient

$$\Delta t \leq \frac{2\varepsilon}{\omega} \tag{42}$$

où

$$\frac{\Delta t}{T} \leq \frac{\varepsilon}{\pi} \tag{43}$$

Par exemple, pour $\varepsilon = 0.01$ il vient $\Delta t/T < 1/(100\,\pi) \approx 3.2*10^{-3}$. Dans le cas d'un système à plusieurs degrés de liberté en posant

$$V_3 = \|V_{3i}\|_\infty,\ v_{3i} = \max(\bar{\ddot{u}}_i(t)) - \min(\bar{\ddot{u}}_i(t)),\ t\in[T_0,TT] \tag{44}$$

une condition suffisante pour qu'on ait

$$\|\dddot{U}(t_1) - \dddot{U}(t_0)\|_\infty \le V_3 \tag{45}$$

est (40) où

$$\|U^{(4)}(t)\| \le M_4,\ t\in[T_0,TT] \tag{46}$$

Pour un système linéaire en vibration libre, les conditions (42,43) ont lieu avec ω = la fréquence minimale et T = la période maximale du système.

II) Lors de la résolution de l'équation (20) par la méthode du point fixe, on doit avoir $\Delta t < \Delta t_c$ où Δt_c est donné par (35).

III) Δt sera toujours plus petit où égal à la limite de stabilité de la méthode - v.[3].

Remarques:

- La condition la plus restrictive est I-b.
- Le choix d'un Δt excessivement petit conduira à un grand nombre de pas d'intégration et par cela, à une propagation accrue des erreurs d'arondissement qui affectera la précision de la réponse calculée.
- Une voie empirique pour choisir un Δt convenable est la suivante: on calcule la réponse sur un petit intervalle $[t_0,TT_1]$, successivement avec des pas Δt et $\Delta t/2$ (le premier Δt soumis aux conditions I-III); on choisit comme Δt, le pas pour lequel les réponses calculées avec Δt et $\Delta t/2$ contient un même nombre de chiffres significatifs identiques - ce nombre étant fixé d'avance.

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THE APPLICATION OF (P,Q)-ANALYTIC FUNCTIONS TO STUDY OF AN AXIALLY-SYMMETRIC IDEAL AND COMPRESSIBLE JET

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REZUMAT. - Aplicarea funcţiilor (*P,Q*) analitice la studiul unui jet axial simetric ideal. Se prezintă o metodă de rezolvare a problemelor spaţiale legate de mişcările fluide compresibile ideale în caz axial simetric (sau plan), utilizând proprietăţile transformărilor cvasiconforme ca soluţii ale unor ecuaţii eliptice de tip Beltrami. Ca aplicaţie este studiat cazul "Cisotti-Villat-Popp".

We consider a stationary, irotationally motion of an ideal barotrop fluid and we suppose that the masic forces are null (I_p).

The isentropic loin is:

$$p = p^1\left(\frac{\rho}{\rho^1}\right) \tag{1.1}$$

where p is the pression, ρ is the fluid's density and p^1, ρ^1 are the characteristic values for p,ρ. Here p^1, ρ^1 may be p_0, ρ_0 in the points of null velocity, or p^0, ρ^0 which are calculated on the free surface of the jet when the velocity is $V_0 = \text{const}$. Then the Bernoulli equation is:

$$\frac{1}{2}V^2 + \int_{p_0}^{p}\frac{dp}{\rho} = 0 \tag{1.2}$$

where V is the algebraic velocity.

The sound velocity C for ρ, p, and C_0 which corresponding for ρ_0, p_0 are

$$C^2 = \frac{dp}{d\rho},\quad C_0^2 = \left(\frac{dp}{d\rho}\right)_0 \tag{1.3}$$

By using the Ciaplighin variable [1]

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$$\tau = \frac{\gamma-1}{2}\cdot\frac{V^2}{V_0^2} = \frac{V^2}{V_{max}^2},\ \text{and}\ \beta = \frac{1}{\gamma-1},$$

where γ is the adiabatic constant, then we obtain:

$$\rho = \rho_0(1-\tau)^\beta,\ p = p_0(1-\tau)^{\beta+1},\ C^2 = C_0^2(1-\tau) \tag{1.4}$$

We will consider the system of cylindrical coordinates (Ox,Or), $z = x + ir$, where Ox is the radial axis, which in the case of plane motions is the Oy axis. Then we have the motion equations:

$$\begin{cases} \frac{\partial}{\partial x}(\rho r^k u) + \frac{\partial}{\partial r}(\rho r^k v) = 0 \\ \frac{\partial u}{\partial r} - \frac{\partial v}{\partial x} = 0 \end{cases} \tag{1.5}$$

where the first equation is the continuity equation and secondly is the irotationality equation (rot $\bar{V} = o$), where $\bar{V} = (u, v)$.

For $k = 1$ the motion is axially symmetric and for $k = 0$ the motion is plane.

By (1.5) we introduce the velocity potential $\varphi(x,r)$ and the stream function $\psi(x,r)$ and we obtain the next system:

$$u = \frac{\partial\varphi}{\partial x} = p\frac{\partial\psi}{\partial r},\ v = \frac{\partial\varphi}{\partial r} = -p\frac{\partial\psi}{\delta x},\ p = r^k\frac{\rho}{\rho^1} \tag{1.6}$$

In this case the complex potential is $f(z) = \varphi(x,r) + i\psi(x,r)$ and the complex velocity is $w = u + iv$, respectively.

By using the hodographic plane (V, θ), where $u = V\cos\theta$, $v = V\sin\theta$, with $V = |w|$ and $\theta = \arg(u + iv)$, or the plane (τ,θ), by (1.3) and (1.6), we have the system:

$$\begin{cases} \frac{\partial\tau}{\partial\psi} = \frac{2\tau}{p}\cdot\frac{\partial\theta}{\partial\varphi} \\ \frac{\partial\tau}{\partial\varphi} = -\frac{2\tau(1-\tau)p}{1-(2\beta+1)\tau}\cdot\frac{\partial\theta}{\partial\psi} - \frac{k}{r}\cdot\frac{1-\tau}{1-(2\beta+1)\tau}\cdot\frac{\sin\theta}{2}\sqrt{\frac{2\tau}{\alpha}} \end{cases} \tag{1.7}$$

For the plane problems is known the hodographic method of Ciaplighin-Iacob-

Falkovici (C-J-F) [3].

Because the system (1.7) is very complicated it is applied the theory of generalized functions or (P,Q) - anaytic functions.

For this aim we consider the canonic domain $D_\zeta = \xi + i\eta$ and we transform conformally the domains D_z, D_f, D_w on D_ζ.

This canonic domain may be semicircle or semiplane.

We prefer the semiplane $\zeta = \xi + i\eta$, $\eta \geq 0$ and we introduce the Jukowski function ω be defined by:

$$\omega = \ln\frac{V_0}{V} + i\theta = t + i\theta\,,\; W = Ve^{i\theta}, \bar{W} = V_0 e^{-\omega(\zeta)} \tag{1.8}$$

DEFINITION [4],[5]. The complex function $f(z) = U(x,y) + i\,V(x,y)$, $z = x + iy$, is called *(P,Q) - analytic* in a domain D_z if it satisfy the Beltramy system:

$$PU_x + QU_y - V_y = 0\,,\quad -QU_x + PU_y + V_x = 0 \tag{1.9}$$

where $P(x,y)$, $Q(x,y)$ such that $P > 0$ and U,V satisfy the regularity conditions. ($U_x = \partial U/\partial x$, etc.)

When $Q = 0$, f is called *P-analytic function* in D_z if it satisfy:

$$PU_x = V_y\,,\; PU_y = -V_x \tag{1.9'}$$

The system (1.6) has the form (1.9'). When the system (1.9) has the elliptical form, then it is equivalent with the equation:

$$W_{\bar{z}} = A\,W_z + B\,\bar{W}_{\bar{z}} + C \tag{1.10}$$

where $W(z) = PV + i(V - QU)$, $W_z = \dfrac{\partial W}{\partial z}$, $W_{\bar{z}} = \dfrac{\partial W}{\partial \bar{z}}$.

In this case the one-to-one maps between D_z and D_f which are defined by the (P,Q)-analytic functions, are called *quasiconform transformations* [4], [5].

Let be the quasiconformal system

$$\begin{cases} -\psi_y + a_{11}\varphi_x + a_{12}\varphi_y + a_0\varphi + b_0\psi = h \\ \psi_x + a_{21}\varphi_x + a_{22}\varphi_y + c_0\varphi + d_0\psi = i \end{cases} \tag{1.11}$$

where a_{ij}, a_0, b_0, c_0, d_0, h, i are measurable, bounded functions by (x,y,φ,ψ). Then the function $f = \varphi + i\psi$, which is solution of this system and an homeomorphism between the planes (x,y) and (φ,ψ), is a quasiconformal transformation or q-quasiconformal if is verified the differential condition [1]:

$$\varphi_x^2 + \varphi_y^2 + \psi_x^2 + \psi_y^2 \leq \left(q + \frac{1}{q}\right)(\varphi_x\psi_y - \varphi_y\psi_x) \tag{1.12}$$

From the geometrical aspect f transform one-to-one the infinitesimal circles into infinitesimal ellipses.

In the case $q = 1$ we obtain the conformally mapping.

The system (1.11) may be written in the complex form

$$f_{\bar{z}} + \mu_1 f_z + \mu_2 \bar{f}_{\bar{z}} = F, \tag{1.13}$$

where the coefficients and F are measurable and analytic functions by z, $\bar{z}$, f, $\bar{f}$ and the q-quasiconformality condition (1.12) becomes:

$$|f_{\bar{z}}| \leq \frac{q-1}{q+1}|f_z|$$

Let the function $f(z) = \varphi + i\psi$, where $z = x + ir$, be defined on a domain D_z.

DEFINITION [1]. The P-derivation operator of f has the form

$$\frac{d_P f(z)}{dz} = \frac{P\varphi_x + \psi_r}{2} + i\frac{\psi_x - P\psi_r}{2} \tag{1.14}$$

The (P,Q)-derivation operator of f has the form:

$$\frac{d_{(P,Q)} f(z)}{dz} = \frac{P\psi_x - Q\varphi_r + \psi_r}{2} + i\frac{\psi_x - Q\varphi_x - P\psi_r}{2} \tag{1.15}$$

In this manner the necessary and sufficient condition that f to be P-analytic (or (P,Q) analytic) on D_z is:

$$\frac{d_P f}{d\bar{z}} = 0 \quad \left(\text{or } \frac{d_{(P,Q)} f}{d\bar{z}} = 0\right) \tag{1.16}$$

We are presenting four immediate results obtained by using the analytic generalized functions.

By using (1.6), (1.9'), we observe that the complex potential f is p-analytic function.

THEOREM 1 [8]. *The velocity $W(z)$ is analytic generalized function of z and W verifies an equation of the form* (1.10):

$$\frac{d\bar{W}}{d\bar{z}} = -\frac{1}{2P}\cdot\frac{d\bar{W}}{dz} - \frac{1}{2P}\cdot\frac{dW}{dz}, \quad P = r^k\frac{\rho}{\rho^1}. \tag{1.17}$$

The solution of this equation realizes the quasi-conformal transformation between D_z and D_w.

THEOREM 2 [8]. *The function $\omega(f)$ defined by* (1.8) *is analytic generalized in D_f and verifies the equation* (1.18) *which is the type* (1.10):

$$\omega_{\bar{f}} - q_1\omega_f - q_2\bar{\omega}_{\bar{f}} = F \tag{1.18}$$

where

$$q_1 = \frac{P^2+D-1}{N}, \quad q_2 = \frac{PD}{N}, \quad N = (P+1)(P-D+1), \quad F = \frac{k}{r}\cdot\frac{\partial r}{\partial\varphi}, \quad D = \frac{2\beta V^2}{V^2_{max} - V^2}$$

THEOREM 3 [8]. *If there is a conformally transformation between D_f and D_ζ, then the function which represents D_z on D_ζ ($z = z(\zeta)$) is (P', Q')-analytic of ζ and hence are verify the Beltrami equations*

$$r_\xi = -P'x_\eta - Q'x_\xi, \quad r_\eta = P'x_\xi - Q'x_\eta$$
$$P' = \frac{PV^2}{P^2u^2+v^2}, \quad Q' = \frac{(1-P^2)uv}{P^2u^2+v^2} \tag{1.19}$$

THEOREM 4 [8]. *We suppose that the conditions of Theorem 3 are satisfied, then the function $\omega = \omega(\zeta)$ is analytic generalized on D_ζ, and satisfies the equation:*

$$\omega_{\zeta} - q_1^*\omega_\zeta - q_2^*\bar{\omega}_\zeta = F^*, \tag{1.24}$$

where

$$q_1^* = \frac{(\varphi_\xi + i\varphi_\eta)(P^2 + D - 1)}{(\varphi_\xi - i\varphi_\eta)}, \quad q_2^* = \frac{DP}{N^*}, \quad N^* = (P+1)(P+D^*),$$

$$D^* = \frac{1-M}{1-\dfrac{M^2}{2\beta+1}}, \quad F^* = \frac{kr}{V^2 N^*} \cdot \frac{\varphi_\xi^2 + \varphi_\eta^2}{\varphi_\xi - i\varphi_\eta}, \quad D = 1 - D^*,$$

and M is the Mach's number.

The "Cisotti-Villat-Popp" Model for Axially-Symmetric Compressible Fluid. We consider an ideal plasma jet which is coming out from a cylindric tube (ADA'D') with the axially section xOr.

In the next we will suppose that the I_p hypothesis is satisfied.

Let V_1 denote the velocity at the infinity (-∞).

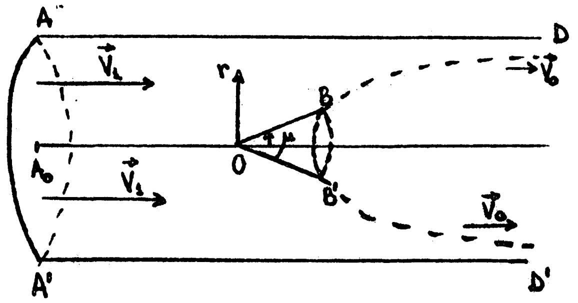
In his motion, the fluid meets the cone BOB' and at B and B' is detached the free surface (BCB'C') where the velocity is V_0, such that $V_1 < V_0$.

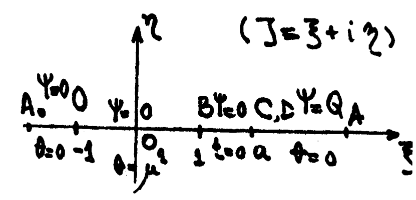
If the free surfaces ended by other cylindric coaxially tube which is situated in the cylinder's inner, then is obtained the "Jukowski-Roshko-Eppler" model [16].

Our model is called "Cisotti [3] - Villat [3] - Popp [10] model" because Cisotti and Villat studied the incompressible scheme and S.Popp studied the compressible plane case.

We are studying the motion in the upper half-plane and we shall note the corresponding flow by $Q = q/2^k\pi^k$, where for $k = 1$ we have the axisymmetric case and for $k = 0$ the plane case.

As we have mentioned above we transform conformally the half-plane D_f ($f = \varphi + i\psi$) from the motion domain $\{0 \le \psi \le Q, -\infty < \varphi < \infty\}$ in the canonical half-plane D_ζ, $\zeta = \xi + i\eta$, $\xi \in (-\infty, \infty)$, $\eta \ge 0$. ($\psi = 0$ on A_0O, OB, BC and $\psi = Q$ on AD).





(fig.1) (fig.2)

It will be determined hence the analytic function $f = f(\zeta)$ on D_ζ ($\eta \geq 0$) by knowing the imaginary part values ψ on the boundary $\eta = 0$, with the following correspondence:

$$\psi|_{\eta=0} = \begin{cases} 0, \ \xi \in (-\infty, a) \\ Q, \ \xi \in (a, +\infty) \end{cases} \quad \text{where } \ a > 1 . \tag{2.1}$$

The solution of this Dirichlet's problem for the half-plane will be determined with Cisotti's formula [3]:

$$f(\zeta) = -\frac{q}{2^k \pi^{k+1}} \log(\zeta - a) + i \frac{\zeta}{2^k \pi^k} \tag{2.2}$$

We observe that for $\eta \to 0$ we have

$$\frac{\partial \varphi}{\partial \xi} = -\frac{q}{2^k \pi^{k+1}} \frac{1}{\xi - a}, \ \frac{\partial \varphi}{\partial \eta} \to 0 \tag{2.3}$$

We transform the motion domain from D_ω, biuniquely on the canonically domain D_ζ and by using the Theorem 4 we get $\omega = \omega(\zeta)$ the analytic generalized in D_ζ. In this way with $\omega = \ln V_0 / V + i\theta = t + i\theta$, we know the values of $\omega = \omega(\zeta)$ on the boundary with the following correspondence as in (2.1)

$(A_0 O)$: $\theta = 0$; (OB): $\theta = \mu = \alpha\pi$; (BC): $t = 0$; (AD): $\theta = 0$.

This is a Volterra's problem.

Next we consider the auxiliary function

$$S: D_\zeta \to C, \ S(\zeta) = R + iT = \frac{\omega(\zeta)}{\sqrt{a-\zeta}\sqrt{1-\zeta}} = \frac{t + i\theta}{\sqrt{a-\zeta}\sqrt{1-\zeta}}$$

hence we must determine an analytic function $S = S(\zeta)$ on the half-plane $\eta \geq 0$ knowing its imaginary part $T(\xi)$ on the boundary

$$(A_0O)\ \ T = 0;\ (OB)\ T = \frac{\alpha\pi}{\sqrt{1-\xi}\,\sqrt{a-\xi}};\ (BC)\ T = 0;\ (AD) T = 0.$$

By applying the Cisotti's formula, we deduce

$$\omega(\zeta) = \alpha\sqrt{1-\zeta}\,\sqrt{a-\zeta}\int_{-1}^{1}\frac{d\xi}{\sqrt{1-\xi}\,\sqrt{a-\xi}\,(\xi-\zeta)} \tag{2.4}$$

By solving this integral we obtain

$$\omega = \omega(\zeta) = 2\alpha\,\log\left[\frac{\sqrt{a+1}\,\sqrt{1-\zeta}+\sqrt{2}\,\sqrt{a-\zeta}}{\sqrt{a-1}\,\sqrt{-1-\zeta}}\right] \tag{2.5}$$

By following the formula (1.8) we obtain the complex velocity

$$\bar{\omega}(\zeta) = u - iv = V_0\left[\frac{\sqrt{a-1}\,\sqrt{1-\zeta}}{\sqrt{a+1}\,\sqrt{1-\zeta}+\sqrt{2}\,\sqrt{a-\zeta}}\right]^{2\alpha} \tag{2.6}$$

With (2.2), (2.5), (2.6) is realized the Theorem 2. Also are satisfied the hypothesis of Theorem 3 and the transition formulas (1.19) $z = z(\zeta)$ become

$$\frac{\partial x}{\partial\xi} = \frac{u}{V^2}\frac{\varphi_\xi}{P},\ \frac{\partial r}{\partial\xi} = \frac{v}{V^2}\frac{\varphi_\xi}{P},\ P = \frac{\rho}{\rho_0}r^k \tag{2.7}$$

In this was is obtained the Theorem 1 which determine the correspondence $D_w \leftrightarrow D_f$, $D_f \leftrightarrow D_z$.

Furthermore we shall specify the distribution of speeds, pressures and densities on the boundary of the motion domain ($\eta = 0$)

(A_0O): $u = V_0 f_1(\xi),\ v = 0,\ \rho = \rho_0\left[1-\tau_0 f_1^2\right]^\beta,\ p = p_0\left[1-\tau_0 f_1^2\right]^{\beta+1}$,

where $f_1(\xi) = \left[\dfrac{\sqrt{a-1}\,\sqrt{|\xi+1|}}{\sqrt{a+1}\,\sqrt{1-\xi}+\sqrt{2}\,\sqrt{a-\xi}}\right]^{2\alpha}$, $\xi\in(-\infty,-1)$

(OB): $u = V_0 f_2\cos\alpha\pi,\ v = V_0 f_2\sin\alpha\pi,\ \rho = \rho_0\left[1-\tau_0 f_2^2\right]^\beta,\ p = p_0\left[1-\tau_0 f_2^2\right]^{\beta+1}$,

where $f_2(\xi) = \left[\dfrac{\sqrt{a-1}\,\sqrt{1+\xi}}{\sqrt{a+1}\,\sqrt{1-\xi}+\sqrt{2}\,\sqrt{a-\xi}}\right]^{2\alpha}$, $\xi\in(-1,1)$

(BC): $u = V_0\cos f_3(\xi),\ v = V_0\sin f_3(\xi),\ \rho = \rho_0[1-\tau_0]^\beta,\ p = P_0(1-\tau_0)^{\beta+1}$,

where $f_3(\xi) = 2\alpha\,\mathrm{arc}tg\sqrt{\dfrac{2}{a+1}}\sqrt{\dfrac{a-\xi}{\xi-1}}$, $\xi\in(1,a)$

(AD): $u = V_0 f_4(\xi),\ v = 0,\ \rho = \rho_0\left[1-\tau_0 f_4^2\right]^\beta,\ p = p_0\left[1-\tau_0 f_4^2\right]^{\beta+1}$, where $f_4(\xi) = \left[\frac{\sqrt{a-1}\sqrt{1+\xi}}{\sqrt{2}\sqrt{\xi-a}+\sqrt{a+1}\sqrt{\xi-1}}\right]^{2\alpha}$, $\xi \in (a,+\infty)$

We observe that:

$V(0) = V(\xi = -1) = 0$; $V(B) = V(\xi = 1) = V_0$, $V(C) = V(D) = V(\xi = a) = V_0$

and

$$V(A) = V(A_0) = V(\xi \to -\infty) = V_1 = V_0\left[\frac{\sqrt{a-1}}{\sqrt{2}+\sqrt{a+1}}\right]^{2\alpha}$$

If we denote $s = V/V_0$, $s \in [0,1]$ and $s_1 = \frac{V_1}{V_0} = \left[\frac{a-1}{\sqrt{2}+\sqrt{a+1}}\right]^{2\alpha}$,

we determine the parameter $a = a(V_1, V_0) = 1 + \frac{8}{\left[s_1^{-\frac{1}{2\alpha}} - s_1^{\frac{1}{2\alpha}}\right]^2}$.

Using the above distributions for speed and density we can obtain the limits equations and other geometric parameters, by integrating the equations (2.7):

$$\begin{cases} r(\xi) = \left[(1\sin\mu)^{k+1} + \frac{(k+1)q}{2^k\pi^{k+1}l^kV_0}\frac{\rho^0}{\rho_0}\int_1^\xi \frac{\sin f_3(\xi)}{a-\xi}d\xi\right]^{\frac{1}{k+1}} \\ x(\xi) = 1\cos\mu + \frac{2^k}{2^k\pi^{k+1}l^kV_0}\frac{\rho^0}{\rho_0}\int_1^\xi \frac{\cos f_3(\xi)}{r^k(\xi)(a-\xi)}d\xi,\ \xi\in(1,\infty), \end{cases} \tag{2.8}$$

where OB = 1, $x(B) = 1\cos\mu$, $r(B) = 1\sin\mu$.

In our problem we must know the parametric equations for OB and the "a priori" calculus of OB = 1.

By using (2.7) we have

$$\begin{cases} r(\xi) = \left[\frac{q(k+1)\sin\mu(1-\tau_0)^\beta}{2^k\pi^{k+1}V_0}\right]^{\frac{1}{k+1}}\left[I(\xi)\right]^{\frac{1}{k+1}} \\ x(\xi) = \frac{q(1-\tau_0)^\beta\cos\mu}{2^k\pi^{k+1}V_0}\int_{-1}^\xi \frac{d\xi}{r^k(\xi)(a-\xi)f_2(\xi)(1-\tau_0 f_2^2(\xi))^\beta} \end{cases} \tag{2.9}$$

$\xi \in (-1,1)$ where $I(\xi) = \int_{-1}^{\xi} \frac{d\xi}{f_2(\xi)(a-\xi)(1-\tau_0 f_2^2(\xi))^\beta}$

If we take $\xi = 1$ in (2.9), we obtain

$$(1 \sin\mu)^{k+1} = \frac{q(k+1)\sin\mu(1-\tau_0)^\beta}{2^k \pi^{k+1} V_0} I(1) \tag{2.10}$$

In the same way, by applying the continuity equation we deduce $V_1 \rho_1 H^{k+1} = V_0 \rho^0 h^{k+1}$, where $AA_0 = H$, $CD = h = H - r_c$, so H/h depend by V_1 and V_0.

The study of the drag coefficient. The pression action on the cone ($k = 1$) or on the plane ($k = 0$) is given by the relation [3]:

$$P = (2\pi)^k \int_{r(-1)}^{r(1)} (p - p^0) r^k dr, \tag{3.1}$$

where $p = p_0 \left[1 - \tau_0 f_2^2 \right]^{\beta+1}$ and $p^0 = p_0 \left[1 - \tau_0 \right]^{\beta+1}$

Here p_0 is the pression on the obstacle where $V = 0$.

Also by (2.10) we have

$$P = \frac{(2\pi)^k p_0 (\beta+1)(1 \sin\mu)^{k+1}}{I(1)} I_2(1) \tag{3.2}$$

where $I_2(1) = I_2 = \int_{-1}^{1} \frac{(1-\tau_0 f_2^2(\xi))^{\beta+1} - (1-\tau_0)^{\beta+1}}{(a-\xi)(1-\tau_0 f_2^2(\xi))^\beta f_2(\xi)} d\xi$

and $I(1) = I_1$, hence

$$P = (2\pi)^k p_0 (\beta+1)(1 \sin\mu)^{k+1} \frac{I_2}{I_1}, \quad \text{for } k \in \{0,1\}. \tag{3.3}$$

With the notation

$$s = \frac{V}{V_0} = \left[\frac{\sqrt{a-1}\,\sqrt{1+\xi}}{\sqrt{a+1}\,\sqrt{1-\xi} + \sqrt{2}\,\sqrt{a-\xi}} \right]^{2\alpha} \leq 1 \tag{3.4}$$

we obtain

$$I_1 = \int_0^1 L(s,a,\alpha,\beta,\tau_0)\, ds \tag{3.5}$$

and

$$I_2 = \int_0^1 f_0(s,\tau_0,\beta)\, L(s,a,\alpha,\beta,\tau_0)\, ds \tag{3.6}$$

where

$$L(s,a,\alpha,\beta,\tau_0) = \frac{8}{\alpha(a^2-1)} \frac{s^{\frac{1}{\alpha}-2}\left[a^2\left(1-s^{\frac{1}{\alpha}}\right)+2s^{\frac{1}{\alpha}}\right]}{\left(1-\tau_0 s^2\right)^{\beta}\left[8s^{\frac{1}{\alpha}}+(a-1)\left(1+s^{\frac{1}{\alpha}}\right)^2\right]}$$

and

$$f_0(s,\tau_0,\beta) = (1-\tau_0 s^2)^{\beta+1} - (1-\tau_0)^{\beta+1}$$

In (3.3) we consider the particular cases $\mu \in \{30^\circ, 45^\circ, 60^\circ, 75^\circ, 90^\circ\}$ or $\alpha \in \{1/6, 1/4, 1/3, 5/12, 1/2\}$.

The drag coefficient is given by

$$C_p = 2P/\rho^0 V_0 S_k \tag{3.7}$$

where $S_k = \pi^k l^{k+1}\sin^k\mu$ represents the area of the cone (k=1) or the length of the plate (k = 0).

By using (3.3) and $\rho^0 = \rho_0(1-\tau_0)^\beta$ we deduce:

$$C_p = \frac{2^k \sin\mu}{(k+1)(1-\tau_0)^\beta}\frac{I_2}{I_1} = \frac{\sin\mu}{(1-\tau_0)^\beta}\frac{I_2}{I_1}. \tag{3.8}$$

When $\beta \to 0$ (the incompressible case) we have $C_p(i) = \frac{I_2(i)}{I_1(i)}\sin\mu$, where

$$I_1(i) = \frac{16}{a^2-1}\int_0^1 \frac{a^2(1-s^2)+2s^2}{8s^2+(a-1)(1+s^2)^2}\,ds = \frac{16}{a^2-1}\int_0^1 L^i ds$$

$$I_2(i) = \frac{16}{a^2-1}\int_0^1 (1-s^2)\,L^i\,ds.$$

We shall indicate several particular and remarkable cases which are obtained by using the general formulas (3.3).

a) The formula (3.8) can be compared with the formula obtained by S.Popp in the plane case [10], [11], when the obstacle is a feather on the channel (C-V-P).

The same problem was studied by Y.Sungurtsev [12] by following the method of integral operators.

For the incompressible case, we obtain the drag coefficient for the Cisotti-Villat model

[14]

$$C_p(i) = \frac{\frac{1}{2}\left(\frac{V_1}{V_0} + \frac{V_0}{V_1}\right) - 1}{\frac{V_0}{V_1} - 1 + \frac{2}{\pi}\left[\frac{V_1}{V_0} - \frac{V_0}{V_1}\right] \operatorname{arc}tg\frac{V_1}{V_0}}$$

When the plate has the incidence, the problem was generalized by M.Lupu [7].

b) If we suppose that $V_1 \to V_0$, $a \to \infty$, we obtain the disc's problem for $k = 1$ and the feather's problem for $k = 0$, in the delimited jet, which was studied by C.Jacob [3].

c) From a) if $V_1 \to 0$ we obtain the cone's problem or disc's problem in the free stream ($k = 1$) and the feather's problem in the free stream ($k = 0$), which were studied by C.Jacob [3]. For $\mu = \pi/2$ and $k = 0$ we obtain the Helmholtz's problem [3].

In the next we will present the values of the drag coefficient C_p for some values of the angle μ ($\mu = \alpha\pi$).

Alpha = 1/2

s1/tau0	0.08	0.09	0.10	0.11	0.12	0.14	0.1667
0.20	0.291525	0.332442	0.374516	0.417806	0.462372	0.555593	0.689400
0.30	0.284736	0.324585	0.365534	0.407635	0.450946	0.541445	0.671117
0.40	0.279669	0.318723	0.358833	0.400050	0.442429	0.530905	0.657510
0.50	0.275991	0.314469	0.353972	0.394550	0.436254	0.523268	0.647659
0.60	0.273418	0.311493	0.350572	0.390703	0.431936	0.517930	0.640776
0.75	0.271112	0.308827	0.347526	0.387257	0.428069	0.513151	0.634616

Alpha = 1/3

s1/tau0	0.08	0.09	0.10	0.11	0.12	0.14	0.17
0.20	0.239573	0.272995	0.307314	0.342572	0.378815	0.454455	0.562635
0.30	0.227299	0.258821	0.291142	0.324300	0.358335	0.429206	0.530203
0.40	0.217168	0.247134	0.277822	0.309267	0.341502	0.408500	0.503626
0.50	0.209167	0.237911	0.267320	0.297423	0.328253	0.392229	0.482899
0.60	0.203172	0.231006	0.259462	0.288568	0.318353	0.380089	0.467417
0.75	0.197476	0.224448	0.252003	0.280166	0.308966	0.368590	0.452776

Alpha = 1/4

s1/tau0	0.08	0.09	0.10	0.11	0.12	0.14	0.17
0.20	0.189594	0.215951	0.242993	0.270752	0.299261	0.358683	0.443489
0.30	0.176091	0.200379	0.225230	0.250732	0.276852	0.331132	0.408233
0.40	0.164239	0.186731	0.209725	0.233243	0.257302	0.307183	0.377724
0.50	0.154278	0.175277	0.196713	0.218605	0.240972	0.287220	0.352390
0.60	0.146370	0.166194	0.186405	0.207021	0.228059	0.271477	0.332472
0.75	0.133404	0.157053	0.176042	0.195388	0.215108	0.255713	0.312582

Alpha = 1/6

s1/tau0	0.08	0.09	0.10	0.11	0.12	0.14	0.17
0.20	0.129735	0.147705	0.166124	0.185014	0.204398	0.244742	0.302190
0.30	0.117604	0.133732	0.150226	0.167102	0.184376	0.220196	0.270901
0.40	0.106300	0.120743	0.135481	0.150525	0.165890	0.197639	0.242331
0.50	0.095078	0.109020	0.122199	0.135625	0.149307	0.177487	0.216950
0.60	0.087264	0.098930	0.110787	0.122843	0.135107	0.160292	0.195396
0.75	0.077441	0.087702	0.098110	0.108669	0.119386	0.141320	0.171727

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CRONICĂ

JÁNOS BOLYAI (1802 - 1860)*

Á. PÁL**

"Ich hatte diesen jungen Geometer
v. Bolyai für eine Genie erster Grösse".
(C. F. GAUSS către Chr. L. GERLING, 1832)

ABSTRACT. Brief presentation of biographical data and genesis of János [Iohann] Bolyai's renowned work "Appendix". The impact produced by the non - Euclidean geometry of Bolyai - Lobacevski on the development of mathematics. The space - an attribute of the matter, with intrinsec properties to be cleared up ("treasures" as J.Bolyai wrote). Aspects of the reflection of basic structural changes in mathematical education. "Appendix" in translations (from Latin in German, French, Italian, English, Hungarian, Serbian, Russian, Romanian). Homages in scientific manifestations held till now in J.Bolyai's native town. References.

* Prezentat la Simpozionul János Bolyai, organizat de Catedra de Geometrie a Universităţii "Babeş-Bolyai" din Cluj-Napoca, la 31 august (Cluj-Napoca) şi 1 septembrie (Tg. Mureş) 1992, ("Prima geometrie neeuclidiană şi influenţa ei asupra dezvoltării ştiinţelor"), precum şi la Sesiunea ştiinţifică a Institutului Astronomic al Academiei Române din Bucureşti, dedicată Anului Internaţional al Spaţiului, "Scientia Spatii absolute vera" de J.Bolyai şi cercetarea Universului", la 17-18 decembrie 1992.

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1. Sunt 190 de ani de la naşterea marelui matematician maghiar János [Johann] Bolyai, care a creat, - concomitent şi independent de marele matematician rus Nikolai Ivanovici Lobacevski (1792 - 1856) -, prima geometrie neeuclidiană, deschizând prin aceasta calea ce avea să ducă la apariţia unor noi geometrii neeuclidiene şi a aplicaţiilor lor în astronomie, mecanică şi fizică.

János Bolyai s-a născut la Cluj, la 15 decembrie 1802. A studiat de timpuriu matematica sub îndrumarea tatălui său, Farkas [Wolfgang] Bolyai, el însuşi un distins matematician, profesor la Colegiul din Târgu-Mureş. A urmat apoi cursurile Academiei Tehnice militare de la Viena, devenind inginer militar. După strălucitele studii, János capătă post la Direcţia fortificaţiilor din Timişoara, ca sublocotenent de geniu, fiind avansat în scurt timp până la gradul de căpitan.

Încă la Viena, János Bolyai a început să vadă clar în "problema paralelelor" (faimoasă problemă a axiomei a XI-a a lui Euclid, numită şi al 5-lea postulat al lui Euclid) pe care a atacat-o sub formă indirectă, la care nu se gândiseră matematicienii până atunci. Într-o comunicare ce i-a făcut-o tatălui său, din Timişoara, în anul 1823, János Bolyai, găsind cheia rezolvării acestei probleme, exclama: "din nimic am creat o altă lume, o lume nouă", aceste cuvinte devenind simbol al marii sale creaţii.

În 1832, János Bolyai publcă lucrarea sa ca un apendice la tratatul tatălui său "Tentamen"; din acest motiv, lucrarea lui János Bolyai este cunoscută sub denumirea de "Appendix", titlul ei adevărat fiind "Scientia Spatii absolute vera". În acestă lucrare el arată cum se poate construi o nouă geometrie, suprimând axioma paralelelor a lui Euclid şi punând în loc trei posibilităţi: o singură paralelă, o infinitate de paralele cuprinse între laturile unui triunghi, nici una; din aceasta triadă alegând-o pe cea mijlocie.

JÁNOS BOLYAI (1802-1860)

Originală în prezentare, cu o simbolică nouă şi extrem de concisă (28 de pagini), această geometrie non-euclidiană, numită "geometrie absolută" a lui János Bolyai, era echivalentă în principiile ei, cu "geometria hiperbolică" a lui Nikolai Lobacevski (apărută în 1829). (De altfel, lucrarea lui János Bolyai a apărut, la îndemnul tatălui lui, în broşură separată, într-un număr redus de exemplare, încă din 1831. Nici unul din cei doi Bolyai nu bănuia însă atunci că un alt geniu, profesorul de matematică de la Universitatea din Kazan', N.I.Lobacevski, publicase deja în limba rusă aceeaşi descoperire).

Descoperirea geometriei neeuclidiene, după o muncă asiduă de 10 ani, a reprezentat nu numai o victorie a cunoaşterii, dar şi una de ordin etic. "A afirma că - scrie istoricul clujean Benkö Samu - la o dreaptă *d*, printr-un punct exterior oarecare *P*, se pot duce un număr infinit de paralele nesecante, a însemnat un curaj intelectual şi o responsabilitate tot atât de mare ca, odinioară, afirmarea sfericităţii Pământului ori a heliocentrismului".

Ultimii 27 de ani ai vieţii (1833 - 1860), János Bolyai, pensionat de boală, şi i-a trăit în mari lipsuri, luptând fără succes pentru recunoaşterea descoperirii sale de a cărei importanţă crucială era pe deplin convins. "Nu din răutate ci din ignoranţă - afirma el - nu mi-au recunoscut rezultatele". De la J.Bolyai ne-au mai rămas încă câteva lucrări (a se vedea Stäckel, Weszely), dar nici una, se pare, nu poate rivaliza cu "Scientia Spatii absolute vera". El a murit la Târgu-Mureş, la 27 ianuarie 1860, fiind reînmormântat în acelaşi loc cu tatăl său.

Întru eternizarea memoriei celor doi Bolyai, posteritatea recunoscătoare le-a ridicat monumente; străzi, societăţi, instituţii ştiinţifice şi culturale le poartă numele, iar Uniunea Astronomică Internaţională (International Astronomical Union) a atribuit numele Bolyai unui crater de pe Lună şi unui asteroid (mica planetă nr. 1441).

2. Geometriile neeuclidiene, în evoluţia ulterioară descoperirii lor, au parcurs trei etape bine conturate: (i) teorii ignorate (până în jurul anului 1867); apoi, (ii) teorii acceptate, dar considerate drept teorii abstracte - rod al unor speculaţii pur formale -; iar mai apoi, (iii) teorii acceptate, cu recunoaşterea calităţii de a descrie spaţiul real (între anii 1913 - 1915).

În corespondenţa sa, celebrul matematician german de la Göttingen K.F.Gauss (1777 - 1855) (numit "princeps mathematicorum" datorită marilor sale realizări în multe domenii ale matematicii, fizicii şi astronomiei), coleg de studii şi prieten al lui Farkas Bolyai, s-a exprimat foarte laudativ despre "Appendix" (precum şi despre lucrarea similară a lui Lobacevski), dar în public nu şi-a dat părerea despre ele.

Timp de 30 de ani de la descoperirea geometriei hiberbolice "Bolyai - Lobacevski", puţini matematicieni i-au acordat atenţie şi se pare că nici unul n-a prevăzut implicaţiile ei asupra dezvoltării matematicii întregi şi asupra aplicaţiilor ei în astronomie, mecanică şi fizică.

Se poate chiar afirma că faţă de noua geometrie a existat o oarecare neîncredere: cercetătorii îşi puneau problema dacă aceasta este "adevărată" în acelaşi sens ca geometria lui Euclid şi nu credeau că ea are vreo valoare ştiinţifică.

În general, datorită neînţelegerii esenţei lor, matematicienii n-au luat în considerare rezultatele lui Bolyai şi Lobacevski, rezultate ce s-au dovedit - repetăm - ulterior ca fiind epocale.

Independent de Bolyai şi Lobacevski, B.Riemann (1826 - 1866) a fost acela care a înţeles, în plenitudinea ei, importanţa problematicii geometriei neeuclidiene. El a creat o teorie generală, apărută în 1854, admiţând nu numai existenţa geometriei hiperbolice, ci chiar a unor geometrii mai generale (numite astăzi geometrii sau spaţii riemanniene). În întregime ei, această teorie a fost înţeleasă de generaţia următoare, care s-a convins de importanţa ei teoretică şi practică.

Dubiile au fost risipite de demonstraţia surprinzător de elegantă a lui E.Beltrami, din 1868, care a arătat că geometria hiperbolică se poate interpreta ca geometria geodezicelor pe o suprafaţă de curbură constantă negativă (numită pseudosferă); la fel, geometria sferică (numită şi eliptică sau riemanniană), se poate interpreta pe o suprafaţă de curbură constantă pozitivă. Deoarece sfera şi pseudosfera sunt suprafeţe în spaţiul euclidian, consistenţa geometriilor neeuclidiene a fost demonstrată cu ajutorul modelelor amintite. Aceasta înseamnă că, deşi geometria euclidiană este cea mai folositoare în practică (în practica actuală), totuşi şi celelalte geometrii sunt la fel de "adevărate" şi vor fi folositoare - după cum vom menţiona mai jos - în practica viitoare.

Realizarea modelelor lui Beltrami a arătat că, de fapt, geometria lui Euclid şi geometriile neeuclidiene clasice sunt concomitent admisibile sau neadmisibile din punct de vedere logic. Dar consistenţa internă (din punctul de vedere al logicii matematice) n-a fost încă demonstrată la nici una dintre ele (!). (Problema consistenţei interne a oricărei geometrii este redusă, după cum se ştie, de către D.Hilbert (1862 - 1943) la consistenţa sistemului numerelor reale, care este o problemă a logicii matematice).

Succesorii lui E.Beltrami au ajuns la un nivel mai evoluat şi au creat metoda axiomatică modernă a geometriei. Progresul rapid în privinţa abstractizării se poate observa la trei mari matematicieni, a căror gândire originală a fost tipică pentru generaţia lor, M.Pasch, G.Peano (1858 - 1932) şi D.Hilbert.

Pasul hotărâtor în acestă direcţie a fost făcut de D.Hilbert în opera sa Bazele geometriei ("Grundlagen der Geometrie"), apărută în 1899, operă devenită clasică într-un scurt timp, inaugurând matematica secolului XX. Cu un minim de simbolism, Hilbert a convins pe geometri de caracterul abstract, pur formalizat, al geometriei şi autoritatea sa a încetăţenit metoda axiomatică nu numai în geometrie, dar în întreaga matematică de după 1900.

3. János Bolyai concepe spaţiul - aşa cum arată şi în titlul lucrării sale - "independent de adevărul sau falsitatyea axiomei a XI-a a lui Euclid (care nu poate fi dedusă niciodată a priori)" ca un atribut al materiei cu proprietăţile intrinseci ce pot fi cunoscute doar a posteriori, pe baza experienţei. (Aceasta în opoziţie cu concepţia lui Kant potrivit căreia spaţiul poate fi conceput aprioric, independent de lumea exterioară şi de materie, ca o formă a sensibilităţii). El însuşi spune: "spaţiul în adâncul său ascunde foarte multe comori pe care cel ce umblă la suprafaţă nu le vede niciodată". Mai mult încă, J.Bolyai scrie: "Şi legea gravitaţiei pare să fie (se prezintă) într-o strânsă legătură, continuare, cu forma, cu esenţa (cu structura), cu felul de a fi, ale spaţiului".

Ideea lui J.Bolyai privind legătura geometriei cu gravitaţia a stat, mai pe urmă, după aproape un secol de la exprimarea acestei conjecturi, la baza teoriei generalizate a relativităţii a lui Albert Einstein (1879 - 1955). Această teorie, bazată pe geometria neeuclidiană, dezvoltată de Riemann, admiţând că spaţiul tridimensional împreună cu timpul formează un tot (spaţiul cu patru dimensiuni), putea explica gravitaţia universală, deci un fenomen fizic, cu toate proprietăţile calitative şi cantitative, ca fiind o manifestare a curburii spaţiului respectiv.

Această teorie a fost strălucit confirmată prin explicarea fenomenului "deplasării periheliului" planetei Mercur şi prin "devierea razei de lumină" în apropierea masei gravitaţionale a Soarelui. De fapt, Farkas Bolyai a fost primul din lume care, în "Tentamen", deci încă în anul 1832, a arătat că din perturbaţiile planetare se pot trage concluzii asupra caracterului neeuclidian al spaţiului. El a raţionat astfel: "mecanica cerească se bazează pe geometrie euclidiană şi dacă mişcarea planetelor ar urma o geometrie neeuclidiană, atunci între orbita calculată şi cea observată cu timpul s-ar prezenta diferenţe esenţiale".

Succesul ideii de "geometrizare a câmpului gravitaţional", realizată de A.Einstein, a dat un imbold puternic pentru a elabora o teorie geometrizată ce cuprinde şi alte câmpuri fizice (câmpul electromagnetic, câmpul mezonic nuclear etc.). Aceste cercetări, iniţiate şi începute chiar de către Einstein, în perioada anilor 1918 - 1920, sunt strâns legate şi de dezvoltarea geometriei diferenţiale moderne ce avea să ducă la o serie de rezultate remarcabile privind generalizările geometriei riemanniene.

Marile progrese obţinute în secolul nostru în ştiinţele Universului, în special în astrometrie, mecanica cerească, geodezie, astrofizică şi cosmologie, n-ar fi fost posibile dacă nu s-ar fi elaborat teoria relativităţii generale, în măsură să explice fenomenul gravitaţiei şi să dezvăluie proprietăţile spaţiu-timpului în care au loc mişcările corpurilor cereşti şi propagarea razelor de lumină şi a celorlalte radiaţii electromagnetice. Reforma bazelor de calcul al efemeridelor şi orbitelor corpurilor cereşti, aflată în plină desfăşurare la ora actuală, care vizează introducerea unor noi sisteme de referinţă din TRG (pentru nevoile practice), de exemplu, pentru nevoile geoaeziei, va inaugura Astronomia secolului XXI - o astronomie care va fi o sinteză a realizărilor tuturor ştiinţelor Universului.

4. Descoperirile unice ale lui Bolyai şi Lobacevski au determinat schimbări structurale fundamentale în învăţământul matematicii moderne, în special în învăţământul superior.

Fiind vorba de un nou sistem geometric, s-a ivit necesitatea de a restructura vechile expuneri de geometrie, mai întâi la nivel academic (universitar), dar apoi şi în învăţământul general, răspunzând exigentelor moderne şi pregătind înţelegerea noilor teorii. Faptul că lucrarea lui J.Bolyai "Appendix" a apărut ca o anexă la lucrarea didactică a lui F.Bolyai "Tentamen..." ("Încercarea de iniţiere a tineretului studios în elementele matematicii pure, elementare şi superioare, printr-o metodă intuitivă, potrivită acestui scop") ne pare deosebit de semnificativ. S-ar putea crede că tânărul Bolyai, prin a sa "Ştiinţă absolută a Spaţiului", ar fi dorit să dea o replică expunerilor clasice didacticiste ale geometriei. Într-adevăr, expunerea sa sistematică şi unitară a elementelor de geometrie, care atestă o mare putere de sinteză, constituie un exemplu de abordare a studiului fundamentelor, dezvăluind într-un mod metodic, natural, alternativa celor două sisteme geometrice, sistemul Σ (euclidian) şi sistemul S (neeuclidian). Această schemă generală de studiu a geometriei, cuprinzând totalitatea proprietăţilor geometrice ale spaţiului independente de postulatul 5 al lui Euclid, se constituie în ceea ce ulterior s-a numit "geometrie absolută".

Pe lângă opera ştiinţifică a lui J.Bolyai, este relevantă, de asemenea, atitudinea sa privind problemele educaţiei tineretului, îndeosebi ale formării sale matemetice. El recomandase Colegiului din Târgu - Mureş să fie procurate operele şi lucrările didactice ale marilor matematicieni, formulând idei realiste privitoare la organizarea învăţământului general.

O prezentare a felului în care influenţa creaţiei lui J.Bolyai asupra geometriei şi a matematicii (schiţată mai sus în punctele 2 şi 3) s-a reflectat în monografii şi lucrări cu caracter didactic, cursuri sau tratate de nivel universitar, dar şi în manuale de liceu, se poate găsi în lucrarea publicată în Gazeta Matematică - Perfecţionarea metodică şi metodologică în matematică şi informatică (Pál, Ţarină, 1983).

Primele traduceri şi comentarii ale operei "Appendix" sunt prezentate în Anexa (1).

Opera creatorilor geometriilor neeuclidiene a fost omagiată la Cluj, cu prilejul diferitelor conferinţe şi simpozioane de specialitate, dintre care amintim pe cele cuprinse în Anexa (2).

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ANEXA (1)

Primele traduceri ale operei "Appendix", apărută în anul 1832, în limba latină, sub formă de adaus la opera "Tentamen" de Farkas Bolyai (cîteva tipărituri separate ale operei "Appendix" au apărut în anul precedent, iulie 1831)*:

1832. J.Bolyai, Raumlehre, Urabhänging von der (a priori nie entschiden werdenden) Wahroder Falschheit des berüchtigen XI Euklid'schen Axioms; für den Fall einer Falschheit desselben geometrische Quadratur des Kreises. (Traducerea autorului, în care cuvântul "Appendix" din titlu este înlocuit prin expresia "Ştiinţa Spaţiului", iar unele capitole conţin abateri esenţiale de la textul iniţial; această traducere s-a păstrat printre hârtiile lui J.Bolyai).

1867. N.Lobatschewsky, Pangeometria. Versione in italiano del Prof. Battaglini, vol. V di questo Giornale, 1867.

1867. J.Bolyai, La science absolue de l'espace... Traduit par G.J. Houel. "Mémoires de la Société des sciences physiques et naturelles de Bordeaux", 5, Bordeaux, 1864; ediţie separată, Paris, Gauthier - Villars, 1868; a doua ediţie, 1895.

1868. J.Bolyai, Sulla scienza dello spazio assolutamente vera... Versione dal latino per G.Battaglini, "Giornale di Matematica", 6, Napoli, 1868.

1872. J.Frischauf, Absolute Geometrie nach Johann Bolyai, Leipzig, 1872, ediţia a 2-a, 1876 (Lucrare ce reproduce "Appendix"-ul, cu câteva completări proprii ale autorului, care au căpătat o dezvoltare însemnată în a doua ediţie).

1891. J.Bolyai, The Science Absolute of Space... Translated from the Latin (1832), by G.B.Halsted, Austin (Texas), 1891; 1892; 1894; 1896. (Traducerea însoţită de un articol introductiv foarte larg). Aceeaşi traducere, în anii 1892 - 1894, a fost retipărită şi în Japonia.

1897. J.Bolyai, A térnek abszolut igaz tudománya (Rados). "Math. Phys. Lit.", 6, Budapest, 1897.
J.Bolyai, Scientia Spatii absolute vera. A tér abszolut igaz tudománya (J.Sutak, F.Schmidt), Budapest, 1897.

1902. Joann Bolyai de Bolya, Appendix... Editio nova oblata ab Academia scientiarium Hungarica ad diem natalem centesimum auctoris concelebrandum. Ediderunt Josephus Kürschák, Mauritius Réthy, Béla Tötössy de Zepethnek. Academiae scientiarum Hungaricae sodales. Budapestini. Sumptibus Academiae Scientiarum Hungaricae, MDCCCCII.

1928. "Appendix" od Ianoşa Bolyai, Belgrad, 1928. (Traducerea de B.Petronievici, însoţită de observaţii ample).

1950. Ianoş Bolyai, Appendix. Prilojenie, soderjasceie nauku o prostranstve absoliutno istinnuiu, Klassiki nauki, Moskva, 1950. (Lucrarea a fost tradusă, comentată şi completată cu texte explicative amănunţite de V.F.Kagan).

1954. Ianoş Bolyai, Appendix. Anexă, în care este expusă ştiinţa absolut adevărată a spaţiului independent de adevărul sau de falsitatea axiomei a XI-a a lui Euclid (care nu poate fi decisă a priori), pentru cazul falsităţii fiind adăugată cuadratura geometrică a cercului. Editura Academiei R.P.Române, Bucureşti, 1954. (Traducerea a fost făcută după originalul în limba latină de J.Toth; comentariile şi observaţiile au fost redactate după textul lui V.F.Kagan din ediţia în limba rusă; prefaţa a fost întocmită de Prof. G.Vrănceanu).

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* În cazul traducerii în limba italiană, se va menţiona şi opera lui N. Lobacevski. Aceasta şi pentru următoarea comparare: "I circoli di raggio infinito furono chiamati da Lobatschewski oricicli e dal Bolyai curve limiti", (Ettore Ricordi), "Giornale di Matematica", vol.XVIII, 1880, p.257.

ANEXA (2) a)

Bolyai János élete és műve (1953)

TARTALOM

ANEXA (2) b)

Lucrările Simpozionului Bolyai J. (1979)

CONTENTS

ANEXA (2) c)

Proceedings of the Nat. Coll. Geom. and Top. (1982)

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