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SUMAR - CONTENTS - SOMMAIRE

A n i v e r s ă r i - A n n i v e r s a r i e s - A n n i v e r s a i r e s

STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXXVI, 2, 1991

SOME IMPROVED INEQUALITIES

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Dedicated to Professor P. T. Mocanu on his 60th anniversary

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REZUMAT. - **Cîteva inegalităţi întărite.** Cîteva inegalităţi cunoscute sînt întărite folosind o metodă din lucrarea [9].

1. Let $0<a<b$ and let n be an integer, $n\ge 2$. Let $x=(x_1,\dots,x_n)\in[a,b]^n$. We shall use the following notation:

$A_n=(x_1+\dots+x_n)/n$, $G_n=(x_1\dots x_n)^{1/n}$, $S_n(x)=\sum_{i<j}(x_j-x_i)^2$

$\log(x)=(\log(x_1),\dots,\log(x_n))$, $\sqrt{x}=(\sqrt{x_1},\dots,\sqrt{x_n})$

Then:

$$\frac{1}{2bn^2}S_n(x)\le A_n-G_n\le\frac{1}{2an^2}S_n(x) \tag{1}$$

For the long history of (1) see [3]-[5], [8]-[10], [12]. Let us remark that the counterexample to (1), given in [12], is inconclusive.

We have also (see[6]):

$$\frac{1}{n(n-1)}S_n(\sqrt{x})\le A_n-G_n\le\frac{1}{n}S_n(\sqrt{x}) \tag{2}$$

2. Let $f\in C^2[a,b]$; let $2m$ and $2M$ be the minimum, respectively the maximum of f'' on $[a,b]$. Then $f(t)-mt^2$ and $Mt^2-f(t)$ are convex functions on $[a,b]$.

This elementary remark, combined with an appropriate choice

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of f, leads to results comparable with (1) and (2). Namely (see [9]):

$$\exp\frac{S_n}{2n^2b^2}\le\frac{A_n}{G_n}\le\exp\frac{S_n}{2n^2a^2} \tag{3}$$

$$\frac{a}{2n^2}S_n(\log(x))\le A_n-G_n\le\frac{b}{2n^2}S_n(\log(x)) \tag{4}$$

$$\frac{2a}{bn^2}S_n(\sqrt{x})\le A_n-G_n\le\frac{2b}{an^2}S_n(\sqrt{x}) \tag{5}$$

Other results obtained by the same method are to be found in [1] and [2].

3. We shall apply the above method in order to improve some results of A.O.Pittenger [8].

Let ϕ be a real-valued function defined on an interval I, possibly unbounded. Let $t_0\in I$. Y will denote a random variable whose range is almost surely in I.

THEOREM ([8]). *Suppose Y has finite mean μ and variance σ^2, and $\phi(Y)$ has finite expectation. If $\mu=t_0$, set $a_0=t_0$ and $p_0=1$; otherwise set $a_0=\mu+\sigma^2/(\mu-t_0)$ and $p_0=(\mu-t_0)^2/(\sigma^2+(\mu-t_0)^2)$. Then if $Y\le t_0$ a.s., and if the function $(\phi(t)-\phi(t_0))/(t-t_0)$ is concave on $(-\infty,t_0)\cap I$, we have*

$$p_0\phi(a_0)\ +\ (1-p_0)\phi(t_0)\le E(\phi(Y)) \tag{6}$$

Equality is attained for the random variable Y_0 which equals a_0 with probability p_0 and t_0 with probability $1-p_0$. If in addition ϕ is convex, the left side of (6) dominates $\phi(\mu)$. All the foregoing hold for $Y\ge t_0$, provided that the function $(\phi(t)-\phi(t_0))/$ $/(t-t_0)$ is convex on $(t_0,\infty)\cap I$.

Remark 1. If I is bounded, the inequality (6) is equivalent to the inequalities given (with different proofs) for $n=2$ in [11,p.279].

Remark 2. It is easy to verify that if ϕ is 3-convex (in particular, if $\phi^{(3)}\geq 0$), then $(\phi(t)-\phi(t_0))/(t-t_0)$ is convex on $(t_0,\infty)\cap I$.

Using Remark 2 it is easy to check the convexity of $(\phi(t)-\phi(t_0))/(t-t_0)$ in all examples considered in [8]: it suffices to verify that $\phi^{(3)}\geq 0$. Moreover, the inequalities given in those examples can be improved.

For example, let Y be a random variable, $0\leq Y\leq 1$, with mean μ and variance σ^2. Using the above theorem for $t_0 = 0$ and $\phi(t) = t\log(t)$, $\phi(0) = 0$ (note that $\phi^{(3)} \leq 0$), Pittenger obtains in [8]

$$E(Y\log(Y))\leq\mu\log(\mu+\sigma^2/\mu) \tag{7}$$

Using the Jensen inequality for ϕ (note that $\phi^{(2)}\geq 0$) he obtains $\mu\log(\mu) \leq E(Y\log(Y))$ and, finally, the following elegant result

$$0 \leq E(Y\log(Y)) - \mu\log(\mu) \leq \mu\log(1+\sigma^2/\mu^2) \tag{8}$$

Now, in the spirit of Section 2, let us consider the functions $\phi_1(t) = \phi(t) + t^3/6$ and $\phi_2(t) = \phi(t) - t^2/2$. Then $\phi_1^{(3)}\leq 0$ and $\phi_2^{(2)}\geq 0$ for $0<t\leq 1$. By using Pittenger's technique with ϕ_1 and ϕ_2 instead of ϕ, we obtain

$$\sigma^2/2 \leq E(Y\log(Y)) - \mu\log(\mu) \leq \mu\log(1+\sigma^2/\mu^2) - \delta/6 \tag{9}$$

where $\delta = E(Y^3) - \mu^3(1+\sigma^2/\mu^2)^2$ is positive by virtue of the last inequality in [8].

A similar treatment can be applied to the other examples discussed in [8].

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STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXXVI, 2, 1991

FUNCTION WITH NEGATIVE COEFFICIENTS n-STARLIKE OF COMPLEX ORDER

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Dedicated to Professor P.T.Mocanu on his 60[th] anniversary

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RESUMAT. - **Funcţii cu coeficienţi negativi *n*-stelate de un ordin complex.** În lucrare se pun în evidenţă unele relaţii între clasa $T_{n,1}$ a funcţiilor cu coeficienţi negativi n-stelate şi clase $T_{n,b}$ de funcţii cu coeficienţi negativi n-stelate de un ordin complex b.

1. Introduction. Let A denote the class of functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ which is analytic in $U = \{z \in C ; |z| < 1\}$. We denote by N the set of nonnegative integers ($N = \{0,1,2,\ldots\}$).

DEFINITION 1 ([3]). We define the operator $D^n : A \to A$, $n \in N$, by : a) $D^0 f(z) = f(z)$; b) $D^1 f(z) = Df(z) = zf'(z)$; c) $D^n f(z) = D(D^{n-1} f(z))$, $z \in U$.

DEFINITION 2 ([3]). A function $f \in A$ is said to be *n-starlike* if $\mathrm{Re}[D^{n+1} f(z) / D^n f(z)] > 0$, $z \in U$, $n \in N$. We denote by S_n the class of n-starlike functions.

We remark that $S_0 = S^*$ is the class of starlike functions and $S_1 = S^c$ is the class of convex functions. In [3] it is proved that all n-starlike functions ($n \in N$) are univalent and $S_n \supset S_{n+1}$.

DEFINITION 3. We say that $f \in A$ is *n-starlike of complex order b* (b is a complex number and $b \neq 0$, $n \in N$) if $D^n f(z) / z \neq 0$,

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($z \in U$) and

$$Re\left[1 + \frac{1}{b}\left(\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right)\right] > 0, z \in U.$$

We denote by $S_{n,b}$ the class of n-starlike functions of complex order b.

M.A.Nasr and M.K.Aouf introduced and studied the class $S_{0,b}$ of starlike functions of complex order b ([1]). We also note that $S_{n,1} = S_n$.

DEFINITION 4. Let $n \in N$ and let b be complex and $b \neq 0$; we define the class $T_{n,b}$ by

$$T_{n,b} = \{f \in S_{n,b};\ f(z) = z - \sum_{k=2}^{\infty} a_k z^k,\ a_k \geq 0,\ k=2,3,\ldots\}.$$

A function $f \in T_{n,b}$ is said to be a function with negative coefficients n-starlike of complex order b.

The classes $T_{0,1-\alpha}$ and $T_{1,1-\alpha}$, $\alpha \in [0,1)$ were introduced and studied by H.Silverman [4] and the classes $T_{n,1-\alpha}$, $\alpha \in [0,1)$, $n \in N$, were defined in [2].

In this paper we give some relationships between the classes $T_{n,b}$ (b complex) and $T_{n,1}$.

We will use the following lemma

LEMMA A. *Let* $n \in N$ *and let* $\alpha \in [0,1)$; *a function* $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ *is in* $T_{n,1-\alpha}$ *if and only if*

$$\sum_{k=2}^{\infty} k^n (k - \alpha) a_k \leq 1 - \alpha.$$

The proof of a more general form of this lemma may be found in

[2].

2. **Main result.** We denote by B the set $\{z \in C, |z - 1/2| \leq 1/2$ and $z \neq 0\} = \{z \in C; Re \frac{1}{z} \geq 1\}$.

THEOREM 1. *Let $n \in N$ and let b be in B; then $T_{n,b} \subset T_{n,1}$.*

Proof. Let f be in $T_{n,b}$ and $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ ($a_k \geq 0$, $k=2,3,\ldots$). Then, by Definition 3, we have

$$Re\left[1 + \frac{1}{b}\left(\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right)\right] > 0$$

or

$$1 + Re\left[\frac{1}{b}\left(\frac{z - \sum_{k=2}^{\infty} k^{n+1} a_k z^k}{z - \sum_{k=2}^{\infty} k^n a_k z^k} - 1\right)\right] > 0.$$

This last inequality is equivalent to

$$1 + Re\left[\frac{1}{b}\frac{-\sum_{k=2}^{\infty} k^n (k-1) a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} k^n a_k z^{k-1}}\right] > 0.$$

By letting $z \to 1^-$, z real, we obtain

$$1 + Re\,\frac{1}{b}\cdot\frac{-\sum_{k=2}^{\infty} k^n (k-1) a_k}{1 - \sum_{k=2}^{\infty} k^n a_k} \geq 0$$

and this inequality can be rewritten as

$$Re\,\frac{1}{b}\cdot\frac{\sum_{k=2}^{\infty}k^n(k-1)a_k}{1-\sum_{k=2}^{\infty}k^n a_k}\le 1. \tag{1}$$

By using the condition $b \in B$ which is equivalent to $Re(1/b) \ge 1$, from (1) we deduce

$$\frac{\sum_{k=2}^{\infty}k^n(k-1)a_k}{1-\sum_{k=2}^{\infty}k^n a_k}\le 1. \tag{2}$$

But we have $1-\sum_{k=2}^{\infty}k^n a_k > 0$ because $D^n f(z)\,/\,z = 1-\sum_{k=2}^{\infty}k^n a_k z^{k-1} \ne 0$ (Definition 3) and $\lim_{z\to 0}[D^n f(z)\,/\,z] = 1$; from (2) we obtain $\sum_{k=2}^{\infty}k^{n+1}a_k \le 1$ which, by Lemma A, implies $f\in T_{n,1}$.

COROLLARY 1. *If f is a function with negative coefficients starlike of complex order b and $b \in B$, then f is starlike ($f\in S^*$).*

COROLLARY 2. *If f is a function with negative coefficients convex of complex order b and $b\in B$, then f is a convex function ($f\in S^c$).*

3. **Remarks.** 1). If $b \in B$ and $b \ne 1$, then we can find functions f in $T_{n,1}$ such that f are not in $T_{n,b}$ (i.e. $T_{n,1}\not\subset T_{n,b}$). Indeed, let $f(z) = z - z^2/2^n$; then f is in $T_{n,1}$, but for $b \in B$, $1/b = p + iq$, we have

$$1+\frac{1}{b}\left(\frac{D^{n+1}f(z)}{D^n f(z)}-1\right)=\frac{(p+1+iq)z-2}{z-2}.$$

and

$$1 + \frac{1}{b}\left(\frac{D^{n+1}f(z_0)}{D^n f(z_0)} - 1\right) = 0$$

for $z_0 = 2/(p+1+iq) \in U$, because $p>1$, and this implies $f \notin T_{n,b}$.

2). Let b be a complex number with $|b| = r > 1$ and for $n \in N$ we consider the functions

$$f_n(z) = z - \frac{r}{2^n(r+1)} z^n;$$

then $f_n \in T_{n,b}$, but $f_n \notin T_{n,1}$ (f_n is not a n-starlike function).

Proof. By using Lemma A we have

$$\sum_{k=2}^{\infty} k^{n+1} a_k = 2^{n+1} a_2 = \frac{2r}{r+1} > 1$$

and this implies $f \notin T_{n,1}$.

Let denote by $U(c;\rho)$ the disc $\{z \in C; |z - c| < \rho\}$. We prove that

$$1 + \frac{1}{b}\left(\frac{D^{n+1}f_n(z)}{D^n f_n(z)} - 1\right) \in U(1;1), \quad z \in U = U(0;1). \tag{3}$$

But (3) is equivalent to $(D^{n+1}f_n(z)/D^n f_n(z) - 1)/b \in U$ and we also have

$$\frac{1}{r}\left(\frac{D^{n+1}f_n(z)}{D^n f_n(z)} - 1\right) = \frac{1}{r}\left(\frac{z - 2rz^2/(r+1)}{z - rz/(r+1)} - 1\right) = \frac{-z}{r+1-rz} \in$$

$$\in U\left(-\frac{r}{2r+1}; \frac{r+1}{2r+1}\right) \subset U(0;1) = U.$$

If we denote by θ the argument of b ($b = r\, e^{i\theta}$), then we also have

$$\frac{1}{b}\left(\frac{D^{n+1}f_n(z)}{D^n f_n(z)} - 1\right) = -\frac{z}{r + 1 - rz}e^{-i\theta} \in U$$

and we obtain that (3) holds.

From (3) we have $f \in T_{n,b}$.

3). For b real the last result (Remark 2) can be extended. So, by a simple calculation, we also can obtain the next result: if $-1/2 < b < -1/3$ and $1/2 < \beta < -b/(1+b)$ or if $b \leq -1/2$ and $1/2 < \beta < 1$, then $f_\beta(z) = z - \beta z^2 \in T_{0,b}$ and $f_\beta \notin T_{0,1}$.

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STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXXVI, 2, 1991

THE RADIUS OF STARLIKENESS FOR THE ERROR FUNCTION

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Rezumat. - Raza de stelaritate a funcţiei eroare. În lucrare se determină razele de stelaritate pentru funcţiile f_n definite prin relaţia de mai jos. Acestea se exprimă cu ajutorul rădăcinii ecuaţiei (3) din intervalul $(\pi/2,\pi)$.

The purpose of this note is to find the radii of starlikeness for the functions

$$f_n(z)=\int_0^z \exp(-t^n)\,dt\ ,\ z\in\mathbf{C}\ ,\ n\in\mathbf{N}^n\ .$$

Particularly, for $n=1$, the result obtained by P.T.Mocanu in [2] which gives the radius of starlikeness for the exponential function will be refind and, for $n=2$, the radius of starlikeness for the error function $\mathrm{Erf}(z)=f_2(z)$ will be obtained.

Let f be an analytic function around the origin, with $f(0)=0$ and $f'(0)\neq 0$. The radius of starlikeness for f is defined as being the radius of the largest disk centered at 0 in which f is starlike. According to [1], this radius equals $\min|z|$ where z is a root of following system:

$$\mathrm{Re}\ [z\ f'(z)/f(z)]=0$$

$$\mathrm{Re}\ [z\ f''(z)/f'(z)]+1=0$$

For the function f_n this system becomes

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$$\mathrm{Re}\left(\int_0^1 \exp[z^n(1-u^n)]\,du\right)^{-1}=0 \tag{1}$$

$$\mathrm{Re}\ z^n=1/n. \tag{2}$$

Denoting by $r=r^*(f_n)$ the radius of starlikeness of f_n relation (2) gives $(\mathrm{Im}\ z^n)^2=r^{2n}-1/n^2$, so, it follows by (1) that r is the smallest positive root of the equation

$$\int_0^1 \exp[(1-u^n)/n]\cos[(1-u^n)(r^{2n}-1/n^2)^{1/2}]\,du=0$$

Consider the equation

$$F_n(x)=\int_0^1 \exp(-u^n/n)\cos[x(1-u^n)]\,du=0 \tag{3}$$

Then $r^{2n}=(x_n)^2+1/n^2$, where x_n is the smallest positive root of the equation (3).

Let now $n=1$. Then repeated integration by parts in (3) gives the following equation for x_1:

$$x\sin(x)+\cos(x) = 1/e\ ,$$

so,as in [2], we obtain $r^*(f_1)=2.83\ldots$

For $n>1$ we have

$$F_n'(x)=-\int_0^1 (1-u^n)\exp(-u^n/n)\sin[x(1-u^n)]\,du=0$$

so F_n is a decreasing function on $[0,\pi]$. It is obvious that $F_n(\pi/2)>0$. We shall show now that $F_n(\pi)<0$.

Let

$$g_n(u)=\exp(-u^n/n)\ \cos[\pi(1-u^n)].$$

Using the sign of g_n and the inequality $\exp(-u^n/n) \le$ $\le \exp[-1/(2n)]$ valid for $u^n \ge 1/2$ we obtain

$$F_n(\pi) < \int_0^{\frac{1}{2}} g_n(u)\,du + \exp[-1/(2n)]\,(1-2^{-\frac{1}{n}}) .$$

But for $u\in[0,1/2]$ the following inequalities hold

$$\exp(-u^n/n) \geq \exp(-u),$$

$$\cos[\pi(1-u^n)] \leq \cos[\pi(1-u)] \leq 0,$$

so

$$g_n(u) \leq \exp(-u)\ \cos[\pi(1-u^n)] \leq g_1(u).$$

Integrating by parts it is easy to obtain the next relation

$$\int_0^{\frac{1}{2}} g_1(u)\,du = \frac{-1+\pi\exp(-1/2)}{1+\pi^2} < -\frac{1}{4} .$$

Finally we get the following inequality

$$F_n(\pi) < -1/4 + \exp[-1/(2n)]\ (1-2^{-1/n}).$$

If $n\geq 3$ using $\exp[-1/(2n)]<1$ it follows that $4\ F_n(\pi) <$ $< 3-4\times 2^{-1/n} < 0$. If $n=2$ we have $\exp(-1/4)\times(1-2^{-1/2}) < 15/64$ so $F_n(\pi) < 0$ for every $n\geq 2$.

We can conclude now that x_n is the unique root of the equation $F_n(x)=0$ situated in $(\pi/2,\pi)$ and

$$r^*(f_n) = [(x_n)^2 + 1/n^2]^{1/(2n)}.$$

By computation the following value is obtained for the radius of starlikeness of the error function:

$$r^*(f_2) = r^*(\mathrm{Erf}) = 1.504...$$

Solving again equation (3) for $n \in \{3,4,5\}$ we obtain

$$r^*(f_3) = 1.268...$$

$$r^*(f_4) = 1.178...$$

$$r^*(f_5) = 1.131...$$

Remark. Since the numbers $r^*(f_n)$ are greater than 1, every function f_n is starlike in the unit disk.

R E F E R E N C E S

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ON A SUBORDINATION BY CONVEX FUNCTIONS

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REZUMAT. - Asupra unei subordonări prin funcţii convexe. În lucrare sînt determinate condiţii pentru ca $g \prec f$, unde f este o funcţie analitică convexă, iar g este dată de (2).

1. **Introduction.** Let A be class of all analytic functions f in the unit disc U normalized by $f(0)=0$, $f'(0)=1$. A function $f \in A$ is said to be convex in U if

$$\operatorname{Re}\left[\frac{zf''(z)}{f'(z)}+1\right]>0 \quad , \ z \in U \tag{1}$$

Let $f \in A$ be convex in U and let

$$g(z)=\frac{\varphi(z)}{z^{\gamma}}\int_0^z f(\zeta)\zeta^{\gamma-1}d\zeta \tag{2}$$

where $\gamma>-1$ and $\varphi(z)$ is analytic in with $\varphi(z)\neq 0$.

In this paper we determine conditions on $\varphi(z)$ so that $g \prec f$. For $\varphi(z)\equiv\lambda$ real or complex number, this problem was solved in [7] for λ real and $\gamma=0$, $\gamma=1$, in [6] for λ real and all $\gamma>-1$ and in [4] for λ complex.

2. **Preliminaries.** We will need the following lemmas to prove our results.

LEMMA 1. ([2]) *Let p be analytic in U, let q be analytic and univalent in $\bar{U}$, with $p(0)=q(0)$. If p is not subordinate to q, then there exists points $z_0 \in U$ and $\zeta_0 \in \partial U$ and an $m \geq 1$ for wich*

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$p(|z|<|z_0|) \subset q(U)$,

(i) $p(z_0) = q(\zeta_0)$ and

(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$

LEMMA 2. ([1] and [8]) *If $f\epsilon A$ satisfies (1), then*

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \frac{1}{2} \quad , \ z\in U \tag{3}$$

LEMMA 3. ([3]) *If $f\epsilon A$ satisfies (3), then the function*

$$g_1(z) = \frac{1}{z^{\gamma}}\int_0^z f(\zeta)\zeta^{\gamma-1}d\zeta \tag{4}$$

satisfies

$$\operatorname{Re}\frac{zg_1'(z)}{g_1(z)} > \delta(\gamma) \quad , \ z\in U \tag{5}$$

where

$$\delta(\gamma) = \frac{\gamma+1}{2F(1,\gamma+1,\gamma+2,-1)} - \gamma \tag{6}$$

and $F(\alpha,\beta,\gamma,z)$ is the hypergeometric function:

$$F(\alpha,\beta,\gamma,z) = \sum_{n=0}^{\infty}\frac{\alpha(\alpha+1)\dots(\alpha+n-1)\beta(\beta+1)\dots(\beta+n-1)}{n!\gamma(\gamma+1)\dots(\gamma+n-1)}z^n$$

3. **Main results.**

THEOREM. *Let $f\epsilon A$ be convex and let g be defined by (2). If $\varphi(z)$ is analytic in U with $\varphi(z) \neq 0$ and satisfies:*

$$\operatorname{Re}\frac{z\varphi'(z)}{\varphi(z)} \geq 0 \quad , \ z\in U \tag{7}$$

$$(\gamma+2\delta(\gamma))\,\mathrm{Re}\frac{1}{\varphi(z)}-\mathrm{Re}\frac{z\varphi'(z)}{(\varphi(z))^2}-\left|\frac{\gamma}{\varphi(z)}-\frac{z\varphi'(z)}{(\varphi(z))^2}-1\right|-1\geq 0,\quad z\in U \tag{8}$$

where $\delta(\gamma)$ is given by (6), then $g(z)\prec f(z)$, $z\in U$.

Proof. Without loss of generality we can assume that f and φ satisfies the condition of the theorem of the closed disc $\bar{U}$. If not, then we can replace $f(z)$ by $f_r(z) = f(rz)$, $\varphi(z)$ by $\varphi(z)=\varphi(rz)$ and hence $g(z)$ by $g_r(z) = g(rz)$, where $0<r<1$. $f_r(z)$ is convex on $\bar{U}$. We would then prove $g_r(z)\prec f_r(z)$ for all $0<r<1$. By letting $r\to 1^-$ we obtain $g(z)\prec f(z)$, $z\in U$.

From (2) we deduce:

$$\left(\frac{\gamma}{\varphi(z)}-\frac{z\varphi'(z)}{(\varphi(z))^2}\right)\cdot g(z)+\frac{zg'(z)}{\varphi(z)}=f(z) \tag{9}$$

If g is not subordinate to f, then by Lemma 1. there exists points $z_0\in U$ and $\zeta_0\in\partial U$ and an $m\geq 1$ such that:

$$g(z_0) = f(\zeta_0) \text{ and } z_0g'(z_0) = m\zeta_0 f'(\zeta_0) \tag{10}$$

From (9) and (10) we obtain:

$$f(z_0)=\left(\frac{\gamma}{\varphi(z_0)}-\frac{z_0\varphi'(z_0)}{(\varphi(z_0))^2}\right)f(\zeta_0)+m\frac{\zeta_0 f'(\zeta_0)}{\varphi(z_0)}$$

hence

$$Q=\frac{f(z_0)-f(\zeta_0)}{\zeta_0 f'(\zeta_0)}=\left(\frac{\gamma}{\varphi(z_0)}-\frac{z_0\varphi'(z_0)}{(\varphi(z_0))^2}-1\right)\frac{f(\zeta_0)}{\zeta_0 f'(\zeta_0)}+\frac{m}{\varphi(z_0)}=$$
$$=m\left[\frac{1}{\varphi(z_0)}+\left(\frac{\gamma}{\varphi(z_0)}-\frac{z_0\varphi'(z_0)}{(\varphi(z_0))^2}-1\right)\frac{g(z_0)}{z_0g'(z_0)}\right].$$

Since $g(z) = \varphi(z)g_1(z)$, where $g_1(z)$ is defined by (4), if we note $w=\dfrac{g(z_0)}{z_0g'(z_0)}$ from (5) and (7) we deduce

$$\mathrm{Re}\frac{1}{w}>\delta(\gamma) \quad \text{or} \quad \left|w-\frac{1}{2\delta(\gamma)}\right|\le\frac{1}{2\delta(\gamma)} .$$

Using this result combined with (8) and $m\geq 1$, we obtain:

$$\mathrm{Re}\,Q=m\Big(\mathrm{Re}\frac{1}{\varphi(z_0)}+\frac{1}{2\delta(\gamma)}\mathrm{Re}\left(\frac{\gamma}{\varphi(z_0)}-\frac{z_0\varphi'(z_0)}{(\varphi(z_0))^2}-1\right)+$$
$$+\mathrm{Re}\left[\left(\frac{\gamma}{\varphi(z_0)}-\frac{z_0\varphi'(z_0)}{(\varphi(z_0))^2}-1\right)\left(w-\frac{1}{2\delta(\gamma)}\right)\right]\Big)\geq m\,\mathrm{Re}\,[\frac{1}{\varphi(z_0)}+$$
$$+\frac{1}{2\delta(\gamma)}\mathrm{Re}\left(\frac{\gamma}{\varphi(z_0)}-\frac{z_0\varphi'(z_0)}{(\varphi(z_0))^2}-1\right)-\frac{1}{2\delta(\gamma)}\left|\frac{\gamma}{\varphi(z_0)}-\frac{z_0\varphi'(z_0)}{(\varphi(z_0))^2}-1\right|]\geq 0 ,$$

which is equivalent to

$$\left|\arg\frac{f(z_0)-f(\zeta_0)}{\zeta_0 f'(\zeta_0)}\right|\le\frac{\pi}{2} \tag{11}$$

Since $\zeta_0 f'(\zeta_0)$ is the outward normal to the boundary of the convex domain $f(U)$ at $f(\zeta_0)$, (11) implies that $f(z_0)\notin f(U)$. This contradiction shows that $g\prec f$.

If we let $\varphi(z)\equiv\lambda$, complex number, in the Theorem, we obtain the result of [4]:

COROLLARY. *Let* $f\in A$ *be convex and let* $g(z)=\frac{\lambda}{z^\gamma}\int_0^z f(\zeta)\zeta^{\gamma-1}d\zeta$. *If* λ *is a complex number which satisfies:*

$$(\gamma+2\delta(\gamma))\mathrm{Re}\frac{1}{\lambda}-\left|\frac{\gamma}{\lambda}-1\right|-1\geq 0$$

where $\delta(\gamma)$ *is given by* (6), *then* $g\prec f$.

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SUBCLASS OF ANALYTIC FUNCTIONS

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REZUMAT. Subclase de funcţii analitice. Scopul acestei lucrări este de a obţine cîteva proprietăţi interesante ale unor subclase de funcţii analitice.

1. Introduction. Let A be the class of analytic functions f in the unit disc U, normalized by $f(0) = f'(0) - 1 = 0$.

Definition [2]. Let Ω be a set in C and let q be analytic and univalent on $\bar{U}$ except for those $\zeta\in\partial U$ for which $\lim_{z\to\zeta} q(z)=\infty$. We define $\psi(\Omega,q)$ to be the class of functions $\psi:C^3 \times U \to U$ for which $\psi(r,s,t;z)\notin\Omega$ when $r = q(\zeta)$ is finite, $s = m\zeta q'(\zeta)$, $\mathrm{Re}\,(1 + \frac{t}{s})\geq m\,\mathrm{Re}\,(1+ \frac{\zeta q''(\zeta)}{q'(\zeta)})$ and $z\in U$, for $m\geq 1$ and $|\zeta| = 1$.

In the special case when Ω is a simply connected domain and h is a conformal mapping of U onto Ω we denote the class by $\psi(h(U),q)$ or $\psi(h,q)$.

If $h(z) = q(z) = \frac{1+z}{1-z}$, then

(1) $q(U) = \Omega = h(U) = \{w;\ \mathrm{Re}\ w>0\}$.

LEMMA A[2] *Let the function $\psi\in\psi(\Omega,q)$, where Ω and q are defined by* (1). *If p is analytic in U, with $p(0)=1$ and if p satisfies*

$$\mathrm{Re}\ \psi(p(z),\ zp'(z),\ z^2p''(z);z)>0,\ z\in U,$$

then $\mathrm{Re}\ p(z)>0$ *for all* $z\in U$.

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2. **Main results**

THEOREM 1. *Let* $M(z)=az^n + a_{n+k}z^{n+k} + \dots$, $N(z) = bz^n + \dots$, *be analytic in the unit disc* U, $a.b\neq 0$, $n,k\geq 1$. *Suppose*

$$\frac{M(z)}{N(z)}\neq 0\ ,\ z\in U\ ,\ \mathrm{Re}\left[\frac{\alpha}{\mu}\frac{N(z)}{zN'(z)}\right]>\delta\ ,\ \text{where } 0\leq\delta<\mathrm{Re}\frac{\alpha}{\mu\cdot n}\ ,$$

$\mu>0$ *and* $\alpha\in C$ *with* Re $\alpha>0$. *If*

$$\mathrm{Re}\left[(1-\alpha)\left(\frac{M(z)}{N(z)}\right)^{\mu}+\alpha\frac{M'(z)}{N'(z)}\left(\frac{M(z)}{N(z)}\right)^{\mu-1}\right]>\beta\ ,\ \beta<\left(\frac{a}{b}\right)^{\mu}\ , \tag{2}$$

then

$$\mathrm{Re}\left(\frac{M(z)}{N(z)}\right)^{\mu}>\frac{2\beta+\delta\left(\frac{a}{b}\right)^{\mu}\cdot k}{2+\delta\cdot k}\ ,\ \text{for } z\in U\ .$$

Proof. Let $p(z) = \left(\frac{M(z)}{N(z)}\right)^{\mu}$, then p is analytic in U and $p(0)=\left(\frac{a}{b}\right)^{\mu}$.

From (2) we deduce that

$$\mathrm{Re}\left[p(z)+\frac{\alpha}{\mu}\frac{N(z)}{zN'(z)}zp'(z)\right]>\beta \tag{3}$$

We will obtain the real number φ, for which (3) implies Re $p(z)>\varphi$, for $z\in U$.

Let $q(z)=\frac{1}{\left(\frac{a}{b}\right)^{\mu}-\varphi}[p(z)-\varphi]$, then q is analytic in U and $q(z) = 1 + C_k z^k + \dots$

If we define the function $\psi:C^2\times U\to C$

$$\psi(w_1,w_2;z)=\left[\left(\frac{a}{b}\right)^{\mu}-\varphi\right]w_1+\frac{\alpha}{\mu}\frac{N(z)}{zN'(z)}\left[\left(\frac{a}{b}\right)^{\mu}-\varphi\right]w_2+\varphi-\beta\ ,$$

then from (3) we deduce Re $\psi(q(z), zq'(z); z) > 0$ and

$\mathrm{Re}\,\psi(1,0;\ z) = \left(\frac{a}{b}\right)^{\mu} - \varphi > 0$, for all $z\in U$.

For $s\le -\frac{k}{2}(1+r^2)$, $r\in R$, we obtain

$$\mathrm{Re}\ \psi(ir,s;z) = \mathrm{Re}\left[\frac{\alpha}{\mu}\,\frac{N(z)}{zN'(z)}\right]\left[\left(\frac{a}{b}\right)^{\mu}-\varphi\right]s+\varphi-\beta\le$$
$$\le -\delta\frac{k}{2}(1+r^2)\left[\left(\frac{a}{b}\right)^{\mu}-\varphi\right]+\varphi-\beta\ ,\ if\ \varphi\le\left(\frac{a}{b}\right)^{\mu}.$$

Since $\max\left\{\varphi:\varphi-\beta-\delta\frac{k}{2}(1+r^2)\left[\left(\frac{a}{b}\right)^{\mu}-\varphi\right]\le 0;\ r\in R\right\} =$

$= \dfrac{2\beta-\delta k\left(\frac{a}{b}\right)^{\mu}}{2+\delta k} = \varphi_0$, we have $\varphi_0<\left(\frac{a}{b}\right)^{\mu}$ and

$\mathrm{Re}\ \psi(ir,s;\ z)\le 0$, for $z\in U$, $s\le -\frac{k}{2}(1+r^2)$ and $\varphi\le\varphi_0$.

From Lemma A we deduce that Re $q(z)>0$, for $z\in U$ and $\forall\varphi\le\varphi_0$.

Hence Re $p(z)>\varphi_0$, $z\in U$.

If we let $\mu=1$ in Theorem 1, we obtain

COROLLARY 1. *Let* $M(z) = az^n + a_{n+k}z^{n+k} + \dots$, $N(z) = bz^n + \dots$ *be analytic in* U, $a.b \ne 0$, $n,k\ge 1$.

Suppose that $\mathrm{Re}\left[\alpha\frac{N(z)}{zN'(z)}\right]>\delta$, where $0\le\delta<\mathrm{Re}\frac{\alpha}{n}$ and

$$\mathrm{Re}\left[(1-\alpha)\frac{M(z)}{N(z)}+\alpha\frac{M'(z)}{N'(z)}\right]>\beta\ ,\ \text{where}\ \alpha\in C\ ,\ \mathrm{Re}\,\alpha>0\ ,\ \beta<\frac{a}{b}\ ,$$

then

$$\mathrm{Re}\frac{M(z)}{N(z)}>\frac{2\beta+\delta k\frac{a}{b}}{2+\delta k}\ ,\ \text{for}\ z\in U.$$

This result was recently obtained by *T.* Bulboacă [1].

If we let $a = b = 1$, $\mu = 1/2$, $N(z) = z$, $M(z) = f(z)$, $n = k = 1$ in Theorem 1, then we deduce

COROLLARY 2. *Let* $f\in A$, $\frac{f(z)}{z}\ne 0$ $z\in U$, $\mathrm{Re}\alpha>0$, $\beta<1$ *and suppose that*

$$\operatorname{Re}\left\{\sqrt{\frac{f(z)}{z}}+\alpha z\left(\sqrt{\frac{f(z)}{z}}\right)'\right\}>\beta\ ,$$

then

$$\operatorname{Re}\sqrt{\frac{f(z)}{z}}>\frac{2\beta+\operatorname{Re}\alpha}{2+\operatorname{Re}\alpha}$$

Proof. From (4) we have

$$\operatorname{Re}\left\{\left(1-\frac{\alpha}{2}\right)\left(\frac{f(z)}{z}\right)^{\frac{1}{2}}+\frac{\alpha}{2}f'(z)\left(\frac{f(z)}{z}\right)^{-\frac{1}{2}}\right\}=\operatorname{Re}\left\{\sqrt{\frac{f(z)}{z}}+\alpha z\left(\sqrt{\frac{f(z)}{z}}\right)'\right\}>\beta$$

By using Theorem 1 we obtain $\operatorname{Re}\sqrt{\frac{f(z)}{z}}>u(\delta)$, where $u(\delta)=\frac{2\beta+\delta}{2+\delta}$, $\delta\in[0,\operatorname{Re}\alpha]$.
But $\sup\{u(\delta)\,;\delta\in[0,\operatorname{Re}\alpha]\}=u(\operatorname{Re}\alpha)=\frac{2\beta+\operatorname{Re}\alpha}{2+\operatorname{Re}\alpha}$ hence $\operatorname{Re}\sqrt{\frac{f(z)}{z}}>\frac{2\beta+\operatorname{Re}\alpha}{2+\operatorname{Re}\alpha}$.

The case $\alpha>0$, $0\le\beta<1$ in Corollary 2 improves the result of Shigeyoshi Owa and C.Y.Shen [3].

If we take $M(z)=zf'(z)$, $f\in A$, $N(z)=z$, $\alpha>0$, $\mu=1/2$, $0\le\beta<1$ in Theorem 1, then we deduce the result of Shigeyoshi Owa C.Y.Shen [3].

THEOREM 2. *Let* $f(z)=z+\sum_{n=k+1}^{\infty}a_nz^n$, $k\ge1$ *be analytic in* U, $\frac{f(z)}{z}\ne0$ *in* U *and suppose that*

$$\operatorname{Re}\left[(1-\alpha)\left(\frac{f(z)}{z}\right)^{\mu}+\alpha f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right]>\beta\ ,\ z\in U\ , \tag{5}$$

where $0\le\beta<1$, $\alpha\ge0$, $\mu>0$.

Then $\operatorname{Re}\sqrt{\left(\frac{f(z)}{z}\right)^{\mu}}>\gamma_1$ *for* $z\in U$, *where*

$$\gamma_1=\frac{\frac{\alpha}{\mu}k+\sqrt{\left(\frac{\alpha}{\mu}k\right)^2+4\beta\left(1+\frac{\alpha}{\mu}k\right)}}{2\left(1+\frac{\alpha}{\mu}k\right)} \tag{6}$$

Proof. Let $p(z)=\sqrt{\left(\frac{f(z)}{z}\right)^{\mu}}$, then p is analytic in U and $p(z) = 1+C_k z^k+\dots$

We will obtain the real number γ for which $\text{Re } p(z)>\gamma$, $z\epsilon U$.

If we set $q(z)=\frac{1}{1-\gamma}[p(z)-\gamma]$, then q is analytic in U and $q(0)=1$.

A simple calculation yields:

$\text{Re } \psi(q(z),\ zq'(z);\ z)>0$, where

$\psi(w_1,w_2;\ z) = (1-\gamma)^2 w_1 + 2\gamma(1-\gamma)w_1 + 2\ \frac{\alpha}{\mu}(1-\mu)w_2\cdot$
$((1-\gamma)w_1 + \gamma) + \gamma^2 - \beta$.

We have

$\text{Re } \psi(ir,s;z)\leq -r^2[(1-\gamma)^2 + \frac{\alpha}{\mu}(1-\gamma)\gamma\cdot k] +$
$+ [\gamma^2 - \beta - \frac{\alpha}{\mu}k\gamma(1-\gamma)] \leq 0$, if $o\leq\gamma\leq 1$

and $(1-\gamma)^2 + \frac{\alpha}{\mu}(1-\gamma)\gamma k\geq 0$

$\gamma^2(1+\frac{\alpha}{\mu}k) - \frac{\alpha}{\mu}k\gamma - \beta \leq 0.$

These inequalities imply $\gamma\epsilon[0,\gamma_1]$ where γ_1 is deined by (6).

For $\gamma=\gamma_1<1$ we deduce $\text{Re } \psi(ir,s;z)\leq 0$, $z\epsilon U$, $s\leq -\frac{k}{2}(1+r^2)$ and by Lemma A we obtain $\text{Re } q(z)>0$, hence $\text{Re } p(z)>\gamma_1$, for $z\epsilon U$.

For $\mu=1/2$, $k=1$ from Theorem 2 we obtain.

COROLLARY 3. *Let* $f(z) = z + a_2z^2 +\dots$ *be analytic in* U. *Suppose that*

$$\text{Re}\left\{\sqrt{\frac{f(z)}{z}}+\alpha z\left(\sqrt{\frac{f(z)}{z}}\right)'\right\}>\beta$$

where $0\leq \beta <1$, $\alpha\geq 0$.

Then $\operatorname{Re}\sqrt[4]{\frac{f(z)}{z}} > \frac{\alpha+\sqrt{\alpha^2+4\beta(1+\alpha)}}{2(1+\alpha)}$, for $z\in U$. for $z\in U$.

This result was recently obtained by Shigeoyshi Owa and Zhworen Wu [4].

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SUFFICIENT CONDITIONS FOR UNIVALENCE OBTAINED BY SUBORDINATION METHOD

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REZUMAT. - Condiţii suficiente de univalenţă obţinute cu metoda subordonării. Sînt găsite mai multe condiţii de univalenţă cu ajutorul metodei subordonării.

1. DEFINITION 1. Let $f(z)$, $g(z)$ be two regular functions in $U=\{z:|z|<1\}$. We say that $f(z)$ is *subordinate* to $g(z)$, written $f(z)\prec g(z)$, if there exists a function $\varphi(z)$ regular in U which satisfies $\varphi(0)=0$, $|\varphi(z)|<1$ and

$$f(z)=g(\varphi(z)) \quad |z|<1 \tag{1}$$

DEFINITION 2. Let $f(z)$ be a regular function in U and $f'(z)\neq 0$ for $z\in U$. The function $f(z)$ is said to be *convex* if

$$Re\frac{zf''(z)}{f'(z)}+1>0, \quad z\in U. \tag{2}$$

Let S denote the class of functions $f(z)$ regular and univalent in the unit disk U, for which $f(0)=0$, $f'(0)=1$.

F.G. Avhadiev and L.A. Aksentiev [1] have proved the following theorem:

THEOREM A. *Let $f(z)=z+\dots$ and $g(z)=z+\dots$ be two regular functions in U. If*

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$$\left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{1}{1-|z|^2} \tag{3}$$

for all $z \in U$ *and* $\operatorname{Log} f'(z) \prec \operatorname{Log} g'(z)$, $\operatorname{Log} f'(0) = \operatorname{Log} g'(0) = 0$, *then the function* $f(z)$ *is univalent in* U.

A generalization of this theorem was obtained in [3]:

THEOREM B. *Let* $f(z)$, $g(z)$ *be regular functions in* U, $f(z) = z + \ldots$, $g(z) = z + \ldots$, *and let* α *be a complex number,* $0 < \operatorname{Re}\alpha \leq 1$. *If* $\operatorname{Log} f'(z) \prec \operatorname{Log} g'(z)$, $\operatorname{Log} f'(0) = \operatorname{Log} g'(0) = 0$ *and*

$$(1-|z|^2)\left|\frac{zg''(z)}{g'(z)}\right| \leq \operatorname{Re}\alpha, \tag{4}$$

for any $z \in U$, *then the function*

$$F_\alpha(z) = [\alpha \int_0^z u^{\alpha-1} f'(u)\, du]^{1/\alpha}$$

is regular and univalent in U.

In [2] it is proved the following theorem:

THEOREM C. *Let* β *and* γ *be complex numbers and let* $h(z) = c + h_1 z + \ldots$ *be convex (univalent) in* U *with*

$$\operatorname{Re}\,[\beta h(z) + \gamma] > 0. \tag{5}$$

If $p(z) = c + p_1 z + \ldots$ *is analytic in* U *then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec h(z). \tag{6}$$

In [5] it is proved the next univalence criterion:

THEOREM D. *Let* α *and* c *be complex numbers for which* $|\alpha| < 1$, $|c| \leq 1$, $c \neq -1$, $\frac{\alpha-1}{\alpha+1} \notin [1,\infty)$.

If $g(z) = z + \dots$ is a regular function in U, and

(i) $\dfrac{g(z)}{z} \neq 0$ *in* U *when* $\dfrac{1}{\alpha + 1} \in N^* = \{1, 2, \dots\}$

(ii) $\left| c|z|^2 + (1 - |z|^2)\left(\alpha \dfrac{zg'(z)}{g(z)} + \dfrac{zg''(z)}{g'(z)}\right) \right| \le 1$ *for all* z *in* U,

then the function $g(z)$ is univalent in U.

2. In this note we obtain by subordination method new conditions for univalence. First we will prove a consequence of Theorem D.

THEOREM 1. *Let $g(z) = z + \dots$ be a regular function in U with $\dfrac{g(z)g'(z)}{z} \neq 0$ for all $z \in U$, and let α, γ be complex numbers. If the regular function*

$$F(z) = \int_0^z \left[\frac{g(u)}{u}\right]^{1/\gamma} [g'(u)]^{1/\alpha\gamma}\, du \tag{7}$$

is univalent in U,

$|\alpha| < 1$, $\qquad \alpha \dfrac{2\gamma + 1}{\alpha + 1} \notin [1, \infty)$,

and

(i) $\qquad |\alpha| \le \dfrac{1}{4|\gamma|}$ *if* $|\gamma| \ge \dfrac{1}{2}$

(ii) $\qquad |\alpha| \le \dfrac{1}{1 + 4|\gamma|^2}$ *if* $|\gamma| < \dfrac{1}{2}$

then the function $g(z)$ is also univalent.

Proof. We will show that the differential equation

$$\alpha \frac{zg'(z)}{g(z)} + \frac{zg''(z)}{g'(z)} = \alpha + \alpha\gamma \frac{zF''(z)}{F'(z)} \tag{8}$$

has a regular solution $F(z)$, $F(0) = 0$, $F'(0) = 1$ in U.

Integrating (8) from 0 to z we obtain:

$$F'(z) = \left[\frac{g(z)}{z}\right]^{1/\gamma}[g'(z)]^{1/\alpha\gamma} \tag{9}$$

The function $\left(\frac{g(z)}{z}\right)^{1/\gamma}$ is regular because $\frac{g(z)}{z} \neq 0$ (we choose the branch which is equal to 1 at the origin)

We have also $g'(z) \neq 0$. Then $F'(z)$ from (9) is regular and the differential equation (8) has the regular solution

$$F(z) = \int_0^z \left[\frac{g(u)}{u}\right]^{1/\gamma}[g'(z)]^{1/\alpha\gamma}du, \tag{10}$$

for which $F(0) = 0$, $F'(0) = 1$.

For $c = -2\alpha\gamma$ the relation (ii) from Theorem D becomes:

$$\left|\alpha\gamma\ [-2|z|^2 + (1 - |z|^2)\frac{zF''(z)}{F'(z)}] + \alpha(1 - |z|^2)\right| \le 1.$$

Because $F(z) = z + \ldots$ is univalent, we have

$$\left|-2|z|^2 + (1 - |z|^2)\frac{zF''(z)}{F'(z)}\right| \le 4|z|, \qquad \text{therefore:}$$

$$\left|\alpha\gamma\ [-2|z|^2 + (1 - |z|^2)\frac{zF''(z)}{F'(z)}] + \alpha(1 - |z|^2)\right| \le$$

$$\le |\alpha\gamma|4|z| + |\alpha|(1 - |z|^2) = |\alpha|[-|z|^2 + 4|\gamma||z| + 1].$$

Calculating the maximum value of expression

$$E = |\alpha|[-|z|^2 + 4|\gamma||z| + 1] \quad \textit{for} \quad |z| < 1 \quad \textit{we obtain}:$$

$$E \le \begin{cases} 4|\alpha\gamma| & \textit{if}\ |\gamma| \ge \frac{1}{2} \\ |\alpha|(1 + 4|\gamma|^2) & \textit{if}\ |\gamma| < \frac{1}{2}. \end{cases}$$

From Theorem D and conditions (i) and, respectively (ii) of Theorem 1 we conclude that $g(z)$ belongs to S.

THEOREM 2. *Let $g(z) = z + \ldots$, $h(z) = z + \ldots$ be regular* *functions in U, and let α,γ complex numbers. If* $\frac{h(z)h'(z)}{z} \neq 0$,

$$\left|\frac{zg''(z)}{g'(z)}\right| \le \frac{1}{1-|z|^2} \quad \text{for all } z \in U, \tag{11}$$

and

$$|\alpha| < 1, \qquad \alpha\,\frac{2\gamma+1}{\alpha+1} \notin [1,\infty), \tag{12}$$

$$\begin{cases} |\alpha| \le \dfrac{1}{4|\gamma|} & \text{if } |\gamma| \ge \dfrac{1}{2} \\ |\alpha| \le \dfrac{1}{1+4|\gamma|^2} & \text{if } |\gamma| < \dfrac{1}{2} \end{cases} \tag{13}$$

$$\operatorname{Log}\left\{\left[\frac{h(z)}{z}\right]^{1/\gamma}[h'(z)]^{1/\alpha\gamma}\right\} \prec \operatorname{Log} g'(z), \tag{14}$$

then the function h(z) belongs to the class S.

(for $[\frac{h(z)}{z}]^{1/\gamma}[h'(z)]^{1/\alpha\gamma}$ we choose the branch which is equal to 1 at the origin, and for logarithmic functions the branches equal to 0 at the origin).

Proof. Let $f'(z) = [\frac{h(z)}{z}]^{1/\gamma}\ [h'(z)]^{1/\alpha\gamma}$.

From (11), (14), Log $f'(0)$ = Log $g'(0) = 0$, and Theorem A we deduce that $f(z) \in S$. Now, applying Theorem 1 with $F(z) = f(z)$ we have:

$$F'(z) = [\frac{h(z)}{z}]^{1/\gamma}[h'(z)]^{1/\alpha\gamma},$$

that is (9) with $g(z) = h(z)$.

Then Theorem 1 shows that $h(z)$ belongs to the class S.

THEOREM 3. *Let* $g(z) = z + \ldots$, Log $F'(z) = a_1 z + \ldots$ *be regular functions in U, let* Log $G'(z) = b_1 z + \ldots$ *be convex (univalent) in U, and let α, β, γ, δ complex numbers.*

If $\frac{g(z)g'(z)}{z} \neq 0$,

$$(1 - |z|^2)\left|\frac{zG''(z)}{G'(z)}\right| \le \frac{1}{1 - |z|^2} \tag{15}$$

$$[F'(z)]^{\alpha\delta} = [\frac{g(z)}{z}]^{\alpha} g'(z) \qquad for \quad all \quad z \in U, \tag{16}$$

and

$$Re\,\gamma > 0, \quad Re[\text{Log } e^{\gamma} G'^{\beta}] > 0, |\alpha| < 1, \ \alpha\frac{2\delta + 1}{\alpha + 1} \in [1, \infty),$$

$$\begin{cases} |\alpha\delta| \le \frac{1}{4} & if \ |\delta| \ge \frac{1}{2} \\ |\alpha| \le \frac{1}{1 + 4|\delta|^2} & if \ |\delta| < \frac{1}{2} \end{cases} \tag{17}$$

$$\text{Log } F'(z) + \frac{zF''(z)}{F'(z)\ \text{Log}(e^{\gamma}F'^{\beta})} \prec \text{Log } G'(z) \tag{18}$$

(for logarithmic functions we choose the branches equal to 0 at the origin), then the function g(z) belongs to S.

Proof. Let $p(z) = \text{Log } F'(z)$, which is regular, and $h(z) = \text{Log } G'(z)$ which is convex (univalent) in U. Then

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} = \text{Log}F'(z) + \frac{zF''(z)}{F'(z)\text{Log}(e^{\gamma}F'^{\beta})}$$

By (18) and Theorem C we obtain

$$\text{Log } F'(z) \prec \text{Log } G'(z) \tag{19}$$

From (15), (19) and Theorem A it follows that $F(z) \in S$. Because all the conditions of Theorem 1 are satisfied, we conclude that $g(z)$ belongs to the class S.

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A NEW GENERALIZATION OF NEHARI'S CRITERION OF UNIVALENCE

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Dedicated to Professor P.T.Mocanu on his 60th anniversary

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REZUMAT: - O nouă generalizare a criteriului de univalenţă al lui Nehari. În această notă se obţine o generalizare a unui bine cunoscut rezultat de univalenţă al lui Nehari.

We denote by U the unit disk $\{ z:|z| < 1 \}$. The aim of this paper is to obtain a generalization of the following well-known result due to Nehari.

THEOREM A [1]. *If* $f(z) = z + a_2z^2 + \dots$ *is a regular function in* U, *and*

$$|\{f;z\}| \le \frac{2}{(1-|z|^2)^2}, \quad \forall z \in U \tag{1}$$

where

$$\{f;z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 \tag{2}$$

then the function $f(z)$ *is univalent in* U.

In the following demonstrations, we shall use the result due to Pommerenke.

THEOREM B [2]. *Let* r_0 *be a real number,* $r_0 \in (0,1]$, $U_{r_0} = \{z:|z| < r_0\}$ *and let* $f(z,t) = a_1(t)z + \dots$, $a_1(t) \neq 0$, *be analytic in* U_{r_0} , *for all* $t \ge 0$ *and locally absolutely continuous in* $I = [0,\infty)$, *locally uniformly with respect to* U_{r_0}

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Supposing that for almost all $t \in I$, $f(z,t)$ *satisfies the equation*

$$z\frac{\partial f(z,t)}{\partial z} = p(z,t)\frac{\partial f(z,t)}{\partial t}, \quad z \in U_{r_0}, \tag{3}$$

where $p(z,t)$ *is analytic in* U *and* $\operatorname{Re} p(z,t) > 0$ *for all* $t \in I$, $z \in U$. *If* $|a_1(t)| \to \infty$ *for* $t \to \infty$ *and if* $\{ f(z,t)/a_1(t) \}$ *forms a normal family in* U_{r_0}, *then for all* $t \in I$, $f(z,t)$ *has an analytic and univalent extension to the whole disk* U.

THEOREM 1. *Let* α *be a real number,* c *be a complex number,* $|c| < 1$ *and* $f(z) = z + a_2z^2 + \ldots$ *a regular function in the unit disk* U. *If*

$$\left|\frac{1-\alpha}{1+\alpha} + \frac{2c}{1+\alpha}\right| < 1 \tag{4}$$

and

$$\left|\frac{1-\alpha}{1+\alpha} + \frac{2}{1+\alpha}ce^{-2t\alpha} + \frac{1}{1+\alpha}\frac{p(e^{-t\alpha}z)}{1+c}z^2(1-e^{-2t\alpha})^2\right| < 1 \tag{5}$$

for all $z \in U$, $t \geq 0$, *where*

$$p(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2, \tag{6}$$

then the function $f(z)$ *is univalent in* U.

Proof. If the function $f(z)$ is regular in U, then

$$f'(z) = \frac{f_1(z)}{f_2(z)} \tag{7}$$

where the functions $f_1(z)$ and $f_2(z)$ verifies the relations

$$f_K''(z) + \frac{p(z)}{2} f_K(z) = 0 \quad (K = 1,2) \tag{8}$$

and

$$\begin{aligned} f_1(0) &= 0, \quad f_1'(0) = 1 \\ f_2(0) &= 1, \quad f_2'(0) = 0. \end{aligned} \tag{9}$$

Let's consider L(z,t) a regular function $L: U_{r_0} \times [0,\infty) \to \mathbf{C}$, $r_0 \in (0,1)$, defined by

$$L(z,t) = \frac{f_1(e^{-t\alpha}z) + \frac{1}{1+c}(e^{t\alpha} - e^{-t\alpha}) z f_1'(e^{-t\alpha}z)}{f_2(e^{-t\alpha}z) + \frac{1}{1+c}(e^{t\alpha} - e^{-t\alpha}) z f_2'(e^{-t\alpha}z)} = \tag{10}$$

$$= a_1(t) z + \ldots$$

where

$$a_1(t) = e^{-t\alpha} + \frac{1}{1+c}(e^{t\alpha} - e^{-t\alpha}). \tag{11}$$

Let's prove that $a_1(t) \neq 0$ for all $t \geq 0$. We observe that if $a_1(t) = 0$, then from (11) results that $c = -e^{-2t\alpha} \in (-\infty,-1]$. Because from hypothesis $c \notin (-\infty,-1]$, it results that $a_1(t) \neq 0$, for all $t \geq 0$.

From (10) we obtain

$$\frac{\partial L(z,t)}{\partial z} = \{[f_1'(e^{-t\alpha}z) f_2(e^{-t\alpha}z) - f_1(e^{-t\alpha}z) f_2'(e^{-t\alpha}z)] \cdot$$

$$\cdot [\frac{e^{t\alpha} + c e^{-t\alpha}}{1+c} + \frac{1}{(1+c)^2} \frac{(1-e^{-2t\alpha})}{(e^{t\alpha}-e^{-t\alpha})^{-1}} z^2 \frac{p(e^{-t\alpha}z)}{2}]\} : \tag{12}$$

$$: \{[f_2(e^{-t\alpha}z) + \frac{1}{1+c}(e^{t\alpha} - e^{-t\alpha}) z f_2'(e^{-t\alpha}z)]^2\};$$

$$\frac{\partial L(z,t)}{\partial t} = \{z[f_1'(e^{-ta}z)f_2(e^{-ta}z) - f_1(e^{-ta}z)f_2'(e^{-ta}z)]\cdot$$

$$\cdot[\frac{\alpha(e^{ta}-e^{-ta})}{1+c} - \frac{1}{(1+c)^2}\frac{(1-e^{-2ta}z)}{(e^{ta}-e^{-ta})^{-1}}z^2\frac{p(e^{-ta}z)}{2}]\}: \qquad (13)$$

$$:\{[f_2(e^{-ta}z) + \frac{1}{1+c}(e^{ta}-e^{-ta})zf_2'(e^{-ta}z)]^2\}.$$

Let's prove that L(z,t) is a Loewner chain in U. It is easy to prove that the function L(z,t) is locally absolutely continuous in I and locally uniformly with respect to U_{r_0}. The family of the functions $\{L(z,t)/a_1(t)\}$ forms a normal family of regular functions in $U_{r_1} = \{z:|z| < r_1\}$, $0 < r_1 < r_0$. From (11) we obtain $a_1(t) \to \infty$, for $t \to \infty$. Let we consider the function $Q : U_{r_0} \times I \to \mathbf{C}$, by

$$Q(z,t) = z\frac{\partial L(z,t)}{\partial z} \Big/ \frac{\partial L(z,t)}{\partial t}, \quad z \in U \qquad (14)$$

From (12), (13) and (14) it results that

$$Q(z,t) = \frac{(e^{ta}+ce^{-ta}) + \dfrac{p(e^{-ta}z)}{2(1+c)}z^2(1-e^{-2ta})(e^{ta}-e^{-ta})}{(e^{ta}-ce^{-ta}) - \dfrac{p(e^{-ta}z)}{2(1+c)}z^2(1-e^{-2ta})(e^{ta}-e^{-ta})} \qquad (15)$$

In order to prove that L(z,t) is a Loewner chain it is sufficient to prove that, there exists a real number $r \in (0,1)$, such that L(z,t) is a regular function in $U_r = \{z:|z| < r\}$, for all $t > 0$, the function Q(z,t) defined from (15) to be regular in U for all $t > 0$ and

$$\text{Re}\, Q(z,t) > 0, \qquad (16)$$

for all $z \in U$ and $t \geq 0$.

Let's consider the function

$$K(z,t) = f_2(e^{-t\alpha}z) + \frac{1}{1+c}(e^{t\alpha} - e^{-t\alpha})zf_2'(e^{-t\alpha}z) \qquad (17)$$

We shall prove that the function $K(z,t) \neq 0$. Because $f_2(0) = 1$ and $f_2(z)$ is regular in U, it results that there exists a number $r\in(0,1)$ such that $K(z,t) \neq o$ for any $z \in U_r$ and hence the function $L(z,t)$ is regular in U for all $t \geq 0$.

In order to prove that the function $Q(z,t)$ is regular in U and with positive real part in U, for all $t \in I$, it is sufficient to prove that

$$|R(z,t)| < 1 \qquad (18)$$

for all $z \in U$ and $t \geq 0$, where

$$R(z,t) = \frac{Q(z,t) - 1}{Q(z,t) + 1} \qquad (19)$$

From (15) and (19) we obtain

$$R(z,t) = \frac{1-\alpha}{1+\alpha} + \frac{2c}{1+\alpha}e^{-2t\alpha} + \frac{p(e^{-t\alpha}z)}{(1+\alpha)(1+c)}z^2(1-e^{-2t\alpha})^2. \qquad (20)$$

By (5) and (20) it results that the inequality (18) holds true for all $z \in U$ and $t \geq 0$.

Using Theorem B, it results that the function $L(z,t)$ is regular and univalence in U for all $z \in U$ and $t \geq 0$.

It results that $L(z,t)$ is a Loewner chain, and hence the function

$$L(z,0) = f_1(z)/f_2(z) = f(z)$$

is univalent in U.

THEOREM 2. *Let* α *be a real number,* c *a complex number,* $|c| < 1$ *and* $f(z) = z + a_2z^2 + \ldots$ *a regular function in* U.

If

$$\left|\frac{1-\alpha}{1+\alpha}+\frac{2c}{1+\alpha}\right|<1 \tag{21}$$

and

$$\left|\frac{1-\alpha}{1+\alpha}+\frac{2c}{1+\alpha}|z|^2+\frac{1}{1+\alpha}\,\frac{p(z)}{1+\alpha}e^{2i\theta}(1-|z|^2)^2\right|<1 \tag{22}$$

for all $z \in U$ *and* θ *a real number,* $p(z) = \{f;z\}$, *then the function* $f(z)$ *is univalent in* U.

Proof. Using the notations from the Theorem 1 it results that the function $R(z,t)$ defined by the relation (20) is regular in U for all $t > 0$. It results that for all $t > 0$, we have

$$\max_{|z|=1} |R(z,t)| = |R(e^{i\theta},t)| =$$
$$= \left|\frac{1-\alpha}{1+\alpha}+\frac{2c}{1+\alpha}e^{-2t\alpha}+\frac{1}{1+\alpha}\,\frac{p(e^{-t\alpha+i\theta})}{1+c}e^{2i\theta}(1-e^{-2t\alpha})^2\right| \tag{23}$$

where $\theta \in \mathbb{R}$. If $\zeta = e^{-t\alpha+i\theta}$, then $|\zeta| = e^{-t\alpha} < 1$ and hence applying the maximum principle to the function $R(z,t)$ we have

$$|R(z,t)| < \max_{|z|=1} |R(z,t)| =$$
$$= \left|\frac{1-\alpha}{1+\alpha}+\frac{2c}{1+\alpha}|\zeta|^2+\frac{1}{1+\alpha}\,\frac{p(\zeta)}{1+c}e^{2i\theta}(1-|\zeta|^2)^2\right|. \tag{24}$$

Because $\zeta \in U$, from (22) and (24) we obtain

$$|R(z,t)| < 1 \tag{25}$$

for all $z \in U$ and $t > 0$.

From hypothesis we observe that for $t = 0$

$$|R(z,0)| = \left|\frac{1-\alpha}{1+\alpha}+\frac{2c}{1+\alpha}\right| < 1 \tag{26}$$

The inequality (25) holds true for all $z \in U$ and for all

$t \geq 0$, and, hence by Theorem 1 it results that the function $f(z)$ is univalent in U.

Remark 1. For $\alpha = 1$ and $c = 0$ we obtain Nehari's criterion of univalence.

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ON CERTAIN ANALYTIC FUNCTIONS WITH POSITIVE REAL PART

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REZUMAT. - Asupra unor funcţii analitice cu partea reală pozitivă. Fie α un număr real şi n un număr întreg pozitiv. Fie P şi Q funcţii analitice în discul unitate U, cu $P(z) \neq 0$, care verifică inegalitatea (3). Se arată că dacă $p(z) = 1 + p_n z^n + \ldots$ este o funcţie analitică în U, care verifică ecuaţia diferenţială (4), atunci $Re\ p(z) > 0$ în U. Acest rezultat este îmbunătăţit în cazul cînd funcţia Q este o constantă reală.

1. Introduction. In this paper we shall show that under certain conditions on α, P and Q the solution $p(z) = 1 + p_n z^n + \ldots$ of the differential equation (4) has positive real part. This result is improved when Q is a real constant and we obtain an extension of the "open door" Theorem in [3].

As a simple application we obtain a sufficient condition of starlikeness. The results are obtained by applying the theory of differential subordination. A survey of this theory and applications may be found in [4].

2. Preliminaries. Let A_n be the class of analytic functions f in the unit disc $U=\{z;\ |z|<1\}$ of the form $f(z)=z+a_{n+1}z^{n+1}+\ldots$, where $n\geq 1$.

Denote $A = A_1$. A function $f \in A$ is said to be starlike if $Re\ [\ z\ f'(z)\ /\ f(z)] > 0$ in U. Denote by S^* the class of the starlike functions.

Let F and G be analytic functions in U. If G is univalent,

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then we say that F is subordinate to G, written $F \prec G$ or $F(z) \prec G(z)$ if $F(0) = G(0)$ and $F(U) \subset G(U)$.

We will need the following lemma to prove our results.

LEMMA A. *Let Ω be a set in the complex plane* **C** *and let n be a positive integer. Suppose that the function Ψ:* $\mathbf{C}^2 \times U \to$ **C** *satisfies the condition*

$$\Psi\,(\,is,\ t;\ z) \notin \Omega \tag{1}$$

for all real s, $t \leq -(n/2)(1 + s^2)$ and $z \in U$.

In the function $p(z) = 1 + p_n z^n + \ldots$ is analytic in U and

$$\Psi\,[p(z),\ z\,p(z)\ ;\ z] \in \Omega, \tag{2}$$

for $z \in U$, then $\operatorname{Re}\rho(z) > 0$ in U.

More general forms of this lemma can be found in [1],[2] and [4].

3. Main results.

THEOREM 1. *Let α be real and let n be a positive integer.*

Let P and Q be analytic functions in U, with $\operatorname{Re} P(z) \neq 0$ and suppose that

$$(2\alpha + n)\left[\frac{\operatorname{Im} Q(z)}{\operatorname{Re} P(z)}\right]^2 - 2\frac{\operatorname{Re}\,[P(z)\ \overline{Q}(z)]}{\operatorname{Re} P(z)} + n > 0 \tag{3}$$

for $z \in U$.

If $p(z) = 1 + p_n z^n + \ldots$ is analytic in U and satisfies the differential equation

$$z\,p'(z) + \alpha p^2(z) + P(z)\,p(z) + Q(z) = 0 \tag{4}$$

then $\operatorname{Re} p(z) > 0$ in U.

Proof. Let $\Psi\,(\,w_1,\ w_2\ ;\ z\,) = w_2 + \alpha w_1^{\,2} + P(z)w_1 + Q(z)$

If we let $\Omega = \{0\}$, then equation (4) can be written as

$$\mathcal{F}\,[p(z)\ ,\ z\,p'(z)\ ;\ z] \in \Omega \tag{5}$$

In order to apply Lemma A we show that $\mathcal{F}$ satisfies condition (1), i.e.

$$t - \alpha s^2 + is\,P(z) + Q(z) \neq 0 \tag{6}$$

for all real s, $t \le -(n/2)(1+s^2)$ and $z \in U$.

If for some s,t and z satisfying the above conditions the equality

$$t - \alpha s^2 + is\,P(z) + Q(z) = 0 \text{ holds, then}$$

$$t - \alpha s^2 + s\,Im\,P + Q = 0 \tag{7}$$

and

$$s\,Re\,P + Im\,Q = 0 \tag{8}$$

From (7) we deduce

$$t = \alpha s^2 + s\,Im\,P - Re\,Q \le -(n/2)(1+s^2)$$

hence s satisfies the inequality

$$\frac{2\alpha + n}{2}\,s^2 + s\,Im\,P - Re\,Q + \frac{n}{2} \le 0 \tag{9}$$

Since $Re\,P(z) \neq 0$, from (8) we deduce

$$s = -\frac{Im\,Q}{Re\,P}$$

and from (8) we obtain the inequality

$$(2\alpha + n)\left(\frac{Im\,Q}{Re\,P}\right)^2 - 2\,\frac{Re\,P\,\overline{Q}}{Re\,P} + n \le 0$$

which contradicts (3). Hence condition (6) is satisfied and by Lemma A we deduce $Re\,p(z) > 0$ in U.

If the function Q is a real constant then Theorem 1 can be improved by the following result.

THEOREM 2. *Let* n *be a positive integer and let* α *and* β

be real numbers, with $2\alpha + n > 0$ *and* $2\beta + n > 0$. *Let* H *be the function*

$$H(z) = \frac{\beta - \alpha + 2(\alpha+\beta+n)z + (\beta-\alpha)z^2}{1 - z^2}, \quad z \in U \tag{10}$$

Let P *be analytic function in* U *satisfying* $P \prec H$.

If $p(z) = 1 + p_n z^n + \dots$ *is analytic in* U *and satisfies the differential equation*

$$z\,p'(z) + \alpha p^2(z) + P(z)p(z) = \beta \tag{11}$$

then $Re\, p(z) > 0$ *in* U.

Proof. As in the proof of Theorem 1 we have to check the condition (1) of Lemma A, i.e.

$$t - \alpha s^2 + isP(z) \neq \beta \tag{12}$$

for all real s, $t \le -(n/2)(1 + s^2)$ and $z \in U$.

If for some s, t and z satisfying the above conditions the equality

$$t - \alpha s^2 + is\, P(z) = \beta$$

holds, then

$$t - \alpha s^2 - s\, Im\, P = \beta \tag{13}$$

and

$$s\, Re\, P = 0 \tag{14}$$

If $Re\, P(z) \neq 0$, then form (14) we deduce $s = 0$ and using (13) we obtain $t = \beta > -n/2$ which contradicts

$$t \le -(n/2)(1 + s^2) \le -n/2.$$

Therefore, in this case condition (12) is satisfied.

Suppose now that $Re\, P(z) = 0$.

If $s > 0$, from (13) we deduce

$$Im\, P(z) = \frac{t}{s} - \alpha s - \frac{\beta}{s} \le -\frac{n}{2s}(1 + s^2) - \alpha s - \frac{\beta}{s} =$$

$$= -\frac{1}{2}\left[(2\alpha+n)s + (2\beta+n)\frac{1}{s}\right] = \varphi(s)$$

It is easy to show that the maximum value of $\varphi(s)$ is given by

$$- \sqrt{(2\alpha + n)(2\beta + n)}\,.$$

Hence $Im\, P(z) \le - \sqrt{(2\alpha + n)(2\beta + n)}$.

Similarly, for $s < 0$ we deduce

$$Im\, P(z) \ge \sqrt{(2\alpha + n)(2\beta + n)}\,.$$

Therefore condition (12) holds if either

$Re\, P(z) \neq 0$ or $Re\, P(z) = 0$ and $|Im\, P(z)| < \sqrt{(2\alpha + n)(2\beta + n)}$.

If we let

$$C = \sqrt{(2\alpha + n)(2\beta + n)} \quad \text{and}$$

$$G(z) = 2\,C\,\frac{z}{1 - z^2}$$

then $H(z) = G\left(\frac{z+a}{1+az}\right)$, where $2\,C\,\frac{a}{1-a^2} = \beta-\alpha$.

We deduce that $H(U) = G(U)$ is the complex plane slit along the half-lines $Re\, w = 0$ and $|Im\, w| \ge C$ and $H(0) = \beta-\alpha$.

From the above analyisis we deduce that condition (12) holds if $P \prec H$. By applying Lemma A we deduce $Re\, p(z) > 0$.

4. A starlikeness condition

THEOREM 3. *Let* $f \in A_n$, *with* $\frac{f(z)\, f'(z)}{z} \neq 0$ *in* U *and suppose that*

$$1 + \frac{z\, f''(z)}{f'(z)} - \frac{f(z)}{z\, f'(z)} \prec \frac{2(2 + n)z}{1 - z^2}\,. \tag{15}$$

Then $f \in S^*$.

Proof. If we let $\alpha = \beta = 1$ then in (10) we have

$$H(z) = \frac{2(2+n)z}{1-z^2}.$$

If we denote $p(z) = \frac{z f'(z)}{f(z)}$ then (15) becomes

$$p(z) + \frac{z p'(z)}{p(z)} - \frac{1}{p(z)} \prec H(z)$$

and if we take $P(z) = \frac{1}{p(z)} - p(z) - \frac{z p'(z)}{p(z)}$
then

$$P(z) \prec H(z) = H(-z)$$

and from Theorem 2 we deduce $Re\ p(z)>0$, which shows that $f \in S^*$.

COROLLARY. *If* $f \in A_n$, $\frac{f(z) f'(z)}{z} \neq 0$
and

$$\left| Im \left[1 + \frac{z f''(z)}{f'(z)} - \frac{f(z)}{z f'(z)} \right] \right| < 2 + n$$

then $f \in S^*$.

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DISTORSION OF LEVEL LINES OF THE CAPACITY FUNCTIONS UNDER K-qc MAPPINGS

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Dedicated to Professor P. T. Mocanu on his 60^{th} anniversary

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REZUMAT. - Deformarea liniilor de nivel ale funcţiilor capacitate prin transformări K-qc. Fie R şi R' două suprafeţe Riemann deschise cu frontierele ideale Γ şi Γ', iar p_Γ şi $p_{\Gamma'}$, funcţiile capacitate ale celor două frontiere. În lucrare se studiază imaginea printr-o funcţie f a liniilor de nivel ale lui $p_\Gamma(\ ,z_0)$ în raport cu cele ale lui $p_{\Gamma'}(\ ,Z_0)$unde $f:R\to R'$, $f(z_0)=Z_0$ este o transformare K-qc (omeomorfism K-cvasiconform).

0. **Introduction.** The capacity functions have introduced by L.Sario [7]. Let R be an open Riemann surface, Γ its ideal boundary, z_0 a point in R and D an arbitrary but fixed parametric disc containing z_0. The **capacity function** of the ideal boundary Γ of R with respect to z_0 and D [7], [8,p.179] is a function $p_\Gamma(\cdot\ ,z_0) = p_R(\cdot\ ,z_0)$ with the following properties:

1) p_Γ is harmonic on $R\backslash z_0$,

2) $p_\Gamma(z,z_0) = \log|z-z_0|+h(z)$, $z\in D$, where $h(z)$ is a harmonic function with $h(z_0)=0$, and

3) p_Γ minimizes the integral

$$\frac{1}{2\pi}\int_\Gamma \varphi * d\varphi := \lim_{n\to\infty}\frac{1}{2\pi}\int_{\Gamma_n}\varphi * d\varphi$$

in the family of the functions $\varphi:R\backslash z_0 \to \mathbf{R}$ which verify 1) and 2), where Γ_n is the boundary of a regular region [1, p.26] π_n from a countable exhaustion of R with $z_0\in\pi_0$.

One knows that

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$$k_{\Gamma}=\frac{1}{2\pi}\int_{\Gamma} p_{\Gamma} * dp_{\Gamma}=\sup_{z\in R} p_{\Gamma}(z,z_0) \le +\infty$$

is the Robin constant of R with respect to z_0 and D, while $c_{\Gamma}=e^{-k_{\Gamma}}$ is the capacity of Γ with respect to z_0 and D, the Riemann surface R being hyperbolic or parabolic according as $k_{\Gamma}<+\infty$ or $k_{\Gamma}=+\infty$.

In the hyperbolic case,

$$p_{\Gamma}(\cdot\ ,z_0) + G_{\Gamma}(\cdot\ ,z_0) = k_{\Gamma}, \qquad (0.1)$$

where $G_{\Gamma}(\cdot\ ,z_0) = G_R(\cdot\ ,z_0)$ is the **Green function** on R with the logarithmic pole z_0, [8, pp.180-181], [10,IX,I], which is characterized by the following properties:

1) $G_R(\cdot\ ,z_0)$ is harmonic on $R\setminus z_0$,

2) $G_R(z,z_0) = \log\frac{1}{|z-z_0|} + v(z)$, $z\in D$, where v is a harmonic function, and

3) $G_R(\cdot\ ,z_0) = \inf\{P:P$ is positive and satisfies to 1) and 2)$\}$.

In what follows we consider two open Riemann surfaces R and R' with the ideal boundaries Γ and Γ' respectively, two arbitrarily fixed points $z_0\in R$ and $z_0'\in R'$, two parametric discs $D\ni z_0$ and $D'\ni z_0'$ and the corresponding capacity functions $p_{\Gamma}(\cdot\ ,z_0)$ and $p_{\Gamma'}(\cdot,z_0')$. We denote by z and z' points in R and R' as well as parameters on these surfaces.

Suppose that there are K-qc mappings (K-quasiconformal homeomorphisms) $f:R\to R'$ with $f(z_0) = z_0'$ and denote by $\mathcal{F}$ their family and $z'=f(z)$.

If f would be a conformal mapping $p_{\Gamma'}(z',z_0')=p_{\Gamma}(z,z_0)$ (by a convenient choice of the parameter of D'), hence level lines of

$p_\Gamma(\cdot\ ,z_0)$ will be mapped by f in level lines of $p_{\Gamma'}(\cdot,z_0')$. Generally this property does not hold for K-qc mappings.

Our aim is to study the image under f of the level lines of $p_\Gamma(\cdot\ ,z_0)$ by means of the level lines of $p_{\Gamma'}(\cdot,z_0')$. For hyperbolic surfaces we first treat this problem by working with the Green functions $G_R(\cdot\ ,z_0)$ and $G_{R'}(\cdot,z_0')$ and taking into account (0.1). This way enlarges the possibilities of application in as much as the form of the results for the Green function is more adequate.

In the proofs we use the following

LEMMA [3]. *Let R and R' be arbitrary Riemman surfaces which are not conformally equivalent to C and $\hat{C}$. If $z_0\epsilon R$ and $z_0'\epsilon R'$, the family $\mathcal{F}$ of the K-qc mappings $f:R\to R'$ with $f(z_0)= z_0'$ is normal and closed.*

Compacity property will play a main role in the paper. Thus we consider only Riemann surfaces of **class** R_p, i.e. Riemann surfaces on which there exists a capacity function with compact level lines [8, p.30], [6, p.231]. As it was proved by M.Nakai, this class contains all parabolic Riemann surfaces [8,IV,§1]. The interior of a compact bordered Riemann surface gives an example of a hyperbolic Riemann surface of class R_p.

1. Level lines of the Green function.

1.1. Let be R and R' two hyperbolic Riemann surfaces of class R_p, z_0 and z_0' two points in R and R', $G_R(\cdot\ ,z_0)$ and $G_{R'}(\cdot,z_0')$ the corresponding Green functions and $\mathcal{F}$ the family defined above.

We designate by C_λ the level line $G_R(z,z_0)\ =\ \lambda$ where $\lambda\epsilon(0,+\infty)$, by Π_λ the regular region $\{z\epsilon R:G_R(z,z_0)>\lambda\}$ and for

$\lambda_2<\lambda_1$ by $C_{\lambda_1\lambda_2}$ the curve family $\{C_\lambda : \lambda\in[\lambda_2,\lambda_1]\}$ and $\Pi_{\lambda_1\lambda_2}=\Pi_{\lambda_2}\setminus\overline{\Pi}_{\lambda_1}$. Further we introduce on R' the notations $C'_{\lambda'}$ - the level line $G_{R'}(z',z_0')=\lambda'$, $\lambda'\in(0,+\infty)$, and similary $\Pi'_{\lambda'}$, $C'_{\lambda_1'\lambda_2'}$ and $\Pi'_{\lambda_1'\lambda_2'}$, $\lambda_2'<\lambda_1'$.

The modulus of $\overline{\Pi}_{\lambda_1\lambda_2}$ defined as the modulus of the curve family separating C_{λ_1} from C_{λ_2} in $\Pi_{\lambda_1\lambda_2}$ is given by the modulus of $C_{\lambda_1\lambda_2}$ [2], namely

$$\operatorname{Mod} C_{\lambda_1\lambda_2}=\frac{\lambda_1-\lambda_2}{2\pi}. \tag{1.1}$$

Since $R\in R_p$, we can define for every $f\in\mathcal{F}$ the functions:

$$\lambda_0'(\lambda,f)=\min\{\lambda'=G_{R'}(z',z_0') : z'\in fC_\lambda\}$$

and

$$\Lambda_0'(\lambda,f)=\min\{\lambda'=G_{R'}(z',z_0') : z'\in fC_\lambda\}.$$

PROPOSITION 1.1. *The functions* $\lambda_0'(\lambda,f)$ *and* $\Lambda_0'(\lambda,f)$ *are strictly increasing with respect to* λ, *and verify the inequalities:*

$$\lambda_0'(\lambda,f) \le \lambda' \le \Lambda_0'(\lambda,f) \tag{1.2}$$

and

$$K^{-1}\lambda_0'(\lambda,f) \le \lambda \le K\,\Lambda_0'(\lambda,f). \tag{1.3}$$

1) *Proof that* $\Lambda_0'(\lambda,f)$ *is a strictly increasing function of* λ. We remark that fC_λ descomposes R' in $f\Pi_\lambda$ and $R'\setminus f\overline{\Pi}_\lambda=:\Omega_\lambda'$, that $\max\{G_{R'}(z',z_0') : z'\in\overline{\Omega}_\lambda'\}=\Lambda_0'(\lambda,f)$ and $C'_{\Lambda_0'(\lambda,f)}\subset f\overline{\Pi}_\lambda$. Further if $\lambda_1>\lambda_2$, then $fC_{\lambda_2}\subset\Omega_{\lambda_1}'$. Suppose that $C'_{\Lambda_0'(\lambda_2,f)}$ does not intersect fC_{λ_1}; since it has at least a common point with fC_{λ_2}, hence with Ω_{λ_1}', it follows that $\Lambda_0'(\lambda_2,f) < \max\{G_{R'}(z',z_0') : z'\in\overline{\Omega}_{\lambda_1}'\}=\Lambda_0'(\lambda_1,f)$. If $C'_{\Lambda_0'(\lambda_2,f)}$ intersects fC_{λ_1} then $\Lambda_0'(\lambda_2,f) \le \Lambda_0'(\lambda_1,f)$; however,

equality cannot occur, since otherwise there would exist a point in $fC_{\lambda_2}\cap C'_{\Lambda'_0(\lambda_1,f)}$, hence in $\Omega'_{\lambda_1}\cap f\overline{\Pi}_{\lambda_1}=\emptyset$.

2) *Proof of (1.3)*. According to (1.1) and to the Grötzsch inequalities

$(2\pi)^{-1}\Lambda'_0(\lambda,f)=\mathrm{Mod}\,C'_{\Lambda'_0(\lambda,f)0}=$ the modulus of the curve family separating $C'_{\Lambda'_0(\lambda,f)}$ from Γ' on $R'\setminus\pi'_{\Lambda'_0(\lambda,f)}\geq\mathrm{Mod}\,fC_{\lambda_0}\geq K^{-1}\mathrm{Mod}C_{\lambda_0}=(2\pi K)^{-1}\lambda$.

Similary,

$(2\pi)^{-1}\lambda'_0(\lambda,f)=\mathrm{Mod}C'_{\lambda'_0(\lambda,f)0}\leq K\mathrm{Mod}f^{-1}C'_{\lambda'_0(\lambda,f)0}\leq K$ the modulus of the curve family separating C_λ from Γ on $R\setminus\Pi_\lambda$ = K Mod $C_{\lambda_0}=(2\pi)^{-1}K\lambda$.

PROPOSITION 1.2. *If* $\lambda_1>\lambda_2$, *then*

$$K^{-1}[\lambda'_0(\lambda_1,f)-\Lambda'_0(\lambda_2,f)]\leq\lambda_1-\lambda_2\leq$$
$$\leq K[\Lambda'_0(\lambda_1,f)-\lambda'_0(\lambda_2,f)]. \qquad (1.4)$$

The inequalities (1.3) are a particular case of the inequalities (1.4). They can be obtained from (1.4) by taking $\lambda_1=\lambda$ and $\lambda_2=0$, since $\Lambda'_0(0,f)=\lambda'_0(0,f)=0$. However the proof of (1.4) is also similar to that of (1.3):

$$(2\pi)^{-1}[\Lambda'_0(\lambda_1,f)-\lambda'_0(\lambda_2,f)]=\mathrm{Mod}C'_{\Lambda'_0(\lambda_1,f)\lambda'_0(\lambda_2,f)}\geq$$
$$\geq\mathrm{Mod}fC_{\lambda_1\lambda_2}\geq K^{-1}\mathrm{Mod}C_{\lambda_1\lambda_2}=(2\pi K)^{-1}(\lambda_1-\lambda_2)$$

and, if $\lambda'_0(\lambda_1,f)>\Lambda'_0(\lambda_2,f)$,

$$(2\pi)^{-1}[\lambda'_0(\lambda_1,f)-\Lambda'_0(\lambda_2,f)]=\mathrm{Mod}C'_{\lambda'_0(\lambda_1,f)\Lambda'_0(\lambda_2,f)}\leq$$
$$\leq K\mathrm{Mod}f^{-1}C'_{\lambda'_0(\lambda_1,f)\Lambda'_0(\lambda_2,f)}\leq K\mathrm{Mod}C_{\lambda_1\lambda_2}=(2\pi)^{-1}K(\lambda_1-\lambda_2).$$

Remark 1.1 The image fC_λ of a level line of $G_R(\ ,z_0)$ is included in $\overline{\Pi}_{\Lambda'_0(\lambda,f)\lambda'_0(\lambda,f)}$ so that its distorsion from the level lines of $G_{R'}(\ ,z'_0)$ could be measured by

$\mathrm{Mod}C'_{\Lambda'_0(\lambda,f)\lambda'_0(\lambda,f)}=(2\pi)^{-1}[\Lambda'_0(\lambda,f)-\lambda'_0(\lambda,f)]$.

In the family $\mathcal{F}$ there are K-qc mappings with the property:

$fC_\lambda = C_{\lambda'}$, for $\lambda' = \lambda'(\lambda, f)$. For such a function, if we write $\lambda'_j = \lambda'(\lambda_j, f)$, $j=1,2$, the inequalities (1.3) and (1.4) become

$$K^{-1}\lambda' \leq \lambda \leq K\lambda' \text{ and} \tag{1.3'}$$

$$K^{-1}(\lambda'_1 - \lambda'_2) \leq \lambda_1 - \lambda_2 \leq K(\lambda'_1 - \lambda'_2) . \tag{1.4'}$$

This case implies the equality in some of the inequalities used to prove (1.3) or (1.4). The results in [2] show that equality in the right- (left-) hand side of (1.3) and (1.4) is assured if we add to this property of f the conditions: the dilatation quotient of f is the constant K and the major axes of the characteristic ellipses are orthogonal (or respectively tangent) to the curves C_λ a.e. in R. Then we have e.g.

$$\lambda' = K^{-1}\lambda \qquad (\text{or } \lambda' = K\lambda, \text{ respectively}). \tag{1.3''}$$

If $K=1$, the equality holds in both sides, $\lambda'=\lambda$, and expresses the conformal invariance of the Green function as in the Lindelöf principle.

Remark 1.2. If we denote by $\lambda_0(\lambda', f^{-1}) = \min\{G_R(z, z_0) : z \in f^{-1}C'_{\lambda'}\}$ and by $\Lambda_0(\lambda'_0, f^{-1}) = \max\{G_R(z, z_0) : z \in f^{-1}C'_{\lambda'}\}$,
then we obtain

$$\lambda_0[\Lambda'_0(\lambda, f), f^{-1}] = \lambda = \Lambda_0[\lambda'_0(\lambda)].$$

1.2. Till now we studied the functions $\lambda'_0(\lambda, f)$ and $\Lambda'_0(\lambda, f)$ which correspond to a K-qc mapping $f \in \mathcal{F}$. We now introduce two functions which delimit the distorsion of the Green level lines with respect to the whole family of mappings $\mathcal{F}$. Namely we define

$$\lambda'_0(\lambda) = \inf\{\lambda'_0(\lambda, f) : f \in \mathcal{F}\}$$

and

$$\Lambda'_0(\lambda) = \sup\{\Lambda'_0(\lambda, f) : f \in \mathcal{F}\}.$$

PROPOSITION 1.3. *If R and R' are hyperbolic Riemann surfaces* *of class* R_p, *there exist extremal mappings* $f_{0,\lambda}$ *and* $F_{0,\lambda}\in\mathcal{F}$ *such that* $\lambda_0'(\lambda)=\lambda_0'(\lambda,f_{0,\lambda})$ and $\Lambda_0'(\lambda)=\Lambda_0'(\lambda,F_{0,\lambda})$.

Proof. Let λ be an arbitrary but fixed positive number and $\{f_n\}$ be a sequence in $\mathcal{F}$ such that $\lambda_0'(\lambda,f_n)\to\lambda_0'(\lambda)$. According to the Lemma quoted in Introduction the family $\mathcal{F}$ is normal and closed, such that $\{f_n\}$ contains a subsequence again denoted by $\{f_n\}$ which uniformly converges in the compact subsets of R to a mapping $f_{0,\lambda}\in\mathcal{F}$. Let us choose for any n a point $z_n\in C_\lambda$ such that $G_{R'}(f_n(z_n),z_0')=\lambda_0'(\lambda,f_n)$. As $R\in R_p$, the sequence $\{z_n\}$ contains a convergent subsequence with a limit $z^*\in C_\lambda$. By a new change of notations we may suppose that $\{f_n\}\subset\mathcal{F}$ uniformly converges on the compact subsets of R to $f_{0,\lambda}$, that $G_{R'}(f_n(z_n),z_0')=\lambda_0'(\lambda,f_n)$ and that $z_n\to z^*$. Since $f_n(z_n)\to f_{0,\lambda}(z^*)$, $\lambda_0'(\lambda)=\lim_{n\to\infty}G_{R'}(f_n(z_n),z_0')=$ $=G_{R'}(f_{0,\lambda}(z^*),z_0')\geq\lambda_0'(\lambda,f_{0,\lambda})$. It follows thus by the definition that $\lambda_0'(\lambda)=\lambda_0'(\lambda,f_{0,\lambda})$. The proof for $\Lambda_0'(\lambda)$ is similar.

PROPOSITION 1.4. *The functions* $\lambda_0'(\lambda)$ *and* $\Lambda_0'(\lambda)$ *are strictly increasing. They verify the inequalities*

$$\lambda_0'(\lambda)\leq\lambda'\leq\Lambda_0'(\lambda), \tag{1.5}$$

where $\lambda'=G_{R'}(z',z_0')$ for $z'=f(z)$ and $z\in C_\lambda$,

$$K^{-1}\lambda_0'(\lambda)\leq\lambda\leq K\Lambda_0'(\lambda),\ \text{and if}\ \ \lambda_1>\lambda_2 \tag{1.6}$$

$$K^{-1}[\lambda_0'(\lambda_1)-\Lambda_0'(\lambda_2)]\leq\lambda_1-\lambda_2\leq K[\Lambda_0'(\lambda_1)-\lambda_0'(\lambda_2)]. \tag{1.7}$$

Proof that $\Lambda_0'(\lambda)$ *is strictly increasing.* Suppose that $\lambda_1>\lambda_2$. From the definition of the function $\Lambda_0'(\lambda)$ and since $\Lambda_0'(\lambda,f)$, $f\in\mathcal{F}$, is strictly increasing, we deduce:

$$\Lambda_0'(\lambda_1)\geq\Lambda_0'(\lambda_1,F_{0,\lambda_2})>\Lambda_0'(\lambda_2,F_{0,\lambda_2})=\Lambda_0'(\lambda_2).$$

The proof for $\lambda_0'(\lambda)$ is similar.

Proof of the inequalities. From (1.2) and (1.3) it follows directly (1.5) and (1.6) respectively, by passing to $\inf_{f\in\mathcal{F}}(\sup_{f\in\mathcal{F}})$ in the left-(right-) hand side. Starting from (1.4) one obtains (1.7) by means of the inequalities

$$\Lambda_0'(\lambda_1,f)-\lambda_0'(\lambda_2,f)\le\Lambda_0'(\lambda_1)-\lambda_0'(\lambda_2) \quad \text{and}$$
$$\lambda_0'(\lambda_1,f)-\Lambda_0'(\lambda_2,f)\ge\lambda_0'(\lambda_1)-\Lambda_0'(\lambda_2) \quad \text{respectively.}$$

Remark 1.3. Proposition 1.3 shows that the functions $\lambda_0'(\lambda)$ and $\Lambda_0'(\lambda)$ are finite and Proposition 1.4. permits us to obtain a uniform majorant. If λ_1, $\lambda_2 \in [m,M]$, $\lambda_1 > \lambda_2$, then

$$\Lambda_0'(\lambda_1)-\lambda_0'(\lambda_2)\le\Lambda_0'(M)-\lambda_0'(m) .$$

2. Level lines of the capacity function.

2.1. We now consider two arbitrary Riemann surfaces R and R' of class R_p, and - as in Introduction - the capacity functions $p_\Gamma(\ ,z_0)$ and $p_{\Gamma'}(\ ,z_0')$ of the ideal boundaries Γ of R and Γ' of R' with respect to $z_0\in R$, the parametric disc D and $z_0'\in R',D'$ respectively.

We denote by c_τ the level line $p_\Gamma(z,z_0)=\tau$, where $\tau\in(-\infty,k_\Gamma)$ and by Π_τ the regular region $\{z\in R:\ p_\Gamma(z,z_0)<\tau\}$. For $\tau_1<\tau_2$ let $c_{\tau_1\tau_2}=\{c_\tau\ :\ \tau\in[\tau_1,\tau_2]\}$ and $\Pi_{\tau_1\tau_2}=\Pi_{\tau_2}\setminus\overline{\Pi}_{\tau_1}$. The modulus of $\overline{\Pi}_{\tau_1\tau_2}$ is now given by

$$\operatorname{Mod} C_{\tau_1\tau_2} = \frac{\tau_2-\tau_1}{2\pi}. \tag{2.1}$$

Further we introduce similar notations $c'_{\tau'}, c'_{\tau'_1\tau'_2}, \Pi_{\tau'}, \Pi_{\tau'_1\tau'_2}$ on R', we consider the family $\mathcal{F}$ and we define as in 1.1. the functions

$$\tau'_0(\tau, f) = \min\{\tau' = p_{\Gamma'}(z', z'_0) \ : \ z' \in fc_\tau\}$$

and

$$T'_0(\tau, f) = \max\{\tau' = p_{\Gamma'}(z', z'_0) \ : \ z' \in fc_\tau\}.$$

By the same device as in 1.1 which is now applied to the capacity function instead of the Green function (in the hyperbolic case by using (0.1)) we prove the following results.

PROPOSITION 2.1. *The functions* $\tau'_0(\tau, f)$ *and* $T'_0(\tau, f)$ *are strictly increasing with respect to* τ. *They verify the inequalities*

$$\tau'_0(\tau, f) \le \tau' \le T'_0(\tau, f), \tag{2.2}$$

and in the hyperbolic case

$$K^{-1}[k_{\Gamma'} - T'_0(\tau, f)] \le k_\Gamma - \tau \le K[k_{\Gamma'} - \tau'_0(\tau, f)]. \tag{2.3}$$

PROPOSITION 2.2. *If* $\tau_1 < \tau_2$, *then*

$$K^{-1}[\tau'_0(\tau_2, f) - T'_0(\tau_1, f)] \le \tau_2 - \tau_1 \le K[T'_0(\tau_2, f) - \tau'_0(\tau_1, f)]. \tag{2.4}$$

Remark 2.1. Once again equality in the right-(left-) hand side of (2.4) takes place for a mapping $f \in \mathcal{F}$ with the properties: $fc_\tau = c'_{\tau'}$, for a function $\tau' = \tau'(\tau, f)$ (then $\tau'_0(\tau, f) = T'_0(\tau, f) = \tau'$), the dilatation quotient of f is the constant K and the major axes of the characteristic ellipses are orthogonal (tangent) to the curves c_τ a.e. in R. Then (2.4) becomes $\tau'_2 - \tau'_1 = K^{-1}(\tau_2 - \tau_1)$, (or $= K(\tau_2 - \tau_1)$ respectively). Inequalities similar to (1.3') and (1.4'), and equalities as in Remark 1.2. are valid.

2.2. As in 1.2. if we introduce the functions

$$\tau_0'(\tau)=\inf\{\tau_0'(\tau,f)\ :\ f\in\mathcal{F}\}$$
and
$$T_0'(\tau)=\sup\{T_0'(\tau,f)\ :\ f\in\mathcal{F}\}$$

we obtain

PROPOSITION 2.3. *Let R and R' be Riemann surfaces of class R_p not conformally equivalent to C. There exist mappings $f_{0,\tau}$ and $F_{0,\tau}\in\mathcal{F}$ such that* $\tau_0'(\tau)=\tau_0'(\tau,f_{0,\tau})$ and $T_0'(\tau)=T_0'(\tau,F_{0,\tau})$.

PROPOSITION 2.4. *The functions* $\tau_0'(\tau)$ *and* $T_0'(\tau)$ *are strictly increasing and verify the inequalities*

$$\tau_0'(\tau)\le\tau'\le T_0'(\tau) \tag{2.5}$$

where $\tau'=p_{\Gamma'}(z',z_0')$, $z'=f(z)$ *and* $z\in c_\tau$,

$$K^{-1}[\tau_0'(\tau_2)-T_0'(\tau_1)]\le\tau_2-\tau_1\le K[T_0'(\tau_2)-\tau_0'(\tau_1)] \tag{2.6}$$

for $\tau_1<\tau_2$, *and*

$$K^{-1}[k_{\Gamma'}-T_0'(\tau)]\le k_\Gamma-\tau\le K[k_{\Gamma'}-\tau_0'(\tau)] \tag{2.7}$$

in the hyperbolic case.

Remark 2.2. If $\tau_1,\tau_2\in[m,M]$, $\tau_1<\tau_2$, then

$$T_0'(\tau_2)-\tau_0'(\tau_1)\le T_0'(M)-\tau_0'(m)$$

so that we have again a uniform majorant.

Remark 2.3. The compact Riemann surfaces can be also studied with this method - as parabolic surfaces hence surfaces of class R_p - namely if S and S' are two such surfaces (for Propositions 2.3 and 2.4 not conformally equivalent to C), one deals with $R=S\setminus z_\infty$ and $R'=S'\setminus z_\infty'$ for two arbitrary points $z_\infty\in S$, $z_\infty'\in S'$. The family $\mathcal{F}$ consists now of all the K-qc mappings $f:S\to S'$ with

$f(z_h) = z'_h$,

$h=0,\infty, z_0 \in R$ and $z'_0 \in R'$.

Remark 2.4. These results have been applied in [4] in order to generalize a Gehring's theorem. The paper [4] contains also proofs of Propositions 2.1.-2.4. Let us mention that our main tools - the functions $\lambda'_0(\lambda, f)$, $\Lambda'_0(\lambda, f)$ and $\tau'_0(\tau, f)$, $T'_0(\tau, f)$ - generalize classical functions considered in the plane by different authors and which have various applications. As an example we quote [9] where for the level lines of the capacity function in the plane (the Evans-Selberg potential) with respect to 0, $c_r: |z| = r$, the function $M(r,f) = \max\{|f(z)| : z \in c_r\}$ is used to thoroughly study the growth of the entire quasiregular functions.

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LOCALLY BILIPSCHITZ MAPPINGS AS A SUBCLASS OF QUASICONFORMAL HOMEOMORPHISM IN A NORMED SPACE

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REZUMAT. - Aplicaţii local biLipschitziene ca o subclasă de homeomorfisme cvasiconforme în spaţii normate. În lucrare se dau două caracterizări ale clasei aplicaţiilor local biLipschitziene.

In this paper, I show in a normed space X, the locally biLipschitz mappings $f:D\to D'$ (D,D' domains in X), the local quasi-isometries and a certain subclass of quasiconformal mappings (considered in my paper [2]) and characterized by the quasi-invariance of a certain kind of module $\text{mod}_{\infty}^{D}\Gamma$ of an arc family Γ, coincide. I show also that the distance $d_D(E_0,E_1)$ between two sets E_0,E_1 relatively to a domain D coincide to the extremal length $\lambda_{\infty}^{D}(E_0,E_1,D)$ of the family $\Gamma(E_0,E_1,D)$ of the arcs γ joining E_0 and E_1 in D and defined as the inverse of the module $\text{mod}_{\infty}^{D}\Gamma(E_0,E_1,D)$. This allows us to give another characterization of the subclass from above by means of the quasi-invariance of the relative distance.

Now let Γ be a family of arcs $\gamma\subset D$ (by abuse and for simplicity sake, I shall denote it by $\Gamma\subset D$) and let

$F^{D}(\Gamma)=\{\rho;\rho\geq 0,\rho|_{CD}=0$, bounded and continuous in D and such that

$$\int_{\gamma}\rho\,ds\geq 1\ \forall\gamma\in\Gamma\}\qquad (\ \forall=\text{"for every"})$$

be the corresponding class of admissible functions. Then, we define the module of Γ as

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$$\text{mod}_\infty^D \Gamma = \inf_{\rho \in F^D(\Gamma)} \sup_X \rho(x) = \inf_{\rho \in F^D(\Gamma)} \|\rho\|_\infty$$

We recall that the origin of this concept of module (cf our paper [1]) is the n-dimensional module

$$\text{mod}\Gamma = \inf_{\rho \in F^D(\Gamma)} \left(\int_{\mathbb{R}^n} \rho^n dm\right)^{\frac{1}{n}},$$

taking into account that, for ρ continuous,

$$\lim_{n\to\infty} \left(\int_X \rho^n dm\right)^{\frac{1}{n}} = \sup_X \rho(x) = \|\rho\|_\infty .$$

In this paper, a map f means a homeomorphism $f:D\to D'$, where D,D' domains of the normed space X.

Let us denote by $\Gamma(E_0,E_1,D)$ the family of open arcs $\gamma\subset D$ - an open arc being the homeomorphic image of the linear open interval (0,1) - such that the closure $\bar{\gamma}$ of γ is a homeomorphic image of the closed linear interval [0,1], the endpoints of $\bar{\gamma}$ belonging to E_0 and E_1, respectively.

A homeomorphism f is K-**quasiconformal** if $\forall$ $E_0,E_1\subset D$, the double inequality

$$\frac{\text{mod}_\infty^D(E_0,E_1,D)}{K} \le \text{mod}_\infty^{D'}\Gamma(E_0',E_1',D') \le K\text{mod}_\infty^D\Gamma(E_0,E_1,D) \tag{1}$$

holds, where $E_k' = f(E_k)$ $(k=0,1)$. A **quasiconformal mapping** is a K-quasiconformal one with non specified K. In this paper, by K-quasiconformal mapping, we understand only the mappings of the subclass characterized by (1).

We recall that the relative distance $d_E(E_0,E_1)$ (with respect to a set E) between two sets E_0,E_1 is

$$d_E(E_0,E_1) = \inf_{\gamma\in\Gamma(E_0,E_1,E)} H^1(\gamma),$$

where H^1 is linear Hausdorff measure. If $\Gamma(E_0,E_1,E)=\emptyset$, then, we consider $d_E(E_0,E_1)=\infty$.

PROPOSITION 1. $E_0,E_1\subset\bar{D}$ →

$$\text{mod}_{\infty}^{D}\Gamma(E_0,E_1,D) = \frac{1}{d_D(E_0,E_1)}$$

(P.Caraman [2], lemma 5).

Taking into account that the extremal length $\lambda_{\infty}^{D}\Gamma=\frac{1}{\text{mod}_{\infty}^{D}\Gamma}$, the preceding proposition yields the following

COROLLARY. $d_D(E_0,E_1)=\lambda_{\infty}^{D}\Gamma(E_0,E_1,D)$.

PROPOSITION 2. *f is K-quasiconformal iff* $\forall\ E_0,E_1\subset D$,

$$\frac{d_D(E_0,E_1)}{K} \le d_{D'}(E_0',E_1') \le Kd_D(E_0,E_1). \tag{2}$$

(P.Caraman [2], lemma 7).

COROLLARY. *f is K-quasiconformal iff* $\forall\ E_0,E_1\subset D$,

$$\frac{\lambda_{\infty}^{D}\Gamma(E_0,E_1,D)}{K} \le \lambda_{\infty}^{D'}\Gamma(E_0',E_1',D') \le K\lambda_{\infty}^{D}\Gamma(E_0,E_1,D).$$

A mapping f is said to be a **local** C**-isometry** with $0<C<\infty$ if $\forall\, x\in D$, there exists a neighbourhood $U_x\subset D$ such that

$$\frac{\|y-z\|}{C} \le \|f(y)-f(z)\| \le C\|y-z\|$$

$\forall\ y,z\in U_x$.

THEOREM 1. *f is K-quasiconformal iff it is a local K-isometry.*

Proof. Suppose f is K-quasiconformal and consider an arbitrary point $x\in D$. Next, let $U_x = B(x,r)\subset D$. Then, on account of the preceding proposition in the particular case $E_0=\{y\}$,

$E_1=\{z\}$,

$$\|f(y)-f(z)\| \le d_{D'}[f(y),f(z)] \le Kd_D(y,z) = K\|y-z\| \quad (3)$$

$\forall\ y,z \in U_x$. But also the converse is true. Indeed, let $x' \in D'$ be an arbitrary point and $V_{x'} = B(x',r') \subset D'$. Then, on account of the preceding proposition, in the particular case $E_0=\{y\}, E_1=\{z\}$,

$$\|y-z\| \le d_D(y,z) \le Kd_{D'}(y',z') = K\|y'-z'\| = K\|f(y)-f(z)\|$$

$\forall\ y,z \in V_x = f^{-1}(V_{x'})$. This relation, together with (3), yields

$$\frac{\|y-z\|}{K} \le \|f(y)-f(z)\| \le K\|y-z\|$$

$\forall\ y,z \in U_x \cap f^{-1}(V_{x'})$.

Now, let us prove also the opposite implication. Assume f is a local K-isometry $(1 \le K < \infty)$, $x_0 \in D$, $U_x = B(x_0,r_0) \subset D$. Then,

$$d_D(x,y) = \|x-y\| \le K\|f(x)-f(y)\| \le Kd_{D'}(x',y') \quad (4)$$

$\forall\ x,y \in U_0$. But, also conversely, if $x_0 \in D$ and V_0 is a neighbourhood of x_0 such that $f(V_0) \subset B[f(x_0), r_0'] \subset D'$, then

$$d_{D'}(x',y') = \|f(x)-f(y)\| \le K\|x-y\| \le Kd_D(x,y)$$

$\forall\ x,y \in V_0$, hence and on account of (4), we obtain

$$\frac{d_D(x,y)}{K} \le d_{D'}(x',y') \le Kd_D(x,y) \quad (5)$$

$\forall\ x,y \in W_0 \subset U_0 \cap V_0$.

Next, we observe that there exist two sequences $\{x_n'\} \subset E_0'$, $\{y_n'\} \subset E_1'$ such that

$$d_{D'}(E_0',E_1') = \inf_{\substack{x' \in E_0' \\ y' \in E_1'}} d_{D'}(x',y') = \lim_{n\to\infty} d_{D'}(x_n',y_n') = \lim_{n\to\infty} \inf_{\gamma_n' \in \Gamma(x_n',y_n',D')} H^1(\gamma_n'). \quad (6)$$

And now, $\forall\ \varepsilon>0$ and $\forall\ n \in N$, there is an arc $\gamma_n' \in \Gamma(y_n',z_n',D')$ such that

$$d_D(x_n', y_n') > H^1(\gamma_n') - \varepsilon \; ; \tag{7}$$

but $\overline{\gamma_n} = f^{-1}(\overline{\gamma_n'})$ is compact, hence $d(\overline{\gamma_n}, \partial D) = d_n > 0$. Then, let $x_n^k \in \gamma_n$ such that $x_n^0 = x_n = f^{-1}(x_n'), x_n^p = y_n = f^{-1}(y_n')$ and $d(x_n^k, x_n^{k+1}) < d_n$ $(k = \overline{0, m-1})$. But, on account of (5) and (7),

$$d_D(x_n, y_n) \le \sum_{k=0}^{p} d(x_n^k, x_n^{k+1}) \le K \sum_{k=0}^{p} d_{D'}(x_n'^k, x_n'^{k+1}) \le$$
$$\le K \sum_{k=0}^{p} H^1(\gamma_n'^k) = K H^1(\gamma_n') < K d_{D'}(x', y') + K\varepsilon,$$

where $\gamma_n'^k$ is the subarc of γ_n' joining $x_n'^k$ and $x_n'^{k+1}$, hence, letting $\varepsilon \to 0$, it follows that

$$d_D(x_n, y_n) \le K d_{D'}(x_n', y_n') \; ,$$

whence and since $x_n \in E_0, y_n \in E_1$, we obtain

$$d_D(E_0, E_1) \le d_D(x_n, y_n) \le K d_{D'}(x_n', y_n') \qquad \forall\; n \in N,$$

so that, taking into account (6), we obtain

$$d_D(E_0, E_1) \le K \lim_{n\to\infty} d_{D'}(x_n', y_n') = K d_{D'}(E_0', E_1') \; . \tag{8}$$

In order to establish the opposite inequality, we use a similar argument. We observe first that there exist two sequences $\{x_n\}, \{y_n\}$ such that

$$d_D(E_0, E_1) = \lim_{n\to\infty} d_D(x_n, y_n) = \lim_{n\to\infty} \inf_{\gamma_n \in \Gamma(x_n, y_n, D)} H^1(\gamma_n) \; , \tag{9}$$

hence, $\forall\; n \in N$, and $\varepsilon > 0$, there exist $\gamma_n \in \Gamma(x_n, y_n, D)$ such that $d_D(x_n, y_n) > H^1(\gamma_n) - \varepsilon$. Since $\overline{\gamma_n'} = \overline{f(\gamma_n)}$ is compact, $d(\overline{\gamma_n'}, \partial D') = d_n' > 0$ so that we may choose $x_n'^k \in \gamma_n'$ $(k = 0, 1, \ldots, p)$ so that $x_n'^0 = x_n', x_n'^p = y_n'$ and $d(x_n'^k, x_n'^{k+1}) < d_n'$. But then, taking into account (5), we get

$$d_{D'}(x_n', y_n') \le \sum_{k=0}^{p} d(x_n'^k, x_n'^{k+1}) \le K\sum_{k=0}^{p} d_D(x_n^k, x_n^{k+1}) =$$
$$= K\sum_{k=0}^{p} H^1(\gamma_n^k) = KH^1(\gamma_n) < Kd_D(x_n, y_n) + \varepsilon K$$

and, letting $\varepsilon \to 0$, we deduce that

$$d_{D'}(x_n', y_n') \le Kd_D(x_n, y_n),$$

which, taking into account (9), yields

$$d_{D'}(E_0', E_1') \le \lim_{n\to\infty} d_{D'}(x_n', y_n') \le K\lim_{n\to\infty} d_D(x_n, y_n) = Kd_D(E_0, E_1),$$

which, together with (8), yields (2), implying (by the preceding proposition) the K-quasiconformality of f, as desired.

Arguing as in the preceding theorem, we obtain the

COROLLARRY. *f is K-quasiconformal iff $\forall\ x, y \in D$,*

$$\frac{d_D(x,y)}{K} \le d_{D'}(x', y') \le Kd_D(x,y).$$

A mapping f is said to be **uniformly locally Lipschitz with the constant** $M>0$ if $\forall\ x\in D$, there exists a neighbourhood $U_x \subset D$ such that $\forall\ y, z \in U_x$, $\|f(y)-f(z)\| \le M\|y-z\|$ f is said **uniormly locally biLipschitz with the constant** $M>0$ if f and f^{-1} are uniformly locally Lipschitz with the constant M.

THEOREM 2. *f is K-quasiconformal iff it is uniformly locally bilipschitz with the constant K.*

Proof. If f is K-quasiconformal, then, according to the preceding theorem, f is K-isometry, hence f and f^{-1} are uniformly locally Lipschitz with the constant K. The converse follows by a similar argument.

COROLLARY. *A K-quasiconformal mapping $f: B(x_0, R) \to D'$ is*

Lipschitz with constant K.

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ON A CONJECTURE OF HORN IN COINCIDENCE THEORY

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REZUMAT. - Asupra unei conjecturi a lui Horn în teoria coincidenţei. Conjectura lui Horn afirmă că doi operatori continui şi comutativi ce invariază un compact convex dintr-un spaţiu Banach, au cel puţin un punct de coincidenţă. In presenta lucrare se dau mai multe propoziţii echivalente cu conjectura lui Horn. In finalul lucrării se introduce noţiunea de structură de coincidenţă şi se stabileşte o teoremă generală de coincidenţă.

1. **Introduction.** Horn's conjecture ([1]) states that if two commutative mappings, onto a compact convex subset of a Banach space into it self are continuous, then this pair of mappings has at least a coincidence point. In this paper we present some equivalent statements with the Horn's conjecture.

2. **Measures of noncompactness.** Let X be a Banach space. By a weak measure of noncompactness on X we mean a mapping, $\alpha: P_b(X) \to R_+$, which satisfies the following conditions:

(i) $\alpha(A)=0$ implies $\bar{A} \in P_{cp}(X)$,

(ii) $\alpha(\overline{co}A)=\alpha(A)$, for all $A \in P_b(X)$.

By definition a weak measure of noncompactness is a measure of noncompactness if satisfies the condition

$\bar{A} \in P_{cp}(X)$ implies $\alpha(A)=0$.

For example, α_k (Kuratowski's measure of noncompactness) and α_H (Hausdorff's measure of noncompactness) are measure of noncompactness and δ is a weak measure of noncompactness.

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3. **Invariant subsets.** Let X be a nonempty set and let $f,g:X\to X$ be two mappings. We denote

$I(f):=\{A\subset X \mid A\neq\phi,\ f(A)\subset A\}$,

$I(f,g):=I(f)\cap I(g)$,

$F_f:=\{x\in X \mid f(x)=x\}$,

$C(f,g):=\{x\in X \mid f(x)=g(x)\}$.

We have

LEMMA 1. *Let X be a nonempty set, $\mu:\mathcal{P}(X)\to\mathcal{P}(X)$ a clasure operator, $Y\in F_\mu$ and $f,g:Y\to Y$ such that $f\circ g=g\circ f$. Let $A_1\subset Y$, $A_1\neq\phi$. Then there exists $A_0\subset Y$ such that*

(i) $A_0\supset A_1$,

(ii) $A_0\in F_\mu$,

(iii) $A_0\in I(f,g)$,

(iv) $\mu(f(A_0)\cup g(A_0)\cup A_1)=A_0$.

Proof. Let $\mathcal{B}:=\{B\subset Y \mid B$ satisfies (i)+(ii)+(iii)$\}$. We have $\cap\mathcal{B}\in\mathcal{B}$. Let $A_0:=\cap\mathcal{B}$. We remark that $\mu(f(A_0)\cup g(A_0)\cup A_1)\in\mathcal{B}$ and $\mu(f(A_0)\cup g(A_0)\cup A_1)\subset A_0$. This implies (iv).

4. **α-condensing pair.** Let X be a Banach space, $Y\subset X$ and $f,g:Y\to Y$. Let $\theta:P_b(x)\to R_+$. The pair (f,g) is θ-condensing if

(i) $A\in P_b(Y)$ implies $f(A)$, $g(A)\in P_b(Y)$

(ii) $\theta(f(A)\cup g(A)<\theta(A)$, $\forall A\in I_b(f,g)$, $\theta(A)\neq 0$.

Example 1. Let $Y\in P_b(X)$ and let $f,g:Y\to Y$ be two compact mapping. Then the pair (f,g) is α_k-condensing.

Example 2. Let $Y\in P_b(X)$ and let $f,g:Y\to Y$ be two δ-condensing mapping. In general, the pair (f,g) is not δ-condensing.

Now we consider

Statement $S(\theta)$. Let Y be a bounded closed convex subset of a Banach space X and let $f,g:Y \to Y$ be commuting continuous mappings. If the pair is θ-condensing, then $C(f,g) \neq \phi$.

The main results of this paper is the following:

THEOREM 1. *The following statements are equivalent:*

(i) *(Horn) Let Y be a compact convex subset of X and let $f,g:Y \to Y$ be commuting continuous mappings. Then $C(f,g) \neq \phi$.*

(ii) *Statement $S(\alpha_k)$.*

(iii) *Statement $S(\alpha)$, for an α - a measure of noncompactness on X.*

(iv) *Statement $S(\alpha)$ for all α - measures of noncompactness on X (i.e., $\{S(\alpha) | \alpha \in$ the set of all measure of noncompactness on $X\}$*

(v) *Statement $S(\alpha)$ for all α - weak measures of noncompactness on X (i.e., $\{S(\alpha) | \alpha \in$ the set of all weak measures of noncompactness on $X\}$.*

Proof. The proof follows from the following implications:

$$(v) \to (iv) \begin{matrix} \nearrow (iii) \searrow \\ \searrow (ii) \nearrow \end{matrix} (i) \to (v).$$

We will prove (i) $\to$ (v). Let $A_1 = F_f$ and $\mu(A) = \overline{co}\ A$. By Schauder's fixed point theorem, $F_f \neq \phi$. We have $f(F_f) = F_f$ and $g(F_f) \subset F_f$. By Lemma 1, there exists $A_0 \subset Y$ such that

$$\overline{co}(f(A_0) \cup g(A_0) \cup F_f \supset = A_0.$$

Since, $F_f \in f(A_0) \cup g(A_0)$, hence $\overline{co}(f(A_0) \cup g(A_0)) = A_0$. We have

$$\alpha(\overline{co}(f(A_0) \cup g(A_0)) = \alpha(f(A_0) \cup g(A_0)) = \alpha(A_0)$$

Thus implies that $A_0 \in P_{cp,cv}(X)$.

From (i), we have that, $C(f,g) \neq \phi$.

5. Coincidence property. Let X be a nonempty set and $Y \in P(X)$. We denote by $M(Y)$ the set of all mappings, $f:Y \to Y$. A triple (X,S,M) is a coincidence structure if

(i) $S \subset P(X)$, $S \neq \phi$,

(ii) $M:P(X) \multimap \bigcup_{Y \in P(X)} M(Y)$, $Y \mapsto M(Y) \subset M(Y)$, is a mapping such that, if $Z \subset Y$, $Z \neq \phi$, then $M(Z) \supset \{f|_Z : f \in M(Y) \text{ and } f(Z) \subset Z\}$,

(iii) ($Y \in S$, $f,g \in M(Y)$, $f \circ g = g \circ f$) imply $C(f,g) \neq \phi$.

For example (see [1]), if $X=R$, $S=\{[a,b] \mid a,b \in R\}$ and $M(Y)=C(Y):=\{f:Y \to Y \mid f\text{-continuous}\}$, then the triple (X,S,M) is a coincidence structure.

Let (X,S,M) be a coincidence structure. A pair (θ,μ) is compatible with (X,S,M) if

(i) $\theta:Z \to R_+$, $S \subset Z \subset P(X)$,

(ii) $\mu:P(X) \to P(X)$ is a clasure operator, $S \subset \mu(Z) \subset Z$, and $\theta(\mu(Y))=\theta(Y)$, for all $Y \in Z$,

(iii) $F_\mu \cap Z_\theta \subset S$.

The Theorem 1 suggests us the following very general results

THEOREM 2. *Let (X,S,M) be a coincidence structure and (θ,μ) a compatible pair with (X,S,M). Let $Y \in \mu(Z)$ and $f,g \in M(Y)$ such that $f \circ g = g \circ f$.*

We suppose that

(i) $\theta(f(A) \cup g(A)) < \theta(A)$, *for all* $A \in I(f,g)$, $\alpha(A) \neq 0$;

(ii) $F_f \neq \phi$.

Then

$C(f,g)\neq\phi$.

Proof. Let $A_1=F_f$. From the Lemma 1 there exists $A_0\subset Y$ such that

$\mu(f(A_0)\cup g(A_0)\cup F_f)=A_0$.

since (θ,μ) is a compatible pair with (X,S,M), it follows

$\theta(\mu(f(A_0)\cup g(A_0)\cup F_f))=\theta(A_0)$.

This implies $\theta(A_0)=0$. Thus, $A_0\in F_\mu\cap Z_\theta$.

$S_0,A_0\in S$, i.e., $C(f,g)\neq\phi$.

Remark 1. In the Theorem 2, insted of the condition (ii), we can take the following

(ii') $x\in Y$ $A\in Z$ implies $A\cup\{x\}\in Z$ and $\theta(A\cup\{x\})=\theta(A)$.

Remark 2. For the θ-condensing mappings see: [2] [3].

Remark 3. For the coincidence theory, see [4].

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BIVARIATE BIRKHOFF INTERPOLATION OF SCATTERED DATA

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REZUMAT. Interpolare Birkhoff bidimensională pentru date arbitrare. Se studiază formule de interpolare de tip Birkhoff pentru funcţii de două variabile definite pe un domeniu plan oarecare, obţinute prin generalizarea cazului rectangular.

0. In a previous paper [1] there was considered the following general scattered data interpolation problem (*SDIP*): let f be a real valued function defined on a given domain $D\subset\mathbb{R}^2$, $\Pi=\{D_k\subset D\,|\,k=1,\ldots,N\}$ a given partition of D and L_kf some given informations on the function f at D_k, $k=1,\ldots,N$. Find a function g, from a given set of functions, say A, such that $L_kg=L_kf$, $k=1,\ldots,N$.

Remark 1. The usual informations are the values or some medium values of the function f and of certain of its derivatives $f^{(\mu,\nu)}$, $(\mu,\nu)\in\mathbb{N}^2$.

Remark 2. If $\Pi=\{D_1,\ldots,D_N\}$ is a set of discret points then the (*SDIP*) is a punctual interpolation problem and it is a transfinite interpolation problem otherwise.

Particularly, if L_kf are the Lagrange informations ($L_kf = f(x_k,y_k)$) then the (*SDIP*) take the clasical fashion (the scattered data-fitting problem).

Remark 3. The (*SDIP*) can be also a deterministic or a non-deterministic problem if L_kf are deterministic or non-

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deterministic informations.

DEFINITION 1. The degree of exactness of the interpolation formula defined by the informations $L_k f$, $k=1,\dots,N$, will be called the exactness degree of these informations.

Remark 4. For the bivariate case we cane have the total degree of exactnees and the degree of exactness on regard with each variable.

We remark two ways for solving a (*SDIP*):

1) to generalize the tensor product or the Boolean sum techniques from a regular domain D (rectangle or triangle) to a unusual shape on.

2) to generalize or to modify the Shepard's method.

The goal of this note is to derive some scattered data interpolation formulas using first way and the Birkhoff informations of the function f.

1. For the begining, one supposes that $L_k f = f(x_k, y_k)$, $k=1,\dots,N$.

Now, if the partition Π is

$$\Pi=\{(x_i,y_j)\in D \mid i=0,1,\dots,m;\ j=0,1,\dots,n\}$$

then the solution of the corresponding (*SDIP*) is given by the tensor product of the univariate Lagrange operators L_m^x and L_n^y corresponding to the nodes x_i, $i=0,1,\dots,m$ respectively $y_j, j=0,1,\dots,n$, i.e.

$$(L_m^x \otimes L_n^y)(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{u(x)}{(x-x_i)\,u'(x_i)} \frac{v(y)}{(y-y_j)\,v'(y_j)} f(x_i,y_j)$$

where $u(x) = (x-x_0)\ldots(x-x_m)$; $v(y) = (y-y_0)\ldots(y-y_n)$.

In [5], J.F.Steffensen had given a first generalization of the Lagrange interpolation problem for the partition

$$\Pi = \{(x_i,y_j) \in D \mid i=0,1,\ldots,m;\ j=0,1,\ldots,n_i \text{ and } n_i \in \mathbb{N}\}$$

One obtains

$$(P_1 f)(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n_i} \frac{u(x)}{(x-x_i)\,u'(x_i)} \frac{v_i(y)}{(y-y_j)\,v_i'(y_j)} f(x_i,y_j)$$

where $v_i(y) = (y-y_0)\ldots(y-y_{n_i})$.

In 1957, D.D.Stancu [3] had given a new extension of the Lagrange interpolation problem, that is also a generalization of the Steffensen problem, taking

$$\Pi = \{(x_i,y_{ij}) \in D \mid i=0,1,\ldots,m;\ j=0,1,\ldots,n_i \text{ with } n_i \in \mathbb{N}\}$$

i.e.

$$(P_2 f)(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n_i} \frac{u(x)}{(x-x_i)\,u'(x_i)} \frac{v_i(y)}{(y-y_{ij})\,v_i'(y_{ij})} f(x_i,y_{ij})$$

with $v_i(y) = (y-y_{i,0})\ldots(y-y_{i,n_i})$.

We note that in both generalization are given expressions for the error functions $(f-P_i f, i=1,2)$ in terms of divided differences.

We, also, remark that $P_2 f$ is a solution for the clasical (*SDIP*), i.e. a Lagrange's scattered data interpolation polynomial.

Next, we consider the *(SDIP)* with the punctual Birkhoff's type informations on f.

2. Let $M=\{(x_k,y_k),\ k=1,\dots,N\}$ be a given set of points in D.

Following [1], on considers the partition M_i $i=0,1,\dots,p$ of the set M, where M_i is the set of all points $(x_k,y_k)\in M$ with $x_k=x_i$, $k=0,1,\dots,q_i$ and $x_i\neq x_j$ for $i\neq j$, i.e. $M_i=\{(x_i,y_{ij})\,|\,j=0,1,\dots,q_i\}$ for $i=0,1,\dots,p$.

Let $L^{\mu,\nu}_{i,i_j}=f^{(\mu,\nu)}(x_i,y_{ij}),\ j=0,1,\dots,q_i;\ i=0,1,\dots,p$ and $(\mu,\nu)\in I_i\times J_{ij}$ with $I_i,J_{ij}\subset\mathbb{N}$, be informations of the Birkhoff type of the function f, while $L^{\mu}_{i}f=f^{(\mu,0)}(x_i,.)$ respectively $L^{\nu}_{j}f=f^{(0,\nu)}(.,y_j)$ will be considered as partial informations of the function f on regard to x and y.

2.1. For the begining one considers the rectangular case, i.e. $D=[x_0,x_p]\times[y_0,y_q]$, $M=\{x_0,\dots,x_p\}\times\{y_0,\dots,y_q\}$ and $I_i,J_j\subset\mathbb{N}$, with $|I_0|+\dots+|I_p|=m+1$, $|J_0|+\dots+|J_q|=n+1$. If B^x_m and B^y_n are the Birkhoff's interpolation operators corresponding to the partial informations $L^{\mu}_{i}f=f^{(\mu,0)}(x_i,.)$, $i=0,1,\dots,p;\ \mu\in I_i$ respectively $L^{\nu}_{j}f=f^{(0,\nu)}(.,y_j),\ j=0,1,\dots,q;\ \nu\in J_j$ then the well known bivariate interpolation formula is

$$f=B^x_m\otimes B^y_n f+R^x_m\oplus R^y_n f\ , \tag{1}$$

where R^x_m and R^y_n are the corresponding remainder operators. More precisely

$$(B^x_m\otimes B^y_n f)(x,y)=\sum_{i=0}^{p}\sum_{j=0}^{q}\sum_{\mu\in I_i}\sum_{\nu\in J_j}b_{i\mu}(x)\,b_{j\nu}(y)\,f^{(\mu,\nu)}(x_i,y_j)$$

and for $f\in C^{m+1,n+1}(D)$,

$$(R_m^x \oplus R_n^y f)(x,y) = \int_{x_0}^{x_p} \phi_m(x,s) f^{(m+1,0)}(s,y)\,ds +$$

$$+\int_{y_0}^{y_q} \psi_n(y,t) f^{(0,n+1)}(x,t)\,dt - \iint_D \phi_m(x,s)\psi_n(y,t) f^{m+1,n+1}(s,t)\,dsdt$$

where $b_{i\mu}$ and $b_{j\nu}$ are the fundamental interpolation polinomials, while ϕ_m and ψ_n are the Peano's kernels.

Remarks 5. It is obviously that the degree of exactness of the formula (1) is (m,n) (m on regard to x and n on regard to y).

2.2. One considers now the general case.

So, $M=M_0\cup\ldots\cup M_p$ with $M_i=\{(x_i,y_{ij})\,|\,j=0,1,\ldots,q_i\}$. Let B_m^x be the same operator that interpolate the data $f^{(\mu,0)}(x_i,.)$ for $i=0,1,\ldots,p$ and $\mu\in I_i$ with $|I_0|+\ldots+|I_p|=m+1$. Using this operator we obtain, in a first level of interpolation, the formula

$$f = B_m^x f + R_m^x f \tag{2}$$

where

$$(B_m^x f)(x,y) = \sum_{i=0}^{p}\sum_{\mu\in I_i} b_{i\mu}(x) f^{(\mu,0)}(x_i,y)$$

Now, let $B_{n_i}^y$ be the Birkhoff's operators which interpolate, respectively the data $f^{(\mu,\nu)}(x_i,y_{ij}),\ j=0,1,\ldots,q_i$ and $\nu\in J_{ij}$ with $|J_{i,0}|+\ldots+|J_{i,q_i}|=n_i+1$ and $R_{n_i}^y$ the corresponding remainder operators, for all $i=0,1,\ldots,p$ and $\mu\in I_i$. Applying these operator, from (2) one obtains, in a second level of interpolation, the final scattered data interpolation formula

$$f(x,y)=\sum_{i=0}^{p}\sum_{j=0}^{q_i}\sum_{\mu\in I_i}\sum_{\nu\in J_{ij}} b_{i\mu}(x)\,b_{ij\nu}(y)\,f^{(\mu,\nu)}(x_i,y_{ij})+(Rf)(x,y) \qquad (3)$$

with

$$(Rf)(x,y)=(R_m^x f)(x,y)\ +\sum_{i=0}^{p}\sum_{\mu\in I_i} b_{i\mu}(x)\,(R_{n_i}^y f)(x_i,y)\,.$$

PROPOSITION 1. *The degree of exactness of the formula* (3) *is* (m,r), *where* $r=\min\{n_0,\dots,n_p\}$.

The proof is a consequence of the theorem 1 from [1].

From the Peano's kernel theorem we also have:

PROPOSITION 2. *If* $f(\cdot,y)\in H^{m+1}[x_0,x_p]$ *and* $f^{(\mu,n_i+1)}(x_i,\cdot)\in H^{n_i+1}[y_{i_0},y_{i,n_i}]$ *for all* $i=0,1,\dots,p$, *then*

$$(Rf)(x,y)=\int_{x_0}^{x_p}\phi_m(x,s)\,f^{(m+1,0)}(s,y)\,ds+$$
$$\sum_{i=0}^{p}\sum_{\mu\in I_i} b_{i\mu}(x)\int_{y_{i_0}}^{y_{i,n_i}}\phi_{n_i}(y,t)\,f^{(\mu,n_i+1)}(x_i,t)\,dt$$

where

$$\phi_m(x,s)=\frac{(x-s)_+^m}{m!}-\sum_{i=0}^{p}\sum_{\mu\in I_i} b_{i\mu}(x)\,\frac{(x_i-s)_+^{m-\mu}}{(m-\mu)!}$$

and

$$\phi_{n_i}(y,t)=\frac{(y-t)_+^{n_i}}{n_i!}-\sum_{j=0}^{q_i}\sum_{\nu\in J_{ij}} b_{ij\nu}(y)\,\frac{(y_{ij}-t)_+^{n_i-\nu}}{(n_i-\nu)!}$$

Remark 6. From the first proposition it follows that the best case, from the degree of exactness point of view, is obtained for $n_0=n_1=\dots=n_p$. In this case (3) is a homogeneous interpolation formula on regard to the variable y [1]. But, the structure of the interpolation formula depend on the given informations. So, if the initial informations do not permit to construct a homogeneous formula (there exists $i,j\in\{0,1,\ldots,p\}$, $i\neq j$ such that $n_i\neq n_j$) then there exist two posibilities: to generate new informations on f or to try to interpolate the function f first on regard with the variable y and than on regard with x. Anyhow, an interpolation formula as closed as possible of a homogeneous one is recomandable.

Summarizing the given procedure we have:

1. Input data:

$$M_i=\{(x_i,y_{ij})\mid j=0,1,\ldots,q_i\}\ ,i=0,1,\ldots,p;$$
$$I_i,J_{ij},;\ j=0,1,\ldots,q_i;i=0,1,\ldots,p;$$
$$f^{(\mu,\nu)}(x_i,y_{ij}),\ \mu\in I_i,\nu\in J_{ij},\ j=0,1,\ldots,q_i;i=0,1,\ldots,p.$$

2. One determines the fundamental interpolation polynomials $b_{i\mu}$ and b_{ij}. solving the linear algebraic systems:

$$b_{kj}^{(r)}(x_\nu) = 0,\ \nu\neq k,\ r\in I_\nu,$$
$$b_{kj}^{(r)}(x_k) = \delta_{jr},\ r\in I_k,$$
$$\text{for } j\in I_k \text{ and } k,\mu=0,1,\ldots,p,$$

respectively

$$b_{ik\mu}^{(s)}(y_{ir}) = 0,\ r\neq k,\ s\in I_{ir},$$
$$b_{ik\mu}^{(s)}(y_{ik}) = \delta_{\mu s},\ s\in I_{ik},$$
$$\text{for } \mu\in I_k;\ k,r=0,1,\ldots,q_i;$$
$$\text{for all } i=0,1,\ldots,p.$$

3. Compute $F(x,y)$;

$$F(x,y) = \sum_{i=0}^{p} \sum_{j=0}^{q_i} \sum_{\mu\in I_i} \sum_{\nu\in J_{ij}} b_{i\mu}(x)\, b_{ij\nu}(y)\, f^{(\mu,\nu)}(x_i, y_{ij}) \tag{4}$$

EXAMPLE. The test function is

$$f(x,y) = \frac{1}{x^2+y^2+1},$$

with the graph in fig.1. The input data are:

$M_0=\{(-1,-1);(-1,0);(-1,1)\}$;

$M_1=\{(-1/2,0)\}$;

$M_2=\{(0,-1);(0,0);(0,1)\}$;

$M_3=\{(1/2,0)\}$;

$M_4=\{(1,-1);(1,0);(1,1)\}$.

$I_0=I_1=I_2=I_3=I_4=\{0\}$;

$J_{00}=\{1\}$; $J_{01}=\{0\}$; $J_{02}=\{1\}$;

$J_{10}=\{0,1,2\}$;

$J_{20}=\{1\}$; $J_{21}=\{0\}$; $J_{22}=\{1\}$;

$J_{30}=\{0,1,2\}$;

$J_{40}=\{1\}$; $J_{41}=\{0\}$; $J_{42}=\{1\}$.

So, it is used a Lagrange's interpolation with regard to x and a Birkhoff's interpolation with regard to y.

The graph of the interpolating surfaces computed by (4) is in fig.2.

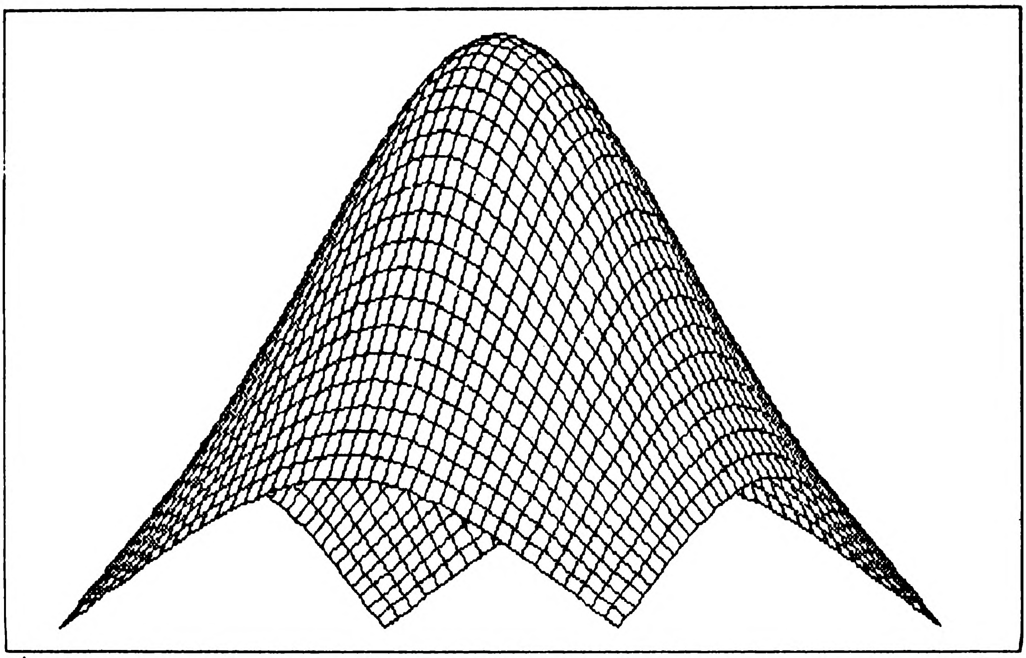

Fig.1.

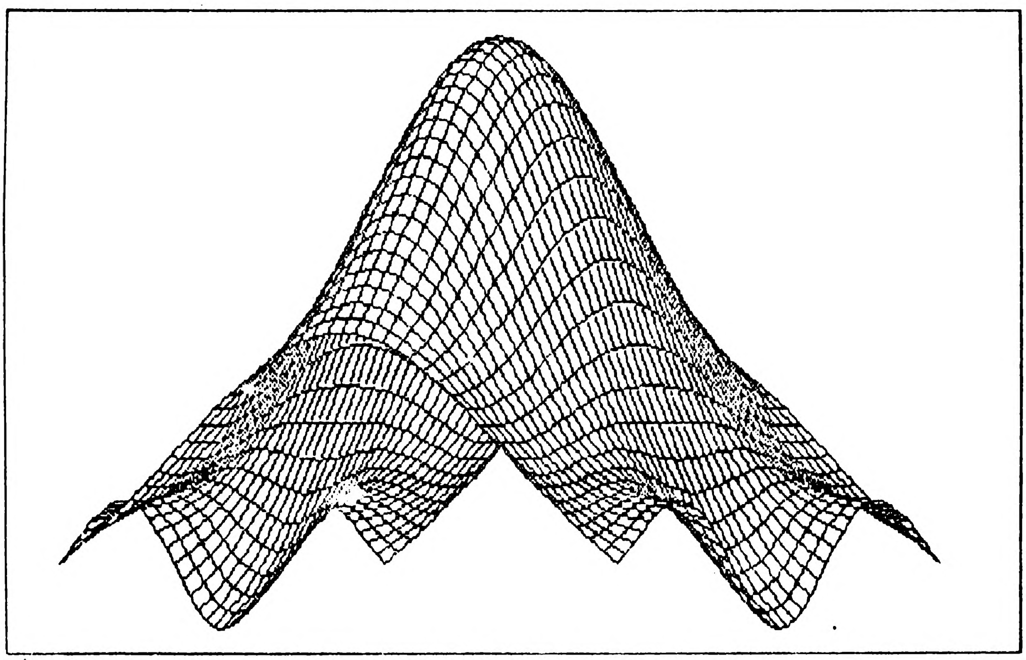

Fig.2.

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ON THE INITIALLY CIRCULAR MOTION AROUND A ROTATION ELLIPSOID

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Dedicated to Professor P.T.Mocanu at his 60^{th} anniversary

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REZUMAT. - Asupra mişcării circulare în jurul unui elipsoid de rotaţie. Se studiază mişcarea iniţial circulară a unei particule test în cîmpul gravitaţional necentral al unui elipsoid de rotaţie. Se stabileşte o formulă analitică pentru perioada mişcării, cu o precizie de ordinul al doilea în raport cu parametrul caracterizînd turtire elipsoidului, generalizîndu-se astfel rezultate anterioare (ale altor autori şi proprii).

1. Introduction. Consider a point mass orbiting an attracting body (under the only gravitational influence of this one) at a distance r. We shall describe the relative motion of the point mass with respect to a Cartesian right-handed frame originated in the mass centre of the attracting body by means of the Keplerian orbital elements $\{ y \in Y ; u \}$, all time-dependent, where:

$$Y = \{p,\ q = e \cos\omega,\ k = e \sin\omega,\ \Omega,\ i\}, \tag{1}$$

and p = semilatus rectum, e = eccentricity, ω = argument of pericentre, Ω = longitude of the ascending node, i = inclination, u = argument of latitude.

Many authors studied such a motion (for a brief survey see e.g. [2]) with very various hypotheses. First and (sometimes) second order perturbations of the orbital parameters were analytically estimated, as well as first order perturbations of the nodal or anomalistic period [1,2,6,7]. We must emphasize the

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fact that the anomalistic period cannot be used to the case of very low eccentric (and especially circular) orbits; that is why we use in this paper the nodal period. Also, as far as we know, nobody determined second order perturbations of the nodal period for a specified perturbing factor.

We shall estimate analytically the nodal period, with a second order accuracy in respect of a small parameter σ on which the perturbing factor is depending, in the following hypotheses:

(i) The attracting body is a rotation ellipsoid with a corresponding mass distribution.

(ii) The initial orbit of the point mass is circular.

(iii) The initial orbital elements are considered in the ascending node of the orbit.

2. **Equations of motion.** Considering hypothesis (i), let us choose the Cartesian right-handed frame mentioned in Section 1 as follows: The basic plane is the equatorial plane of the ellipsoid, while the third axis (normal to this plane) is the rotation axis. Since we study the nodal period, we describe the perturbed motion with respect to this frame by means of the Newton- Euler system written in the form (e.g. [3,5]):

$$
\begin{aligned}
dp/du &= 2(Z/\mu)r^3T,\\
dq/du &= (Z/\mu)(r^3kBCW/(pD) + r^2T(r(q + A)/p + A) + r^2BS),\\
dk/du &= (Z/\mu)(-r^3qBCW/(pD) + r^2T(r(k + B)/p + B) - r^2AS),\\
d\Omega/du &= (Z/\mu)r^3BW/(pD), \qquad (2)\\
di/du &= (Z/\mu)r^3AW/p,\\
dt/du &= (Zr^2(\mu p)^{-1/2},
\end{aligned}
$$

where μ=gravitational parameter of the dynamic system, $A=\cos u$, $B = \sin u$, $C = \cos i$, $D = \sin i$, $Z = (1 - r^2 C\dot{\Omega}/(\mu p)^{1/2})^{-1}$, while S, T, W stand respectively for the radial, transverse, and binormal components of the perturbing acceleration.

For the needs of Section 4, it is to be specified that we consider, as usually, that the elements (1) have small variations over one revolution, such that they may be taken as constant and equal to $y_0 = y(u_0) = y(u(t_0))$, $y \in Y$, in the right-hand side of equations (2), and these ones can be separately considered. So, we can write $y = y_0 + \Delta y$, where, according to hypothesis (iii):

$$\Delta y = \int_0^u (dy/du)\, du, \quad y \in Y. \tag{3}$$

These integrals are estimated from (2) by successive approximations, with $Z \approx 1$, limiting the process to the first order approximation.

In what follows, for simplicity, we shall no longer use the subscript "0" to mark the initial values of elements (1) and of functions of them. In fact, every quantity which does not depend on u (explicitly or through A, B) will be considered constant over one revolution.

3. Perturbing acceleration. Since the gravitational field generated by the attracting body is not Newtonian, the point mass will undergo a perturbing acceleration. Having in view the hypothesis (i), the components of this acceleration are [2,7]:

$$S = -(3/2)c_{20}\mu R^2 r^{-4}(3D^2B^2 - 1),$$
$$T = 3c_{20}\mu R^2 r^{-4} D^2 AB, \tag{4}$$

$$W = 3c_{20}\mu R^2 r^{-4} CDB,$$

where R = equatorial radius of the ellipsoid, while c_{20} is a small parameter featuring the oblateness.

4. Variations of orbital elements. Firstly remind the orbit equation in polar coordinates: $r = p / (1 + e \cos v)$, where v = true anomaly, or:

$$r = p / (1 + qA + kB). \tag{5}$$

Replacing (4) and (5) in (2), taking into account hypothesis (ii), in other words $q(0) = k(0) = 0$, then performing integrals (3) as we showed in Section 2, we obtain:

$$\begin{aligned}
\Delta p &= 3c_{20}\ (R/p)^2 pD^2B^2,\\
\Delta q &= (c_{20}/2)(R/p)^2(7D^2AB^2 + (2C^2 + 1)(1 - A)),\\
\Delta k &= (c_{20}/2)(R/p)^2(7D^2B^3 - 3B),\\
\Delta\Omega &= (3c_{20}/2)(R/p)^2C(u - AB),\\
\Delta i &= (3c_{20}/2)(R/p)^2CDB^2.
\end{aligned} \tag{6}$$

5. Nodal period. As we showed in [4], the nodal period can be written as:

$$T_N = T_0 + \Delta_1 T_N + \Delta_2 T_N, \tag{7}$$

where T_0 is the Keplerian period for $u = 0$; with hypothesis (ii):

$$T_0 = 2\pi p^{3/2}\mu^{-1/2}. \tag{8}$$

The first order (in σ) perturbation is [4]:

$$\Delta_1 T_N = p^{3/2}\mu^{-1/2}\int_0^{2\pi}[\ -2(J_q + J_k) + (3/2)p^{-1}J_p + P^2\mu^{-1}J_e]\ du, \tag{9}$$

where, with hypothesis (ii):

$$J_p = \Delta p, \quad J_q = A\Delta q, \quad J_k = B\Delta k, \quad J_\sigma = B\sigma(CW/D)_\sigma. \qquad (10)$$

As to the second order (in σ) perturbation, this one has the expression, according to [4]:

$$\Delta_2 T_N = p^{3/2}\mu^{-1/2}\int_0^{2\pi} [3(J_{qq} + J_{kk} + 2J_{qk}) - 3p^{-1}(J_{pq} + J_{pk}) + + (3/8)p^{-2}J_{pp} + (7/2)p\mu^{-1}J_{p\sigma} + + p^2\mu^{-1}\cdot(-5J_{q\sigma} - 5J_{k\sigma} + J_{\Omega\sigma} + J_{i\sigma}) + + (1/2)p^4\mu^{-2}J_{\sigma\sigma}]\,du, \qquad (11)$$

where, with hypothesis (ii):

$$J_{xy} = J_x J_y, \quad x \in \{p,q,k\}, \quad y \in \{p,q,k,\sigma\}, \qquad (12)$$

$$J_{\Omega\sigma} = \Delta\Omega(J_\sigma)_\Omega, \quad J_{i\sigma} = \Delta i(J_\sigma)_i, \quad J_{\sigma\sigma} = B^2\sigma^2(C^2W^2/D^2)_{\sigma\sigma}. \qquad (13)$$

We must emphasize the fact that the subscript σ in the right-hand side of the last formula (10), and the subscripts Ω, i, and $\sigma\sigma$ in the right-hand sides of (13) mark the respective partial derivatives. As to the subscripts added to J in (9) - (13), they are simple identifying notations.

6. Results. Substituting W from (4) in the last formula (10) and calculating the required partial derivative (the part of σ is played by c_{20}), then substituting (6) in (10) and the results in (9), and finally performing the integral (9), we obtain:

$$\Delta_1 T_N = 3\pi c_{20} R^2 p^{-1/2}\mu^{-1/2}(3 - 5D^2/2). \qquad (14)$$

Analogously, replacing W in the last formula (13) and calculating the partial derivative ($\sigma = c_{20}$, too), then introducing (6) and the previously calculated (10) in (12) -

(13), substituting the results in (11) and performing the integral, we obtain:

$$\Delta_2 T_N = (\pi/32) c_{20}^2 R^4 p^{-5/2} \mu^{-1/2} (1527D^4 - 3180D^2 + 1620). \quad (15)$$

We must mention that (14) confirms the results of [2,7], while the result (15) is entirely new. Moreover, this result constitutes a first application of our formulae given in [4] to the case of a concrete perturbation.

With (8), (14), and (15), the nodal period (7) can be written as:

$$T_N = T_0(1 + Kf_1(D) + K^2 f_2(D)), \quad (16)$$

where $K = c_{20}(R/p)^2$, and:

$$f_1(D) = (18-15D^2)/4, \quad f_2(D) = (1527D^4-3180D^2+1620)/64. \quad (17)$$

This new, better approximation for the real (perturbed) nodal period could be very useful in the case in which the ellipsoid is strongly oblate and the point mass orbits in its immediate neighbourhood. According to K and to the orbital inclination, the contribution of f_2 (which can act as f_1 or inversely) in altering the period could be sensible.

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ON A COMPLEX BOUNDARY ELEMENT METHOD FOR THE "WALL EFFECT"

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Dedicated to Professor P.T.Mocanu at his 60[th] anniversary

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REZUMAT. - Asupra unei metode de element pe frontieră cu valori compexe pentru "efectul de perete". Prima parte a lucrării conţine o trecere în revistă a unor consideraţii matematice legate de mişcaroa fluidă generată de deplasarea unui profil în prezenţa unui perete nelimitat, rezultate ale autorului care au fost deja prezentate pe larg în [2]. Partea a doua dezvoltă o metodă de element pe frontieră cu valori complexe (CVBEM), pentru care se stabilesc o schemă de utilizare în problema propusă ca şi un rezultat final de convergenţă.

1. Let us consider as given a plane incompressible, potential, inviscid fluid "basic" flow of complex velocity $w_B(z)$. This fluid flow could have some singularities, too, takes place in the presence of an unlimited fixed wall δ.

Let now the plane fluid flow, produced by a general displacement (rototranslation) in the mass of an arbitrary profile (C), in the presence of the same wall δ and which superposes over the basic flow. We assume that during its displacement the profile (C) doesn't cross the singularities of the given basic flow.

A general method to determine the fluid flow which results from the mentioned superposition, method establishing also the existence and the uniqueness of the solution of the joined mathematical model, has been already developed by us [2]. In what follows we intend to make a sketch of a complex variable boundary element method (CVBEM) which could easily be used for the studied

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problem and whose convergence has been already established in the case of the unbounded flow [3, 4].

Concerning the unlimited wall δ and the contour C, we suppose that their parametrical equations $z=\alpha(\varphi)$ and respectively $z=\beta(\psi)$, defined for $\varphi,\psi\in E_1$ versus a fixed, rectangular, Cartesian system of axes, are 2π periodical functions on the interval $(0,2\pi)$, with $\alpha(0) = \infty$ and $\beta(0)$ taking a finite value, which define Jordan positively oriented curves with continuous curvature[1].

In what concerns the given function $w_B(z)$, it belongs to a class (a) of functions having the properties [2]:

1a) They are holomorphic functions in the domain D, bounded by the wall δ, except a finite number of points $\{z_k\}_{k=\overline{1,q}}$ placed at a finite distance, and which represent singular points of these functions; let $D_1^*=D_1\setminus\{z_k\}_{k=\overline{1,q}}$;

2a) They are continuous bounded functions in $V(\infty)$; let $\lim_{|z|\to\infty} w_B(z) = w_B(\infty)$;

3a) They are hölderian functions in the points of $\delta\setminus\{\infty\}$ satisfying also the following boundary condition:

$\exists\ V_B:(0,2\pi)\to\mathbf{R}$ so that $\overline{w_B(\alpha(\varphi))}=V_B(\varphi)\dot{\alpha}(\varphi)/|\dot{\alpha}(\varphi)|$, $\forall\varphi\in(0,2\pi)$

With regard to the unknown function $w(z)$, the complex velocity of the fluid resulting by the considered superposition, must be determined among the functions of class (b), i.e.:

1b) They are holomorphic functions in the domain $D=D_1\setminus\{\overline{\mathrm{Int}\,C}\}$

[1] This last condition is equivalent to the assumption that the functions $1/(\alpha(\varphi)-z_0)$ (where z_0 is a point placed on the "right" side of δ) and $\beta(\psi)$ are from $C^2[0,2\pi)$ having also a nonvanishing first derivative.

except the same points $\{z_k\}_{k=\overline{1,q}}$ which are singular points of the same nature as for $w_B(z)$;

2b) They are continuous bounded functions in $V(\infty)$ where they have an identical behaviour with $w_B(z)$ and consequently

$$\lim_{|z|\to\infty} w(z) = w_\infty \equiv w_B(\infty);$$

3b) They are holomorphic functions in the points of $C \cup \delta\setminus\{\infty\}$ where these functions also satisfy the following boundary conditions:

$$\exists\, V_1:(0,2\pi) \to \mathbf{R} \text{ so that } \overline{w(\alpha(\varphi))} = V_1(\varphi)\dot{\alpha}(\varphi)/|\dot{\alpha}(\varphi)|, \quad \forall\varphi\in(0,2\pi);$$

$$\exists\, V_2:[0,2\pi) \to \mathbf{R} \text{ so that } \overline{w(\beta(\psi))} = V_2(\psi)\dot{\beta}(\psi)/|\dot{\beta}(\varphi)| + 1 + im + i\omega(\beta(\psi) - z_A)$$

$$\forall\psi\in[0,2\pi),$$

where $(1, m, \omega)$ are the given functions of time corresponding to the components of the rototranslation of the profile (C) evaluated in the point $z_A \in \{\text{Int } C\}$;

4b) They satisfy the equality:

$$\int_C w(z)\, dz = \Gamma,$$

where Γ is an "a priori" given constant (circulation).

2. As a consequence of the requirements imposed on the functions $w_B(z)$ and $w(z)$ we remark that the function $g(z) \equiv w(z) - w_B(z)$, known together with $w(z)$, is:

- holomorphic in the fluid flow domain D which also contains the points $\{z_k\}_{k=\overline{1,q}}$;

- continuous and bounded in $\bar{D}$ (the point of infinity included, where $\lim_{|z|\to\infty} g(z) = 0$);

- hölderian on $C \cup \delta\setminus\{\infty\}$;

- satisfying the condition $|z^{\tau}g(z)|_{\delta} < A$, where (A,τ) is a suitable pair of real numbers.

The last condition will ensure the existence (in the Cauchy sense) of the integral taken on the unlimited curve δ [2].

Let us consider now Cauchy's formula for the function $g(z)$ and the domain D. According to the behaviour at far field of this function, we can write [1]:

$$g(z) = -\frac{1}{2\pi i}\int_C \frac{g(\zeta)}{\zeta - z}\,d\zeta + \frac{1}{2\pi i}\int_\delta \frac{g(\tau)}{\tau - z}\,d\tau \ .$$

This formula, which is in fact the integral representation associated with the proposed boundary problem, allows us to determine $g(z)$ (i.e. $w(z)$) once found out its values on the boundary $C \cup \delta$. But the determination of these values through a classical BEM requires the construction and, obviously, the solution of an integral equation on boundary, which could be obtained, for instance, making $z \to \zeta_0 \in C$ and $z \to \tau_0 \in \delta$, respectively.

In what follows we shall succeed to avoid the construction and solution of the boundary integral equations mentioned above, which means a serious and essential step in simplifying all algorithms. The technique used and described by us in the case of the unbounded fluid [3, 4] represents a so-called "improved" CVBEM.

Let d and d' be two divisions of the curves C and δ, consisting of the nodes $z_0, z_1, \ldots, z_n$ $(z_0 = z_n)$ on C (counterclockwise oriented) and, respectively, $z_1', z_2', \ldots, z_n'$ on δ (clockwise oriented). We denote by C_j $(j=1,\ldots,n)$ the corresponding boundary elements (arcs) on C, and by $C_{-\infty}', C_j'$

($j=1,\ldots,n$), C'_{*} the boundary elements on δ. Let us consider now the approximations $\mathcal{g}_d(\zeta)$ in the points of C, and, respectively, $\mathcal{g}_{d'}(\tau)$ in the points of $\delta\setminus\{C'_{*}\cup C'_{*}\}$ of the function $g(z)$, where $\mathcal{g}_d(\zeta)$ and $\mathcal{g}_{d'}(\tau)$ are suitable interpolating spline functions related to the divisions d and d' accordingly.

At once, for every $z \in D$, we have the approximation $g^*(z)$ of $g(z)$, i.e.

$$g^*(z)=-\frac{1}{2\pi i}\int_C\frac{\mathcal{g}_d(\zeta)}{\zeta-z}d\zeta+\frac{1}{2\pi i}\int_{[z'_1,z'_n]}\frac{\mathcal{g}_{d'}(\tau)}{\tau-z}d\tau\ .$$

where the right side could be calculated explicitly sometimes even precisely [3, 4]. Accepting then the existence of

$\lim\limits_{z\to z_k} g^*(z)\,(=g^*(z_k))$ and $\lim\limits_{z\to z'_k} g^*(z)\,(=g^*(z'_k))$,

by separating the real and imaginary parts of the approximate equalities

$g_k=g(z_k)\approx g^*(z_k)$ and $g'_k=g(z'_k)\approx g^*(z'_k)$,

we finally get an agebraic system in the unknowns u_k, v_k, u'_k, v'_k, i.e. the real and imaginary parts of g_k and g'_k. Obviously, while solving this system we should take into account the data connected with the values of $g(z)$ on $C \cup \delta$ (in fact it is a boundary value problem of Hilbert type for $g(z)$) and, of course, the circulation given "a priori"[2]. Once solved this system, via the already written Cauchy formula, one gets the approximate solution $g^*(z)$ valid in all the points of the flow domain.

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[2] In the particular case of a "piecewise" Lagrange interpolating system, which is essentially an approximation by spline functions of first order, the algebraic system becomes linear and it has a unique solution, due to the "a priori" given circulation.

In respect of the convergence of the method, if we admit the "acceptability" of the divisions d and d' (i.e. for every $t \in C_j$ or $t \in C_j'$ there exists $\max\{|t-z_j|, |t-z_{j-1}|\} < |z_j - z_{j-1}|, j=1,\dots,n$), the uniform continuousness of the approximation $\mathcal{g}$ in the points of $C \cup [z_1', z_n']$ allows to prove, like in the case of the unbounded flow [3, 4], the following final result:

THEOREM. *For every point $z \in D$,*

$$\lim_{\substack{n\to\infty \\ (\delta,\delta'\to 0)}} g^*(z) = g(z) ,$$

where δ and δ' are the norms of the acceptable divisions d and d', respectively.

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A N I V E R S Ă R I

PROFESSOR PETRU T. MOCANU AT HIS 60th ANNIVERSARY

Professor Petru T. Mocanu was born in June 1st 1931 in Brăila, Romania. He attended primary and secondary school in Brăila, then universitary studies (1950-1953) and higher studies (1953-1957) at the Faculty of Mathematics, University of Cluj. In 1959 he defended his doctoral thesis "Variational methods in the theory of univalent functions" (under the supervision of the great romanian mathematician G. Călugăreanu). He worked at the University of Cluj ("Babeş-Bolyai University) as assistant professor (1953-1957), lecturer (1957-1962), associate professor (1962-1966 and 1967-1970), full professor (since 1970) and he was visiting professor at Conakry (1966-1967). He has tought the basis course of Complex Analysis and many special cources (Univalent Functions, Measure Theory, Hardy Spaces etc.).

Professor P.T.Mocanu obtained scientific results in the following domains (see "List of publications, Scientific papers"): **extremal problems in the theory of univalent functions** [1-5, 8, 9, 12, 13, 15, 17, 18, 20-22, 24, 25, 27, 37, 42, 47, 72, 91], **new classes of univalent functions** [7, 10, 14-16, 19, 23, 26-32, 34-36, 38, 40, 41, 43, 46, 65, 70, 95, 99, 102, 103, 109], **integral operators on classes of univalent functions** [44, 45, 48, 55-58, 62-64, 66, 67, 69-71, 73, 76, 78, 80-83, 88-90, 92-94, 96-98, 100, 102, 103, 106, 110, 111], **differential subordinations** [49, 50, 52, 53, 61, 64, 68, 74, 77, 79, 85-87, 101, 104, 105, 107], **extentions of certain geometric conditions for injectivity to the case of nonanalytic functions** [51-54, 59, 60, 75, 95]. Some of this results are cited by about 150 mathematicians in more then 300 papers.

Professor P.T.Mocanu was appointed dean of the Faculty of Mathematics (1968-1976 and 1984-1987), head of the chair of Functions Theory (1976-1984 and since 1990) and he is vice-rector of Babeş-Bolyai University since 1990. He is also editor of Mathematica (Cluj) and member of the editorial board of Studia Univ. Babeş-Bolyai, Bulletin de Mathématiques and Gazeta Matematică.

Since 1972 Professor P.T.Mocanu has been a guide of doctorands (twelve students had taken Ph.D. degrees and other ten are prepering their dissertations). He is the chairman of the Seminar on Geometric Function Theory at Babeş-Bolyai University and the head of the Romanian School of Univalent Functions.

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