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MATHEMATICA

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GENERIC AND SPECIFIC

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REZUMAT. - Generic şi specific. Considerind definiţia inductivă a formulelor predicative (cu sau fără predicatul egalităţii) se disting două categorii de formule: generice şi specifice. Se dau exemple şi se pregăteşte cadrul general pentru distingerea celor două categorii.

In each domain of the knowledge, definitions, properties, reasonings and so on, give a general logical formulation over a specific contents. The general formulation is realized within the framework of predicate logic (with or without equality) and the specific part is expressed in the language of terms (see [3]). Thus the proof of theorems is performed within a deductive theorie, whose formulas have both logic and specific aspects.

Example 1. The transitivity of (generalized) parallelism:

$$a \parallel b \wedge b \parallel c \Rightarrow a \parallel c$$

may be formulated by

$$\forall x(\mathcal{P}(x,a) \supset \mathcal{P}(x,b)) \wedge \forall x(\mathcal{P}(x,b) \supset \mathcal{P}(x,c)) \supset \forall x(\mathcal{P}(x,a) \supset \mathcal{P}(x,c)) \qquad (1)$$

where $\mathcal{P}(u,v)$ = "$u \perp v$" (perpendicularity)

Remark 1. If we put in (1) $\mathcal{P}(u,v)$ = "$u \leq v$", $u,v \in Z$, then we obtain the formulation of order-transitivity. Thus, the same logical formula may express similar properties in distinguished domains.

Remark 2. As (1) represents a true predicative formula, the transitivity is a general (not specifical) property.

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Example 2. Consider the predicative formula on Z :

$$\forall x(\wp(x,a) \wedge \wp(x,b) \supset \mathscr{E}(x,1)) \supset \exists y \exists z\, \mathscr{E}(s(p(a,y),p(b,z)),1) \qquad (2)$$

where $1,a,b,x,y,z \in Z$, s, p are terms (the sum and the product respectively) and $\mathscr{E}, \wp$ are the equality and the divisibility respectively. Thus (2) express a theorem in number theory: "If a,b are relative primes then there exist y,z so that $ay + bz = 1$".

Remark 3. If denote $R(a,b)$ = "a,b relative primes" and $\mathscr{E}_1(a,y,b,z)$ = "$ay + bz = 1$", then (2) becomes:

$$R(a,b) \supset \exists y \exists z\, \mathscr{E}_1(a,y,b,z) \qquad (2')$$

which is not a true predicative formula, although she have a (true) model on Z.

In the preceding examples we may observe that there are two tipes of (true) predicative formulas: "generical" true (as (1)) and "specifical" true (as (2')).

The main purpose of this paper is to characterize the two above mentioned notions.

In the following we sketch necessary preliminaries and give some characterizations.

GENERIC FORMULAS

Let $\mathcal{P}, Q, R, \ldots$ be **predicative symbols**, $x_i, y_j, z_k, \ldots$ ($i = \overline{1,n}$; $j = \overline{1,m}$; $k = \overline{1,p}, \ldots$), **individual symbols** and $\mathcal{P}(x_1,\ldots,x_n)$, $Q(y_1,\ldots,y_m)$, $R(z_1,\ldots,z_p), \ldots$ be **elementary** predicates.

We define, by induction, predicative formulas (see [3]):

(i). Every elementary predicate is a (elementary) predicative formula.

(ii). If F, G are predicative formulas then

$\bar{F}, (F)\wedge(G)$, $(F)\vee(G)$, $(F)\supset(G)$ are predicative formulas.

(iii). If the predicative formula $F(x)$ contains "free" variable x then $\forall x F(x)$, $\exists x F(x)$ are predicative formulas.

(iv). There is not other predicative formula.

To emphasize that the formula F contains predicates $\mathcal{P}_i$ from variables x_j ($i = \overline{1,m}$; $j = \overline{1,n}$), we write

$$F(\mathcal{P}_1,\ldots,\mathcal{P}_m; x_1,\ldots,x_n).$$

We call the predicative formulas above defined by (i) - (iii), **generic formulas.**

Remarks. 4. The variable x in $\forall xF(x)$ and $\exists xF(x)$ (see (iii)) is "bounded".

5. The formulas F,G are called **parts (subformulas)** of the formulas defined in (ii) and (iii).

As in [2] we define the **order**, $\acute{O}(H)$, of the part H of a predicative formula:

($\acute{O}_0$). F is a part of order zero of F,

($\acute{O}_1$). F,G are parts of order 1 of the formulas defined in (ii) and (iii).

($\acute{O}_2$). F is a part of order n+1 of G if it is part of order 1 of a part of order n of the formula G.

If F is a part of order n of G, we write $F \underset{n}{\leq} G$.

LEMMA. *If* $F \underset{p}{\leq} H$ *and* $G \underset{q}{\leq} H$ *then* $F \underset{p+q}{\leq} H$.

Proof. By induction, using the definition ($\acute{O}_0$ - $\acute{O}_2$).

Denote $\Pi(F)$ the set of parts of the formula F and define in $\Pi(F)$ the relation

($\leq$). $H \leq K$ iff there is $n \in \mathbf{N}$ so that $H \underset{n}{\leq} K$.

PROPOSITION. ($\Pi(F)$, $\leq$) is an ordered set.

Proof. (re). $H \leq H \Rightarrow H \underset{0}{\leq} H$ (Conf. ($\acute{O}_0$))

(tra). $H \leq K$, $K \leq L \Rightarrow$ there is $m,n \in \mathbf{N}$ so that

$H \underset{m}{\leq} K$ and $K \underset{n}{\leq} L \Rightarrow H \underset{m+n}{\leq} L$ (Lemma) $\Rightarrow H \leq L$.

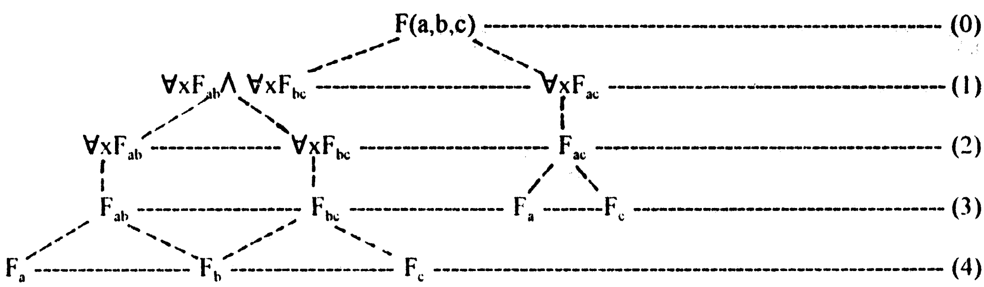
(an). $H \leq K$, $K \leq H \Rightarrow$ there is $m,n \in \mathbf{N}$ so that

$H \underset{m}{\leq} K$ and $K \underset{n}{\leq} H \Rightarrow H \underset{m+n}{\leq} H \Rightarrow m+n = 0 \Rightarrow m = n = 0 \Rightarrow H \underset{o}{\leq} K \Rightarrow H = K$.

COROLLARY 1. *The elementary parts of the formula F are minimals in* $(\Pi(F), \leq)$.

We call **edges** of the formula F, the minimals from $(\Pi(F)), \leq)$. This name derives from the graph of the ordered set $(\Pi(F), \leq)$.

Example 3. The graph of $(\Pi(F), \leq)$, where F is the formula (1) (see Example 1) is given below:



where $F_h = \mathfrak{P}(x,h)$ and $F_{hk} = F_h \supset F_k$. The oders are denoted by (0) - (4).

SPECIFIC FORMULAS

Let M be a set, $\phi = \left(f_i^{\alpha}\right)$, $i \in \mathbf{N}$, $\alpha \in \mathbf{R}$, a family of maps $f_i^{\alpha}: M^{\alpha} \to M$. Define the notion of **term on** M, by:

(t_0). Each (generic) element of M is a term

(t_1). If $f \in \phi$ and $x^{\alpha} \in M^{\alpha}$ then $f(x^{\alpha})$ is a term

(t_2). There are not other terms.

Now recall the definition of generic formulas (i) - (iii).

Denote P the set of bivalent propositions and associate to each elementary predicate $\mathcal{P}(x_1,...,x_n)$ a map $\mathcal{P} : M^n \rightarrow P$, which will be called ***n*-ary predicate on** M.

Remark 6. Between the predicates (on M) we consider also the binary predicate of equality $\mathcal{E}$.

Replace (i), in the definition of generic formulas, by (i_0). If $\mathcal{P}(x_1,...,x_n)$ is n-ary predicate on M and $t_1,...,t_n$ are terms, then $\mathcal{P}(t_1,...,t_n)$ is a **specific formula** (on M), and everywhere in (ii) and (iii), replace the "predicative formula" with "specific formula".

In this way, we obtain the notion of specific formula (on M).

Remark 7. To each generic formula F corresponds a class (F^M) of generic formulas on M.

Using the valuation map v: $P \rightarrow V = \{0,1\}$, we may define the notion of specifical and generical true formulas.

Let $F = F(\mathcal{P}_1,...,\mathcal{P}_m; x_1,...,x_n)$ a predicative formula with n-ary predicates $\mathcal{P}_i$ and $\mathcal{P}_i^M$: $M^n \rightarrow P$ the corresponding n-ary predicates on M.

Analoguosly as in the definition of the generic formula F, we define the **specific formula** $F^M = F^M(\mathcal{P}_1^M,...,\mathcal{P}_m^M ; t_1,...,t_n)$ on M, starting from the correspnding predicates $\mathcal{P}_i^M$ on M, $t_j \in M$.

The specific formula $F^M \in (F^M)$ will be called **specifical true** if is identical true on M (see [1]).

The generic formula F is called **generical true** if every $F^M \in (F^M)$ is specifical true. That is, a predicative formula F is generical true if and only if $F(\mathcal{P}_1,...,\mathcal{P}_m; x_1,...,x_n)$ is a

COROLLARY 2. *The generic true implies the specific one.*

The Example below shows that the converse of Corollary above is false.

Example 4. The formula $F = R(x,t) \supset \exists y \exists z\ \mathscr{E}_1(x,y,t,z) = F(R,\mathscr{E}_1; x,y,z,t)$ (see Example) is not generical true, but there is a specific formula on Z, F^Z = "$(a,b) = 1 \supset \exists t_1 \exists t_2 (at_1 + bt_2 = 1)$", which is specifical true.

The problem arises, in this context, to establish the cases in which the converse of the above affirmation holds.

REFERENCES

1. N.Both, *Algebra logicii cu aplicații*, Dacia, Cluj 1984.
2. N.Both, *Monotonia în calculul propozițional*, Stud. Cerc. Mat.4 Tom 21/1969, 543-551.
3. S.C.Kleene, *Introduction to metamathematics*, D. van Nostr. Co. Inc. New York-Toronto, 1952.

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RECURRENCE RELATIONS FOR SOLUTIONS OF $X^2 - DY^2 = N$ DIOPHANTINE EQUATION AND ITS APPLICATION

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REZUMAT. - Relaţii de recurenţă pentru soluţiile ecuaţiei diophantice $X^2 - DY^2 = N$ şi unele aplicaţii. Se stabileşte formula de recurenţă pentru soluţiile ecuaţiei $X^2 - DY^2 = N$ care generalizează ecuaţia lui Pell şi unele aplicaţii ale ei.

Abstract. The equation $X^2 - DY^2 = 1$ is generally called Pell's equation. In this paper the recurrence relation for solutions of $X^2 - DY^2 = N$ Diophantine equation are given. Moreover some more recurrence relations are investigated beside those that are given by Copley. Finally by using these recurrence relations it is shown that the set {1,2,287} can not be extended.

1. Introduction. Let D be a positive integer which is not perfect square. Then the equation $X^2 - DY^2 = 1$ is usually called Pell's equation and it has always an infinity of solutions, which may be found from the continued fraction for $\sqrt{D}$ [1]. It is well known that if x_1 , y_1 is the smallest positive integral solution of the Pellian Equation, then the general solution X_r, Y_r is given by

$$X_r + Y_r \sqrt{D} = (x_1 + y_1 \sqrt{D})^r \qquad (1)$$

where r = 1,2,3,...[2]. But this is unwieldy for the computation of other solutions and it is more convenient to use recurrence relations which can readily be shown to follows from (1).

G.N.Copley [3] give the following recurrence relations for solutions of Pell's equation. From (1) one gets

$$X_{r+s} + Y_{r+s}\sqrt{D} = (X_r + Y_r\sqrt{D})(X_s + Y_s\sqrt{D}) \tag{2}$$

$$X_{nr} + Y_{nr}\sqrt{D} = (x_r + Y_r\sqrt{D})^n \tag{3}$$

Equating rational and irrational coefficients in (2) gives the general recurrence relations:

$$X_{r+s} = X_rX_s + DY_rY_s,\ Y_{r+s} = X_rY_s + Y_rX_s \tag{4}$$

$$X_{2r} = X_r^2 + DY_r^2 = -1 + 2X_r^2,\ Y_{2r} = 2X_rY_r \tag{5}$$

$$X_{3r} = X_r(4X_r^2 - 3),\ Y_{3r} = Y_r(4X_r^2 - 1) \tag{6}$$

For the solutions of Pell's equation, we also have the following relations:

$$X_{n+2r} \equiv -X_n \pmod{X_r} \tag{7}$$

$$X_{n+2r} \equiv X_n \pmod{Y_r} \tag{8}$$

$$Y_{n+2r} \equiv -Y_n \pmod{X_r} \tag{9}$$

2. Diophantine Equation $X^2 - DY^2 = N$. Now we consider the Diophantine equation

$$X^2 - DY^2 = N \tag{10}$$

where D is a given square-free natural number and N is a given non-zero integer. Suppose that (10) is solvable, and x and y be two integers satisfying (10). Then $x + y\sqrt{D}$ is called a solution of (10). It will be called positive solution if $x > 0$ and $y > 0$. Let $u + v\sqrt{D}$ be the smallest positive solution of the Pell equation

$$U^2 - DV^2 = 1$$

then all of the solutions of (10) are given by

$$X_r + Y_r\sqrt{D} = (x + y\sqrt{D})(u + v\sqrt{D})^r \tag{11}$$

where $r = 0,1,2,3,\ldots$ [2].

So, we have the following relations:

$$X_r + Y_r\sqrt{D} = (x + y\sqrt{D})(u + v\sqrt{D})^r$$

$$= (x + y\sqrt{D})(U_r + V_r\sqrt{D})$$

$$= xU_r + xV_r\sqrt{D} + yU_r\sqrt{D} + DyV_r$$

$$= xU_r + DyV_r + (xV_r + yU_r)\sqrt{D}$$

equating rational and irrational coefficients we have

$$X_r = xU_r + DyV_r \tag{12}$$

$$Y_r = xV_r + yU_r \tag{13}$$

similarly,

$$X_{r+s} = X_rU_s + DY_rV_s \tag{14}$$

$$Y_{r+s} = X_rV_s + Y_rU_s \tag{15}$$

$$X_{r+2s} \equiv -X_r(\text{mod } U_s) \tag{16}$$

$$X_{r+2s} \equiv X_r(\text{mod } V_s) \tag{17}$$

$$Y_{r+2s} \equiv -Y_r(\text{mod } U_s) \tag{18}$$

$$Y_{r+2s} \equiv Y_r(\text{mod } V_s) \tag{19}$$

Some more relations could be found.

3. An Application. We say that two integers α and β have the property P_2 if $\alpha\cdot\beta + 2$ is a perfect square. A set of numbers has the property P_2 if every pair of distinct elements of the set has this property [4]. The set of numbers $\{1,2,287\}$ has the property P_2. We show that this property does not hold for the set $\{1,2,287,c\}$. Where c is any other positive integer.

It is sufficient to prove that there exist no positive integers x,y and z satisfying the following equations:

$$c + 2 = x^2 , \tag{20}$$

$$2c + 2 = y^2 \tag{21}$$

$$287c + 2 = z^2 \tag{22}$$

from (21) we see that $2|y$. Hence, on putting $y = 2Y$, where Y is an integer, equating (21) gives

$$c + 1 = 2Y^2 \tag{23}$$

Now eliminating of c between these equations, we have

$$x^2 - 2Y^2 = 1 \tag{24}$$

and

$$z^2 - 287x^2 = -572 \tag{25}$$

So we must show that the Diophantine equations (24) and (25) do not hold simultaneously.

The general solution of the Pell equation (24) in positive integers is given by

$$X_n + Y_n\sqrt{2} = (3 + 2\sqrt{2})^n$$

where $n = 0,1,2,3,...$

Hence, we have the following table of values:

n	X_n	Y_n
0	1	0
1	3	2
2	17	12
3	99	70
4	577	408
5	3363	2378
6	19601	13860
7	114243	80782

We perform the calculations in four stages. From (25) we have

$$z_n^2 = 287x_n^2 - 572$$

i) If $n \equiv 0 \pmod 4$, then from (8) we have

$$x_n \equiv X_0 \pmod{Y_2}$$

$$\equiv 1 \pmod{12}$$

Hence $z_n^2 \equiv 3 \pmod{12}$ which is impossible since 3 is not quadratic rezidu modulo 12

ii) If $n \equiv 1 \pmod 4$, then by (8) we have

$$x_n \equiv X_1 \pmod{Y_2}$$

$$\equiv 3 \pmod{12}$$

Thus, $z_n^2 \equiv 7 \pmod{12}$ which is impossible since 7 is not quadratic rezidu modulo 12.

iii) Similarly if $n \equiv 3 \pmod 4$, then we have

$$x_n \equiv 99 \equiv 3 \pmod{12}$$

which leads to a contradiction again.

iv) If $n \equiv 2 \pmod 4$, then by (8) we have

$$x_n \equiv 17 \equiv 5 \pmod{12}$$

and $x_n^2 \equiv 1 \pmod{12}$. So $z_n^2 \equiv 3 \pmod{12}$ which leads to a contradiction again.

Thus, we have shown that the Diophantine equations (24) and (25) can not simultaneously. This completes the proof.

REFERENCES

1. L.J.Mordell, *Diophantine Equations*, Academic Press, London. 1969.
2. T.Nagell, *Introduction to Number Theory*, Wiley, New York, 1951.
3. G.N.Copley, *Recurrence Relations for Solutions of Pell's Equation*, Amer. Math. Monthly 66, 288-290
4. S.P.Mohanty, A.M.S.Ramasamy, *The Simultaneous Diophantine Equations* $5Y^2 - 20 = X^2$ *and* $2Y^2 + 1 = Z^2$, Journal of Number Theory **18**(1984), 356-359.

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Π - CLOSURE AND THE P PROPERTY

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REZUMAT. - **Π - Închidere şi proprietatea P.** În lucrare sunt demonstrate câteva proprietăţi echivalente cu proprietatea *P*. Acestea sunt folosite la stabilirea unei legături între Π-închiderea unui omomorf şi proprietatea *P*. Aceasta ne conduce la o nouă formă a teoremei principale din [1], furnizând informaţii suplimentare despre grupurile Π-rezolubile în care subgrupurile acoperitoare şi proiectorii corespunzători unei clase Schunck coincid.

Abstract. Some properties equivalent with the *P* property given in [1] are proved. They are used to establish some connection between the Π-closure of a homomorph and the *P* property. This leads to a new form of the main theorem from [1], giving further informations about Π-solvable groups in which covering subgroups and projectors respecting to a Schunck class coincide.

1. Preliminaries. All groups considered in this paper are finite. We denote by Π a set of primes, Π' the complement to Π in the set of all primes and $0_{\Pi'}(G)$ the largest normal Π'-subgroup of a group G.

DEFINITION 1.1. a) A class χ of groups is a *homomorph* if χ is closed under homomorphisms.

b) A homomorph χ is a *Schunck class* if χ is primitively closed, i.e. if any group G, all of whose primitive factor groups are in χ, is itself in χ.

DEFINITION 1.2. Let χ be a class of groups, G a group and H a subgroup of G.

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a) H is an χ-*covering subgroup* of G if : (i) $H \in \chi$; (ii)

$H \leq K \leq G$, $K_0 \trianglelefteq K$, $K/K_0 \in \chi$ imply k HK_0.

b) H is an χ-projector of G if for any normal subgroup N of G, HN/N is χ-maximal in G/N.

Respecting to a fixed set of primes Π we define the following classes of groups.

DEFINITION 1.3. Let χ be a class of groups.

a) χ is Π-*closed* if:

$G/0_{\Pi'}(G) \in \chi \Rightarrow G \in \chi$.

b) A Π-closed homomorph is called Π-*homomorph* and a Π-closed Schunck class will be caled Π-*Schunck class*.

2. Π-Homomorphs with the P property. Let Π be a set of primes and χ a class of groups. The following three properties are defined in [1]:

DEFINITION 2.1. a) A class χ has the *P property* if for any Π-solvable group G we have:

N minimal normal subgroup of G and N Π'-group $\Rightarrow G/N \in \chi$. (1)

b) χ has the *P' property* if for any Π-solvable group G we have:

$N \trianglelefteq G$, $N \neq 1$ and N is a Π'-group $\Rightarrow G/N \in \chi$. (2)

c) χ has the *P'' property* if for any Π-solvable group G we have:

$0_{\Pi'}(G) \neq 1 \Rightarrow G \in \chi$. (3)

First let us compare conditions (1), (2) and (3).

LEMMA 2.2. *Let G be a group.*

a) *If* χ *is an arbitrary class of groups, then* (2) *implies* (1).

b) *If* χ *is a homomorph, then* (1) *and* (2) *are equivalent.*

Proof. a) Let N be a minimal normal subgroup of G which is a Π'-group. Then w are in the hypothesis of (2) and so $G/N \in \chi$.

b) Suppose (1) is true and let us prove (2). Let N be a normal subgroup of G, $N \neq$ and N Π'-group. There is a minimal normal subgroup M of G, such that $M \subseteq N$. Clearly is also a Π'-group. Applying (1), $G/M \in \chi$. But then $G/N \cong (G/M)/(N/M)$ is in χ, because is a homomorph.■

LEMMA 2.3. *Let G be a group.*

a) *If* χ *is a homomorph, then* (3) *implies* (1).

b) *If* χ *is a* Π*-homomorph, then* (1) *and* (3) *are equivalent.*

Proof. a) Let N be a minimal normal subgroup of G and N Π'-group. Then $N \leq 0_{\Pi'}(G)$ and so $0_{\Pi'}(G) \neq 1$. By (3), $G \in \chi$, hence, χ being homomorph, $G/N \in \chi$.

b) We prove that (1) implies (3). Let $0_{\Pi'}(G) \neq 1$. If follows that there is a minima normal subgroup N of G such that $N \leq 0_{\Pi'}(G)$. Clearly N is a Π'-group. By (1), we have $G/$ $\in \chi$. But

$$G/0_{\Pi'}(G) \cong (G/N)/(0_{\Pi'}(G)/N)$$

and using that χ is a homomorph we deduce that $G/0_{\Pi'}(G) \in \chi$. But χ is Π-closed and so ($\in \chi$.■

LEMMA 2.4. *Let G be a group.*

a) *If* χ *is a homomorph, then* (3) *implies* (2).

b) *If* χ *is a* Π*-homomorph, then* (2) *and* (3) *are equivalent.*

Proof. a) By 2.3.a), (3) implies (1) and by 2.2.b), (1) implies (2). So (3) implies (2).

b) By 2.2.a), (2) implies (1) and by 2.3.b), (1) implies (3). It follows that (2) implies

3).■

Lemmas 2.2., 2.3. and 2.4. lead to the following two theorems:

THEOREM 2.5. *Let G be a group.*

a) *If* χ *is a homomorph, then conditions* (1) *and* (2) *are equivalent.*

b) *If* χ *is a* Π*-homomorph, then conditions* (2) *and* (3) *are equivalent. In this case,*

onditions (1), (2) *and* (3) *are equivalent.*

THEOREM 2.6. *Let* χ *be a class of groups.*

a) *If* χ *is a homomorph, then the following statements are equivalent:*

(i) χ *has the P property;*

(ii) χ *has the P' property.*

b) *If* χ *is a* Π*-homomorph, then the statements* (i), (ii) *and* (iii) *are equivalent, where*

ii) *is given below:*

(iii) χ *has the P'' property.*

Finally, we establish some connection between the Π-closure of a homomorph and the

' property.

THEOREM 2.7. *If* χ *is a class of groups having the P'' property respecting to a set*

I *of primes, then* χ *is* Π*-closed.*

Proof. Let $G/0_{\Pi'}(G) \in \chi$. There are two possibilities:

1) $0_{\Pi'}(G) = 1$. Then $G = G/0_{\Pi'}(G)$ and so $G \in \chi$.

2) $0_{\Pi'}(G) \neq 1$. If follows, by the P'' property, that $G \in \chi$.■

Theorems 2.6.. and 2.7. give the following result:

THEOREM 2.8. *Let* χ *be a homomorph. The following two conditions are equivalent:*

a) χ *is* Π-*closed and* χ *has the P property;*

b) χ *has the P" property,*

where properties P and P" are respecting to the same set Π *of primes.*

Proof. (a) implies (b). It follows from 2.6.b).

(b) implies (a). From 2.7. follows that χ is Π-closed. Hence χ is a Π-homomorph and we can apply 2.6.b). So χ has the *P* property. ■

Theorem 2.8. leads to a new form of theorem 3.3. from [1] which we give below.

THEOREM 2.9. ([1)] *Let* $\mathcal{F}$ *be a Schunck class with the P" property respecting to a set* Π *of primes. If the* Π-*solvable group G has a chain of normal subgroups:*

$$1 = N_0 \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq N_r = G$$

with every factor N_{i+1}/N_i *nilpotent,* $i = 0, 1, \ldots, r-1$, *then the following conditions on a subgroup F of G are equivalent:*

(i) *F is an* $\mathcal{F}$-*projector of G;*

(ii) *F is an* $\mathcal{F}$-*covering subgroup of G.*

REFERENCES

1. Covaci,R., *On* Π-*Schunck classes with the P property*, "Babeş-Bolyai" University Research Seminars, Preprint No. 5(1988), 22-34.
2. Huppert,B., *Endliche Gruppen I*, Berlin - New York, Springer-Verlag, 1967.

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SOME INEQUALITIES FOR m-CONVEX FUNCTIONS

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REZUMAT. - Câteva inegalităţi pentru funcţii *m*-convexe. Funcţiile *m*-convexe au fost definite în [4]. Ele au alură intermediară celei de convexitate şi celei de stelaritate. Pentru aceste funcţii în lucrare se demonstrează inegalităţi de tip Jensen şi de tip Hermite-Hadamard.

1. Introduction. We will follow the paper [5].

Let X be a real linear space, $I = [0,1]$ and $m \geq 0$ a fixed real number.

DEFINITION 1. A set $D \subseteq X$ will be called m-convex if for any $x, y \in D$ and any $t \in I$ we have $tx + m(1-t)y \in D$.

The following two lemmas which describe some properties of m-convex sets hold.

LEMMA 1. *If $m > 1$, $0 \in D$ and D is m-convex, then for any $x \in D$, $t \geq 0$ we have $tx \in D$.*

Taking into account this property, in what follows we shall consider only $m \in I$. The value $m = 1$ corresponds to convexity and $m = 0$ to starshapendness.

LEMMA 2. *If D is m-convex and $0 \leq n < m \leq 1$, then D is also n-convex.*

Now, let D be a m-convex set in the linear space X with $m \in I$. Transposing the ideea from [3] to the real case, in [4] it was introduced the following class of functions.

DEFINITION 2. A function $f : D \to \mathbf{R}$ is said to be m-convex if for every $x, y \in D$ and $t \in I$ it verifies the condition:

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$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y).$$

Here again, $m = 1$ gives convex functions and $m = 0$ starshaped functions.

As it is shown in [5], it is natural to suppose $0 \in D$ and $f(0) \le 0$.

Now we recall some fundamental properties of m-convex functions (see [5]).

LEMMA 3. *The function $f : D \to \mathbf{N}$ is m-convex if and only if the set:*

$$epi(f) = \{(x, y) \in D \times \mathbf{R}, y \ge f(x)\}$$

is m-convex.

LEMMA 4. *If f is m-convex then it is starshaped.*

THEOREM 1. *If f is m-convex and $0 \le n < m \le 1$ then f is n-convex.*

2. Jensen's inequality for m-convex functions. We will prove the following inequality of Jensen's type.

THEOREM 2. *Let X be a linear space, $m \in [0,1]$ and $D \subseteq X$ is a m-convex set in X. If $f : D \to \mathbf{R}$ is a m-convex function, then for all $p_i > 0$ and $x_i \in D$ $(i = 1,...,n)$ we have:*

$$\sum_{i=1}^{n} p_i m^{i-1} x_i / P_n \in D, \quad \text{where} \quad P_n = \sum_{i=1}^{n} p_i$$

and the following inequality:

$$f\left(\sum_{i=1}^{n} p_i m^{i-1} x_i / P_n\right) \le \sum_{i=1}^{n} p_i m^{i-1} f(x_i) / P_n \tag{1}$$

holds.

Proof. We proceed by mathematical induction. If $n = 2$, the statement follows by the defintion. Suppose that (1) holds for "n - 1", i. e.

$$f\left(\sum_{i=1}^{n-1} q_i m^{i-1} y_i / Q_{n-1}\right) \le \sum_{i=1}^{n-1} q_i m^{i-1} f(y_i) / Q_{n-1}$$

where $\sum_{i=1}^{n-1} q_i m^{i-1} y_i / Q_{n-1}$ is assumed to be in D, provided that $q_i > 0$, $y_i \in D$ and

$Q_{n-1} = \sum_{i=1}^{n-1} q_i$. Now:

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i m^{i-1} x_i = \frac{p_1}{P_n} x_1 + m\left(1 - \frac{p_1}{P_n}\right)\sum_{i=2}^{n} p_i m^{i-2} x_i \Big/ \sum_{i=2}^{n} p_i$$

and since:

$$\sum_{i=2}^{n} p_i m^{i-2} x_i \Big/ \sum_{i=2}^{n} p_i \in D \quad \text{it follows} \quad \sum_{i=1}^{n} p_i m^{i-1} x_i / P_n \in D.$$

By the above considerations we have that:

$$f\left(\sum_{i=1}^{n} p_i m^{i-1} x_i / P_n\right) = f\left(\frac{p_1}{P_n} x_1 + m\left(1 - \frac{p_1}{P_n}\right)\sum_{i=2}^{n} p_i m^{i-2} x_i \Big/ \sum_{i=2}^{n} p_i\right) \le$$

$$\le \frac{p_1}{P_n} f(x_1) + \left(1 - \frac{p_1}{P_n}\right) f\left(\sum_{i=2}^{n} p_i m^{i-2} x_i \Big/ \sum_{i=2}^{n} p_i\right) \le \frac{p_1}{P_n} f(x_1) +$$

$$+ m\frac{1}{P_n}\sum_{i=2}^{n} p_i \sum_{i=2}^{n} p_i m^{i-2} f(x_i) \Big/ \sum_{i=2}^{n} p_i = \sum_{i=1}^{n} p_i m^{i-1} f(x_i) / P_n .$$

and the theorem is proved.

COROLLARY 1. *In the above assumptions for D, f, m and x_i ($i = 1,\dots,n$) we have that* $\sum_{i=1}^{n} m^{i-1} x_i / n \in D$ *and*:

$$f\left(\sum_{i=1}^{n} m^{i-1} x_i / n\right) \le \sum_{i=1}^{n} m^{i-1} f(x_i) / n .$$

Application 1. Let $m \in I$ and x_i, $p_i > 0$ for $i = 1,\dots,n$.

The one has the inequalities:

$$\left(\sum_{i=1}^{n} p_i m^{i-1} x_i\right)^q \le p_n^{q-1} \sum_{i=1}^{n} p_i m^{i-1} x_i^q , \ \forall\, q \ge 1$$

and

$$1 + \frac{1}{P_n}\sum_{i=1}^{n} p_i m^{i-1} x_i \ge \left(\prod_{i=1}^{n} (x_i + 1) m^{i-1} p_i\right)^{1/P_n}$$

The proof of the above inequalities follows by (1) choosing the functions $f: [0, \infty) \to [0, \infty)$, $f(x) = x^q$, respectively $f: [0, \infty) \to (-\infty, 0]$, $f(x) = -\ln(x+1)$ which are m-convex.

A second result is contained in the next theorem.

THEOREM 3. *Let X be a linear space, $m \in I$ and D a m-convex set in X. If $f: D \to$ **R** is a m-convex function, then for all $p_i > 0$, $x_i \in D$, one has the inequalities:*

$$f\left((t+m(1-t))\frac{1}{P_n}\sum_{i=1}^{n} p_i m^{i-1}\frac{1}{P_n}\sum_{i=1}^{n} p_i m^{i-1}x_i\right) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i m^{i-1}\cdot$$
$$\cdot f\left(t\frac{1}{P_n}\sum_{j=1}^{n} p_j m^{j-1}x_i + m(1-t)\frac{1}{P_n}\sum_{j=1}^{n} p_j m^{j-1}x_j\right) \le \frac{1}{P_n^2}\sum_{i,j=1}^{n} p_i p_j \cdot$$
$$\cdot m^{i+j-2} f(tx_i + m(1-t)x_j) \le (t+m(1-t))\frac{1}{P_n^2}\sum_{i=1}^{n} m^{i-1}p_i\sum_{i=1}^{n} m^{i-1}p_i f(x_i).$$

Proof. By the definition of m-convex functions, one has:

$f(tx_i + m(1-t)x_j) \le t\cdot f(x_i) + m(1-t)\cdot f(x_j)$ for all $i,j \in \{1,...,n\}$.

By multiplying with $m^{j-1}p_j \ge 0$ and summing over j to 1 at n, one has:

$$\frac{1}{P_n}\sum_{j=1}^{n} p_j m^{j-1} f(tx_i + m(1-t)x_j) \le$$
$$\le t\frac{1}{P_n}\sum_{j=1}^{n} m^{j-1}p_j\cdot f(x_i) + m(1-t)\frac{1}{P_n}\sum_{j=1}^{n} p_j m^{j-1} f(x_j).$$

Using Jensen's inequality for m-convex functions, we get:

$$f\left(t\frac{1}{P_n}\sum_{j=1}^{n} p_j m^{j-1}\cdot x_i + m(1-t)\frac{1}{P_n}\sum_{j=1}^{n} p_j m^{j-1}x_j\right) = f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j m^{j-1}(tx_i +\right.$$
$$\left.+m(1-t)x_j)\right) \le \frac{1}{P_n}\sum_{j=1}^{n} p_j m^{j-1} f(tx_i + m(1-t)x_j) \le$$
$$\le t\frac{1}{P_n}\sum_{j=1}^{n} m^{j-1}p_j\cdot f(x_i) + m(1-t)\frac{1}{P_n}\sum_{j=1}^{n} p_j m^{j-1} f(x_j)$$

for all $i \in \{1, 2,...,n\}$.

Multipying this inequality with $p_i m^{i-1}$ and summing over i to 1 at n one has:

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i m^{i-1} f\left(t\frac{1}{P_n}\sum_{j=1}^{n} p_j m^{j-1}\cdot x_i + m(1-t)\frac{1}{P_n}\sum_{j=1}^{n} p_j m^{j-1}x_j\right) \le$$
$$\le \frac{1}{P_n^2}\sum_{i,j=1}^{n} p_i p_j m^{i+j-2} f(tx_i + m(1-t)x_j) \le t\frac{1}{P_n}\sum_{j=1}^{n} m^{j-1}p_j\cdot$$
$$\cdot\frac{1}{P_n}\sum_{i=1}^{n} p_i m^{i-1} f(x_i) + m(1-t)\frac{1}{P_n}\sum_{j=1}^{n} p_j m^{j-1} f(x_j)\frac{1}{P_n}\sum_{i=1}^{n} p_i m^{i-1} =$$
$$= (t+m(1-t))\frac{1}{P_n}\sum_{i=1}^{n} p_i m^{i-1}\frac{1}{P_n}\sum_{i=1}^{n} p_i m^{i-1} f(x_i).$$

On the other hand, by Jensen's inequality for m-convex functions, we deduce

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i m^{i-1} f\left(t\frac{1}{P_n}\sum_{j=1}^{n} p_j m^{j-1}\cdot x_i + m(1-t)\frac{1}{P_n}\sum_{j=1}^{n} p_j m^{j-1}x_j\right) \ge$$

$$\geq f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i m^{i-1}\left(t\frac{1}{P_n}\sum_{j=1}^{n} p_j m^{j-1}\cdot x_j + m(1-t)\frac{1}{P_n}\sum_{j=1}^{n} p_j m^{j-1} x_j\right)\right) =$$

$$= f\left((t+m(1-t))\frac{1}{P_n}\sum_{i=1}^{n} p_i m^{i-1}\frac{1}{P_n}\sum_{i=1}^{n} p_i m^{i-1} x_i\right)$$

and the theorem is proved.

Remark 1. If we assume that $m = 1$, we obtain a refinement of Jensen's inequality established in [2].

COROLLARY 2. *In the above assumptions for D, f and x_i, we have for all $m \in [0,1)$ the inequalities:*

$$f\left(\frac{(t+m(1-t))(1-m^n)}{1-m}\cdot\frac{1}{n}\sum_{i=1}^{n} m^{i-1}x_i\right) \leq$$

$$\leq \frac{1}{n}\sum_{i=1}^{n} m^{i-1} f\left(t\frac{m^n-1}{m-1}x_i + m(1-t)\frac{1}{n}\sum_{j=1}^{n} m^{j-1}x_j\right) \leq$$

$$\leq \frac{1}{n^2}\sum_{i,j=1}^{n} m^{i+j-2} f(tx_i + m(1-t)x_j) \leq \frac{(t+m(1-t))(1-m^n)}{n(1-m)}\sum_{i=1}^{n} m^{i-1} f(x_i).$$

The following applications also hold.

Application 2. Let x_i, $p_i > 0$, $q \geq 1$ and $m \in I$. Then one has the inequalities:

$$(t+m(1-t))^q\left(\sum_{i=1}^{n} p_i m^{i-1}\right)^q\left(\sum_{i=1}^{n} p_i m^{i-1}x_i\right)^q \leq$$

$$\leq P_n^{q-1}\sum_{i=1}^{n} p_i m^{i-1}\left(t\sum_{j=1}^{n} p_j m^{i-1}x_i + m(1-t)\sum_{j=1}^{n} p_j m^{j-1}x_j\right)^q \leq$$

$$\leq P_n^{2q-2}\sum_{i,j=1}^{n} p_i p_j m^{i+j-2}(tx_i + m(1-t)x_j)^q \leq$$

$$\leq (t+m(1-t))P_n^{2q-2}\sum_{i=1}^{n} m^{i-1}p_i\sum_{j=1}^{n} m^{j-1}p_j x_j^q.$$

Application 3. Let x_i, $p_i > 0$ and $m \in (0, 1]$. Then one has the inequalities:

$$(t+m(1-t))\frac{1}{P_n^2}\sum_{i=1}^{n} p_i m^{i-1}\sum_{i=1}^{n} p_i m^{i-1}x_i + 1 \geq$$

$$\geq \left(\prod_{i=1}^{n}\left(\frac{tx_i}{P_n}\sum_{j=1}^{n} p_j m^{j-1} + \frac{m(1-t)}{P_n}\sum_{j=1}^{n} p_j m^{j-1}x_j + 1\right)^{p_i m^{i-1}}\right)^{1/P_n} \geq$$

$$\geq \left(\prod_{i,j=1}^{n} \left(t x_i + m(1-t) x_j + 1 \right)^{p_i p_j m^{i+j-2}} \right)^{1/P_n^2} \geq$$

$$\geq \left(\prod_{i=1}^{n} (x_i + 1)^{p_i m^{i-1}} \right)^{(t+m(1-t))\frac{1}{P_n^2}\sum_{i=1}^{n} m^{i-1} p_i}.$$

The proofs follow by Theorem 3 applied to the m-convex functions $f: [0, \infty) \to [0, \infty)$, $f(x) = x^q$ $(q \geq 1)$ respectively $f: [0, \infty) \to (-\infty, 0]$, $f(x) = -\ln(x+1)$.

Note that Theorem 2 and Theorem 3 give also some interesting inequalities in a normed linear space.

Application 4. Let $(X, \|\cdot\|)$ be a normed space, $p_i \geq 0$ with $p_n > 0$, $x_i \in X$, $m \in I$ and $q \geq 1$. Then one has the inequalities:

$$\left\| \sum_{i=1}^{n} p_i m^{i-1} x_i \right\|^q \leq P_n^{q-1} \sum_{i=1}^{n} p_i m^{i-1} \|x_i\|^q$$

and

$$(t + m(1-t))^q \left(\sum_{i=1}^{n} p_i m^{i-1} \right)^q \left\| \sum_{i=1}^{n} p_i m^{i-1} x_i \right\|^q \leq$$

$$\leq P_n^{q-1} \sum_{i=1}^{n} p_i m^{i-1} \left\| t \sum_{j=1}^{n} p_j m^{j-1} \cdot x_i + m(1-t) \sum_{j=1}^{n} p_j m^{j-1} x_j \right\|^q \leq$$

$$\leq P_n^{2q-2} \sum_{i,j=1}^{n} p_i p_j m^{i+j-2} \| t x_i + m(1-t) x_j \|^q \leq$$

$$\leq (t + m(1-t)) P_n^{2q-2} \sum_{i=1}^{n} m^{i-1} p_i \sum_{i=1}^{n} m^{i-1} p_i \|x_i\|^q.$$

The proofs follow by the above theorems for the m-convex function $f: X \to \mathbb{R}$, $f(x) = \|x\|^q$.

3. Some integral inequalities for m-convex functions. In what follows we consider only functions defined on the real interval $[0, b]$ and denote by $K_m(b)$ the set of m-convex functions on $[0, b]$ such that $f(0) \leq 0$ (see also [5]).

The following lemmas hold:

LEMMA 5. *The function f is in $K_m(b)$ if and only if:*

$$f_m(x) = \frac{f(x) - mf(y)}{x - my}$$

is increasing on $(my, b]$ for $y \in [0, b]$.

LEMMA 6. *If f is differentiable in $[0, b]$ then $f \in K_m(b)$ if and only if:*

$$f'(x) \geq \frac{f(x) - mf(y)}{x - my} \quad \text{for} \quad x > my.$$

The following integral inequality for m-convex functions holds.

THEOREM 4. *Let $f : [0, \infty) \to \mathbf{R}$ be a m-convex integrable function with $m \in (0, 1]$ and $0 \leq a < b < \infty$. Then one has the inequality:*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{1}{4}\left(f(a) + f(b) + m(f(a/m) + f(b/m))\right) \tag{2}$$

Proof. Since f is m-convex, we have:

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y), \ \forall\, x, y > 0$$

which gives:

$$f(ta + (1-t)b) \leq tf(a) + m(1-t)f(b/m)$$

and

$$f(tb + (1-t)a) \leq tf(b) + m(1-t)f(a/m)$$

for all $t \in I$. Integrating on I we get:

$$\int_0^1 f(ta + (1-t)b)\,dt \leq (f(a) + mf(b/m))/2$$

and

$$\int_0^1 f(tb + (1-t)a)\,dt \leq (f(b) + mf(a/m))/2.$$

But

$$\int_0^1 f(ta + (1-t)b)\,dt = \int_0^1 f(tb + (1-t)a)\,dt = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Thus, adding the above inequalities, we obtain (2).

Remark 2. From the proof we deduce that holds also a better evaluation:

$$\frac{1}{b-a}\int_a^b f(x)\,dx \le \min\left\{(f(a)+mf(b/m))/2;\,(f(b)+mf(a/m))/2\right\}.$$

THEOREM 5. *Let* $f:[0,\infty)\to\mathbf{R}$ *be a m-convex differentiable function with* $m\in(0,1]$. *Then for all* $0\le a<b$ *one has the inequalities:*

$$\frac{f(mb)}{m}-\frac{b-a}{2}f'(mb)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{(b-ma)f(b)-(a-mb)f(a)}{2(b-a)}. \tag{3}$$

Proof. Using Lemma 6, we have for all x, $y\ge 0$ with $x\ge my$ that:

$$(x-my)f'(x)\ge f(x)-mf(y). \tag{4}$$

Choosing in the above inequality $x=mb$ and $a\le y\le b$, then $x\ge my$ and:

$$(mb-my)f'(mb)\ge f(mb)-mf(y).$$

Integrating over y on $[a,b]$, we get:

$$m\frac{(b-a)^2}{2}f'(mb)\ge(b-a)f(mb)-m\int_a^b f(y)\,dy$$

hus the first inequality of (3). Putting in (4) y a and then integrating on $[a,b]$ one gets he second inequality of (3).

Remark 3. The second inequality from (3) is also valid for $m=0$, while (2) is not. For $m=1$ it is identical with (2) and represents a part of Hermite-Hadamard's inequality.

REFERENCES

1. Dragomir, S. S., *An improvement of Jensen's inequality*, Mat. Bilten (Skopje), **15**(1991), 35 - 37.
2. Dragomir, S. S., *Some refinements of Jensen's inequality*, J. Math. Anal. Appl. **168**(1992), 518 - 522.
3. Mocanu, P. T., *Une propriété de convexité généralisée dans la théorie de la representation conforme*, Mathematica **11**(34) (1969), 127 - 133.
4. Toader, Gh., *Some generalizations of the convexity*, Proc. Colloq. Approx. Optim., Cluj-Napoca, 1984, 329 - 338.
5. Toader, Gh., *On a generalization of the convexity*, Mathematica 30(53) (1988), 83 - 87.

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A PROBABILISTIC ARGUMENT FOR THE CONVERGENCE OF SOME SEQUENCES ASSOCIATED TO HADAMARD'S INEQUALITY

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REZUMAT. - O metodă probabilistică pentru studiul unor şiruri asociate inegalităţii lui Hadamard. În lucrare sunt calculate limitele unor şiruri asociate inegalităţii lui Hadamard pentru funcţii convexe.

0. Abstract. The limits for some sequences associated with the well know Hadamard's inequality for convex functions are pointed out.

1. Introduction. Let $f: I \to \mathbf{R}$ be a convex mapping on the interval of real numbe I, $\mathring{I} \neq \varnothing$ and $a, b \in I$ with $a<b$. The following integral inequality is well known literature as Hadamard's inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \qquad ($$

For some recent refinements, counter parts and generalizations of this classic fact s the papers [1 - 5] and [7 - 10] where further references are given.

In [5], S.S.Dragomir, J.E.Pečarić and J.Sándor proved the following refinement of (1

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{(b-a)^n}\int_a^b \dots \int_a^b f\left(\frac{x_1+\dots+x_n}{n}\right)dx_1\dots dx_n \le$$

$$\le \frac{1}{(b-a)^{n-1}}\int_a^b \dots \int_a^b f\left(\frac{x_1+\dots+x_{n-1}}{n-1}\right)dx_1\dots dx_{n-1} \le \dots \le \frac{1}{b-a}\int_a^b f(x)dx \qquad ($$

where $n \in \mathbf{N}$ and $n \ge 2$. Some applications for Γ-function with interesting connections Number Theory are also made.

Now let consider the following sequence of real numbers:

$$H_n(f):=\frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b f\left(\frac{x_1+\ldots+x_n}{n}\right)dx_1\ldots dx_n,\ n\geq 1.$$

obvious that $H_n(f)$ is monotonous nonincreasing and bounded and, thus, convergent. It natural to point out its limit.

2. The main results. We will start with the following theorem:

THEOREM 1. *Let $f: I\subseteq\mathbf{R}\to\mathbf{R}$ be convex and $a,b\in \mathring{I}\neq\varnothing$, with $a<b$. Then:*

$$\lim_{n\to\infty} H_n(f)=\inf\{H_n(f): n\in\mathbf{N}^*\}=f\left(\frac{a+b}{2}\right). \tag{3}$$

Firstly, we will state the next lemma which is also interesting in itself.

LEMMA. *Let $g:\mathbf{R}\to\mathbf{R}$ be a bounded, Lebesgue measurable function and $a,b\in\mathbf{R}$ th $a<b$. If g is continuous in $\frac{a+b}{2}$ then*

$$\lim_{n\to\infty}\frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b g\left(\frac{x_1+\ldots+x_n}{n}\right)dx_1\ldots dx_n=g\left(\frac{a+b}{2}\right). \tag{4}$$

Proof. We will give a probabilistic argument.

Let $(X_n)_n, X_n:(\Omega,\mathcal{F},p)\to\mathbf{R}$ be a sequence of independent random variables which are iiformly distributed on the interval $[a,b]$. By the use of the "strong law of large numbers" ,p. 216] we have:

$$\frac{X_1+X_2+\ldots+X_n}{n}\xrightarrow{a.e.} M(X_1)=\frac{a+b}{2}.$$

The mapping g being Lebesque measurable on $\mathbf{R}$ and continuous in $\frac{a+b}{2}$ we otain:

$$g\left(\frac{X_1+\ldots+X_n}{n}\right)\xrightarrow{a.e.} g\left(\frac{a+b}{2}\right).$$

sing the dominated convergence theorem of Lebesque we obtain

$$\int_\Omega g\left(\frac{X_1+X_2+\ldots+X_n}{n}\right)dp\to\int_\Omega g\left(\frac{a+b}{2}\right)dp=g\left(\frac{a+b}{2}\right), \tag{5}$$

But $X_1, \ldots, X_n$ are independent and then the repartition $p \circ (X_1,\ldots,X_n)^{-1}$ of the vector $(X_1,\ldots,X_n)$ has the density $p(x_1,\ldots,x_n) = p_1(x_1)\ldots p_n(x_n)$ [6,p. 14] where $p_i(x_i)$ is the densi... ...e random variable X_i.

Since X_i ($i \in \mathbf{N}$) are uniformly distributed, $p_i(x) = (b-a)^{-1}$ if $x \in [a,b]$ and $p_i(x) = 0$ if $x \in \mathbf{R} - [a,b]$, $i \geq 1$, then

$$\int_{\Omega} g\left(\frac{X_1+\ldots+X_n}{n}\right)dp = \int_{\mathbf{R}^n} g\left(\frac{x_1+\ldots+x_n}{n}\right)d(p\circ(X_1,\ldots,X_n)^{-1})(x_1,\ldots,x_n)$$
$$= \frac{1}{(b-a)^n}\int_a^b\ldots\int_a^b g\left(\frac{x_1+\ldots+x_n}{n}\right)dx_1\ldots dx_n \tag{6}$$

Now, by (5) and (6) we obtain the desired result embodied in (4).

Proof of Theorem. If $f: I \subseteq \mathbf{R} \to \mathbf{R}$ is convex on I, then the mapping $g: \mathbf{R} \to \mathbf{R}$, $g(x) = f(x)$ if $x \in I$ and $g(x) = 0$ if $x \in \mathbf{R} - I$, satisfies the conditions in the above lemma and then

$$\lim_{n\to\infty} H_n(f) = f\left(\frac{a+b}{2}\right).$$

The fact that $\lim_{n\to\infty} H_n(f) = \inf_{n\in\mathbf{N}^*} H_n(f)$ follows by the monotonicity of $H_n(f)$.

As above, it is also natural to consider the following sequences associated with a convex mapping $f: I \subseteq \mathbf{R} \to \mathbf{R}$

$$H_n^{(1)}(f) := \frac{1}{(b-a)^{n-1}}\int_a^b\ldots\int_a^b f\left(\frac{x_1+x_2+\ldots+x_{n-1}+(a+b)/2}{n}\right)dx_1\ldots dx_{n-1},$$

$$H_n^{(2)}(f) := \frac{1}{(b-a)^{n-2}}\int_a^b\ldots\int_a^b f\left(\frac{x_1+x_2+\ldots+x_{n-2}+2(a+b)/2}{n}\right)dx_1 dx_2\ldots dx_{n-2}$$

$$\ldots$$

$$H_n^{(n-1)}(f) := \frac{1}{(b-a)}\int_a^b f\left(\frac{x_1+(n-1)(a+b)/2}{n}\right)dx_1$$

for $n \geq 2$.

The next theorem contains some properties of these sequences:

THEOREM 2. *Suppose that* $f: I \subseteq \mathbb{R} \to \mathbb{R}$ *is a convex mapping on* I *and* $a, b \in \mathring{I} \neq \varnothing$ *with* $a < b$. *Then*

(i) *we have the following refinements of Hadamard's inequality:*

$$f\left(\frac{a+b}{2}\right) \le H_n^{(n-1)}(f) \le \dots \le H_n^{(1)}(f) \le H_n(f), \tag{7}$$

for all $n \ge 2$;

(ii) *we have the limits:*

$$\lim_{n\to\infty} H_n^{(n-1)}(f) = \dots = \lim_{n\to\infty} H_n^{(1)}(f) = f\left(\frac{a+b}{2}\right).$$

Proof. (i). By Jensen's integral inequality we have:

$$\frac{1}{b-a}\int_a^b f\left(\frac{x_1+(n-1)(a+b)/2}{n}\right)dx_1 \ge f\left[\frac{1}{b-a}\left(\int_a^b \frac{x_1+(n-1)(a+b)/2}{n}dx_1\right)\right] =$$

$$= f\left(\frac{a+b}{2}\right)$$

which proves the first inequality in (7).

Now, by Jensen's integral inequality we also have:

$$H_n^{(i)}(f) = \frac{1}{(b-a)^{n-i}}\int_a^b \dots \int_a^b f\left(\frac{x_1+\dots+x_{n-i}+i(a+b)/2}{n}\right)dx_1 \dots dx_{n-i} \ge$$

$$\ge \frac{1}{(b-a)^{n-i-1}}\int_a^b \dots \int_a^b f\left[\frac{1}{b-a}\int_a^b \left(\frac{x_1+\dots+x_{n-i-1}+x_{n-i}+i(a+b)/2}{n}\right)dx_{n-i}\right]dx_1 \dots dx_{n-i-1} =$$

$$= \frac{1}{(b-a)^{n-i-1}}\int_a^b \dots \int_a^b f\left(\frac{x_1+\dots+x_{n-i-1}+(i+1)(a+b)/2}{n}\right)dx_1 \dots dx_{n-i-1} =$$

$$= H_n^{(i+1)}(f)$$

for all $1 \le i \le n-2$.

The last inequality is also obvious by Jensen's integral inequality. Indeed, we have:

$$H_n(f)=\frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b f\left(\frac{x_1+\ldots+x_n}{n}\right)dx_1\ldots dx_n\geq$$

$$\geq\frac{1}{(b-a)^{n-1}}\int_a^b\cdots\int_a^b f\left(\frac{1}{b-a}\int_a^b\frac{x_1+\ldots+x_{n-1}+x_n}{n}dx_n\right)dx_1\ldots dx_{n-1}=$$

$$=\frac{1}{(b-a)^{n-1}}\int_a^b\cdots\int_a^b f\left(\frac{x_1+\ldots+x_{n-1}+(a+b)/2}{n}\right)dx_1\ldots dx_{n-1}=H_n^{(1)}(f).$$

(ii). Follows by the inequality (7) and by Theorem 1.

REFERENCES

1. H.Alzer, *A note on Hadamard's inequalities*, C.R.Math. Rep. Acad. Sci. Canada **11**(1989), 255-258.
2. S.S.Dragomir, *Some refinements of Hadamard's inequalities*, G.M. Metodica, **11**(1990), 189-191.
3. S.S.Dragomir, *Two refinements of Hadamard's inequalities*, Coll. of Sci. Pop. of the Fac. of Sci. Kragujevac, **11**(1990), 23-26.
4. S.S.Dragomir, *Two mappings in connection to Hadamard's inequalities*, J.Mat. Anal. Appl. **167**(1992), 49-56.
5. S.S.Dragomir, J.E.Pečarić and J.Sándor, *A note on the Jensen-Hadamard inequalities*, Anal. Num. Theor. Approx. **19**(1990), 29-43.
6. B.U.Gnidenko, *The Theory of Probability (English translation)*, Mir Publishers, 1976.
7. J.E.Pečarić, *A note on Hadamard's inequalities*, Macedonian Acad. of Sci. and Arts, Sec. of Math. **10**(1989), 21-23.
8. J.E.Pečarić and S.S.Dragomir, *A generalization of Hadamars's inequality for isotonic linear functionals*, Radovi Mat. (Sarajevo) **7**(1991), 103-107.
9. J.Sándor, *Some integral inequalities*, Elem. Math. **43**(1988), 177-180.
10. P.M.Vasić and I.B Lacković, *Notes on convex functions*, Univ. Beograd Publ. Elektr. Fak. Sec. Mat. Fiz. No. 577-598 (1977), 21-24.

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INEQUALITIES FOR A CLASS OF MEANS

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REZUMAT. - Inegalităţi pentru o clasă de medii. În lucrare sunt prezentate câteva inegalităţi pentru o clasă de medii studiate în [3] şi [4].

1. Let $n \geq 2$, $a = (a_1,...,a_n)$, $0 < \min a_i < \max a_i$. Let $p \neq 1/(n-1)$. Denote

$$B_n^{(p)}(a) = \left(\frac{G_n^{np}(a)}{H_n(a^p)A_n(a)} \right)^{\frac{1}{p(n-1)-1}}$$

where $a^p = (a_1^p,...,a_n^p)$ and A_n, G_n, H_n are the arithmetic, geometric and harmonic means, respectively.

$B_n^{(p)}(a)$ coincides with the mean $B_n^{[p,n-1;1,1]}(a)$ studied in [3] and [5,p.101]. A consequence of the results of H.Bauer [2] is that $\min a_i < B_n^{(p)}(a) < \max a_i$ if $p \leq 0$ or $p > 1$, and

$$B_n^{(p)}(a) < A_n(a) \text{ for all } p<0. \tag{1}$$

Let us consider also

$$M_n^{(p)}(a) = \left(\frac{G_n^{np}(a)}{H_n^p(a)A_n(a)} \right)^{\frac{1}{p(n-1)-1}}$$

We shall study some properties of $B_n^{(p)}(a)$ and $M_n^{(p)}(a)$ as functions of p. They will enable us to extend (1). Some inequalities related to previously known results will be also given.

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2. For $q = (q_1,...,q_n)$, $q_i > 0$, let $M_n^{[r]}(a;q)$ be, as usually, the weighted mean of order $r \in R$ (see, for example, [5]).

If $q_1 = ... = q_n$ we denote simply $M_n^{[r]}(a)$ instead of $M_n^{[r]}(a;q)$.

It is known that $M_n^{[r]}(a;q)$ is strictly increasing as function of $r \in R$.

It is easy to see that $M_2^{(p)}(a) = A_2(a)$ and $B_2^{(p)}(a) = M_2^{[p-1]}(a;a)$.

In the sequel let $n \geq 3$.

THEOREM. (i) *The functions* $p \to B_n^{(p)}(a)$ and $p \to M_n^{(p)}(a)$ *are strictly increasing on* $\left(-\infty, \frac{1}{n-1}\right)$ *and on* $\left(\frac{1}{n-1}, +\infty\right)$.

(ii) $B_n^{(p)}(a) < M_n^{(p)}(a) < A_n(a)$ *for all* $p < 0$

$B_n^{(p)}(a) > M_n^{(p)}(a) > H_n(a)$ *for all* $p > 1$.

(iii) $B_n^{(p)}(a) < M_n^{(p)}(a)$ *for* $\frac{1}{n-1} < p < 1$

$B_n^{(p)}(a) > M_n^{(p)}(a)$ *for* $0 < p < \frac{1}{n-1}$

Proof. We shall use the following inequality of Sierpinski (see [4], [6], [7]):

$$H_n^{n-1}(a)A_n(a) < G_n^n(a) < A_n^{n-1}(a)H_n(a) \tag{2}$$

Since

$$\frac{d}{dp}\log M_n^{(p)}(a) = (p(n-1)-1)^{-2}\log\left(A_n^{n-1}(a)H_n(a)/G_n^n(a)\right)$$

we conclude that the function $p \to M_n^{(p)}(a)$ is strictly increasing on $\left(-\infty, \frac{1}{n-1}\right)$ and on $\left(\frac{1}{n-1}, +\infty\right)$.

Now set $b_i = ((a_1 \ldots a_n)/a_i)^{1/(n-1)}$. Then

$$B_n^{(p)}(a) = M_n^{[p(n-1)-1]}(b;b)K^{-1/(p(n-1)-1)}$$

where

$$K = \frac{G_n^n(b)}{A_n(b)H_n(b^{n-1})}.$$

We have

$$(H_n(b^{n-1}))^{1/(n-1)} = M_n^{[-n+1]}(b) < M_n^{[-1]}(b) = H_n(b)$$

hence $H_n(b^{n-1}) < H_n^{n-1}(b)$. It follows that

$$K > G_n^n(b)/(A_n(b)H_n^{n-1}(b)) > 1$$

We conclude that the function $p \to B_n^{(p)}(a)$ is strictly increasing on $\left(-\infty, \frac{1}{n-1}\right)$ and on $\left(\frac{1}{n-1}, +\infty\right)$.

So (i) is proved.

Now $H_n(a) = M_n^{[-1]}(a)$ and $(H_n(a^p))^{1/p} = M_n^{[-p]}(a)$; it follows that for $p<0$ we hav[e] $H_n(a^p) < H_n^p(a)$ and therefore

$$B_n^{(p)}(a) < M_n^{(p)}(a) < M_n^{(0)}(a) = A_n(a).$$

The other inequalities of (ii) and (iii) can be proved similarly.

3. With usual notation we recall the following inequality of D.S.Mitrinović an[d] P.M.Vasić [4]:

$$A_n^{n-1}(a)H_n(a)/G_n^n(a) \geq A_{n-1}^{n-2}(a)H_{n-1}(a)/G_{n-1}^{n-1}(a) \qquad (3$$

Using (3) it is easy to derive, for $p \leq 0$,

$$(M_n^{(p)}(a)/A_n(a))^{p(n-1)-1} \geq (M_{n-1}^{(p)}(a)/A_{n-1}(a))^{p(n-2)-1} \qquad (4$$

For $p > 0$, $p \neq \frac{1}{n-1}$, the inequality is reversed.

4. Let $t \in (0, +\infty)$, $a+t = (a_1+t, \ldots, a_n+t)$. In [6] it is shown that the function

$$t \to A_n^{n-1}(a+t)H_n(a+t)/G_n^n(a+t)$$

is nonincreasing on $(0, +\infty)$. Consequently, for $t < s$ we have

$$\frac{d}{dp} \log M_n^{(p)}(a+t) \geq \frac{d}{dp} \log M_n^{(p)}(a+s)$$

Now let $p, q < \frac{1}{n-1}$ or $p, q > \frac{1}{n-1}$; if $p < q$, it follows that

$$M_n^{(q)}(a+t)/M_n^{(p)}(a+t) \geq M_n^{(q)}(a+s)/M_n^{(p)}(a+s) \tag{5}$$

5. Finally, let $0 < a_i \leq \frac{1}{2}$, $i = 1,\ldots,n$. H.Alzer [1] has proved the following inequality

$$(A_n(a)/A_n(1-a))^{n-1} H_n(a)/H_n(1-a) \geq G_n^n(a)/G_n^n(1-a) \tag{6}$$

Using (6) it is easy to obtain for $p \leq 0$ or $p > \frac{1}{n-1}$

$$A_n(a)/A_n(1-a) \geq M_n^{(p)}(a)/M_n^{(p)}(1-a) \tag{7}$$

For $0 < p < \frac{1}{n-1}$ the inequality is reversed.

Moreover, using (6) we deduce

$$\frac{d}{dp} \log M_n^{(p)}(a) \geq \frac{d}{dp} \log M_n^{(p)}(1-a)$$

It follows that the function $p \to M_n^{(p)}(a)/M_n^{(p)}(1-a)$ is increasing on $\left(-\infty, \frac{1}{n-1}\right)$ and on $\left(\frac{1}{n-1}, +\infty\right)$.

REFERENCES

1. Alzer,H., *Verschärfung einer Ungleichung von Ky Fan*, Aeq. Math. **36**(1988), 246-250.
2. Bauer,H., *A class of means and related inequalities*, Manuscripta Math. **55**(1986), 199-211.
3. Gini,C., *Le medie*, Milano 1958.
4. Mitrinović,D.S., Vasić,P.M., *On a theorem of W.Sierpinski concerning means*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 544 - No. 576 (1976), 113-114.
5. Mitrinović,D.S., Bullen,P.S., Vasić,P.M., *Sredine i sa njima povezane nejednakosti*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 600(1977). See also: Bullen,P.S., Mitrinović,D.S., Vasić,P.M., *Means and their inequalities*, Dordrecht 1988.
6. Pečarić,J.E., *On an inequality of W.Sierpinski*, Punime Mat. **3**(1988), 9-11.
7. Pečarić,J.E., Wang,C.L., *An extension of a Sierpinski inequality*, Congressus Numerantium **61**(1988), 35-38.

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SUBCLASSES OF STARLIKE FUNCTIONS WITH $\mathrm{Re} f'(z) > 0$

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REZUMAT. - Subclase de funcţii stelate cu $\mathrm{Re} f'(z) > 0$. Fie A_n clasa funcţiilor $f(z) = z + a_{n+1}z^{n+1} + \ldots$, $n \geq 1$ care sunt analitice în discul unitate $U = \{ z;\ |z| < 1 \}$. Pentru $f \in A_n$ se obţin condiţii asupra funcţiei $zf'(z)/f(z)$, care să implice $\mathrm{Re} f'(z) > 0$.

1. Introduction. Let A_n denote the set of functions

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots,\ n \geq 1$$

that are analytic in the unit disc $U = \{ z \in \mathbf{C};\ |z| \leq 1 \}$. Let S^* be the usual class of starlike (univalent) functions in U, i.e.

$$S^* = \{ f \in A;\ \mathrm{Re}\,[\, zf'(z) / f(z)\,] > 0,\ z \in U \}$$

Let

$$R = \{ f \in A,\ \mathrm{Re} f'(z) > 0,\ z \in U \}.$$

In [2] P.T.Mocanu obtained subsets E of the right half-plane, such that $f \in S^*$, whenever $f'(z) \in E$, for all $z \in U$. On the other hand it is obvious that $S^* \not\subset R$ and a natural problem is to find certain subsets E of the right half-plane, such that $f \in R$, whenever $\dfrac{zf'(z)}{f(z)} \in E$, for all $z \in U$.

In this paper we obtain some conditions in term of $\dfrac{zf'(z)}{f(z)}$ such that $f \in R$.

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2. Preliminaries. If f and g are analytic in the unit disc U, then we say that f is subordinate to g, written $f \prec g$, or $f(z) \prec g(z)$, if g is univalent, $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $\mathcal{H}[a,n]$ denote the set of functions

$f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots$, $n \geq 1$ (where n is a positive integer) that are analytic in U.

We shall use the following lemmas to prove our results.

LEMMA 1. [3]. *Let h, with $h(0) = 0$ be starlike in U and let $p \in \mathcal{H}[a,n]$ in U. If* $\frac{zp'(z)}{p(z)} \prec h(z)$, *then* $p(z) \prec q(z)$, *where* $q(z) = a \exp \frac{1}{n}\int_0^z \frac{h(t)}{t}\,dt$

LEMMA 2. [1]. *Let E be a set in the complex plane $\mathbf{C}$ and let q be an analytic and univalent function on $\bar{U}$. Suppose that the function $H : \mathbf{C} \times U \to \mathbf{C}$ satisfies*

$H[\,q(\zeta), m\,\zeta\, q'(\zeta); z\,] \notin E,$

whenever $m \geq n$, $|\zeta| = 1$ and $z \in U$. If p is analityc on U of the form $p(z) = q(0) + p_n z^n + \dots$, and p satisfies $H[\,p(z), zp'(z); z\,] \in E$, for $z \in U$, then $p \prec q$.

3. Main results.

THEOREM 1. *If $f \in A_n$ satisfies*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq M, \quad \text{for } z \in U \tag{1}$$

where $M = M_n$ is the solution of the equation

$$M \tan \frac{M}{n} = \sqrt{1 - M^2} \tag{2}$$

then $f \in R$.

Proof. If we let $P(z) = f(z) / z$, then $P \in \mathcal{H}[1,n]$, $f'(z) = zP'(z) + P(z)$ and we have

$$\frac{zf'(z)}{f(z)} = \frac{1}{P(z)}\cdot[zP'(z) + P(z)] = 1 + \frac{zP'(z)}{P(z)}. \tag{3}$$

From (1) and (3) we deduce

$$\frac{zf'(z)}{f(z)} \prec 1 + Mz \quad \text{or} \quad \frac{zP'(z)}{P(z)} \prec Mz,$$

and by Lemma 1 we obtain

$$P(z) \prec e^{\frac{M}{n}z}. \tag{4}$$

If we let $P(z) = f'(z)$ then we have $\dfrac{zf'(z)}{f(z)} = \dfrac{p(z)}{P(z)}$

and (1) becomes

$$\left|\frac{p(z)}{P(z)} - 1\right| < M. \tag{5}$$

In order to show that (5) implies $\operatorname{Re} p(z) > 0$ in U, according to Lemma 2 it is sufficient to check the inequality

$$\left|\frac{is}{P(z)} - 1\right|^2 \geq M^2 \tag{6}$$

for all real s and all $z \in U$. The inequality (6) can be rewritten as

$$s^2 - 2s\operatorname{Im}P + (1 - M^2)\,|P|^2 \geq 0$$

and this inequality holds for all real s if

$$|\operatorname{Im}P|^2 \leq (1 - M^2)\,|P|^2 \text{ which becomes}$$

$$|\operatorname{Im} P(z)| \leq \frac{\sqrt{1-M^2}}{M}\operatorname{Re} P(z). \tag{7}$$

On the other hand from (4) we deduce

$$|\arg P(z)| < \frac{M}{n}, \text{ i.e.}$$

$$|\operatorname{Im} P(z)| < \tan\frac{M}{n}\operatorname{Re} P(z) \tag{8}$$

and from (7) and (8) we deduce

$$M \tan\frac{M}{n} = \sqrt{1-M^2}$$

hence if $M = M_n$ satisfies (2) then the inequality holds and we deduce $\operatorname{Re} p(z) > 0$ in U.

which shows that $f \in R$. ■

We note that $M_1 = 0.739.....$ and $M_2 = 0.9003....$

COROLLARY 1. *If $g \in \mathcal{H}[0,n]$ satisfies $|g(z)| < M$, $z \in U$, where $M = M_n$ is the solution of the equation* (2), *then*

$$\operatorname{Re}\left\{[1 + g(z)] \exp \int_0^z \frac{g(t)}{t}\, dt\right\} > 0, \text{ for } z \in U.$$

Proof. If we let

$$f(z) = z \exp \int_0^z \frac{g(t)}{t}\, dt,$$

then

$$\frac{zf'(z)}{f(z)} = 1 + g(z),\ f'(z) = [1 + g(z)] \exp \int_0^z \frac{g(t)}{t}\, dt$$

and the result follows from Theorem 1. ■

THEOREM 2. *If $f \in A_n$ satisfies*

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \theta \tag{9}$$

where $\theta = \theta_n$ is the solution of the equation

$$\frac{\pi}{2n} \tan\theta + \theta = \frac{\pi}{2} \tag{10}$$

then $f \in R$.

Proof. Let

$$Q(z) = \left(\frac{1 + z}{1 - z}\right)^{\beta} \quad 0 < \beta \leq 1$$

and let

$$H(z) = 1 + n\frac{zQ'(z)}{Q(z)} = 1 + \frac{2n\beta z}{1 - z^2}.$$

If we put $P(z) = f(z) / z$ then we have $P \in \mathcal{H}[1,n]$

and

$$\frac{zf'(z)}{f(z)} = 1 + \frac{zP'(z)}{P(z)}$$

From Lemma 1 we easily deduce that

$$1 + \frac{zP'(z)}{P(z)} \prec H(z) \Rightarrow P \prec Q .$$

In particular the inequality (9) implies

$$|\arg P(z)| < \beta \frac{\pi}{2}, \tag{11}$$

where

$$\tan \theta = n\,\beta. \tag{12}$$

If we let $p(z) = f'(z)$ then we have

$$\frac{zf'(z)}{f(z)} = \frac{p(z)}{P(z)}$$

and

$$\arg p(z) = \arg P(z) + \arg \frac{zf'(z)}{f(z)}$$

Hence by using (9), (10), (11) and (12) we deduce

$$|\arg p(z)| \leq |\arg P(z)| + \left|\arg \frac{zf'(z)}{f(z)}\right| < \beta \frac{\pi}{2} + \theta = \frac{\pi}{2}$$

i.e. $\operatorname{Re} p(z) = \operatorname{Re} f'(z) > 0$ which shows that $f \in R$. ■

We note that

$$\theta_1 = 0.568\ldots \quad (32^o 54\ldots) \quad \text{and}$$

$$\theta_2 = \frac{\pi}{2} = 0.785\ldots \quad (45^o)$$

COROLLARY 2. *If* $h \in \mathcal{H}[1,n]$ *satisfies* $|\arg h(z)| < \theta$, $z \in U$ *where* $\theta = \theta_n$ *is the solution of the equation* (10) *then*

$$\operatorname{Re} \left\{ h(z) \exp \int_0^z \frac{h(t) - 1}{t}\, dt \right\} > 0, \quad \textit{for} \quad z \in U.$$

THEOREM 3. *If* $f \in A_n$ *satisfies*

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < N, \quad z \in U, \tag{13}$$

where $N = N_n$ *is the solution of the equation*

$$(1+N)e^{\frac{N}{n}} = 2 \tag{14}$$

then

$$|f'(z)-1| < 1, \quad z \in U. \tag{15}$$

Proof. If we let $P(z) = f(z) / z$, then by using (3), (4) and (13) we obtain

$$P(z) \prec e^{\frac{N}{n}z} \tag{16}$$

If we let $p = f'$ then the inequality (13) becomes

$$\left|\frac{p(z)}{P(z)} - 1\right| < N . \tag{17}$$

Since (15) is equivalent to

$$f'(z) \prec 1 + z,$$

according to Lemma 2, in order to show that (13) implies (15) it is sufficient to check the inequality

$$\left|\frac{1+\zeta}{P(z)} - 1\right| \geq N \tag{18}$$

for all complex ζ, with $|\zeta| = 1$ and all $z \in U$. The inequality (18) can be rewritten as

$$|1 - P + \zeta|^2 \geq N^2 |P|^2. \tag{19}$$

We have

$$|1-P+\zeta|^2 = |1-P|^2 + 2\,\mathrm{Re}[\bar{\zeta}(1-P)] + 1 \geq |1-P|^2 - 2|1-P| + 1.$$

Hence the inequality (19) becomes

$$||P-1|-1| \geq N|P|. \tag{20}$$

By using (16) and (14) we easily deduce that

$$|P-1| < 1,$$

therefore (20) becomes

$$1 - |P-1| \geq N|P|$$

which yields

$$N \leq \frac{1}{|P|} - \left|1 - \frac{1}{P}\right|.$$

We note that (16) is equivalent to

$$\frac{1}{P} \prec e^{\frac{N}{n}z}.$$

Hence the best bound N in (13), which can be obtained by our method is given by

$$N = \min_{|z|=N/n}\left\{|e^z| - |e^z - 1|\right\}.$$

If we let $z = x + iy$, with $x^2 + y^2 = \dfrac{N^2}{n^2}$, then

$$\varphi(x,y) = |e^z| - |e^z - 1| = e^x - \sqrt{e^{2x} - 2e^x \cos y + 1}\ .$$

It is easy to show that the system

$$\frac{\partial\varphi}{\partial x} = 0\ ,\quad \frac{\partial\varphi}{\partial y} = 0$$

yields

$$\cos^2 y = 1 \quad \text{and} \quad \sin y = 0$$

Hence the minimum φ occors for $y = 0$.

For $-\dfrac{N}{n} \leq x \leq \dfrac{N}{n}$ we have

$$\varphi(x,0) = e^x - |e^x - 1| = \begin{cases} 1 & ,e^x > 1 \\ 2e^x - 1 & ,e^x < 1 \end{cases}$$

and we deduce

$$\min \varphi = 2e^{-\frac{N}{n}} - 1$$

hence $N = 2e^{-\frac{N}{n}} - 1$, which is the equation (14). ■

We note that

$$N_1 = 0.374....$$

$$N_2 = 0.532....$$

If we put $zf'(z) / f(z) = 1 - g(z)$, then from Theorem 3 we obtain

COROLLARY 3. *If* $g \in \mathcal{H}[0,n]$ *satisfies* $|g(z)| < N$, $z \in U$, *where* $N = N_n$ *is the solution of the equation* (14), *then*

$$\operatorname{Re}\left\{\frac{1}{1-g(z)}\exp\int_0^z \frac{g(t)}{t}\,dt\right\} > \frac{1}{2},\quad \textit{for } z \in U.$$

4. Examples.

Example 1. If we let $g(z) = \lambda z$, then from Corollary 1 we obtain

$$\operatorname{Re}\,[(1+\lambda z)\,e^{\lambda z}] > 0,\ z \in U$$

if $|\lambda| \le M_1 = 0.739....$

Example 2. If we let $g(z) = \lambda \sin z^2$, then $g \in \mathcal{H}[0,2]$ and $|g(z)| < |\lambda|$ sh 1. Hence by Corollary 1 we deduce that

$$\operatorname{Re}\left\{(1+\lambda \sin z^2)\exp \lambda\int_0^z \frac{\sin t^2}{t}\,dt\right\} > 0,\quad z \in U,$$

whenever

$$|\lambda| \le \frac{M_2}{sh\ 1} = 0.765....$$

Example 3. If we let $h(z) = e^{\lambda z}$, then $h \in \mathcal{H}[1,1]$ and

$$|\arg h(z)| = |\operatorname{Im}(\lambda z)| \le |\lambda|$$

and from Corollary 2 we deduce that

$$\operatorname{Re}\left\{e^{\lambda z}\exp\int_0^z \frac{e^{\lambda t}-1}{t}\,dt\right\} > 0,\quad z \in U,$$

if

$$|\lambda| \le \theta_1 = 0.568....$$

Example 4. If we let $h(z) = e^{\lambda z^2}$, then $h \in \mathcal{H}[1,2]$ and from Corollary 2 we deduce that

$$\operatorname{Re}\left\{e^{\lambda z^2}\exp\int_0^z \frac{e^{\lambda t^2}-1}{t}\,dt\right\} > 0,\quad z \in U,$$

whenever $|\lambda| \le \pi/4$.

REFERENCES

1. Miller,S.S., Mocanu,P.T., *Differential subordinations and inequalities in the complex plane*, J. of Differential Equations 67, **2**(1987), 199-211.
2. Mocanu,P.T., *Some starlikeness conditions for analytic functions*, Rev. Roumaine Math. Pure., **33**(1988), 1-2, 117-124.
3. Suffridge,T.J., *Some remarks on convex maps of the unit disc*, Duke Math. J. **37**(1970), 775-777.

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ON THE APPROXIMATION OF FUNCTIONS AND THEIR DERIVATIVES BY BERNSTEIN POLYNOMIALS

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REZUMAT. - Asupra aproximării funcţiilor şi a derivatelor lor prin polinoame Bernstein. În lucrare se îmbunătăţesc unele rezultate obţinute recent de C. Badea, I. Badea şi H. H. Gonska în [1].

1. Let $C[0,1]$ be the space of all continuous and real-valued functions on $[0,1]$, and, for every positive integral n, $\Pi_n = \Pi_n[0,1]$ denote the space of all polynomials of degree n on $[0,1]$. We shall consider the Bernstein Operators $B_n : C[0,1] \to \Pi_n$ defined by

$$B_n(f,x) = \sum_{i=0}^{n} p_{n,i}(x) f\left(\frac{i}{n}\right), \quad x \in [0,1]$$

where

$$p_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}$$

Denoting by $\|\cdot\|$ the sup-norm on $C[0,1]$ we know that

$$\lim_{n\to\infty} \|f - B_n f\| = 0$$

In 1930 I. Chlodovscky proved that B_n has the property of simultaneous approximation that is for all elements f in the space $C^r[0,1]$, $r \in \mathbf{N}$, of all real-valued and r-times continuously differentiable functions on $[0,1]$, one has

$$\lim_{n\to\infty} \|D^r(f - B_n f)\| = 0$$

where D^r is the r-th differential operator.

In 1937 T. Popoviciu ([2]) obtained the following est-

$$\|D^r(f-B_nf)\| \le A_{n,r}\,\omega_1\left(f^{(r)},\delta_{n,r}\right) + \frac{r(r-1)}{2n}\|f^{(r)}\| \tag{1}$$

for all positive integers r and $n \ge r+1$, where

$$A_{n,r} = 3/2 + \frac{2r(n-r)^{1/2}}{n},\ \delta_{n,r} = \frac{1}{\sqrt{n-r}}$$

and $\omega_1(\cdot)$ is the first order modulus of smoothness. A history of the esimates of the form (1) was given in [1]. Assuming that $f \in C^{(r+1)}[0,1]$ D.D.Stancu [5] showed that

$$\|D^r(f-B_nf)\| \le M_{n,r}\,\omega_1\left(f^{(r+1)},\delta_{n,r}\right) + \frac{r}{n}\|f^{(r+1)}\| + \frac{r(r-1)}{2n}\|f^{(r)}\| \tag{2}$$

where $\delta_{n,r} = n^{-1/2}$, $M_{n,r} = (1+\varphi(n,r))\,\varphi(n,r)\,n^{-1/2}$ with

$$\varphi(u,v) := u^{-1/2}\left[2v + (u+4v^2-v)^{1/2}\right]/2.$$

We mention here the strong the result of F.Schurer and F.W.Steutel [3] which proved that:

$$\|f - B_nf\| \le \frac{1}{4}\frac{1}{\sqrt{n}}\,\omega_1\left(f', \frac{1}{\sqrt{n}}\right). \tag{3}$$

Here $\frac{1}{4}$ is the best possible constant in front of $n^{-1/2}$.

The aim the present paper is to prove news estimates of the above types by a method wich uses the inequality (3) and the estimate given bt Sikkema [4]

$$\|f - B_nf\| \le K\,\omega_1\left(f, \frac{1}{\sqrt{n}}\right), \tag{4}$$

where $K = (4306 + 837\sqrt{6})/5832 = 1.0898873...$ Here K is the best constant in front of $\omega_1\left(f, \frac{1}{\sqrt{n}}\right)$.

2. For $a \in R$ and $b \in N$ we denote

$$(a)_b := \prod_{k=0}^{b-1}(a-k).$$

The r-th derivative of a Bernstein polynomial is given by

$$\left(D^rB_nf\right)(x) = \frac{(n)_r}{n^r}\,r!\sum_{k=0}^{n-r}\left[\frac{k}{n}, \frac{k+1}{n}, \dots, \frac{k+r}{n}; f\right]p_{n-r,k}(x). \tag{5}$$

We denote by g_r the function $g_r : [0,1] \to \mathbb{R}$ defined by

$$g_r(x) = \left[\frac{x(n-r)}{n}, \frac{x(n-r)+1}{n}, \ldots, \frac{x(n-r)+r}{n}; f\right]$$

and we obtain

$$\left(D^r B_n f\right)(x) = \frac{(n)_r}{n^r} r! \, (B_{n-r} g_r)(x) \tag{6}$$

LEMMA 1. *Let $f \in C^r[0,1]$. Then for $n \geq r+1$ we have*

$$\| D^r(f - B_n f)\| \leq \frac{(n)_r}{n^r} r! \, K \omega_1\left(g_r, \frac{1}{\sqrt{n-r}}\right) + \left\| \frac{(n)_r}{n^r} r! \, g_r - f^{(r)} \right\| \tag{7}$$

where K is Sikkema's constant.

Proof. From (6) we obtain:

$$|D^r(B_n f - f)(x)| \leq \frac{(n)_r}{n^r} r! \, |(B_{n-r} g_r)(x) - g_r(x)| + \left| \frac{(n)_r}{n^r} r! \, g_r(x) - f^{(r)}(x) \right| \tag{8}$$

Using inequality (4) we obtain (7) from (8).

LEMMA 2. *Let $f \in C^{r+1}[0,1]$. Then for $n \geq r+1$*

$$\left\| D^r(B_n f - f)\right\| \leq \frac{1}{4} \frac{1}{\sqrt{n-r}} \frac{(n)_r}{n^r} r! \, \omega_1\left(g_r', \frac{1}{\sqrt{n-r}}\right) + \left\| \frac{(n)_r}{n^r} r! \, g_r - f^{(r)} \right\|. \tag{9}$$

Proof. We obtain the inequality (9) from the inequalities (3) and (6).

THEOREM 1. *Let $f \in C^r[0,1]$. Then for $n \geq r+1$ we have:*

$$\begin{aligned} \| D^r(B_n f - f)\| \leq \frac{(n)_r}{n^r} K \omega_1\left(f^{(r)}, \frac{n-r}{n} \frac{1}{\sqrt{n-r}}\right) + \\ + \frac{(n)_r}{n^r} \omega_1\left(f^{(r)}, \frac{r}{n}\right) + \left(1 - \frac{(n)_r}{n^r}\right) \| f^{(r)} \|. \end{aligned} \tag{10}$$

Proof. We have

$$g_r(x+\delta) - g_r(x) = \left[0, \frac{1}{n}, \ldots, \frac{r}{n}; f\left(\frac{(x+\ )(n-r)}{n} + t\right) - f\left(\frac{x(n-r)}{n} + t\right)\right] \tag{11}$$

Using Cauchy's formula we ootain from (11):

$$g_r(x+\delta) - g_r(x) = \frac{1}{r!} f^{(r)}\left(\frac{(x+\delta)(n-r)}{n} + \frac{c}{n}\right) - f^{(r)}\left(\frac{x(n-r)}{n} + \frac{c}{n}\right) \tag{12}$$

where $c = c(f, x, n) \in [0, r]$, $\delta = \dfrac{1}{\sqrt{n-r}}$.

From (12) we obtain that:

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$$\omega_1\left(g_r, \frac{1}{\sqrt{n-r}}\right) \le \frac{1}{r!}\,\omega_1\left(f^{(r)}, \frac{n-r}{n}\frac{1}{\sqrt{n-r}}\right). \tag{13}$$

Because we have

$$\left|\frac{(n)_r r!}{n^r} g_r(x) - f^{(r)}(x)\right| \le \frac{(n)_r}{n^r} r!\left|g_r(x) - \frac{f^{(r)}(x)}{r!}\right| + \left(1 - \frac{(n)_r}{n^r}\right)|f^{(r)}(x)|$$

this inequality together with the inequality (13) and (4) implies (10).

THEOREM 2. *Let* $f \in C_{r+1}[0,1]$. *Then for* $n \ge r+1$ *we have:*

$$\begin{aligned}\|D^r(B_n f - f)\| \le \frac{1}{4}\frac{1}{\sqrt{n-r}}\frac{(n)_{r+1}}{n^{r+1}}\,\omega_1\left(f^{(r+1)}, \frac{n-r}{n}\frac{1}{\sqrt{n-r}}\right) + \\ + \frac{(n)_r}{n^r}\,\omega_1\left(f^{(r)}, \frac{r}{n}\right) + \left(1 - \frac{(n)_r}{n^r}\right)\|f^{(r)}\|\end{aligned} \tag{14}$$

Proof. The inequality (14) will result if we observe that we can write

$$g_r'(x) = \left[\frac{n-r}{n}, \frac{x(n-r)}{n}, \ldots, \frac{x(n-r)+r}{n}; f'\right]$$

and then we use Lemma 2.

COROLLARY. *Let* $r, n \in N, n \ge r+1$ *and let* $f \in C^{r+1}[0,1]$. *Then we have:*

$$\begin{aligned}\|D^r(f - B_n f)\| \le \frac{1}{\sqrt{n}}\frac{(n)_r}{4n^r}\,\omega_1\left(f^{(r+1)}, \frac{1}{\sqrt{n}}\right) + \\ + \frac{r}{n}\frac{(n)_r}{n^r}\|f^{(r+1)}\| + \frac{r(r-1)}{2n}\|f^{(r)}\|\end{aligned} \tag{15}$$

Proof. We have:

$$\frac{(n-r)}{n}\frac{1}{\sqrt{n-r}} < \frac{1}{\sqrt{n}}, \quad n \ge r+1$$

$$\frac{(n)_{r+1}}{n^{r+1}}\frac{1}{\sqrt{n-r}} < \frac{(n)_r}{n^r}\frac{1}{\sqrt{n}}$$

$$\omega_1\left(f^{(r)}, \frac{r}{n}\right) \le \frac{r}{n}\|f^{(r+1)}\|$$

Now, the inequality (15) result from the Theorem 2.

Remark. In [1] the authors proved that:

$$\|D^r(f-B_n f)\| \le \frac{1}{4}\frac{1}{\sqrt{n-r}}\,\omega_1\left(f^{(r+1)}, \frac{1}{\sqrt{n-r}}\right) + \frac{r}{n}\|f^{(r+1)}\| + \frac{r(r-1)}{2n}\|f^{(r)}\| \tag{16}$$

We observe the inequality (15) improves the inequalities (16) and (2).

REFERENCES

1. C.Badea, I.Badea, H.H.Gonska, *Improved on simultaneos approximation by Bernstein operators*. To appear in Anal. Numer. Theor. Approx.
2. T.Popoviciu, *Despre cea mai bună aproximaţie a funcţiilor continue prin polinoame*, Cluj: Inst. Arte Grafice Ardealul, 1937.
3. F.Schurer, F.W Steutel, *The degree of local approximation of functions in* $C^1[0,1]$ *by Bernstein polynomials*, J.Approx. Theory 19(1977), 69-82.
4. P.C.Sikkema, *Der Wert einiger Konstanten in der Theorie der Approximation mit Bernstein-Polynomen*, Numer. Math. 3(1961), 107-116.
5. D.D.Stancu, *Generalizations of an inequality of G.G.Lorentz*, An. Ştiinţ. Univ. "Al.I.Cuza" Iaşi (N.S) Secţ. I. Mat. Fiz. Chim. 9(1963), 49-58.

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REMARKS ON TWO THEOREMS OF CIRIC AND MAITI AND BABU

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REZUMAT. - Observaţii asupra a două teoreme ale lui Ciric, Maiti şi Babu. În lucrare sunt studiate aplicaţii care satisfac condiţia de contracţie generalizată (2). Se demonstrează două teoreme de punct fix pentru astfel de aplicaţii.

Abstract. Let f be a self-map of a metric space (X,d), $F(f)$ the set of fixed points of f, $L(x)$ the set of subsequential limit points of $\{f^n x\}_{n=0}^{\infty}$. We obtain two results on $L(x) \subset F(f)$, one of which extends properly a theorem of Ciric.

1. **Introduction.** Let f be a self-map of a metric space (X,d). Ciric [1] proved the existence of fixed point for f provided it satisfies

$$\min \{d(fx,fy),d(x,fx),d(y,fy)\} - \min \{d(x,fy),d(y,fx)\} < d(x,y) \quad (1)$$

for $x \neq y$. Maiti and Babu [2] studied the structure of the subsequential limit points of a sequence of iterates of maps which are contractive over two consecutive elements of an orbit.

Let R_0 be the subspace $[0,\infty)$ of the real line with usual topology, h a continuous function from $X \times X$ into R_0. The purpose of this paper is to consider maps f on X which satisfy

$$\min \{h(fx,fy),h(x,fx),h(y,fy)\} - \min \{h(x,fy),h(y,fx)\} <$$

$$< \max \{h(x,y), \min \{h(x,fx), h(y,fy)\}\} \quad (2)$$

for $x \neq y$. A fixed point theorem is obtained which is a proper generalization of Theorem 3

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of Ciric [1]. We arrive at the same conclusion as in [2] about the set of subsequential limit points of a sequence of iterates of f, where f satisfies (2).

In what follows $F(f)$ denotes the set of fixed points of f. The orbit of $x \in X$ generated by f is denoted by $O(x,f)$ and its closure by $\bar{O}(x,f)$. $L(x)$ denotes the set of subsequential limit points of $\{f^n x\}_{n=0}^{\infty}$ For $z \in X$ and $A,B \subset X$, define $d(A,B) = \inf\{d(x,y) \mid x \in A \text{ and } y \in B\}$ and $d(z,B) = d(\{z\},B)$. A self-map f of X is called orbitally continuous if $\lim_{i\to\infty} f^{n_i} x = w$ implies $\lim_{i\to\infty} f f^{n_i} x = fw$ for each $x \in X$.

2. Results and example.

THEOREM 1. *Let f be an orbitally continuous self-map of a metric space (X,d), h a continuous function from $X\times X$ into R_0 such that $h(x,y) = 0$ if and only if $x = y$. Assume that there exists $x_0 \in X$ such that $L(x_0) \neq \varnothing$. If f satisfies (2) for all distinct $x,y \in \bar{O}(x_0,f)$, then $F(f)$ is nonempty and $L(x_0)$ is a closed subset of $F(f)$.*

Proof. Let $x_n = f^n x_0$ for $n \geq 1$. If $x_{n-1} \neq x_n$, by (2) we have

$$\min\{h(x_n,x_{n+1}),h(x_{n-1},x_n),h(x_n,x_{n+1})\} - \min\{h(x_{n-1},x_{n+1}),h(x_n,x_n)\} <$$

$$< \max\{h(x_{n-1},x_n),\min\{h(x_{n-1},x_n),h(x_n,x_{n+1})\}\}$$

which implies that $h(x_n,x_{n+1}) < h(x_{n-1},x_n)$. Thus $h(x_n,x_{n+1}) \leq h(x_{n-1},x_n)$ for all $n \geq 1$. Hence $h(x_{n-1},x_n) \to r$ as $n \to \infty$. For each $p \in L(x_0)$, there exists a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$ such that $x_{n_i} \to p$ as $i \to \infty$. Since f is orbitally continuous, $x_{n_i+1} \to fp$ and $x_{n_i+2} \to f^2 p$ as $i \to \infty$. By the continuity of h it follows that

$$h(p,fp) = \lim_{i\to\infty} h(x_{n_i},x_{n_i+1}) = r = \lim_{i\to\infty} h(x_{n_i+1},x_{n_i+2}) = h(fp,f^2p)$$

We assert that $p = fp$. Otherwise, by (2) we obtain

$$\min\{h(fp, f^2p), h(p,fp), h(fp,f^2p)\} - \min\{h(p,f^2p), h(fp,fp)\}$$
$$< \max\{h(p,fp), \min\{h(p,fp), h(fp,f^2p)\}\}$$

which implies that $h(fp,f^2p) < h(p,fp)$. This is a contradiction. Hence $p \in F(f)$; i.e., $L(x_0) \subset F(f) \neq \emptyset$. It is easy to see that $L(x_0)$ is a closed subset of $F(f)$. This completes the proof.

Note that Theorem 3 of Ciric [1] is a special case of our Theorem 1. The following example shows that our Theorem 1 is a genuine generalization of Theorem 3 of Ciric [1].

Example. Let $X = \{0,5\} \cup \left\{\frac{1}{n} \mid n \geq 1\right\}$ with the usual metric. Define $f: X \to X$ by $f0 = 0$, $f1 = 5$, $f5 = \frac{1}{2}$ and $f\frac{1}{n} = \frac{1}{n+1}$ for $n \geq 2$. The Theorem 3 of Ciric [1] is not applicable since f does not satisfy (1) for $x = 1$ and $y = 5$. Let $h(x,y) = |x-y|(1+x)$ for $(x,y) \in X \times X$ and $x_0 = \frac{1}{2}$. Then $\bar{O}(x_0,f) = \{0\} \cup \left\{\frac{1}{n} \mid n \geq 2\right\}$. To prove that f satisfies (2) we consider three cases:

(i) $x = 0$ and $y = \frac{1}{n}$. Clearly $h(x,fx) = 0$ and $h(x,y) = \frac{1}{n}$. Hence (2) holds.

(ii) $x = \frac{1}{n}$ and $y = 0$. Similarly we can prove that (2) holds also.

(iii) $x = \frac{1}{n}$, $y = \frac{1}{m}$ and $n \neq m$. It is easy to see that

$$h(fx,fy) = \frac{|m-n|}{(n+1)(m+1)} \cdot \frac{n+2}{n+1} < \frac{|m-n|}{nm} \cdot \frac{n+1}{n} = h(x,y)$$

which implies that (2) holds.

Obviously, the assumptions of our Theorem 1 are satisfied.

THEOREM 2. *Let f be a continuous self-map of a metric space (X,d), h a continuous function from $X \times X$ into R_0 such that $h(x,y) = 0$ if and only if $x = y$. Assume there exists $x_0 \in X$ such that $\bar{O}(x_0,f)$ is compact and* (2) *holds for all distinct $x,y \in \bar{O}(x_0,f)$. Then $L(x_0)$ is a nonempty, closed and connected subset of $F(f)$, and either*

$L(x_0)$ is a singleton, and $\lim_{n\to\infty} f^n x$ exists and belongs to $F(f)$, or (3)

$L(x_0)$ *is uncountable, and it is contained in the boundary of* $F(f)$. (4)

Proof. The compactness of $\bar{O}(x_0,f)$ implies that $L(x_0)$ is nonempty. By Theorem 1 it follows that $L(x_0)$ is a closed subset of $F(f)$. Put $x_n = f^n x_0$ for $n \geq 1$.

We show that $d(x_n,x_{n+1}) \to 0$ as $n \to \infty$. Otherwise there exists an $\varepsilon > 0$ and a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$ such that $d(x_{n_i}, x_{n_i+1}) \geq \varepsilon$ for $i \geq 1$. Since $\bar{O}(x_0,f)$ is compact, we can find a convergent subsequence $\{x_{m_i}\}_{i=1}^{\infty}$ of $\{x_{n_i}\}_{i=1}^{\infty}$. Let $x_{m_i} \to p$ as $i \to \infty$. Then $p \in L(x_0) \subset F(f)$. By the continuity of f, we have $d(x_{m_i}, x_{m_i+1}) \to d(p,fp) = 0$ as $i \to \infty$, which is impossible. Hence $d(x_n,x_{n+1}) \to 0$ as $n \to \infty$.

We now show that $d(x_n,L(x_0)) \to 0$ as $n \to \infty$. If not, there exists an $\varepsilon > 0$ and a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ of $\{x_n\}_{i=0}^{\infty}$ such that $d(x_{n_i}, L(x_0)) \geq \varepsilon$ for $i \geq 1$. As above we can show that there exists a subsequence $\{x_{m_i}\}_{i=1}^{\infty}$ of $\{x_{n_i}\}_{i=1}^{\infty}$ such that $x_{m_i} \to p \in L(x_0)$ as $i \to \infty$. Thus $d(x_{m_i}, L(x_0)) \leq d(x_{m_i}, p) \to 0$ as $i \to \infty$; i.e., $d(x_{m_i}, L(x_0)) \to 0$ as $i \to \infty$, which is impossible. Hence $d(x_n, L(x_0)) \to 0$ as $n \to \infty$.

We next show that $L(x_0)$ is connected. Suppose the contrary. Then there exist two nonempty, closed and disjoint subsets A and B of $L(x_0)$ such that $L(x_0) = A \cup B$. Note that $L(x_0)$ is a closed subset of the compact set $\bar{O}(x_0,f)$. Then $L(x_0)$ is also compact. This implies that A and B are compact and $d(A,B) > 0$. Put $d(A,B) = 3t$. By the above results there exists an integer N such that $\max\{d(x_n,x_{n+1}), d(x_n,A \cup B)\} < t$ for $n \geq N$. Since $L(x_0)$ is compact, there exists $p \in A \cup B$ such that $d(x_n,A \cup B) = d(x_n,p)$. If $p \in A$, then $d(x_n,A) < t$. Thus, for any $n \geq N$, either $d(x_n,A) < t$ or $d(x_n,B) < t$. But both these inequalities cannot hold simultaneously for the same n because in that case

$$d(A,B) \leq d(x_n, A) + d(x_n, B) < 2t < d(A,B)$$

which is impossible. The set of positive integers $n \geq N$, such that $d(x_n, A) < t$, is not empty, because $\emptyset \neq A \subset L(x_0)$. Similarly, the set of positive integers $n \geq N$, such that $d(x_n, B) < t$, it also not empty. Consequently there exists a positive integer $k \geq N$ such that both $d(x_n, A) <$ $< t$. Hence

$$d(A,B) \leq d(x_k, A) + d(x_k, x_{k+1}) + d(x_{k+1}, B) < 3t = d(A,B)$$

which is a contradinction. Hence $L(x_n)$ is connected.

By Theorem 1 of Berge [3,p.96] it follows that $L(x_0)$ is either a singleton or uncountable. Note that $d(x_n, L(x_0)) \to 0$ as $n \to \infty$. Therefore, if $L(x_0)$ is a singleton, then $\lim_{n\to\infty} x_n$ exists and belongs to $F(f)$. In case that $L(x_0)$ is uncountable, it is contained in the boundary of $F(f)$ by the argument in [4,p.469]. This completes the proof.

REFERENCES

1. L.B. Ciric, *On some maps with a nonunique fixed point*, Publ. Inst. Math. 17(31) (1974), 52-58.
2. M. Maiti and A.C. Babu, *On subsequential limit points of a sequence of iterates*, Proc. Amer. Math. Soc. 82(1981), 377-381.
3. C. Berge, *Topological spaces*, Oliver & Boyd, London, 1963.
4. J.B. Diaz and F.T. Metcalf, *On the set of subsequential limit points of successive approximations*, Trans. Amer. Math. Soc. 135(1969), 459-485.

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ONE-STEP METHODS FOR THE NUMERICAL SOLUTION OF STIFF ORDINARY DIFFERENTIAL SYSTEMS

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REZUMAT. - Metode cu un singur pas pentru rezolvarea numerică a unor sisteme diferenţiale. În lucrare sunt examinate unele metode cu un singur pas pentru integrarea numerică a unor sisteme diferenţiale de ordinul întâi.

Abstract. Some one-step methods suitable for the approximate numerical integration of stiff systems of first-order ordinary differential equations are examinated. The given formula are A-stable. The generating idea is to build one-leg methods associated to the second derivative multistep methods.

1. Introduction. This paper deals with the numerical solution of the initial values problem for stiff systems of ordinary differential equations. Throughout we shall use

$$y'(t) = f(y(t)), \quad 0 \le t \le T, \quad y(0) = y_0 \tag{1.1}$$

to denote problems or classes of problems under consideration. Here $y(t)$ is a real vector of N elements and f, a real-valued vector nonlinear function. We assume that f is a Lipschitz function. This implies that for all initial vectors y_0, the problem possesses an unique solution for all $t \in [0,T]$.

Stiff problems occur in many fields of applications including chemical kinetics, reactor kinetics, control theory, dynamics of missile guidance, electronic circuit theory,

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mathematics, etc.

The essence of stiffness is the solution to be computed is slowly varying, but perturbations exist which are rapidly damped. The presence of such perturbations complicates the numerical computation of the solution.

Example 1: We consider the scalar equation:

$$y'(t) = \lambda y(t) + g'(t) - \lambda g(t), \ t \geq 0, \ y(0) = y_0, \ \lambda << 0$$

where g is slowly varying function on t only, the solution $y(t)$ is given by

$$y(t) = g(t) + e^{\lambda t}[y_0 - g(y_0)]$$

Because $\lambda << 0$, after a very short time distance the transient $e^{\lambda t}[y_0 - g(y_0)]$, which is also called the stiff component or strongly varying solution component, is no longer present in the solution $y(t)$. The function $g(t)$ dominates the solution to be computed on the larger part of the integration interval $[0,T]$. The explicit Euler method

$$y_{n+1} = y_n + hf(y_n), \ n = 0(1)M, \ \ Mh = T$$

is damped only if $-2 < h\lambda < 0$. This condition of numerical stability imposes a severe restriction of the stepsize h if $\lambda << 0$, even when $y_n - g(t_n)$ is negligible small. This situation is typicaly when we apply an explicit linear to a stiff problem. The stepsize is restricted by numerical stability rather than by accuracy.

Example 2: For the general linear problem

$$y'(t) = Ay(t) + r(t), \ \ t \geq 0, \ \ y(0) = y_0$$

where A is a constant $N \times N$ matrix and r is a time dependent forcing term, the most obvious way of defining stiffness is to impose conditions on the eigenvalues of A. Different solution components occur when the Jacobian matrix possesses eigenvalues which differ greatly in magnitude. Let $\lambda_1,\ldots,\lambda_N$ denote these eigenvalues. Then (1.5) may be called *stiff* if

(i) exit λ_i with $\mathrm{Re}(\lambda_i) << 0$;

(ii) exit λ_i of moderate size, i.e., $|\lambda_i|$ is small when compared with the modulus of the eigenvalues satisfying (i);

(iii) no λ_i exist with a large positive real part;

(iv) no λ_i exist with a large imaginary part, unless $\mathrm{Re}(\lambda_i) << 0$.

It is assumed here that the forcing term $r(t)$ is a smooth as the slowly varying exponentials in the solution.

Stiffness for a nonlinear problem is usually described in terms of the eigenvalues of the Jacobian matrix. The argument is based on local linearization.

In the literature, stiff problems are also called problems with large Lipschitz constants, because the property of those is the presence of a large classical Lipschitz constant

$$TL = T \sup_x \|f_y'(u)\| >> 1$$

The main requirement for a good stiff method is that it should have strong stability properties. The concept of absolute stability is connected with the scalar equation

$$y'(t) = \lambda y(t),\ \lambda \in \mathbf{C},\ t \geq 0,\ y(0) = y_0 \tag{1.2}$$

Though this equation is very simple, its use as a model to predict the stability behaviour of numerical methods for general nonlinear systems. An one-step method applied to this test equation reduces to

$$y_{n+1} = R(z)y_n,\quad z = h\lambda \tag{1.3}$$

where R is called the *stability function.* The method is said to be *absolutely stable* at $z \in \mathbf{C}$ if, for this z, $|R(z)| \leq 1$. The set of all points z which satisfies this requirement is called *the* *absolute stability region*. If the left half-plane $\mathrm{Re}(z) \leq 0$ is contained in the absolute stability region, the method is said to be *A-stable*. A condition which ensures that the method has the correct damping at $z = -\infty$, is *stability at infinity*: $\lim_{z\to-\infty} R(z) = 0$. Then, a one-step numerical method is said to be L-stable if it is A-stable and stable at infinity. In the case of A-stability and only $\lim_{z\to-\infty} |R(z)| < 1$, we have *strong A-stability*. It is well known from a famous result of Dahlquist, that the highest attainable order of an *A-stable* implicit linear multistep method is limited to two. The explicit methods do not have property of A-stability. Among the class of implicit Runge-Kutta formula we have the possibility of deriving *A-stable* methods of high order. One of the main arguments against the use of these was on the grounds of the amount of computational effort required to solve the resulting systems of algebraic equations.

Research into finding efficient stiff methods has followed some main directions:

(1) inside the Runge-Kutta class, the investigation of the use of transformation methods to obtain a solution of algebraic equations in an efficient manner or the derivation of different classes of implicit Runge-Kutta formula which do not call for the solution of a system of simultaneous equations;

(2) the generalization of linear methods by formula with high derivations, cyclic and composite methods, hybrid methods, pseudo Runge-Kutta etc;

(3) the construction of nonlinear methods such that of rational Runge-Kutta type (see on this subject an author's paper [12]);

The aim of this paper is to analyse the possibility to replace second derivative multistep formula by hybrid methods with same stability properties. We make a first step with the support of one-step methods. These are examples for some classes given in the following sections.

In section 3 we deal with second derivative methods and in section 4 with hybrid methods connected to those. We examine carefully the stability properties and the performance. **Our purpose is to derivate A-stable formula.** Finally, in section 5, numerical comparisions of some new methods with the classical ones are given. The efficiency of the new integrations is demonstrated by solving a series of challenging test problems.

2. Backgrounds. One-leg methods were introduced by Dahlquist in 1975 (to see reference [7]). The characteristic of these is the presence of only one value of f in each step. This made possible a certain theoretical stability analysis for stiff nonlinear problems (*G*-stability, contractivity). Every linear *k-step* method

$$\sum_{i=0}^{k} \alpha_{k-i} y_{n+1-i} = h\sum_{i=0}^{k} \beta_{k-i} f_{n+1-i}, \; f_{n+i} = f(t_n + ih, y_{n+i}), \; \sum_{i=0}^{k} \beta_{k-i} = 1$$

has a "one-leg twin"

$$\sum_{i=0}^{k} \alpha_{k-i} y_{n+1-i} = hf\left(\sum_{i=0}^{k} \beta_{k-i} t_{n+1-i}, \sum_{i=0}^{k} \beta_{k-i} y_{n+1-i}\right) \tag{2.1}$$

The point in which the function is evaluated is named *collocation point.*

For linear autonomous problem the one-leg difference equation is identical to the linear multistep equation. Hence, the stability regions are the same for a linear multistep method and its one-leg twin.

It was realized that, for fixed step size, the one-leg implementation of the equivalent linear *k*-step method would be advantageous with regard to storage economy. For variable step size, in which case they are no equivalent, the one-leg methods seem to have superior stability properties, in stiff problems. On other side, Dahlquist methods are particulary easy to apply to implicit differential equations.

The disadvantage is the decrease of the maximum order versus the multistep formula. In [7], Dahlquist shows that the maximum order of a one-leg formula with k steps is $k + 1$. Therefore, in the case of a one-step method the maximum order is 2.

Second derivative multistep formula. Iteration schemes that have been proposed for stiff equations are usually based on a modified Newton-Raphson technique. The usual predictor-corector iteration scheme is not feasible since $\|h\ df/dy\|$ must remain small to ensure the convergence. Realizing that the Jacobian matrix might be used for the iteration scheme, Enright in 1974 (see reference [9]) consider the possibility of developing a class of formula that explicitly uses the Jacobian matrix. Since $y'' = (df/dy)y'$ for autonomous systems, the above mention author considers the following class of second derivative formula:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} - h\sum_{i=0}^{k} \beta_i f_{n+i} - h^2 \sum_{i=0}^{k} \gamma_i f'_{n+i} = 0, f'_{n+i} = J_{n+i} f_{n+i}, J_{n+i} = \frac{\partial f}{dy}(y_{n+i}) \tag{2.2}$$

If the coefficients γ_i are zero, except for the last, the condition of stability at infinity is ensured and the maximum order of the formula with k steps is $k + 2$.

Example: For the one-step case, the method is *L-stable* as well and of order 3:

$$y_{n+1} = y_n + \frac{h}{3}(2f_{n+1} + f_n) - \frac{h^2}{6} f'_{n+1} \tag{2.3}$$

The local truncation error $TE = y(t_{n+1}) - y(t_n)$ is

$$TE = \frac{h^4}{72} \frac{d^3 f}{dt^3}(y(t_n)) + O(h^5)$$

Problem 1: *Is it possible to build similar one-leg formula for second derivative multistep methods? How great is the lose in the accuracy order of such a formula?*

England's hibrid methods. England (see reference [8]) gives a partial answers to this question. He builp up a θ-class of hybrid methods with the same stability properties as those of the Enright's formula.

Example: For the one-step case, the methods of order 3 have the following form:

$$\begin{cases} y_{n+1} = y_n + h\dfrac{3\theta-1}{6\theta}f_n + h\dfrac{3\theta-2}{6(\theta-1)}f_{n+1} - \dfrac{\theta}{6\theta(\theta-1)}f_{n+\theta} \\ y_{n+\theta} = (\theta-1)^2 y_n - \theta(\theta-2)y_{n+1} + h\theta(\theta-1)f_{n+1} \end{cases} \tag{2.4}$$

The particular member taken into implementation by the author of the above mention paper is in accordance to the condition of a zero coefficient in the first equation for f_n:

$$\begin{cases} y_{n+1} = y_n + \dfrac{h}{4}f_{n+1} + \dfrac{3h}{4}f_{n+1/3}\,, \\ y_{n+1/3} = \dfrac{5}{9}y_{n+1} + \dfrac{4}{9}y_n - \dfrac{2h}{9}f_{n+1} \end{cases}$$

$$TE = -\frac{h^4}{216}\left(\frac{d^3f}{dt^3} - 4\frac{d^2f}{dt^2}\frac{df}{dt}\right)(y(t_n)) + O(h^5) \tag{2.5}$$

The formula certainly is *L-stable* and of order 3.

The derivative calculus is replaced by a function evaluation in a new point.

Highest order. It is possible to derive a second derivative method with upper order that 3. The Obrechkoff's formula is of order 4, but it is only *A-stable:*

$$y_{n+1} = y_n + \frac{h}{2}(f_{n+1} + f_n) - \frac{h^2}{12}(f'_{n+1} - f'_n) \tag{2.6}$$

$$TE = \frac{h^5}{720}\frac{d^4f}{dt^4}(y(t_n)) + O(h^6)$$

Problem 2: *Is it possible to build up hybrids methods of order 4 similar to the Obrechkoff's second derivative formula?* Preserving the convention of only one implicit equation to solve, in y_{n+1} *is it possible to reach better stability properties?*

Exponential fitting. The idea of using exponentially fitted formula for the approximate numerical integration of certain classes of stiff systems has received considerable attention. The basis is to derive integration formula containing free parameters, other than the steplength of integration, and then choose these parameters so that a given exponential function satisfies the integration formula exactly. It needs to be emphasized that exponential fitting is really applicable to only a limited class of stiff systems, i.e.., to systems having an Jacobian which is in some sense slowly varying, with all the eigenvalues of large modulus, lying in two or fewer clusters. However, for systems for which exponential fitting integration formula are substantially more efficient than conventional ones.

When the method is applied to the test equation, the approximation error is related to

$$T(z) = R(z) - e^{-z}, \quad z = h\lambda$$

If for some $q = h\lambda_0$ we have $T(q) = 0$, then the numerical solution of the test equation is exact in the discret meaning. If q is a zero of $d + 1$ multiplicity, we note that $R(z)$ is exponential fitted of order d at $z = q$. In the stiff case the exponential fitting points are the bigest eigenvalues multiplied by the stepsize.

Pseudo-Runge-Kutta methods. Bokhonven's formula. Pseudo Runge-Kutta process are generalizations of the classical methods with the same name. Bokhoven in 1980 (see reference [1]) describes such methods named by the author *Implicit Endpoint Quadrature Formula*. They have the following form:

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i k_i, \; k_i = (1-\theta_i)y_n + \theta_i y_{n+1} + h\sum_{j=1}^{s} a_{ij} k_j, \; i = l(1)s \tag{2.7}$$

Also, the same author established the order conditions.

Example 1: Among others formula, we see some *A-stable* ones of order 3 and 4. The formula of order 4 is the following:

$$\begin{cases} y_{n+1} = y_n + \frac{h}{6}(f_{n+1} + f_n) + \frac{2h}{3} f_{n+\frac{1}{2}}, \\ y_{n+\frac{1}{2}} = \frac{1}{2}(y_{n+1} + y_n) - \frac{h}{8}(f_{n+1} - f_n) \end{cases} \tag{2.8}$$

$$TE = -\frac{h^5}{2880}\left(\frac{d^4 f}{dt^4} - 5\frac{d^3 f}{dt^3}\frac{df}{dt}\right)(y(t_n)) + O(h^6)$$

Example 2: For order 3 are given the England's one-step scheme and:

$$\begin{cases} y_{n+1} = y_n + \frac{h}{2}(f_{n+\theta_1} + f_{n+\theta_2}), \\ y_{n+\theta_1} = \frac{2+\sqrt{3}}{6} y_n + \frac{4-\sqrt{3}}{6} y_{n+1} - \frac{h}{6} f_{n+1}, \ \theta_1 + \frac{3-\sqrt{3}}{6}, \ \theta_2 = \frac{3+\sqrt{3}}{6} \\ y_{n+\theta_2} = \frac{4-\sqrt{3}}{6} y_n + \frac{2+\sqrt{3}}{6} y_n + 1 + \frac{h}{6} f_n \end{cases} \tag{2.9}$$

$$TE = -\frac{h^3}{24}\left(\frac{d^2 f}{dt^2}\frac{df}{dt}\right)(y(t_n)) + O(h^5)$$

3. One leg methods associated to second derivative formula. We consider the class of integration formula

(OLS 1)

$$\sum_{i=0}^{k} \alpha_{k-i} y_{n+1-i} - hf\left(\sum_{i=0}^{k} \beta_{k-i} t_{n+1-i}, \sum_{i=0}^{k} \beta_{k-i} y_{n+1-i}\right) -$$

$$-\frac{h^2}{2}\delta f'\left(\sum_{i=0}^{k} \gamma_{k-i} t_{n+1-i}, \sum_{i=0}^{k} \gamma_{k-i} y_{n+1-i}\right) = 0$$

where $\sum_{i=0}^{k} \gamma_{k-i} = \sum_{i=0}^{k} \beta_{k-i} = 1$.

Order. A statement about the order can be proved similar to that for one-leg methods (like im [7]).

PROPOSITION 1. *The maximum order of a method (OLS) is k + 2 and there are at least k + 1 distinct methods with this property. For these only one coefficient β_i is not zero.*

Proof. We introduce operator notation

$$\rho y_n = \sum_{i=0}^{k} \alpha_{k-i} y_{n+1-i},\ \sigma y_n = \sum_{i=0}^{k} \beta_{k-i} y_{n+1-i},\ \gamma y_n = \sum_{i=0}^{k} \gamma_{k-i} y_{n+1-i}$$

The differentiation error operator L_i and the interpolation error L_i are

$$(L_i\varphi)(t_n) = \rho\varphi(t_n) - h\varphi'(\sigma t_n) - \frac{\delta h^2}{2}\varphi''(\gamma t_n)$$

$$(L_i{}'\varphi)(\sigma t_n) = \sigma\varphi(t_n) - \varphi(\sigma t_n),\ (L_i{}'\varphi)(\gamma t_n) = \gamma\varphi(t_n) - \varphi(\gamma t_n)$$

where φ is a sufficiently smooth function. Then, the local truncation error is

$$Ly(t_n) = \rho y(t_n) - hf(\sigma t_n, \sigma y(t_n)) - \frac{h^2\delta}{2}f'(\gamma t_n, \gamma y(t_n)) =$$

$$= (L_i y)(t_n) + h[f(\sigma t_n, y(\sigma t_n)) - f(\sigma t_n, \sigma y(t_n))] +$$

$$+\frac{h^2\delta}{2}[f(\gamma t_n, y(\gamma t_n)) - f(\gamma t_n, \gamma y(t_n))] \approx$$

$$\approx (L_i y)(t_n) - hf'(\sigma t_n, y(\sigma t_n))(L_i{}'y)(\sigma t_n) - \frac{h^2\delta}{2}f''(\gamma t_n, Y(\gamma t_n))(L_i{}'y)(\gamma t_n)$$

If

$$L\varphi = O(h^p),\ L_{i\varphi} = O(h^{pd}),\ L_i{}'\varphi = O(h^{pI}),\ l_i{}'\varphi = O(h^{pI})$$

then

$$p = \min\{p_i, p_1 + 1, p_2 + 2\}$$

Dahlquist shows in [7] that

$$\max p_1 = \begin{cases} \infty, & \text{if } \exists j : \sigma t_n = t_{n+1-j},\ 0 \le j \le k \\ k, & \text{otherwise} \end{cases}$$

A similar statement holds for p_2. If $p_1 = p_2 = \infty$ we don't have an one-leg methods. Then the maximum order is obtained when $p_1 = \infty$ and $p_2 = k$. If we note $\theta = (t_{n+1} - \gamma t_n)/h$, p_2 is maximized when

$$\gamma_{k-i}(\theta) = \frac{\omega(\gamma t_n)}{(\omega t_n - t_{n+1-i}\omega')(t_{n+1-i})},\quad \omega(t) = \prod_{j=0}^{k}(t - t_{n+1-i})$$

For each j we have k+2 free parameters: $\theta,\delta,\alpha_1,\ldots,\alpha_k$. In these conditions, max $p_d = k+2$ and, in conclusion, max $p = k+2$. For each j, the linear system of condition for order $p_d = k+2$ has at least one solution.

One-step method. We study the particular case of the *one-step formula* with minimum order 2:

$$y_{n+1} = y_n + hf\left(\frac{1+u}{2}y_{n+1} + \frac{1-u}{2}y_n\right) - h^2\frac{u}{2}f'\left(\frac{1+v}{2}y_{n+1} + \frac{1-v}{2}y_n\right)$$

Maximum order. We note that it is possible to eliminate the $O(h^3)$ terms from the truncation error by choosing $u = 1$, $v = 1/3$ or $u = -1$, $v = -1/3$. Then the formula have the optimal order 3. One from these is the following (for an autonomous system):

$$y_{n+1} = y_n + hf(y_{n+1}) - \frac{h^2}{2}f'\left(\frac{2}{3}y_{n+1} + \frac{1}{3}y_n\right) \qquad (3.1)$$

$$TE = -\frac{h^4}{72}\left[\frac{d^3f}{dy^3}f^3 - 3\left(\frac{df}{dy}\right)^3 f\right](y(t_n)) + O(h^5)$$

This truncation error is comparable with the error produced by Enright's one-step method.

Stability properties: In order to examine the stability properties of our formula, we use the maximum modulus theorem. Applying the method of order 2 to the scalar test equation, we obtain the stability function:

$$R(z) = \frac{1 + z(1-u)/2 - z^2u(1-v)/4}{1 - z(1+u)/2 + z^2u(1+v)/4}$$

We notice that $|R(ip)| \le 1$, $\forall\, p \in \mathbb{R}$ if and only if $v \ge 0$. Under this circumstance, the requirement that $R(z)$ may be analytic in $\mathrm{Re}(z) < 0$, i.e., that there are not zero of the denominator of $R(z)$ in the left complex half-plane, is equivalent with $u > 0$. The inequality $\lim_{z\to-\infty}|R(z)| < 1$ is reduced to $v > 0$. Thus, the method is strong *A-stable if and only if* $u > 0$, $v > 0$ and *L-stable when* $v = 1$, $u > 0$. The method of maximum order 3 is only *strong A-stable*.

One-leg associated to second derivative formula. If we consider a second derivative formula of minimum order 2:

$$y_{n+1} = y_n + h\left(\frac{1+a}{2}f_{n+1} + \frac{1-a}{2}f_n\right) - h^2\left(\frac{b+a}{4}f'_{n+1} - \frac{b-a}{4}f'_n\right)$$

we can associated a twin formula of the new class with the same stability function for $u=a$, $v = b/a$. We exclude the case of $a = 0$. Thus, if the second derivative method is L-stable and order 2, we can associate to it a formula (3.1) also L-stable and of order 2.

Example: In the particular case of the 3^{th}-order Enright's method and for an autonomous system, the associated formula is of order 2 and has the following form:

$$y_{n+1} = y_n + hf\left(\frac{2}{3}y_{n+1} + \frac{1}{3}y_n\right) - \frac{h^2}{6}f'(y_{n+1}) \qquad (3.2)$$

The error produced at each step by this method is:

$$TE = \frac{h^3}{9}\left(\frac{d^2f}{dy^2}f^2\right)(y(t_n)) + O(h^4)$$

In the above class of methods the maximum order is obtain when only one β_i is **not** zero. The following class of methods don't suffer this restriction.

$$\text{(OLS 2)} \quad \sum_{i=0}^{k} \alpha_{k-i} y_{n+1-i} - h\sum_{i=0}^{k} \beta_{k-i} f_{n+1-i} - \frac{h^2\delta}{2}f'\left(\sum_{i=0}^{k} \gamma_{k-i} t_{n+1-i}, \sum_{i=0}^{k} \gamma_{k-i} y_{n+1-i}\right) = 0$$

The maximum order is also $k+2$, but there are k free parameters in the set of α_i or β_i.

One-step case. If we take into account the one-step case, we can write the method of minimum order 2:

$$y_{n+1} = y_n = h\left(\frac{1+u}{2}f_{n+1} + \frac{1-u}{2}f_n\right) - h^2\frac{u}{2}f'\left(\frac{1+v}{2}y_{n+1} + \frac{1-v}{2}y_n\right) \qquad (3.3)$$

Order. Unfortunately, it is not possible to reach order 4 with such a method, but the formula of order 3 form a class depending on a free parameter since the unique restriction is $uv = \frac{1}{3}$.

Stability. The A-and L-stability conditions are the same for the above mention class, because the stability function is the same. If the parameter is chosen in such a way that the method is stable at infinity, we get the Enright's one-step formula. If the free parameter is chosen for exponential fitting at $q \in C_- = \{z \in C | Re z < 0\} : R(q) = e^q$, then:

$$u = w = \frac{1}{3v} = \frac{1}{3}\frac{(-q^2+6q-12)e^q+q^2+6q+12}{(q^2-2q)e^q+q^2+2q}$$

The Enright's method can be seen thus as an exponential fitting to -∞ of the formula in discussion, because $\lim_{q\to-\infty} u(q) = \frac{1}{3}$. The exponential fitting to zero is not possible since $\lim_{q\to 0} u(q) = 0$. It is easy to verify analitically that the exponentialy fitted formula is *A-stable* for any $q \in \mathbf{R}^{\bullet}$ because $u(q) > 0$.

4. Hibrid methods. We pose the problem to find new hybrid methods which can replace the Jacobian, which is necessary to calculate for a formula of the above section. We search for a scheme with the following form:

(HM 1)
$$\begin{cases} y_{n+1} = y_n + \frac{1+n}{2} hf(vy_{n+1} + (1-v)y_n) + \frac{1-x}{2} hf_{n+1} \\ y_{n+1} = wy_{n+1} + (1-w)y_n + \frac{1}{2}(\theta - w + x)f_{n+1} + \frac{1}{2}(\theta - w - x)f_n \end{cases}$$

The first equation is known as the quadrature formula and the second, as the interpolation formula.

Order. The maximum order of these schemes is 3. If we take into account the conditions of order 3 for the quadrature formula and of order 2 for the interpolation formula, the hybrid scheme preserves the order 3 of the quadrature formula. We get t[illegible]sses that depend each on a parameter of the second equation. The coefficients are given by $n = -\frac{1}{2}$,

$x=-\frac{2}{9}$ and in a *first case* $v=0$, $\theta=\frac{2}{3}$ or in a *second case* $v=1$, $\theta=\frac{1}{3}$, where w remain a free parameter. The condition of order 3 for the interpolation formula give two methods with the error coefficient equal to the quadrature formula. One, for $\theta=\frac{2}{3}$, is

$$\begin{cases} y_{n+1} = y_n + \frac{h}{4}f_n + \frac{3h}{4}f_{n+2/3}, \\ y_{n+2/3} = \frac{20}{27}y_{n+1} + \frac{7}{27}y_n - \frac{4h}{27}f_{n+1} + \frac{2h}{27}f_n \end{cases} \tag{4.1}$$

$$TE = \frac{h^4}{216}\left(\frac{d^3f}{dt^3}\right)(y(t_n)) + O(h^4)$$

The local truncation error is lower that the one of the Enright,s one-step formula and some system functions f, lower that the one of England's method.

Stability. Strong *A-stability* take place, in the first case, if $w > \frac{2}{3}$, and, in the second case, if $w > \frac{1}{3}$. Thus, the above method is an example of strongly A-stable class member.

Exponential fitting. In the case of order 2, the free parameter w is possible to be used for exponential fitting. In the first case we get:

$$w(q) = \frac{4}{9}\frac{(q^2-6)e^q + 2q^2 + q + 6}{(q^2-2q)e^q + q^2 + 2q}, \quad \lim_{q\to-\infty} w(q) = \frac{8}{9}, \quad \lim_{q\to 0} w(q) = \frac{2}{3}$$

and the formula is *strongly A-stable* for any $q \in \mathbf{R}^{*}$. In the second case:

$$w(q) = \frac{1}{9}\frac{(q^2+6q-24)e^q + 5q^2 + 18q + 24}{(q^2-2q)e^q + q^2 + 2q}, \quad \lim_{q\to-\infty} w(q) = \frac{5}{9}, \quad \lim_{q\to 0} w(q) = \frac{1}{3}$$

Also in this case the corresponding formula is *strongly A-stable* for any $q \in \mathbf{R}^{*}$.

If, instead of the exponential fitting, we put the condition of stability at infinity then we get two methods. One England's one-step method for $w = \frac{5}{9}$ and is the last formula exponential fitted at $-\infty$. Another, in the case $\theta = \frac{2}{3}$, $w = \frac{8}{9}$ is also a formula exponential fitted at $-\infty$:

$$\begin{cases} y_{n+1} = y_n + \frac{h}{4} f_n + \frac{3h}{4} f_{n+2/3}, \\ y_{n+2/3} = \frac{8}{9} y_{n+1} + \frac{1}{9} y_n - \frac{2h}{9} f_{n+1} \end{cases} \tag{4.2}$$

$$TE = \frac{h^4}{216}\left(\frac{d^3 f}{dt^3} + 2\frac{d^2 f}{dt^2}\frac{df}{dt}\right)(y(t_n)) + O(h^5)$$

Thus, the formula is comparable with England's one step scheme and Enright's formula. It is a better alternative as (4.1), because it has the property of *L-stability.* The formula (4.2) is also included in the θ-class described by England, which replace the Enright's method and contains only L-stabe methods.

The England's θ-class contains all L-stable methods with the minimum order 1 in both equations of the following formula class:

(HM 2)
$$\begin{cases} y_{n+1} = y_n + h\left(\frac{u+v}{2} f_{n+1} + \frac{u-v}{2} f_n\right) - hv f_{n+1} \\ y_{n+1} = w y_{n+1} + (1-w) y_n + h\frac{q-v+g}{2} f_{n+1} + h\frac{q-v-g}{2} f_n \end{cases}$$

It is easier to see that for this class there is an increasing in maximum order of accuracy versus the class (HM 1). The conditions of order 4 lead to the Bokhoven's formula (2.8), which has the same stability function as the Obrechkoff's method, thus it is only *A-stable.*

To improve the performance of Bokhoven's 3^{th}-order scheme (2.9), we study the class:

(HM 3)
$$\begin{cases} y_{n+1} = y_n + h\frac{l+n}{2} F_{n+l_1} + h\frac{l-n}{2} f_{n+l_2} \\ y_{n+l_1} = w_1 y_{n+1} + (1-w_1) y_n + \frac{h}{2}(\theta_1 - w_1 + x_1) f_{n+1} + \frac{h}{2}(\theta_1 - w_1 - x_1) f_n \\ y_{n+l_2} = w_2 y_{n+1} + (1-w_2) y_n + \frac{h}{2}(\theta_2 - w_2 + x_2) f_{n+1} + \frac{h}{2}(\theta_2 - w_2 - x_2) f_n \end{cases}$$

Order. If we require the conditions of order 3, the methods depend on three free parameters: u, w_1, w_2:

$$\theta_1 = \frac{1}{2} \pm \frac{\sqrt{3(1-u^2)}}{6(1+u)},\ \theta_2 = \frac{1}{2} \mp \frac{\sqrt{3(1-u^2)}}{6(1-u)},\ x_1 = -\frac{1+2u}{6(1+u)},\ x_2 = \frac{2u-1}{6(1-u)}$$

Stability. We consider the case of order 3. The stability function is the following:

$$R(z) = \frac{1+(1-2t)z+(1/3-t)z^2}{1-2tz-(1/6-t)z^2},\quad t = \frac{1+u}{4}w_1 + \frac{1-u}{4}w_2$$

The condition of *strong A-stability is* $t > \frac{1}{4}$. Bokhoven's method of order 3 is only *A-stable*, because $t = \frac{1}{4}$. When $t = \frac{1}{3}$, we get a class which depends on two parameters (for example w_1,w_2) with the same linear stability properties as the Enright's method. If we take in account the case $w_1 = w_2 = \frac{2}{3}$, then u is a free parameter and it can be choose for minimising error.

Example 1. In the case of L-stability, if the parameters are choosed such that all equations of (HM 3) are of order 3, then one of the solutions is the next:

$$\begin{cases} y_{n+1} = y_n + \frac{2+\sqrt{3}}{4}hf_{n+\sqrt{3}/3} + \frac{2-\sqrt{3}}{4}hf_{n-\sqrt{3}/3} \\ y_{n+\sqrt{3}/3} = \left(1-\frac{2\sqrt{3}}{9}\right)y_{n+1} + \frac{2\sqrt{3}}{9}y_n - \frac{\sqrt{3}}{9}(\sqrt{3}-1)hf_{n+1} + \frac{2\sqrt{3}}{9}(2-\sqrt{3})hf_n \\ y_{n-\sqrt{3}/3} = \left(1+\frac{2\sqrt{3}}{9}\right)y_{n+1} - \frac{2\sqrt{3}}{9}y_n - \frac{\sqrt{3}}{9}(\sqrt{3}+1)hf_{n+1} - \frac{2\sqrt{3}}{9}(2+\sqrt{3})hf_n \end{cases} \quad (4.3)$$

$$TE = \frac{h^4 d^3 f}{72\, dt^3}(y(t_n)) + O(h^5)$$

This method is an important one. We observe the identity on the local truncation error and on the form of stability function with the Enright's method. Similar efects with the classical one-step method are obtain, but the evaluation of the Jacobian matrix to each Newton iteration step is replaced with 2 functions evaluations. Here we did not find in the form of the local trucation error some perturbation produced by the interpolation equations, like in Bokhoven's one-step method.

Example 2. For such methods it is possible to reach order 4. The unique method is the following:

$$\begin{cases} y_{n+1} = y_n + \frac{h}{2} f_{n+1/2+\sqrt{3}/6} + \frac{h}{2} f_{n+1/2-\sqrt{3}/6} \\ y_{n+1/2\pm\sqrt{3}/6} = \left(\frac{1}{2} \pm \frac{2\sqrt{3}}{9}\right) y_{n+1} + \left(\frac{1}{2} \mp \frac{2\sqrt{3}}{9}\right) y_n - \frac{h}{6}\left(\frac{1}{2} \pm \frac{\sqrt{3}}{6}\right) f_{n+1} + \frac{h}{6}\left(\frac{1}{2} \mp \frac{\sqrt{3}}{6}\right) f_n \end{cases} \tag{4.4}$$

$$TE = \frac{h^5}{4320}\left(\frac{d^4 f}{dt^4} + 5\frac{d^3 f}{dt^3}\frac{df}{dt}\right) + O(h^6)$$

Analysing the stabil function, we can see that this is the same as for Obrechkoff's formula. Thus, the method is only *A-stable*. We observe that, for the most of the system functions f, the level of local truncation error is lower than the one of Obrechkoff's formula or Bokhoven's scheme (2.8). The computational effort is the same like for (2.8).

In scheme Bokhoven's (2.9) are needed 3 functions evaluation per Newton step: in $y_{n+\theta_1}$, $y_{n+\theta_2}$, y_{n+1}. One step needs $3m + 1$ function evaluations, where m is the iteration number taken to solve the implicit equation; m is relatively small if the starting value is good.

We consider now a different class for which are also necessary 3 evaluations:

(HM 4)
$$\begin{cases} y_{n+1} = y_n + ahf(uy_{n+1} + (1-u)y_n) + bhf(y_{n+1}) + chf(vy_{n+1} + (1-v)y_n) \\ y_{n+\theta} = wy_{n+1} + (1-w)y_n + hdf(uy_{n+1} + (1-u)y_n) + hef(vy_{n+1} + (1-v)y_n) \end{cases}$$

Order. The conditions of 3-order are:

$a + b + c = 1, \quad \theta = w + d + e, \quad \theta^2 = w + 2(du + ev)$

$$au + b\theta + cv = \frac{1}{2}, \quad au^2 + b\theta^2 + cv^2 = \frac{1}{3}, \quad au + b\theta^2 + cv = \frac{1}{3}$$

Under these circumstances, a formula of order 3 has three free parameters, u, v, θ.

The local error produced by the quadrature formula is:

$$TE_1 = -\frac{h^4}{36}\left[\left(\frac{1}{2}-\theta\right)\frac{d^3f}{dt^3} + uv\frac{3\theta^2-4\theta+1}{\theta^2-\theta}\frac{d^3f}{dy^3}f^3\right](y(t_n)) + O(h^5)$$

To this is added the error produced by the approximation of y_{n+1} with the integration formula:

$$TE_2 = \frac{h^4}{36}\left[(1-\theta)\frac{d^2f}{dt^2} - \frac{(u+v-1)\theta-[6(\theta^2-\theta)+1]uv}{\theta^2-\theta}\left(\frac{df}{dy}\right)^2 f\right]\frac{df}{dy}f(y(t_m)) + O(h^5)$$

We observe that the error formula introduced by England's one-step method is a particular of the above. The adventage of using this class is the dependence on more parameters and the possibility to reach order 4 of formula. The method of maximum order in this class is the Bokhoven's formula (2.8).

Stability. If we impose the suplementary condition of stability at infinity, the new equation is:

$$d(1-u) + e(1-v) = 0$$

Then the coefficients are:

$$a = \frac{v-\frac{1}{2}}{v-u} + \frac{v-\theta}{6(\theta^2-\theta)(v-u)}, \quad c = \frac{u-\frac{1}{2}}{u-v} + \frac{u-\theta}{6(\theta^2-\theta)(u-v)}, \quad b = -\frac{1}{6(\theta^2-\theta)}$$

$$d = \frac{(\theta^2-\theta)(v-1)}{v-u}, \quad e = \frac{(\theta^2-\theta)(u-1)}{u-v}, \quad w = 2\theta-\theta^2$$

where θ is the solution of the equation

$$[6uv-3(u+v)+3]\theta^2 - [6uv-2(u+v)+2]\theta + uv = 0$$

These methods, depending on u and v, are all *L-stable* because the stability function is the same as that the Enright's method and that of the θ-*class* of England.

Examples:

1) $u = 0$, $v = 1$ or $u = 1$, $v = 0$ gives the England's θ-class;

2) $v = 0$, $\theta = \frac{2}{3}$ gives a class of formula depending on u:

$$\begin{cases} y_{n+1} = y_n + \frac{3h}{4} f_{n+2/3} + \frac{h}{4} f_n, \\ y_{n+2/3} = \frac{8}{9} y_{n+1} + \frac{1}{9} y_n - \frac{2h}{9n} f(u y_{n+1} + (1-u) y_n) + \frac{2(1-n)h}{9n} f_n, \end{cases} \tag{4.5}$$

$$TE = \frac{h^4}{216}\left(\frac{d^3 f}{dt^3} + 2\frac{d^2 f}{dt^2}\frac{df}{dt} + 18(u-1)\frac{d^2}{dy^2} f^2 \frac{df}{dt}\right)(y(t_n)) + O(h^5)$$

For $u = 1$ we get scheme (4.2).

3) $v = 1$, $\theta = \frac{1}{3}$ gives the England's one-step method;

4) if $u + v = 1$, then

$$\theta = \frac{1}{2} \pm \frac{\sqrt{3}}{6},\ b = 1,\ d = \frac{u}{6(u-v)},\ e = \frac{v}{6(u-v)},\ w = \frac{1}{6} + \theta,\ a = \frac{\theta - \frac{1}{2}}{v-u},\ c = \frac{\theta - \frac{1}{2}}{u-v}$$

The local truncation error is:

$$TE = -\frac{h^4}{36}\left[\pm\frac{\sqrt{3}}{6}\left(\frac{d^3 f}{dt^3} + 6u(1-u)\frac{d^3 f}{dy^3} f^3\right) - \left(\frac{1}{2} \mp \frac{\sqrt{3}}{6}\right)\frac{d^2 f}{dt^2}\frac{df}{dy} f\right](y(t_n)) + O(h^5)$$

If we consider $1 - 6u(1-u) = 0$ and if by convention $u > v$, then for $u = \frac{1}{2} + \frac{\sqrt{3}}{6}$ the formula is the following:

$$\begin{cases} y_{n+1} = y_n + hf\left(y_{n+\frac{1}{2} \pm \frac{\sqrt{3}}{4}}\right) \mp \\ \mp \frac{1}{2} hf\left(\left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right) y_{n+1} + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right) y_n\right) \pm \frac{1}{2} hf\left(\left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right) y_{n+1} + \left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right) y_n\right) \\ y_{n+\frac{1}{2} \pm \frac{\sqrt{3}}{4}} = \left(\frac{2}{3} \pm \frac{\sqrt{3}}{6}\right) y_{n+1} + \left(\frac{1}{3} \mp \frac{\sqrt{3}}{6}\right) y_n - \\ - \frac{1+\sqrt{3}}{12} hf\left(\left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right) y_{n+1} + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right) y_n\right) + \frac{\sqrt{3}-1}{12} hf\left(\left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right) y_{n+1} + \left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right) y_n\right) \end{cases} \tag{4.6}$$

$$TE = \mp\frac{\sqrt{3}h^4}{216}\left(\frac{d^4f}{dt^4}+\frac{d^3f}{dy^3}f^3\right)(y(t_n)) + \frac{3\mp\sqrt{3}}{216}\left(\frac{d^2f}{dt^2}\frac{df}{dy}f\right)(y(t_n))+O(h^5)$$

which for some functions f is possible to produce a lower error that the one of the Enright's one-step method.

4. Numerical results.

We have been testing the following schemes:

No.	Scheme	Stability	Effort at one step
		Order 3	
(2.3)	-Enright's second derivative formula	L-stable	1 Function, 1 Derivative
(2.5)	-England's hybrid one-step scheme	L-stable	2 Function, 0 Derivative
(3.1)	-from class (OLS 1)	stongly A-stable	1 Function, 1 Derivative
(4.1)	-from class (HM 1)	stongly A-stable	2 Function, 0 Derivative
(4.2)	-from class (HM 1)	L-stable	2 Function, 0 Derivative
4.6.1)	-for $\theta=1/2+\sqrt{3}/6$, from class (HM 4)	L-stable	3 Function, 0 Derivative
4.6.2)	-for $\theta=1/2-\sqrt{3}/6$, from class (HM 4)	L-stable	3 Function, 0 Derivative
		Order 4	
(2.6)	-Obrechkoff's second derivative formula	A-stable	1 Function, 1 Derivative
(2.8)	-Bokhoven's hybrid scheme	A-stable	2 Function, 0 Derivative
(4.4)	-from class (HM 3)	A-stable	3 Function, 0 Derivative

These formula have been implemented in a constant stepsize method.

The above methods suppose to solve some equation in y_{n+1}:

$$F(y_{n+1}) = 0$$

where F depends on the choosed method. The iteration scheme adapted to solve the implicit set of equations is a modified Newton-Raphson technique:

$$F'(y_n)(y_{n+1}^{(i-1)} - y_n^{(i)}) = -F(y_n^{(i)}),\quad i \geq 0,\quad y_{n+1}^{(0)} = y_n + hf_n$$

The starting value is given by the Euler explicit formula.

The numerical process consists of the following stages at each step:

Stage 1.1. The evaluation of the specific linear system matr

$F'(y_n) = I - ahJ_n + bh^2J_n^2$, with specific couple (a,b): $a = 2/3$, $b = 1/6$ f

(2.3), (2.5), (4.2), (4.6), $a = 5/9$, $b = 1/9$ for (4.1), $a = 1$, $b = 1/3$ for (3.1) ar

$a = 1/2$, $b = 1/12$ for (2.6), (2.8), (4.4).

Stage 1.2. Compute $F(y_{n+1}^{(i)})$;

Stage 2.2. Solve the linear system:

(0) the decomposition $LU = F'(y_n)$, U upper triangular matrix, L lower triangular matrix;

(1) solve the system $Lx = -F(t_{n+1}^{(i)})$;

(2) solve the system $Ud = x$;

(3) compute $y_{n+1}^{(i+1)} = y_n^{(i)} + d$, $i \leftarrow i+1$;

Stage 2.3. If $\|d\|_2 \geq$ *tolerance* and if the iteration number exceed a certain limit the GoTo *stage 2.1.*; in the opposite case, if the maximal number of iteratio steps has been overpassed, an error mesage is printed and *finish step*, otherwise continue;

Stage 3. Evaluate function at the approximation $y_{n+1}^{(i)}$ necessary for the following ste and store the values and the Jacobian if the method asks for it.

Finish step. Continue with the next step.

The efficiency of the methods has been measured by independent machine statistic like the number of function calls, Jacobian evaluations, and matrix inversions.

The numerical results appear in the following tables. We have noted: *xxF* = the function evaluation number, *xxD* = the derivative evaluation number, *xxS* = the number of linear systems solved. By * we have indicated the method with the lowest error for a certain choise of the step and system component.

The comparison was drawn between the exact solution and the solutions given by the methods for each system, at the point $t = 1$.

The number of iterations depends on the chosen steplength. The stepsize is indicated in the table head. The possible values are 0.1, 0.05 or 0.01, depending on the required condition of convergence of the methods in discussion.

The following testing systems are known to be stiff:

System (S1):

$$\begin{cases} y_1'(t) = -4498y_1(t) - 5996y_2(t) + 0.006 - t \\ y_2'(t) = 2248.5y_1(t) + 2997y_2(t) - 0.503 + 3t, \quad y(0) = \begin{pmatrix} 25498/1500 \\ -16499/1500 \\ 1 \end{pmatrix}, \\ y_3'(t) = -y_3(t) \end{cases}$$

$$\textit{Exact:} \begin{cases} y_1(t) = -2e^{-t} + 7e^{-1500t} + \dfrac{17998-14991t}{1500} \\ y_2(t) = 1.5e^{-t} - 3.5e^{-1500t} - \dfrac{13499-11245.5t}{1500}, \quad y(1) = \begin{pmatrix} 1.268908 \\ -0.9505142 \\ 0.3678795 \end{pmatrix} \\ y_3(t) = e^{-t} \end{cases}$$

System (S2):

$$\begin{cases} y_1'(t) = -6y_1(t) + 5y_2(t) + 2\sin t \\ y_2'(t) = 94y_1(t) - 95y_2(t) \qquad , \quad y(0) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}; \\ y_3'(t) = -1000y_3(t) - y_3^2(t) \end{cases}$$

$$\textit{Exact}: \begin{cases} y_1(t) = \dfrac{94}{99}e^{-t} + \dfrac{1}{10001}\left(\dfrac{10}{99}e^{-1000t} - 9496\cos t + 9506\sin t\right) \\ y_2(t) = \dfrac{94}{99}e^{-1} + \dfrac{1}{10001}\left(-\dfrac{108}{99}e^{-1000t} - 9494\cos t + 9306\sin t\right) \\ y_3(t) = \dfrac{1000}{1-(1+1000)e^{1000t}} \end{cases} \quad y(1) = \begin{pmatrix} 0.6361023 \\ 0.6193826 \\ 0 \end{pmatrix}$$

System (S3):

$$\begin{cases} y_1'(t) = -0.013y_1(t) - 1000y_1(t)y_3(t) \\ y_2' = -2500t_2(t)y_3(t) \\ y_3'(t) = -0.013y_1(t) - 1000y_1(t)y_3(t) - 2500y_2(t)y_3(t) \end{cases}, \quad y(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\textit{Exact}\ [10]: y(1) = \begin{pmatrix} 0.99073192 \\ 1.00926441 \\ -0.00000367 \end{pmatrix}$$

Analysing the results, we can see that the proposed methods in the present paper have all good stability properties and implementation performances. Some method are indicated for solving special stiff systems like methods (4.1) and (4.4) for the system (S1) or (S3) and (4.2) the system (S2).

Although it is difficult to draw any definite conclusions from these limited results, a general pattern is indicated. It appears that our methods are at least comparable to the classical ones and therefore worth considering in a comprehensive comparison. However, we do feel that our results indicate that a properly implementation version of our algorithms should be useful for the numerical integration of stiff diferential systems. We expect that, in the case of a variable steplength, those new methods have better properties that classical methods.

Method	(S1)	Effort	(S2)	Effort	(S3)	Effort
	h=0.01		h=0.05		h=0.1	
	1.195369	302F	.6904017	64F	.9916564	23F
Enright (2.3)	- .8953443	302D	.6731095	64D	1.00834	23D
	.3690993	202S	-6.539034E-30*	44S	-3.670194E-6*	14S
	1.205755	504F	.6834461	114F	.9916568	39F
England (2.5)	- .9031347	100D	.6661996	20D	1.00834	9D
	.3678792	202S	-6.546161E-30	47S	-3.670197E-6	15S
	1.174317	510F	.7059933	124F	.9916567	67F
OLS 1 (3.1)	- .8795553	305D	.6886706	72D	1.008339	38D
	.3715501	205S	-2.77061E-7	52S	-3.676911E-6	29S
	1.205853*	506F	.6830078	128F	.9916446*	67F
HM 1 (4.1)	- .9032077*	100D	.6657606	20D	1.008352*	9D
	.3678793*	203S	-4.812774E-9	54S	-3.662948E-6	29S
	1.205848	504F	.683004*	114F	.9916566	41F
HM 1 (4.2)	- .9032041	100D	.6657568*	20D	1.00834	9D
	.3678792	202S	-6.539737E-30	47S	-3.670196E-6	16S
	1.205755	706F	.6834486	158F	.9916787	48F
HM 4 (4.6.1)	- .9031349	100D	.6661998	20D	1.008318	9D
	.3678792	202S	-6.579986E-30	46S	-3.670312E-6	9S
	1.205758	706F	.6834486	158F	.9916565	54F
HM 4 (4.6.2)	- .9031366	100D	.6661998	20D	1.00834	9S
	.3678792	202S	-6.568837E-30	46S	-3.670195E-6	15S
	h=0.01		h=0.01		h=0.01	
	1.196627	304F	.6464161	296F	.9907243	202F
Obrechkoff	- .8962881	304D	.6295791	296D	1.009272	202D
(2.6)	.3678732	204S	0*	196S	1.009272	202D
	1.205815*	510F	.645691*	464F	-3.665286E-6	102S
Bokhoven	- .931796*	100D	.6288624*	100D	.9907317	310F
(2.8)	.3678794*	205S	0*	182S	-3.665327E-6*	105S
	1.205753	712F	.6457027	646F	.9907318*	415F
HM 3 (4.4)	- .9031334	100D	.6288741	100D	1.009264*	100D
	.3678794*	204S	0*	182S	-3.665327E-6*	105S

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REFERENCES

1. van Bokhoven, W.M.G., *Efficient Higher Order Implicit One-step Methods for Integration of Stiff Differential Equation*, BIT **20**, (1980), 34-43.
2. Caira R., Costabile C., Costabile F., *A Class of Pseudo Runge-Kutta Methods*, BIT **30**, 1990, 642-649.
3. Carrol J., *A Composite Integration Scheme for the Numerical Solution of System of Ordinary Differential Equations*, Journal of Computational and Applied Mathematics **33**, 1990, 245-259.
4. Cash J.R., Singhal A., *Mono-implicit Runge-Kutta for the Numerical Integration of Stiff Differential Systems*, IMA Journal of Numerical Analysis **2**, 1982, 211-227.
5. Cash J.R., *A Class of Implicit Runge-Kutta Methods for the Numerical Integration of Stiff Ordinary Differential Equation*, Journal of the Association for Computing Machinery **22**, 1975, 504-511.
6. Cooper G.J., Sayfy A., *Additive Runge-Kutta Methods for Stiff Differential Equations*, Mathematics of Computation 40, no. **161**, 1983, 207-218.
7. Dahlquist G., *On One-leg Multistep Methods*, SIAM Journal of Numerical Analysis 20, no. **6**, 1983, 1130-1138.
8. England R., *Some Hybrid Implicit Stiffy Stable Methods for Ordinary Differential Equation*, in Numerical Analysis. Proceedinds of the Third IIMAS Workshop held at Cocoyoc, Mexico, January 1981, Lectures Notes in Mathematics **909**, 1982.
9. Enright H.W., *Second Derivative Multistep for Stiff Ordinary Differential Equation*, SIAM Journal of Numerical Analysis 11, no. **2**, 1974, 321-331.
10. Lautsch M., *An Implicit Off-step Point Method for the Numerical Integration of Stiff Differential Equation*, Computing **31**, 1983, 177-183.
11. Patricio F., *A Class of Hybrid Formula for the Numerical Integration of Stiff Systems*, BIT **23**, 1983, 360-369.
12. Petcu D., *New Methods for Solving Stiff Differential Equation Systems*, Analele Universităţii din Timişoara, vol XXVI, fasc. **3**, 1988, Seria Ştiinţe Matematice, 67-72.
13. Petcu D., *Hibrid Methods for Stiff Differential Equations*, Preprint 92-20, Universitat Heidelberg, Heidelberg 1992.

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SPLINE APPROXIMATIONS FOR SECOND ORDER NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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REZUMAT. - Aproximaţii spline pentru ecuaţii diferenţiale de ordinul al doilea cu argument întârziat de tip neutral. Se consideră un procedeu de colocaţie cu funcţii spline polinomiale de grad coborât pentru rezolvarea numerică a ecuaţiilor diferenţiele de ordinul al doilea cu argument întârziat de tip neutral. Legătura cu metodele discrete ale multipaşilor este punctul cheie în studiul convergenţei metodelor spline. Se studiază estimarea erorii şi convergenţa metodelor spline pentru aproximaţiile spline de gradul trei şi patru.

Abstract. A collocation procedure with polynomial spline functions of low degree is considered for numerical solution of a second order initial value problem for neutral delay differential equations. A connection with discrete multistep methods is the ingredient in the study of the convergence of the spline methods. The estimation of the error as well as the convergence of cubic and quartic spline approximations is investigated.

1. Introduction. In recent years, there has been a growing interest in the numerical treatment of differential equations with deviating argument; see [5] - [11]. Because of the versatility of such equations in the modelling of processes in various applications, especially physics, engineering, biomathematics, medical science, economics, etc., neutral delay differential equations provide the best and some times the only realistic simulation of observed phenomena.

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Recently many authors [13,14,19,20,22] have proposed methods to approximate the solution of differential equations with deviating argument by means of spline functions. It seems that the spline approximating solutions for such kinds of equations possess some advantages over other methods.

In this paper, we consider a spline approximation for the numerical solution of second order neutral delay differential equations with given initial conditions. The purpose of the present study is to extend some results from the ordinary case to the second order neutral one. In a slightly modified manner, we shall construct a spline approximation solution and also we shall investigate the estimation of the errors and the convergence of the given procedure for cubic and quartic splines. The notation used is this paper is taken from [7], [9] and [18]-[20].

2. Description of the spline collocation method. Consider the following second order initial value problem for neutral delay differential equations:

$$\begin{aligned} &y''(t) = f(t,y(t), y(g(t)), y'(g(t))), \quad t \in [a,b], \\ &y(t) = \phi(t),\ y'(t) = \phi'(t), \quad t \in [\alpha,a],\ \alpha \le a < b. \end{aligned} \tag{2.1}$$

The function g, called the delay function, is assumed to be continuous on the interval $[\alpha,b]$, and to satisfy the inequality $\alpha \le g(t) < t$, $t \in [a,b]$, and $\phi \in C^{m-1}[\alpha,a]$, where $m > 2$. Assume that the functional

$$f: [a,b] \times C^1[a,b] \times C^1[\alpha,b] \times C[\alpha,b] \to R$$

satisfies the following conditions H_1 and H_2:

H_1. For any $x \in C^1[\alpha,b]$, the mapping $t \to f(t,x(t),x(\cdot),x'(\cdot))$ is continuous on $[a,b]$.

H_2. The following Lipschitz condition holds:

$$\|f(t,x_1(t),y_1(\cdot),z_1(\cdot)-f(t,x_2(t),y_2(\cdot),z_2(\cdot))\|$$
$$\leq L_1(\|x_1-x_2\|_{[\alpha,t]}+\|y_1-y_2\|_{[\alpha,t-\delta]}+\|z_1-z_2\|_{[\alpha,t-\delta]})$$
$$+L_2\|z_1-z_2\|_{[\alpha,t]}\ ,$$

with $L_1 \geq 0$, $0 \leq L_2 < 1$, $\delta > 0$, for any $t \in [a,b]$, $x_1,x_2 \in C^1[a,b]$, $y_1,y_2,z_1,z_2 \in C[\alpha,b]$. Under conditions H_1 and H_2, the problem (2.1) has a unique solution $y \in C^2[a,b] \cap C[\alpha,b]$; see [7,12]. For a discussion of the qualitative behaviour of the solution y, in particular the presence of jump discontinuities in the higher derivatives caused by the delay function g, the reader is referred to [4], for example. Jump discontinuities can occur in various higher derivatives of the solution even if f,g,ϕ are analytic in their arguments. Such jump discontinuities are caused by the delay function g and propagate from the point a as the order of derivative increases. We denote the jump discontinuities by $\{\xi_i\}$ which are the roots of the equations $g(\xi_i) = \xi_{i-1}$; $\xi_0 = a$ is the jump discontinuity of ϕ. Since in this paper g does not depend on y (no statedependent delay) we can consider the jump discontinuities to be known for sufficiently high order derivatives and to be such that

$$\xi_0 < \xi_1 <\ldots< \xi_{k-1} < \xi_k < \ldots < \xi_M.$$

We shall construct a spline approximating function $s \to R$, $s \in S_m$ (the polynomial spline function space of degree m and continuity class C^{m-1}) which is defined on each interval $[\xi_j,\xi_{j+1}]$, $j = 0,\ldots,M-1$. For this construction, we use the modified collocation method as in [2] and [18]-[20].

Consider the interval $[\xi_j,\xi_{j+1}]$, $0 \leq j \leq M-1$, subdivided by a uniform partition defined by the knots

$$\xi_j < t_0 < t_1 < \ldots < t_k < t_{k+1} <\ldots < t_N = \xi_{j+1},$$

where $t_k = t_0 + kh$ and $h = (\xi_{j+1} - \xi_j)/N$. The spline function s approximating the solution of (2.1) isdefined on each subinterval $[t_k,t_{k+1}]$ by

$$s(t) = \sum_{i=0}^{m-1} \frac{s^{(i)}(t_k)}{i!}(t-t_k)^i + \frac{a_k}{m!}(t-t_k)^m, \tag{2.2}$$

where $s^{(i)}(t_k)$, $0 \le i \le m-1$, are left-hand limits of the derivatives as $t \to t_k$ of the segment of s defined on $[t_{k-1},t_k]$, and the parameter a_k is determined from the following collocation conditions:

$$\begin{gathered} s_j''(t_{k+1}) = f(t_{k+1}, s_j(t_{k+1}, s_{j-1}(g(t_{k+1})), s_{j-1}'(g(t_{k+1}))), \\ J = 0,\dots,M, \; k = 0,\dots,N-1, \end{gathered} \tag{2.3}$$

where $s_j = s|I_j$, $I_j = [t_{j-1},t_j]$, and $s_{-1} = \phi$. This procedure yields a spline approximating function $s \in S_m$ over the entire interval $[\alpha,b]$. It remains to show that, for h sufficiently small, the parameter a_k, $0 \le k \le N$, can be uniquely determined from (2.3).

THEOREM 2.1. *If f satisfies conditions H_1 and H_2, $\phi \in C^{m-1}[\alpha,a]$, $\alpha \le g(t) \le t$, $t \in [\alpha,b]$, and if h is sufficiently small, then there exists a unique spline approximating solution of problem* (2.1) *given by* (2.2)-(2.3).

Proof. It suffices to prove that a_k can be uniquely determined from (2.3). Substituting (2.2) in (2.3), we have

$$a_k = \frac{(m-2)!}{h^{m-2}} \left\{ f(t_{k+1}, A_k(t_{k+1}) + \frac{a_k}{m!}h^m, s_{k-1}(g(t_{k+1})), s_{k-1}'(g(t_{k+1}))) - A_k''(t_{k+1}) \right\}, \tag{2.4}$$

where

$$A_k(t) = \sum_{i=0}^{m-1} \frac{s^{(i)}(t_k)}{i!}(t-t_k)^i.$$

If brevity we denote (2.4) by

$$a_k = G_k(a_k), \tag{2.5}$$

using assumption H_2 for $h < [m(m-1)/L_1]^{1/2}$ the application $G : R \to R$ is a contraction and (2.4) has a unique solution a_k which can be found by iteration. This completes the proof of the theorem.

In order to make a connection between the spline method and discrete multistep methods (see), we present the following theorem which gives the relationship between the value of any spline function $s \in S_m$ and its second derivative at the knots ("consistency relation").

THEOREM 2.2. [17] *For any spline function $s \in S_m$, $m \geq 3$, there exists a unique linear consistency relation between the quantities $s(t_k)$ and $s''(t_k)$, $k = 0,1,\ldots,m-1$, namely*

$$\sum_{j=0}^{m-1} a_j^{(m)} s(t_{j+v}) = h^2 \sum_{j=0}^{m-1} b_j^{(m)} s''(t_{j+v}), \quad 0 \leq v \leq N-m+1 \tag{2.6}$$

where

$$a_j^{(m)} = (m-1)![Q_{m-1}(j+1) - 2Q_{m-1}(j) + Q_{m-1}(j-1)],$$

$$b_j^{(m)} = (m-1)!Q_{m-1}(j+1), \tag{2.7}$$

and

$$Q_k(x) = \frac{1}{(k-1)!}\sum_{i=0}^{k}(-1)^i \binom{k}{i}(x-i)_+^{k-1}.$$

THEOREM 2.3. *The values $s(t_k)$, $k = 0,1,\ldots,N$, of the spline function are precisely the values furnished by the discrete multistep method defined by*

$$\sum_{j=0}^{m-1} a_j^{(m)} y_{j+k} = h^2 \sum_{j=0}^{m-1} b_j^{(m)} y_{j+k}'', \quad k = 0,1,2,\ldots, \tag{2.8}$$

if the starting values

$$y_0 = s(t_0),\ y_1 = s(t_0+h), \ldots, y_{m-2} = s(t_0+(m-2)h), \tag{2.9}$$

are used.

Proof. For $h < [m(m-1)/L_1]^{1/2}$, only $\{y_j\}_{j=0}^{N}$ satisfies (2.8) with the starting values(2.9). By (2.6), the sequence $\{s(t_j)\}, j = 0,1,\ldots,$ satisfies (2.8) and obviously has starting values (2.9). Thus the values $s(t_j)$ must coincide with the values y_j, $j = m-1,\ldots,N$, generated by the corresponding multistep method.

In the sequel, we shall be concerned with estimating the error in the approximation of the solution of (2.1) by splines as well as with the convergence of the approximation s to the exact solution y as $h \to 0$.

We now define the step function $s^{(m)}$ at the knots $\{t^k\}_{k=1}^{N-1}$ by the ususal arithmetic mean:

$$s^{(m)}(t_k) = \frac{1}{2}\left[s^{(m)}\left(t_k - \frac{1}{2}h\right) + s^{(m)}\left(t_k + \frac{1}{2}h\right)\right] \tag{2.10}$$

We need the following lemmas:

LEMMA 2.1. *If*

$$|s(t_k) - y(t_k)| < Kh^p, \quad |s(g(t_k)) - y(g(t_k))| < Kh^p,$$

where K is a constant independent of h, and

$$s''(t_k) = f(t_k, s(t_k), s(g(t_k)), s'(g(t_k))),$$

then there exists a constant K_1 *such that*

$$|s(t_k) - y(t_k)| < K_1 h^p, \quad |s''(t_k) - y''(t_k)| < K_1 h^p.$$

Using the Lipschitz condition H_2, the proof is simply a modification of Lemma 4.1 of [17].

LEMMA 2.2. [20] *Let* $y \in C^{m+1}[a,b]$ *and* $s \in S_m$ *such that*

$$|s^{(r)}(t_k) - y^{(r)}(t_k)| = O(h^{p_r}), \quad |s^{(r)}(g(t_k)) - y^{(r)}(g(t_k))| = O(h^{p_r}), \tag{2.11}$$

for $r = 0,1,\dots,m-1$, $k = 0,1,\dots,N-1$, *and*

$$|s^{(m)}(t) - y^{(m)}(t)| = O(h), \quad t \in [t_k, t_{k+1}]. \tag{2.12}$$

Then

$$|s(t) - y(t)| = O(h^p), \quad t \in [a,b], \tag{2.13}$$

where $p = \min_{r=0,1,\dots,m}(r+p_r)$, $p_m = 1$, *and furthermore*

$$|s^{(m)}(t) - y^{(m)}(t)| = O(h), \quad t \in [a,b]. \tag{2.14}$$

In the following sections, we shall investigate the cubic (m=3) and the quartic (m=4) spline approximations of the solution of (2.1).

3. Cubic spline function approximating the solution. By Theorem 2.3 for m=3, the cubic approximating spline function yields the same values at the knots as discrete multistep method based on the following recurrence formula:

$$y_{k+1}-2y_k+y_{k-1} = \frac{h^2}{6}[y''_{k+1}+4y''_k+y''_{k-1}] = \frac{h^2}{6}[f_{k+1}+4f_k+f_{k-1}] \tag{3.1}$$

where

$$f_j = f(t_j,y(t_j),y(g(t_j)),y'(g(t_j))),$$

if starting values $y_0 = s(t_0)$ and $y_1 = s(t_0+h)$ are used. The discrete method (3.1) has degree of exactness two provided that the starting values y_0 and y_1 have second ordsr accuracy. As in [17], it is easy to prove that the starting values $y_0 = s(t_0)$ and $y_1 = s(t_0+h)$ have the same order of exactness as the recurrence formula (3.1); therefore we can conclude that

$$|s(t_k)-y(t_k)| = O(h^2), \quad |s''(t_k)-y''(t_k)| = O(h^2). \tag{3.2}$$

The second relation follows from Lemma 2.1 for p = 2.

LEMMA 3.1. [17] *Let $y \in C^4[a,b]$ and assume $t_k,t_{k+1} \in [a,b]$. If P_3 is the unique polynomial of degree 3 satisfying the Hermite-Birkhoff interpolating conditions*

$$\begin{aligned} y(t_k) &= P_3(t_k), \quad y(t_{k+1}) = P_3(t_{k+1}) \\ y''(t_k) &= P_3''(t_k), \quad y''(t_{k+1}) = P_3''(t_{k+1}), \end{aligned} \tag{3.3}$$

then there exists a constant K_3 such that

$$|y'''(t_k)-P_3'''(t_k)| < K_3h.$$

THEOREM 3.1. *If $f \in C^1([a,b] \times C^1[a,b] \times C^1[\alpha,b] \times C[\alpha,b])$ and s is the cubic spline function approximating the solution of (2.1), then there exists a constant K, independent* *of h, such that, for h sufficiently small and* $t \in [a,b]$,

$$|y^{(i)}(t) - s^{(i)}(t)| < Kh^2, \quad i = 0,1,2,$$

and

$$|y'''(t) - s'''(t)| < Kh,$$

provided $s'''(t_k)$ *is given by* (2.10) *with* $m = 3$.

Proof. Denote the cubic component of s over $[t_k, t_{k+1}]$ by

$$s(t) = b_k + c_k(t-t_k) + d_k(t-t_k)^2 + e_k(t-t_k)^3, \quad t \in [t_k, t_{k+1}].$$

Solving a system similar to (3.3) for s we obtain

$$e_k = \frac{1}{6h}[s''(t_{k+1}) - s''(t_k)] = \frac{1}{6h}[y''(t_{k+1}) - y''(t_k)] + O(h),$$

since

$$s''(t_k) = y''(t_k) + O(h^2).$$

Now let $t \in (t_k, t_{k+1})$. We have $s'''(t_k) = 6e_k$ and Lemma 3.1 implies that

$$s'''(t) = P_3'''(t_k) + O(h) = y'''(t_k) + O(h) = y'''(t) + (t_k - t)y^{(4)}(c) + O(h).$$

Since $|t_k - t| < h$, we have

$$s'''(t) = y'''(t) + O(h), \quad t \in (t_k, t_{k+1}), \quad k = 0,1,\ldots,N-1. \tag{3.4}$$

Hence it follows that the condition (2.14) of Lemma 2.2 is satisfied for $m = 3$. Since the function s''' is constant on (t_k, t_{k+1}), we may write

$$y(t_{k+1}) = y(t_k) + hy'(t_k) + \frac{h^2}{2}y''(t_k) + \frac{h^3}{6}y'''(c),$$

and

$$s(t_{k+1}) = s(t_k) + hs'(t_k) + \frac{h^2}{2}s''(y_k) + \frac{h^3}{6}s'''(c),$$

where $c \in (t_k, t_{k+1})$. Subtracting we obtain

$$|s(t_{k+1})-y(t_{k+1})| = s(t_k)-y(t_k)+h[s'(t_k)-y'(t_k)]+ \\ +\frac{h^2}{2}[s''(t_k)-y''(t_k)]+\frac{h^3}{6}[s'''(c)-y'''(c)],$$

which implies that

$$s'(t_k) = y'(t_k)+O(h^2). \tag{3.5}$$

From (3.2) and (3.5) it follows that the conditions (2.11) of Lemma 2.2 are fulfilled for $m = 3$, $p_0 = 2$, $p_1 = 2$, $p_2 = 2$. Applying Lemma 2.2 three times successively, first for s then for s' and s'' in the role of s, the first three inequalities follow. The last inequality follows from (3.4), and thus the theorem is proved.

4. Quartic spline function approximating the solution. According to Theorem 2.3 for $m = 4$, the quartic spline function approximating the solution of (2.1) furnishes values at the knots which coincide with the values of the discrete multistep method defined by

$$\begin{aligned} y_{k+1}-y_k-y_{k-1}+y_{k-2} &= \frac{h^2}{12}[y''_{k+1}+11y''_k+11y''_{k-1}+y''_{k-2}] \\ &= \frac{h^2}{12}[f_{k+1}+11f_k+11f_{k-1}+f_{k-2}], \end{aligned} \tag{4.1}$$

provided the initial values are $y_0 = s(t_0)$, $y_1 = s(t_0 + h)$, $y_2 = s(t_0 + 2h)$. The multistep method (4.1) has degree of exactness 4 if the starting values have the same exactness. Also for this case, as has been shown in [17], it is easy to conclude that the starting values have degree of exactness 4. From this fact and by Lemma 2.1 for $p = 4$, it follows that

$$|s(t_k)-y(t_k)| = O(h^4),\quad |s''(t_k)-y''(t_k)| = O(h^4),\quad k = 0,1,\dots,N. \tag{4.2}$$

In a similar manner as in the above theorem, it may be shown that the following relations hold:

$$|s'''(t_k)-y'''(t_k)| = O(h^3),\ |s'(t_k)-y'(t_k)| = O(h^3),\ k = 0,1,\dots,N. \tag{4.3}$$

and also that

$$|s^{(4)}(t)-y^{(4)}(t)| = O(h),\ t\in(t_k,t_{k+1}),\ k = 0,1,\dots,N-1. \tag{4.4}$$

The relations (4.2) - (4.4) show that the conditions of Lemma 2.2 are satisfied for $m = 4$, $p_0 = 4$, $p_1 = 3$, $p_2 = 4$, $p_3 = 3$. Applying Lemma 2.2 for s, then successively for s', s'', s''' in the role of s, we have following theorem:

THEOREM 4.1. *If $f \in C^3([a,b] \times C^1[a,b] \times C^1[\alpha,b] \times C[\alpha,b])$ and s is the quartic spline function approximating the solution of* (2.1), *then there exists a constant K, independent of h, such that, for h sufficiently small and $t \in [a,b]$,*

$$|y^{(i)}-s^{(i)}(t)| < Kh^{4-i},\ i = 0,1,2,3,$$

and

$$|y^{(4)}(t)-s^{(4)}(t)| < Kh,$$

provided $s^{(4)}$ is given by (2.10) *with $m = 4$.*

The methods of approximating the solutions of neutral delat differential equations by spline functions, given here for $m = 3,4$ have advantages over the other methods in that they give a global approximation of the solution, are convergent, and also permit the study of the behaviour of the derivetives of the aproximate solutions.

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REFERENCES

1. H.T.Bank and J.A.Burns, *Hereditary control problems: numerical methods based on averaging approximations*, SIAM J. Control Optim., **16**(1978), 169-208.
2. A.Bellen, G.Micula and M.Zennaro, *Spline approximations for neutral delay equations*, to appear.
3. A.Bellen and M.Zennaro, *Numerical solution of delay differential equations by uniform correction to an implicit Runge-Kutta method*, Numer. Math., **47**(1985), 301-316.
4. R.D.Driver, *Ordinary and Delay Differential Equations*, Applied Math. Science, Vol. **20**, Springer Verlag, New York, 1977.
5. J.K.Hale, *Theory of Functional Differential Equations*, Springer Verlag, New York, 1977.
6. U.Hornung, *Euler Verfavren fur neutrale Funktional-Differential Gleichungen*, Numer. Math., **24**(1975), 233-240.
7. Z.Jackiewicz, *One step methods for any order for neutral functional differential equations*, SIAM J.Numer. Anal., **21**(1984), 486-511.
8. Z.Jackiewicz, *Quasilinear multistep methods, and variable step predictor corrector methods fr neutral functional differential equations*, SIAM J. Numer. Anal., **23**(1986), 4223-452.
9. Z.Jackiewicz, *Variable-step variable-order algorithm for the numerical solution of neutral functional differential quations*, Appl. Numer. Math., **3**(1987), 317-329.
10. Z.Jackiewicz, *One step methods for neutral delay-differential equations with state dependent delays*, Zastos. Math., **20**(1990), 445-463.
11. Z.Jackiewicz, *The numerical solution of neutral functional differential equations by Adams predictor-corrector methods*. Appl. Numer. Math., **8**(1991), 477-491.
12. Z.Kamout and M.Kwapisk, *On the Cauchy problem for differential-delay equations in a Banach space*, Math. Nachr., **74**(1976), 173-190.
13. F.Kappel and K.Kunisch, *Spline approximations for neutral functional-differential equations*, SIAM J. Numer. Anal., **18**(1981), 1058-1080.
14. A.G.Kemper, *Spline function approximation for solutions of functional differential equations*, SIAM J. Numer. Anal., **12**(1975), 73-88.
15. G.Meinardus and G.Numberger, *Delay Equations, Approximation and Applications*, ISNM **74**, Birkhäuser Verlag, PLACE, 1985.
16. G.Micula, *Spline Functions and Applications*, Ed. Tehnica, Bucharest, 1978. (Romanian).
17. G.Micula, *Die numerische Losung nichtlinearer Differentialgleicheungen unter Verwendung von Spline-Funktionen*, Lecture Notes in Mathematics 393, Springer-Verlag, New York, 1974, 57-83.
18. G.Micula. *Uber die numerishe Losung gewohnlicher Differentialgleichungen zweiter Ordnung mit nacheilendem Argument durch Spline-Funktionen*, Rev. Roumain de Math. Pures et Appliquées, **34**(1989), 890-909.
19. G.Micula and H.Akca, *Numerical solution of differential equationswith deviating argument using spline functions*, Studia Univ. Babeş-Bolyai, Mathematica, **33**(1988), 45-57.
20. G.Micula and H.Akca, *Aproximate solutions of the secnd order differential equations with deviating argument by spline functions*, Mathematica-Revue d'Analyse Numérique et de Théorie de l'Approximation, Tome **30**(53) 1, (1988), 37-46.
21. A.D.Myshkis, *On some problems of the theory of differential equations with deviating arguments*, Usp. Mat. Nauk, **32**(1977), 173-202, (Russian).
22. T.S.Nikolova and D.D.Bainov, *Application of spline functions for the construction of an approximate solution of boundary value problems for a class of functional differential equations*, Yokohoma Math. J., **29** (1981), 108-122.
23. H. J. Oberle and J. Pesch, *Numerical treatment of delay differential equations by Hermite interpolation*, Numer. Math., **37** (1981), 235-255.
24. K. Schmidth, *Delay and Functional Differential Equations and Their Applications*, Academic Press, New York, 1972.

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FIRST ORDER EFFECTS OF LENSE-THIRRING PRECESSION IN QUASI-CIRCULAR SATELLITE ORBITS

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REZUMAT. - Efecte de ordinul întâi ale precesiei Lense-Thirring asupra orbitelor cvasicirculare de sateliţi. Se studiază mişcarea unui satelit artificial al Pământului, având orbita iniţială cvasicirculară, sub influenţa perturbatoare a rotaţiei terestre, descrisă de acceleraţia Lense-Thirring. Se determină perturbaţiile de ordinul întâi în cinci elemente orbitale independente, pe durata unei perioade nodale. Efectele constatate constau în mişcarea apsidală, precesia orbitei şi variaţia înclinării planului orbitei. Se atrage atenţia asupra posibilităţii determinării pe această bază a perturbaţiilor aceloraşi elemente pe intervale mari de timp.

1. Introduction. The effect of "inertial frame dragging" on an orbit in the gravitomagnetic field of a rotating central body was discussed since 1918 (see e.g. [5]). A description of this relativistic phenomenon, also called Lense-Thirring precession, characteristic to a rotating, gravitomagnetic field-generating body, can be found e.g. in [3].

Let such a body be the Earth (of gravitational parameter μ and rotating uniformly with the angular velocity ω_E), and let an artificial satellite be orbiting the Earth at geocentric distance r. Let the relative motion of the satellite be described (with respect to a Cartesian right-handed frame originated in the Earth's mass centre) by means of the Keplerian orbital elements $\{y \in Y;\ u\}$, all time-dependent, where:

$$Y = \{p, q = e\cos\omega,\ k = e\sin\omega, \Omega, i\}, \tag{1}$$

and p = semilatus rectum, e = eccentricity, ω = argument of perigee, Ω = longitude of

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ascending node, i = inclination, u = argument of latitude. Since the Lense-Thirring precession influence on the satellite orbit can be treated perturbatively, we shall estimate analytically the first order variations of the orbital elements (1), due to the mentioned influence, over one nodal period. The results will be established under the following hypotheses:

(i) The elements (1) have small variations over one revolution of the satellite (this is likely since the Lense-Thirring precession is a relativistic effect).

(ii) The initial orbit of the satellite is quasi-circular (accordingly, expansions to first order in e, through q and k will be used).

2. Basic equations. Since the nodal period was chosen as basic time interval, we shall describe the satellite motion by means of the Newton-Euler system written with respect to u as (e.g. [1,2]):

$$
\begin{aligned}
dp/du &= 2(Z/\mu)r^3T,\\
dq/du &= (Z/\mu)(r^3kBCW/(pD) + r^2T(r(q+A)/p + A) + r^2BS),\\
dk/du &= (Z/\mu)(-r^3qBCW/(pD) + r^2T(r(k+B)/p + B) - r^2AS),\\
d\Omega/du &= (Z/\mu)r^3BW/(pD),\\
di/du &= (Z/\mu)r^3AW/p,\\
dt/du &= Zr^2(\mu p)^{-1/2},
\end{aligned}
\tag{2}
$$

where $Z = (1 - r^2C\dot{\Omega}/(\mu p)^{1/2})^{-1}$, $A = \cos u$, $B = \sin u$, $C = \cos i$, $D = \sin i$, S, T, W = radial, transverse, and binormal components of the perturbing acceleration, respectively.

By virtue of hypothesis (i), the elements (1) can be considered constant and equal to their initial values $y_0 = y(u_0) = y(u(t_0))$, $y \in Y$, in the right-hand side of equations (2), and these ones can be separately integrated. So, we can write $y = y_0 + \Delta y$, where the first order variations are found from:

$$\Delta y = \int_0^{2\pi} (dy/du)\, du, \quad y \in Y, \tag{3}$$

with the integrands provided by (2). The integrals are therefore estimated by successive approximations, with $Z \approx 1$, limiting the process to the first order approximation.

In what follows, for simplicity, we shall omit the factor Z in the motion equations. Also, we shall no longer use the subscript "0" to mark initial values of $y \in Y$. In fact, every quantity which does not depend on u (explicitly or through A, B) will be considered constant over one revolution of the satellite.

3. Perturbing acceleration. Under the influence of the Lense-Thirring precession, the satellite undergoes a perturbing acceleration whose components are [5]:

$$\begin{aligned} S &= KhC/r^4, \\ T &= -KhCe \sin v/(pr^3), \\ W &= KhD(2B + Are \sin v/p)r^4, \end{aligned} \tag{4}$$

where v = true anomaly, $h = (\mu p)^{1/2}$, and:

$$K = 2(\gamma + 1)\mu \omega_E R^2/(5c^2), \tag{5}$$

in which $\gamma = 1.000 \pm 0.002$ is the space curvature parameter, R = mean equatorial terrestrial radius, c = speed of light.

Taking into account the fact that $v = u - \omega$, and the definition of q and k, formulae (4) can also be written as:

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$$S = KhC/r^4,$$
$$T = -K(h/p)C(Bq - Ak)/r^3, \tag{6}$$
$$W = KhD(2B + r(Bq - Ak)/p)/r^4.$$

4. Variations of orbital elements. Consider the orbit equation in polar coordinates:

$$r = p/(1 + e\cos v) = p/(1 + Aq + Bq), \tag{7}$$

which, by virtue of hypothesis (ii), leads to:

$$r^n = p^n(1 - nAq - nBk). \tag{8}$$

Replacing (6) in (2), then substituting (8) in the resulting expressions, and observing the considerations made in Section 2, the first five equations (2) acquire the form:

$$dp/du = -2KpbC(Bq - Ak),$$
$$dq/du = KbC(B + 2(1 + B^2)k),$$
$$dk/du = -KbC(A + 2(1 + B^2)q), \tag{9}$$
$$d\Omega/du = Kb(2B^2 + B^2(1 + 2A)q + B(2B^2 - A)k),$$
$$di/du = KbD(2AB + AB(1 + 2A)q + A(2B^2 - A)k),$$

in which we denoted $b = (ph)^{-1} = p^{-3/2}\mu^{-1/2}$.

So, the expressions in the right-hand side of (9) contain only explicit functions of u (through A, B) and quantities considered constant over one revolution of the satellite.

Performing now the integrals (3) with the integrands provided by (9), we obtain the first order variations of the orbital elements due to the Lense-Thirring acceleration over one nodal period:

$$
\begin{aligned}
\Delta p &= 0, \\
\Delta q &= 6\pi KbCk, \\
\Delta k &= -6\pi KbCq, \\
\Delta\Omega &= \pi Kb(2+q), \\
\Delta i &= -\pi KbDk.
\end{aligned} \tag{10}
$$

5. Comments. Examining these results, we observe that, in a first order approximation, the Earth's rotation (by means of the Lense-Thirring acceleration) does non affect the shape and dimensions of the initially quasi-circular orbit over one nodal period. Indeed, taking into account the definition of q,k, and the second and third formulae (10), we easily get:

$$\Delta e = \Delta q \cos\omega + \Delta k \sin\omega = 0. \tag{11}$$

Considering the relation $p = a(1 - e^2)$, too, where a is the semimajor axis, one sees immediately that $\Delta a = 0$.

Taking again into account the definition of q,k, and the second and third expression (10), we obtain:

$$\Delta\omega = (\Delta k \cos\omega - \Delta q \sin\omega)/e = -6\pi KbC, \tag{12}$$

that is, the Lense-Thirring acceleration causes apsidal motion. By (12) and $C = \cos i$, one sees that the value of ω decreases if the satellite motion is direct, and increases if this motion is retrograde.

The last two expressions (10) show that the position of the orbit plane is affected by the Lense-Thirring acceleration. The value of Ω increases (hence the satellite orbit undergoes a precession), while the value of i increases or decreases as the initial value of the product sin i sin ω is negative or positive, respectively (remind that D = sin i).

According to hypothesis (ii), the results (10) are obtained with a first order accuracy as regards the eccentricity (through q and k). Observe that only the longitude of ascending node undergoes perturbations of zeroth order in eccentricity. Subsequently, in the particular case of initially circular orbit, the only first order effect on the Lense-Thirring acceleration is the precession of the orbit.

As a final remark, if the integrals (3) are performed between the initial (u_0) and current (u) positions, the variations of the orbital elements in the interval $[u_0,u]$ can be used to determine the first order perturbations in the nodal period (e.g.[1,2]). Also, starting from the formulae (10), the evolution of these elements can be studied over large time intervals, using either an averaging-type method or the numerical integration, or a mixed method (e.g.[4]).

REFERENCES

1. Blaga,P., Mioc,V., *Elliptic-Type Motion in a Schwarzschild-De Sitter Gravitational Field*, Europhys. Lett., 17(1992), 275-278.
2. Mioc,V., Pál,Á., *Nodal Period Perturbations due to the Fifth Zonal Harmonic of the Geopotential*, Studia Univ. Babeş-Bolyai, ser. Math., 30(1985), 55-60.
3. Misner,C.W., Thorne,K.S., Wheeler,J.A., *Gravitation*, W.H.Freeman and Co., San Francisco, London, 1973.
4. Pál,Á., *On the Integration of the Artificial Satellite Motion Equations over Large Time Intervals*, Babeş-Bolyai Univ. Fac. Math. Res. Sem., Preprint 2, 1980, p.171-176.
5. Soffel, M.H., *Relativity in Astrometry, Celestial Mechanics and Geodesy*, Springer-Verlag, Berlin, Heidelberg, New York, 1989.

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CRONIC

NICOLAE COPERNIC (1473 - 1543)*

Á. PÁL**

ABSTRACT. Copernicus' life and fundamental work "De revolutionibus orbium coelestium" are briefly presented according to the references placed at the end of this paper. Two actions devoted to the spreading of Copernicus' revolutionary ideas are pointed out, the 2nd General Assembly of the European Astronomical Society at Torun, and the celebration of 400 years since the inauguration of Galileo's lectures at University of Padua.

Calendarul UNESCO pe anul 1993 cuprinde, printre evenimentele ştiinţifice şi culturale ce trebuie comemorate pe plan mondial, împlinirea a 520 de ani de la naşterea marelui astronom polonez Nicolae Copernic (Mikolaj Kopernik, Nicolaus Copernicus) şi 450 de ani de la publicarea operei sale capitale, "De revolutionibus orbium coelestium".

Cu 20 de ani în urmă, în anul 1973, sub egida UNESCO, a fost marcată, în cadrul unor ample manifestări ştiinţifice şi culturale, împlinirea a 500 de ani de la naşterea acestui geniu al Renaşterii, care, prin teoria sa heliocentrică a Sistemului solar-planetar - ce era total opusă teoriei geocentrice dominante de peste un mileniu şi jumătate, a deschis o nouă eră în cunoaşterea Universului - era revoluţiilor ştiinţifice ale secolelor următoare. Dintre manifestările ce au avut loc ne reamintim cu emoţie Adunarea Generală Extraordinară a Uniunii Astronomice Internaţionale din Polonia, precum şi manifestările organizate în ţara noastră de către Comitetul pentru sărbătorirea a 500 de ani de la naşterea lui Copernic.

Astăzi, în cadrul simpozionului nostru, ne-am propus o scurtă evocare a momentului Copernic din istoria cunoaşterii Universului. Dar noi, astronomii, putem cel mai bine cinsti memoria acestui geniu al omenirii prin promovarea îndrăzneaţă a învăţămîntului astronomiei şi a cercetării în domeniu.

N.Copernic s-a născut la 19 februarie 1473, la Torun (pe Vistula, la 170 km nord-est de Varşovia), ca fiu al negustorului şi consilierului comunal Nicolae Copernic şi al soţiei sale Barbara, născută Watzenrode. La vârsta de 10 ani, Copernic pierde pe tatăl său, în timpul epidemiei de ciumă, cei 4 copii orfani au fost ajutaţi de fratele mamei lor, canonicul Lukasz Watzenrode, membru al Societăţii Literare Vistulane, doctor in drept canonic al Universităţii din Bologna, episcop de Warmin din 1489.

N.Copernic şi-a însuşit cultura, formându-şi personalitatea prin studii la Universitatea di Cracovia (1491-1495), - matematica, astronomia, clasicii latini -, la Universitatea din Bologna (1496 1501) - dreptul canonic, filosofia, matematica, medicina - şi la Universitatea din Padova (1501-1503 - medicina şi filosofia (diferite sisteme ale lumii: Pitagora, Eudox, Aristarh, ..., Hiparh). Doctoratu în dreptul canonic şi l-a luat la Ferrara.

Canonic, a fost secretar şi medic al unchiului său, episcopul L.Watzenrode la Lidzbark, î anii 1504-1512, când s-a mutat şi a locuit tot restul vieţii la Frombork. Aici a fost inspector ş cancelar (1511-1512), iar în anii 1512-1521 a fost administrator al bunurilor întreprinderilc economice ale Sfatului Canonicilor din Warmia. Ca cetăţean credincios Warmiei, el a contribuit l apărarea cetăţii Olsztin, care era atacată de o armată a cavalerilor teutoni.

Despre Universitatea din Cracovia la epoca respectivă, Hartmann Schedl a scris: "Lîng biserica Sf.Ana se află universitatea, cunoscută prin învăţaţii ei mari şi slăviţi, unde se înva retorica, poetica, filosofia şi fizica. Dar din toate ştiinţele, astronomia înfloreşte mai mult acolo. Î această privinţă, după câte ştiu, în toată Germania nu există o şcoală mai renumită."

N.Copernic a studiat astronomia, predată de Adalbert Budzew după Almageste (traducer in limba arabă a operei lui Ptolemeu "Sintaxa matematica").

Adâncind studiul astronomiei, Copernic constată că diferenţele între poziţiile Lunii calcula cu teoria lui Ptolemeu şi poziţiile observate ajungeau până la 15' (jumătate din diametrul apare al Lunii), instrumentele de observare (fără lunetă) din timpul lui asigurînd o precizie de 10 Diferenţele mari O-C au determinat pe Copernic să pună sub semnul întrebării teoria geocentri a lui Ptolemeu. Copernic avea atunci doar 19 ani!

Copernic avea şi alte argumente care se adăugau spre a-i întări neîncrederea în sistemul l Ptolemeu şi anume: el observase, printre altele, că la momentul opoziţiei planetele Marte, Jupi şi Saturn străluceau mai puternic, fapt care îi arată că aceste planete nu puteau să se rotească în jurul Pământului ca centru al traiectoriilor lor!

La Bologna, Copernic calculează eclipsarea (ocultaţia) de către Lună a stelei Aldebaran (α Tauri), folosind ideea sa asupra mişcării Pământului în jurul Soarelui şi cea a mişcării Lunii în jurul Pământului. Conform prevederii, eclipsarea s-a produs în seara zilei de 9 martie 1497. În a sa "De revolutionibus...", Copernic scrie: "Aşteptându-ne să observăm acest fenomen, am văzut steaua atingând partea întunecată a sferei lunare şi dispărând între coarnele ei."

Revenind în patrie, în Polonia, în castelul de la Lidzbark, Copernic cultiva mai departe observarea cerului şi studiile privind sistemul mişcărilor, studii care se apropiau de faza elaborării finale.

În 1507, N.Copernic compune lucrarea "Mic comentariu despre ipotezele mişcărilor corpurilor cereşti", care nu a fost tipărită, dar a avut o largă circulaţie în forma de manuscris. În această lucrare se arată, în cadrul a 7 axiome, că:

- nu există un centru unic pentru toate orbitele;

- Pământul nu este centrul Universului, ci numai centrul său de gravitaţie şi centrul orbitei lunare;

- Soarele este centrul Universului;

- distanţa de la Pământ la Soare este neânsemnată în raport cu distanţa la stele;

- mişcarea diurnă a sferei cereşti este rezultatul rotaţiei Pământului în jurul axei polilor, împreună cu atmosfera ce-l înconjoară, în timpul unei rotaţii polii păstrând o poziţie neschimbată faţă de stele;

- deplasarea anuală a Soarelui faţă de stele este rezultatul mişcării Pământului în jurul Soarelui, la fel cu mişcarea oricărei alte planete;

- mişcările planetelor în sens direct, staţiile şi retrogradaţiile acestora sunt aparenţe cauzate de mişcările planetelor şi Pământului în jurul Soarelui.

Astfel au fost formulate în "Micul comentariu..." ideile de bază pe care le va dezvolta Copernic în opera sa magistrală "De revolutionibus...".

N.Copernic a ezitat să-şi publice lucrarea, gândindu-se la reacţiile negative care ar fi putut să se dezlănţuie pe nedrept contra adepţilor noului sistem al lumii - sistem ce se afla în contrazicere cu scrierile sfinte.

N.Copernic a fost sigur că structura heliocentrică corespundea adevărului şi că prin perfecţionarea ulterioară a observaţiilor se vor preciza şi legile mişcărilor, ceea ce s-a şi întâmplat în realitate, astfel:

- J.Kepler, (între anii 1609 - 1619) a descoperit legile mişcării eliptice neuniforme a planetelor şi sateliţilor - făcând pasul decisiv de la mişcările circulare şi uniforme din sistemul copernician;

- G.Galilei a îndreptat spre cer (în seara de 7 ianuarie 1610) luneta construită de el şi a observat, printre altele, fazele planetei Venus, care dovedeau tocmai mişcarea acesteia în jurul Soarelui, aşa cum prevăzuse Copernic;

- J.Bradley, în 1727, a descoperit aberaţia luminii, dovadă elocventă a mişcării Pământului în jurul Soarelui; se ştie că el dorea să pună în evidenţă paralaxa anuală a stelelor, dar precizia de 1" a lunetei de care dispunea nu era suficientă (paralaxele anuale ale stelelor, o ştim bine astăzi, sunt toate mai mici de 1").

- Primele paralaxe stelare au fost descoperite mai târziu;

- V.Struve, în 1837, descoperă paralaxa anuală a stelei Vega (α Lyrae), iar

- F.W.Bessel, în 1838, descoperă paralaxa anuală a stelei 61 Cygni.

În vara anului 1541, un tânăr şi renumit profesor de la Universitatea din Wittemberg, elev al lui Copernic, Georgius Joachimus Rheticus, obţinând un concediu, îl ajută pe maestru la grăbirea redactării şi apariţiei lucrării "De revolutionibus..." care intră în tipografia lui Petreius din Nürnberg în anul 1542. Primele exemplare ale operei apar în martie 1543 - acum exact 450 de ani. Copernic se stinge din viaţă la 24 mai 1543 şi se crede că un exemplar din carte a ajuns în mâinile muribundului.

Ca o măsură de precauţie, N.Copernic dedică lucrarea papei Paul al III-lea, cu următoarele cuvinte: "Dedic cartea mea sanctităţii-tale, pentru ca savanţii şi ignoranţii să vadă că eu nu fug de judecată. Dacă unii oameni uşuratici şi ignoranţi ar voi să abuzeze de pasajele din Sfânta Scriptură şi, schimbând sensul lor, să abuzeze contra mea, eu dispreţuiesc atacul lor temerar. Adevărurile matematice trebuie judecate numai de matematicieni".

"De revolutionibus..." a fost interzisă la 5 martie 1616 de Congregaţia Indexului, când lui Galilei, profesor la Universitatea din Pisa, i se interzice să propage ideile lui Copernic.

Faptul că a criticat în mod curajos şi constructiv atât ştiinţa "oficială" a timpului său, cât şi aparenţele imediate ale simţului comun, îl situează pe Copernic printre marii inovatori ai ştiinţei, fiind considerat - pe drept cuvânt - precursorul primei revoluţii ştiinţifice a secolului al XVII-lea.

În perspectiva timpului ne dăm seama şi mai bine că fără drumul deschis de el, fără revoluţia copernicană, n-ar fi fost posibile progresele spectaculoase ale astronomiei, ale ştiinţei în general, pentru a căror ilustrare este suficient să ne referim la realizările obţinute de zborurile cosmice şi astronomia spaţială.

Simbol al puterii înnoitoare a minţii omeneşti în căutarea adevărului, Copernic rămâne în amintirea colectivă a umanităţii un titan al gândirii şi un binefăcător al omenirii.

În încheiere, remarcăm două recente evenimente ştiinţifice semnificative pentru promovarea în zilele noastre a ideilor lui Copernic:

i) Cea de a doua Adunare Generală a Societății Europene de Astronomie (Torun, Polonia, 18-20 august 1993) pe tema: "Astronomia extragalactică și Cosmologia observațională";

ii) Celebrarea a 400 de ani de la inaugurarea lecțiilor lui Galilei la Universitatea din Padova (manifestare organizată în cadrul Anului Internațional al Spațiului 1992, la 7 decembrie).

BIBLIOGRAFIE

1. Simpozion Copernic, 19 februarie 1973, organizat de Comitetul pentru sărbătorirea a 500 de ani de la nașterea lui Copernic, Editura Academiei R.S.România, București, 1973.

Contents

2. European Astronomical Society. Newsletter, Geneva Observatory Issue 4, December 1992, Imprimerie Louis-Jean, France
3. On the occasion of the celebration on 7 December 1992 of the 400-th anniversary of Galileo's lectures at the University of Padua, Pontifical Academy of Science, Vatican Observatory Annual Report 1992, Vatican Observatory Publications, Vol.2, No. 17.

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