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TRIUNITARY DIVISOR FUNCTIONS

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REZUMAT. - **Funcţii divizor triunitar.** Noţiunea de divizor "triunitar" a fost intodusă recent de G.L.Cohen. În prezenta lucrare se introduc analoagele funcţiilor Mobius ($\mu(n)$) şi Euler ($\varphi(n)$) referitoare la divizorii "triunitari" şi se stabilesc cîteva proprietăţi ale acestora.

1. Introduction. It is well known that a divisor $d > 0$ of a positive integer n is called unitary if the greatest common divisor d and n/d is 1. $\left(\left(d, \frac{n}{d}\right)_0 = 1\right)$.

Analogous D.Suryanarayana [4], [5] introduced the notion of bi-unitary divisor.

A divisor $d > 0$ of the positive integer n is called bi-unitary if the greatest common unitary divisor of d and $\frac{n}{d}$ is 1. $\left(\left(d, \frac{n}{d}\right)_1 = 1\right)$.

We denote by $(a,b)^{**} = (a,b)_1$ the greatest unitary divisor of both a and b.

Alladi [1] introduced the r-th order divisors which is a generalization of unitary divisors but not of bi-unitary divisors. Other generalization introduced by G.L.Cohen [2] is the concept of triunitary divisor of n. We may call d a triunitary divisor of n if the greatest common bi-unitary divisor of d and $\frac{n}{d}$ is 1. $\left(\left(d, \frac{n}{d}\right)_2 = 1\right)$.

It is easily seen that, for a prime power p^y, the unitary divisors are 1 and p^y, the bi-unitary divisors are all the powers $1, p, p^2, \ldots, p^y$ except for $p^{y/2}$ when y is even and the triunitary

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divisors are 1 and p^y except if $y = 3$ or 6; those of p^3 are $1, p, p^2$ and p^3; and those of p^6 are $1, p^2, p^4$ and p^6. In this paper we introduced some functions related to triunitary divisors and we established some properties.

We write $d|_1 n, d|_2 n$ and $d|_3 n$ if d unitary divisor of n, d biunitary divisor of n and d triunitary divisor of n respectively.

2. Functions of triunitary divisors. Let $\mu(n)$ the Mobius function and $\mu^*(n) = \mu_1(n)$ the unitary analogue of $\mu(n)$:

$$\mu(n) = \begin{cases} 1 & \text{if } n=1 \\ (-1)^k & \text{if } n=p_1p_2\ldots p_k \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_1(n) = \begin{cases} 1 & \text{if } n=1 \\ (-1)^{a_1+a_2+\ldots+a_k} & \text{if } n=p_1^{a_1}\ldots p_k^{a_k} \end{cases}$$

We define the function $\mu(n)$ in the following way:

DEFINITION 1.

$$\mu_2(n) = \begin{cases} 1 & \text{if } n=1 \\ (-1)^{\sum\limits_{p^a \| n}\left(p^a - \sum\limits_{k=1}^{\infty}\left[\frac{p^a}{2^k}\right]\right)} & \text{if } n>1 \end{cases}$$

It is easy to observe that the function $\mu_2(n)$ is a multiplicative function.

If $n \geq 1$ we have:

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } n>1 \end{cases}$$

and for $\mu^*(n)$:

LEMMA 1 ([3]), LEMMA 2.4).

$$\sum_{\substack{d|_1 m\\ d|_1 n}} \mu_1(d) = \begin{cases} 1 & if\ (m,n)_1 = 1 \\ 0 & otherwise \end{cases}$$

The motivation of definition is a following theorem:

THEOREM 1. *For* $n \geq 1$:

$$\sum_{d|_3 n} \mu_2(d) = \begin{cases} 1 & if\ n=1 \\ 0 & if\ n>1 \end{cases}$$

Proof. The formula is clearly true if $n = 1$. Assume, then, that $n > 1$. We have:

$$\sum_{d|_3 n} \mu_2(d) = \sum_{d\cdot\delta=n} \mu_2(d) \cdot \sum_{\substack{t|d\\ t|\delta}} \mu_1(t) =$$
$$= \sum_{\substack{t^2\cdot t_1\cdot t_2=n\\ (t,t_1t_2)=1}} \mu_2(t)\cdot\mu_1(t)\cdot\mu_2(t_1)$$

by Lemma 1. Because $\mu_2(n)$ and $\mu_1(n)$ are multiplicative functions it is enough to prove for prime powers.

But:

$$p^{\alpha} - \sum_{k=1}^{\infty}\left[\frac{p^{\alpha}}{2^k}\right] = p^{\alpha} - 2\left[\frac{p^{\alpha}}{2}\right] + \left[\frac{p^{\alpha}}{2}\right] - 2\left[\frac{p^{\alpha}}{2^2}\right] + \ldots = d_0 + d_1 + \ldots + d_j$$

where $\overline{d_0 d_1 \ldots d_j}$ is a binary form of α.

Let:

$$\mu_2(p^{\alpha}) = a_{\alpha}$$

It's easy to prove by mathematical induction that

$$\begin{cases} a_{2\alpha+1} + a_{\alpha} = 0 \\ a_{2\alpha+2} - a_{\alpha+1} = 0 \end{cases} \quad \forall\ \alpha \in \mathbb{N}^*$$

which implies that

$$\sum_{k=1}^{2\alpha+1} a_k = 0$$

and

$$\sum_{k=1}^{2\alpha+2} a_k - a_{\alpha+1} = 0$$

Thus:

$$\sum_{t_1 \cdot t_2 = p^{2\alpha+1}} \mu_2(t_1) = 0 \tag{1}$$

and

$$\sum_{t_1 \cdot t_2 = p^{2\alpha+2}} \mu_2(t_1) - \mu_2(p^{\alpha+1}) = 0 \qquad \forall\ \alpha \in \mathbb{N}^* \tag{2}$$

(1) and (2) implies that:

$$\sum_{\substack{t^2 \cdot t_1 \cdot t_2 = p^\alpha \\ (t, t_1 \cdot t_2) = 1}} \mu_2(t) \cdot \mu_1(t) \cdot \mu_2(t_1) = 0$$

This complete the proof.

COROLLARY 1. *For $m, n \geq 1$:*

$$\sum_{\substack{t|_2 m \\ t|_2 n}} \mu_2(t) = \begin{cases} 1 & \text{if } (m,n)_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

THEOREM 2. *The equation*

$$g(n) = \sum_{d|_2 n} f(d) \tag{3}$$

implies

$$f(n) = \sum_{d|_2 n} \mu_2(d) \cdot g\left(\frac{n}{d}\right) \tag{4}$$

Conversely (4) *implies* (3).

Proof. We have

$$\sum_{d|_2 n} \mu_2(d)\cdot g\left(\frac{n}{d}\right) = \sum_{d|_2 n} \mu_2(d)\cdot \sum_{\delta|_2 \frac{n}{d}} f(\delta) = \sum_{d\cdot\delta|_2 n} \mu_2(d)\, f(\delta) =$$

$$= \sum_{\delta|_2 n} f(\delta)\cdot \sum_{d|_2 \frac{n}{d}} \mu_2(d) = f(n)$$

by Theorem 1. Conversely:

$$\sum_{d|_2 n} f(d) = \sum_{d|_2 n} f\left(\frac{n}{d}\right) = \sum_{d|_2 n} \sum_{\delta|_2 \frac{n}{d}} \mu_2\left(\frac{n}{d\cdot\delta}\right)\cdot g(\delta) = \sum_{d\cdot\delta|_2 n} \mu_2\left(\frac{n}{d\cdot\delta}\right) g(\delta) =$$

$$= \sum_{\delta|_2 n} g(\delta)\cdot \sum_{d|_2 \frac{n}{\delta}} \mu_2\left(\frac{n}{\delta\cdot d}\right) = g(n)$$

by Theorem 1.

DEFINITION 2. *For* $n \geq 1$:

$$\varphi_2(n) = \sum_{d|_2 n} \mu_2(d)\cdot\delta$$

COROLLARY 2. *If* $n \geq 1$ *we have*

$$\sum_{d|_2 n} \varphi_2(d) = n$$

Proof. The Corollary follow by Definition 2 and Theorem 2 for $f(n) = \varphi(n)$ and $g(n) = n$.

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LOCAL SEPARATION PROPERTIES IN CATEGORIES OF CONVERGENCE SPACES

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REZUMAT. - **Proprietăţi de separare locală în categorii ale spaţiilor de convergenţă.** În lucrare sînt date caracterizări ale proprietăţilor de separare T_0, T_1, $PreT_2$ şi T_2 în categorii topologice ale unor spaţii de convergenţă. O serie de relaţii între aceste proprietăţi sînt de asemenea studiate.

Abstract. In this paper, an explicit characterization of each of the separation properties T_0, T_1, $PreT_2$, and T_2 at a point is given in the topological categories of (Local) Filter Convergence Spaces, Limit Spaces, Pseudotopological Spaces,and Pretopological Spaces. Moreover, speciffic relationships that arise among the various T_0, $PreT_2$, and T_2 structures are examined in these categories.

Introduction. In [1], various generalizations of the separation properties are defined for an arbitrary topological category over Sets, the category of sets. These generalizations are given at a point i.e. locally, then they are generalized to point free definitions by using the generic element, [6] p. 39, method of topos theory. One of the other use of local separation properties is to define the notion of closed subsets of an object of a topological category which is studied in [1].

General results, involving relationships among these generalized separation properties at a point as well as interrelationships among their various forms will be established

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in a subsequent paper.

One of the separation properties, namely Pre T_2'[1], has already appeared in [5] as a generalized Hausdorff condition arising in the study of geometric realization functors that preserve finite limits. Furthermore, some of our T_2 structures ($\overline{T}_2, T_2'$) have appeared in [9] under the name of "Hausdorff convergence spaces" in the case of local filter convergence spaces, limit spaces, pseudo topological spaces, and pretopological spaces. Also, our $\overline{T}_0$ has appeared in [11] under the name of "T_0 objects" in the above categories.

In this paper, we give explicit characterizations of the generalized separation properties at a point as well as we examine the specific relationships that arise betweed the various forms of "T_0, *Pre*T_2", and "T_2" structures in these categories.

Let *E* be a category and *Sets* be the category of sets. A functor $U:E \to$ *Sets* is said to be concrete if it is faithful (i.e *U* is mono on hom sets) and amnestic (i.e if $U(f) = id$ and *f* is an isomorphism then $f = id$). The functor *U* is said to be topological if it is concrete, has small (i.e sets) fibers, and for which every *U*-source has an initial lift or, equivalently, for which every *U*-sink has a final lift [4] p.125 or [8] p. 279.

Let *A* be a set and *K* be a function on *A* whose value $K(a)$ at each a in *A* is a set of nonempty filters on *A*.

1.1. DEFINITION. A pair (A,K) is said to be a *Filter Convergence Space* if for each a in *A*.

1. [a] belongs to $K(a)$, where $[a] = \{B \subset A$ / a is in $B\}$.
2. If α and β are filters on *A* and $\alpha \subset \beta$, then $\beta \in K(a)$ if

$\alpha \in K(a)$. A morphism $(A,K) \to (B,L)$ is a function $f : A \to B$ such that $f\alpha \in L(f(a))$ if $\alpha \in K(a)$, where $f\alpha$ denotes the filter $\{U | U \subset B$ and $U \supset f(C)$ for some $C \in \alpha\}$. We denote by *FCO*, the category so formed. See [10] p. 354.

1.2 DEFINITION. A Filter Convergence Space (A,K) is said to be a *Local Filter Convergence Space* if $\alpha \cap [a]$ belongs to $K(a)$ whenever α belongs to $K(a)$, [9] p. 1374, a *Limit Space* if $\alpha \cap \beta$ belongs to $K(a)$ whenever α and β do, [9] p. 1374, a *Pseudotopological Space* if a filter α belongs to $K(a)$ whenever all the ultrafilters containing α belongs to $K(a)$, [9] p. 1374, *Pretopological Space* if the intersections, N_a, of all filters in $K(a)$ belongs to $K(a)$, [9]. These spaces are the objects of the full subcategories, *LFCO, Lim, PsT,* and *PrT,* of *FCO.*

1.3. The discrete structure (A,K) on A in *FCO, LFCO, Lim, PsT, PrT* is given by $K(a) = \{[a], PA = [\phi]\}$ for all a in A. See [7] p. 528.

1.4. A source $\{f_i : (A,K) \to (A_i,K_i)\ i \in I\}$ is an initial lift in *FCO, LFCO, Lim, PsT, PrT,* if and only if $\alpha \in K(a)$ precisely when $f_i\alpha \in K_i(f_i(a))$ for all i in I. See [9] p. 1374.

1.5. An epi sink $\{i_1,i_2 : (A,K) \to (A_1,K_1)\}$ is final in *FCO* and *LFCO* iff for each a_1 in A_1, $\alpha \in K_1(a_1)$ implies there exists a in A and β in $K(a)$ such that for some $k = 1,2$, $i_k a = a_1$ and $i_k\beta \subset \alpha$. These are special cases of [9] p. 1375.

1.6. An epi sink $\{i_1,i_2 : (A,K) \to (A\vee_p A,K_1)\}$, where $A\vee_p A$ is a wedge and i_1,i_2 denote the canonocal injections, in *Lim* is final iff for each $a \neq p$ in $A\vee_p A$, $\alpha \in K_1(a)$ implies there exists b in A and β in $K(b)$ such that $i_k b = a$ and $i_k\beta \subset \alpha$ for some $k =$

= 1,2. For $a = p$, $\alpha \in K_1(p)$ implies there exist β, γ in $K(p)$ such that $i_1\beta \cap i_2\gamma \subset \alpha$. These are special cases of [9] p. 1375.

1.7. An epi sink $\{i_1, i_2 : (A,K) \to (A_1,K_1)\}$ in *PsT* is final if for each a in A_1 and each ultrafilter α on A_1 $\alpha \in K_1(a)$ implies $\alpha = i_k\beta$ for $K = 1,2$ (an ultrafilter $\alpha \in K_1$ implies $\alpha = i_k\beta$ for some ultrafilter β in K and some $k = 1,2$). These are special cases of [9] p. 1376.

1.8. An epi sink $\{i_1, i_2 : (A,K) \to (A\vee_p A, K_1)\}$ in *PrT* is final iff for each $a \neq p$ in $A\vee_p A$, $\alpha \in K_1(a)$ implies there exists b in A and N_b in $K(b)$ such that $i_k N_b \subset \alpha$ and $i_k(b) = a$ or some $k = 1,2$. For $a = p$, $\alpha \in K_1(p)$ implies there exists N_p in $K(p)$ such that $i_1N_p \cap i_2N_p \subset \alpha$. It is easy to see that these are a generalization of quotient maps in *PrT* given in [9].

Let X be a set and p a point in X. Let $X\vee_p X$ be the wedge product of X with itself, i.e. two distinct copies of X identified at the point p. A point x in $X\vee_p X$ will be denoted by $x_1(x_2)$ if x is in the first (resp. second) component of $X\vee_p X$. Let $X^2 = X \times X$ be the cartesian product of X with itself.

1.9. DEFINITION. The *principal p axis map*, A_p: $X\vee_p X \to X^2$ is defined by $A_p(x_1) = (x_1, p)$ and $A_p(x_2) = (p, x_2)$.

1.10. DEFINITION. *The skewed p axis map*, S_p: $X\vee_p X \to X^2$ is defined by $S_p(x_1) = (x_1, x_1)$ and $S_p(x_2) = (p, x_2)$.

1.11. DEFINITION. The *fold map at p*, $\nabla_p : X\vee_p X \to X$ is given by $\nabla_p(x_i) = x$ for $i = 1,2$.

Let $U : E \to$ Sets be a topological functor, X an object in E, and p a point in $UX = B$.

1.12. DEFINITIONS.

1. X is $\overline{T_0}$ *at* p iff the initial lit of the U-source $\{A_p : B\bigvee_p B \to U(X^2) = B^2$ and $\nabla_p : B\bigvee_p B \to UDB = B\}$ is discrete, where DB is a discrete structure on B.

2. X is T_0' at p iff the initial lift of the U source $\{\mathrm{id} : B\bigvee_p B \to U(X\bigvee_p X) = B\bigvee_p B$ and $\nabla_p : B\bigvee_p B \to UDB = B\}$ is discrete, where $X\bigvee_p X$ is the wedge in E i.e. the final lift of the U-sink $\{i_1, i_2 : UX = B \to B\bigvee_p B\}$ where i_1, i_2 denote the canonical injections.

3. X is *Pre* $\overline{T_2}$ *at* p iff teh initial lift of the U-source $\{S_p : B\bigvee_p B \to U(X^2) = B^2\}$ and the initial lift of the U-source $\{A_p : B\bigvee_p B \to U(X^2) = B^2\}$ agree.

4. X is T_1 at p iff the initial lift of the U-source $\{S_p : B\bigvee_p B \to U(X^2) = B^2$ and $\nabla_p : B\bigvee_p B \to UDB = B\}$ is discrete.

5. X is *Pre* T_2' *at* p iff the initial lift of the U-source $\{S_p : B\bigvee_p B \to U(X^2) = B^2$ and the final lift of the U-sink $\{i_1, i_2 : UX = B \to B\bigvee_p B\}$ agree.

6. X is $\overline{T_2}$ *at* p iff X is $\overline{T_0}$ at p and *Pre* $\overline{T_2}$ at p.

7. X is T_2' *at* p *iff* X is T_0' at p *and* *Pre*T_2' at p [1] p. 15 and 16.

1.13 *Remark*. We define p_1, p_2, ∇_p by $1 + p$, $p + 1$, $1 + 1 : B\bigvee_p B \to B$, respectively where $1 : B \to B$ is the identity map, $p : B \to B$ is constant map at p, and $\pi_i : B^2 \to B$ is the projection $i = 1,2$. Note that $\pi_1 A_p = p_1 = \pi_1 S_p$, $\pi_2 A_p = p_2$, $\pi_2 S_p = \nabla_p$.

1.14 COROLLARY. *Let* α_i, $i = 1,2,3$ *be proper filters on* B. *If* $\sigma = p_1^{-1}\alpha_1 \cup p_2^{-1}\alpha_2 \cup \nabla^{-1}\alpha_3$, *then* σ *is a proper filter iff either* (a) $\alpha_2 \subset [p]$ *and* $\alpha_1 \cup \alpha_3$ *is proper or* (b) $\alpha_1 \subset [p]$ *and* $\alpha_2 \cup \alpha_3$ *is proper* [2] p. 95.

1.15 THEOREM. *Let* α_i, $i = 1,2,3$ *be proper filter on* B. *There* *exists a proper filter* σ *on* $B\bigvee_p B$ *such that* $p_1\sigma = \alpha_1$, $p_2\sigma = \alpha_2$, *and* $\nabla\sigma = \alpha_3$ *iff*

1. *if* (a) *of* 1.14 *fails, then* $\alpha_2 = \alpha_3$ *and* $\alpha_1 = [p]$
2. *If* (b) *of* 1.14 *fails, then* $\alpha_1 = \alpha_3$ *and* $\alpha_2 = [p]$.
3. *If neither* (a) *nor* (b) *of* 1.14 *fails, then* $\alpha_1 \cap \alpha_2 = \alpha_3 \cap [p]$ [2] p. 96.

1.16 THEOREM. *Let* α_1 *and* α_3 *be proper filters on* B. *there exists a proper filter* σ *on* $B\bigvee_p B$ *such that* $p_1\sigma = \alpha_1$ *and* $\nabla\sigma = \alpha_3$ *iff*

1. *If* $\alpha_1 \cup \alpha_3$ *is improper, then* $\alpha_1 = [p]$ *and* $p_2\sigma = \alpha_3$.
2. *If* $\alpha_1 \not\subset [p]$, *then* $\alpha_1 = \alpha_3$ *and* $p_2\sigma = [p]$.
3. *If* $\alpha_1 \subset [p]$ *and* $\alpha_1 \cup \alpha_3$ *is proper, then* $\alpha_1 \supset \alpha_3 \cap [p]$ *and* $p_2\sigma = \alpha_3 \cap [p]$.

Proof. [2] p. 105.

2. Separation Properties at p. In this section, we give explicit characterizations of the generalized separation properties at p for the topological categories of *FCO, LFCO, Lim, PsT,* and *PrT*.

2.1 THEOREM. $X = (B,K)$ *in FCO, LFCO, Lim, PsT, or PrT is* $\overline{T}_0$ *at* p *iff for each* $x \neq p$ $[x] \notin K(p)$ *or* $[p] \notin K(x)$ [3].

2.2 THEOREM. $X = (B,K)$ *in FCO, LFCO, Lim, PsT, or PrT is* T_1 *at* p *iff for each* $x \neq p$ $[x] \notin K(p)$ *and* $[p] \notin K(x)$ [3].

2.3 THEOREM. *All objects in FCO, LFCO, Lim, PsT, PrT are* T_0' *at* p.

Proof. $X = (B,K)$ is T_0' at p means for any σ and z in the wedge, $\sigma \supset i_k\sigma_1$, some $\sigma_1 \in K(x)$ with $i_k = z$ for $k = 1,2$ and

$\nabla\sigma = [x]$ or $[\phi]$ iff $\sigma = [z]$ or $[\phi]$. If $\sigma \supset i_1\sigma_1$ for some $\sigma_1 \in K(x)$ and $\nabla\sigma = [x]$ or $[\phi]$, then if follows easily that $\sigma = [(x,p)]$, $[(p,x)]$, $[\phi]$ or $\sigma \supset [(x,p)] \cup (p,x)]$. Since $\sigma \supset i_1\sigma_1$ for some

$\sigma_1 \in K(x)$, it follows that $\sigma = [(x,p)]$ or $[\phi]$. Similarly if $\sigma \supset i_2\sigma_1$ for some $\sigma_1 \in K(x)$, then $\sigma = [(p,x)]$ or $[\phi]$ since $i_2(x) = (p,x)$. If $\sigma \supset i_k\sigma_1$ or $i_1\sigma_1 \cap i_2\sigma_2$ (in the case of *Lim*, *PsT*, or *PrT*, 1.6, 1.7, 1.8) for some $\sigma_1,\sigma_2 \in K(p)$, $i_k(p) = (p,p)$, $k = 1$ or 2, and $\nabla\sigma = [p]$ or $[\phi]$, then it follows that $\sigma = [(p,p)]$ or $[\phi]$. Hence X is T_0' at p.

2.4 THEOREM. *$X = (B,K)$ is $Pre\overline{T_2}$ at p iff for each x in B, condition* (1) *holds for X if X is Lim, PsT, or PrT and conditions* (1) *and* (2) *hold for X in LFCO, and conditions* (1), (2), *and* (3) *hold for X if X is in FCO, where the conditions are:*

1. *If $K(x) \cap K(p) \neq \{[\phi]\}$, then $K(x) = K(p)$.*
2. *$Kp(x) = \{\alpha | \alpha \subset [p]$ and $\alpha \in K(x)\}$ is closed under finite intersection i.e. if $\alpha,\beta \in K(x)$ and $\alpha \subset [p]$, $\beta \subset [p]$, then $\alpha \cap \beta \in K(x)$.*
3. *For any $\alpha \in K_p(p)$ and $\beta \in K(p)$ if $\alpha \cup \beta$ is proper and*

$\beta \cap [p] \subset \alpha$, then $\beta \cap [p] \in K(p)$.

Proof. Suppose X is $Pre\overline{T_2}$ at p i.e for any filter σ and any point z in the wedge, if $p_1\sigma \in K(p_1z)$, then $p_2\sigma \in K(p_2z)$ iff $\nabla\sigma \in K(\nabla z)$ (1.4, and 1.13). Assume that $x \neq p$ and $K(x) \cap K(p) \neq \{[\phi]\}$. We show that $K(x) = K(p)$. To show $K(x) \subset K(p)$, let $\beta \in K(x)$. If β is improper, then $\beta \in K(p)$. Assume β is proper. Since $K(x) \cap K(p) \neq \{[\phi]\}$, there exists α in $K(x) \cap K(p)$ such that $\alpha \neq [\phi]$. We consider two cases foe α: namely $\alpha \subset [p]$ and

$\alpha \not\subset [p]$.

Case 1. If $\alpha \subset [p]$, then $[p] \in K(x)$ (since $\alpha \in K(x)$). In 1.15, let $\alpha_1 = [p]$ and $\alpha_2 = \beta = \alpha_3$. Note that $\alpha_1 \subset [p]$ and $\alpha_2 \cup \cup \alpha_3 = \beta$ is proper. further, if $\alpha_1 \cup \alpha_2 = [p] \cup \beta$ is improper, then by 1.15 (1), there exists a proper filter σ on the wedge such that $p_1\sigma = \alpha_1 = [p]$, $p_2\sigma = \alpha_2 = \beta = \alpha_3 = \nabla\sigma$. Now $p_1\sigma = = [p] \in K(x)$ and $\nabla\sigma = \beta \in K(x)$. Since X is $Pre\overline{T_2}$ at p, it follows that $p_2\sigma = \beta \in K(p)$. If $\alpha_1 \cup \alpha_3 = [p] \cup \beta$ is proper, then $\beta \subset [p]$. Note that $\alpha_1 \cap \alpha_2 = [p] \cap \beta = \beta = \alpha_3 \cap [p]$. Hence by 1.15 (3), there exists a proper filter σ on the wedge such that $p_1\sigma = \alpha_1 = [p]$, and $p_2\sigma = \alpha_2 = \beta = \alpha_3 = \nabla\sigma$, and consequently $p_2\sigma = \beta \in K(p)$ (since X is $Pre\overline{T_2}$ at p).

Case 2. If $\alpha \not\subset [p]$, then in 1.15, let $\alpha_1 = \alpha = \alpha_3$ and $\alpha_2 = = [p]$. Note that $\alpha_2 \subset [p]$, $\alpha_1 \cup \alpha_3 = \alpha$ is proper and $\alpha_1 \not\subset [p]$. Hence by 1.15 (2), there exists a proper filter σ on the wedge such that $p_1\sigma = \alpha_1 = \alpha = \alpha_3 = \nabla\sigma$ and $p_2\sigma = \alpha_2 = [p]$. Since $p_1\sigma = = \alpha \in K(p)$ and $\nabla\sigma = \alpha \in K(x)$, it follows that $p_2\alpha = [p] \in K(x)$ (since X is $Pre\overline{T_2}$ at p) and consequently the proof follows as in the first case. Hence $K(x) \subset K(p)$.

We next show that $K(p) \subset K(x)$. Let $\beta \in K(p)$. If β is improper, then $\beta \in K(x)$. Assume that β is proper. $K(x) \cap K(p) \neq \neq \{[\phi]\}$ implies that there exists α in $K(x) \cap K(p)$ such that $\alpha \neq [\phi]$. We consider two cases for α again: namely $\alpha \subset [p]$ and $\alpha \not\subset [p]$.

Case 1. If $\alpha \subset [p]$, then $[p] \in K(x)$ (since $\alpha \in K(x)$). In 1.15, take $\alpha_1 = [p]$ and $\alpha_2 = \beta = \alpha_3$. Note that $\alpha_1 \subset [p]$ and $\alpha_2 \cup \cup \alpha_3 = \beta$ is proper. Further if $\alpha_1 \cup \alpha_3 = [p] \cup \beta$ is improper, then by 1.15 (1), there exists a proper filter σ on the wedge such that $p_1\sigma = \alpha_1 = [p]$ and $p_2\sigma = \alpha_2 = ß = \alpha_3 = \nabla\sigma$. Since $p_1\sigma = [p] \in K(x)$ and $p_2\sigma = ß \in K(p)$, it follows that $\nabla\sigma = ß \in K(x)$ i.e. $ß \in K(x)$. If $\alpha_1 \cup \alpha_3 = [p] \cup ß$ is proper, (then in particular $ß \subset [p]$). Note that $\alpha_1 \cap \alpha_2 = [p] \cap ß = ß = \alpha_3 \cap [p]$. Hence by 1.15 (3), there exists a proper filter σ on the wedge such that $p_1\sigma = \alpha_1 = [p]$ and $p_2\sigma = \alpha_2 = ß = \alpha_3 = \nabla\sigma$. Since $p_1\sigma = [p] \in K(x)$ and $p_2\sigma = ß \in K(p)$, it follows that $\nabla\sigma = ß \in K(x)$ (since X is $Pre\overline{T}_2$ at p).

Case 2. If $\alpha \not\subset [p]$, then in 1.15, let $\alpha_1 = \alpha = \alpha_3$, $\alpha_2 = [p]$ and note that $\alpha_2 \subset [p]$, $\alpha_1 \cup \alpha_3 = \alpha$ is proper and $\alpha_2 \cup \alpha_3 = [p] \cup \alpha$ is improper (since $\alpha \not\subset [p]$). Hence by 1.15 (2), there exists a proper filter σ on the wedge such that $p_1\sigma = \alpha_1 = \alpha = \alpha_3 = \nabla\sigma$ and $p_2\sigma = \alpha_2 = [p]$. Since $\alpha \in K(x) \cap K(p)$ i.e. $p_1\sigma \in K(p)$ and $\nabla\sigma = \alpha \in K(x)$, it follows that $p_2\sigma = [p] \in K(x)$ (since X is $Pre\overline{T}_2$ at p) and the proof follows as in case 1. Therefore $K(p) \subset K(x)$.

If $x = p$, then $K(x) \cap K(p) = K(p) \neq \{[\phi]\}$ since $[p] \in K(p)$ and clearly $K(x) = K(p)$. This shows condition (1) holds.

We next show that condition (2) holds i.e $K_p(x)$ is closed under finite intersection i.e if $\alpha, ß \in K(x)$ and α, $ß \subset [p]$, then $\alpha \cap ß \in K(x)$. Assume that $x \neq p$. Since $\alpha \in K(x)$ and $\alpha \subset [p]$, $[p] \in K(x) \cap K(p)$ and as a consequence of condition (1) holding, $K(x) = K(p)$. If $\alpha_1 = \alpha$, $\alpha_2 = ß$ and $\alpha_3 = \alpha \cap ß$, then $\alpha_1 \cup \alpha_3 = \alpha$ and $\alpha_2 \cup \alpha_3 = ß$ are proper, α_1, $\alpha_2 \subset [p]$, $\alpha_1 \cap \alpha_2 = \alpha \cap ß$ and $\alpha_3 \cap [p] = \alpha \cap ß$ (since $\alpha \cap ß \subset [p]$) and consequently $\alpha_1 \cap \alpha_2 = \alpha_3 \cap [p]$. Hence by 1.15 (3) there exists a proper filter σ on the wedge such that $p_1\sigma = \alpha_1 = \alpha$, $p_2\sigma = \alpha_2 = \beta$ and $\nabla\sigma = \alpha_3 = \alpha \cap \beta$. Since $p_1\sigma = \alpha \in K(x)$ and $p_2\sigma = \beta \in K(p) = K(x)$, it follows that $\nabla\sigma = \alpha \cap \beta \in K(x)$ (since X is $Pre\overline{T_2}$ at p). Hence $\alpha \cap \beta \in K_p(x)$. Suppose now that $x = p$. If $\alpha, \beta \in K_p(p)$, then clearly $\alpha \cap \beta \subset [p]$. It remains to be shown that $\alpha \cap \beta \in K(p)$. If $\alpha_1 = \alpha$, $\alpha_2 = \beta$ and $\alpha_3 = \alpha \cap \beta$ in 1.15, then $\alpha_1, \alpha_2 \subset [p]$, $\alpha_1 \cup \alpha_3 = \alpha$ and $\alpha_2 \cup \alpha_3 = \beta$ are proper, and $\alpha_1 \cap \alpha_2 = \alpha \cap \beta = \alpha_3 \cap [p]$ (since $\alpha \cap \beta \subset [p]$). Hence by 1.15 (3) there exists a proper filter σ on the wedge such that $p_1\sigma = \alpha$, $p_2\sigma = \beta$, and $\nabla\sigma = \alpha \cap \beta$. Since $p_1\sigma = \alpha \in K(p)$ and $p_2\sigma = \beta \in K(p)$, it follows that $\nabla\sigma = \alpha \cap \beta \in K(p)$ (since X is $Pre\overline{T_2}$ at p). Thus $\alpha \cap \beta \in K_p(p)$. This shows condition (2) holds.

Finally, we show that (3) holds i.e. for any $\alpha \in K_p(p)$ and $\beta \in K(p)$ if $\alpha \cup \beta$ is proper and $\beta \cap [p] \subset \alpha$, then $\beta \cap [p] \in K(p)$. If $\alpha_1 = \alpha$, $\alpha_2 = \beta \cap [p]$, and $\alpha_3 = \beta$ in 1.15, then $\alpha_1, \alpha_2 \subset [p]$, $\alpha_1 \cup \alpha_3 = \alpha \cup \beta$, and $\alpha_2 \cup \alpha_3 = \beta$ are proper (the former by the assumption). Further, since $\alpha_1 \cap \alpha_2 = \alpha \cap (\beta \cap [p]) = \beta \cap [p]$ ($\beta \cap [p] \subset \alpha$ by the assumption) and $\alpha_3 \cap [p] = \beta \cap [p]$, it follows that $\alpha_1 \cap \alpha_2 = \alpha_3 \cap [p]$. Hence by 1.15 (3), there exists a proper filter σ on the wedge such that $p_1\sigma = \alpha$, $p_2\sigma = \beta \cap [p]$, and $\nabla\sigma = \beta$. Since $p_1\sigma = \alpha \in K(p)$ and $\nabla\sigma = \beta \in K(p)$, it follows that $p_2\sigma = \beta \cap [p] \in K(p)$ (since X is $Pre\overline{T_2}$ at p). This shows condition (3) holds.

To prove the converse, we must show that X is $Pre\overline{T_2}$ at p i.e. by 1.4 and 1.13, for any filter σ on the wedge and any point z in the wedge, if $p_1\sigma \in K(p_1 z)$, then $p_2\sigma \in K(p_2 z)$ iff $\nabla\sigma \in K(\nabla z)$, if (1) holds when X is in *Lim*, *PsT*, or *PrT*, and if (1) and

(2) hold when X is in *LFCO*, and if (1), (2), and (3) hold when X is in *FCO*. we begin by showing that for any filter σ on the wedge and any point z in the wedge if $p_1\sigma \in K(p_1z)$ and $\nabla\sigma \in K(\nabla z)$, then $p_2\sigma \in K(p_2z)$. There are three possibilities for z: namely $z = (x,p)$, (p,x), and (p,p), we first assume $z = (x,p)$. If σ is improper, then clearly $p_2\sigma \in K(p)$. If σ is proper, then we have $p_1\sigma \in K(x)$ and $\nabla\sigma \in K(x)$. We are now applying theorem 1.15 with $\alpha_1 = p_1\sigma$, $\alpha_2 = p_2\sigma$, and $\alpha_3 = \nabla\sigma$. In case 1 of theorem 1.15, $p_1\sigma = \nabla\sigma$ and $p_2\sigma = [p]$ and consequently $p_2\sigma \in K(p)$. In case 2 of theorem 1.15, $p_1\sigma = [p]$ and $p_2\sigma = \nabla\sigma$, and consequently $p_1\sigma = [p] \in K(x) \cap K(p)$. Hence from the assumption, we get $K(x) = K(p)$ and consequently $p_2\sigma \in K(p) = K(x)$ (since $\nabla\sigma \in K(x)$ and $\nabla\sigma = p_2\sigma$). In case 3 of theorem 1.15, $p_1\sigma \cap p_2\sigma = \nabla\sigma \cap [p]$. Since $p_1\sigma \subset [p]$ and $p_1\sigma \in K(x)$, it follows that $[p] \in K(x) \cap K(p)$ and consequently by assumption (1) $K(x) = K(p)$. If further X is in *LFCO, Lim, PsT,* or *PrT*, R_0PrT, then $\nabla\sigma \cap [p] \in K(p) = K(x)$ since $\nabla\sigma \in K(x) = K(p)$ and consequently $p_1\sigma \cap p_2\sigma \in K(p)$. Hence it follows that $p_2\sigma \in K(p)$. If X is in *FCO*, then let $\alpha = p_1\sigma$ and $\beta = \nabla\sigma$. Note that $\alpha \in K_p(p)$, $\beta \in K(p)$, $\alpha \cup \beta = p_1\sigma \cup \nabla\sigma$ is proper and $\beta \cap [p] = \nabla\sigma \cap [p] \subset p_1\sigma = \alpha$. Hence from assumption (3) we have $\beta \cap [p] \in K(p)$ and consequently $p_2\sigma \in K(p)$ (since $\beta \cap [p] \subset p_2\sigma$). We next assume $z = (p,x)$ with $x \neq p$. Now $p_1\sigma \in K(p)$ and $\nabla\sigma \in K(x)$. We show that $p_2\sigma \in K(x)$. We again apply theorem 1.15 with $\alpha_1 = p_1\sigma$, $\alpha_2 = p_2\sigma$, and $\alpha_3 = \nabla\sigma$. In case 1 of theorem 1.15, $p_2\sigma = [p]$ and $\nabla\sigma = p_1\sigma$, and consequently $p_1\sigma \in K(x) \cap K(p)$. Hence by assumption (1), $K(x) = K(p)$ and consequently $p_2\sigma = [p] \in K(p) = K(x)$. In case 2 of theorem 1.15, $p_1\sigma = [p]$ and $p_2\sigma = \nabla\sigma$ and

consequently $p_2\sigma \in K(x)$ since $\nabla\sigma \in K(x)$. In case 3 of theorem 1.15, $p_1\sigma \cap p_2\sigma = = \nabla\sigma \cap [p]$. We have $p_1\sigma \cup \nabla\sigma$ is proper and is in $K(x) \cap K(p)$, and thus, by assumption (1), $K(x) = K(p)$.

If X is in *LFCO, Lim, PsT, PrT*, then $p_2\sigma \supset (\nabla\sigma \cap [p]) \in \in K(p) = K(x)$ (since $\nabla\sigma \in K(p) = K(x)$) and consequently $p_2\sigma \in \in K(x)$. If X is in *FCO*, then let $\alpha = p_1\sigma$ and $\beta = \nabla\sigma$. Note that $\alpha \in K_p(p)$, $\beta \in K(p)$, $\alpha \cup \beta = p_1\sigma \cup \nabla\sigma$ is proper (since case 3 of 1.15 holds) and $\beta \cap [p] \subset p_1\sigma = \alpha$. Hence by assumption (3) $\beta \cap \cap [p] \in K(p) = K(x)$ and consequently $p_2\sigma \in K(x)$ (since $\beta \cap [p] \subset \subset p_2\sigma$). Finally we assume $z = (p,p)$. Then we have $p_1\sigma \in K(p)$ and $\nabla\sigma \in K(p)$. We must show that $p_2\sigma \in K(p)$. We again apply theorem 1.15 with $\alpha_1 = p_1\sigma$, $\alpha_2 = p_2\sigma$, and $\alpha_3 = \nabla\sigma$. In case 1 of theorem 1.15, we have $p_2\sigma = [p]$ and $\nabla\sigma = p_1\sigma$, and consequently $p_2\sigma \in \in K(p)$. In case 2 of theorem 1.15, $p_1\sigma = [p]$ and $\nabla\sigma = p_2\sigma$. Hence $p_2\sigma \in K(p)$ since $\nabla\sigma \in K(p)$. In case 3 of theorem 1.15, $p_1\sigma \cap p_2\sigma = = \nabla\sigma \cap [p]$. If X is in *LFCO, Lim, PsT*, or *PrT*, then $p_2\sigma \supset (\nabla\sigma \cap \cap [p]) \in K(p)$ (since $\nabla\sigma \in K(p)$) and consequently $p_2\sigma \in K(p)$. If X is in *FCO*, then let $\alpha = p_1\sigma$ and $\beta = \nabla\sigma$. Note that $\alpha \in K_p(p)$, $\beta \in K(p)$, $\alpha \cup \beta = p_1\sigma \cup \nabla\sigma$ is proper, and $\beta \cap [p] \subset p_1\sigma = \alpha$. Hence by assumption (3), $\beta \cap [p] \in K(p)$ and consequently since $p_2\sigma \supset \beta \cap [p]$, $p_2\sigma \in K(p)$.

We next show that for any filter σ on the wedge and any point z in the wedge if $p_1\sigma \in K(p_1z)$ and $p_2\sigma \in K(p_2z)$, then $\nabla\sigma \in \in K(\nabla z)$. If σ is improper, then clearly $\nabla\sigma \in K(\nabla z)$. We may assume σ is proper. There are three possibilities for the point z: namely (x,p), (p,x) and (p,p). We first assume $z = (x,p)$ with $x \neq p$. Hence $p_1\sigma \in K(x)$ and $p_2\sigma \in K(p)$ and we must show that

$\nabla\sigma \in K(x)$. To this end, we apply theorem 1.15 with $\alpha_1 = p_1\sigma$, $\alpha_2 = p_2\sigma$ and $\alpha_3 = \nabla\sigma$. In case 1 of 1.15, $p_2\sigma = [p]$ and $p_1\sigma = \nabla\sigma$ and consequently $\nabla\sigma \in K(x)$. In case 2 of 1.15, $p_1\sigma = [p]$ and $p_2\sigma = \nabla\sigma$. Hence $p_1\sigma = [p] \in K(x) \cap K(p)$ and thus, by assumption (1) $K(x) = K(p)$. Consequently, $\nabla\sigma \in K(x)$ (since $p_2\sigma \in K(p) = K(x)$ and $p_2\sigma = \nabla\sigma$). In case 3 of 1.15, $p_1\sigma \cap p_2\sigma = \nabla\sigma \cap [p]$. Since $p_1\sigma \in K(x)$ and $p_1\sigma \subset [p]$, $[p] \in K(x)$ and thus from assumption (1) $K(x) = K(p)$ and consequently if X is in *Lim, PsT,* or *PrT*, then $\nabla\sigma \supset (p_1\sigma \cap p_2\sigma) \in K(p) = K(x)$ and thus $\nabla\sigma \in K(x)$. If X is in *FCO* or *LFCO*, then by assumption (2) $p_1\sigma \cap p_2\sigma \in K_p(x)$ (since $p_1\sigma$, $p_2\sigma \subset [p]$ and $p_1\sigma$, $p_2\sigma$ are in $K(x) = K(p)$) and consequently $\nabla\sigma \in$ $\in K(x)$. We next assume that $z = (p,x)$, $x \neq p$. Then $p_1\sigma \in K(p)$, $p_2\sigma \in K(x)$, and we must show that $\nabla\sigma \in K(x)$. To show this we apply 1.15 with $\alpha_1 = p_1\sigma$, $\alpha_2 = p_2\sigma$, and $\alpha_3 = \nabla\sigma$. In case 1 of 1.15, we have $p_2\sigma = [p]$ and $p_1\sigma = \nabla\sigma$. Since $p_2\sigma \in K(x)$ and $p_2\sigma =$ $= [p]$, it follows from assumption (1) that $K(x) = K(p)$. Hence $\nabla\sigma \in K(x)$ because $p_1\sigma = \nabla\sigma$, and $p_1\sigma \in K(p) = K(x)$. In case 2 of 1.15, $p_1\sigma = [p]$ and $p_2\sigma = \nabla\sigma$, and consequently $\nabla\sigma \in K(x)$. In case 3 of 1.15, we have $p_1\sigma \cap p_2\sigma = \nabla\sigma \cap [p]$. Now $p_2\sigma \subset [p]$ and $p_2\sigma \in$ $\in K(x)$ and thus by assumption (1), $K(x) = K(p)$ and consequently further, if X is in *Lim, PsT*, or *PrT*, then $\nabla\sigma \supset (p_1\sigma \cap p_2\sigma) \in$ $\in K(p) = K(x)$ and consequently $\nabla\sigma \in K(x)$. If X is in *FCO* or *LFCO*, then by assumption (2), $p_1\sigma \cap p_2\sigma \in K_p(x)$ since $p_1\sigma$, $p_2\sigma \subset [p]$ and $p_1\sigma$, $p_2\sigma$ are in $K(x) = K(p)$. Hence $\nabla\sigma \in K(x)$. Finally, if $z=$ $= (p,p)$, then $p_1\sigma \in K(p)$ and $p_2\sigma \in K(p)$. We must show that $\nabla\sigma \in$ $\in K(p)$. To this end, we apply 1.15 with $\alpha_1 = p_1\sigma$, $\alpha_2 = p_2\sigma$, and $\alpha_3 = \nabla\sigma$. In case 1 of 1.15, $p_2\sigma = [p]$ and $p_1\sigma = \nabla\sigma$. Hence $\nabla\sigma \in$

$\in K(p)$. In case 2 of 1.15, we have $p_1\sigma = [p]$ and $p_2\sigma = \nabla\sigma$ and, thus $\nabla\sigma \in K(p)$. In case 3 of 1.15, we have $p_1\sigma \cap p_2\sigma = \nabla\sigma \cap [p]$ and if X is in *Lim, PsT,* or *PrT,* then $p_1\sigma \cap p_2\sigma \in K(p)$ and consequently $\nabla\sigma \in K(p)$. If X is in *FCO* or *LFCO,* then by assumption (2) $p_1\sigma \cap p_2\sigma \in K_p(p)$ since $p_1\sigma, p_2\sigma \subset [p]$ and $p_1\sigma, p_2\sigma$ are in $K(p)$, and consequently $\nabla\sigma \in K(p)$. This completes the proof.

2.5 THEOREM. *$X = (B,K)$ is $PreT_2'$ at p iff condition* (1) *holds for X when X is in Lim, PsT or PrT and conditions* (1) *and* (2) *hold for X when X is in FCO or LFCO, where the conditions are:* (1) *for each $x \neq p$, $K(x) \cap K(p) = \{[\phi]\}$ and* (2) *$K_p(p) =$ $= \{[p]\}$.*

Proof. Suppose X is $PreT_2'$ at p i.e. by 1.4, 1.13, 1.5, and definition 1.12 for any filter σ on the wedge and point z in the wedge (a) $p_1\sigma \in K(p_1z)$ and $\nabla\sigma \in K(\nabla z)$ iff (b) $\sigma \supset i_k\sigma_1$ for some σ_1 in $K(x)$ where $i_k(x) = z$, $k = 1$ or 2 for X in *FCO* or *LFCO*. If X in *Lim, PsT* or *PrT,* then by 1.6, 1.7, or 1.8, condition (b) must be replaced by (c): $\sigma \supset i_k\sigma_1$ for some $\sigma_1 \in K(x)$ where $i_k(x) = z$, $k = 1$ or 2 if $x \neq p$ and $\sigma \supset i_1\sigma_1 \cap i_2\sigma_2$ for some σ_1,σ_2 in $K(p)$ where $i_1(p) = (p,p) = i_2(p)$. We begin by showing that if X is $PreT_2'$ at p, then (1) holds. To this end, suppose there exists a proper filter $\alpha \in K(x) \cap K(p)$ for some $x \neq p$. We consider the two cases $\alpha \subset [p]$ and $\alpha \not\subset [p]$. if $\alpha \subset [p]$, then $[p]$ $\in K(x)$. If $\alpha = [(p,x)]$, then clearly $p_1\sigma = [p] \in K(x)$ and $\nabla\sigma =$ $[x] \in$

$\in K(x)$. Since X is $PreT_2'$ at $p, \sigma \supset i_k\sigma_1$ for some $\sigma_1 \in K(x)$ and $i_k(x) = (x,p)$ i.e. $k = 1$. Note that $p_2\sigma = [x] \supset p_2 i_1\sigma_1 = [p]$, a contradiction since $x \neq p$. Hence $\alpha \not\subset [p]$. However, if $\alpha \not\subset [p]$, then in 1.16, let $\alpha_1 = \alpha = \alpha_3$ and note that $\alpha_1 \cup \alpha_3 = \alpha$ is proper and $\alpha_1 = \alpha \not\subset [p]$. Hence by 1.16 (2) there exists a proper filter σ on the wedge such that $p_1\sigma = \alpha_1 = \alpha = \alpha_3 = \nabla\sigma$ since $\alpha \in K(x) \cap \cap K(p)$, $p_1\sigma \in K(p)$, and $\nabla\sigma \in K(x)$, it follows that $\sigma \supset i_2\sigma_1$ for some $\sigma_1 \in K(x)$ and $i_2(x) = (p,x)$ (since X is $PreT_2'$ at p) and $x \neq p$. Now $p_1\sigma = \alpha \supset [p] = p_1 i_2 \sigma_1$, a contradiction since $\alpha \not\subset [p]$ and α is proper. Hence $K(x) \cap K(p) = \{[\phi]\}$ for all $x \neq p$. This shows condition (1) holds.

We next show that (2) holds i.e. $K_p(p) = \{[p]\}$ if X is *FCO* or *LFCO*. Suppose there exists $\alpha \in K(p)$ with $\alpha \subset K(p)$ and $\alpha \neq [p]$. If we take $\alpha_1 = \alpha = \alpha_3$ in 1.16, then $\alpha_1 \cup \alpha_3 = \alpha$ is proper, $\alpha_1 \subset \subset [p]$, and $\alpha_3 \cap [p] = \alpha \subset \alpha_1 = \alpha$. Hence by 1.16 (3) there exists a proper filter σ on the wedge such that $p_1\sigma = \alpha_1 = \alpha = \alpha_3 = \nabla\sigma$ and $p_2\sigma = \alpha_3 \cap [p] = \alpha \cap [p] = \alpha$. Since $p_1\sigma \in K(p)$ and $\nabla\sigma \in K(p)$, it follows that $\sigma \supset i_1\sigma_1$ or $i_2\sigma_1$ for some $\sigma_1 \in K(p)$ since X is $PreT_2'$ at p. If $\sigma \supset i_2\sigma_1$, then $p_1\sigma = \alpha \supset p_1 i_2 \sigma_1 = [p]$, a contradiction. If $\sigma \supset i_1\sigma_1$, then by 1.16 (3) $p_2\sigma = \alpha_3 \cap [p] = = \alpha \supset p_2 i_1 \sigma_1 = [p]$ i.e $\alpha = [p]$, a contradiction. This shows condition (2) hold.

On the other hand, suppose conditions (1) and (2) hold. We must show that X is $PreT_2'$ at p i.e (a) holds iff either (b) holds for X if X is in *FCO* or *LFCO* or (c) holds for X if X is in *Lim*, *PsT* or *PrT*. By [1] (b) implies (a) and (c) implies (a) since all of the categories *FCO*, *LFCO*, *Lim*, *PsT*, and *PrT* are normalized. Thus it remains to show that (a) implies both (b) and (c). Suppose for any point z in the wedge and any filter σ on the wedge $p_1\sigma \in K(p_1z)$ and $\nabla\sigma \in K(\nabla z)$. There are three possibilities for z: namely $z = (x,p)$, (p,x) and (p,p). If σ is improper, then clearly $p_1\sigma$ is improper and so $i_1p_1\sigma \subset \sigma$ is improper. If σ is proper and $z = (x,p)$, $x \neq p$, then $p_1\sigma \in K(x)$ and $\nabla\sigma \in K(x)$. We now apply 1.16 with $\alpha_1 = p_1\sigma$ and $\alpha_3 = \nabla\sigma$. In case (1) of 1.16, we have $p_1\sigma = p$ and consequently $[p]$ in $K(x) \cap K(p)$, a contradiction (assumption (1)). In case 2 of 1.16, we have $p_1\sigma = \nabla\sigma$ and $p_1\sigma \not\subset$ $\not\subset [p]$. We show that $\sigma \supset i_1p_1\sigma$. If $U \in i_1p_1\sigma$, then $U \supset i_1p_1W$ for some $W = U_1\bigvee_p U_2$ in σ. We may assume that $U_2 = \phi$ since $p_1\sigma \not\subset [p]$ (because (b) of 1.16 fails). Hence $U \supset i_1p_1W = U_1 = W$ and consequently $U \in \sigma$ i.e. $\sigma \supset i_1p_1\sigma_1$. In case 3 of 1.16 we have $p_1\sigma \supset \nabla\sigma \cap [p]$. Since $p_1\sigma \in K(x)$ and $p_1\sigma \subset [p]$, it follows that $[p] \in K(x) \cap K(p)$, a contradiction. Therefore only case 2 of 1.16 holds and in that case $\sigma \supset i_1p_1\sigma$. If $z = (p,x)$ and $x \neq p$, then we have $p_1\sigma \in K(p)$ and $\nabla\sigma \in K(x)$. We show that $\sigma \supset i_2\sigma_1$ for some $\sigma_1 \in K(x)$ with $i_2(x) = (p,x)$. To this end we apply theorem 1.16 with $\alpha_1 = p_1\sigma$ and $\alpha_3 = \nabla\sigma$. In case 1 of 1.16 $p_1\sigma = [p]$. Clearly $\sigma \supset i_2\nabla\sigma$. To see this note that if $U \in i_2\nabla\sigma$, then $U \supset i_2\nabla W$ for some $W = U_1\bigvee_p U_2$ in σ. We may assume that $U_1 = \phi$ since $p_1\sigma \cup \nabla\sigma$ is proper ((a) of 1.16 fails) and $p_1\sigma \subset [p]$. Hence $U \supset i_2\nabla W = U_2 =$ $= W$ and consequently $U \in \sigma$ i.e. $\sigma \supset i_2\nabla\sigma$. In case 2 of 1.16, $p_1\sigma = \nabla\sigma \in K(x) \cap K(p)$, a contradiction. In case 3 of 1.16 $p_1\sigma \cup$ $\cup \nabla\sigma$ is proper and is in $K(x) \cap K(p)$, a contradiction. Finally we assume that $z = (p,p)$. For this case we have $p_1\sigma \in K(p)$ and $\nabla\sigma \in K(p)$. We apply 1.16 with $\alpha_1 = p_1\sigma$ and $\alpha_3 = \nabla\sigma$. In case 1 of 1.16 we have $p_1\sigma = [p]$. Clearly $\sigma \supset i_2\nabla\sigma$ (the proof is given above). In case 2 of 1.16, we have $p_1\sigma = \nabla\sigma$ and consequently

$\sigma \supset i_1p_1\sigma$ (the proof is given above). In case 3 of 1.16, we have $p_1\sigma \supset \nabla\sigma \cap [p]$ and if X is in *Lim*, *PsT* or *PrT*, then $\sigma \supset i_1\nabla\sigma \cap \cap i_2\nabla\sigma$. To see this note that if $U \in i_1\nabla\sigma \cap i_2\nabla\sigma$, then $U \supset i_1V_1 \cup \cup i_2V_2$ for some $V_1,V_2 \in \nabla\sigma$. Since $\nabla\sigma$ is a filter, $V = V_1 \cap V_2 \in \in \nabla\sigma$ and consequently $V \supset \nabla W$ for some $W = U_1\bigvee_p U_2 \in \sigma$. Since $\nabla W = = U_1 \cup U_2$ and $U \supset i_1V \cup i_2V \supset \nabla W\bigvee_p\nabla W = (U_1 \cup U_2)\bigvee_p(U_1 \cup U_2) \supset \supset U_1\bigvee_p U_2 = W$, it follows that $U \in \sigma$ and consequently $\sigma \supset i_1\nabla\sigma \cap \cap i_2\nabla\sigma$. If X is in *FCO* or *LFCO*, then we have $p_1\sigma \subset [p]$ and $p_1\sigma \cup \cup \nabla\sigma$ is proper (since case 3 of 1.16 holds). Hence from assumption (2), $p_1\sigma = [p]$ (since $p_1\sigma \subset K_p(p)$) and consequently $\nabla\sigma = [p]$ (since $p_1\sigma \cup \nabla\sigma = [p] \cup \nabla\sigma$ is proper iff $\nabla\sigma \subset [p]$). Thus $\sigma = [(p,p)] = i_1[p] = i_2[p]$. This completes the proof.

2.6 THEOREM. *$X = (B,K)$ is $\bar{T}_2$ at p iff for each x in B, condition* (1) *holds for X if X is Lim, PsT, or R_0PrT and conditions* (1) *and* (2) *hold for X in LFCO, and conditions* (1), (2), *and* (3) *hold for X if X is in FCO, where the conditions are:*

1. *For each $x \neq p$, $K(x) \cap K(p) = \{[\phi]\}$.*
2. *$K_p(p)$ is closed under finite intersection.*
3. *For any $\alpha \in K_p(p)$ and $\beta \in K(p)$ if $\alpha \cup \beta$ is proper and $\beta \cap [p] \subset \alpha$, then $\beta \cap [p] \in K(p)$.*

Proof. Suppose X is $\bar{T}_2$ at p i.e. by definition 1.12, X is $\bar{T}_0$ at p and *Pre*$\bar{T}_2$ at p. If $K(x) \cap K(p) \neq \{[\phi]\}$, for some $x \neq p$, then by 2.4, $K(x) = K(p)$ (since X is *Pre*$\bar{T}_2$ at p) and consequently $[x] \in K(p)$ and $[p] \in K(x)$. This is a contradiction since X is $\bar{T}_0$ at p (2.1). Hence for each $x \neq p$ $K(x) \cap K(p) = \{[\phi]\}$ and (1) holds. If X is in *LFCO*, then clearly $K_p(p)$ is closed under finite intersection since X is *Pre*$\bar{T}_2$ at p

(2.4). If further X is in *FCO*, then condition (3) holds since X is $Pre\overline{T}_2$ at p (2.4).

On the other hand, if conditions (1), (2), and (3) hold in the appropriate cases then, we will show that X is $\overline{T}_0$ at p and $Pre\overline{T}_2$ at p. To this end, it is clear that for $x \neq p$, $[x] \notin K(p)$ or $[p] \notin K(x)$ since $K(x) \cap K(p) = \{[\phi]\}$. Hence by 2.1 X is $\overline{T}_0$ at p. To show X is $Pre\overline{T}_2$ at p, since $K(x) \cap K(p) = \{[\phi]\}$ condition (1) of 2.4 is trivially true. It remains to be shown that (2) and (3) of 2.4 hold. If $\alpha,\beta \in K_p(x)$ and $x \neq p$ i.e. $\alpha,\beta \in K(x)$ and $\alpha,\beta \subset [p]$, then $[p] \in K(x) \cap K(p)$, a contradiction. Hence $x = p$ and we are reduced to showing that $K_p(p)$ is closed under finite intersection. But this is true by assumption (2) if X is in *LFCO*. Condition (3) of 2.4 is the same as the given condition (3) if X is *FCO*. This completes the proof.

2.7 THEOREM. *$X = (B,K)$ is T_2' at p iff condition* (1) *holds for X when X is in Lim, PsT, or PrT and condition* (1) *and* (2) *hold for X when X is in FCO or LFCO, where the conditions are:* (1) *for each $x \neq p$, $K(x) \cap K(p) = \{[\phi]\}$ and* (2) *$K_p(p) = \{[p]\}$.*

Proof. Since all X's in the given categories are T_0' at p (2.3), the result follows from definition 1.12 and theorem 2.5.

2.8 *Remark*. For the categories *FCO* and *LFCO*, $\overline{T}_0$ at p, $PreT_2'$ at p, and T_2' at p imply T_0' at p, $Pre\overline{T}_2$ at p and $\overline{T}_2$ at p respectively. But the converse is not true. For the categories *Lim*, *PsT*, and *PrT*: $\overline{T}_0$ at p and $PreT_2'$ at p imply T_0' at p and $Pre\overline{T}_2$ at p, respectively. further in these categories $\overline{T}_2$ at p and T_2' at p are identical. (2.6 and 2.7)

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UNIFORM BOUNDEDNESS FROM ABOVE OF FAMILIES OF NUMERICAL FUNCTIONS

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REZUMAT. - Mărginirea superioară uniformă a unor familii de funcţii numerice. Se indică condiţii suficiente pentru ca o familie de funcţii numerice, fie preaditive, fie presubtractive pe o submulţime nevidă închisă a unui grup abelian topologic, să fie uniform mărginită superior pe o vecinătate a elementului neutru al acestui grup.

1. Let X be a topological abelian additive group, whose zero-element is denoted by o, and let F be a family of functions from X to the extended real axis. When Y is a subset of X, then we say that F is:

i) **preadditive** on Y if there exists a real number $c > 0$ such that

$$f(x + y) \leq c \max \{f(x), f(y)\} \tag{1.1}$$

for each $f \in F$ and all $x, y \in Y$;

ii) **presubtractive** on Y if there exists a real number $c>0$ such that

$$f(x - y) \leq c \max \{f(x), f(y)\} \tag{1.2}$$

for each $f \in F$ and all $x, y \in Y$;

iii) **pointwise bounded from above** on Y if

$$\sup \{f(x) : f \in F\} < \infty \text{ for every } x \in Y;$$

iv) **uniformly bounded from above** on Y if

$$\sup \{M(f, Y) : f \in F\} < \infty,$$

where

$$M(f, Y) = \sup \{f(x) : x \in Y\} \text{ for every } f \in F.$$

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The purpose of the present paper is to point out conditions which, together with the assumption on F to be either preadditive or presubtractive on a nonempty closed subset of X, assure that F is uniformly bounded from above on a neighbourhood of o. The results, we shall have obtained, generalize not only the classical uniform boundedness principle concerning a family of continuous linear mappings from a normed linear space to another one (see for instance E.Hille and R.S.Phillips [2, p. 26, Theorem 2.5.5]), but also several other extensions or variants of this principle given in the framework of normed linear spaces by E.Hille and R.S.Phillips [2, p. 26, Theorem 2.5.4], S.B.Stečkin [4], and W.Smajdor [3, p. 49, Theorem 3.10]. Other uniform boundedness principles for families of numerical functions defined on a topological group have been proved by J.Daneš [1] as well as by P.P.Zabreĭko and E.I.Smirnov [5].

2. In this section we establish sufficient conditions for the existence of a neighbourhood of o on which the family F is uniformly bounded from above. First we deal with the case when F is preadditive on a certain subset of X.

THEOREM 2.1. *Let U be a closed symmetric subset of X, and let F satisfy the following conditions:*

(i) *F is preadditive on U;*

(ii) *each $f \in F$ is lower semicontinuous on U;*

(iii) *there exists a subset T of U which is of the second category in X, dense in U, and on which F is pointwise bounded from above.*

Then there exists a neighbourhood of o on which F is uniformly bounded from above.

Proof. For each positive integer n set

$$Y_n = \{ x \in U : f(x) \leq n \text{ for every } f \in F \}. \qquad (2.1)$$

Note that all the sets Y_n ($n \in \mathbb{N}$) are closed. If Y is defined to be the union of the family (Y_n), then we have $T \subseteq Y$. Therefore Y must be of the second category in X. Consequently, there exists a positive integer m for which Y_m has interior points. Let x_0 be any interior point of Y_m. After that select a neighbourhood W of o such that $x_0 + W - W \subseteq Y_m$. Since $-x_0 \in U$ and T is dense in U, we have $(W - x_0) \cap T \neq \phi$. So there exists a $y_0 \in W$ such that $y_0 - x_0 \in T$. Taking into account that F is pointwise bounded from above on T, it results that there is a real number c_0 such that

$$\sup \{ f(y_0 - x_0) : f \in F \} \leq c_0.$$

We claim that

$$\sup \{M(f, W) : f \in F\} \leq c \max \{m, c_0\}, \qquad (2.2)$$

where $c > 0$ is a real number satisfying (1.1) for each $f \in F$ and all $x, y \in U$.

To see this, let f be any function in F and let x be any point in W. Then we have $x_0 + x - y_0 \in x_0 + W - W \subseteq Y_m$. Therefore $x_0 + x - y_0$ lies in U and satisfies $f(x_0 + x - y_0) \leq m$. But, because T is a subset of U, the point $y_0 - x_0$ lies also in U. By applying the preadditivity of F on U, it follows that

$$f(x) = f((x_0 + x - y_0) + (y_0 - x_0)) \leq$$

$$\leq c \max \{f(x_0 + x - y_0), f(y_0 - x_0)\} \leq c \max \{m, c_0\}.$$

Thus (2.2) holds, as claimed.

So it has been proved that F is uniformly bounded from above on W.∎

THEOREM 2.2. *Let U be a closed symmetric neighbourhood of o, and let F satisfy the following conditions:*

(i) *F is preadditive on U;*

(ii) *there exists a real number c_0 such that for each $f \in F$ there is a neighbourhood V of o for which $M(f, V) \leq c_0$;*

(iii) *there exists a subset T of U which is of the second category in X, dense in U, and on which F is pointwise bounded from above.*

Then there exists a neighbourhood of o on which F is uniformly bounded from above.

Proof. For each positive integer n define the set Y_n by (2.1) and then denote by Y the union of the family (Y_n). Since $T \subseteq Y$, the set Y must be of the second category in X. Consequently there exists a positive integer m such that the closure cl Y_m of Y_m has interior points. Since U is closed, we have cl $Y_m \subseteq U$. Furthermore, cl Y_m satisfies

$$\sup \{M(f, \text{cl } Y_m) : f \in F\} \leq c \max \{m, c_0\}, \tag{2.3}$$

where $c > 0$ is a real number such that (1.1) holds for each $f \in F$ and all $x, y \in U$. Indeed, let f be any function belonging to F, and let x be any element belonging to cl Y_m. Due to condition (ii) there is a neighbourhood V of o such that $M(f, V) \leq c_0$. Since

$$(x - U \cap V) \cap Y_m \neq \phi,$$

there exists a $y \in U \cap V$ such that $x - y \in Y_m$. The points $x - y$ and y lie in U, and so the preadditivity of F on U yields

$$f(x) = f((x - y) + y) \leq c \max \{f(x - y), f(y)\} \leq c \max \{m, c_0\}.$$

Thus (2.3) holds, as claimed. For short we put $c_1 = c \max\{m, c_0\}$.

Now choose any interior point x_0 of cl Y_m. After that select a neighbourhood W of o for which $x_0 + W - W \subseteq \text{cl}\, Y_m$. Since $-x_0 \in U$ and T is dense in U, we have $(W - x_0) \cap T \neq \phi$. So there exists a $y_0 \in W$ such that $y_0 - x_0 \in T$. Taking into account that F is pointwise bounded from above on T, it results that there is a real number c_2 such that

$$\sup\{f(y_0 - x_0) : f \in F\} \le c_2.$$

We claim that

$$\sup\{M(f, W) : f \in F\} \le c \max\{c_1, c_2\}. \tag{2.4}$$

To see this, let f be any function in F and let x be any point in W. Then $x_0 + x - y_0 \in x_0 + W - W \subseteq \text{cl}\, Y_m$. Therefore $x_0 + x - y_0$ lies in U and, according to (2.3), it satisfies $f(x_0 + x - y_0) \le c_1$. But, because T is a subset of U, the point $y_0 - x_0$ lies also in U. By applying the preadditivity of F on U, it follows that

$$\begin{aligned} f(x) &= f((x_0 + x - y_0) + (y_0 - x_0)) \le \\ &\le c \max\{f(x_0 + x - y_0), f(y_0 - x_0)\} \le c \max\{c_1, c_2\}. \end{aligned}$$

Thus (2.4) holds, as claimed.

So it has been proved that F is uniformly bounded from above on W. ∎

Next, we assume that F is presubtractive on a given subset of X and show that under this assumption theorems similar with the preceding two ones are true.

THEOREM 2.3. *Let U be a closed subset of X, and let F satisfy the following conditions:*

(i) *F is presubtractive on U;*

(ii) *each $f \in F$ is lower semicontinuous on U;*

(iii) *there exists a subset T of U which is of the second category in X and on which F is pointwise bounded from above.*

Then there exists a neighbourhood of o on which F is uniformly bounded from above.

Proof. For each positive integer n define the set Y_n by (2.1). As in the proof of Theorem 2.1 we can conclude that there is a positive integer m for which Y_m has interior points. Select any interior point x_0 of Y_m. Then $W = Y_m - x_0$ is a neighbourhood of o. Provided that $c > 0$ is a real number satisfying (1.2) for each $f \in F$ and all $x, y \in U$, it follows that

$$f(x) = f((x + x_0) - x_0) \leq c \max \{f(x + x_0), f(x_0)\} \leq cm$$

for every $f \in F$ and every $x \in W$. Hence F is uniformly bounded from above on W.∎

THEOREM 2.4. *Let U be a closed neighbourhood of o, and let F satisfy the following conditions:*

(i) *F is presubtractive on U;*

(ii) *there exists a real number c_0 such that for each $f \in F$ there is a neighbourhood V of o for which $M(f, V) \leq c_0$;*

(iii) *there exists a subset T of U which is of the second category in X and on which F is pointwise bounded from above.*

Then there exists a neighbourhood of o on which F is uniformly bounded from above.

Proof. For each positive integer n define the set Y_n by (2.1). As in the proof of Theorem 2.2 we can conclude that there is a positive integer m for which cl Y_m has interior points. The set cl Y_m is again contained in U and satisfies (2.3), where $c > 0$ is a real number such that (1.2) holds for each $f \in F$ and all $x, y \in U$. In this case, however the proof of (2.3) must be performed as follows. Let f be any function belonging to F, and let x be any element belonging to cl Y_m. Due to condition (ii) there is a neighbourhood V of o such that $M(f, V) \leq c_0$. Since $(x + U \cap V) \cap Y_m \neq \phi$, there exists a $y \in U \cap V$ such that $x + y \in Y_m$. Taking into account that F is presubtractive on U and that the points $x + y$, y lie in U, it results that

$$f(x) = f((x + y) - y) \leq c \max \{f(x + y), f(y)\} \leq c \max \{m, c_0\}.$$

Thus (2.3) holds, as claimed.

Now choose any interior point x_0 of cl Y_m. Then x_0 lies in U and W = cl $Y_m - x_0$ is a neighbourhood of o. Since F is presubtractive on U and satisfies (2.3), it follows that

$$f(x) = f((x + x_0) - x_0) \leq c \max \{f(x + x_0), f(x_0)\} \leq c^2 \max \{m, c_0\}$$

for every $f \in F$ and every $x \in W$. Hence F is uniformly bounded from above on W.∎

The above-stated theorems assure merely that F is uniformly bounded from above on some neighbourhood of o. But, as the next theorem shows, there are cases when this behaviour of F near o implies that each $x \in X$ possesses a neighbourhood on which F is uniformly bounded from above.

THEOREM 2.5. *Let F satisfy the following conditions:*

(i) *F is either preadditive or presubtractive on X;*

(ii) *there exists a subset T of X which is dense in X and on which F is pointwise bounded from above;*

(iii) *there exists a neighbourhood V of o on which F is uniformly bounded from above.*

Then there exists a neighbourhood W of o such that F is uniformly bounded from above on $x + W$ for any $x \in X$.

Proof. Let W be a neighbourhood of o such that $W - W \subseteq V$. We claim that F is uniformly bounded from above on $x + W$ for any $x \in X$.

To see this, pick any $x \in X$. Since T is dense in X, we can find a point $y_0 \in W$ for which $x + y_0 \in T$. Taking into account that F is pointwise bounded from above on T, it results that there is a real number c_1 such that

$$\sup \{f(x + y_0) : f \in F\} \le c_1. \tag{2.5}$$

Further there exists a real number c_2 such that

$$\sup \{M(f, V) : f \in F\} \le c_2. \tag{2.6}$$

Since $W - y_0$ and $y_0 - W$ are subsets of V, it follows from (2.6) that

$$\sup \{M(f, W - y_0) : f \in F\} \le c_2 \tag{2.7}$$

as well as that

$$\sup \{M(f, y_0 - W) : f \in F\} \le c_2. \tag{2.8}$$

If F is preadditive on X and $c > 0$ is a real number satisfying

$f(t + u) \le c \max \{f(t), f(u)\}$ whenever $f \in F$ and $t, u \in X$,

then we obtain from (2.5) and (2.7) that

$$f(x + y) = f((x + y_0) + (y - y_0)) \le$$
$$c \max \{f(x + y_0), f(y - y_0)\} \le c \max \{c_1, c_2\}$$

for any $f \in F$ and any $y \in W$. Similarly, if F is presubtractive on X and $c > 0$ is a real number satisfying

$f(t - u) \le c \max \{f(t), f(u)\}$ whenever $f \in F$ and $t, u \in X$, then we obtain from (2.5) and (2.8) that

$$f(x + y) = f((x + y_0) - (y_0 - y)) \leq c \max \{f(x + y_0), f(y_0 - y)\} \leq c \max \{c_1, c_2\}$$

for any $f \in F$ and any $y \in W$. Consequently, in both cases the inequality

$$\sup \{M(f, x + W) : f \in F\} \leq c \max \{c_1, c_2\}$$

holds, that is, F is uniformly bounded from above on $x + W$.■

3. By using the theorems established in the preceding section we state now sufficient conditions for F to be uniformly bounded from above on a given bounded neighbourhood of o. We recall that a subset Y of X is said to be **bounded** if for each neighbourhood V of o there exists a positive integer n such that

$$Y \subseteq \underbrace{V + \ldots + V}_{n \text{ terms}} .$$

THEOREM 3.1. *Let U be a bounded neighbourhood of o, and let F satisfy the following conditions:*

(i) *F is preadditive on U;*

(ii) *there exists a neighbourhood W of o on which F is uniformly bounded from above.*

Then F is uniformly bounded from above on U.

Proof. Let $c_0 > 0$ be a real number such that

$$\sup \{M(f, W) : f \in F\} \leq c_0.$$

Further, there exists a positive integer n such that

$$U \subseteq \underbrace{U \cap W + \ldots + U \cap W}_{n \text{ terms}}, \tag{3.1}$$

because U is bounded. Finally, choose a real number $c > 0$ such that (1.1) holds for each $f \in F$ and all $x, y \in U$.

Now pick any $f \in F$ and any $x \in U$. In view of (3.1) x can be written under the form $x = x_1 + \ldots + x_n$, where $x_1, \ldots, x_n \in U \cap W$. The repeated application of (1.1) yields $f(x) \le c_0$ when $c < 1$, and $f(x) \le c_0 c^{n-1}$ when $c \ge 1$, respectively. Hence we have

$$f(x) \le \max\,\{c_0,\ c_0 c^{n-1}\}.$$

Since f and x were arbitrarily chosen in F and in U, respectively, the family F is uniformly bounded from above on U.■

COROLLARY 3.2. *Let U be a bounded, closed and symmetric neighbourhood of o, and let F satisfy the following conditions:*

(i) *F is preadditive on U;*

(ii) *either each $f \in F$ is lower semicontinuous on U, or there exists a real number c_0 such that for each $f \in F$ there is a neighbourhood V of o such that $M(f, V) \le c_0$;*

(iii) *there exists a subset T of U which is of the second category in X, dense in U, and on which F is pointwise bounded from above.*

Then F is uniformly bounded from above on U.

Proof. Depending on the part of condition (ii) that is true, first we apply either Theorem 2.1 or Theorem 2.2 and we conclude that F is uniformly bounded from above on a neighbourhood of o. Next we apply Theorem 3.1.■

THEOREM 3.3. *Let U be a bounded symmetric neighbourhood of*

o, and let F satisfy the following conditions:

(i) *F is presubtractive on U;*

(ii) *there exists a neighbourhood W of o on which F is uniformly bounded from above;*

(iii) *$f(x) \geq 0$ for every $f \in F$ and every $x \in U$.*

Then F is uniformly bounded from above on U.

Proof. In view of the conditions (i) and (iii) there exists a real number $c \geq 1$ such that (1.2) holds for each $f \in F$ and all $x, y \in U$. Fix any function $f \in F$. If $x \in U$, then (1.2) implies

$$f(-x) = f(o - x) \leq c \max\{f(o), f(x)\}$$

as well as

$$f(o) = f(x - x) \leq cf(x).$$

Thus we have

$$f(-x) \leq c^2 f(x) \text{ for all } x \in U. \tag{3.2}$$

From (1.2) and (3.2) it follows that

$$f(x + y) \leq c \max\{f(x), f(-y)\} \leq c \max\{f(x), c^2 f(y)\} \leq$$
$$\leq c^3 \max\{f(x), f(y)\}$$

whenever $x, y \in U$. Since f was arbitrarily chosen in F, we have proved that each $f \in F$ satisfies

$$f(x + y) \leq c^3 \max\{f(x), f(y)\} \text{ whenever } x, y \in U.$$

This means that F is preadditive on U. Hence we can apply Theorem 3.1 and conclude that F is uniformly bounded from above on U.∎

COROLLARY 3.4 *Let U be a bounded, closed and symmetric neighbourhood of o, and let F satisfy the following conditions:*

(i) *F is presubtractive on U;*

(ii) *either each $f \in F$ is lower semicontinuous on U, or there exists a real number c_0 such that for each $f \in F$ there* *is a neighbourhood V of o such that $M(f, V) \le c_0$;*

(iii) *there exists a subset T of U which is of the second category in X and on which F is pointwise bounded from above;*

(iv) *$f(x) \ge 0$ for every $f \in F$ and every $x \in U$.*

Then F is uniformly bounded from above on U.

Proof. Depending on the part of condition (ii) that is true, first we apply either Theorem 2.3 or Theorem 2.4 and we conclude that F is uniformly bounded from above on a neighbourhood of o. Next we apply Theorem 3.3.■

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ENTIRE FUNCTIONS WITH SOME UNIVALENT GELFOND-LEONTEV DERIVATIVES

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REZUMAT. - **Funcţii întregi cu derivate Gelfond - Leontev univalente.** Se obţin margini superioare pentru funcţiile întregi aparţinînd claselor $E(n_p)$ şi $E_c(n_p)$.

Abstract. Let $\{n_p\}_{p=1}^{\infty}$ be a strictly increasing sequence of positive integers. Let $E(n_p)$ (resp. $E_c(n_p)$) denote the class of entire functions f for which the Gelfond-Leontev derivatives $D^{n_p}f$ of f are analytic and univalent (resp. convex) in the unit disc. In the present paper, we found upper bounds for the type of functions belonging to the class $E(n_p)$ and $E_c(n_p)$ respectively.

1. Introduction. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1.1}$$

be analytic in the disc $\{ z \in \mathbf{C}: |z| < R \}$, $0 < R \leq \infty$. For a nondecreasing sequence $\{d_n\}_{n=1}^{\infty}$ of positive number, the Gelfond-Leontev derivative (GLD) Df of f is defined as [3]

$$Df(z) = \sum_{n=1}^{\infty} d_n a_n z^{n-1} \tag{1.2}$$

For $p = 2,3,\ldots$, the pth iterate $D^p f$ of Df is given by

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$$D^p f(z) = \sum_{n=p}^{\infty} d_n \ldots d_{n-p+1} a_n z^{n-p}$$

It is readily seen that for $d_n = n$, $n = 1,2,\ldots$, Df is the ordinary derivative of f; where as, if $d_n = 1$, $n = 1,2,\ldots,D$ is the shift operator L which transforms f, given by (1.1), into $Lf(z) = \sum_{n=1}^{\infty} a_n z^{n-1}$. If f, defined by (1.1), is analytic and univalent in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ then [2]

$$|a_n| \le n|a_1| \tag{1.3}$$

for $n = 2,3,\ldots$.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of order α, $0 < \alpha < \infty$, and type T. In [1, p. 11], it is proved that

$$\limsup_{n\to\infty} n\, |a_n|^{\alpha/n} = e\alpha T \tag{1.4}$$

Further, if

$$\limsup_{n\to\infty} n\, |a_n|^{1/n} \le \tau e \tag{1.5}$$

then f is of exponential type no bigger that τ.

Juneja and Shah [5] proved that if f is an entire function of finite order α and type T, then

$$\liminf_{n\to\infty} \left[\frac{d_n \rho_{n-2}}{n^{1/\alpha}}\right]^{\alpha} \le \frac{d_2^{\alpha}}{\alpha T} \tag{1.6}$$

where ρ_n is the radius of univalence of $D^n f$. We note that if all the *GLD*'s $D^n f$ are analytic and univalent in the unit disc U and $(d_n^{\alpha}/n) \to \infty$ as $n \to \infty$ then f is of minimal type, i.e., $T = 0$. However, if infinitely many ρ_n's are zero, (1.6) does not give any non-trivial information about the type of the entire function

f.

In this paper, we partially solve this problem by considering some of the *GLD*'s of f to be analytic and univalent in the unit disc U. In the process, we find a number of new results.

To simplify notations, we shall write $a(n_p)$ for a_{n_p} and $d(n_p)$ for d_{n_p}. We shall assume throughout in the sequel that $d_p \to \infty$ as $p \to \infty$.

2. Entire functions of finite order. For a strictly increasing sequence $\{n_p\}_{p=1}^{\infty}$ of positive integers, we denote by $E(n_p)$ the class of entire functions for which $D^{n_p}f$ are analytic and univalent in the unit disc U. Likewise, let $E_c(n_p)$ be the class of functions in $E(n_p)$ such that $D^{n_p}f$ are convex in U.

We now prove

THEOREM 1. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in E(n_p)$ *have order* α, $0 < \alpha < \infty$, *and type* T. *Let the sequence* $\{d_n\}_{n=1}^{\infty}$ *in* (1.2) *satisfy*

$$\frac{d(n_p+k)}{d(k)} \geq \frac{k}{k-1}, \quad k = 2, 3, \ldots, p \geq p_0. \tag{2.1}$$

Then

$$\liminf_{p\to\infty}\left[\frac{\{d(n_1)\,.\,d(n_2)\ldots d(n_p)\}^{\alpha/n_{p+1}}}{n_{p+1}}\right] \leq \frac{\limsup_{p\to\infty}(2d_2)^{\alpha p/n_p}}{e\alpha T} \leq \frac{(2d_2)^{\alpha}}{e\alpha T}. \tag{2.2}$$

Proof. Since $f \in E(n_p)$, we have that

$$D^{n_p}f(z) = \sum_{k=0}^{\infty} d(n_p+k)\ldots d(k+1)\,a(n_p+k)\,z^k$$

is analytic and univalent in U. In view of (1.3), this implies that for k [illegible],3,... and $p = 1,2,\ldots,$

$$|a(n_p+k)| \le \frac{k\,d(n_p+1)\ldots d(2)\,|a(n_p+1)|}{d(n_p+k)\ldots d(k+1)} \tag{2.3}$$

Due to (2.1), the function $k/(d(n_p+k)\ldots d(k+1))$ decreases with k for every $p \ge p_0$ so that (2.3) yields for $k = 2,3,\ldots$ and $p \ge$ $\ge p_0$

$$\begin{aligned} |a(n_p+k)| &\le \frac{2d(n_p+1)\ldots d(2)\,|a(n_p+1)|}{d(n_p+2)\ldots d(2)} \\ &< \frac{2d(2)\,|a(n_p+1)|}{d(n_p)}. \end{aligned} \tag{2.4}$$

Now, letting $k = n_{p+1} - n_p + 1$ and using inductive argument on p in (2.4), we get for $p > p_0$

$$|a(n_p+1)| \le \frac{(2d(2))^{p-p_0}\,|a(n_{p_0}+1)|}{d(n_{p_0})\ldots d(n_{p-1})}.$$

Thus if, $2 \le k \le n_{p+1} - n_p + 1$, it follows by using the above inequality in (2.4) that for $p > p_0$

$$\begin{aligned} |a(n_p+k)| &\le \frac{(2d(2))^{p-p_0+1}\,|a(n_{p_0}+1)|}{d(n_{p_0})\ldots d(n_p)} \\ &= \frac{A(2d(2))^p}{d(n_1)\,d(n_2)\ldots d(n_p)} \end{aligned}$$

where A is an absolute constant. From the above inequality, we deduce that

$$\limsup_{k\to\infty} k\,|a_k|^{\alpha/k} =$$

$$= \limsup\left\{(n_p+k)\ |a(n_p+k)|^{\alpha/(n_p+k)} : 2\le k\le n_{p+1}-n_p+1,\ p>p_0\right\}$$

$$\le \frac{\limsup_{p\to\infty} (2d_2)^{\alpha p/n_p}}{\liminf_{p\to\infty}\left[\frac{\{d(n_1)\,d(n_2)\ldots d(n_p)\}^{\alpha/n_p+1}}{n_{p+1}}\right]}.$$

Thus, in view of (1.4), the first inequality in (2.2) follows. The second inequality in (2.2) is obivious.

COROLLARY 1. *Assume all the hypothesis of Theorem 1. If*

$$\liminf_{p\to\infty}\left[\frac{\alpha}{n_{p+1}}\sum_{i=1}^{p}\log d(n_i) - \log n_{p+1}\right] = \mu$$

then $T \le (2d_2)^{\alpha}/e^{1+\mu}$.

With $d_n = n$, Theorem 1 gives

COROLLARY 2. *Let f, defined by* (1.1), *be an entire function of finite order* α *and type T. Let* $\{n_p\}_{p=1}^{\infty}$ *be a strictly increasing sequence of positive integers such that* $f^{(n_p)}$ *are univalent in the unit disc U. Then*

$$\liminf_{p\to\infty}\left[\frac{(n_1.n_2\ldots n_p)^{\alpha/n_p}}{n_p}\right] \le \frac{\limsup_{p\to\infty} 4^{\alpha p/n_p}}{e\alpha T} \le \frac{4^{\alpha}}{e\alpha T}. \tag{2.5}$$

Remarks 1. We note that, besides many other choices of the sequence $\{d_n\}_{n=1}^{\infty}$, (2.1) holds if $d(n) = n^{\beta}$, $e^{n^{\beta}}$ ($\beta > 0$), $n^{\beta}\log n$ ($\beta \ge 1$) for $n = 1,2,\ldots$. In general, (2.1) is satisfied if $d(k+2)/d(k) \ge k/(k-1)$, $k = 2,3,\ldots$. This inequality is easier to check that (2.1) and it also generates a number of simple admissible sequence $\{d_n\}_{n=1}^{\infty}$ for Theorem 1.

2. We observe that the inequality (2.5) establishes a relation between the exponents 'n_p' for which $f^{(n_p)}$ are univalent in U and the type of the entire function f.

3. If $\limsup_{p\to\infty}\left(\frac{n_{p-1}}{n_p}\right) < 1$, then $\limsup_{p\to\infty}\left(\frac{p}{n_p}\right) = 0$ so that (2.2) becomes

$$\liminf_{p\to\infty}\left[\frac{\{d(n_1)\,d(n_2)\,...\,d(n_p)\}^{\alpha/n_{p+1}}}{n_{p+1}}\right] \le \frac{1}{e\alpha T}$$

which on putting $d_n = n$ gives

$$\liminf_{p\to\infty}\left[\frac{(n_1 . n_2 ... n_p)^{\alpha/n_p}}{n_p}\right] \le \frac{1}{e\alpha T}.$$

4. We have not been able to obtain an entire function f and a non-decreasing sequence $\{d_n\}_{n=1}^{\infty}$ of positive numbers satisfying (2.1) for which equality holds in (2.2). Even for $d_n = n$, the estimate (2.5) is not sharp.

We now prove a result which is more general than corollary 1. We need the following lemma.

LEMMA [7]. *Let $\{n_p\}_{p=1}^{\infty}$ be a strictly increasing sequence of positive integers. Then for each $p \ge 2$,*

$$\prod_{j=1}^{p-1} (n_{j+1} - n_j + 1)^{1/(n_p+2)} \le \left(1 + \frac{n_p}{p}\right)^{p/n_p} \le 2. \qquad (2.6)$$

THEOREM 2. *Let f be an entire function of order α and type T. Let $\{n_p\}_{p=1}^{\infty}$ be a strictly increasing sequence of positive integers such that $f^{(n_p)}$ are univalent in U. If $(n_{p+1} - n_p) \le \lambda$ for large p and $\limsup_{p\to\infty}(p/n_p) = \beta$, then $\alpha \le 1$ and f is of* *exponential type no bigger than* $\left[\dfrac{\sqrt{2\pi}\, e^{1/24}}{e^{(1+\beta)/\beta}}\ (1+1/\beta)^{5/2}\right]^{\beta}\ (\lambda+1)$.

Proof. Since $(n_{p+1} - n_p) \le \lambda$ for large p, we have $(n_{p+1} - n_p) = o(\log n_{p+1})$ so that it follows directly from Corollary 1 [8, p.398] that $\alpha \le 1$. To prove the second part, we assume, without loss of generality, that $(n_{p+1} - n_p) \le \lambda$ for all p (otherwise, we let $m_p = n_{p+Q}$ and work with $\{m_p\}_{p=1}^{\infty}$). Since $f^{(n_p)}(z)$ is univalent in U, it follows from (1.3) that

$$|a(n_p+k)| \le \frac{k\, k!\ (n_p+1)!\ |a(n_p+1)|}{(n_p+k)!} \tag{2.7}$$

Setting $k = n_{p+1} - n_p + 1$ and inducting on p in (2.7), we get for $p = 2,3,\ldots$

$$|a(n_p+1)| \le \frac{D^*}{(n_p+1)!}\prod_{j=1}^{p-1}\ (n_{j+1}-n_j+1)\,(n_{j+1}-n_j+1)!$$

where $D^* = (n_1 + 1)!\ |a(n_1 + 1)|$. Using the above inequality in (2.7), we conclude that for $p \ge 2$ and $2 \le k \le n_{p+1} - n_p+1$,

$$|a(n_p+k)| \le \frac{D^*\, k k!}{(n_p+k)!}\prod_{j=1}^{p-1}\ (n_{j+1}-n_j+1)\,(n_{j+1}-n_j+1)! \tag{2.8}$$

If n is a positive integer greater than 1, then [6, p.183] there exist two positive numbers A and B such that

$$A\ n^{1/2}(n/e)^n < n! < B\ n^{1/2}(n/e)^n$$

where $A = Be^{-1/24} = \sqrt{2\pi}$. Using this in the right hand side of (2.8) and taking the (n_p+k) - th root of both sides of the resulting inequality and applying (2.6) to part of right hand side of this, it follows that for $2 \le p$ and for $2 \le k \le n_{p+1} - n_p+1$,

$$(n_p+k)\ |a(n_p+k)|^{1/(n_p+k)} \le \left[\frac{D^* B^p e^{n_1+1-p}}{A}\left(\frac{k^3}{n_p+k}\right)^{1/2}\right]^{1/(n_p+k)} \times$$
$$\times\left(1+\frac{n_p}{p}\right)^{5p/2n_p}\left[k^{k/(n_p+k)}\prod_{j=1}^{p-1}(n_{j+1}-n_j+1)^{(n_{j+1}-n_j)/(n_p+k)}\right] \tag{2.9}$$

Since $(n_p - n_{p-1}) \le \lambda$ for all p, it follows that for $2 \le p$ and for $2 \le k \le n_{p+1} - n_p+1$.

$$\left[\left(\frac{B}{e}\right)^p \prod_{j=1}^{p-1}(n_{j+1}-n_j+1)^{(n_{j+1}-n_j)}\right]^{1/(n_p+k)}$$
$$\le (B/e)^{p/n_p} \exp\left[\frac{1}{n_p}\sum_{j=1}^{p-1}(n_{j+1}-n_j)\log(n_{j+1}-n_j+1)\right]$$
$$\le (B/e)^{p/n_p}(\lambda+1)\ .$$

Further, as $k^{k/(n_p+k)}$ is an increasing function of k, we have

$$k^{k/(n_p+k)} \le (n_{p+1}-n_p+1)^{(n_{p+1}-n_p+1)/n_p}$$
$$\le (\lambda+1)^{(\lambda+1)/n_p} = o(1)$$

and

$$\left[\frac{D^* e^{n_1+1}}{A}\left(\frac{k^3}{n_p+k}\right)^{1/2}\right]^{1/(n_p+k)} = 1 + o(1)\ .$$

Hence,

$$\lim_{k\to\infty} \sup k|a_k|^{1/k} = \lim \sup \{(n_p+k)\ |a(n_p+k)|^{1/(n_p+k)} :$$
$$2 \le p,\ 2 \le k \le n_{p+1} - n_p+1\}$$
$$\le (B/e)^{\beta}\ (1+1/\beta)^{5\beta/2}\ (\lambda+1)\ .$$

Substituting $\sqrt{2\pi}\, e^{1/24}$ for B, we finally get

$$\lim_{k\to\infty} \sup k\,|a_k|^{1/k} \le e\left[\frac{\sqrt{2\pi}\, e^{1/24}}{e^{(1+\beta)/\beta}} (1+1/\beta)^{5/2}\right]^{\beta} (\lambda+1)$$

$$\le e\left[\frac{\sqrt{2\pi}\, e^{1/24} 2^{5/2}}{e^2}\right] (\lambda+1)$$

Remark. We note that the estimate obtained in Theorem 2 improves the corresponding estimate obtained by Shah and Trimble [8, p.398].

We next prove results in other direction.

THEOREM 3. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in E(n_p)$ *have order* $\alpha (0 < \alpha < \infty)$ *and type* T. *Let the sequence* $\{d_n\}_{n=1}^{\infty}$ *satisfy* (2.1). *If there is a positive integer* $M > 1$ *such that* $\limsup_{p\to\infty} (n_{p+1} - n_p) \ge M$, *then*

$$\liminf_{p\to\infty} \left[\frac{\{d(n_1)\, d(n_2) \ldots d(n_{p-1})\}^{(M-1)/n_{p+1}}}{n_{p+1}}\right] \le \frac{1}{e\alpha T} \tag{2.10}$$

Proof. From (2.3), we get for $k = 2,3,\ldots,$ and $p = 1,2,\ldots$

$$|a(n_p+k)| \le \frac{k.d(n_p+M) \ldots d(2)\ |a(n_p+1)|}{(d(n_p+M) \ldots d(n_p+2))\,(d(n_p+k) \ldots d(k+1))} \tag{2.11}$$

Since, by (2.1), $k/(d(n_p+k)\ldots d(k+1))$ is a decreasing function of k for all $p \ge p_0$, for $k \ge M+1$, there exists a positive integer Q_1 such that for $p \ge Q_1$

$$\frac{kd(n_p+M) \ldots d(2)}{d(n_p+k) \ldots d(k+1)} \le \frac{(M+1)\, d(n_p+M) \ldots d(2)}{d(n_p+M+1) \ldots d(M+2)}$$

$$= \frac{(M+1)\, d(M+1) \ldots d(2)}{d(n_p+M+1)}$$

$$< 1.$$

Thus, for $p \ge \max\{p_0, Q_1\}$ and $k \ge M+1$, (2.11) yields

$$|a(n_p+k)| \le \frac{|a(n_p+1)|}{d(n_p+M)\ldots d(n_p+2)} < \frac{|a(n_p+1)|}{d(n_p)^{M-1}}. \tag{2.12}$$

Also, there is a positive integer Q_2 such that for $p \ge Q_2$ and $k \ge 2$

$$\frac{k\, d(n_p+1)\ldots d(2)}{d(n_p+k)\ldots d(k+1)} < 1.$$

Using this in (2.11) again, it follows that for $p \ge Q_2$ and $k \ge 2$

$$|a(n_p + k)| < |a(n_p + 1)| \tag{2.13}$$

Since, $\liminf_{p\to\infty} (n_{p+1} - n_p) \ge M$, there is a positive integer Q_3 such that for $p \ge Q_3$, $(n_{p+1} - n_p) \ge M$. Let $Q = \max\{p_0, Q_1, Q_2, Q_3\}$. Then $(n_{p+1} - n_p + 1) \ge M + 1$ for $p \ge Q$ and so the inequality (2.12) implies that for $p \ge Q$

$$|a(n_{p+1}+1)| < \frac{|a(n_p+1)|}{d(n_p)^{M-1}}.$$

An inductive argument on p in the above inequality shows that for $p > Q$

$$|a(n_p+1)| < \frac{|a(n_Q + 1)|}{\{d(n_k)\ldots d(n_{p-1})\}^{M-1}} = \frac{C_1}{\{d(n_1)\, d(n_2)\ldots d(n_{p-1})\}^{M-1}}$$

where C_1 is an absolute constant. If $2 \le k \le n_{p+1} - n_p + 1$, then using the above inequality in (2.13), we have for $p > Q$

$$|a(n_p+k)| < \frac{C_1}{\{d(n_1)\,d(n_2)\,...\,d(n_{p-1})\}^{M-1}}.$$

Thus,

$$\begin{aligned} e\alpha T &= \limsup_{k\to\infty} k|a_k|^{\alpha/k} \\ &= \lim sup\{(n_p+k)\ \ |a(n_p+k)|^{\alpha/(n_p+k)} : 2 \le k \le n_{p+1}-n_p+1,\ p>Q\} \\ &\le \frac{1}{\liminf_{p\to\infty}\left[\dfrac{\{d(n_1)\,d(n_2)\,...\,d(n_{p-1})\}^{\alpha(M-1)/n_{p+1}}}{n_{p+1}}\right]}. \end{aligned}$$

This completes the proof of Theorem 3.

COROLLARY 4. *Assume all the hypothesis of Theorem 3. If* $\lim_{p\to\infty}(n_{p+1}-n_p) = \infty$ *and*

$$\liminf_{p\to\infty}\left[\frac{\{d(n_1)\,d(n_2)\,...\,d(n_{p-1})\}^{\alpha/n_{p+1}}}{n_{p+1}}\right] > 1,$$

then $T = 0$.

Putting $d_n = n$ in Theorem 3, we have the following:

COROLLARY 5. *Let f defined by* (1.1), *be an entire function of finite order* α *and type* T. *Let* $\{n_p\}_{p=1}^{\infty}$ *be a strictly increasing sequence of positive integers such that* $f^{(n_p)}$ *are univalent in* U.

If there is a positive integer $M > 1$ *with* $\liminf_{p\to\infty}(n_{p+1}-n_p) \ge M$, *then*

$$\liminf_{p\to\infty}\left[\frac{(n_1 . n_2 ... n_p)^{\alpha(M-1)/n_p}}{n_p}\right] \le \frac{1}{e\alpha T} \qquad (2.14)$$

Remark. We have not been able to obtain an entire function and a non-decreasing sequence $\{d_n\}_{n=1}^{\infty}$ of positive numbers for which equality holds in (2.10). Even for $d_n = n$, the estimate (2.14) is not sharp.

The following results for functions belonging to the class $E_c(n_p)$ can be obtained by a proof similar to that used in Theorem 1 and Theorem 3. At this place, we do not need the hypothesis (2.1) on the sequence $\{d_n\}_{n=1}^{\infty}$.

THEOREM 4. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in E_c(n_p)$ *have order* $\alpha (0 < \alpha < \infty)$ *and type* T. *Then*

$$\liminf_{p\to\infty} \left[\frac{\{d(n_1)\, d(n_2) \ldots d(n_p)\}^{\alpha/n_{p+1}}}{n_{p+1}} \right] \le \frac{\limsup_{p\to\infty} d_2^{\alpha p/n_p}}{e\alpha T} \le \frac{d_2^{\alpha}}{e\alpha T}.$$

THEOREM 5. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in E_c(n_p)$ *have order* $\alpha (0 < \alpha < \infty)$ *and type* T. *If there is a positive integer* $M > 1$ *such that*

$\liminf_{p\to\infty} (n_{p+1} - n_p) \ge M$, *then*

$$\liminf_{p\to\infty} \left[\frac{\{d(n_1)\, d(n_2) \ldots d(n_{p-1})\}^{\alpha(M-1)/n_{p+1}}}{n_{p+1}} \right] \le \frac{1}{e\alpha T}.$$

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LOBATTO-TURÁN QUADRATURE RULES AND CAUCHY PRINCIPAL VALUE INTEGRALS*

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REZUMAT. - Reguli de cuadratură Lobatto-Turán şi valoarea principală Cauchy a integralelor. Se deduc anumite reguli, bazate pe formulele de cuadratură Lobatto-Turán, pentru evaluarea numerică a valorii principale Cauchy a integralelor unidimensionale şi se demonstrează convergenţa unui subşir de astfel de reguli. În continuare se prezintă inegalitatea Posse - Markov - Stieltjes pentru sume de cuadratură Lobatto-Turán. De asemenea se dau anumite inegalităţi pentru zerourile polinoamelor s-ortogonale.

Abstract. Some rules, based on Lobatto-Turán quadrature formulae, are derived for the numerical evaluation of one-dimensional Cauchy principal value (C.P.V.) integrals, and the convergence of a subsequence of such rules is proved. Further, the Posse-Markov-Stieltjes inequality for Lobatto-Turán quadrature sums is presented, and some inequalities for the zeros of s-orthogonal polynomials are given.

1.Introduction. In recent years, several papers have been written about convergence of Gaussian type integration rules for singular integrands [2-5, 14, 15, 21-22]; further, some results of the above papers have been extended to the case when Turán type quadrature rules [12] (that is integration rules with multiple nodes) are considered to evaluate C.P.V. integrals

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$$I(f;t) = \int_{-1}^{1} \frac{f(x)}{x-t} w(x)\,dx = \\ = \lim_{\epsilon\to 0}\left[\int_{-1}^{t-\epsilon} + \int_{t+\epsilon}^{1}\right] \frac{f(x)}{x-t} w(x)\,dx, \quad |t|<1 \tag{1.1}$$

where w is a suitable weight function, nonnegative on $A:=[-1,1]$, f is, at least, Hölder continuous of order μ, $0 < \mu \le 1$.

The convergence of Lobatto and Radau integration rules with simple nodes for singular integrands has also been analysed in [21]; generalizing this approach we present here some results concerning the convergence of the Lobatto-Turán quadrature rules, (which are a particular case of rules studied in [23]), when applied to the evaluation of (1.1).

In Section 2 we introduce some notations and the integration rule; in Section 3 we give some results concerning properties of the zeros of s-orthogonal polynomials, as well as an extension of the Posse-Markov-Stieltjes inequality for Lobatto-Turán rules. This inequality, in fact, provides an important tool to investigate convergence rates of Gaussian or Turán quadrature for singular integrands [15,12] and the same does in the case of Lobatto-Turán rules.

In section 4, the rate of convergence of the latter rules for evaluating C.P.V. integrals is studied, for particular weight functions, belonging to the class of GSJ (generalized smooth Jacobi) functions [17].

2. Some basic properties. Let w denote a positive weight function on $A = [-1,1]$, given a nonnegative integer s, it is well known [7, pg. 74] that there exists a unique sequence $\{P_{ms}\}$ of monic polynomials s-orthogonal in A, with respect to w, that is satisfying the conditions

$$\int_A x^k [P_{ms}(x)]^{2s+1} w(x)\,dx = 0, \quad k = 0,1,\ldots,m-1 \qquad (2.1)$$

These polynomials are polynomials of least L_{2s+2} weighted norm [7,25].

The zeros of the polynomial P_{ms} of degree m, are real, simple, and lie in $(-1,1)$; they depend on s, but for the sake of simplicity we shall denote them only by x_{mi}.

For the evaluation of the zeros of s-orthogonal polynomials, and some monotonicity results we refer the interested reader to [8, 10, 16].

Based on the zeros of s-orthogonal polynomials are the Turán quadrature rules [25],

$$\int_{-1}^{1} f(x)\,w(x)\,dx = \sum_{i=1}^{m}\sum_{h=0}^{2s} C_{hi} f^{(h)}(t_{mi}) + r_{ms}(f),$$

exact for $f \in P_{m(2s+2)-1}$ and the Lobatto-Turán quadrature rules (or generalized Lobatto quadrature rules):

$$\int_{-1}^{1} f(x)\,w(x)\,dx = \sum_{i=0}^{m+1}\sum_{h=0}^{r_i} C_{hi} f^{(h)}(x_{mi}) + R_{ms}(f) \qquad (2.2)$$

where fw is integrable on A and f has the required derivatives, more

$$x_0=-1;\ x_m=1;\ r_0=p;\ r_i=2s,\ i=1,2,\ldots,m;\ r_{m+1}=q; \qquad (2.3)$$

If in (2.3) $p=-1$, or $q=-1$, formula (2.2) reduces to a Radau-Turán type rule, which, in this paper, will be considered as a particular case o the Lobatto-Turán ones.

The degree of exactness of (2.2) is $\nu = m(2s+2)+p+q+1$ when the nodes x_{mi}, $i=1,2,\dots,m$, are the zeros of polinomial Π_{ms} of degree m, s-orthogonal in A, with respect to the weight function

$$W(x) = (1+x)^{p}(1-x)^{q}w(x). \tag{2.4}$$

The zeros x_{mi} are so ordered: $-1<x_{m1}<x_{m2}<\dots<x_{mm}<1$.

Rules (2.2) are of particular interest when f and its derivatives vanish at -1 and/or 1.

Some properties of the rules (2.2) have been pointed out in [13], where the following estimates have also been established

$$\begin{cases} |C_{hi}| \le \dfrac{d_i^h}{h!}\displaystyle\int_{x_{m,i-1}}^{x_{m,i+1}} w(x)\,dx; \\ i = 0(1)m+1, \quad h = 0(1)r_i\,, \\ \begin{cases} x_{m,-1}=x_{m0}=-1, \ x_{m,m+1}=x_{m,m+2}=1, \\ d_i=\max\left(|x_{m,i+1}-x_{mi}|,\ |x_{mi}-x_{m,i-1}|\right) \end{cases} \end{cases} \tag{2.5}$$

Convergence and rate of convergence of integration rules with preassigned node have been analysed in [9].

Turning to (1.1), we remark that it is known that $I(f;t)$ exists when $f \in H_\mu(A)$, $0 < \mu \le 1$, and $w \in$ GSJ; that is

$$w(x) = \psi(x)\,(1+x)^{\alpha}(1-x)^{\beta}\prod_{j=1}^{\rho}|x-t_j|^{\gamma_j}, \quad |x| \le 1, \tag{2.6}$$

where $-1 < t_1 < \dots < t_\rho < 1$, $\gamma_j > -1$, $j = 1,2,\dots,\rho$, $\alpha > -1$, $\beta > -1$, ψ is positive and continuous and its modulus of continuity $\omega(\psi;\cdot)$ satisfies the condition $\int_A \omega(\psi;\delta)\,\delta^{-1}d\delta < \infty$.

Let

$$\tilde{A} = \bigcup_{k=0}^{\rho}[a_k, a_{k+1}], \quad \forall\ [a_k, a_{k+1}] \subset (t_k, t_{k+1}), \quad k = 0(1)\rho,$$

$$t_0 = -1, \quad t_{\rho+1} = 1.$$

In the following we shall also denote by GJ the subset of the functions (2.6) whith $\gamma_j = 0$, $j = 1(1)p$.

The use of (2.2) for evaluating C.P.V. integral (1.1) yields the following integration rule

$$I_m(f;t) = f(t)A_{ms}(t) + \sum_{i=0}^{m+1}\sum_{h=0}^{r_i} D_{hi} f^{(h)}(x_{mi}) + E_{ms}(f;t) = \\ =: H_{ms}(f;t) + E_{ms}(f;t), \tag{2.8}$$

where

$$\begin{cases} A_{ms}(t) = I(1;t) - L_{ms}\left(\dfrac{1}{x-t}\right), \\ E_{ms}(f;t) = R_{ms}\left(\dfrac{f-f(t)}{x-t}\right), \\ D_{hi} = \displaystyle\sum_{k=h}^{r_i} \binom{k}{n} C_{ki} \left[D^{k-h}\frac{1}{x-t}\right]_{x=x_{mi}}, \quad i=0,1,\ldots,m+1. \end{cases} \tag{2.9}$$

Formula (2.8) is obtained integrating the Lagrange-Hermite polynomial which interpolates f at the points:

$$x = t, \; x = x_{mi}, \; i = 0,1,\ldots,m+1;$$

under the assumption $\Pi_{ms}(t) \neq 0$, that is $m \in N^*$ with

$$N^* := \{m \in N |\; x_{mi} \neq t, \; i = 1,2,\ldots,m\}. \tag{2.10}$$

Seen the interlacing property of the zeros of s-orthogonal polynomials [20], the set is infinite.

We remark that, if w is a weight function $\in$ GSJ, such is also the weight function W given by (2.4), whose zeros are the nodes of (2.8).

The remainder $E_{ms}(f,t)$ in the integration rule (2.8) can be written as

$$E_{ms}(f,t) = \int_A f\left[\begin{matrix} x_{m0} & x_{m1} & \cdots & x_{m,m+1} \\ p+1 & 2s+1 & \cdots & q+1 \end{matrix}\; t,\, x\right]\cdot \pi(x)\cdot w(x)\,dx,$$

$$\pi(x) := \prod_{i=0}^{m+1} (x-x_{mi})^{r_i+1}$$

where f[ ] is the divided difference based on the simple node t, the multiple nodes x_{mi} and x [23]. The degree of exactness of (2.8) is $\nu + 1 = m(2s + 2) + p + q + 2$.

We conclude this section with some more notations.

The zero of Π_{ms}, closest to t will be denoted by $x_{c(m)}$

The symbol " ~ " will be used to compare sequences and functions; if A and B are two expressions depending on some variables, then

$$A \sim B$$

will mean

$$|AB^{-1}| \le \text{const}, \qquad |A^{-1}B| \le \text{const}$$

uniformly for the variables in question; "const" denotes a positive constant, which may take on a different value each time it is used.

A function f is said to be *k-absolutely monotone* (k-A.M.) in an interval E if

$$f^{(i)}(x) \ge 0, \quad x \in E, \quad i = 1,2,\ldots,k; \qquad (2.11)$$

f is said to be strictly *k-absolutely monotone* (k-S.A.M.) if the strict inequality holds in (2.11).

A function f is said to be *k-completely monotone* in E (k-C.M.) if

$$(-1)^i f^{(i)}(x) \ge 0, \quad x \in E, \quad i = 1,2,\ldots,k; \qquad (2.12)$$

f is said to be strictly *k-completely monotone* (k-S.C.M.) if the strict inequality holds in (2.12).

If (2.11) or (2.12) hold for every k, we shall omit the index k in the short notation.

3. Auxiliary results. It is well known that the Posse-Markov-Stieltjes inequality holds for Gaussian quadrature sums [6,15]; recently it has been proved that even for quadrature sums of Turán type the above inequality is valid [11]. Here, the latter result is extended to quadrature sums of Lobatto-Turán or Radau-Turán type.

The following lemma [6, pg.30] plays a role in our proof:

LEMMA 1. *Let f be (m+1)-S.A.M. in* $(-1,\xi]$; *let* $P \in P_m$ *and let*

m_1 = *total multiplicity of zeros of* $f-P$ *in* $(-1,\xi]$

m_2 = *total multiplicity of zeros of* P *in* (ξ,∞).

Then $m_1 + m_2 \leq m + 1$.

Now, let us turn to the quadrature sums in (2.2); the following theorem can be stated.

THEOREM 1. *Let* $m \in N^+$ *and* $s \in N$; *let* $\nu = (2s+2)m+p+q+1$. *If f is* ν-*A.M. in* $[-1,x_{mk})$ *for a given k,* $1 \leq k \leq m+1$, *then*

$$\sum_{i=0}^{k-1} \sum_{h=0}^{r_i} C_{hi} f^{(h)}(x_{mi}) \leq \int_{-1}^{x_{mk}} f(x)\, w(x)\, dx \tag{3.1}$$

If, in addition, f is ν-*A.M. in* $[-1,x_{mk}]$, *then*

$$\sum_{i=0}^{k} \sum_{h=0}^{r_i} C_{hi} f^{(h)}(x_{mi}) \geq \int_{-1}^{x_{mk}} f(x)\, w(x)\, dx. \tag{3.2}$$

Proof. First, we assume f is S.A.M. in $[-1,x_{mk})$; let $p \in$ $\in P_{\nu-1}$, satisfy the interpolation conditions

$$\begin{cases} p^{(h)}(x_{m0}) = f^{(h)}(x_{m0}) \ , & h=0,1,\ldots,p; \\ p^{(h)}(x_{mi}) = f^{(h)}(x_{mi}) \ , & h=0,1,\ldots,2s+1; \quad i=1,2,\ldots,k-1; \\ p^{(h)}(x_{mk}) = 0 \ , & h=0,1,\ldots,2s; \\ p^{(h)}(x_{mi}) = 0 \ , & h=0,1,\ldots,2s+1; \quad i=k+1,\ldots,m; \\ p^{(h)}(x_{m,m+1}) = 0 \ , & h=0,1,\ldots,q. \end{cases} \tag{3.3}$$

Let ξ be an arbitrary point in $(x_{m,k-1}, x_{mk})$; then

$f-p$ has $m_1 \geq p+1+(2s+2)(k-1) =: n_1$ zeros in $[-1,\xi]$

p has $m_2 \geq q+1+(2s+2)(m-k)+2s+1 =: n_2$ zeros in $[\xi,\infty)$

with

$$n_1 + n_2 = \nu, \qquad m_1 + m_2 \geq n_1 + n_2,$$

but, by Lemma 1, $m_1 + m_2 \leq \nu$, thus $m_1 = n_1$, $m_2 = n_2$.
Hence $f-p$ has only zeros of even multiplicity in $(-1,\xi]$, from which it follows that $f-p$ does not change sign in $(-1,\xi]$ for any $\xi \leq x_{mk}$; then $f-p$ does not change sign in $[-1,x_{mk})$. Since $p(x_{mk}) = 0$ and f is S.A.M. in $[-1,x_{mk})$, then

$$f(x) \geq p(x), \qquad x \in [-1,x_{mk}).$$

Moreover, as $\xi > x_{m,k-1}$ is arbitrary, then p has in $(x_{m,k-1},+\infty)$ only the n_2 zeros listed above. Now, $p(x_{m,k-1}) = f(x_{m,k-1}) > 0$ and p has a zero of odd order in x_{mk}, and zeros of even order at x_{mi}, $i = k+1,\ldots,m$; then p changes sign at x_{mk} and

$$p(x) \leq 0, \qquad x \in [x_{mk},1].$$

So, from (2.2), and (3.3) one gets

$$\int_{-1}^{x_{mk}} f(x)\ w(x)\ dx \geq \int_{-1}^{x_{mk}} p(x)\ w(x)\ dx + \int_{x_{mk}}^{1} p(x)\ w(x)\ dx = \sum_{i=0}^{k-1} \sum_{h=0}^{r_i} C_{hi} p^{(h)}(x_{mi}) = \sum_{i=0}^{k-1} \sum_{h=0}^{r_i} C_{hi} f^{(h)}(x_{mi}).$$

If f is ν-A.M., let us put $f_\varepsilon(x) = f(x) + \varepsilon e^x$; so we obtain a function ν-S.A.M. for any $\varepsilon > 0$.

Repeating the above reasoning, when f is replaced by f_ε, and passing to the limit as ε tends to zero, we get (3.2).

If, in addition, f is ν-A.M. in $[-1,x_{mk}]$, we introduce a polynomial $q \in P_{\nu-1}$, satisfying the conditions

$$q^{(h)}(x_{m0}) = f^{(h)}(x_{m0}), \qquad h = 0,1,\dots,p;$$

$$q^{(h)}(x_{mi}) = f^{(h)}(x_{mi}), \qquad h = 0,1,\dots,2s+1; \; i = 1,2,\dots,k-1;$$

$$q^{(h)}(x_{mk}) = f^{(h)}(x_{mk}), \qquad h = 0,1,\dots,2s;$$

$$q^{(h)}(x_{mi}) = 0, \qquad h = 0,1,\dots,2s+1; \; i = k+1,\dots,m;$$

$$q^{(h)}(x_{m,mi}) = 0, \qquad h = 0,1,\dots,q.$$

By Lemma 1, reasoning analogously as above, we get

$$f(x) \le q(x), \qquad x \in [-1,x_{mk}]$$

$$q(x) \ge 0, \qquad x \in [x_{mk},+\infty]$$

from which (3.2) is derived.■

For ν-C.M. functions, the following Theorem 2 holds.

THEOREM 2. *Let f be ν-C.M. in $(x_{mk},1]$ for a given k, $0 \le k \le m$, then*

$$\sum_{i=k+1}^{m+1} \sum_{h=0}^{r_i} C_{hi} f^{(h)}(X_{mi}) \le \int_{x_{mk}}^{1} f(x)\, w(x)\, dx.$$

If, in addition, f is ν-C.M. in $[x_{mk},1]$, then

$$\sum_{i=k}^{m+1} \sum_{h=0}^{r_i} C_{hi} f^{(h)}(x_{mi}) \ge \int_{x_{mk}}^{1} f(x)\, w(x)\, dx.$$

Proof. The proof follows from the one of theorem 1, making the change of variable $x \to -x$ (See also [11]).■

We remark that Theorems 1 and 2 can be stated in a more general form, related to a measure $d\alpha(x)$ instead of $w(x)dx$; and when one considers some other prefixed nodes external to $[-1,1]$, or when both prefixed and Gaussian nodes have different multiplicity.

COROLLARY. *Let* $f \in \nu$*-A.M. in* $[-1,x_{mk}]$ *for any given* k, $1 \le k \le m+1$, *then*

$$\sum_{h=0}^{r_i} C_{hi} f^{(h)}(x_{mi}) \ge 0, \quad i=0,1,\ldots,k. \tag{3.4}$$

If $f \in \nu$*-C.M. in* $[x_{mk},1]$ *for any given* k, $1 \le k \le m+1$, *then*

$$\sum_{h=0}^{r_i} C_{hi} f^{(h)}(x_{mi}) \ge 0, \quad i=k,k+1,\ldots,m+1. \tag{3.5}$$

The proof of (3.4) is obtained by applying (3.1) and (3.2) to the intervals $[-1,x_{mi}]$; $i = 0,1,2,\ldots,k$; similarly (3.5) follows from the results of Theorem 2 applied to the intervals $[x_{mi},1]$, $i = k,\ldots,m+1$.■

Now, we shall present some new results on the behavior of the zeros of polynomials s-orthogonal with respect to GSJ weight functions.

While the properties of polynomials orthogonal (s=0) with respect to GSJ weight functions, have been studied by Badkov [1] and especially by Nevai [17], who also pointed out the behavior of the corresponding zeros and Christoffel functions, little is known about s-orthogonal polynomials, and asymptotic properties of their zeros.

Hence Lemmas 2 should be of interest not in the context of this paper, but also in itself, since it provide some new "interlacing" type properties for the zeros of s-orthogonal polynomials.

In the analysis of the convergence of Turán [12] or Lobatto-Turán quadrature rules for evaluating C.P.V. integrals we need the assumpion of the existence of a set

$$\bar{N} := \left\{m \in N \middle| \; |x_{c(m)} - t| \sim \frac{1}{m}\right\} \text{ infinite,} \tag{3.6}$$

where $x_{c(m)}$, the node of the integration rule closest to the singularity t, is a zero of a polynomial s-orthogonal with respect to the weight function w or W (see (2.4)) respectively in the case of Turán or Lobatto-Turán rules.

For the latter rules, then, we can consider the following weight functions W

$$\begin{aligned} W(x) &= (1-x^2)^{s+1/2}, \\ W(x) &= (1+x)^{s+1/2}(1-x)^{-1/2}, \\ W(x) &= (1+x)^{-1/2}(1-x)^{s+1/2}, \end{aligned} \tag{3.7}$$

for which it is known [19] that the corresponding s-orthogonal polynomials reduce to polynomials orthogonal with respect to suitable Jacobi weighs, and for Jacobi weights it has been proved in [3] that (3.6) holds.

Beyond the weights (3.7), we shall prove that (3.6) holds for a large class of weight functions. To be more precise, let us assume W is an even function $\in$ GJ satisfying the conditions

$$\begin{cases} W(x) \,/\, (1-x^2)^{s+1/2} & \text{is increasing in } [0,1], \\ W(x)\sqrt{1-x^2} & \text{is decreasing in } [0,1]. \end{cases} \tag{3.8}$$

We shall denote by G^* the set of functions just introduced (examples of weights $\in G^*$ are provided by $W(x) = (1-x^2)^\alpha$, with $-1/2 < \alpha < s + 1/2$). Then the zeros x_{mi} of the polynomial Π_{ms} of degree m, s-orthogonal with respect to $W \in$ GJ, are symmetric with respect to the origin and the positive ones satisfy the relation [19]

$$v_{mi} < x_{mi} < \tau_{mi}\ ,\ i = \left[\frac{m+1}{2}\right] + 1, \ldots, m, \tag{3.9}$$

where v_{mi} and τ_{mi} denote the zeros of the Chebyshev polynomials of the second and the first kind, respectively:

$$v_{mi} = \cos\frac{m-i+1}{m+1}\pi\ ;\quad \tau_{mi} = \cos\frac{2(m-i)+1}{2m}\pi \tag{3.10}$$

As regards to (3.6), we now state the following theorem:

THEOREM 3. *Let $W \in G^*$ and $t = 0$, then (3.6) holds true.*

Proof. Assume

$$\bar{N} = N_2 = \{m \in N \mid m = 2n\}.$$

then $\Pi_{2n,s}(0) \neq 0$, and

$$|x_{c(m)} - t| = |x_{2n,n}| = |x_{2n,n+1}|\ ,$$

hence (3.9) yields (see [3,(6.2.7)])

$$|x_{c(m)} - t| \geq v_{2n,n+1} \geq 1/(2n+1).\ ■$$

In order to show that the result of the above theorem can be extended to values of $t \neq 0$, we pass to prove the mentioned interlacing properties of the zeros of weight functions $\in G^*$, restricting ourselves to consider only the positive ones, due to the symmetry.

LEMMA 2. *Let $w \in G^*$, then the following relations*

$$|x_{mi} - x_{m+1,i}| \sim m^{-1}, \tag{3.11}$$

$$|x_{m+1,i+1} - x_{mi}| \sim m^{-1} \tag{3.12}$$

hold for $i = [(m+I)/2] + I, \ldots, i_0$, where $i_0 = [(7m+5)/2]$.

Proof. By (3.9), (3,10) one has

$$x_{mi} - x_{m+1,i} > v_{mi} - \tau_{m+1,i}$$

and a straightforward calculation yields

$$v_{mi} - \tau_{m+1,i} = 2\sin[(4(m+1-i)+1)/(2m+2))\pi/2]\sin[\pi/(4m+4)] \geq$$

$$\geq 1/(m+1) \sin 3\pi/10 \geq 1/2(m+1),$$

moreover

$$x_{mi} - x_{m+1,i} \leq \tau_{mi} - v_{m+1,i} \leq 3\pi/2(m+2).$$

Analogously, for the given values of i, one has $v_{m+1,i+1} > \tau_{mi}$ and then

$$v_{m+1,i+1} - \tau_{mi} < x_{m+1,i+1} - x_{mi} < \tau_{m+1,i+1} - v_{mi}$$

which, reasoning as before, yields

$$1/5(m+2)\sin 17\pi/30 \leq x_{m+1,i+1} - x_{mi} \leq \pi/2(m+2). \qquad (3.13)$$

Now, let $t \in D :=[-0.5,0.5]$, $w \in G^*$, then, again restricting to consider the situation in the interval $[0,1]$, it is easy to check that the zeros $v_{c(m)}$, $x_{c(m)}$, $\tau_{c(m)}$ closest to t correspond to values of $i \geq i_0$, for every m; more, the zeros of Π_{ms} and $\Pi_{m+1,s}$ interlace. Under the above assumptions on t and W, Lemma 3 can be stated.

LEMMA 3. *The set* $\bar{N}$ *is infinite.*

Proof. Let $x_{c(m)} = x_{mi}$, for large m one has

$$x_{mi} \leq t < x_{m,i+1}, \qquad (3.14)$$

and

$$0 \leq x_{mi} \leq t < x_{m+1,i+1} < x_{m,i+1} \leq \tau_{m,i+1} \leq \cos 3\pi/10,$$

(or $0 \leq x_{m+1,i} < x_{m+1,i+1} \leq t < x_{m,i+1} \leq \cos 3\pi/10$).

If $\bar{N}$ were finite, there would result, for some m

$$|x_{c(m)} - t| = o(m^{-1}), \qquad (3.15)$$

passing to $m+1$, one should have: $x_{c(m+1)} = x_{m+1,i+1}$ or $x_{c(m+1)} = x_{m+1,i}$; in the first case (3.15) and (3.12) yield

$$|x_{m+1,i+1} - t| \geq \| x_{m+1,i+1} - x_{mi}| - |t - x_{mi}\| \geq \text{const } m^{-1},$$

which gives rise to a contradiction.

For the remaining cases, the reasoning is similar.■

Finally, we remark [9], that (3.10) implies

$$|x_{mi} - x_{m,i+1}| \leq \text{const}/m. \tag{3.16}$$

4. A convergence result. We shall deal with the convergence of rule (2.8) under the following assumption H: the weight function W (2.4) and the singularity t of $I(f;t)$ fulfil the conditions (3.6), (3.16) and

$$|C_{hi}| \leq \text{const } m^{-h-1} \tag{4.1}$$

As it was shown in the previous section, conditions (3.6), (3.16) are satisfied in several cases; as for (4.1), it follows from (2.5) at least for any weight function $\in$ GJ.

For $m \in \bar{N}$, we have $x_{mk} < t < x_{mk+1}$, for some k; without loss of generality we may assume $x_{c(m)} = x_{mk}$.

Now, consider the following sums of the weights of (2.8)

$$\sum_{i=0}^{m+1} |D_{hi}| , \quad h = 0,1,\ldots,r_i .$$

under the assumpion H, the asymptotic behavior of the above sums is described by Theorems 4 and 5.

THEOREM 4. *The relation*

$$\sum_{i=0}^{m+1} |D_{0i}| \sim \log m \tag{4.2}$$

holds uniformly on $\tilde{A}$, $m \in \bar{N}$.

Proof. Recalling (2.9), we may write, with $m \in \bar{N}$

$$\sum_{i=0}^{m+1} |D_{0i}| = \sum_{i=0}^{k-1} |D_{0i}| + |D_{0k}| + \sum_{i=k+1}^{m+1} |D_{0i}| =$$

$$= \sum_{i=0}^{k-1} \left| \sum_{h=0}^{r_i} C_{hi} D^h \left[\frac{1}{x-t} \right]_{x=x_{mi}} \right| + \left| \sum_{h=0}^{r_k} C_{hk} \left[D^h \frac{1}{x-t} \right]_{x=x_{mk}} \right| + \tag{4.3}$$

$$+ \sum_{i=k+1}^{m+1} \left| \sum_{h=0}^{r_i} C_{hi} \left[D^h \frac{1}{x-t} \right]_{x=x_{mi}} \right|.$$

Consider first $i = 1,2,\dots,k$, and observe that

$$|D_{0i}| = \left| \sum_{h=0}^{r_i} C_{hi} \left[D^h \frac{1}{x-t} \right]_{x=x_{mi}} \right| = \left| \sum_{h=0}^{r_i} C_{hi} \left[D^h \frac{1}{t-x} \right]_{x=x_{mi}} \right|$$

Since the function $\frac{1}{t-x}$ is ν-absolutely monotone in $[-1,x_{mk}]$ (see (2.11)), then the corollary of section 3 yields

$$\sum_{h=0}^{r_i} C_{hi} \left[D^h \frac{1}{t-x} \right]_{x=x_{mi}} \geq 0,$$

so one has

$$|D_{0i}| = \sum_{h=0}^{r_i} C_{hi} \left[D^h \frac{1}{t-x} \right]_{x=x_{mi}}, \qquad i=0,1,\dots,k.$$

Analogously, since the function $\frac{1}{x-t}$ is ν-completely monotone in $[x_{k+1},t]$ we have

$$|D_{0i}| = \sum_{h=0}^{r_i} C_{hi} \left[D^h \frac{1}{x-t} \right]_{x=x_{mi}}, \qquad i = k+1,\dots,m+1.$$

Making use of Theorems 2 and 3, one gets

$$\int_{-1}^{x_{m,k-1}} \frac{w(x)}{t-x} dx \leq \sum_{i=0}^{k-1} |D_{0i}| \leq |D_{0,k-1}| +$$

$$+ \sum_{i=0}^{k-2} \sum_{h=0}^{r_i} C_{hi} \left[D^h \frac{1}{t-x} \right]_{x=x_{mi}},$$

hence

$$\int_{-1}^{x_{m,k-1}} \frac{w(x)}{t-x}\,dx \le \sum_{i=0}^{k-1} |D_{0i}| \le |D_{0,k-1}| + \int_{-1}^{x_{m,k-1}} \frac{w(x)}{t-x}\,dx.$$

Similarly,

$$\int_{x_{m,k+1}}^{1} \frac{w(x)}{x-t}\,dx \le \sum_{i=k+1}^{m+1} |D_{0i}| \le |D_{0,k+1}| + \int_{x_{m,k+1}}^{1} \frac{w(x)}{x-t}\,dx$$

then, from the last two relations we get

$$\int_{E_k} \frac{w(x)}{|x-t|}\,dx \le \sum_{i=0}^{k-1} |D_{0i}| + \sum_{i=k+1}^{m+1} |D_{0i}| \le |D_{0,k-1}| + |D_{0,k+1}| + \int_{E_k} \frac{w(x)}{|t-x|}\,dx \tag{4.4}$$

where $E_k = A\backslash[x_{mk-1} - x_{mk+1}]$.

Now, since $m \in \overline{N}$, we have $|x_{mi}-t| \sim m^{-1}$, $j = k-1,k,k+1$; thus, taking into account (4.1), and (2.9), we find

$$|D_{0,j}| \le \text{const}, \qquad j = k-1,k,k+1. \tag{4.5}$$

Furthermore it is not difficult to prove that

$$\int_{E_k} \frac{w(x)}{|x-t|}\,dx = w(t)\int_{E_k} \frac{dx}{|x-t|} + \int_{E_k} \frac{w(x)-w(t)}{|x-t|}\,dx \sim \log n$$

then (4.3), (4.4), (4.5) yield (4.2).■

THEOREM 5. *The relation*

$$\sum_{i=0}^{m+1} |D_{hi}| \le const.\, m^{1-h}, \qquad h = 1,2,\dots,r_i, \tag{4.6}$$

holds uniformly on $\tilde{A}$, $m \in \overline{N}$.

Proof. From (2.9), (4.1) one gets

$$\sum_{i=0}^{m+1}|D_{hi}| = \sum_{i=0}^{m+1}\left|\sum_{k=h}^{r_i}\frac{k!}{h!}\frac{(-1)^{k-h}C_{ki}}{(x_{mi}-t)^{k-h+1}}\right| \le$$
$$\le\sum_{i=1}^{m}\sum_{k=h}^{r_i}\frac{k!}{h!}\frac{|C_{ki}|}{|x_{mi}-t|^{k-h+1}} \le \text{const}\,.\,m^{1-h}. \blacksquare$$

Now, we turn to the problem of the convergence of the sequence $\{H_{ms}(f;t)\}$, and prove the Theorem 6 below. We recall the following well known result [24, pag.6];

if $f \in C^r(A)$, for any $m \ge r$ there exists a polynomial ϕ_m of degree m, and a constant C, such that

$$\|q_m^{(i)}\| \le C\left[\frac{1}{m}\right]^{r-i}\omega\left[f^{(r)},\frac{1}{m}\right], \quad i=0,1,\ldots,r. \tag{4.7}$$

where $q_m = f - \phi_m$, and $\|\cdot\|$ is the uniform norm.

We state the theorem below, where $d = \max(p,q,2s)$.

THEOREM 6. *Let $f \in C^d(A)$, $f^{(d)} \in H_\mu(A)$, $d < \mu \le I$, and let assumption H hold, then the sequence $\{H_{ms}(f;t)\}$, $m \in \overline{N}$ converges to $I(f;t)$ uniformly on $\tilde{A}$.*

Proof. For every $m \in \overline{N}$, seen the degree of exactness of (2.8), one has

$$E_{ms}(f;t) = R_{ms}\left[\frac{f-f(t)}{x-t}\right] = R_{ms}\left[\frac{q_m-q_m(t)}{x-t}\right] =$$
$$= I(q_m;t) - H_{ms}(q_m;t)$$

Then, from (2.2), (2.8), one gets

$$|E_{ms}(f;t)| \le \left|\int_{-1}^{1}\frac{q_m(x)-q_m(t)}{x-t}w(x)\,dx\right| +$$
$$+\sum_{i=0}^{m+1}|D_{0i}||q_m(x_{mi}) - q_m(t)| +$$
$$+\sum_{i=0}^{m+1}\sum_{h=1}^{r_i}|D_{hi}||q_m^{(h)}(x_{mi})| := P_1+P_2+P_3.$$

We have

$$P_1 < \int_A \left|\frac{q_m(x)-q_m(t)}{x-t}\right| |w(x)-w(t)|\,dx + w(t)\left|\int_A \frac{q_m(x)-q_m(t)}{x-t}\,dx\right|$$

$$\le 2\|\,q_m\|\left[\int_A \frac{|w(x)-w(t)|}{|x-t|}dx + 2w(t)\cdot\log m\right] +$$

$$+\left|\int_{t-1/m}^{t+1/m} \frac{q_m(x)-q_m(t)}{x-t} w(t)\,dx\right| \le 2C\left[\frac{1}{m}\right]^d \omega\left[f^{(d)};\frac{1}{m}\right]\cdot$$

$$\cdot\left[\int_A \frac{|w(x)-w(t)|}{|x-t|}dx \quad +2\,w(t)\cdot\log m\right] + J(f;t)$$

with

$$J(f;t) = \left|\int_{t-1/m}^{t+1/m} \frac{q_m(x)-q_m(t)}{x-t}\, w(t)\,dx\right|.$$

Then (4.2), (4.6) yield

$$P_2 \le 2\,\|q_m\|\log m \le 2\,\text{const}\left[\frac{1}{m}\right]^d \cdot \omega\left[f^{(d)};\frac{1}{m}\right]\cdot\log m$$

$$P_3 \le \sum_{h=1}^{d} \|q_m^{(h)}\|\cdot m^{1-h} \le \text{const}\ \omega\left[f^{(d)};\frac{1}{m}\right] \sum_{h=1}^{d} \left[\frac{1}{m}\right]^{d-1},$$

where, as usual, $\sum_{h=1}^{d}$ is to be considered = 0.

Now, the assumption $f^{(d)} \in H_\mu(A)$ implies

$$\omega\left[f^{(d)};\frac{1}{m}\right]\cdot\log m = o(1)\ ,\quad J(f;t) = o(1)\ ,$$

from which the claim follows.∎

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EXISTENCE THEOREMS FOR SOME OPERATORIAL EQUATIONS IN HILBERT SPACES

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REZUMAT. - Teoreme de existenţă pentru ecuaţii operatoriale în spaţii Hilbert. În lucrare sînt folosite cîteva rezultate de cea mai bună aproximare în spaţii normate netede pentru a obţine teoreme de existenţă pentru ecuaţii asociate unei clase de operatori pe un spaţiu Hilbert, care conţine operatorii liniari strict pozitivi. Sînt date aplicaţii la ecuaţii cu derivate parţiale.

Summary. In this paper we will use some results of best approximation theory in smooth normed spaces [3] to give existence theorems for certain equations associated to a class of operators defined on a Hilbert space and containing the strictly positive linear operators. Some applications to partial differential equations are also given.

0. Introduction. We recall the concept of semi-inner product on a linear space E over real or complex number field K.

DEFINITION 0.1. A mapping $[,]$ of $E \times E$ into K is *semiinner product* (s.i.p.) on E if the following conditions (P1)-(P4) are satisfied (see [5] and [6]):

(P1) $[x,x] \geq 0$ for all $x \in E$ and $[x,x] = 0$ implies $x = 0$;

(P2) $[\lambda x,y] = \lambda[x,y]$ and $[x,\lambda y] = \bar{\lambda}[x,y]$ for all $\lambda \in K$ and x,y in E;

(P3) $[x + y,z] = [x,z] + [y,z]$ for all $x,y,z \in E$:

(P4) $|[x,y]|^2 \leq [x,x][y,y]$ for all $x,y \in E$.

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It is easy to see that the mapping $E \ni x \to [x,x]^{1/2} \in \mathbb{R}^+$ is a norm on E and if E is a normed space, then every s.i.p. on E which generates the norm is of the form

$$[x,y] = \langle \tilde{J}(y), x \rangle \text{ for all } x,y \in E,$$

where $\tilde{J}$ is a section of normalized dual mapping [3]. It is also known that a normed space E is smooth iff there exists a continuous s.i.p. on it which generates its norm, i.e., a s.i.p. satisfying the condition

$$\lim_{t\to 0} \text{Re}[y, x + ty] = \text{Re}[y,x] \text{ for all } x,y \in E \text{ (see [5])}.$$

Let two elements x,y in E and a s.i.p. $[,]$ on E which generates the norm of E. The element x is said to be Lumer-orthogonal over y or L-orthogonal over y, for short, if $[y,x] = 0$. We denote this fact by $x \; L \; y$. For some properties of s.i.p., L-orthogonality, representation of continuous linear functionals in terms of s.i.p. we send to [1-6] where further references are given.

Now, let $(H;(,))$ be a Hilbert space over the real or complex number field K. An operator $A : D(A) \subset H \to H$, $D(A)$ is a subspace in H, will be called of Lumer-type or of L-type, for short, if the following conditions are satisfied

(L1) $D(A)$ is dense in H and $(u,Au) > 0$ for all $u \in D(A)\setminus\{0\}$;

(L2) $A(\alpha u) = \alpha Au$ for all $u \in D(A)$, $\alpha \in K$;

(L3) $|(u,Av)|^2 \le (u,Au)(v,Av)$ for all $u,v \in D(A)$;

(L4) $\lim_{t\to 0} \text{Re}(u,A(v + ut)) = \text{Re}(u,Av)$ for all $u,v \in D(A)$.

As examples of L-type operators, we can give the class of strictly positive linear operators on a dense subspace of a

Hilbert space H.

Now, let consider the operatorial equation

$(A;y)$ $\quad Ax = y,\ x \in D(A)$ and y is given in H,

where A is an operator as above. Further on, we shall give some existence theorems for this equation in terms of best approximation in smooth normed spaces.

1. Some preliminary results. In this section we shall give some concepts and results in best approximation theory that will be used in the sequel.

Let E be a normed space and x,y be two elements in E. The vector x is called orthogonal in the sense of Birkhoff over the vector y if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in K$. We denote this $x \perp y$.

If G is a nondense linear subspace in E and

$$P_G(x_0) := \{y_0 \in G \mid \|y_0 - x_0\| = \inf_{y \in G} \|y - x_0\|\}$$

denotes the set of best approximation elements refering to $x_0 \in E \setminus \bar{G}$ in G, then the following simple characterization lemma in terms of Birkhoff's orthogonality holds [7, Lemma 1.14]:

LEMMA 1.1. *Let E, G, x_0 be as above and $g_0 \in G$. Then $g_0 \in P_G(x_0)$ if and only if $x_0 - g_0 \perp G$.*

For other characterizations of best approximation element in normed spaces see [7] and [2] where further references are given.

In what follows, E will be a smooth normed linear space and $[,]$ will be the unique s.i.p. which generates its norm. The following characterization of L-orthogonality in smooth normed spaces holds (see for example [3, Lemma 1.1]):

LEMMA 1.2. *Let E, $[,]$ be as above and x,y two given elements in E. Then $x \mathrel{L} y$ iff $x \perp y$.*

By the use of this lemma, we have the following proposition [3, Theorem 1.2]:

PROPOSITION 1.3. *Let G be a nondense linear subspace in E, $x_0 \in E \setminus \bar{G}$ and $g_0 \in G$. Put $G^L := \{w \in E \mid w \mathrel{L} g,\ g \in G\}$. Then $g_0 \in P_G(x_0)$ if and only if there exists an element $w_0 \in G^L$ so that $x_0 = g_0 + w_0$.*

Another result which will be used in the following is embodied in (see [3, Theorem 1.5]):

PROPOSITION 1.4. *Let f be a nonzero continuous linear functional on smooth normed space E, $x_0 \in E \setminus \mathrm{Ker}(f)$ and $g_0 \in$ $\in \mathrm{Ker}(f)$. Then $g_0 \in P_{\mathrm{Ker}(f)}(x_0)$ if and only if the following representation holds*

$$f(x) = [x, \overline{f(x_0)}\,(x_0 - g_0)/\|x_0 - g_0\|^2] \text{ for all } x \in E. \tag{1}$$

Now, recall the concept of proximinal linear subspaces in a normed space. The linear subspace G is called proximinal in E if for all $x_0 \in E$ the set $P_G(x_0)$ contains at least one element. The following characterization of proximinal linear subspaces in smooth normed spaces holds [3, Corollary 1.3]:

PROPOSITION 1.5. *Let E be a smooth normed space, $[,]$ be the s.i.p. which generates its norm and G be its linear subspace. Then G is proximinal if and only if the following decomposition*

$$E = G + G^L \tag{2}$$

holds.

CONSEQUENCES 1.6.1. *Let E be a smooth normed space and G be a linear subspace in E so that $\bar{B}_G := \{g \in G \mid \|g\| \le 1\}$ is weakly* *sequentially compact in E. Then the decomposition (2) holds.*

The proof follows by V.Klee's theorem (see [3] or [7, p.91]) and by the above proposition.

2. If G is finite-dimensional, then (2) is valid too.

3. If E is a smooth reflexive Banach space, then for every closed linear subspace G in E the decomposition (2) also holds.

The following proposition is important in the sequel.

PROPOSITION 1.7. *Let E be a smooth normed space, $[,]$ be the s.i.p. which generates its norm and f be a continuous linear functional on it. Then the following statements are equivalent:*

(i) $\mathrm{Ker}(f)$ *is proximinal in E;*

(ii) *there exists at least one u_f in E such that the following representation*

$$f(x) = [x, u_f] \quad \text{for all } x \text{ in } E \tag{3}$$

holds

For the proof of this fact see [3, Corollary 1.6].

The following consequences are interesting in themselves too.

CONSEQUENCES 1.8.1. *Let E be a smooth normed linear space, f be a continuous linear functional on it such that $\bar{B}_{\mathrm{Ker}(f)} :=$ $:= \{g \in \mathrm{Ker}(f) \mid \|g\| \le 1\}$ is weakly sequentially compact in E. Then there exists at least one u_f in E such that (3) holds.*

The proof follows by V.Klee's theorem and by the above proposition.

2. If $f \in E^*$, then for every finite-dimensional subspace G in E there exists an element $u_{f,G} \in G$ so that

$$f(x) = [x, u_{f,G}] \text{ for all } x \in G.$$

3. If E is reflexive, then for every continuous linear functional f on it, there exists at least one element u_f in E so that the representation (3) holds.

2. Existence theorems. Further on, $(H;(,))$ will be a Hilbert space over the real or complex number field K.

Let reconsider the operatorial equation:

$(A;y) \qquad Ax = y, \quad x \in D(A) \quad \text{and} \quad y \in H,$

where A is an operator of L-type and y is a given element in H.

We observe that, the mapping $[,]_A : D(A)\times D(A) \to K$, $[x,y]_A := (x,Ay)$ is a continuous s.i.p. on $D(A)$, i.e., the pair $(D(A); \|\cdot\|_A)$ where $\|x\|_A := (x,Ax)^{1/2}$ is a smooth normed space.

Consider the following linear subspace of $D(A)$:

$$y_A^{\perp} := \{x \in D(A) \mid (x,y) = 0\}.$$

The following characterization theorem for the solutions of equation $(A;y)$ holds.

THEOREM 2.1. *If $y_A^{\perp}$ is a nondense linear subspace of $(D(A), \|\cdot\|_A)$, $x_0 \in D(A) \setminus \overline{y_A^{\perp}}$ (where $\overline{y_A^{\perp}}$ is the closure of $y_A^{\perp}$ in the space $(D(A); \|\cdot\|_A)$) and $g_0 \in y_A^{\perp}$, then the following statements are equivalent*

(i) $g_0 \in P_{y_A^{\perp}}(x_0)$ *in* $(D(A); \|\cdot\|_A)$;

(ii) *there exists an element* $z_0 \in (y_A^{\perp})^{\perp_A}$ *such that*

$$x_0 = g_0 + z_0;$$

(iii) *the following relation is valid*

$$(g, A(x_0 - g_0)) = 0 \text{ for all } y \in y_A^{\perp};$$

(iv) *the element* $u_0 \in D(A)$ *given by*

$$u_0 := \frac{(y, x_0)\ (x_0 - g_0)}{(x_0 - g_0, A(x_0 - g_0))} \tag{4}$$

is a solution of equation $(A;y)$.

Proof. "(i) ⟷ (ii)". Follows from Proposition 1.3.

"(i) ⟷ (iii)". Follows from Lemma 1.1 and Lemma 1.2.

"(i) ⟷ (iv)". Let consider the linear functional $f_y : D(A) \to K$ given by $f_y(x) := (x,y)$. Then $\mathrm{Ker}(f_y) = y_A^{\perp}$ and since $y_A^{\perp}$ is nondense in $(D(A), \|\cdot\|_A)$ hence f_y is continuous in this normed space.

On the other hand, because $g_0 \in P_{Ker(f_y)}(x_0)$ in $(D(A), \|\cdot\|_A)$ then by Proposition 1.4, we have the representation:

$$\begin{aligned}(x,y) &= f_y(x) = \left[x, \overline{f_y(x_0)}\,(x_0 - g_0)/\|x_0 - g_0\|_A^2\right]_A = \\ &= (x, A((\overline{x_0, y})\,(x_0 - g_0)/(x_0 - g_0, A(x_0 - g_0))\end{aligned}$$

for all x in $D(A)$, and by the density of $D(A)$ in $(H;(,))$ We derive that $A(u_0) = y$ where u_0 is given in (4).

"(iv) ⟷ (i)". If $u_0 := (y,x_0)(x_0 - g_0)/(x_0 - g_0, A(x_0 - g_0))$ is a solution of $(A;y)$, then

$$(x,y) = (x, A((y,x_0)(x_0 - g_0)/(x_0 - g_0, A(x_0 - g_0))$$

for all x in $D(A)$. Consequently

$$f_y(x) = \left[x, \overline{f_y(x_0)}\,(x_0 - g_0)/\|x_0 - g_0\|_A^2\right]_A$$

for all x in $D(A)$, and by Proposition 1.4 we deduce that $g_0 \in P_{Ker(f_y)}(x_0)$ in the smooth normed space $(D(A); \|\cdot\|_A)$.

The proof is finished.

Remark 2.2. The condition : $y_A^{\perp}$ is nondense in $D(A)$ with the norm $\|\cdot\|_A$ is equivalent with the existence of a constant $k > 0$

so that

$$|(x,y)| \le k(x,Ax)^{1/2} \quad \text{for all } x \text{ in } D(A). \tag{5}$$

An operator of L-type will be called positive definite if the following condition holds

$$(x,Ax) \ge m\|x\|^2 \quad \text{for all } x \text{ in } D(A),$$

where m is a positive number. As examples of such operators we can give the positive definite linear operators defined on a dense linear subspace in a Hilbert space.

The following corollary holds.

COROLLARY 2.3. *Let $(H;(,))$ be a Hilbert space, $A : D(A) \to H$ be a positive definite operator of L-type and y a given nonzero element in H. If $x_0 \in D(A)\setminus y_A^{\perp}$ and $g_0 \in y_A^{\perp}$, then the previous conditions* (i), (ii), (iii) *and* (iv) *are equivalent.*

Proof. If $y \neq 0$, we have $\|y\| \neq 0$ and

$$|(x,y)| \le \|x\|\ \|y\| \le (\|y\|/m^{1/2})'(x,Ax)^{1/2} \text{ for all } x \in D(A).$$

Putting $k = \|y\|/m^{1/2} > 0$, the condition (5) holds and then $y_A^{\perp}$ is nondense in $(D(A);\|\cdot\|_A)$ and the argument follows from the above theorem.

The following existence theorem is also valid.

THEOREM 2.4. *Let $(H;(,))$ be a Hilbert space, $A : D(A) \to H$ be an operator of L-type and y a nonzero element in H. Then the following statements are equivalent*

(i) *$y_A^{\perp}$ is proximinal in $D(A)$ endowed with the norm $\|\cdot\|_A$;*

(ii) *the equation $(A;y)$ has at least one solution.*

Proof. "(i) $\to$ (ii)". If $y_A^{\perp} := \operatorname{Ker}(f_y)$ $(f_y(x) = (x,y)$, $x \in D(A))$ is proximinal $(D(A);\|\cdot\|_A)$, then, by Proposition 1.7, there exists an element $u \in D(A)$ such that

$$(x,y) = f_y(x) = [x,u]_A = (x,Au) \text{ for all } x \in D(A),$$

i.e., u is a solution of equation $(A;y)$.

"(ii) → (i)". If u is a solution of $(A;y)$, then, as above, we have the representation

$$f_y(x) = [x,u]_A \quad \text{for all} \quad x \in D(A),$$

i.e., by Proposition 1.7, $\mathrm{Ker}(f_y)$ is proximal.

The proof is finished.

CONSEQUENCES 2.5. *Let $(H;(,))$ be a Hilbert space, $A : D(A) \subset \subset H \to H$ be an operator of L-type and y be a nonzero given element in H.*

1. *If the ball $\bar{B}_y(A) := y_A^\perp \cap \{x \in D(A) \mid \|x\|_A \le 1\}$ is weakly sequentially compact in $(D(A), \|\cdot\|_A)$, then the equation $(A;y)$ has at least one solution.*

2. *If $y_A^\perp$ is finite-dimensional in $D(A)$, then the equation $(A;y)$ also has at least one solution.*

Now, we will apply the above results to differential equations as follows.

3. Applications to partial differential equations. Let us consider the following partial differential equation

$$(\Delta, f) \qquad \begin{cases} -\Delta u = -\sum_{i=1}^{n} \partial^2 u/\partial x_i^2 = f \in L^2(\Omega), \Omega \subset \mathbb{R}^n \\ u|_{\partial\Omega} = 0. \end{cases}$$

Putting $\phi^{(2)}(\Omega) := \{u \in C^2(\Omega) \mid \ u|_{\partial\Omega} = 0\}$, then $\phi^{(2)}(\Omega)$ is a dense linear subspace in $L^2(\Omega)$ and if we define the mapping:

$$(,)_\Delta : \phi^{(2)}(\Omega) \times \phi^{(2)}(\Omega) \to \mathbb{R},\ (u,v)_\Delta := -\int_\Omega \frac{\partial^2 u}{\partial x_i^2}\, v\, dx$$

then $(,)_\Delta$ is an inner-product on $\phi^{(2)}(\Omega)$, we have

$$(u,v) = \int_\Omega \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_i}\, dx, \quad u, v \in \phi^{(2)}(\Omega),$$

and there exists a real number k so that:

$$(u,u)_\Delta \geq k^2 \|u\|_2^2, \quad u \in \phi^{(2)}(\Omega),$$

where $\|\cdot\|_2$ is the usual norm in $L^2(\Omega)$ [1, p.189].

PROPOSITION 3.1. *Let $w_0 \in \phi^{(2)}(\Omega)\backslash f^\perp$ and $g_0 \in f^\perp$ where*

$$f^\perp := \left\{ u \in \phi^{(2)}(\Omega) \mid \int_\Omega u(x) f(x)\, dx = 0 \right\}.$$

Then the following statements are equivalent

(i) $P_{f^\perp}(w_0) = \{g_0\}$ in $(\phi^{(2)}(\Omega); (,)_\Delta)$;

(ii) *the following relation holds:*

$$\int_\Omega \frac{\partial(w_0 - g_0)}{\partial x_i} \cdot \frac{\partial g}{\partial x_i}\, dx = 0 \text{ if } g \in f^\perp;$$

(iii) *the element $u_0 \in \phi^{(2)}(\Omega)$ given by*

$$u_0 := \frac{\int_\Omega \sum_{i=1}^{n} (\partial f/\partial x_i \cdot \partial w_0/\partial x_i)\, dx}{\int_\Omega \sum_{i=1}^{n} (\partial(w_0 - g_0)/\partial x_i)^2\, dx} (w_0 - g_0)$$

is the unique solution of the equation $(\Delta; f)$.

The existence follows by Theorem 2.1 for $H = L^2(\Omega)$, $A = -\Delta$, $D(A) = \phi^{(2)}(\Omega)$, $y = f$ and $y_A^\perp = f^\perp$ and observing that the subspace $y_A^\perp$ is closed in $D(A)$.

The unicity is clear by the linearity of A.

PROPOSITION 3.2. *Let f be a nonzero element in $L^2(\Omega)$. Then the following sentences are equivalent:*

(i) *$f^{\perp}$ is chebychefian in $(\phi^{(2)}(\Omega);(,)_{\Delta})$, i.e. the set of best approximation $P_{f^{\perp}}(x_0)$ is a singleton for all x_0;*

(ii) *the equation $(\Delta;f)$ has a unique solution.*

The proof is obvious from Theorem 2.4.

Similar results can be stated for the following differential equations:

$$1. \quad \begin{cases} -\Delta u = f \in L^2(\Omega), \ \Omega \subset \mathbf{R}^n, \\ \dfrac{\partial u}{\partial n} + \sigma(P)\, u|_{\partial\Omega} = 0, \ \sigma \in C(\partial\Omega), \ \sigma(P) \geq \sigma_0 = \text{constant} > 0. \end{cases}$$

and $\partial\Omega$ is lipschitzian.

$$2. \quad \begin{cases} \sum_{k=0}^{m} (-1)^k d^k [p_K(x)\, d^k u/dx^k] / dx^k = f \in L^2(0,1) \\ u(0) = u^{(1)}(0) = \ldots = u^{(m-1)}(0) = 0 \\ u(1) = u^{(1)}(1) = \ldots = u^{(m-1)}(1) = 0 \end{cases}$$

and p_k $(k = 0,\ldots,m)$ satisfy the conditions

a. $p_k(x) \geq 0$ for all $x \in [0,1]$ and there exists k_0 such that

$$p_{k_0}(x) \geq p_0 > 0 \quad \text{for all} \quad x \in [0,1];$$

b. $p_k \in C^{(k)}[0,1]$, $k = 0,1,\ldots,m$.

For other examples of this type we send to [1, pp. 167-235] where further references are given.

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NOTE ON AN ABSTRACT CONTINUATION THEOREM

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REZUMAT. -Notă asupra unei teoreme abstracte de continuare. În această notă se arată că teorema de transversalitate topologică a lui Granas şi proprietatea de invarianţă la omotopie a indexului de punct fix, pot fi deduse dintr-o teoremă abstractă de continuare demonstrată în [6], [8]. Aceste aplicaţii ca şi aplicaţia privind proprietatea de omotopie a aplicaţiilor zero-epi în sensul lui Furi-Martelli-Vignoli [9] arată că teorema noastră de continuare permite o abordare unitară a unor principii de bază ale analizei neliniare. În context, teorema de transversalitate topologică a lui Granas pentru aplicaţii cu valori într-o submulţime inchisă şi convexă a unui spaţiu Banach *E*, este extinsă la cazul aplicaţiilor cu valori într-un retract al lui *E*.

1. Introduction. In this note our general continuation theorem given in [6], [8] is used in order to derive two useful principles in nonlinear analysis, namely, the topological transversality theorem of A.Granas [2] and the homotopy invariance property of the fixed point index. These applications, together with that in [9] concerning the homotopy property of zero-epi maps in the sense of Furi-Martelli-Vignoli, show that our continuation theorem permits an unified approach to some basic principles in nonlinear analysis. In context, Granas transversality theorem for maps with values into a closed convex subset of a Banach space *E* is extended to maps with values into a retract of *E*.

For other consequences of the abstract continuation theorem, several applications and related topics we send to [3], [4], [5] and [7].

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2. The abstract continuation theorem. Let X be a normal topological space, A a proper closed subset of X, Y a set, B a proper subset of Y,

$$H: [0,1] \times X \to Y$$

a map and let d be a certain function which is defined at least on the following family of subsets of X:

$$\{H(a(.),.)^{-1}(B);\ \ a \in C(X;\ [0,1]) \text{ constant on } A\} \cup \{\phi\}.$$

The nature of the values of d is not important.

THEOREM 1. *Assume that the following conditions are satisfied:*

(i) $cl(\cup\{H(t,.)^{-1}(B);\ t \in [0,1]\} \cap A = \phi;$

(ii) *the map* $F = H(0,.)$ *satisfies*

$$d(H(a(.),.)^{-1}(B)) = d(F^{-1}(B)) \neq d(\phi) \qquad (1)$$

for any function $a \in C(X;\ [0,1])$ *constant on* A *and such that*

$$H(a(.),.)|_A = F|_A.$$

Then there exists at least one $x \in X \setminus A$ *solution to* $H(1,x) \in B$. *Moreover,* $F = H(1,.)$ *also satisfies condition* (1) *and*

$$d(H(1,.)^{-1}(B)) = d(H(0,.)^{-1}(B)). \qquad (2)$$

The proof of the first part was given in [6]. For the last part, formula (2), we use the same argument as in the proof of the second part of Theorem 1 in [8].

The meaning of Theorem 1 is that property (1), which is stronger that $F^{-1}(B) \neq \phi$, is invariable to homotopy. In applications, Theorem 1 ensures the solvability of the inclusion $H(1,x) \in B$ when it is known that $H(0,.)$ satisfies (1).

A map f in the class of all maps of the form $H(a(.),.)$, where $a \in C(X;\ [0,1])$ is constant on A, is said to be ***d*-essential** if it satisfies condition (1). Therefore, Theorem 1 saies tnat the d-essentiality property is invariable to homotopy.

3. **Applications.** Let E be a real Banach space and K a retract of E (this means that there is a continuous map $R: E \to K$ such that $R(x) = x$ on K). All topological notions referring to subsets of K will be understood with respect to the topology induced on K.

Let U be an open bounded subset of K and $h: [0,1] \times \overline{U} \to K$ be compact.

a) A first application depends upon the concept of **fixed point index.** For a compact map $f: \overline{U} \to K$ such that $\mathrm{Fix}(f) \cap \partial U = \phi$ the fixed point index is the integer number $D_{LS}(I - fR, R^{-1}(U), 0)$ where D_{LS} is the Leray-Schauder degree. We shall denote it by $i(f,U,K)$ (see [1, pp 238]).

COROLLARY 1 (Leray-Schauder). *Assume that the following conditions are satisfied:*

(i) $h(t,x) \neq x$ *for all* $t \in [0,1]$ *and* $x \in \partial U$;

(ii) $i(h(0,.),U,K) \neq 0$.

Then there exists at least one fixed point of $h(1,.)$ *in* U. *Moreover,*

$$i(h(1,.),U,K) = i(h(0,.),U,K).$$

Proof. Apply Theorem 1 to: $X = \overline{U}$, $A = \partial U$, $Y = E$, $B = \{0\}$, $H(t,x) = x - h(t,x)$, $d(\phi) = 0$,

$$d(H(a(.),.)^{-1}(B)) = i(h(a(.),.),U,K).$$

In this case, condition (1) is satisfied and its equality part expresses just the boundary value dependence of the degree.

Remark. Condition (ii) in Corollary 1 clearly holds if $h(0,x) = x_0$ for all $x \in \bar{U}$ (recall $x_0 \in U$).

b) The next application depends upon the concept of **essential map.** A compact map $f: \bar{U} \to K$ is said to be admissible if it is fixed point free on ∂U. An admissible map is essential if each admissible extention g of $f|_{\partial U}$ has at least one fixed point in U.

COROLLARY 2 (Granas). *Assume that the following conditions are satisfied:*

(i) $h(t,x) \neq x$ *for all* $t \in [0,1]$ *and all* $x \in \partial U$;

(ii) $h(0,.)$ *is essential.*

Then there exists at least one fixed point of $h(1,.)$ *in* U. *Moreover, the map* $h(1,.)$ *is essential too.*

Proof. The conclusion follows from Theorem 1 if for each admissible extension g of $h(1,.)|_{\partial U}$ (in particular for $g = h(1,.)$) we set: $X = \bar{U}$, $A = \partial U$, $Y = E$, $B = \{0\}$,

$$H(t,x) = x - h(2t,x) \qquad \text{for } t \in [0,1/2]$$

$$= x - 2(1-t)h(1,x) - (2t-1)g(x) \qquad \text{for } t \in [1/2,1]$$

and $d(\phi) = 0$, $d(C) = 1$ for $C \neq \phi$.

Remark 2. Condition (ii) in Corollary 2 also holds for $h(0,x) = x_0$, $x \in \bar{U}$. This follows by Schauder's fixed point theorem.

Remark 3. Recall that every closed convex subset is a retract and that every retract is closed but not necessarily convex; for instance, $\partial B_1(0)$ is a retract of E if $\dim E = \infty$.

Remark 4. There are examples of compact maps having null index but which are essential. Here is one from [10]: Let E be a real Hilbert space, U a bounded open subset of E with $0 \in U$ and let $f: \overline{U} \to E$ be compact such that $f(x) \neq x$ on ∂U and

$$(f(x), x) \geq (x, x) \quad \text{for all } x \in \partial U.$$

If E is infinite dimensional, one has

$$D_{LS}\ (I - f,\ U,\ 0) = 0$$

and f is essential. Therefore, such a map can stand for $h(0,.)$ in Corollary 2, but not in Corollary 1 if E is infinite dimensional.

Remark 5. The main ingredient in the proof of Theorem 1 is Urysohn's extension theorem in normal (T_4) topological spaces. Using the extension argument in a way adequated to the separation properties, we are able to prove Theorem 1 even for more general T_n spaces ($n < 4$); in particular, for Hausdorff locally convex spaces.

Remark 6. Note that the properties in Corollary 1 and Corollary 2 can be derived from Theorem 1 even for more general maps.

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FIXED POINT THEOREMS OF KRASNOSELSKII TYPE FOR ε-LOCALLY CONTRACTIVE MULTIVALUED MAPPINGS

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REZUMAT. - Teoreme de punct fix de tip Krasnoselskii pentru ε-contracţii locale multivoce. Scopul acestei lucrări este de a demonstra teoreme de punct fix de tip Krasnoselskii pentru ε-contracţii locale multivoce, extinzînd astfel unele rezultate obţinute în [5], [6].

1. Preliminaries. In [6] B.Rzepecki proved that if (X,d) is a compact metric space, Y is a nonempty, closed and convex subset of a Banach space and F is a multivalued mapping from $X \times Y$ into Y, H-continuous in the first variable and k-contraction $(0 < k < 1)$ in the second one, then there is a continuous mapping $g: X \times Y \to Y$ such that $g(x) \in F(x,g(x))$, for every $x \in X$. As a consequence a fixed point principle ot Krasnoselskii type is given.

In 1985, L.Rybinski generalized these results to the case of a multivalued φ-contraction (see [6]). The purpose of this paper is to prove fixed point theorems of Krasnoselskii type for ε-locally contractive multivalued mappings.

Throughout this paper, if it is not assumed otherwise, (X,d) is a metric space, $(Y, \|.\|)$ is an uniformly convex Banach space. Let $P_{cl}(Y)$ be the collection of all nonempty, closed subsets of Y endowed with the generalized Hausdorff metric H.

$(P_{cl}(Y),H)$ is a generalized metric space (see [7]). Let $P_{cl,cv}(Y)$ be the family of convex elements of $P_{cl}(Y)$. In this

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paper a multivalued mapping $G: X \to P_{cl}(Y)$ is called H-continuous if it is continuous in the sense of the generalized Hausdorf metric.

Let us assume that $\varphi: X \times \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that:

(1) φ is continuous

(2) $\varphi(x,.): \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing, for each $x \in X$

(3) $\sum_{n=0}^{\infty} \varphi^n(x,t) < \infty$, for each $x \in X$ and $t \in \mathbb{R}_+$ where $\varphi^0(x,t) = t$, $\varphi^n(x,t) = \varphi(x,\varphi^{n-1}(x,t))$

(4) for any continuous function a: $X \to \mathbb{R}_+$ the function

$$S: X \to \mathbb{R}_+,\ S(x) = \sum_{n=0}^{\infty} \varphi^n(x,a(x)) \text{ is continuous.}$$

If $y \in Y$ and $A \subset Y$, denote $D(y,A) = \inf\{d(y,a) \mid a \in A\}$.

2. Basic results. The following definition is modeled after Nadler-Covitz's definition of a locally contractive multivalued mapping (see [2]).

DEFINITION 1. *A multivalued mapping* $G: X \times Y \to P_{cl}(Y)$ *is said to be* ε*-locally contractive* ($\varepsilon > 0$) *if there exists* $\varphi: X \times \mathbb{R}_+ \to \mathbb{R}_+$ *such that:* $u,v \in Y$, $\|u - v\| < \varepsilon \Rightarrow H(G(x,u), G(x,v)) \le \varphi(x, \|u - v\|)$ *for every* $x \in X$.

For the proof of the main result we need some lemmas.

LEMMA 1. ([5]) *If* $\varphi: X \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies* (2) *and* (3) *then* $\varphi(x,0) = 0$ *and* $\varphi(x,t) < t$, *for each* $x \in X$, $t > 0$.

LEMMA 2. ([1]) *Let* (X,d) *be a metric space,* $(Y,\|.\|)$ *an uniformly convex Banach space,* $g: X \to Y$ *a continuous singlevalued mapping and* $G: X \to P_{cl,cv}(Y)$ *a* H*-continuous multivalued mapping. Then there exists a continuous mapping* $h: X \to Y$ *such that* $h(x) \in G(x)$ *and* $\| h(x) - g(x) \| = D(g(x),G(x))$, *for every* $x \in X$.

We will prove now the main result of this paper.

THEOREM 1. *Suppose that* (X,d) *is a compact metric space,* $(Y,\|.\|)$ *is an uniformly convex Banach space and* $G: X \times Y \to$ $\to P_{cl,cv}(Y)$ *is a multivalued mapping such that:*

(i) $G(.,y): X \to P_{cl,cv}(Y)$ *is H-continuous, for each* $y \in Y$

(ii) $G(x,.): Y \to P_{cl,cv}(Y)$ *is a* ε*-locally contractive multivalued mapping, with a function* $\varphi: X \times \mathbb{R}_+ \to \mathbb{R}_+$ *which satisfies* (1), (2), (3) *and* (4), *for each* $x \in X$.

(iii) *there is an element* $u_0 \in Y$ *such that* $D(u_0,G(x,u_0)) <$ $< \varepsilon$ *for each* $x \in X$.

Then, there exists a continuous function $g: X \to Y$ *such that* $g(x) \in G(x,g(x))$, *for every* $x \in X$.

Proof. Let $u_0 \in Y$ be such that $D(u_0,G(x,u_0)) < \varepsilon$, $x \in X$. Since the mapping $G(.,u_0)$ is H-continuous, the function $a: X \to$ $\to \mathbb{R}_+$, $a(x) = D(u_0,G(x,u_0))$, $x \in X$ is continuous.

By Lemma 2. we can find a continuous selection of $G(.,u_0)$, call it $g_1: X \to Y$ such that $g_1(x) \in G(x,u_0)$ and $\|g_1(x) - u_0\| =$ $= D(u_0,G(x,u_0)) = a(x) < \varepsilon$, for every $x \in X$.

Invoking again Lemma 2, there is a continuous mapping $g_2: X \to Y$ such that $g_2(x) \in G(x,g_1(x))$ and $\|g_2(x) - g_1(x)\| =$ $= D(g_1(x),G(x,g_1(x))$, for every $x \in X$. Using Lemma 1 we have: $\|g_2(x) - g_1(x)\| = D(g_1(x),G(x,g_1(x))) \le H(G(x,u_0),G(x,g_1(x))) \le$ $\le \varphi(x,\|g_1(x) - u_0\|) \le \varphi(x,a(x)) < a(x) < \varepsilon$, for every $x \in X$.

By induction we get an interative sequence of continuous mappings $(g_n)_{n\in N}$, $g_n: X \to Y$ satisfying

(A) $g_0(x) = u_0$, $g_{n+1}(x) \in G(x,g_n(x))$, for every $x \in X$

(B) $\|g_{n+1}(x) - g_n(x)\| \le \varphi^n(x,a(x)) < \varepsilon$, for every $x \in X$

and $n \in \mathbb{N}$.

From (B) $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Y so there is $g(x) = \lim_{n\to\infty} g_n(x)$, for each $x \in X$. Denote $r_n(x) = \varphi^k(x, a(x))$ for $n = 2k$ or $n = 2k + 1$, $n,k = 0,1,2,\ldots$, and $x \in X$. Obviously $\sum_{n=0}^{\infty} r_n(x) = 2S(x)$, for each $x \in X$. Thus by (B) we get

$$\|g_{n+p}(x) - g_n(x)\| \le \sum_{i=n}^{n+p-1} r_i(x), \ x \in X.$$

Denote $R_n(x) = 2S(x) - \sum_{i=0}^{n-1} r_i(x)$, for each $x \in X$.

By (1) and (4), $R_n\colon X \to \mathbb{R}_+$ is continuous for $n = 1,2,3,\ldots$, (X,d) is a compact metric space and so we have that $R_n \rightrightarrows 0$, as $n \to \infty$.

Therefore, for $\varepsilon > 0$ there is $n(\varepsilon) \in \mathbb{N}$ such that $R_n(x) < \varepsilon$, for every $n > n(\varepsilon)$ and $x \in X$.

Since $\|g(x) - g_n(x)\| \le R_n(x) < \varepsilon$, or every $n > n(\varepsilon)$ and $x \in X$ we obtain:

$$D(g(x), G(x,g(x))) \le \|g(x) - g_n(x)\| + D(g_n(x), G(x,g(x))) \le$$
$$\le \|g(x) - g_n(x)\| + H(G(x,g_{n-1}(x)), G(x,g(x))) \le R_n(x) +$$
$$+ \varphi(x, \|g_{n-1}(x) - g(x)\|) \le R_n(x) + \varphi(x, R_{n-1}(x)),$$ for every $n > n(\varepsilon) + 1$ and every $x \in X$.

Therefore $g(x) \in G(x,g(x))$, $x \in X$.

It remains to show that g is continuous on X. We following the technique given in [5]. Fix an arbitrary $x_0 \in X$. For $\varepsilon_1 > 0$ let m be a positive integer such that $R_m(x_0) < \frac{1}{4}\varepsilon_1$.

Choose an open neighborhood U_m of x_0 such that $\|g_m(x) - g_m(x_0)\| < \frac{1}{4}\varepsilon_1$, $|R_m(x) - R_m(x_0) < \frac{1}{4}\varepsilon_1$, whe never $x \in U_m$. Since $\|g(x) - g(x_0)\| \le \|g(x) - g_m(x)\| + \|g_m(x) - g_m(x_0)\| + \|g_m(x_0) - g(x_0)\| \le R_m(x) + \|g_m(x) - g_m(x_0)\| + R_m(x)$ for each

$x \in U_m$, we have:

$$\|g(x) - g(x_0)\| \leq R_m(x_0) + 2\frac{1}{4}\varepsilon_1 + R_m(x_0) < \varepsilon_1.$$

Therefore g is continuous on X. Q.E.D.

As a consequence, we obtain the following fixed point theorem of Krasnoselskii type for ε-locally contractive multivalued mappings.

THEOREM 2. *Let $T:Y \to X$ be a completely continuous operator, $G:X \times Y \to P_{cl,cv}(Y)$ be as in Theorem 1.*

Then there is an element $y_0 \in Y$ such that: $y_0 \in G(T(y_0),y_0)$.

Proof. By Theorem 1 (with $\overline{T(Y)}$ instead of X) there exists a continuous function $g:\overline{T(Y)} \to Y$ such that $g(x) \in G(x,g(x))$, for $x \in \overline{T(Y)}$.

In virtue of the Schauder fixed point principle there is an $y_0 \in Y$ with $y_0 = g(T(y_0))$.

Therefore $y_0 = g(T(y_0)) \in G(T(y_0),g(T(y_0))) = G(T(y_0),y_0)$ Q.E.D.

Remark 1. One can get another version of Theorem 2 with the assumption that for every function $g:\overline{T(Y)} \to Y$ satisfying $g(x) \in G(x,g(x))$, for $x \in \overline{T(Y)}$ the superposition $g \circ T : Y \to Y$ has a fixed point.

Remark 2. An application of Theorem 2 to functional-differentional inclusions can be given in the same way as in [3].

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ON THE PROCEDURE OF THE CONSTRUCTION OF SUFFICIENT STATISTICS WITH APPLICATION ON THE DARMOIS-KOOPMAN FAMILY DISTRIBUTION*

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REZUMAT. - Asupra procedeului de construire de statistici suficiente cu aplicaţie la familia de distribuţii Darmois-Koopman. Obiectivul acestei lucrări este de a găsi legea de probabilitate a variabilei $Y = mkB^2$ unde B este coeficientul generalizat de concordanţă al lui Kendall şi asemenea de a calcula statisticele suficiente pentru estimatorul y^*, adică $y^* = mkb^{*2}$.

Résumé. Les applications de test non paramétriques sont de plus en plus répandues. Nous cherchons ici a améliorer la précision de ces tests basés sur le coeficient de concordance de Kendall. Par suite de travaux antérieurs du second auteur, l'objectif de notre travail est de trouver la loi de probabilité de la variable $Y = mkB^2$ où B est le coeficient généralisé de concordance de Kendall et aussi de calculer les statistiques suffisantes pour l'estimateur y^*, c.à.d. $y^* = mkb^{*2}$. En encadrant les distributions trouvées dans la famille de Darmois-Koopman, nous avons envisagé des procédures numériques spécifiques à cette famille. Dans le cas de volume de l'échantillon donné n_0, nous donnons la procédure pour construire les statistiques suffisantes pour les paramètres y^*, en se servant de la méthode de Monte-Carlo dans le calcul numériques des intégrales.

I. The s-dimension parameter Darmois-Koopman family has the

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density function of probability given (see [2]) by:

$$f(u/\beta) = A(\beta)D(u)\exp\left[\sum_{j=1}^{s} H_j(\beta)G_j(u)\right];$$

where A and $H_1, H_2, \ldots, H_s$ are the arbitrary functions of the s-parameter β, and D, G_1, G_2, $\ldots, G_s$ are the arbitrary functions of the random variable u. If we have a sample U_1, U_2, $\ldots, U_n$ of the random variable u, we know that it is possible to construct a s-sufficient statistics given by the formula:
$T_i = \sum_{j=1}^{n} G_i(U_j)$; $i = 1, 2, \ldots, s$ for the s-parameter β. In our work (see [6]) we have $y^* = mkb^{*2}$ as a estimator of the $Y = mkB^2$ and the following density bivariate of probability function:

$$d(u/y^*) = \frac{(y^*)^{-1/2} \cdot e^{-y^*/(2mk)}}{2^{(k-1)/2}\,\Gamma\left[\frac{k-1}{2}\right](2\pi)^{\frac{1}{2}}} \cdot g(u/y^*); \tag{2}$$

where m and k are the finite integral numbers and the $g(u/y^*)$ is the function of the random variable u and of the appraiser (estimateur) y^* is given by:

$$g(u/y^*) = \frac{e^{-\frac{u}{2}-\frac{1}{2}\left[\frac{u^2}{4m^2(k-1)^2} - \frac{u(y^*)^{\frac{1}{2}}}{m^{\frac{3}{2}}k^{\frac{1}{2}}(k-1)}\right]} \cdot u^{\frac{k-1}{2}-1}}{\left\{\Phi\left[\frac{\frac{3}{2}-\frac{u}{2m(k-1)}}{\sigma_{b_*}}\right] - \Phi\left[\frac{\frac{-u}{2m(k-1)}}{\sigma_{b_*}}\right]\right\}\sigma_{b_*}} \tag{3}$$

where ϕ is the function of Laplace and $\sigma_{b_*} = [2m^2(k-1)+1] / 2m^2(k-1)$ is the variance of b_*.

In [6] we have $b_* = z + \frac{m(k-1)w}{2m(k-1)}$, where z is the normal random variable with the variance = 1 and $m(k-1)w$ is independent from z and is chi-square distributed with k-1 degrees of freedom, therefore with the variance $2(k-1)$, see [2]. Consequently the variance of $b*$ is $\sigma^2_{b_*} = [2m^2(k-1)+1]/2m^2(k-1)$.

The formulas (2) and (3) are result of formula (2.19) from [6] which is the following:

$$h(b_*,u) = \frac{\exp\left\{-\frac{1}{2}\left[b_* - \frac{u}{2m(k-1)}\right]^2\right\}\cdot\sigma_{b_*}}{(2\pi)^{\frac{1}{2}}\cdot\sigma_{b_*}}\cdot\frac{u^{\frac{k-1}{2}-1}\cdot e^{-\frac{u}{2}}}{2^{\frac{k-1}{2}}\Gamma\left(\frac{k-1}{2}\right)} \qquad (2.19)$$

The second factor is Chi-square distribution.

The first factor (when u is constant) is normal distribution (without the factor σ_{b_*}). We have normalised the first factor with respect to $b*$, i.e. we divided by integral with respect $b*$ of this factor (when u is constant), which is just the Laplace Function ϕ.

All this is possible because we know the interval of variation for $b*$ i.e. $(0,3/2)$ (see proof after Theorem which is below).

Next to all this, we used also the transformation $y^* = mkb*^2$. The factor $1/2(mk)^{1/2}$ which becomes visible after being multiplied by $b* = (y^*)^{-1/2}/2(mk)^{1/2}$ have been ignored. This factor is independent from the variable u and y^*.

In this case the formula (2) obtained from (2.19) represents the probability density function for random variable u and y^*. This result is possible because y^* is an estimator i.e. is a random variable. But for each value of y^* the distribution (2) is conditional probability density for the random variable u. In this second interpretation we have connection the Darmois-Koopman family distribution when it has a single parameter (see explanation below by J.Johnston).

II. The object of our work is to prove the connections between the Darmois-Koopman family $f(u/\beta)$ and our family distribution $d(u/y^*)$; and also the procedure for the construction of a sufficient statistics for the parameter y^*.

In order to identify the best procedure which helps our object, we make the following remarks:

a) The random variable $u = m(k-1)w$ (see [4] and [6]) where w is the value of the Kendall's coefficient of concordance: $w = S/S_{max} \leq 1$.

According to [4] we have also:

$$w = [(m-1)\cdot\mu_s+1]/m; \tag{4}$$

where μ_s the average of the Spearman coefficients r_s, and then we have

$$w \leq 1 \tag{5}$$

because $r_s \leq 1$.

From (5) we infer also that there is $u = m(k-1)w \leq m(k-1)$ which is the least upper bound of the variable w.

b) We have a single parameter y^* which is the estimator of $Y = mkB^2$; (see [6]) because the $b*$ is an estimator of the general Kendall coefficient of concordance B. We consider here $b*$ as a parameter in the same way as is considered the variance s^2 of the sample in Student distribution t (see J.Johnston "Econometric Methods" French Edition-Economica 1985 page 41). When we replace σ^2 by his estimator s^2 brings us to replace the normal distribution by a Student distribution (after J.Johnston). The variance s^2 has its oun distribution function and s^2 is also the parameter in the Student distribution. In our case all formulas from [6] are asymtotically satisfied when $k \to \infty$; the estimator $b_* \to B$ (real); so that in our case the distribution given by (2) is a likely approximation of the distribution of a function in random variable u.

We have $s = 1$, a single dimension by random variable u, and then it is necessarry to construct a single sufficient statistics, which have the following form:

$$T_1 = \sum_{j=1}^{n} G_1(U_j) \tag{6}$$

when we have the sample $U_1, U_2, \ldots, U_n$.

III. The method. Our density function (2) is the same that the density function (1) when we consider: $\beta = y^*$,

$$A(\beta) = A(y^*) = \frac{(y^*)^{-\frac{1}{2}} e^{-\frac{y^*}{2mk}}}{2^{\frac{k-1}{2}} \cdot \Gamma\left(\frac{k-1}{2}\right)} \tag{7}$$

$$D(u) = \frac{1}{\left[\Phi\left(\dfrac{\frac{3}{2} - \frac{u}{2m(k-1)}}{\sigma_{b_*}}\right) - \Phi\left(\dfrac{-\frac{u}{2m(k-1)}}{\sigma_{b_*}}\right)\right] \cdot \sigma_{b_*}} \tag{8}$$

and also if it is possible to identify the functions $H_1(y^*)$ and $G_1(u)$ such that we have:

$$\exp\left[H_1(y^*) \cdot G_1(u)\right] = T_1(u/y^*) = e^{-\frac{u}{2} - \frac{1}{2}\left[\frac{u^2}{4m^2(k-1)^2} - \frac{u(y^*)^{\frac{1}{2}}}{m^{\frac{3}{2}} k^{\frac{1}{2}} (k-1)}\right]} \cdot u^{\frac{k-1}{2} - 1} \tag{9}$$

THEOREM. *If* $H_1(y^*) = 1$ *in* (9) *then there is a function* $G_1(u)$ *such that the distribution* (2) *is a particular case of the distribution* (1) *i.e. the distribution* (2) *belongs of the Darmois-Koopman family.*

Proof. From [6] we have: $b_* = \frac{\sum_{j=1}^{k} \frac{1}{1+p_j}}{k} + \frac{w}{2}$; where $p_j > 0$ for $j = 1,2,3,\ldots,k$ are random integers, then the inequality $\sum_{j=1}^{k} \frac{1}{1+p_j} < k$ gives us $0 < b_* < 1 + 1/2 = 3/2$ because $w < 1$. For $y^* = m(k-1)b*^2$ we have $0 < y^* < 9/4m(k-1)$. Now, for $H(y^*) = 1$ and by integration in (9) with respect to y^* we have the following:

$$\int_0^{\frac{9}{4m(k-1)}} e^{G_1(u)} dy^* =$$

$$= e^{-\frac{u}{2} - \frac{1}{2}\left[\frac{u^2}{4m^2(k-1)^2}\right]} U^{\frac{k-1}{2}-1} \int_0^{\frac{9}{4m(k-1)}} e^{\frac{u}{2}\frac{(y^*)^{\frac{1}{2}}}{m^{\frac{3}{2}}k^{\frac{1}{2}}(k-1)}} dy^*.$$

The calculation of the second integral can be done by the substitution $v = (y^*)^{1/2}$ and by parts. After other calculation, we have the following:

$$G_1(U) = L_n\, e^{-\frac{u}{2} - \frac{1}{2}\left[\frac{u^2}{4m^2(k-1)^2}\right]} \cdot U^{\frac{k-1}{2}-3} \cdot \left[\frac{8(3u\sqrt{m(k-1)}-4}{9m^{\frac{5}{2}}k^{\frac{1}{2}}(k-1)^2}\right] \qquad (10)$$

The function under the logarithm in (10) is positive for $k>1$ and this involve there is the function $G_1(u)$. If we consider (7), (8), (9) and (10) it is clear that the distributions (2) belong to the Darmois-Koopman family, q.e.d. The sufficient statistics will be calculated by the formula (6).

Remark: The hypothesis $k > 1$, (under the function in (10) is finite) is met in most of applications in non parametric statistics.

In many applications (for example in order to calculate the moments of $d(u/y^*)$), one must calculate the integral from the function $F(u)$ as under Ln in (10) in the interval $[0,m(k-1)]$. In this case it is possible to apply a quadrature formula whose knots are the values of the order statistics $u_1, u_2, \ldots, u_n$, obtained from the sample $U_1, U_2, \ldots, U_n$. We consider this procedure because it is the same which makes possible to construct the sufficient statistics for the parameters of Darmois-Koopman family (see [2]). We have the formula

$$\int_0^{m(k-1)} F(u)\,du = \sum_{l=1}^{n} C_l e^{-\frac{u_l}{2}-\frac{1}{2}\left[\frac{u_l^2}{4m^2(k-1)^2}\right]} \cdot \left[3u_l\sqrt{m(k-1)}-4\right]\cdot u^{\frac{k-1}{2}-3} + R \qquad (11)$$

where R is the remainder of the quadrature formula and C_l for $l = 1,2,\ldots,n$ are the coefficients of this formula.

The delimitation of R is possible because the function to integrate without the constant $8/9m^{5/2}k^{1/2}(k-1)^2$ which is given by:

$$F(u) = e^{-\frac{u}{2}-\frac{1}{2}\left[\frac{u^2}{4m^2(k-1)^2)}\right]}\cdot U^{\frac{k-1}{2}-3}\,[3\,U\sqrt{m(k-1)}-4] \qquad (12)$$

is simple, continuous differentiable and bounded in random variable u.

In conclusion it is possible to find suitable quadrature formula (optima) in spite of that $U_1, U_2, \ldots, U_n$ are the values of random variable u in the interval $[0,m(k-1)]$, where the integers m and k are finite.

When $n = n_0$ (fixed) the remainder R will be delimited by:

$$|R| \le C(n_0)\cdot |F(u)^{(p)}| \qquad (13)$$

where only $C(n_0) = 1/[m(k-1)]^{n_0}$; while the quantity $|F(u)^{(p)}|$ is bounded because $|F(u)|$ is bounded; p = order of the derivative of F, and $|R| \to 0$ when $n = n_0 \to \infty$.

When n is random it is possible to apply the sequential procedures because the statistics T_1 is sufficient by means of Frasser (see [1]).

Because the distribution of variable $u = m(k-1)w$ is specified (see [4] and [6]), namely the Chi-Square distribution with k-1 degrees of freedom, the calculation of (11) by the Monte-Carlo method it is immediatly clear (see [3]).

The interval of integral from (11), being $[0,m(k-1)]$, then immediately we have the following knots

$$u_l - w_{l\mu} \cdot m(k-1), \text{ for } l = 1,2,3,\ldots,n; \qquad (14)$$

where $w_{l\mu}$ are the random numbers from the Chi-Square distribution with k-1 degrees of freedom in the range 0-1. By means of the relations (7), (8), (9), (10), (11), (12) all the criteria for the construction of the sufficient statistics for family (2) such as the Darmois-Koopman family (1) are satisfied (see [4]), including the Sobel-Wald test with three hypothesis:

$$H_0{:}\phi=\phi_0; \qquad H_1{:}\phi=\phi_1; \qquad H_2{:}\phi=\phi_2.$$

IV. The conclusions. The construction of the general confidence intervals for the parameter distribution of type (2) is very important in applicable statistics, including the non parametric statistics. These sayings are incontestable if we analyze the means and the applications where we meet the indicators as in the families (1) and (2).

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