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A HIERARCHY OF SUPERMULTIPLICITY OF SEQUENCES IN A SEMIGROUP

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ABSTRACT. -- In this paper are defined the classes of convex, starshaped and supermultiplicative sequences in a semigroup. Also some relations among them are proved.

1. **Introduction.** In [1] it was proved that all the convex functions are shaped and these are all superadditive. This property was named in [4] archy of convexity of functions. In [7] we have proved a similar property real sequences and now we want to transpose it to sequences from a semi-up. But it seems to me to be more adequate to call the property „hierarchy upermultiplicity" because all the sequences are supermultiplicative.

In the next paragraph we shall give the notions relative to semigroups ch we need in what follows. Some of them are taken from [3] but the others be new because we couldn't find them in the accessible literature. Then define the classes of convex, starshaped and supermultiplicative sequences a semigroup and prove some relations among them.

2. **Semigroups.** By a *semigroup* $(G,\cdot)$ we shall mean a non-empty set G which is defined an associative binary operation. We suppose that the semi-up is commutative and has an identity e, thus:

$$ex = x, \ \forall x \in G.$$

A semigroup can have a zero, the is an element z with the property:

$$zx = z, \ \forall x \in G.$$

$x = x$, the element x is called idempotent.

A basic relation which we need in what follows is the divisibility:

$$a \mid b \Leftrightarrow \exists\, x \in G, \ b = ax.$$

Also, we shall consider semigroups in which some kinds of reductions are d.

DEFINITION 1. The semigroup $(G,.)$ is *cancellative* if:

$$xa = xb \Rightarrow a = b. \tag{1}$$

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We remind that for the product of n elements, each equal to x, it is us[ed] the notation x^n.

DEFINITION 2. The semigroup $(G,.)$ has *radicals* if:

$$x^n = y^n \Rightarrow x = y.$$

DEFINITION 3. The semigroup $(G,.)$ *preserves the divisibility* if:

$$x^n \mid y^n \Rightarrow x \mid y.$$

Remark 1. Some results are immediate. If $(G,.)$ is a group then it is ca[n]cellative and preserves the divisibility. If every element of $(G,.)$ is idempot[ent] the semigroup has radicals and preserves the divisibility but is non cancellat[ive]. Also, if $(G,.)$ has a zero element it is non cancellative.

We show by examples some relations between the three definitions.

Example 1. The ensemble of subsets of a non-empty set X with res[pect] to intersection is a non cancellative semigroup but which has radicals and [pre]serves the divisibility (any element is idempotent).

Example 2. The set of classes $\hat{k}$ of integers congruent modulo 4 with res[pect] to addition represents a cancellative semigroup which preserves the divisib[ility] (in fact it is a group) but which has no radicals as:

$$\hat{2} + \hat{2} = \hat{0} + \hat{0} = \hat{0}.$$

Example 3. In the semigroup generated by the transformation:

$$t = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 3 \end{pmatrix}$$

in respect to composition, we have: $(t^2)^2 t = t^2$ and $(t^2)^3 = t^3$, but $t^2 t^k \neq$ [...] any $k \geqslant 0$, thus the semigroup is non cancellative, has no radical and do[es not] preserves the divisibility.

To give examples of semigroups which satisfy all the conditions (1)- we consider also the next notion:

DEFINITION 4. The semigroup $(G,.)$ has the *base* B if it is satisfied [the] following condition:

$$\forall x \in G - \{e\}, \ \exists! n \geqslant 1, \ \exists! p_1, \ldots, p_n \in B,$$

$$\exists!\, k_1, \ldots, k_n \geqslant 1 : x = \prod_{i=1}^{n} p_i^{k_i}.$$

Remark 2. As in the case of a vectorial space, a base of a semig[roup] generates him. From the unicity of the representation in (4), it follows that the base is an independent set in the sense that if $p, q \in B$ then [...] for no $n \geqslant 1$. We must suppose $p^0 = e$, $\forall p \in B$. So we can represent two elements of G by the same elements of the base B but with non-neg[ative] powers:

$$x = \prod_{i=1}^{n} p_i^{k_i}, \ y = \prod_{i=1}^{n} p_i^{h_i}; \ k_i, \ h_i \geqslant 0.$$

remains true that $x = y$ if and only if $k_i = h_i$, $i = 1, \ldots, n$. We have also me consequences:

LEMMA 1. *If the semigroup $(G,.)$ has a base, then hold the following implitions for every $x, y, z \in G$, $m, n \geq 1$:*

a) $x^n = x^m \Rightarrow n = m$;

b) $xz = yz \Rightarrow x = y$;

c) $x^n = y^n \Rightarrow x = y$;

d) $x^n \mid y^n \Rightarrow x \mid y$.

Proof. All the affirmations follow from the unicity of the representation the base B. For example, for the last implication, we suppose that $y^n = x^n z$ where:

$$x = \prod_{i=1}^{n} p_i^{k_i}, \quad y = \prod_{i=1}^{n} p_i^{h_i}, \quad z = \prod_{i=1}^{n} p_i^{j_i}.$$

en $nh_i = nk_i + j_i$, for $i = 1, \ldots, n$. So $j_i = ng_i$, where $g_i = h_i - k_i$, and tting:

$$w = \prod_{i=1}^{n} p_i^{g_i}$$

have $y = xw$, thus $x \mid y$.

Remark 3. Thus, if the semigroup has a base it is cancellative, has radis and preserves the divisibility. But, because of the property a), it cannot finite. Examples of semigroups with base are $(N, +)$ with $B = \{1\}$ and , .) with the base consisting of all prime numbers.

3. **Sequences in a semigroup.** Let $(x_n)_{n \geq 1}$ be a sequence of elements of the igroup $(G,.)$.

DEFINITION 5. The sequence $(x_n)_{n \geq 1}$ is *convex* if it verifies the relation:

$$x_n^2 \mid x_{n+1} x_{n-1}, \quad \forall n \geq 1. \tag{5}$$

LEMMA 2. a) *If:*

$$x_n = \prod_{i=1}^{n} y_i^{n-i+1}, \quad n \geq 1 \tag{6}$$

re $(y_n)_{n \geq 1}$ is arbitrary, then the sequence $(x_n)_{n \geq 1}$ is convex.

b) *If $(G, .)$ is cancellative, then every convex sequence may be represented (6) with adequate $(y_n)_{n \geq 1}$.*

Proof. a) From (6) we deduce:

$$x_{n+1} x_{n-1} = x_n^2 y_{n+1}$$

b) For $n = 1$, (6) is $x_1 = y_1$, which we consider. Suppose (6) vali
$n \leqslant m$. Then (5) gives an y_{m+1} such that:

$$\prod_{i=1}^{m} y_i^{2(m-i+1)} y_{m+1} = x_{m+1} \prod_{i=1}^{m-1} y_i^{m-i}$$

and by cancellation we get (6) for $n = m + 1$.

DEFINITION 6. A sequence $(x_n)_{n \geqslant 1}$ is called *starshaped* if:

$$x_n^{n+1} \mid x_{n+1}^n, \quad \forall n \geqslant 1.$$

LEMMA 3. a) *If the sequence $(x_n)_{n \geqslant 1}$ is starshaped, then it may be repre*
by:

$$x_n^{(n-1)!} = z_1^{n!} \prod_{i=2}^{n} z_i^{n!/i(i-1)}$$

with an adequate sequence $(z_n)_{n \geqslant 1}$.

b) *If $(G,.)$ has radicals and the sequence $(x_n)_{n \geqslant 1}$ is represented by* (8)
it is starshaped.

Proof. a) We take $z_1 = x_1$. Suppose (8) be valid for $m - 1$. Then (7)
a z_{m+1} such that:

$$x_{m+1}^m = x_m^{m+1} z_{m+1}.$$

So:

$$x_{m+1}^{m!} = z_1^{(m+1)!} \prod_{i=2}^{m} z_i^{(m+1)!/i(i-1)} \cdot z_{m+1}^{(m-1)!}$$

that is (8).

b) From (8) we have:

$$x_{n+1}^{n!} = x_n^{(n+1)(n-1)!} \cdot z_n^{(n-1)!}$$

and taking the $(n - 1)!$ — th radical we get (7).

DEFINITION 7. A sequence $(x_n)_{n \geqslant 1}$ is called *supermultiplicative* if it v
the relation:

$$x_n x_m \mid x_{n+m}, \quad \forall n, m \geqslant 1.$$

Remark 4. We can consider also sequences $(x_n)_{n \geqslant 0}$ but then the r
(7) must be replaced by:

$$x_n^{n+1} \mid x_{n+1}^n x_0, \quad \forall n \geqslant 0$$

and (9) by:

$$x_n x_m \mid x_{n+m} x_0, \quad \forall n, m \geqslant 0.$$

Otherwise we must suppose $x_0 = e$.

LEMMA 4. *If the sequence* $(x_n)_{n\geq 1}$ *is given by:*

$$x_n = \prod_{i=1}^{n} w_i^{[n/i]}, \quad n \geqslant 1 \tag{10}$$

here $(w_n)_{n\geqslant 1}$ *is an arbitrary sequence and* $[x]$ *represents the integer part of* x, *en it is supermultiplicative.*

Proof. We can write:

$$x_n x_m = \prod_{i=1}^{n+m} w_i^{[n/i]+[m/i]}$$

cause $[n/i] = 0$ for $i > n$. As:

$$[(n + m)/i] \geqslant [n/i] + [m/i]$$

follows (9).

Remark 5. In [6] we have stated that for $(N,.)$ the representation (10) also necessary for supermultiplicity. The problem was also posed in [5] and result analogous with that from [6] was „proved" in [2]. But as we have narked in [7], a condition like (10) is not necessary even in the case of the nigroup $(\mathbf{R}, +)$. The affirmation is valid in all the cases.

Example 4. For a fixed $p \in G$ and the sequence of integers $(c_n)_{n\geqslant 1}$ we usider:

$$x_n = p^{\sum_{i=1}^{n} c_i [n/i]} .$$

$c_i \geqslant 0$, $\forall i$, it follows that it is represented by (10) with:

$$w_i = p^{c_i}.$$

t the sequence $(x_n)_{n\geqslant 1}$ can be supermultiplicative also for some negative ues of c_i and then it cannot be represented by (10). For example we can :e $c_1 = c_2 = c_3 = 1 = -c_4$ and then:

$$c_n \geqslant - \min_{p=1,\dots,n-1} \prod_{k=2}^{n-1} \left(\left[\frac{n}{k}\right] - \left[\frac{p}{k}\right] - \left[\frac{n-p}{k}\right]\right) c_k$$

get a supermultiplicative sequence.

4. **A hierarchy of supermultiplicity of sequences.** Let us denote by K, S^* l S the set of convex, starshaped respectively supermultiplicative sequences m $(G,.)$. Let also denote by K' and $S^{*\prime}$ the set of sequences from $(G,.)$ ich may be represented by (6) respectively by (8).

THEOREM. a) *For every semigroup* $(G, .)$ *hold the inclusions:*

$$K' \subset S^* \subset S^{*\prime} \tag{11}$$

b) *If* $(G,.)$ *preserves the divisibility, then holds also:*

$$S^{*\prime} \subset S. \tag{12}$$

Proof. a) If the sequence $(x_n)_{n\geq 1}$ is in K', it may be represented by (6) and so:

$$x_n^{n-1} = x_{n-1}^{n} \cdot \prod_{k=2}^{n} y_k^{k-1}$$

thus it belongs to S^*. The second inclusion follows from Lemma 3.

b) From (8) we get:

$$x_{m+n}^{(m+n-1)!} = x_m^{(m+n-1)!} \cdot x_n^{(m+n-1)!} \cdot$$
$$\cdot \prod_{k=m+1}^{m+n-1} z_k^{m(m+n-1)!/k(k-1)} \cdot \prod_{k=n+1}^{n+m-1} z_k^{n(m+n-1)!/k(k-1)} \cdot z_{n+m}^{(m+n-2)!}$$

and as the semigroup preserves the divisibility, we deduce that the sequence $(x_n)_{n\geq 1}$ is supermultiplicative, that is we get (12).

COROLLARY *If the semigroup* $(G,.)$ *is cancellative and preserves the divisibility, then hold the inclusions:*

$$K \subset S^* \subset S.$$

Proof. Indeed, then $K = K'$ and (11) and (12) are valid.

Remark 6. In the case $(\mathbf{R}, +)$ more results may be found in [7].

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IDÉAUX EN TREILLIS NON-COMMUTATIFS DE TYPE (G)

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REZUMAT. — Ideale în reticule necomutative de tip (G). În lucrare sînt definite şi sînt studiate proprietăţi ale idealelor şi idealelor duale intr-un reticul necomutativ de tip (G).

Le triplet $(L, \wedge, \vee)$, où L est un ensemble et $\wedge$ et $\vee$ sont deux opéra- ns binaires définies en L, s'appelera de *treillis non-commutatif de type (G)* pour tout $a, b, c, \in L$ vérifie les axiomes :

$$(A)\ \begin{cases}(a \wedge b) \wedge c = a \wedge (b \wedge c)\\(a \vee b) \vee c = a \vee (b \vee c)\end{cases}$$

$$(B)\ \begin{cases}a \wedge (a \vee b) = a\\a \vee (a \wedge b) = a\end{cases}$$

$$(G)\ \begin{cases}a \wedge b = (a \wedge b \wedge c) \vee (b \wedge a)\\a \vee b = (a \vee b \vee c) \wedge (b \vee a)\end{cases}$$

En [2] on montre que, si $(L, \wedge, \vee)$ est treillis non-commutatif de type (G), rs pour tous $a, b, c \in L$ sont vraies les égalités :

$$(i)\ \begin{cases}a \wedge a = a\\a \vee a = a\end{cases}$$

$$(ii)\ \begin{cases}a \wedge b = (a \wedge b) \vee (b \wedge a)\\a \vee b = (a \vee b) \wedge (b \vee a)\end{cases}$$

$$(iii)\ \begin{cases}a \wedge (b \vee a) = a\\a \vee (b \wedge a) = a\end{cases}$$

$$(iv)\ \begin{cases}a \wedge (b \vee c) = a \wedge (c \vee b)\\a \vee (b \wedge c) = a \vee (c \wedge b)\end{cases}$$

$$(v)\ \begin{cases}a \wedge (b \wedge c) = a \wedge (c \wedge b)\\a \vee (b \vee c) = a \vee (c \vee b)\end{cases}$$

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(vi) $\begin{cases} a \wedge b \wedge a = a \wedge b \\ a \vee b \vee a = a \vee b \end{cases}$

(vii) $\begin{cases} (a \vee b) \wedge a = a \\ (a \wedge b) \vee a = a \end{cases}$

On obtient un exemple de treillis non-commutatif de type (G) si dans le produit cartésien $P(M) \times P(M) = \{(A, B)/A \subseteq M \text{ et } B \subseteq M\}$, où M est un ensemble non-vide, on définie les opérations binaires $\wedge$ et $\vee$ ainsi:

$$(A_1, B_1) \wedge (A_2, B_2) = (A_1 \cap A_2, B_1)$$

$$(A_1, B_1) \vee (A_2, B_2) = (A_1 \cup A_2, B_1)$$

On constate que dans $(P(M) \times P(M), \wedge, \vee)$ les lois de commutativité ne sont pas verifiées.

Dans ce travail on définie et on étudie les propriétés des idéaux et des idéaux duals d'un treillis non-commutatif de type (G).

Soit $(L, \wedge, \vee)$ un treillis non-commutatif de type (G). Les notions de idéal et idéal dual se définissent également comme dans le cas des treillis, c'est-à-dire le sous-ensemble non-vide I de L est idéal dans le $(L, \wedge, \vee)$ si pour tous $a, b \in L$ est vraie l'équaivalence:

$$a \in I \text{ et } b \in I \Leftrightarrow a \vee b \in I,$$

et si pour tous $a, b \in L$ est vraie l'équivalence:

$$a \in I \text{ et } b \in I \Leftrightarrow a \wedge b \in I$$

alors I sera nommé idéal dual en $(L, \wedge, \vee)$.

(1.1) *Si $(L, \wedge, \vee)$ est le treillis non-commutatif de type (G), alors le sous-ensemble non-vide I de L est idéal dans le $(L, \wedge, \vee)$ si et seulement si pour tous $a, b, c \in L$ nous avons:*

$$a \in I \text{ et } b \in I \Rightarrow a \vee b \in I \quad (1)$$

$$a \in I \text{ et } c = c \wedge a \Rightarrow c \in I. \quad (2)$$

Démonstration. Soit I un idéal dans $(L, \wedge, \vee)$. Si $a \in I$ et $b \in I$, alors de la définition de idéal on a $a \vee b \in I$, mais si $a \in I$ et $c = c \wedge a$, alors $a \vee c = a \vee (c \wedge a)$, donc $c \in I$.

Inversement, soit I un sous-ensemble non-vide de L qui possède les propriétés (1) et (2). Si $a \in I$ et $b \in I$, alors $a \vee b \in I$, mais si $a \vee b \in I$, alors de $a = a \wedge (a \vee b)$ et $b = b \wedge (a \vee b)$ on obtient que $a \in I$ et $b \in I$, en conséquence I est idéal dans le $(L, \wedge, \vee)$.

En prenant en considération aussi le fait que le système d'axiomes définissant les treillis non-commutatifs de type (G) est autodual on peut formuler le suivant théorème sans démonstration:

(1.1′) *Si* $(L, \wedge, \vee)$ *est le treillis non-commutatif de type (G), alors le sous-*
semble non-vide I de L est idéal dual en $(L, \wedge, \vee)$ *si et seulement si pour*
us a, b, c $\in$ *L nous avons:*

$$a \in I \text{ et } b \in I \Rightarrow a \wedge b \in I \tag{3}$$

$$a \in I \text{ et } c = c \vee a \Rightarrow c \in I. \tag{4}$$

(1.2) *Si* $(L, \wedge, \vee)$ *est le treillis non-commutatif de type (G), alors le sous-*
semble non-vide I de L est idéal en $(L, \wedge, \vee)$ *si et seulement si pour tous*
, b, c $\in$ *L nous avons:*

$$a \in I \text{ et } b \in I \Rightarrow a \vee b \in I \tag{5}$$

$$a \in I \text{ et } a = a \vee c \Rightarrow c \in I. \tag{6}$$

Démonstration. Soit I un idéal dans le $(L, \wedge, \vee)$. Si $a \in I$ et $b \in I$, alors
e la définition de idéal on a $a \vee b \in I$, mais si $a \in I$ et $a = a \vee c$, alors
$\vee\, c = a \wedge (a \vee c)$, donc $c \in I$.

Inversement, soit I un sous-ensemble non-vide de L qui possède les pro-
riétés (5) et (6). Si $a \in I$ et $b \in I$, alors $a \vee b \in I$, mais si $a \vee b \in I$, alors
ı utilisant les propriétés (vi) et (i) on obtient que $a \vee b = (a \vee b) \vee a$ et
$\vee\, b = (a \vee b) \vee b$, donc $a \in I$ et $b \in I$, en conséquence I est idéal dans
$(L, \wedge, \vee)$.

(1.2′) *Si* $(L, \wedge, \vee)$ *est le treillis non-commutatif de type (G), alors le sous-*
semble non-vide I de L est idéal dual en $(L, \wedge, \vee)$ *si et seulement si pour*
us, a, b, c $\in$ *L nous avons:*

$$a \in I \text{ et } b \in I \Rightarrow a \wedge b \in I \tag{7}$$

$$a \in I \text{ et } a = a \wedge c \Rightarrow c \in I. \tag{8}$$

(1.3) *Si* $(L, \wedge, \vee)$ *est le treillis non-commutatif ae type (G), alors le sous-*
semble non-vide I de L est idéal en $(L, \wedge, \vee)$ *si et seulement si pour tous*
b, c $\in$ *L nous avons:*

$$a \in I \text{ et } b \in I \Rightarrow a \vee b \in I \tag{9}$$

$$a \in I \Rightarrow c \wedge a \in I. \tag{10}$$

Démonstration. Soit I un idéal dans le $(L, \wedge, \vee)$. Si $a \in I$ et $b \in I$
ors de la définition de l'idéal on a $a \vee b \in I$, mais si $a \in I$, alors en utili-
nt la propriété (iii) on obtient que $a = a \vee (c \wedge a)$, donc $c \wedge a \in I$.

Inversement, soit I un sous-ensemble non-vide de L qui possède les pro-
iétés (9) et (10). Si $a \in I$ et $b \in I$, alors $a \vee b \in I$, mais si $a \vee b \in I$,
ors en utilisant (10) on obtient que $a \wedge (a \vee b) = a \in I$ et $b \wedge (a \vee b) \in I$,
nc I est idéal dans le $(L, \wedge, \vee)$.

(1.3′) *Si* $(L, \wedge, \vee)$ *est le treillis non-commutatif de type (G), alors le sous-*
semble non-vide I de L est idéal dual en $(L, \wedge, \vee)$ *si et seulement si pour*
ıs a, b, c $\in$ *L nous avons:*

$$a \in I \text{ et } b \in I \Rightarrow a \wedge b \in I \tag{11}$$

$$a \in I \Rightarrow c \vee a \in I. \tag{12}$$

(1.1′) *Si* $(L, \wedge, \vee)$ *est le treillis non-commutatif de type (G), alors le sous-nsemble non-vide I de L est idéal dual en* $(L, \wedge, \vee)$ *si et seulement si pour us* $a, b, c \in L$ *nous avons:*

$$a \in I \text{ et } b \in I \Rightarrow a \wedge b \in I \tag{3}$$

$$a \in I \text{ et } c = c \vee a \Rightarrow c \in I. \tag{4}$$

(1.2) *Si* $(L, \wedge, \vee)$ *est le treillis non-commutatif de type (G), alors le sous-nsemble non-vide I de L est idéal en* $(L, \wedge, \vee)$ *si et seulement si pour tous* , $b, c \in L$ *nous avons:*

$$a \in I \text{ et } b \in I \Rightarrow a \vee b \in I \tag{5}$$

$$a \in I \text{ et } a = a \vee c \Rightarrow c \in I. \tag{6}$$

Démonstration. Soit I un idéal dans le $(L, \wedge, \vee)$. Si $a \in I$ et $b \in I$, alors e la définition de idéal on a $a \vee b \in I$, mais si $a \in I$ et $a = a \vee c$, alors $\vee c = a \wedge (a \vee c)$, donc $c \in I$.

Inversement, soit I un sous-ensemble non-vide de L qui possède les pro-riétés (5) et (6). Si $a \in I$ et $b \in I$, alors $a \vee b \in I$, mais si $a \vee b \in I$, alors 1 utilisant les propriétés (vi) et (i) on obtient que $a \vee b = (a \vee b) \vee a$ et $\vee b = (a \vee b) \vee b$, donc $a \in I$ et $b \in I$, en conséquence I est idéal dans $(L, \wedge, \vee)$.

(1.2′) *Si* $(L, \wedge, \vee)$ *est le treillis non-commutatif de type (G), alors le sous-semble non-vide I de L est idéal dual en* $(L, \wedge, \vee)$ *si et seulement si pour us,* $a, b, c \in L$ *nous avons:*

$$a \in I \text{ et } b \in I \Rightarrow a \wedge b \in I \tag{7}$$

$$a \in I \text{ et } a = a \wedge c \Rightarrow c \in I. \tag{8}$$

(1.3) *Si* $(L, \wedge, \vee)$ *est le treillis non-commutatif ae type (G), alors le sous-semble non-vide I de L est idéal en* $(L, \wedge, \vee)$ *si et seulement si pour tous* $b, c \in L$ *nous avons:*

$$a \in I \text{ et } b \in I \Rightarrow a \vee b \in I \tag{9}$$

$$a \in I \Rightarrow c \wedge a \in I. \tag{10}$$

Démonstration. Soit I un idéal dans le $(L, \wedge, \vee)$. Si $a \in I$ et $b \in I$ ›rs de la définition de l'idéal on a $a \vee b \in I$, mais si $a \in I$, alors en utili-nt la propriété (iii) on obtient que $a = a \vee (c \wedge a)$, donc $c \wedge a \in I$.

Inversement, soit I un sous-ensemble non-vide de L qui possède les pro-iétés (9) et (10). Si $a \in I$ et $b \in I$, alors $a \vee b \in I$, mais si $a \vee b \in I$, ›rs en utilisant (10) on obtient que $a \wedge (a \vee b) = a \in I$ et $b \wedge (a \vee b) \in I$, nc I est idéal dans le $(L, \wedge, \vee)$.

(1.3′) *Si* $(L, \wedge, \vee)$ *est le treillis non-commutatif de type (G), alors le sous-semble non-vide I de L est idéal dual en* $(L, \wedge, \vee)$ *si et seulement si pour ts* $a, b, c \in L$ *nous avons:*

$$a \in I \text{ et } b \in I \Rightarrow a \wedge b \in I \tag{11}$$

$$a \in I \Rightarrow c \vee a \in I. \tag{12}$$

(1.4) *Si* $(L, \wedge, \vee)$ *est le treillis non-commutatif de type (G), alors le sous-ensemble non-vide* I *de* L *est idéal en* $(L, \wedge, \vee)$ *si et seulement si pour tous* $a, b, c \in L$ *nous avons :*

$$a \in I \text{ et } b \in I \Rightarrow a \vee b \in I \tag{13}$$

$$a \vee c \in I \Rightarrow c \in I. \tag{14}$$

Démonstration. Soit I un idéal dans le $(L, \wedge, \vee)$. Si $a \in I$ et $b \in I$, alors de la définition de l'idéal on a $a \vee b \in I$, mais si $a \vee c \in I$, alors $c \in I$.

Inversement, soit I un sous-ensemble non-vide de L qui possède les propriétés (13) et (14). Si $a \in I$ et $b \in I$, alors $a \vee b \in I$, mais si $a \vee b \in I$, alors en utilisant (2) on obtient que $b \in I$, mais si en utilise (1) on obtient que $b \vee (a \vee b) = b \vee a \in I$, donc $a \in I$, en conséquence I est idéal dans le $(L, \wedge, \vee)$.

(1.4') *Si* $(L, \wedge, \vee)$ *est le treillis non-commutatif de type (G), alors le sous-ensemble non-vide* I *de* L *est idéal dual en* $(L, \wedge, \vee)$ *si et seulement si pour tout* $a, b, c \in L$ *nous avons :*

$$a \in I \text{ et } b \in I \Rightarrow a \wedge b \in I \tag{15}$$

$$a \wedge c \in I \Rightarrow c \in I. \tag{16}$$

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ON THE WEAKLY CONVEX SETS

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REZUMAT. — Asupra mulţimilor slab convexe. În lucrare sînt studiate cîteva proprietăţi ale funcţiilor convexe definite pe mulţimi slab convexe. Rezultatele sînt în legătură cu noţiunile de funcţie conjugată şi subdiferenţială.

Introduction. In the last years several authors investigated the optimization problem of the functions defined on a σ algebra $\mathcal{A}$ of an atomless finite measure space $(X, \mathcal{A}, \mu)$ (see Morris [8], Lai [4], [5], [6], [7], Chou [2]). In these papers necessary and sufficient conditions for optimality are given, an analogous to the Fenchel duality theorem for convex functions is proved and also various properties for the subdifferential of convex functions and their conjugates are given.

In this note we investigate some properties of the convex functions defined on weakly convex sets. A set is said to be weakly convex if its closure is convex. For example, the set of the characteristic functions of measurable sets of an atomless finite measure space is weakly convex (see Ex. 2 below).

In this way the theory of convex set functions and also the theory of ordinary convex functions, naturally are included in an unitary theory. The p-convex functions which were studied for example in [4], [9], [1] and [3] can also be included in the class of the convex functions defined on weakly convex sets.

The results of this note relate to conjugate functions and subdifferentials of convex functions defined on weakly convex sets. In a further note we shall investigate the optimization problems for such functions.

The authors are grateful to Professor I. Muntean for his useful suggestions concerning this paper.

1. Weakly convex sets and convex functions.

DEFINITION 1. Let E be a real locally convex space. A subset M of E is said to be *weakly convex* if $\lambda x + (1 - \lambda)y \in cl\,M$ for any $x, y \in M$ and $\lambda \in [0, 1]$, where $cl\,M$ is the closure of M.

THEOREM 1. *A set $M \subseteq E$ is weakly convex if and only if $cl\,M$ is convex. Moreover it holds:*

$$cl\,M = cl\,co\,M,$$

where $co\,M$ is the convex hull of M.

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Proof. For the first we assume that $M \subseteq E$ is weakly convex. We have

$$M \subseteq co\, M$$

and thus

$$cl\, M \subseteq cl\, co\, M.$$

We will show by induction that

$$co\, M \subseteq cl\, M.$$

Actually, by definition, for any $x_1,\ x_2 \in M$ and $\lambda \in [0,\ 1]$, we have

$$(1 - \lambda)x_1 + \lambda x_2 \in cl\, M$$

Let us suppose that any convex combination of the most $n - 1$ elements of M belongs to $cl\, M$ and let be $x_0 \in co\, M$, $x_0 = \sum_{i=1}^{n} \lambda_i x_i$, with $\lambda_i > 0$, $i = \overline{1, n}$, $\sum_{i=1}^{n} \lambda_i = 1$ and $x_i \in M$, $i = \overline{1, n}$. Now, notting by $\alpha = \sum_{i=2}^{n} \lambda_i$ and $y = \sum_{i=2}^{n} \frac{\lambda_i}{\alpha} x_i$, we have $\lambda_1 + \alpha = 1$ and $x_0 = \lambda_1 x_1 + \alpha y$.

By the induction hypothesis $y \in cl\, M$. Then there exists the net $(z_i')_{i\in I}$ with $z_i' \in M$ and $z_i' \to y$ $(i \in I)$. Let us note by $z_i'' := \lambda_1 x_1 + \alpha z_i'$, $i \in I$. We have $z_i'' \in cl\, M$, $i \in I$ by induction hypothesis. This implies that there exists a net $(z_{i,j})_{j\in J}$ with $z_{i,j} \in M$ for any $j \in J$ and $i \in I$ fixed, so that $z_{i,j} \to z_i''$.

On the other hand it is clear that $z_i'' \to x_0$.

Let V be a neighborhood of x_0. The convergence $z_i'' \to x_0$ implies that $z_i'' \in V$ for sufficiently great i, $i \in I$. Much more, from $z_{i,j} \in M$ and $z_{i,j} \to z_i''$ we have that the neighborhood V of x_0, arbitrarily choosed, intersects M and thus x_0 belongs to the closure of M.

The implication: $(cl\, M = cl\, co\, M \Rightarrow M$ is weak convex) is immediately if it takes into account that $co\, M \subseteq cl\, co\, M$.

Example 1. Let E be a real normed space and E' its dual space. Then the unit ball B in E is weakly convex in the $\sigma(E'', E')$ — topology. Much more $cl\, B = B''$ where E'' is the bidual space of the space E and B'' is the unit ball in E''.

Example 2. Let $(X, \mathcal{A}, \mu)$ be an atomless finite measure space with $L_1(X, \mathcal{A}, \mu)$ separable. Let be E and E' defined by $E = L_1(X, \mathcal{A}, \mu)$, $E' = L_\infty(X, \mathcal{A}, \mu)$ and let E' be endowed with $\sigma(E', E)$ topology.
Then

$$M := \{\chi_A : A \in \mathcal{A}\}$$

is weakly convex.

Indeed, by Definition 1 we must show that $\lambda\chi_A + (1 - \lambda)\chi_B \in cl\, M$ for any $\chi_A,\ \chi_B \in M$ and $\lambda \in [0,\ 1]$.

It follows from the atomless measure space that (see [8]) for given A, B in $\mathcal{A}$ and $\lambda \in [0,\ 1]$ there exist $L_\infty(X, \mathcal{A}, \mu)$ —sequences (χ_{A_n}) and (χ_{B_n}) such that

$$\chi_{A_n} \to \lambda\chi_{A-B} \text{ and } \chi_{B_n} \to (1 - \lambda)\chi_{B-A}$$

Hence by [2], Lemma 2, it holds

$$\chi_{A_n \cup B_n \cup (A \cap B)} \to \lambda \chi_A + (1 - \lambda)\chi_B$$

nd this complets the proof.

In the next we will use the notation $\bar{\mathbf{R}} = \mathbf{R} \cup \{+\infty\}$.

DEFINITION 2. Let E be a real locally convex space and M be a weakly onvex subset of E. A function $f: M \to \bar{\mathbf{R}}$ is called *convex* if for any $x, y \in M$, $\in [0, 1]$ and for any net $(z_i)_{i \in I}$ with $z_i \in M$ and $z_i \to \lambda x + (1 - \lambda)y$, $i \in I$, he following inequality holds:

$$\limsup_{i \in I} f(z_i) \leqslant f(x) + (1 - \lambda)f(y).$$

DEFINITION 3. Let $M \subseteq E$ be weakly convex and $f: M \to \mathbf{R}$ a function.

(i) f is called *lower semicontinuous* at $x \in M$ if

$$f(x) \leqslant \liminf_{i \in I} f(x_i)$$

r any net $(x_i)_{i \in I}$ with $x_i \in M$ and $x_i \to x (i \in I)$.

(ii) f is called *upper semicontinuous* at $x \in M$ if

$$f(x) \geqslant \limsup_{i \in I} f(x_i)$$

r any net $(x_i)_{i \in I}$ with $x_i \in M$ and $x_i \to x(i \in I)$.

(iii) f is called *continuous* at $x \in M$ if

$$f(x) = \lim_{i \in I} f(x_i)$$

r any net $(x_i)_{i \in I}$ with $x_i \in M$ and $x_i \to x$ $(i \in I)$.

THEOREM 2. *Let $M \subseteq E$ be weakly convex and $f: M \to \bar{\mathbf{R}}$. If f is convex then is upper semicontinuous on M.*

Proof. If $f(x) = +\infty$ for any $x \in M$, then the statement of the theorem obvious.

For any $x \in M$ let $(x_i)_{i \in I}$ be a net in M such that $x_i \to x(i \in I)$. Let $\in M$ so that $f(y) < \infty$ and $\lambda = 1$. It holds

$$\limsup_{i \in I} f(x_i) \leqslant 1 \cdot f(x) + (1 - 1)f(y) = f(x),$$

hich shows that f is upper semicontinuous on M.

Remark. It is obvious that any convex set $M \subseteq E$ is also weakly convex. ıe convexity of the function $f: M \to \mathbf{R}$ introduced by the Definition 2 implies upper semicontinuity. For convenience, we maintain the name „convexity" th the hope that this fact will not produce confusion.

COROLLARY 1. *Let $M \subseteq E$ be weakly convex. Then any convex and lower micontinuous function $f: M \to \bar{\mathbf{R}}$ is continuous.*

DEFINITION 4. Let $M \subseteq E$ be weakly convex and $f: M \to \mathbf{R}$.

The set $epi\, f := \{(\alpha, x) \in \mathbf{R} \times M : f(x) \leqslant \alpha\}$ is called the *epigraph* of f.

THEOREM 3. *Let $M \subseteq E$ be weakly convex and $f: M \to \mathbf{R}$. If f is convex then $cl\, epi\, f$ is convex in $\mathbf{R} \times E$.*

Proof. We shall prove that $epi\, f$ is weakly convex. For this let be (r, x), $(s, y) \in epi\, f$ and $\lambda \in [0, 1]$. M being weakly convex, there exists a net $(z_i)_{i \in I}$ such that $z_i \in M$, $i \in I$ and

$$z_i \to \lambda x + (1 - \lambda)y.$$

From the convexity of f it follows:

$$\limsup_{i \in I} f(z_i) \leqslant \lambda f(x) + (1 - \lambda)f(y) \leqslant \lambda r + (1 - \lambda)s$$

Hence, for any $n \in \mathbf{N}^*$ there exists an index $i_n \in I$ such that the following inequality holds:

$$f(z_i) \leqslant \lambda r + (1 - \lambda)s + \frac{1}{n} =: t_n$$

for any $i \in I$ with $i \geqslant i_n$, therefore $(t_n, z_i) \in epi\, f$. So $(\lambda r + (1 - \lambda)s, \lambda x + (1 - \lambda)y) \in cl\, epi\, f$ and by Theorem 1, follows the convexity of $cl\, epi\, f$ in $\mathbf{R} \times E$.

DEFINITION 5. Let $f: M \to \overline{\mathbf{R}}$, $M \subseteq E$. The set

$dom\, f := \{x \in M : f(x) \in \mathbf{R}\}$ is called the *effective domain* of the function f.

f is said to be *proper* if $dom\, f \neq \emptyset$.

THEOREM 4. *Let $M \subseteq E$ be weakly convex and $f: M \to \mathbf{R}$. If f is convex then $dom\, f$ is weakly convex.*

Proof. Let be $x, y \in dom\, f$ and $\lambda \in [0, 1]$. By the Definition 5 we have $x, y \in M$ and $f(x) \in \mathbf{R}$, $f(y) \in \mathbf{R}$. By the definition of convexity of M there exists a net $(z_i)_{i \in I}$ with $z_i \in M$, $z_i \to (1 - \lambda)x + \lambda y$ $(i \in I)$ and by the convexity of f it follows

$$\limsup_{i \in I} f(z_i) \leqslant (1 - \lambda)f(x) + \lambda f(y).$$

Without loss of generality we may assume that $f(z_i) < \infty$ for every $i \in I$. This implies that $z_i \in dom\, f$, $i \in I$ and thus

$$(1 - \lambda)x + \lambda y \in cl\, \mathrm{dom}\, f,$$

hence $dom\, f$ is weakly convex.

2. The conjugates of convex functions.

DEFINITION 6. Let E' be the dual space of E.

(i) The function $f^*: E' \to \mathbf{R}$ defined by

$$f^*(x') = \sup\{\langle x, x'\rangle - f(x) : x \in M\},\ x' \in E'$$

is called the *conjugate* function of f.

(ii) The function $f^{**} : E \to \mathbf{R}$ defined by

$$f^{**}(x) = \begin{cases} \sup \{\langle x, x' \rangle - f^*(x') : x' \in E'\}, & \text{if } x \in dom\, f \\ +\infty, & \text{if } x \notin dom\, f \end{cases}$$

called the *biconjugate* function of f.

By the definitions of f^* and f^{**} we have the inequalities

$$f^*(x') + f(x) \geqslant \langle x, x' \rangle \text{ (Young's inequality)}, \tag{1}$$

$$f^{**}(x) \leqslant f(x) \tag{2}$$

r all $x \in dom\, f$ and $x' \in dom\, f^*$.

The next theorem will show that if f is lower semicontinuous then the uality in (2) holds.

THEOREM 5. *Let $M \subseteq E$ be weakly convex and $f: M \to \overline{\mathbf{R}}$ a lower semicon- uous convex function on $dom\, f$. Then*

(i) $dom\, f^* \neq \emptyset$

(ii) $f(x) = f^{**}(x)$ *for all* $x \in M$.

Proof. It follows from Corollary 1 that f is continuous on $dom\, f$.

Let $D = dom\, f$ and $F = f/D$ be the restriction of the function f to the t D. It has been proved that the set D is weakly convex (Theorem 4). It obvious that the function F is convex and $cl\, epi\, F$ is convex in $\mathbf{R} \times E$.

We have $(f(x), x) \in epi\, f$ for all $x \in dom\, f$. Let be $x \notin dom\, f$ and $< f(x)$. Then $(r, x) \notin cl\, epi\, f$. Applying the separation theorem we can find nonzero functional $(\alpha, x') \in \mathbf{R} \times E'$ which strictly separates the point (r, x) d the set $cl\, epi\, F$. Thus there exists $\varepsilon > 0$ such that

$$\sup \{\alpha\lambda + \langle y, x' \rangle : (\lambda, y) \in epi\, F\} \leqslant \alpha r + \langle x, x' \rangle - \varepsilon$$

It follows that

$$\langle y, x' \rangle + \alpha\lambda \leqslant \langle x, x' \rangle + \alpha r - \varepsilon \tag{3}$$

r all $y \in D$ and $\lambda \geqslant F(y)$.

Note that $\alpha \leqslant 0$; otherwise, letting $\lambda \to \infty$, we obtain a contradiction. tually $\alpha < 0$. For putting in (3) $y = x$ and $\lambda = F(x)$ we obtain that

$$-\alpha(F(x) - r) \geqslant \varepsilon > 0$$

Since $r < F(x)$ it follows that $\alpha < 0$. Now, in (3) put $\lambda = F(y)$, $y' = -\frac{1}{\alpha}\, x'$ d then divide both sides of (3) by $-\alpha$. We obtain

$$\langle y, y' \rangle - F(y) \leqslant \langle x, y' \rangle - r + \frac{\varepsilon}{\alpha}$$

By taking the supremum over $y \in D$, we obtain

$$F^*(y') \leqslant \langle x, y' \rangle - r + \frac{\varepsilon}{\alpha} < +\infty$$

d so $y' \in dom\, F^*$. This establishes the statement (i).

But from the upper inequality it follows that

$$r < r - \frac{\varepsilon}{\alpha} \leqslant \langle x, y' \rangle - F^*(y') \leqslant F^{**}(x)$$

This shows that for any $r < F(x)$ we have

$$r < F^{**}(x)$$

and even more:

$$F(x) \leqslant F^{**}(x), \text{ for all } x \in D \qquad (4)$$

Consequently, from (4) and (2) we obtain

$$F^{**}(x) = F(x) \text{ for all } x \in D$$

Thus $f^{**}(x) = f(x)$ for all $x \in dom\, f$.

If $x \notin dom\, f$, then $f(x) = f^{**}(x) = +\infty$. Hence

$$f(x) = f^{**}(x) \text{ for all } x \in M.$$

THEOREM 6. *Let $M \subseteq E$ be weakly convex, $f: M \to \mathbf{R}$ convex and f^* its conjugate function. Then*

(i) *f^* is convex (in the usual sense);*

(ii) *$epi\, f^*$ is a convex and closed set.*

Proof. For any $x_1', x_2' \in E'$ and any $\lambda \in [0, 1]$, we have

$$\begin{aligned} f^*[(1-\lambda)x_1' + x_2'\lambda] :&= \sup\{\langle x, (1-\lambda)x_1' + \lambda x_2'\rangle - f(x) : x \in M\} = \\ &= \sup\{(1-\lambda)(\langle x, x_1'\rangle - f(x)) + \\ &+ \lambda(\langle x, x_2'\rangle - f(x)) : x \in M\} \leqslant \\ &\leqslant (1-\lambda)\sup\{\langle x, x_1'\rangle - f(x) : x \in M\} + \\ &+ \lambda \sup\{\langle x, x_2'\rangle - f(x) : x \in M\} = \\ &= (1-\lambda)f^*(x_1') + \lambda f^*(x_2') \end{aligned}$$

from which it results that f^* is convex.

To establish the statement (ii) it is sufficient, by virtue of the Theorem 3 and convexity of f^*, to show the closedness of epi f^*.

Let $(\alpha_i, x_i') \in epi\, f^*$ be such that

$$(\alpha_i, x_i') \to (\alpha, x').$$

We shall show that $(\alpha, x') \in epi\, f^*$.

For every $i \in I$, $(\alpha_i, x_i') \in epi\, f^*$ implies that

$$\alpha_i \geqslant f_i^*(x_i') \geqslant \langle x, x_i'\rangle - f(x), \text{ for any } x \in dom\, f.$$

Then holds

$$\alpha \geqslant \langle x, x'\rangle - f(x) \text{ for any } x \in dom\, f \text{ and thus}$$

$$\alpha \geqslant \sup\{\langle x, x'\rangle - f(x) : x \in dom\, f\} = f^*(x').$$

It shows that $(\alpha, x') \in epi\, f^*$. Hence $epi\, f^*$ is a closed set.

3. The subdifferential of convex functions.

DEFINITION 7. Let $M \subseteq E$ be a weakly convex subset of E. An element $\in E'$ is called a *subgradient* of a convex function $f: M \to \mathbf{R}$ at $x_0 \in M$ if satisfies the inequality

$$f(x) \geqslant f(x_0) + \langle x - x_0, x' \rangle \text{ for all } x \in M.$$

The set of all subgradients of the function f at x_0 (denoted by $\partial f(x_0)$), is led the *subdifferential* of f at x_0. The next theorem gives an answer whether equality in (1) holds or not.

THEOREM 7. *Let $M \subseteq E$ be weakly convex, $f: M \to \overline{\mathbf{R}}$ proper convex, f^* its ijugate and $x_0 \in dom\, f$, $x'_0 \in dom\, f^*$. Then*

(i) $x'_0 \in \partial f(x_0)$ *if and only if* $f(x_0) + f^*(x'_0) = \langle x_0, x'_0 \rangle$;

(ii) $x_0 \in \partial f^*(x'_0)$ *if and only if* $f(x_0) + f^*(x'_0) = \langle x_0, x'_0 \rangle$.

Proof. (i) If $x'_0 \in \partial f(x_0)$ then by definition, for any $x \in M$, we have

$$f(x) \geqslant f(x_0) + \langle x - x_0, x'_0 \rangle$$

is implies that

$$\langle x_0, x'_0 \rangle \geqslant f(x_0) + \langle x, x'_0 \rangle - f(x), \text{ for all } x \in dom\, f$$

l hence

$$\langle x_0, x'_0 \rangle \geqslant f(x_0) + \sup \{\langle x, x'_0 \rangle - f(x) : x \in M\}.$$

It follows that

$$\langle x_0, x'_0 \rangle \geqslant f(x_0) + f^*(x'_0)$$

l by Young's inequality this implies the equality

$$\langle x_0, x'_0 \rangle = f(x_0) + f^*(x'_0)$$

Conversely, if $\langle x_0, x'_0 \rangle = f(x_0) + f^*(x'_0)$, then by the definition of conjugate ctional, we have

$$\langle x_0, x'_0 \rangle = f(x_0) + f^*(x'_0) \geqslant f(x_0) + \langle x, x'_0 \rangle - f(x)$$

all $x \in dom\, f$, and hence

$$f(x) - f(x_0) \geqslant \langle x - x_0, x'_0 \rangle \text{ for all } x \in dom\, f.$$

This implies that $x'_0 \in \partial f(x_0)$.

(ii) For $x_0 \in \partial f^*(x'_0)$ we have

$$f^*(x') \geqslant f^*(x'_0) + \langle x_0, x' - x'_0 \rangle \text{ for all } x' \in dom\, f^*$$

$$\langle x_0, x'_0 \rangle \geqslant f^*(x'_0) + \langle x_0, x' \rangle - f^*(x') \text{ for all } x' \in dom\, f^*.$$

ollows that

$$\langle x_0, x'_0 \rangle \geqslant f^*(x'_0) + \sup \{\langle x_0, x' \rangle - f^*(x') : x' \in E'\}.$$

That is, by the Theorem 5

$$\langle x_0, x_0' \rangle \geqslant f^*(x_0') + f(x_0).$$

Hence by Young's inequality we obtain

$$f^*(x_0') + f(x_0) = \langle x_0, x_0' \rangle.$$

Conversely, if $f^*(x_0') + f(x_0) = \langle x_0, x_0' \rangle$, then

$$\langle x_0, x_0' \rangle = f(x_0) + f^*(x_0') \geqslant \langle x_0, x' \rangle - f^*(x') + f^*(x_0')$$

(see Theorem 5) for $x' \in E'$, or

$$f^*(x') \geqslant f^*(x_0') + \langle x_0, x' - x_0' \rangle.$$

This means that $x_0 \in \partial f^*(x_0')$.

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SOME REMARKS ON A CONVERSE OF KY FAN'S INEQUALITY

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REZUMAT. — Citeva remarci asupra unei inegalităţi inverse pentru inegalitatea lui Ky Fan. În această lucrare se rafinează un rezultat al lui H. Alzer, rezultat ce conţine o inegalitate inversă celei binecunoscute datorată lui Ky Fan.

1. In paper [1], by the use of some technique of majorization and Schur-functions [3, p. 207], H. Alzer established the following inegality:

$$\sum_{i=1}^{n} a_i / \sum_{i=1}^{n} (1 - a_i) \leqslant \prod_{i=1}^{n} (a_i/(1 - a_i))^{a_i / \sum_{j=1}^{n} a_j} \tag{1}$$

$a_i \in (0, 1)$, $i = 1, \ldots, n$; where the equality holds iff $a_1 = \ldots = a_n$.

Further on, we shall improve this fact using a recently refinement of Jensen's inequality due to J. E. Pečarić and S. S. Dragomir (see [5, Theorem 1.1]):

$$\begin{aligned} f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) &\leqslant (\geqslant) \frac{1}{P_n^{k+1}} \sum_{i_1, \ldots, i_{k+1}=1}^{n} p_{i_1} \ldots p_{i_{k+1}} f\left(\frac{1}{k+1}\sum_{j=1}^{k+1} x_{i_j}\right) \leqslant \\ &\leqslant (\geqslant) \frac{1}{P_n^k} \sum_{i_1, \ldots, i_k=1}^{n} p_{i_1} \ldots p_{i_k} f\left(\frac{1}{k}\sum_{j=1}^{k} x_{i_j}\right) \leqslant (\geqslant) \ldots \leqslant (\geqslant) \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \end{aligned} \tag{2}$$

where $f: I \subseteq \mathbf{R} \to \mathbf{R}$ is convex (concave) on interval I, $x_i \in I$, $p_i \geqslant 0$, $x_{i_j} \in \{x_i\}_{i=\overline{1,n}}$, $j = \overline{1, k+1}$, $P_n := \sum_{i=1}^{n} p_i > 0$ and k is a positive integer with $1 \leqslant k \leqslant n - 1$.

2. The following refinement of (1) holds.

THEOREM 1. *Let $a_i \in (0, 1)$, $i = 1, \ldots, n$, $a_{i_j} \in \{a_i\}_{i=\overline{1,n}}$ for $j = 1, \ldots, k$ and k is a positive integer with $1 \leqslant k \leqslant n - 1$. Then one has the inequalities:*

$$\sum_{i=1}^{n} a_i / \sum_{i=1}^{n} (1 - a_i) \leqslant$$

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$$\leqslant \left[\prod_{i_1, \ldots, i_{k+1}=1}^{n} \left(\sum_{j=1}^{k+1} a_{i_j} \Big/ \sum_{j=1}^{k+1} (1 - a_{i_j})\right)^{1/(k+1)\sum_{j=1}^{k+1} a_{i_j}}\right]^{1\Big/\left(n^k \sum_{i=1}^{n} a_i\right)} \leqslant$$

$$\leqslant \left[\prod_{i_1, \ldots, i_k=1}^{n} \left(\sum_{j=1}^{k} a_{i_j} \Big/ \sum_{j=1}^{k} (1 - a_{i_j})\right)^{1/k\sum_{j=1}^{k} a_{i_j}}\right]^{1\Big/\left(n^{k-1} \sum_{i=1}^{n} a_i\right)} \leqslant$$

$$\leqslant \ldots \leqslant \prod_{i=1}^{n} (a_i/(1 - a_i))^{a_i \Big/ \sum_{j=1}^{n} a_j},$$

where the equality holds in all inequalities iff $a_1 = \ldots = a_n$.

Proof. Consider the mapping (see also [1])

$$g: (0, 1) \to \mathbf{R}, \ g(a) = (a/(1 - a))^a.$$

Simple computations yield that:

$$\frac{d^2(\ln g(a))}{da^2} = \frac{1}{a(1 - a)} + \frac{1}{(1 - a)^2} > 0.$$

Hence $\ln g$ is strictly convex on the interval $(0, 1)$. Applying inequality (2 for $f = \ln g$, $p_1 = \ldots = p_n = 1$, we derive:

$$\ln\left(\frac{1}{n}\sum_{i=1}^{n} a_i \Big/ \left(1 - \frac{1}{n}\sum_{i=1}^{n} a_i\right)\right)^{\frac{1}{n}\sum_{i=1}^{n} a_i} \leqslant$$

$$\leqslant \frac{1}{n^{k+1}} \sum_{i_1, \ldots, i_{k+1}=1}^{n} \ln\left[\frac{1}{k+1}\sum_{j=1}^{k+1} a_{i_j} \Big/ \left(1 - \frac{1}{k+1}\sum_{j=1}^{k+1} a_{i_j}\right)\right]^{\frac{1}{k+1}\sum_{j=1}^{k+1} a_{i_j}} \leqslant$$

$$\leqslant \frac{1}{n^k} \sum_{i_1, \ldots, i_k=1}^{n} \ln\left[\frac{1}{k}\sum_{j=1}^{k} a_{i_j} \Big/ \left(1 - \frac{1}{k}\sum_{j=1}^{k} a_{i_j}\right)\right]^{\frac{1}{k}\sum_{j=1}^{k} a_{i_j}} \leqslant$$

$$\leqslant \ldots \leqslant \frac{1}{n}\sum_{i=1}^{n} \ln(a_i/(1 - a_i))^{a_i},$$

which implies:

$$\left(\sum_{i=1}^{n} a_i \Big/ \sum_{i=1}^{n} (1 - a_i)\right)^{\frac{1}{n}\sum_{j=1}^{n} a_j} \leqslant$$

$$\leqslant \prod_{i_1, \ldots, i_{k}=1}^{n} \left[\left(\sum_{j=1}^{k+1} a_{ij} / \sum_{j=1}^{k+1} (1 - a_{ij}) \right)^{\frac{1}{k+1} \sum_{j=1}^{k+1} a_{ij}} \right]^{1/n^{k+1}} \leqslant$$

$$\leqslant \prod_{i_1, \ldots, i_{k}=1}^{n} \left[\left(\sum_{j=1}^{k} a_{ij} / \sum_{j=1}^{k} (1 - a_{ij}) \right)^{\frac{1}{k} \sum_{j=1}^{n} a_{ij}} \right]^{1/n^{k}} \leqslant \ldots \leqslant \prod_{i=1}^{n} [(a_i/(1 - a_i))^{a_i}]^{1/n}$$

m where results the desired inequality.

The case of equality holds from [1].

REMARK. The above theorem also contains a new proof of inequality (1) mpare with [1]) using the well-known Jensen's inequality for the convex pping $f = \ln g$.

In the recent paper [2] we proved between other, the following refine- nt of Jensen's inequality:

$$f\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i\right) \leqslant (\geqslant) \frac{1}{P_n^2} \sum_{i, j=1}^{n} p_i p_j f(t x_i + (1 - t) x_j) \leqslant (\geqslant)$$

$$\leqslant (\geqslant) \frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) \tag{3}$$

ere f, p_i, x_i are as above and t is in $[0, 1]$.

Now, we can give another refinement of (1).

THEOREM 2. *Let $a_i \in (0, 1)$, $i = 1, \ldots, n$ and t be given in $[0, 1]$. Then : has the inequalities:*

$$\sum_{i=1}^{n} a_i / \sum_{i=1}^{n} (1 - a_i) \leqslant$$

$$\leqslant \left\{ \prod_{i, j=1}^{n} [(t a_i + (1 - t) a_j)/(1 - t a_i - (1 - t) a_j)]^{(t a_i + (1 - t) a_j)} \right\}^{\frac{1}{n \sum_{k=1}^{n} a_k}} \leqslant$$

$$\leqslant \prod_{i=1}^{n} (a_i/(1 - a_i))^{a_i / \sum_{k=1}^{n} a_k} \tag{4}$$

ere the equality holds in all inequalities iff $a_1 = \ldots = a_n$.

Proof. The argument is obvious from (3) for the convex mapping $f \colon (0, 1) \to \mathbf{R}$, $f(x) := \ln (x/(1 - x))^x$ and we shall omit the details.

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IE USE OF QUADRATURE FORMULAE IN OBTAINING INEQUALITIES

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REZUMAT. — Folosirea formulelor de cuadratură la obţinerea de inegalităţi. În lucrare sînt prezentate două procedee de utilizare a formulelor de cuadratură la obţinerea de inegalităţi.

If $[a, b]$ is a finite interval, we shall call $L[a, b]$ the class of functions) Lebesgue — integrable (summable) in $[a, b]$ and $AC^k[a, b]$ the class of ıctions $f(x)$ whose k-th derivative $f^{(k)}(x)$ is absolutely continuous in $[a, b]$ $= 0, 1, 2, \ldots$).

We shall call *quadrature formula* relative to the function $f \in L[a, b]$ and the nodes $x_1, x_2, \ldots, x_n$, any formula of the type

$$\int_a^b f(x)dx = \sum_{i=1}^{n} A_i f(x_i) + R_n(f). \tag{1}$$

e constants A_i are the coefficients and $R(f)$ is the remainder of the quaıture formula (1).

The number $\gamma \in \mathbf{N}$ with the property that

$$R_n(f) = 0 \text{ for any } f \in P_r$$
$$\text{and there is } g \in P_{r+1} \text{ so that } R(g) \neq 0$$

named the *degree of exactness* of the quadrature formula.

The problem which arises regarding to a quadrature formula is to deterne the parameters A_i, $i = \overline{1, m}$, and x_i, $i = \overline{1, k}$ in some given conditions 1 to study the remainder term $R(f)$ for the obtained values of A_i and x_i.

For a broader information on quadrature formulae the works [5 – 9, 11] ı be consulted.

Further on we shall present two procedures of using the quadrature forılae in obtaining inequalities.

I. If in a quadrature formula (1) all coefficients are non-negative, then $f \in L[a, b]$, removing the quadrature sums, we obtain inequalities of the)e

$$\int_a^b f(x)dx \geqslant R_n(f) \text{ if } f(x) \geqslant 0 \text{ for any } x \in [a, b] \tag{2}$$

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or

$$\int_a^b f(x)dx \leqslant R_n(f) \text{ if } f(x) \leqslant 0 \text{ for any } x \in [a, b]. \tag{3}$$

The equal sign is reached for functions of the form

$$f(x) = h(x) \prod_{i=1}^{n} (x - x_i)^{2k}, \; k \in \mathbf{N}, \; h \in L[a, b] \text{ with } h(x) \geqslant 0$$

for $x \in [a, b]$ in the case of the inequalities (2) and $h(x) \leqslant 0$ for $x \in [a, b]$ in the case of the inequalities (3).

This method has been used by F. Locher [10] to obtain certain inequalities with polynomials, inequalities used in solving some extremal problems for the quadrature formulae.

Let's consider Gauss—Jacobi's quadrature formula [8]:

$$\int_{-1}^{1} (1 - x)^\alpha (1 + x)^\beta f(x)dx = \sum_{i=1}^{m} A_m^{(\alpha, \beta)} f(x_{mi}^{(\alpha, \beta)}) + R_{2m-1}^{(f)}, \tag{4}$$

$\alpha > -1$, $\beta > -1$, in which the nodes $x_{mi}^{(\alpha, \beta)}$, $i = \overline{1, m}$, are the zeros of Jacobi's polynomial $P_m^{(\alpha, \beta)}(x)$, and the coefficients $A_{mi}^{(\alpha, \beta)}$, $i = \overline{1, m}$ are positive.

For a function $f \in AC^{2m-1}[-1, 1]$ the remainder is written under the form

$$R_{2m-1}(f) = \frac{2^{\alpha+\beta+2m+1} \cdot m!\, \Gamma(\alpha + m + 1)\, \Gamma(\beta + m + 1)\, \Gamma(\alpha + \beta + m + 1)}{(\alpha + \beta + 2m + 1)(\Gamma(\alpha + \beta + 2m + 1))^2} \frac{f^{(2m)}(\xi)}{(2m)!},$$

$$\xi \in (-1, 1).$$

For $\alpha = \beta = 0$ we obtain Gauss quadrature formula.

If in (4) we take as f an arbitrary polynomial $p_{2m}(x)$ of the degree $2m$, $p_{2m}(x) \geqslant 0$ on $[-1, 1]$ and with the dominant coefficient equal to 1, then the inequality

$$\int_{-1}^{1} (1 - x)^\alpha (1 + x)^\beta p_{2m}(x)dx \geqslant R_{2m-1}(p_{2m}).$$

Thus we have obtained the inequality given by F. Locher in [10].

PROPOSITION 1. *For any polynomial $p_{2m}(x) \geqslant 0$, $x \in [-1, 1]$, with the dominant coefficient equal to 1, the inequality*

$$\int_{-1}^{1} (1 - x)^\alpha (1 + x)^\beta p_{2m}(x)dx \geqslant \frac{2^{\alpha+\beta+2m+1} m!\, \Gamma(\alpha + m + 1)\, \Gamma(\beta + m + 1)\, \Gamma(\alpha + \beta + m + 1)}{(\alpha + \beta + 2m + 1)(\Gamma(\alpha + \beta + 2m + 1))^2}$$

true, in which the equal sign is reached only if

$$p_{2m}(x) = \frac{2^m m!\,\Gamma(\alpha+\beta+m+1)}{(\alpha+\beta+2m+1)}\,(p_m^{(\alpha,\beta)}(x))^2.$$

For $\alpha = \beta = 0$ we obtain the inequality

$$\int_{-1}^{1} p_{2m}(x)dx \geqslant \frac{2}{2m+1}\left(\frac{2^m(m!)^2}{(2m)!}\right)^2,$$

h the equal sign reached for

$$p_{2m}(x) = \left(\frac{2^m(m!)^2}{(2m)!}\right)^2 (p_m(x))^2.$$

Locher [10] also demonstrates that:

PROPOSITION 2. *For any polynomial* $p_{2m+1}(x) \geqslant 0$, $x \in [-1, 1]$, *of the degree* $+\,1$ *and with the dominant coefficient equal to* 1, *the inequality*

$$\int_{-1}^{1} (1-x)^\alpha(1+x)^\beta p_{2m+1}(x)dx \geqslant$$

$$\geqslant \frac{2^{\alpha+\beta+2m+2} m!\,\Gamma(\alpha+m+1)\,\Gamma(\beta+m+2)\,\Gamma(\alpha+\beta+m+2)}{(\alpha+\beta+2m+2)(\Gamma(\alpha+\beta+2m+2))^2}$$

valid, with the equal sign reached for

$$p_{2m+1}(x) = \left(\frac{2^m m!\,\Gamma(\alpha+\beta+m+2)}{\Gamma(\alpha+\beta+2m+2)}\right)^2 (x+1)(p_m^{(\alpha,\beta+1)})^2.$$

PROPOZITION 3. *For any polynomial* $p_{2m+1}(x) \leqslant 0$, $x \in [-1, 1]$ *of the degree* $+\,1$ *and with the dominant coefficient equal to* 1, *the inequality*

$$\int_{-1}^{1} (1-x)^\alpha(1+x)^\beta p_{2m+1}(x)dx \leqslant$$

$$\leqslant -\frac{2^{\alpha+\beta+2m+2} m!\,\Gamma(\alpha+m+2)\Gamma(\beta+m+1)\,\Gamma(\alpha+\beta+m+2)}{(\alpha+\beta+2m+2)(\Gamma(\alpha+\beta+2m+2))^2}$$

true, with the equal sign reached for

$$p_{2m+1}(x) = \left(\frac{2^m m!\,\Gamma(\alpha+\beta+m+2)}{\Gamma(\alpha+\beta+2m+2)}\right)^2 (x-1)(p_m^{(\alpha+1,\beta)}(x)).$$

PROPOSITION 4. *For any polynomial $p_{2m}(x) \leqslant 0$, $x \in [-1, 1]$, of the degree $2m$ and with the dominant coefficient equal to 1, we have the inequality*

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} p_{2m}(x)dx \leqslant$$

$$\leqslant -\frac{2^{\alpha+\beta+2m+1}(m-1)!\,\Gamma(\alpha+m+1)\Gamma(\beta+m+1)\Gamma(\alpha+\beta+m+2)}{(\alpha+\beta+2m+1)(\Gamma(\alpha+\beta+2m+1))^2},$$

with equality only if

$$p_{2m}(x) = \left(\frac{2^{m-1}(m-1)!\,\Gamma(\alpha+\beta+m+2)}{\Gamma(\alpha+\beta+2m+1)}\right)^2 (x^2-1)(p_{m-1}^{(\alpha+1,\,\beta+1)}(x))^2.$$

In [3] we have demonstrated that:

PROPOSITION 5. *For each polynomial $p_{2m}(x) \geqslant 0$ of degree $2m$ and with the dominant coefficient equal to 1 the inequality*

$$\int_0^{\infty} x^{\alpha}e^{-x}p_{2m}(x)dx \geqslant m!\,\Gamma(\alpha+m+1), \ \alpha > -1,$$

is valid, with equality only if

$$p_{2m}(x) = (m!)^2(p_m^{(\alpha)}(x))^2,$$

where $p_m^{(\alpha)}(x)$ is the Legendre polynomial.

PROPOSITION 6. *For each polynomial $p_{2m+1}(x) \geqslant 0$, $x \in (0, \infty)$, with the dominant coefficient equal to 1, the inequality*

$$\int_0^{\infty} x^{\alpha}e^{-x}p_{2m+1}(x)dx \geqslant m!\,\Gamma(\alpha+m+1),$$

is valid, with equality only if

$$p_{2m+1}(x) = (m!)^2 x(p_m^{(\alpha+1)}(x))^2,$$

where $p_m^{(\alpha+1)}(x)$ is the polynomial of degree m out of the system of orthogonal polynomials on the interval $(0, \infty)$ refering to the weight $x^{\alpha+1}e^{-x}$.

The proof of Proposition 5 is obtained from the Gauss—Laguerre quadrature formula [8]

$$\int_0^{\infty} x^{\alpha}e^{-x}f(x)dx = \sum_{i=1}^{m} A_i f(x_i) + R_{2m-1}(f)$$

which the coefficients A_i, $i = \overline{1, m}$, are positive and the remainder is given
y

$$R_{2m-1}(f) = \frac{m!\,\Gamma(\alpha + m + 1)}{(2m)!} f^{(2m)}_{(\xi)}, \quad \xi \in (0, \infty).$$

he proof of Proposition 6 is based on the Gauss—Radau quadrature formula
[13])

$$\int_0^{\infty} x^{\alpha} e^{-x} f(x) dx = Bf(0) + \sum_{i=1}^{m} A_i f(x_i) + R_{2m}(f)$$

here

$$A_i > 0, \ i = \overline{1, m}, \ B > 0$$

nd

$$R_{2m}(f) = \frac{m!\,\Gamma(\alpha + m + 2)}{(2m + 1)!} f^{(2m+1)}_{(\xi)}, \quad \xi \in (0, \infty).$$

he generalizations for the results from the Propositions 1—6 were presented
[4].

II. If in a quadrature formula (1) we have $R_n(f) \geqslant 0$ respectively $R_n(f) \leqslant 0$
r $f \in L[a, b]$, then we may write

$$\int_a^b f(x)dx \geqslant \sum_{i=1}^{n} A_i f(x_i)$$

spectively

$$\int_a^b f(x)dx \leqslant \sum_{i=1}^{n} A_i f(x_i).$$

urther on we present applications.

1. Let's consider the trapezoidal quadrature formula

$$\int_a^b f(x)dx = \frac{b - a}{2} [f(a) + f(b)] - \frac{(b - a)^3}{12} f''(c), \ c \in (a, b), \tag{5}$$

here $f \in AC^1[a, b]$. This quadrature formula has the degree of exactness
ual to one.

If $f''(x) \geqslant 0$, for any $x \in [a, b]$, then from (5) we obtain the inequality

$$\int_a^b f(x)dx \leqslant \frac{b - a}{2} [f(a) + f(b)] \tag{6}$$

and if $f''(x) \leqslant 0$ for any $x \in [a, b]$, we obtain the inequality

$$\int_a^b f(x)dx \geqslant \frac{b-a}{2}[f(a) + f(b)]. \tag{7}$$

Both in (6) and in (7) the equal sign is reached for polynomial of the degree one.

If in (7) we insert $f = 1/x$, $x \in [a, b]$, $0 < a < b$, then we find the inequality

$$\ln \frac{b}{a} > 2\frac{b-a}{b+a} \quad ([12],\ 3.6.17)$$

2. Let's consider the rectangular quadrature formula [5—9, 11])

$$\int_a^b f(x)dx = (b-a)f\left(\frac{a+b}{2}\right) + \frac{(b-a)^3}{24} f''(c),\ c \in (a, b) \tag{8}$$

where $f \in AC^1[a, b]$. The formula has the degree of exactness equal to one.

If $f''(x) \geqslant 0$ on $[a, b]$, then from (8) we obtain the inequality

$$\int_a^b f(x)dx \geqslant (b-a)f\left(\frac{a+b}{2}\right) \tag{9}$$

which is known as Hadamard's inequality [12].

If $f''(x) \leqslant 0$ on $[a, b]$, than we have the inequality

$$\int_a^b f(x)dx \leqslant (b-a)f\left(\frac{a+b}{2}\right). \tag{10}$$

In (9) and (10) the equal sign is reached for the polynomial of degree one.

3. Let's consider Simpson's quadrature formula ([5—9, 11]):

$$\int_a^b f(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] + R(f) \tag{11}$$

where

$$R(f) = -\frac{(b-a)^5}{2880} f^{(4)}(c).\ c \in (a, b),$$

and the degree of exactness is three.

If $f \in AC^3[a, b]$ with $f^{(4)}(x) \geqslant 0$ on $[a, b]$, then from (11) results the quality

$$\int_a^b f(x)dx \leqslant \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]. \tag{12}$$

$f^{(4)}(x) \leqslant 0$ on $[a, b]$, then from (11) results the inequality

$$\int_a^b f(x)dx \geqslant \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]. \tag{13}$$

(12) and (13) the equal sign is reached for polynomials of at most the third gree.

4. Now let's consider the generalized formula of the rectangular ([1]):

$$\int_a^b f(x)dx = \sum_{j=1}^{r} d_j f\left(a + d_1 + \ldots + d_{j-1} + \frac{d_j}{2}\right) +$$

$$+ \frac{1}{24}\left(\sum_{j=1}^{r} d_j^3\right) f''(c), \quad c \in (a, b), \tag{14}$$

ere $d_1, d_2, \ldots, d_r$ are the lengths of the r subintervals in which the erval $[a, b]$ has been divided.

If we choose f so that $f''(x) \geqslant 0$ on $[a, b]$, then from (14) results the quality

$$\int_a^b f(x)dx \geqslant \sum_{j=1}^{r} d_j f\left(a + d_1 + d_2 + \ldots + d_{j-1} + \frac{d_j}{2}\right) \tag{15}$$

ich generalizes Hadamard's inequality.

If $f''(x) \leqslant 0$ on $[a, b]$, then we obtain the inequality

$$\int_a^b f(x)dx \leqslant \sum_{j=1}^{r} d_j f\left(a + d_1 + \ldots + d_{j-1} + \frac{d_j}{2}\right). \tag{16}$$

the inequalities (15) and (16) the equal sign is reached for the polynomial degree one.

For $f(x) = 1/x$, $x \in [a, b]$, $0 < a < b$, from (15) we find the inequality

$$\ln \frac{b}{a} \geqslant \sum_{j=1}^{r} d_j 1/\left(a + d_1 + \ldots + d_{j-1} + \frac{d_j}{2}\right).$$

For $f(x) = e^x$, $x \in [a, b]$, from (15) results the inequality

$$e^b - e^a \geqslant \sum_{j=1}^{\gamma} d_j e^{a+d_1+\ldots+d_{j-1}+d_j/2} \tag{17}$$

and from here for $d_1 = d_2 = \ldots = d_\gamma = (b - a)/\gamma$ results

$$e^b - e^a \geqslant \frac{b-a}{\gamma} \sum_{j=1}^{\gamma} e^{a+(2j-1)(b-a)/2\gamma}. \tag{18}$$

From (18) for $a = 0$ and $b = x \geqslant 0$ results the inequality

$$e^x \geqslant \frac{x}{\gamma} \sum_{j=1}^{\gamma} e^{(2j-1)x/2\gamma} + 1.$$

We notice that the quadrature formulae are a rich source of inequalities.

We notice that, by using cubature formulae, we can build up inequalities for functions of two or more variables.

For examples, going out from the formula of the rectangular extended to functions of two variables ([6]), we can demonstrate that:

1) if $f''_{x^2}(x, y) \geqslant 0$ $f''_{y^2}(x, y) \geqslant 0$ and $f^{(4)}_{x^2y^2}(x, y) \geqslant 0$

on the rectangular $[a, b] \times [c, d]$, then we find the inequality

$$\int_a^b \int_c^d f(x, y)dxdy \geqslant (b-a)(d-c) f\left(\frac{a+b}{2}, \frac{a+d}{2}\right);$$

2) if $f''_{x^2}(x, y) \leqslant 0$, $f''_{y^2}(x, y) \leqslant 0$ and $f^{(4)}_{x^2y^2}(x, y) \leqslant 0$

on the reactangular $[a, b] \times [c, d]$, then we obtain the inequality

$$\int_a^b \int_c^d f(x, y)dx\,dy \leqslant (b-a)(d-c) f\left(\frac{a+b}{2}, \frac{c+d}{2}\right).$$

If we use Simpson's formula extended to functions of two variables ([6]):

$$\int_a^b \int_c^d f(x, y)dx\,dy = \frac{(b-a)(d-c)}{36}\Big[f(a, c) + f(a, d) + f(b, c) + f(b, d) +$$

$$+ 4f\left(\frac{a+b}{2}, c\right) + 4f\left(\frac{a+b}{2}, d\right) + 4f\left(a, \frac{c+d}{2}\right) + 4f\left(b, \frac{c+d}{2}\right) +$$

$$+ 16f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\Big] + R(f),$$

ıere

$$R(f) = -\frac{(b-a)^5(d-c)}{2880}\cdot f^{(4)}_{(\xi,\eta)} - \frac{(b-a)(d-c)^5}{2880} f^{(4)}_{(\xi_1\eta_1)} - $$
$$-\frac{(b-a)^5(d-c)^5}{2880^2} f^{(8)}_{x^4y^4}(\xi_2, \eta_2).$$

$f^{(4)}_{x^4}(x, y) \geqslant 0$, $f^{(4)}_{y^4}(x, y) \geqslant 0$ and $f^{(8)}_{x^4y^4}(x, y) \geqslant 0$

the rectangular $[a, b] \times [c, d]$, the inequality

$$\int_a^b\int_c^d f(x, y)dxdy \leqslant \frac{(b-a)(d-c)}{36}\Big[f(a, c) + f(a, d) + f(b, c) +$$
$$+ f(b, d) + 4f\left(\frac{a+b}{2}, c\right) + 4f\left(\frac{a+b}{2}, d\right) + 4f\left(a, \frac{c+d}{2}\right) +$$
$$+ 4f\left(b, \frac{c+d}{2}\right) + 16f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\Big]. \tag{19}$$

$f^{(4)}_{x^4}(x, y) \leqslant 0$, $f^{(4)}_{y^4}(x, y) \leqslant 0$ and $f^{(8)}_{x^4y^4}(x, y) \leqslant 0$ on the rectangular $[a, b] \times [c, d]$ in (19) the sens of the inequality changes.

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ON SOME IMPROVEMENTS OF ČEBYŠEV'S INEQUALITY FOR SEQUENCES AND INTEGRALS

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REZUMAT. — Citeva imbunătăţiri ale inegalităţii lui Cebîşev pentru şiruri şi integrale. În lucrare se stabilesc cîteva inegalităţi care rafinează binecunoscuta inegalitate a lui Cebîşev.

1. Let $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ be n-tuples of real numbers. pair (a, b) is called *synchrone* if $(a_i - a_j)(b_i - b_j) \geq 0$ for all $i, j = 1, \ldots, n$. We shall introduce the following subset of $\mathbf{R}^n$ for a, b synchrone:

$S(a, b) := \{x \in \mathbf{R}^n | (a + x, b)$ and $(a - x, b)$ are synchrone$\}$.

The following simple lemma of characterization holds.

LEMMA 1. *Let $a, b \in \mathbf{R}^n$, $a, b \neq 0$ such that (a, b) is synchrone. Then the wing statements are equivalent:*

(i) $x \in S(a, b)$;

(ii) *For all $i, j = 1, \ldots, n$; the following inequality holds:*

$$|(b_i - b_j)(x_i - x_j)| \leq (a_i - a_j)(b_i - b_j). \tag{1}$$

Proof. It is clear that $x \in S(a, b)$ if

$$(a_i + x_i - a_j - x_j)(b_i - b_j) \geq 0 \text{ and } (a_i - x_i - a_j + x_j)(b_i - b_j) \geq 0$$

all $i, j = 1, \ldots, n$; what is equivalent to:

$$(a_i - a_j)(b_i - b_j) + (b - b_j)(x_i - x_j) \geq 0 \text{ and } (a_i - a_j)(b_i - b_j) -$$

$- (b_i - b_j)(x_i - x_j) \geq 0$ for all $i, j = 1, \ldots, n$; what is equivalent to (1).

Denote $S(a) = S(a, a)$. Then we have the following:

COROLLARY. *If $a \in \mathbf{R}^n$, $a \neq 0$, then the following sentences are equivalent:*

(i) $x \in S(a)$;

(ii) *For all $i, j = 1, \ldots, n$; we have:*

$$|(a_i - a_j)(x_i - x_j)| \leq (a_i - a_j)^2. \tag{2}$$

Now, we shall point out some fundamental properties of the set $S(a, b)$.

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PROPOSITION 1. *Let $a, b \in \mathbf{R}^n$, $a, b \neq 0$ and the pair (a, b) is synchrone. Then*

(i) *$S(a, b)$ is convex and balanced;*

(ii) *$S(a, b)$ is symmetric and $a, -a \in S(a, b)$;*

(iii) *If $x \in S(a, b)$ then $|x| \in S(a, b)$ where $|x| = (|x_1|, \ldots, |x_n|)$ and $|a| \in S(a, b)$.*

Proof. (i). Let $x, y \in S(a, b)$ and $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$. Then we have:

$$|(b_i - b_j)(\alpha x_i + \beta y_i - \alpha x_j - \beta y_j)| \leqslant \alpha |(b_i - b_j)(x_i - x_j)| + \\ + \beta |(b_i - b_j)(y_i - y_j)| \leqslant (a_i - a_j)(b_i - b_j)$$

for all $i, j = 1, \ldots, n$; i.e., $\alpha x + \beta y \in S(a, b)$.

If $|\lambda| \leqslant 1$ and $x \in S(a, b)$ then $\lambda x \in S(a, b)$.

(ii). It's abvious by Lemma 1.

(iii). If $x \in S(a, b)$ then:

$$|(b_i - b_j)(|x_i| - |x_j|)| \leqslant |(b_i - b_j)(x_i - x_j)| \leqslant (a_i - a_j)(b_i - b_j)$$

for all $i, j = 1, \ldots, n$; i.e., $|x| \in S(a, b)$.

Remark 1. The above properties are also valid for $S(a)$.

Now, let us consider the following linear functional on $\mathbf{R}^n$, $T_n(\cdot, a, p)$ with $a \in \mathbf{R}^n \setminus \{0\}$ and $p \in \mathbf{R}^n_+ \setminus \{0\}$ and:

$$T_n(x, a, p) := \sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i a_i x_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i x_i .$$

The next result holds.

THEOREM 1. *Let $a, b \in \mathbf{R}^n \setminus \{0\}$ and the pair (a, b) is synchrone. Then we have the inequality:*

$$0 \leqslant \sup_{x \in S(a, b)} |T_n(x, b, p)| \leqslant T_n(a, b, p). \tag{3}$$

Proof. We shall give two arguments of this fact.

1. If $x \in S(a, b)$ then $(a + x, b)$ and $(a - x, b)$ are synchrone and by Čebyšev's inequality we have:

$$T_n(a + x, b, p) \geqslant 0 \text{ and } T_n(a - x, b, p) \geqslant 0.$$

By linearity of T_n in the first argument we obtain:

$$T_n(a, b, p) + T_n(x, b, p) \geqslant 0 \text{ and } T_n(a, b, p) - T_n(x, b, p) \geqslant 0$$

what implies (3).

2. If $x \in S(a, b)$ we have:

$$p_i p_j |(b_i - b_j)(x_i - x_j)| \leqslant p_i p_j (a_i - a_j)(b_i - b_j)$$

for all $i, j = 1, \ldots, n$.

Summing these inequalities, we deduce:

$$\left|\sum_{i,j=1}^{n} p_i p_j (b_i - b_j)(x_i - x_j)\right| \leqslant \sum_{i,j=1}^{n} p_i p_j |(b_i - b_j)(x_i - x_j)| \leqslant$$

$$\leqslant \sum_{i,j=1}^{n} p_i p_j (a_i - a_j)(b_i - b_j),$$

d since:

$$T_n(x, b, p) = \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j (b_i - b_j)(x_i - x_j),$$

deduce the desired inequality.

COROLLARY 1. *Let* $a, b \in \mathbf{R}^n \setminus \{0\}$ *and* (a, b) *be synchrone. Then*

$$0 \leqslant \max \{ |T_n(|a|, b, p)|, |T_n(a, |b|, p)|, |T_n(|a|, |b|, p)| \} \leqslant$$

$$\leqslant T_n(a, b, p). \tag{4}$$

Proof. By Theorem 1 it is clear that:

$$|T_n(|a|, b, p)|, |T_n(a, |b|, p)| \leqslant T_n(a, b, p).$$

On the other hand:

$$|(|a_i| - |a_j|)(|b_i| - |b_j|)| \leqslant (a_i - a_j)(b_i - b_j)$$

all $i, j = 1, \ldots, n$; what implies, by a similar argument to that in the oof 2 of the above theorem:

$$|T_n(|a|, |b|, p)| \leqslant T_n(a, b, p)$$

d the proof is finished.

Remark 2. For another proof of this fact for isotonic functionals see [2] ere further consequences are given.

COROLLARY 2. *Let* $a \in \mathbf{R}^n \setminus \{0\}$. *Then the followuing inequalities are valid:*

$$0 \leqslant \sup_{x \in S(a)} |T_n(x, a, p)| \leqslant T_n(a, a, p) \tag{5}$$

d

$$0 \leqslant |T_n(|a|, a, p)| \leqslant T_n(a, a, p) \tag{6}$$

Further on, we shall consider another functional defined on $\mathbf{R}^n$ in connec- n to Čebyšev's functional T_n.

Let $a \in \mathbf{R}^n \setminus \{0\}$, $p \in \mathbf{R}^n_+ \setminus \{0\}$ and put:

$$\tilde{T}_n(x, a, p) := T_n(x, a, p) - T_{n-1}(x, a, p), \ x \in \mathbf{R}^n.$$

It is well-known that (see for example [4]) if (x, a) is synchrone, then $(x, a, p) \geqslant 0$.

THEOREM 2. *Let $a, b \in \mathbf{R}^n \setminus \{0\}$ such that (a, b) is synchrone. Then we have the inequality:*

$$0 \leqslant \sup_{x \in S(a, b)} |\tilde{T}_n(x, b, p)| \leqslant \tilde{T}_n(a, b, p). \tag{7}$$

Proof. We also give two arguments of this fact.

1. If $x \in S(a, b)$ then $(a + x, b)$ and $(a - x, b)$ are synchrone and by the above inequality we have:

$$\tilde{T}_n(a + x, b, p) \geqslant 0 \text{ and } \tilde{T}_n(a - x, b, p) \geqslant 0$$

what easely imply (7).

2. We use the identity:

$$T_n(x, a, p) = \sum_{i=1}^{n-1} p_i p_n (x_i - x_n)(a_i - a_n).$$

The proof follows as in Proof 2 of the above theorem and we omit the details.

COROLLARY 3. *Let a, b be as above. Then:*

$$0 \leqslant \max \{ |T_n(|a|, b, p)|, \ |T_n(a, |b|, p)|, \ |T_n(|a|, |b|, p)|\} \leqslant$$
$$\leqslant T_n(a, b, p). \tag{8}$$

COROLLARY 4. *Let $a \in \mathbf{R}^n \setminus \{0\}$. Then the following inequalities hold:*

$$0 \leqslant \sup_{x \in S(a)} |\tilde{T}_n(x, a, p)| \leqslant \tilde{T}_n(a, a, p) \tag{9}$$

and

$$0 \leqslant |\tilde{T}_n(|a|, a, p)| \leqslant \tilde{T}_n(a, a, p). \tag{10}$$

The proof is abvious and we omit the details.

2. Let f, g be two continuous functions on $[a, b]$. The pair (f, g) is called *synchrone* on $[a, b]$ if:

$$(f(x) - f(y))(g(x) - g(y)) \geqslant 0 \text{ for all } x, y \in [a, b].$$

We shall introduce the following subset of $C[a, b]$ for f, g given as above:

$$S(f, g) := \{k \in C[a, b] \mid (f + k, g) \text{ and } (f - k, g) \text{ are synchrone on } [a, b]\}.$$

The following simple lemma of characterization holds.

LEMMA 2. *Let $f, g \in C[a, b] \setminus \{0\}$ such that f, g are synchrone. Then the following sentences are equivalent:*

(i) $k \in S(f, g)$;

(ii) *For all $x, y \in [a, b]$ we have:*

$$|(g(x) - g(y))(k(x) - k(y))| \leqslant (f(x) - f(y))(g(x) - g(y)).$$

The proof is similar to that of Lemma 1 and we omit the details.

As in the discrete case we have the following proposition which contains properties of $S(f, g)$.

PROPOSITION 2. *Let f, g be as above. Then:*

(i) *$S(f, g)$ is convex and balanced;*

(ii) *$S(f, g)$ is symmetric and $f, -f \in S(f, g)$;*

(iii) *If $k \in S(f, g)$ then $|k| \in S(f, g)$ and $|f| \in S(f, g)$.*

Now, let us consider the following linear functional on $C[a, b]$, $T(\cdot, f, p)$ re f, g belong $C[a, b]$ and $p \geqslant 0$, given by:

$$T(k, f, p) := \int_a^b p(x)\mathrm{d}x \int_a^b k(x)f(x)p(x)\mathrm{d}x - \int_a^b f(x)p(x)\mathrm{d}x \int_a^b k(x)p(x)\mathrm{d}x.$$

It is well-known that if k, f are synchrone, then $T(k, f, p) \geqslant 0$. Further we shall improve this fact.

THEOREM 3. *Let $f, g \in C[a, b] \setminus \{0\}$ such that f, g are synchrone. Then we e the inequality:*

$$0 \leqslant \sup_{k \in S(f,g)} |T(k, g, p)| \leqslant T(f, g, p) \tag{11}$$

Proof. 1. If $k \in S(f, g)$ then $(f + k, g)$ and $(f - k, g)$ are synchrone and Čebyšev's inequality we have:

$$T(f + k, g, p) \geqslant 0 \text{ and } T(f - k, g, p) \geqslant 0.$$

The inequality (11) follows by the linearity of T in the first variable.

2. We shall use the following integral identity:

$$T(k, g, p) = \frac{1}{2} \int_a^b \int_a^b p(x)p(y)(k(x) - k(y))(g(x) - g(y))dxdy \tag{12}$$

Now, suppose that $k \in S(f, g)$. Then by Lemma 2 we have:

$$p(x)p(y)\,|(k(x) - k(y))(g(x) - g(y))| \leqslant$$
$$\leqslant p(x)p(y)(f(x) - f(y))(g(x) - g(y))$$

all $x, y \in [a, b]$.

Integrating this inequality in rectangle $[a, b] \times [a, b]$ and using identity we derive easely inequality (11).

COROLLARY 5. *Let $f, g, p \in C[a, b]$, f, g be synchrone and $p \geqslant 0$, then*

$$0 \leqslant \max\{|T(|f|, g, p)|, |T(f, |g|, p)|, |T(|f|, |g|, p)|\} \leqslant$$
$$\leqslant T(f, g, p). \tag{13}$$

COROLLARY 6. *Let $f, p \in C[a, b]$, $p \geq 0$. Then the following two inequalities are valid:*

$$0 \leq \sup_{k \in S(f)} |T(k, f, p)| \leq T(f, f, p) \tag{14}$$

and

$$0 \leq |T(|f|, f, p)| \leq T(f, f, p) \tag{15}$$

where $S(f) := S(f, f)$.

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SUFFICIENT CONDITIONS FOR UNIVALENCE OF A LARGE CLASS OF FUNCTIONS

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REZUMAT. — Condiţii suficiente de univalenţă pentru o clasă largă de funcţii. În lucrare se prezintă o condiţie suficientă de univalenţă pentru funcţii complexe de forma:

$$g(z) = \left[\alpha \int_0^z \frac{f^\alpha(u)}{u}\, du\right]^{\frac{1}{\alpha}}$$

unde f este o funcţie de variabilă complexă cu proprietatea $f(0) = 0$, $f'(0) = 1$.

1. **Introduction.** Let A be the class of functions f, which are analytic in the unit disc $U = \{z \in \mathbf{C} : |z| < 1\}$, with $f(0) = 0$ and $f'(0) = 1$.

In order to prove our main results we shall need the theory of Loewner and a theorem due to Christian Pommerenke [1].

A function $L(z, t)$, $z \in U$, $t \geqslant 0$ is called a *Loewner chain* or a *subordination chain* if $L(z, t)$ is analytic and univalent in U for all $t \geqslant 0$, is continuously differentiable on $[0, \infty)$ for all z in U, and for all s, t, $0 \leqslant s < t : L(z, t) \prec \prec L(z, t)$ [by $\prec$ we denote the relation of subordination].

2. **Preliminaries.** We denote by U_r, $0 < r \leqslant 1$ the disc of the z-plane $\{z \in \mathbf{C} : |z| < r\}$.

Theorem (Pommerenke) ([1], [2]). *Let* $r_0 \in (0, 1]$ *and let* $L(z, t) = a_1(t) \cdot z + a_2(t) \cdot z^2 + \ldots$, $a_1(t) \neq 0$, *be analytic in* U_{r_0} *for all* $t \in [0, \infty)$ *and locally absolutely continuous in* $[0, \infty)$, *locally uniform with respect to* U_{r_0}. *For almost all* $t \in [0, \infty)$ *suppose*

$$z \cdot \frac{\partial L(z, t)}{\partial z} = p(z, t) \cdot \frac{\partial L(z, t)}{\partial t}, \quad z \in U_{r_0} \tag{1}$$

where $p(z, t)$ *is analytic in* U *and* $\operatorname{Re} p(z, t) > 0$, $z \in U$, $t \in [0, \infty)$.

If $|a_1(t)| \to \infty$ *for* $t \to \infty$ *and* $\{L(z, t)/a_1(t)\}$ *forms a normal family in* U_{r_0}, *then, for each* $t \in [0, \infty)$, $L(z, t)$ *has an analytic and univalent extension to the whole disc* U [*and is, consequently, a Loewner chain*].

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3. Main result.

THEOREM 1. *Let $f(z) = z + a_2 z^2 + \ldots$ be analytic in the unit disc U and let α be a complex number. If the following conditions are satisfied:*

$$|\alpha - 1| < 1 \tag{2}$$

$$\left|\alpha(1 - z^2)\left(\frac{zf'(z)}{f(z)} - 1\right) + \alpha - 1\right| \leqslant 1 \tag{3}$$

for all z in U, then the function

$$g_\alpha(z) = \left[\alpha \int_0^z \frac{f^\alpha(u)}{u}\, du\right]^{\frac{1}{\alpha}} \tag{4}$$

is analytic and univalent.

Proof. The function

$$h(u) = \frac{f(u)}{u} = 1 + a_1 u + \ldots + a_n u^n + \ldots \tag{5}$$

is analytic in U and $h(0) = 1$. Then, it exists r_0, $0 < r_0 \leqslant 1$, so that $h(u)$ does not vanish in U_{r_0}. In this case we denote by $h_1(u)$ the uniform branch of $[h(u)]^\alpha$ which satisfies $h_1(0) = 1$ and is analytic in U_{r_0}.

Let

$$h_2(z,\ t) = \alpha \int_0^{e^{-t}z} h_1(u) \cdot u^{-1} \cdot du = (e^{-t}z)^\alpha + \frac{a_1}{\alpha + 1}(e^{-t}z)^{\alpha+1} +$$

$$+ \ldots + \frac{a_n}{\alpha + n}(e^{-t}z)^{\alpha+n} + \ldots \text{ for } t \geqslant 0 \tag{6}$$

It is clear that, if $z \in U_{r_0}$, then $e^{-t}z \in U_{r_0}$, and, because h is analytic in U_{r_0} and $\lim\limits_{n\to\infty} \sqrt[n]{\left|\frac{\alpha a_n}{\alpha + n}\right|} = \lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$ it follows that $h_2(z, t)$ is also analytic in U_{r_0} for all $t \geqslant 0$, and

$$h_2(z,\ t) = (e^{-t}z)^\alpha \cdot h_3(z,\ t) \text{ where} \tag{7}$$

$$h_3(z,\ t) = 1 + \frac{a_2}{1 + \alpha} e^{-t}z + \ldots \tag{8}$$

Let

$$h_4(z,\ t) = h_3(z,\ t) + (e^{2t} - 1)h_1(e^{-t}z) \tag{9}$$

Because $h_4(0, t) = e^{2t} \neq 0$, we can choose r_1, $0 < r_1 \leqslant r_0$ so that $h_4(z, t) \neq 0$ in U_{r_1}, for all $t \geqslant 0$. Now, for $t \geqslant 0$, denote by $h_5(z, t)$ the uniform branch of $[h_4(z, t)]^{1/\alpha}$ which satisfies $h_5(0) = e^{2t/\alpha}$.

It follows, by this construction, in analytical steps, that the function

$$L(z, t) = e^{-t}z \cdot h_5(z, t) \tag{10}$$

is analytic in U_{r_1} and $L(0, t) = 0$.

It is clear that $e^{-t} \cdot h_5(0, t) = e^{\frac{(2-\alpha)t}{\alpha}}$

Now, we can formally write [using (6), (7), (8), (9), (10)]

$$L(z, t) = \left[\alpha \int_0^{e^{-t}z} f^\alpha(u) \cdot u^{-1} \cdot du + (e^{2t} - 1) f^\alpha(e^{-t}z)\right]^{\frac{1}{\alpha}} =$$

$$= z \cdot e^{\frac{(2-\alpha)t}{\alpha}} + \dots \tag{11}$$

Denote $a_1(t) = e^{(2-\alpha)t/\alpha}$. It is clear that $a_1(t) \neq 0$ for all $t \geqslant 0$.

From (2) it follows immediately that $\operatorname{Re} \frac{(\alpha - 1) + 1}{1 - (\alpha - 1)} > 0$, and, hence $\operatorname{Re} \frac{2 - \alpha}{\alpha} > 0$.

Then $\lim\limits_{t\to\infty} |a_1(t)| = \lim\limits_{t\to\infty} \left| e^{\frac{(2-\alpha)t}{\alpha}} \right| = \lim\limits_{t\to\infty} e^{t \cdot \operatorname{Re} \frac{2-\alpha}{\alpha}} = \infty$

$L(z, t)/a_1(t)$ is analytic in U_{r_1} for all $t \geqslant 0$ and, hence, the family $\{L(z, t)/a_1(t)\}$ is uniformely bounded in $U_{\frac{r_1}{2}}$. From Montel's theorem, it follows that $[L(z, t)/a_1(t)\}$ is a normal family in $U_{\frac{r_1}{2}}$.

Using (9), (10) it follows that

$$\frac{\partial L(z, t)}{\partial t} = e^{-t}z \left[\frac{1}{\alpha} (h_4(z, t))^{\frac{1}{\alpha} - 1} \frac{\partial h_4(z, t)}{\partial t} + (h_4(z, t))^{\frac{1}{\alpha}}\right] \tag{12}$$

Because $h_4(0, t) = e^{2t} \neq 0$, it follows that we can define an uniform branch for $[h_4(z, t)]^{\frac{1}{\alpha} - 1}$ which is analytic in U_{r_2} $(0 < r_2 \leqslant \frac{r_1}{2}$ and $[h_4(0, t)]^{\frac{1}{\alpha} - 1} = e^{2t\left(\frac{1}{\alpha} - 1\right)}$. r_2 is choosen so that $[h_4(z, t)]^{\frac{1}{\alpha} - 1}$ do not vanish in U_{r_2}.

It is clear that $\partial h_4(z, t)/\partial t$ is analytic in U_{r_2} and then, $\frac{\partial L(z, t)}{\partial t}$ is also. It follows that $L(z, t)$ is locally absolutely continuous.

Let

$$p(z, t) = \frac{z\partial L(z, t)}{\partial z} \Big/ \frac{\partial L(z, t)}{\partial t} \tag{13}$$

In order to prove that the function $p(z, t)$ has an analytic extension with positive real part in U, for all $t \geqslant 0$, it is sufficient to prove that the function

$$w(z, t) = \frac{p(z, t) - 1}{p(z, t) + 1} \text{ is analytic in } U, \text{ for } t \geqslant 0 \tag{14}$$

and

$$|w(z, t)| < 1 \text{ for all } z \in U \text{ and } t \geqslant 0. \tag{15}$$

Using (5), after simple computations, we obtain:

$$w(z, t) = \frac{(\alpha - e^{2t}) \cdot h(e^{-t}z) + \alpha(e^{t} - e^{-t})f'(e^{-t}z) \cdot e^{t}}{e^{2t} \cdot h(e^{-t}z)}. \tag{16}$$

Because $h(e^{-t}z)$ do not venish in U_{r_1} and is analytic, it follows that $w(z, t)$ is analytic in U_{r_1}, for all $t \geqslant 0$, Then, $w(z, t)$ has an analytic extension in U denoted also by $w(z, t)$.

For $t = 0$, $|w(0, t)| = |\alpha - 1| < 1$ (from (2))

Let $t > 0$. In this case, the function $w(z, t)$ is analytic in $\bar{U}$ because $|e^{-t} \cdot z| \leqslant e^{-t} < 1$ for any $z \in \bar{U}$. Then:

$$|w(z, t)| < \max_{|z|=1} |w(z, t)| = |w(e^{i\theta}, t)| \text{ where } \theta \text{ is real.} \tag{17}$$

In order to prove (15) it is sufficient that

$$|w(e^{i\theta}, t)| \leqslant 1 \text{ for all } t > 0. \tag{18}$$

If $u = e^{-t} \cdot e^{i\theta}$, $u \in U$, then $|u| = e^{-t}$, and from (16) we have:

$$|w(e^{i\theta}, t)| = \left| \frac{\left(\alpha - \frac{1}{|u|^2}\right) f(u) + \alpha\left(\frac{1}{|u|} - |u|\right) e^{i\theta} \cdot f'(u)}{\frac{1}{|u|^2} \cdot f(u)} \right| = \tag{19}$$

$$= \left| \frac{(\alpha|u|^2 - 1)f(u) + \alpha e^{t}u(1 - |u|^2)f'(u)}{f(u)} \right| = \left| \alpha \cdot |u|^2 - 1 + \alpha(1 - |u|^2) \frac{uf'(u)}{f(u)} \right| =$$

$$= \left| \alpha|u|^2 - \alpha + \alpha - 1 + \alpha(1 - |u|^2) \frac{uf'(u)}{f(u)} \right| = \left| \alpha(1 - |u|^2)\left(\frac{uf'(u)}{f(u)} - 1\right) + \alpha - 1 \right|$$

and inequality (18) becomes

$$\left| (1 - |u|^2)\left(\frac{uf'(u)}{f(u)} - 1\right) + \alpha - 1 \right| < 1 \tag{20}$$

Because $u \in U$, relation (3) implies (20). Combining (17), (18), (19) and (20) it follows that $|w(z, t)| < 1$ for all $z \in U$ and $t \geqslant 0$.

Then, from Pommerenke's theorem it follows that $L(z, t)$ is a subordination chain and, hence, the function $L(z, 0) = g_{\alpha}(z)$ defined by (4) is analytic and univalent in U.

COROLLARY 1. *Let $f(z) = z + a_2z^2 + \ldots$ be analytic in the unit disc U and let α be a complex number. If the following conditions are satisfied:*

(i) $|\alpha - 1| < 1$,

(ii) $(1 - |z|^2) \cdot \left|\frac{zf'(z)}{f(z)} - 1\right| \leqslant \frac{1 - |\alpha - 1|}{|\alpha|}$ *in U,*

then $g(z) = \left[\alpha \cdot \int_0^z \frac{f^\alpha(u)}{u}\, du\right]^{\frac{1}{\alpha}}$ *is analytic and univalent in U.*

Proof. Condition (i) is similar to (2) in Theorem 1 and (ii) is equivalent to:

$$\left|\alpha\left(\frac{zf'(z)}{f(z)} - 1\right)\right|(1 - |z|^2) + |\alpha - 1| \leqslant 1.$$

But (3) is a weaker condition because

$$\left|\alpha\left(\frac{zf'(z)}{f(z)} - 1\right)(1 - |z|^2) + \alpha - 1\right| \leqslant \left|\alpha\left(z\,\frac{zf'(z)}{f(z)} - 1\right)(1 - |z|^2)\right| + |\alpha - 1| \leqslant 1$$

and then, from Theorem 1 it follows the assertion.

COROLLARY 2. *Let $f(z) = z + a_2z^2 + \ldots$ be analytic in the unit disc U and let α be a real number with*

(i) $0 < \alpha \leqslant 1$,

If

(ii) $(1 - |z|^2)\left|\frac{zf'(z)}{f(z)} - 1\right| \leqslant 1$ *for all $z \in U$,*

then $g_\alpha(z) = \left[\alpha \int_0^z \frac{f^\alpha(u)}{u}\, du\right]^{\frac{1}{\alpha}}$ *is analytic and univalent in U.*

Proof. If α is real and $0 < \alpha \leqslant 1$ it follows easily that $\frac{1 - |\alpha - 1|}{|\alpha|} = 1$ and $|\alpha - 1| < 1$. Then, by corrolary 1 it follows the assertion.

4. **Some particular cases.** *Example 1. If α is real, $0 < \alpha \leqslant 1$, then the function*

$$f(z) = \left[\alpha\, 2^\alpha \int_0^z \frac{u^{\alpha-1}}{(2 + u)^\alpha}\, \mathrm{d}u\right]^{\frac{1}{\alpha}}$$ *is analytic and univalent in U.*

Proof. $z = x + iy$ where x and y are real numbers. $|z| < 1$ implies $x > -1$ and consequently

$$4 + 4x + x^2 + y^2 > x^2 + y^2,$$ which is equivalent to

$|z + 2|^2 > |z|^2$ and hence $\frac{|z|}{|z + 2|} < 1$, which means

(a) $\left|\frac{zf_1'(z)}{f_1(z)} - 1\right| < 1$ where $f_1(z) = \frac{2z}{2+z} = z + \dots$ is analytic in U. But $|z| < 1$ and then $(1 - |z|^2) \leqslant 1$, and because of (a), it follows that

$$(1 - |z|^2) \cdot \left|\frac{zf_1'(z)}{f_1(z)} - 1\right| < 1.$$

By applying Corollary 2 it follows that

$$f(z) = \left[\alpha \int_0^z \frac{f_1^\alpha(u)}{u}\, du\right]^{\frac{1}{\alpha}} \text{ si univalent.}$$

Remark 1. By the demonstration of P. T. Mocanu, the function f from the previous example is starlike and, then, univalent, This result was obtained using another method.

THEOREM 2. *Let b be a real number, $b > 1$ and let a be a complex number. We denote by m the maximum between* Re a *and* Im a. *If the following assertions hold:*

(i) $\alpha \in \mathbb{C}$ *and* $|\alpha - 1| \leqslant \frac{b-1}{b+1}$

(ii) $m \in (-\infty, -b-1] \cup [b+1, +\infty)$

then f, defined in the unit disc U by

$$f(z) = \left[\alpha a^\alpha \cdot \int_0^z \frac{u^{\alpha-1}}{(a+u)^\alpha}\, du\right]^{\frac{1}{\alpha}}$$

is analytic and univalent in U.

Proof. $|\alpha - 1| \leqslant \frac{b-1}{b+1}$ is equivalent to $\frac{1-|\alpha-1|}{1+|\alpha-1|} \geqslant \frac{1}{b}$, and because $|\alpha| \leqslant 1 + |\alpha - 1|$ it follows that:

$$\frac{1-|\alpha-1|}{|\alpha|} \geqslant \frac{1-|\alpha-1|}{1+|\alpha-1|} \geqslant \frac{1}{b}. \tag{21}$$

If we define $g(z) = \frac{az}{a+z} = z + \dots$, we obtain that g is analytic in U and $\left|\frac{zg'(z)}{g(z)} - 1\right| = \frac{|z|}{|a+z|}$, If $z = x + iy$, where x and y are real, from (ii) and from $-1 < x < 1$, $-1 < y < 1$ it follows, after simple computations that $|a + z| \geqslant b$. Taking into account that $|z| < 1$ it follows immediately that $\frac{|z|}{|a+z|} \leqslant \frac{1}{b}$ which is equivalent to $\left|\frac{zg'(z)}{g(z)} - 1\right| \leqslant \frac{1}{b}$. Because $1 - |z|^2 \leqslant 1$ and from (21) we obtain that:

$$(1 - |z|^2)\left|\frac{zg'(z)}{g(z)} - 1\right| \leqslant \frac{1}{b} \leqslant \frac{1-|\alpha-1|}{|\alpha|}$$

It follows from Corollary 1 that

$$f(z) = \left[\alpha \int_0^z \frac{g^{\alpha}(zu)}{u}\, du\right]^{\frac{1}{\alpha}} = \left[\alpha a^{\alpha} \cdot \int_0^z \frac{u^{\alpha-1}}{(a+u)^{\alpha}}\, du\right]^{\frac{1}{\alpha}} \text{ is}$$

analytic and univalent in U and then, the theorem is proved.

Remark 2. In his papers, related to hypergeometric functions, P. T. Mocanu proved, using another method, that the functions defined in theorem 2 are univalent, and more, starlike, but conditions (i) and (ii) are different from those used by the above-mentioned author.

Remark 3. In theorem 2 we must take $b > 1$ because, for all $\alpha \in \mathbb{C}$ with $|\alpha - 1| < 1$ we have $\frac{1 - |\alpha - 1|}{|\alpha|} \leqslant 1$. It is easy to show that equality holds if and only if α is real and $0 < \alpha \leqslant 1$. Example 1 is not a particular case of theorem 2 because in the theorem we need for the proof that $b > 1$.

Example 2. The function

$$f(z) = \left[\left(1 + \frac{i}{2}\right)(4 - i)^{1+\frac{i}{2}} \cdot \int_0^z \frac{u^{\frac{i}{2}}}{(4 - i + u)^{1+\frac{i}{2}}}\, du\right]^{\frac{4-2i}{5}}$$

is analytic and univalent in U.

Proof. If we apply theorem 2 with $b = 3$ and $\alpha = 1 + \frac{i}{2}$ which satisfies $|\alpha - 1| = \frac{1}{2} = \frac{b-1}{b+1}$ and $m = 4 = b + 1$, we obtain immediately the result (taking $a = 4 - i$).

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THE SHARP UPPER BOUND FOR THE MODULUS OF THE SCHWARZIAN IN THE CLASS OF CONVEX FUNCTIONS

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'S subject classification: 30C45

REZUMAT. — Marginea superioară a modulului derivatei lui Schwarz în clasa funcţiilor convexe. Fie f o funcţie univalentă în discul unitate U din planul complex. Atunci are loc binecunoscuta delimitare a modulului derivatei lui Schwarz a lui f

$$|\{f;\ z\}| \leqslant \frac{6}{(1-r^2)^2},\ z \in U,\ |z| = r,$$

marginea fiind atinsă în cazul funcţiei lui Koebe.

În această notă este demonstrată următoarea

Teoremă. Dacă f este o funcţie convexă în U şi $z \in U$, $|z| = r$, atunci

$$|\{f;\ z\}| \leqslant \frac{2}{(1-r^2)^2},$$

marginea fiind atinsă.

1. **Introduction.** Let S be the class of analytic and univalent functions f ı the unit disk U, normalized by the conditions $f(0) = 0, f'(0) = 1$. Let $K \subset S$: the class of convex functions on U. It is well known that for every analytic ıd univalent function f on U the following sharp estimate holds:

$$|\{f;\ z\}| \leqslant \frac{6}{(1-r^2)^2},\ \forall z \in U,\ |z| = r,$$

here $\{f;\ z\}$ denotes the Schwarzian derivative of f in z

$$\{f;\ z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

In this note is obtained the sharp upper bound for $|\{f;\ z\}|$ in the class f convex functions. This result is given by the following.

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THEOREM *If f is a convex function on U and $z \in U$ with $|z| = r$, then*

$$|\{f;\ z\}| \leqslant \frac{2}{(1-r^2)^2}$$

and the bound is sharp.

2. **Preliminaries.** Let P be the class of analytic functions p on U with positive real part and with $p(0) = 1$. It is well known that $p \in P$ if and only if p has the form $p = (1+\varphi)/(1-\varphi)$, where φ is analytic on U, $\varphi(0) = 0$ and $|\varphi(z)| < 1$, $z \in U$.

We will also use the following

LEMMA ([1], p. 319). *If ψ is an analytic function on U with $|\psi(z)| \leqslant 1$ for every $z \in U$, then*

$$|\psi'(z)| \leqslant \frac{1-|\psi(z)|^2}{1-|z|^2}.$$

3. *Proof of the theorem.* We may presume that $f \in K$. Then $p \in P$, where $p(z) = 1 + zf''(z)/f'(z)$. If we put $p = (1+\varphi)/(1-\varphi)$ we obtain

$$\frac{f''(z)}{f'(z)} = 2\,\frac{\varphi(z)}{z(1-\varphi(z))},$$

where $\varphi(0) = 0$ and $|\varphi(z)| < 1$ in U. It follows immediately that the Schwarzian of f has the form

$$\{f;\ z\} = \frac{2}{(1-\varphi(z))^2}\left(\frac{\varphi(z)}{z}\right)'.$$

Let $\psi(z) = \frac{\varphi(z)}{z}$. Then ψ satisfies the hypothesis of the lemma and

$$\{f;\ z\} = \frac{2}{(1-z\psi(z))^2}\,\psi'(z).$$

Applying the lemma we obtain

$$|\{f;\ z\}| \leqslant \frac{2}{1-r^2}\,\frac{1-|\psi(z)|^2}{(1-r|\psi(z)|)^2},\quad |z| = r.$$

But

$$\max\left\{\frac{1-x^2}{(1-rx)^2} : x \in [0,\ 1]\right\} = \frac{1}{1-r^2}$$

so

$$|\{f;\ z\}| \leqslant \frac{2}{(1-r^2)^2}.$$

We will show now that the bound is sharp by considering the function

$$f_a(t) = \frac{1}{2}\log\frac{1+at}{1-at},\quad |a| = 1,$$

which maps U on the band $\left\{w : -\frac{\pi}{4} < \operatorname{Im} w < \frac{\pi}{4}\right\}$. By direct computation we find out that $\{f_a ; t\} = 2a^2(1 - a^2t^2)^{-2}$, so $|\{f_a ; z\}| = 2(1 - t^2)^2$ if we take $a = r/z$. The proof is finished.

Remark. The sharp bound given by the theorem is actually valid for the larger class

$$\left\{g : g \text{ analytic on } U,\ g = \frac{af + b}{cf + d},\ a, b, c, d \in \mathbf{C},\ f \in K\right\}.$$

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A SUFFICIENT CONDITION FOR UNIVALENCE

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REZUMAT. — O condiţie suficientă de univalenţă. În lucrare este dată o condiţie suficientă pentru univalenţa unei funcţii regulare în discul unitate.

In this note we prove a sufficient condition for univalence of a regular ıction in the unit disc $U = \{z : |z| < 1\}$.

Pommerenke has proved the next lemma:

LEMMA ([1]). *Let r_0 be a real number, $r_0 \in (0, 1]$ and $U_{r_0} = \{z : |z| < r_0\}$. t $f(z, t) = a_1(t)z + \ldots$, $a_1(t) \neq 0$ be a regular function for all $t \in I = [0, \infty)$ $z \in U_{r_0}$ and locally absolutely continuous in I, locally uniform with respect U_{r_0}. For almost all $t \in I$ suppose*

$$z \frac{\partial f(z, t)}{\partial z} = p(z, t) \frac{\partial f(z, t)}{\partial t}, \quad z \in U_{r_0} \tag{1}$$

ıere $h(z, t)$ is regular in U and satisfies $\operatorname{Re} h(z, t) > 0$, $z \in U$, $t \in I$.

If $|a_1(t)| \to \infty$ for $t \to \infty$ and $f(z, t)/a_1(t)$ forms a normal family in U_{r_0}, then each $t \in I$, $f(z, t)$ can be continued regularly in U and gives a univalent nction.

THEOREM 1. *Let m be a real number, $m > 0$, α be a complex number such at*

$$\left|\alpha - \frac{m+1}{2}\right| < \frac{m+1}{2} \tag{2}$$

ıd $f(z) = z + \ldots$ be a regular function in U. If there exists a regular function $U g(z) = z + \ldots$ such that

(i) $\frac{g(z)}{z} \neq 0$, $\forall z \in U$ *if $\frac{1}{\alpha}$ is a positive integer*

(ii) $\left|\left(\frac{f(z)}{g(z)}\right)^{\alpha-1} \frac{f'(z)}{g'(z)} - \frac{m+1}{2}\right| < \frac{m+1}{2}$, $\forall z \in U$

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(iii) *for all* $z \in U$:

$$\left| |z|^{m+1} \left[\left(\frac{f(z)}{g(z)} \right)^{\alpha-1} \frac{f'(z)}{g'(z)} - \frac{m+1}{2} \right] + (1 - |z|^{m+1}) \left[(\alpha - 1) \frac{zg'(z)}{g(z)} + \right. \right. \tag{3}$$

$$\left. \left. + \frac{zg''(z)}{g'(z)} + 1 - \frac{m+1}{2} \right] \right| \leqslant \frac{m+1}{2} \cdot$$

then the function $f(z)$ *is univalent in* U.

Proof. If there exists $z_0 \in U$ such that $g(z) = (z - z_0)^p \varphi(z)$, where p is a positive integer and $\varphi(z)$ is a regular function, $\varphi(z_0) \neq 0$, then

$$(\alpha - 1) \frac{zg'(z)}{g(z)} + \frac{zg''(z)}{g'(z)} = \frac{(p\alpha - 1)z}{z - z_0} + \psi(z), \tag{4}$$

where $\psi(z)$ is a regular function in U.

If $\frac{1}{\alpha}$ isn't a positive integer, then $p\alpha - 1 \neq 0$ and hence

$$\lim_{z \to z_0} \left[(\alpha - 1) \frac{zg'(z)}{g(z)} + \frac{zg''(z)}{g'(z)} \right] = \infty \tag{5}$$

Because (5) is in contradiction with (3), it results that $g(z)/z \neq 0$ for all $z \in U$.

By (ii) it results that $f(z) \cdot f'(z)/z \neq 0$ for all $z \in U$ and hence for the function $\left(\frac{f(z)}{g(z)} \right)^{\alpha-1}$ we can choose the regular branch in U, equal to 1 at the origin.

Let a and b be positive numbers and $\frac{b}{a} = m$. Because the function $(g(z)/f(z))^{\alpha-1} = 1 + c_1 z + \ldots$ is regular in U it results that there exists r_0, $0 < r_0 \leqslant 1$ such that

$$h(z, t) = 1 + \alpha(e^{bt} - e^{-at}) \left[\frac{g(e^{-at}z)}{f(e^{-at}z)} \right]^{\alpha-1} \frac{zg'(e^{-at}z)}{f(e^{-at}z)} \neq 0 \tag{6}$$

for all $t \in I$ and $z \in U_{r0} = \{z : |z| < r_0\}$ and hence the function

$$f(z, t) = f(e^{-at}z) [h(z, t)]^{1/\alpha} = a_1(t)z + \ldots \tag{7}$$

is regular in U_{r0} (in (6) and (7) we choose the branch equal to 1 at the origin).

In (7) $a_1(t)$ is given by

$$a_1(t) = e^{\frac{(a+b-\alpha a)t}{\alpha}} \left[\alpha + (1 - \alpha) e^{-(a+b)t} \right]^{1/\alpha} \tag{8}$$

By (2) it results that $\operatorname{Re} \alpha > 0$ and hence
$\operatorname{Re} [\alpha + (1 - \alpha)e^{-(a+b)t}] > 0$ and $a_1(t) \neq 0$ for all $t \in I$.

We observe that

$$\operatorname{Re} \frac{(a + b - \alpha a)}{\alpha} = a \operatorname{Re} \frac{m + 1 - \alpha}{\alpha} = \frac{a}{|\alpha|^2} \operatorname{Re} [- |\alpha|^2 + \alpha(m + 1)] > 0.$$

Then, from (2) we conclude that $\lim_{t \to \infty} a_1(t) = \infty$ (we have chosen a fixed branch for $a_1(t)$).

It follows that $f(z, t)/a_1(t)$ forms a normal family of regular fuctions in U_{r1}, $r_1 = r_0/2$.

Let $p(z, t)$ be the function defined in $U_{r1} \times [0, \infty)$ by

$$p(z, t) = z \frac{\partial f(z, t)}{\partial z} \Big/ \frac{\partial f(z, t)}{\partial t}, \ t \in I. \tag{9}$$

In order to prove that the function $p(z, t)$ has an analytic extension with pozitive real part in U, for all $t \in I$, it is sufficient to prove that the function

$$w(z, t) = \frac{p(z, t) - 1}{p(z, t) + 1} \tag{10}$$

is regular in U for $t \in I$ and

$$|w(z, t)| < 1, \ \forall z \in U, \ \forall t \in I. \tag{11}$$

By (6), (7), (9) and (10) we obtain

$$w(z, t) = \frac{(1 + a)A(z, t) + 1 - b}{(1 - a)A(z, t) + 1 + b}, \tag{12}$$

where

$$A(z, t) = e^{-(a+b)t} \left[\left(\frac{f(e^{-at}z)}{g(e^{-at}z)} \right)^{\alpha - 1} \frac{f'(e^{-at}z)}{g'(e^{-at}z)} - 1 \right] + \tag{13}$$

$$+ (1 - e^{-(a+b)t}) \left[(\alpha - 1) \frac{e^{-at}z \cdot g'(e^{-at}z)}{g(e^{-at}z)} + \frac{e^{-at}z \cdot g''(e^{-at}z)}{g'(e^{-at}z)} \right]$$

and hence the function $w(z, t)$ is regular in U for all $t \in I$.

Let $X = \operatorname{Re} A(z, t)$, $Y = \operatorname{Im} A(z, t)$. Then from (12) we have

$$|w(z, t)|^2 = \frac{|(1 + a)(X + iX) + 1 - b|^2}{|(1 - a)(X + iY) + 1 + b|^2} = \frac{[(1 + a)X + 1 - b]^2 + [(1 + a)Y]^2}{[(1 - a)X + 1 + b]^2 + [(1 - a)Y]^2} < 1$$

if $X^2 + Y^2 + \frac{a - b}{a} X - \frac{b}{a} < 0$, or

$$\left| A(z, t) + 1 - \frac{m + 1}{2} \right| < \frac{m + 1}{2}, \text{ that is} \tag{14}$$

$$\left| \frac{2}{m + 1} (A(z, t) + 1) - 1 \right| < 1 \tag{15}$$

If $A^*(z, t)$ is the function defined by

$$A^*(z, t) = \frac{2}{m+1}\,[A(z, t) + 1] - 1 =$$

$$= \frac{2e^{-(a+b)t}}{m+1}\left[\left(\frac{f(e^{-at}z)}{g(e^{-at}z)}\right)^{\alpha-1}\frac{f'(e^{-at}z)}{g'(e^{-at}z)} - \frac{m+1}{2}\right] +$$

$$+ \frac{2(1 - e^{-(a+b)t}}{m+1}\left[(\alpha - 1)\,\frac{e^{-at}z\cdot g'(e^{-at}z)}{g(e^{-at}z)} + \frac{e^{-at}z\cdot g''(e^{-at}z)}{g'(e^{-at}z)} + 1 - \frac{m+1}{2}\right]$$

then from (15) it results that the inequality $|w(z, t)| < 1$ is equivalent to

$$|A^*(z, t)| < 1$$

From the condition (ii) of the Theorem 1 we obtain

$$|A^*(z, 0)| < 1 \text{ for all } z \in U.$$

Let's observe that if $t > 0$ then the function $A^*(z, t)$ is regular in $\bar{U}$ and hence:

$$|A^*(z, t)| < \max_{|z|=1} |A^*(z, t)| = |A^*(e^{i\theta}, t)|, \tag{16}$$

where θ is a real number.

If $\xi = e^{-at}\cdot e^{i\theta}$, then $|\xi| = e^{-at}$ and

$$e^{-(a+b)t} = |\xi|^{m+1}, \text{ where } m = \frac{b}{a}.$$

Replacing in (16) we obtain

$$|A^*(z, t)| < \frac{2}{m+1}\left|\,|\xi|^{m+1}\left[\left(\frac{f(\xi)}{g(\xi)}\right)^{\alpha-1}\frac{f'(\xi)}{g'(\xi)} - \frac{m+1}{2}\right] +\right.$$

$$\left. + (1 - |\xi|^{m+1})\left[(\alpha - 1)\,\frac{\xi g'(\xi)}{g(\xi)} + \frac{\xi g''(\xi)}{g'(\xi)} + 1 - \frac{m+1}{2}\right]\right| \tag{17}$$

for all $z \in U$ and $t > 0$.

Because $\xi \in U$ for $t > 0$, by condition (iii) of the Theorem 1 and by inequality (17) it results that $|A^*(z, t)| < 1$ for all $z \in U$ and $t > 0$ and hence $|w(z, t)| < 1$ for all $z \in U$ and $t > 0$. From Lemma it results that the function $f(z, t)$ is regular and univalent in U for all $t \in I$ and hence for $t = 0$ we conclude that the function $f(z)$ is univalent in U.

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AN ELEMENTARY DERIVATION OF STIRLING'S ASYMPTOTIC SERIES

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S subject classification: 10A25

REZUMAT. — O deducere elementară a seriei asimptotice a lui Stirling. În lucrare se prezintă o metodă elementară care permite deducerea seriei asimptotice a lui Stirling (2).

1. Introduction. Stirling's asymptotic formula

$$n! \sim \sqrt{2\pi n}\, n^n e^{-n} \tag{1}$$

an important tool in various branches of mathematics, for a detailed proof : for example [3, pp. 181—184] or [5]. Using the Euler summation formula e can deduce Stirling's asymptotic series

$$\ln n! = \ln \sqrt{2\pi} + \left(n + \frac{1}{2}\right) \ln n - n + \sum_{k=1}^{\infty} \frac{B_{k+1}}{k(k+1)n^k} \quad \text{or}$$

$$n! = \sqrt{2\pi n}\, n^n e^{-n} \exp \left\{ \sum_{k=1}^{\infty} \frac{B_{k+1}}{k(k+1)n^k} \right\}, \tag{2}$$

ıere B_m is the m-th Bernoulli number, giving more precise informations abouf : behaviour of $n!$ for n tending to infinity than formula (1), cf. [1, ch. 14]

Recently V. Namias [4] gave a simple proof of formula (2) without :ourse to Euler's summation formula but applying Legendre's duplication :mula and Gauss' multiplication formula, respectively, concerning the gamma nction.

The purpose of this note is to present another simple elementary method obtain the asymptotic series (2). Our treatment uses only the series expan- ›n of the logarithmic function and the binomial series and it does not refer any property of the gamma function but as a starting point of our proof : use formula (1).

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2. **The proof.** Let's consider the sequence (x_n) defined by $x_n = \frac{n!\,e^n}{\sqrt{2\pi n}\, n^n}$ for any $n \geqslant 2$. We shall assume that x_n possesses an asymptotic series representation

$$x_n = \exp\left\{\sum_{k=0}^{\infty} \frac{a_k}{n^k}\right\}, \text{ or} \tag{3}$$

$$\ln x_n = \sum_{k=0}^{\infty} \frac{a_k}{n^k},$$

meaning that there exist the limits

$$\lim_{n\to\infty} \ln x_n = a_0,$$

$$\lim_{n\to\infty} n(\ln x_n - a_0) = a_1.$$

$$\lim_{n\to\infty} n^2\left(\ln x_n - a_0 - \frac{a_1}{n}\right) = a_2,$$

Here $a_0 = 0$, as it is mentioned above and $a_1, a_2, \ldots$ are constant coefficients to be determinated. We have to obtain, of course, $a_k = \frac{B_{k+1}}{k(k+1)}$ for any $k \geqslant 1$, since a function can have at most one asymptotic series.

We have

$$\frac{x_n}{x_{n+1}} = \frac{1}{e}\left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}} = \exp\left\{\left(n + \frac{1}{2}\right)\ln\left(1 + \frac{1}{n}\right) - 1\right\}. \tag{4}$$

Using the expansion

$$\ln(1 + x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}$$

for $x = \frac{1}{n} < 1$ and inserting it into (4) we obtain

$$\frac{x_n}{x_{n+1}} = \exp\left\{\sum_{k=2}^{\infty} (-1)^{k-1}\left(\frac{1}{k} - \frac{1}{2(k-1)}\right)\cdot\frac{1}{n^{k-1}}\right\} \text{ or}$$

$$\frac{x_n}{x_{n+1}} = \exp\left\{\sum_{k=1}^{\infty} (-1)^k \frac{k-1}{2k(k+1)}\cdot\frac{1}{n^k}\right\}. \tag{5}$$

On the other hand, according to (3) we get

$$\frac{x_n}{x_{n+1}} = \exp\left\{\sum_{k=1}^{\infty}\frac{a_k}{n^k} - \sum_{k=1}^{\infty}\frac{a_k}{(n+1)^k}\right\}$$

$$= \exp\left\{\sum_{k=1}^{\infty}\frac{a_k}{n^k} - \sum_{m=1}^{\infty}\frac{a_m}{n^m\left(1+\frac{1}{n}\right)^m}\right\}. \tag{6}$$

Applying now the binomial series

$$(1+x)^{-s} = 1 + \sum_{k=1}^{\infty}(-1)^k\frac{s(s+1)\ldots(s+k-1)}{k!}x^k$$

$s = m$ and $x = \frac{1}{n} < 1$ we immediately obtain

$$\frac{1}{\left(1+\frac{1}{n}\right)^m} = 1 + \sum_{k=m+1}^{\infty}(-1)^{k-m}\frac{m(m+1)\ldots(k-1)}{(k-m)!\,n^{k-m}} =$$

$$1 + \sum_{k=m+1}^{\infty}(-1)^{k-m}\frac{(k-1)!}{(m-1)!\,(k-m)!\,n^{k-m}} = \frac{(-1)^m}{(m-1)!}\sum_{k=m}^{\infty}\frac{(-1)^k(k-1)!}{(k-m)!\,n^{k-m}}. \tag{7}$$

nce by (6) and (7) we conclude

$$\frac{x_n}{x_{n+1}} = \exp\left\{\sum_{k=1}^{\infty}\frac{a_k}{n^k} - \sum_{m=1}^{\infty}\frac{(-1)^m a_m}{(m-1)!\,n^m}\sum_{k=m}^{\infty}\frac{(-1)^k(k-1)!}{(k-m)!\,n^{k-m}}\right\} =$$

$$= \exp\left\{\sum_{k=1}^{\infty}\frac{a_k}{n^k} + \sum_{k=1}^{\infty}(-1)^k(k-1)!\sum_{m=1}^{k}\frac{(-1)^{m-1}a_m}{(k-m)!\,(m-1)!\,n^k}\right\}.$$

us

$$\frac{x_n}{x_{n+1}} = \exp\left\{\sum_{k=1}^{\infty}\left[a_k + (-1)^k(k-1)!\sum_{m=1}^{k}\frac{(-1)^{m-1}a_m}{(k-m)!\,(m-1)!}\right]\frac{1}{n^k}\right\}. \tag{8}$$

The identity of expansions (5) and (8) enables us to write the equality the coefficients of $1/n^k$:

$$a_k + (-1)^k(k-1)!\sum_{m=1}^{k}\frac{(-1)^{m-1}a_m}{(k-m)!\,(m-1)!} = (-1)^k\frac{k-1}{2k(k+1)},\quad k \geqslant 1.$$

we separate the a_k term in the summation we obtain

$$(k-2)!\cdot\sum_{m=1}^{k-1}\frac{(-1)^{m-1}a_m}{(k-m)!\,(m-1)!} = \frac{1}{2k(k+1)!},\quad k \geqslant 2.$$

By $k := k + 1$ and solving the equation for a_k we obtain the recurrence relation

$$a_k = (-1)^k \left[(k-1)! \sum_{m=1}^{k-1} \frac{(-1)^{m-1} a_m}{(k-m+1)!\,(m-1)!} - \frac{1}{2(k+1)(k+2)}\right], \tag{9}$$

valid for any $k \geqslant 1$. From here we easily find that $a_1 = \frac{1}{12}$, $a_2 = 0$, $a_3 = \frac{-1}{360}$, $a_4 = 0$, $a_5 = \frac{1}{1260}$, ... and we obtain the series

$$n! = \sqrt{2\pi n}\, n^n e^{-n} \exp\left\{\frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} + \dots\right\}.$$

Now we show the connection of the coefficients a_k with the Bernoulli numbers, namely we establish starting with (9) that $a_k = \frac{B_{k+1}}{k(k+1)}$ for any $k \geqslant 1$, as it is well known.

From recurrence relation (9) we get

$$k(k+1)a_k = (-1)^k \left[(k+1)! \sum_{m=1}^{k+1} \frac{(-1)^{m-1} m(m+1) a_m}{(k-m+1)!\,(m+1)!} - \frac{k}{2(k+2)}\right].$$

Let's denote $k(k+1)a_k = b_{k+1}$ for any $k \geqslant 1$, hence

$$b_{k+1} = (-1)^k \left[\frac{1}{k+2} \sum_{m=1}^{k-1} (-1)^{m-1} \binom{k+2}{m+1} b_{m+1} - \frac{k}{2(k+2)}\right].$$

By $k := k - 2$ and $m := m - 1$ this can be written in the form

$$b_{k-1} = (-1)^k \left[\frac{1}{k} \sum_{m=2}^{k-2} (-1)^m \binom{k}{m} b_m - \frac{k-2}{2k}\right], \quad k \geqslant 3,$$

or in the more compact form

$$b_{k-1} = (-1)^k \left[\frac{1}{k} \sum_{m=0}^{k-2} (-1)^m \binom{k}{m} b_m - 1\right], \quad k \geqslant 1, \tag{10}$$

where $b_0 = 1$, $b_1 = -\frac{1}{2}$.

For $k = 1$, 2, 3 and 4 relation (10) yields $b_0 = 1 = B_0$, $b_1 = -\frac{1}{2} = B_1$, $b_2 = \frac{1}{6} = B_2$ and $b_3 = 0 = B_3$. We now prove by induction that $b_n = B_n$ for any $n \geqslant 0$. Suppose $b_n = B_n$ for any $n \leqslant k-2$, $k \geqslant 4$, then one obtains from (10)

$$b_{k-1} = (-1)^k \left[\frac{1}{k} \sum_{m=0}^{k-2} (-1)^m \binom{k}{m} B_m - 1\right] = (-1)^k \left[\frac{1}{k} \left(\sum_{m=0}^{k-2} \binom{k}{m} B_m - 2\binom{k}{1} B_1\right) - 1\right] = (-1)^k \left[\frac{1}{k}\left(-\binom{k}{k-1} B_{k-1} + k\right) - 1\right] = (-1)^{k-1} B_{k-1},$$

ing the following familiar properties of the Bernoulli numbers:

$$B_{2m+1} = 0 \text{ for any } m \geqslant 1 \text{ and } \sum_{m=0}^{k-1} \binom{k}{m} B_m = 0 \text{ for any } k \geqslant 2.$$

If $k - 1$ is even we have $b_{k-1} = B_{k-1}$, and if $k - 1$ is odd then $B_{k-1} = 0 = b_{k-1}$ and obtain that $b_n = B_n$ for any $n \geqslant 0$, as required.

3. Final remarks. Using the series (2) and the series expansion of the expo-ntial function one can derive the following asymptotic series of n!

$$n! \doteq \sqrt{2\pi n}\, n^n e^{-n} \left\{ 1 + \sum_{k=1}^{\infty} \frac{c_k}{n^k} \right\}, \tag{11}$$

here the first four coefficients c_k are $c_1 = \frac{1}{12}$, $c_2 = \frac{1}{288}$, $c_3 = \frac{-139}{51840}$, $c_4 = \frac{-571}{2488320}$. ite recently G. Marsaglia and J. C. W. Marsaglia [2] gave an teresting direct derivation of the series (11). They call the attention to the ct, and we also want to underline this, that the asymptotic series (2) and 1) do not converge, although they provide valuable informations about n!. r further general properties of the asymptotic series we refer once again to e book of K. Knopp [1].

I am grateful to Professor József Kolumbán for useful discussions.

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ON THE SOLUTIONS OF A FUNCTIONAL EQUATION USING PICARD MAPPINGS

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REZUMAT. — Asupra soluţiilor unei ecuaţii funcţionale folosind aplicaţii Picard. În lucrare sînt prezentate soluţii ale ecuaţiei funcţionale (1), în cazul în care f este o aplicaţie Picard. Acestea sînt exprimate prin intermediul unei serii care poate fi convergentă dar şi divergentă dar sumabilă printr-o metodă de sumare dată (Césaro, Abel, Toeplitz).

0. **Introduction.** This work is concerned with the solutions g of the functional equation (1). The solutions are given in the form of convergent series or divergent series but summable by some method of summability.

1. **Preliminaries.** Let (X, d) be a complete metric space, Y a real Banach space and let us consider the following functional equation:

$$g(f(x)) + g(x) = F(x), \text{ for all } x \in X \tag{1}$$

where $f: X \to X$, $F: X \to Y$ are given mappings and $g: X \to Y$ is an unknown mapping.

When f, F and g are functions of a single real variable $x \in [a, b]$, the solutions of equation (1) have been studied by many authors [1], [3], [5]. Thus, the work [3] of Kuczma extends some results due to Hardy, respectively Steinhaus, [1] extends [3], [5] extends [1] etc. (for more details see the first section in [5] and also [4], [3]).

The present work may be regarded as an extension of a part of the results given in the papers quoted above, especially in [1]. One can treat similarly other results in [1] and [4], e.g. theorem 3.1 and theorem 4.1 from [5].

In the papers [1], [3], [5] f is assumed to satisfy the following conditions: f is a continuous and strictly increasing function on $[a, b]$; $f(a) = a$, $f(b) = b$ and $f(x) > x$, for all $x \in (a, b)$.

Under thase assumptions it can be readly verified [2] that the sequence of the iterates of f, $(f^n(x))_{n \geq 0}$, defined, as usually, by $f^{(0)}(x) = x$ and $f^n(x) = f(f^{n-1}(x))$, for $n \geqslant 1$ and for all $x \in [a, b]$, converges to b, i.e.

$$\lim_{n\to\infty} f^n(x) = b, \text{ for all } x \in (a, b] \tag{2}$$

Let us observe that only the condition (2) is essentially in the proof of theorem I from [1], and, consequently, we may consider a slower condition on f, i.e., f is a Picard mapping [6].

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This weakness is effective, as is shown by

Example 1.1. The function $f\colon [0, 1] \to [0, 1]$, $f(0) = 0$,

$$f(x) = 6x^2 - \frac{7}{2}x + 1, \text{ for } x \in \left(0, \frac{1}{2}\right) \text{ and } f(x) = 1, \text{ for}$$

$x \in \left[\frac{1}{2}, 1\right]$ is not continuous, also not strictly increasing on $[0, 1]$, but it can be easily verified that

$$\lim_{n\to\infty} f^n(x) = 1, \text{ for all } x \in (0, 1].$$

The aim of this note is to show that the theorem I in [1] remains valid under these weak conditions.

2. **Picard mappings.** For the definitions, examples, properties and other results concerning Picard mappings we refer to [6].

As usually, we denote by F_f the set of all fixed points of a mapping f.

DEFINITION 2.1. Let (X, d) be a metric space. A mapping $f\colon X \to X$ is said to be a *Picard mapping* if there exists $b \in X$ such that

$$F_f = \{b\}$$

and

$$(f^n(x))_{n\in\mathbf{N}} \text{ converges to } b, \text{ for all } x \in X.$$

Example 2.1 Let (X, d) be a complete metric space. A contraction mapping $f\colon X \to X$ is a Picard mapping.

Example 2.2. Let (X, d) be a compact metric space. A contractive mapping $f\colon X \to X$ is a Picard mapping.

Example 2.3. [3] Let $f\colon [a, b] \to [a, b]$ be a continuous and strictly increasing function on $[a, b]$, $f(b) = b$ and $f(x) > x$, for all $x \in [a, b)$. Then f is a Picard mapping.

DEFINITION 2.2. A function $\varphi\colon \mathbf{R}_+ \to \mathbf{R}_+$ is a *comparison function* if satisfies the conditions

i) φ is monotone increasing.

ii) $(\varphi^n(t))_{n\in\mathbf{N}}$ converges to 0, for all $t \geqslant 0$.

DEFINITION 2.3. Let (X, d) be a metric space. A mapping $f\colon X \to X$ is a *φ-contraction* if φ is a comparison function and

$$d(f(x), f(y)) \leqslant \varphi(d(x, y)), \text{ for all } x, y \in X.$$

Now let us recall (see [6], p. 33–34) some results which give sufficient conditions for the mapping f be a Picard mapping.

THEOREM 2.1. *Let (X, d) be a complete metric space and $\varphi\colon \mathbf{R}_+ \to \mathbf{R}_+$ a function which satisfies condition* i) *and the following two conditions*

iii) *$\varphi(t) < t$, for all $t > 0$;*

iv) *φ is right continuous.*

If $f: X \to X$ *is a mapping such that*
$d(f(x), f(y)) \leqslant \varphi(d(x, y))$, *for all* $x, y \in X$, *then* f *is a Picard mapping.*

THEOREM 2.2. *Let* (X, d) *be a complete metric space and* $f: X \to X$ *a* φ*-contraction. Then* f *is a Picard mapping.*

THEOREM 2.3. *Let* (X, d) *be a complete metric space and* $\varphi: \mathbf{R}_+ \to [0, 1)$ *a monotone decreasing function. If* $f: X \to X$ *is such that*

$$d(f(x)), f(y)) \leqslant \varphi(d(x, y)) \cdot \mathrm{d}(x, y), \text{ for all } x, y \in X,$$

then f *is a Picard mapping.*

THEOREM 2.4. *Let* (X, d) *be a complete metric space and* $\varphi: \mathbf{R}_+ \to \mathbf{R}_+$ *a function such that* $\varphi(r) > 0$, *for* $r > 0$.

If $f: X \to X$ *is a mapping such that*

$$d(f(x), f(y)) \leqslant d(x, y) - \varphi(d(x, y)), \text{ for all } x, y \in X,$$

then f *is a Picard mapping.*

3. **Sequences and series.** In this section we recall some definitions and properties concerning sequences in a metric space, series in a real Banach space and we also introduce some concepts of summability of divergent series in Banach spaces, analogous to those for number series in [2].

For simplicity, in the sequel we shall assume, without any special mention, that (X, d) is a complete metric space and Y is a real Banach space.

DEFINITION 3.1. A series $\sum_{n=0}^{\infty} a_n$ in Y is said to be *convergent* with the sum S, and we write $S = \sum_{n=0}^{\infty} a_n$ if the sequence of partial sums $(S_n)_{n \in \mathbf{N}}$ converges to S (in norm).

A series of operators $\sum_{n=0}^{\infty} f_n$, $f_n: X \to Y$, is *convergent* on X to f if, for all $x \in X$, the series

$$\sum_{n=0}^{\infty} f_n(x) \text{ converges to } f(x) \in Y.$$

DEFINITION 3.2. Let $\sum_{n=0}^{\infty} x_n$ be a series in Y. We denote, as usually, by $(S_n)_{n \in \mathbf{N}}$ the sequence of partial sums, and let us consider the sequence $(S_n^{(k)})_{k \in \mathbf{N}}$, defined by $S_n^{(0)} = S_n$ and, for $k \geqslant 1$,

$$S_n^{(k)} = S_0^{(k-1)} + S_1^{(k-1)} + \ldots + S_n^{(k-1)}, \quad (n = 0, 1, 2, \ldots)$$

If, for some k, the sequence $(C_n^{(k)})_{n\in\mathbf{N}}$

$$C_n^{(k)} = \frac{1}{\binom{n+k}{k}} S_n^{(k)}, \ n \geqslant 0,$$

(where $\binom{n+k}{k}$ is the binomial coefficient) converges to S, we say that t the series $\sum_{n=0}^{\infty} x_n$ is *Césaro-summable* or C_k — *summable* with the sum S.

DEFINITION 3.3. Let $(S_n)_{n\in\mathbf{N}}$ be the sequence of partial sums of the ser $\sum_{n=0}^{\infty} x_n$ in Y. Let us consider an infinite matrix $T = (a_{kn})$ of real numbers a let us construct the following series

$$\sum_{n=0}^{\infty} a_{kn} S_n, \ (k = 0, 1, 2, \ldots)$$

We assume that the seies (3) is convergent, with the sum S'_k, $k = 0, 1, 2,$ If the sequence $(S'_k)_{k\in\mathbf{N}}$ converges to S, then the series $\sum_{n=0}^{\infty} x_n$ is said to be T *plitz-summable* or *T-summable* with the sum S.

DEFINITION 3.4. A series $\sum_{n=0}^{\infty} x_n$ in Y is *summable by Abel's method* of su mability, or *A-summable*, with the sum S, if the series $\sum_{n=0}^{\infty} t^n x_n$ converges for $t \in [-r, r]$, $r \geqslant 1$, and there exists

$$\lim_{t\to 1-0} \left(\sum_{n=0}^{\infty} t^n x_n\right) = S.$$

(If $h : \mathbf{R} \times Y \to Y$, the limit

$$\lim_{t\to 1-0} h(t, x) = l_x$$

must be understood as

$$\lim_{t\to 1-0} || h(t, x) - l_x || = 0).$$

Since theorems 1 and 2 [2] pp. 404—405 are important for this work quote them here, adapted to Banach spaces.

THEOREM 3.1. *Let* $(z_n)_{n\in\mathbf{N}}$ *be a sequence,* $z_n \in Y$, *for every* $n = 0, 1, 2, \ldots$ *and* $\lim_{n\to\infty} z_n = 0 \in Y$. *If* $T = (a_{kn})$ *is an infinite regular matrix whose eleme* $a_{kn} \in \mathbf{R}$ *satisfy the conditions* t1) *For each* $n \geqslant 0$, $a_{kn} \to 0$ *as* $k \to \infty$,

t2) *There exists a constant* C *such that*

$$|a_{k_0}| + |a_{k_1}| + \ldots + |a_{kn}| < C, \ (k \geqslant 0,\ n \geqslant 0),$$

then the series $\sum_{n=0}^{\infty} a_{kn} z_n$ *converges for all* $k \geqslant 0$.

Moreover, the sequence $(z'_k)_{k \in \mathbf{N}}$

$$z'_k = \sum_{n=0}^{\infty} a_{kn} z_n, \quad k = 0, 1, 2, \ldots \tag{4}$$

is convergent in Y *and* $\lim_{k \to \infty} z'_k = 0$.

THEOREM 3.2. *If* $T = (a_{kn})$ *is a regular matrix whose elements* $a_{kn} \in \mathbf{R}$ *satisfy the conditions* t1), t2) *from theorem* 3.1 *and the following*

t3)
$$\lim_{k \to \infty} \left(\sum_{n=0}^{\infty} a_{kn} \right) = 1,$$

then, for any sequence $(z_n)_{n \in \mathbf{N}}$ *converging to* z *in* Y, *as* $n \to \infty$, *the sequence* $(z'_k)_{k \in \mathbf{N}}$, *given by* (4), *is convergent and* $\lim_{k \to \infty} z'_k = z$.

4. Solutions of the functional equation. Now we return to equation (1). Let $f: X \to X$ be a Picard mapping and $F_f = \{b\}$. As in [1], [3], [5] we consider the following series

$$\frac{1}{2} F(b) + \sum_{n=0}^{\infty} (-1)^n \{F[f^n(x)] - F(b)\}. \tag{5}$$

The aim of this section is to give sufficient conditions for the existence of a solution of equation (1), by using the convergence or the summability of the series (5).

The main result is stated in

THEOREM 4.1. *Let* (X, d) *be a complete metric space,* Y *a real Banach space,* $F: Y \to Y$ *a given mapping and* $f: X \to X$ *a given Picard mapping.*

a) *If the series* (5) *converges on* X, *its sum* $g(x)$ *is a solution of equation* (1).

b) *If the series* (5) *is* T — *summable with the sum* $g(x)$, *where* $T = (a_{kn})$ *is a regular matrix transformation whose elements* $a_{kn} \in \mathbf{R}$ *satisfy the conditions* t1)–t3), *then* $g(x)$ *is a solution of the equation* (1) *if* F *is continuous in* b.

c) *If the series* (5) *is* C_k — *summable with the sum* $g(x)$, *then* $g(x)$ *is a solution of* (1).

d) *If the series* (5) *is* A — *summable with the sum* $g(x)$ *then* $g(x)$ *is a solution of* (1).

Proof.

a) Substituting $g(x)$ given by the series (5) in (1), we obtain that t first statement holds.

b) Putting

$$S_n(x) = \frac{1}{2} F(b) + \sum_{i=0}^{n} (-1)^i [F(f^i(x)) - F(b)], \; n = 0, 1, 2, \dots$$

and

$$S'_k(x) = \sum_{n=0}^{\infty} a_{kn} S_n(x), \; (k = 0, 1, 2, \dots)$$

it results from the T -- summability of the series (5) that

$$g(x) = \lim_{k \to \infty} S'_k(x).$$

On the other hand, we have

$$S_n[f(x)] = F(x) - S_n(x) - (-1)^{n+1}[F(f^{n+1}(x)) - F(b)]$$

and consequently

$$S'_k[f(x)] = \sum_{n=0}^{\infty} a_{kn}F(x) - S'_k(x) - \sum_{n=0}^{\infty} (-1)^{n+1} a_{kn}[F(f^{n+1}(x)) - F(b)]. \quad (6)$$

From t1), t2) and the continuity of F in b, it follows that

$$\lim_{n \to \infty} \{(-1)^{n+1}[F(f^{n+1}(x)) - F(b)]\} = 0$$

hence, applying theorem 3.1 we obtain

$$\lim_{k \to \infty} \left(\sum_{n=0}^{\infty} (-1)^{n+1} a_{kn} [F(f^{n+1}(x)) - F(b)] \right) = 0$$

It is also obvious, using theorem 3.2 that

$$\lim_{k \to \infty} \left(\sum_{n=0}^{\infty} a_{kn} F(x) \right) = F(x), \text{ for all } x \in X$$

and therefore, taking $k \to \infty$, (6) becomes

$$g[f(x)] = F(x) - g(x).$$

c) The proof is similar with that of the third part of theorem 1 [1] and the unnecessary details will be omitted.

If we denote by $S_n(x)$, $S_n(f(x))$ the partial sum of the series (5), respectively of the series obtained from (5) replacing x by $f(x)$, it results by an elementary calculation

$$C_n^{(k)}(x) = \frac{n}{n+k} C_{n-1}^{(k)}(x) + \frac{1}{\binom{n+k}{k}} S_n^{(k-1)}(x),$$

and

$$C_n^{(k)}(x) + C_n^{(k)}(f(x)) = F(x) - \frac{k}{2(n+k)} F(b) + \frac{1}{\binom{n+k}{k}} S_n^{(k-1)}(f(x)). \qquad (7)$$

Now, in (7) we take $n \to \infty$ and follows, from the C_k — summability of the series (5), i.e.

$$\lim_{n\to\infty} C_n^{(k)}(x) = g(x),$$

that

$$g(x) + g[f(x)] = F(x).$$

d) This follows from theorem 1 [1], d).

The proof of theorem is now complete.

Finally, let us observe that all results in this paper remain valid if X and Y are complex rather than real spaces.

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T-BIRECURRENT AFFINE CONNECTIONS

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REZUMAT. — Conexiuni afine T-birecurente. În lucrare se introduce noţiunea de T-birecurenţă în spaţii cu conexiune afină prin (1), punîndu-se în evidenţă în şapte propoziţii cîteva proprietăţi şi relaţii ce există în aceste spaţii în general, precum şi în cazul conexiunilor semi-simetrice şi E-conexiunilor semi-simetrice.

Let A_n be a space with affine connection. We denote by Γ^i_{jk} the components of the connection in a local map, by $T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$ the components of the torsion tensor and by $T_k = T^i_{ik}$ the components of the torsion vector, or the Vrânceanu's vector.

DEFINITION 1. An A_n space is called *T-birecurrent* if there exists a covariant tensor of second order φ_{rs} so that

$$T^i_{jk,rs} = \varphi_{rs} T^i_{jk} \quad , \tag{1}$$

where comma denotes the covariant derivation with respect to Γ.

Contracting (1) in i and j we get

$$T_{k,rs} = \varphi_{rs} T_k \tag{2}$$

therefore:

PROPOSITION 1. *The T-birecurrent A_n spaces have also the Vrânceanu's vector T-birecurrent with the same T-birecurrency tensor.*

The A_n spaces for which exists a convector φ_r so that

$$T^i_{jk,r} = \varphi_r T^i_{jk} \tag{3}$$

were called spaces with recurrent torsion, or *T-recurrent* [2] if in (3) we derivate covariantly with respect to Γ and we take it into account, we have

$$T^i_{jk,rs} = \varphi_{r,s} T^i_{jk} + \varphi_r T^i_{jk,s} = (\varphi_{r,s} + \varphi_r\varphi_s) T^i_{jk} \tag{4}$$

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and the space A_n is T-birecurrent. Therefore we have:

PROPOSITION 2. *The T-recurrent A_n spaces are T-birecurrent with T-birecurrency tensor*

$$\varphi_{rs} = \varphi_{r,s} + \varphi_r\varphi_s. \tag{5}$$

Suppose now that the connection of the A_n space is semisymmetric, therefore [3], [4]

$$T^i_{jk} = \frac{1}{n-1}\,(\delta^i_j T_k - \delta^i_k T_j) \tag{6}$$

we remark that from (6) and (2) it follows (1), therefore, from the T-birecurrency of the Vrânceanu's vector it follows the T-birecurrency of the A_n space. We have therefore:

PROPOSITION 3. *The A_n spaces with semi-symmetric connection with the Vrânceanu's vector T-birecurrent, are T-birecurrent.*

From the relation of S. G o l a b [4] for spaces with semi-symmetric connection

$$T^i_{sj}T^s_{kh} + T^i_{sk}T^s_{hj} + T^i_{sh}T^s_{jk} = 0 \tag{7}$$

by a two-times covariant derivation and taking count of (1) and (7) it follows:

$$T^i_{sj,r}T^s_{kh,p} + T^i_{sj,p}T^s_{kh,r} + T^i_{sk,r}T^s_{hj,p} + T^i_{sk,p}T^s_{hj,r} + \\ + T^i_{sh,r}T^s_{jk,p} + T^i_{sh,p}T^s_{jk,r} = 0 \tag{8}$$

We have therefore:

PROPOSITION 4. *In an A_n space with T-birecurrent semi-symmetric connection, the torsion tensor verifies (8).*

In the A spaces with semi-symmetric connection, take place [3], [4]:

$$T^i_{jk}T_i = 0. \tag{9}$$

Derivating covariantly two-times the relations (9) and taking count of (1) and (9) it follows:

$$T^i_{jk,r}T_{i,s} + T^i_{jk,s}T_{i,r} = 0 \tag{10}$$

if in (8) we contract with respect to i and j and take count of (10) we have

$$T^i_{sk,r}T^s_{hi,p} + T^i_{sk,p}T^s_{hi,r} + T^i_{sh,r}T^s_{ik,p} + T^i_{sh,p}T^s_{ik,r} = 0 \tag{11}$$

and therefore:

PROPOSITION 5. *In an A_n space with T-birecurrent semi-symmetric connection, take place the relations (10) and (11).*

If the connection of the A_n space is an semi-symmetric E-connection [3], that is

$$T_{i,j} = T_{j,i} \tag{12}$$

derivating covariantly with respect to Γ, and taking count of (1) we have

$$\varphi_{jr} T_i - \varphi_{ir} T_j = 0 \tag{13}$$

therefore

PROPOSITION 6. *In an A_n space endowed with a T-birecurrent semi-symmetric E-connection, the T-birecurrency tensor verifies (13).*

From the vanishing of the divergence of the torsion in an A_n space with semi-symmetric E-connection [3], [4]

$$T^i_{jk,\,i} = 0 \tag{14}$$

by covariant derivation and taking count of (1) we have

$$T^i_{jk}\varphi_{is} = 0 \tag{15}$$

and therefore

PROPOSITION 7. *In the A_n spaces endowed with a T-birecurrent semi-symmetric E-connection the T-birecurrency tensor is a solution of the homogeneous systems (15)*

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BOUNDARY ELEMENT TECHNIQUES WITH COMPLEX VALUES FOR PLANE HYDRODYNAMICS

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REZUMAT. — Tehnici de element frontieră cu valori complexe în hidrodinamica plană. În lucrare se foloseşte o nouă metodă numerică, metoda elementului frontieră cu valori complexe, în studiul unor probleme de hidrodinamică plană.

This paper is devoted to the use of a new numerical technique — boundary elements' method with complex values (CVBEM) — in some problems of the plane hydrodynamics.

More precisely it will be considered namely the fluid flow produced by a general rototranslation in the mass of an ideal fluid of an airfoil with an angular point. The technique used could be easily extended to a system of profiles, performing independent displacements in the mass of the fluid, in the possible presence of some fixed walls. For more generality the ideal fluid is supposed having an „apriori" given basical flow which could present singularities as vortices, sources, etc.

It is known that all the boundary elements methods require an integral representation of the approached boundary problem, and a corresponding integral equation on the boundary. But in the case of the plane hydrodynamics even the Cauchy's formula — used for the solution of the involved boundary problem — is in fact such an integral representation which leads automatically to an integral equation with Cauchy singularity on the boundary. An appropiate interpolating system of functions allows then a quite accurate solving of this integral equation on the boundary. The procedure doesn't use any approximation of the boundary or any numerical quadrature.

Let now be a plane incompressible potential inviscid fluid flow. It is well known that it is always possible to join to such a flow an analytic function $f(z)$ — called the complex potential of the flow whose knowledge is entirely equivalent with the complete determination of the flow.

Conversely any holomorphic function in a given domain could be interpreted as a complex potential of a plane incompressible potential inviscid flow pending addition of some logarithmic terms (multiform functions) in the case of multiply — connected domains.

If we consider only a simple connected domain — like the outside of an obstacle (C) — the complex potential of a fluid flow with the qualities men-

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tioned above, around the obstacle (C), this will be an analytical function in every finite point, having in the neighborhood of infinity the development

$$f(z, t) = w_{\infty} z + \frac{\Gamma}{2\pi i} \ln z + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots$$

We denoted here by

$$w_{\infty} = \lim_{|z| \to \infty} \frac{df}{dz}$$

the complex velocity of the fluid at great distances, by Γ a real function of time (which could be a constant or even zero) called the circulation of the flow and which represents the multiformity period of the real part of the complex potential f, and by t, the time which could explicity appear, the flow being then a nonstationary one.

Additionaly the imaginary part of the values of the function $f(z, t)$ (i.e. the stream function ψ) are given along the contour C. Supposing that the obstacle (C) is performing a general rototranslation in the mass of fluid then, if $l(t)$, $m(t)$ are the components of the translation velocity in a point $z_A \in (C)$ — evaluated in a mobile system of coordiantes Oxy centered in $z_A = 0$ — and ω the instantaneous rotation of the profile, the boundary condition for the function ψ in the points of C is

$$\psi \Big|_C = ly - mx + \frac{\omega}{2} (x^2 + y^2) + \textit{arbitrary function of time} \Big|_C$$

We remark that if instead of the complex potential $f(z, t)$ we would construct the complex velocity

$$w(z; t) = \frac{df}{dz},$$

this will be holomorphic function in the whole outside of the profile (C) which also includes the point at infinity. In the neighborhood of this point the function $w(z; t)$, has a development of the type

$$w(z; t) = w_{\infty} + \frac{\Gamma}{2\pi i} \frac{1}{z} + \frac{b_2}{z^2} + \frac{b_3}{z^3} + \dots$$

Just this regularity of the complex velocity determine us to use the (above mentioned) CVBEM for this function $w(z; t)$ and not for the complex potential as we would have been tempted.

Concerning the boundary conditions in the points of the contour C it will be written for the function $w(z; t)$ under the form:

There is a real function $V(\beta)$ so that for every $\beta \in [0, 2\pi)$ we have

$$w(\zeta(\beta)) = V(\beta) \frac{\dot{\zeta}(\beta)}{|\dot{\zeta}(\beta)|} + 1 + im + i\omega[\zeta(\beta) - z_A],$$

where $\zeta = \zeta(\beta)$ the parametrical equation of the Jordan rectifiable curve C, is a 2π-periodical function, bounded and derivable in $[0, 2\pi)$ so that $\dot{\zeta}(\beta) \neq 0$ and $\dot{\zeta}(\beta) < M$ when M is a finite constant.

Finally, the posible multiformity of the function f leads to the fulfilling of the equality

$$\int_C w(z, t)dz = \Gamma(t)$$

where $\Gamma(t)$ is the circulation „apriori" given. In the case of the profiles with an angular point in $z_p = z \in C$ where the semitangent's angle is equal to $\pi - \mu\pi$ $(-1 \leqslant \mu < 0)$, the behaviour of the complex velocity in this point i.e. $w(z; t) = o\left[(z - z_p)^{\frac{\mu}{1-\mu}}\right]$ imposes — to avoid the unboundness of w in z_p — to choose the circulation such that $\Gamma = Ll + Mm + N\omega$ where the coefficients L, M, N, uniquely determined depend upon the considered profile (C).

In what follows we want to illustrate how CVBEM works for determining the fluid flow induced by a displacement (rototranslation) in the mass of the fluid, of a profile (C), the fluid having already a given basical flow of complex velocity $w_B(z)$ and which superposes on the first flow.

For more generality we shall suppose that the profile (C) has an angular point and the basical flow presents some given singularities (vortices, sources, etc). Obviously the envisaged problem contains also the particular case of a flow past a fixed profile (C), the condition with an „apriori" given circulation becoming the famous Jukovski condition. More, the same method could be used to an arbitrary system of profiles performing independent displacements in the mass of the fluid, in the possible presence of some walls, i.e. practically to the majority of the models of plane hydrodynamics.

Retaking for sake of simplicity the case of only one profile (C), the proposed problem can be formulated as follows.

Let give the function $w_B(z)$, the complex velocity of the basical flow, function which belongs to a class (a) of functions having the properties:

1a) they are holomorphic fonctions in the domain D_1 (the whole plane Oxy, the point of infinity being included) except a bounded number (q) of points z_r placed at a finite distance and which represent singular points for these functions; let D_1^* be the domain D_1 from which one has taken off the singular points $\{z_r\}_{r=\overline{1,q}}$ and let $w_{-B}(\infty)$ be the value of the limit $\lim_{|z|\to\infty} w_B(z)$ which obviously exists and it is finite.

2a) if Γ_B is the circulation of the basical flow, this is equal to

$$\sum_{r=1}^{q} \Gamma_r$$

i.e. with the summ of the circulations of all the given singularities of the flow.

Concerning the unknown function $w(z)$ — the complex velocity of the resultant flow obtained by the above mentioned superposition it will be looked for in a class of functions (b) which satisfy the properties:

1b) they are holomorphic functions in the domain $D = D_1\backslash(\bar{C})$ except the same points $\{z_r\}_{r=\overline{1,q}}$ which are the singular points of the same nature as for

$$w_B(z);$$

at infinity the behaviour of them is identical with that of $w_B(z)$ i.e. $\lim_{|z|\to\infty} w(z) = w(\infty) = w_B(\infty)$;

2b) in the neighborhood of the trailing edge $z_p = \zeta(\beta_0) \in C$ where the semi-tangent's angle is $\pi - \mu\pi$, we have

$$w(z) = (z - z_p)^{\frac{\mu}{1-\mu}} g(z), \; g(z_p) \neq 0;$$

3b) in the points of the curve C the functions $w(\zeta(\beta))$ belong to the class H^* i.e. they are Hölderian functions on C except the angular point $z_p = \zeta(\beta_0)$ in whose neighborhood one has

$$w(\zeta(\beta)) = \frac{w^*(\zeta(\beta))}{[\zeta(\beta) - \zeta(\beta_0)]^{\frac{\mu}{\mu-1}}}.$$

where $w^* \in H_0$ in the same neighborhood/that means $w^*(\zeta(\beta))$ is separately Hölderian on the upper side and on the lower side of the profile in the neighborhood of $z_p = \zeta(\beta_0)$/;

4b) in the points of the curve C they satisfy, except the angular point the following boundary condition:

There is a real continuous function $V(\beta)$ such that for every $\beta \in [0, 2\pi)\backslash\{\beta_0\}$ one has

$$w(\zeta(\beta)) = V(\beta)\frac{\zeta(\beta)}{\dot{\zeta}(\beta)} + 1 + im + i\omega[\zeta(\beta) - z_A] \text{ where}$$

$z_A \in (C)$ and $l(t)$, $m(t)$, $\omega(t)$ are **the given** functions of time determining the rototranslation of the profile (C);

5b) they fulfil the equality

$$\int_C w(z)dz = \Gamma$$

where the circulation of the flow Γ is choosen so that one has the boundness of the velocity in z_p i.e. $\Gamma = L \cdot l + M \cdot m + N \cdot n$ where the coefficients L, M, N are given with the obstacle (C).

Let now the function $w(z) - w_B(z)$ be. This function known together with $w(z)$ being holomorphic in the outside of (C) the Cauchy formula is valid in D and we immediately have

$$w(\xi) = w_B(\xi) - \frac{1}{2\pi i}\int_C \frac{w(z)}{z-\xi}\,dz + \frac{1}{2\pi i}\int_C \frac{w_B(z)}{z-\xi}\,dz \quad \text{for } \xi \in D$$

Finnaly in order to use the boundary condition on C we perform $\xi \to \to \zeta = \zeta(\beta^*) \in C\backslash\{z_p\}$ and so we get

$$w(\zeta(\beta^*)) = w_B(\zeta(\beta^*)) - \frac{1}{\pi i}\oint_0^{2\pi} \frac{w(\zeta(\beta))\cdot\dot{\zeta}(\beta)}{\zeta(\beta)-\zeta(\beta^*)}\,d\beta + \frac{1}{\pi i}\oint_0^{2\pi} \frac{w_B(\zeta(\beta))\cdot\zeta(\beta)}{\zeta(\beta)-\zeta(\beta^*)}\,d\beta$$

This is the boundary integral equation which will be used for the effective construction of an approximative solution by CVBEM. Considering then a system of nodal points $z_0, z_1, \ldots, z_{p-1}, z_p, z_{p+1}, \ldots, z_n = z_0$ on the curve C all together with the system of the piecewise interpolating Lagrange functions on each arc C_j (system which takes into account the behaviour in the neighborhood of z_p) we can write

$$\tilde{w}(\zeta\beta)) - w_B(\zeta(\beta)) = \sum_{j=1}^{n} (w_j - w_{Bj})L_j \quad \text{where}$$

$$\underset{j \neq p-1,\, p,\, p+1}{L_j(\zeta)} = \begin{cases} \dfrac{\zeta - z_{j-1}}{z_j - z_{j-1}} & \text{for } \zeta \in C_j \\[2ex] \dfrac{\zeta - z_{j+1}}{z_j - z_{j+1}} & \text{for } \zeta \in C_{j+1} \\[2ex] 0 & \text{otherwise.} \end{cases}$$

While for the cases $j = p-1,\, p,\, p+1$ we will now have

$$L_{p-1}(\zeta) = \begin{cases} \dfrac{\zeta - z_{k-2}}{z_{k-1} - z_{k-2}}, & \text{for } \zeta \in C_{p-1}, \\[2ex] \left(\dfrac{\zeta - z_k}{z_{k-1} - z_k}\right)^{\frac{\mu}{1-\mu}}, & \text{for } \zeta \in C_p, \\[2ex] 0 & \text{otherwise.} \end{cases}$$

$$L_p(\zeta) = \begin{cases} 1 - \left(\dfrac{\zeta - z_p}{z_{p-1} - z_p}\right)^{\frac{\mu}{1-\mu}}, & \text{for } \zeta \in C_p, \\[2ex] 1 - \left(\dfrac{\zeta - z_p}{z_{p+1} - z_p}\right)^{\frac{\mu}{1-\mu}}, & \text{for } \zeta \in C_{p+1}, \\[2ex] 0 & \text{otherwise.} \end{cases}$$

$$L_{p+1}(\zeta) = \begin{cases} \dfrac{\zeta - z_{p+2}}{z_{p+1} - z_{p+2}} & , \text{ for } \zeta \in C_{p+2}, \\ \left(\dfrac{\zeta - z_p}{z_{p+1} - z_p}\right)^{\frac{\mu}{1-\mu}} & , \text{ for } \zeta \in C_{p+1}, \\ 0 & \text{otherwise.} \end{cases}$$

At once we also obtain:

$$\tilde{L}_j(z) = \frac{1}{2\pi i} \int_{\vec{C}} \frac{L_j(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2i} \left(\frac{z - z_{j-1}}{z_j - z_{j-1}} \ln \frac{z - z_j}{z - z_{j-1}} + \frac{z - z_{j+1}}{z_j - z_{j+1}} \ln \frac{z - z_{j+1}}{z - z_{j+1}} \right)$$

While for $j = p - 1,\ p,\ p + 1$

$$\tilde{L}_{p-1}(z) = \frac{1}{2\pi i} \left\{ \frac{z - z_{p-2}}{z_{p-1} - z_{p-2}} \ln \frac{z - z_{p-1}}{z - z_{p-2}} + 1 - F_{1/1-\mu}\left(\frac{z - z_p}{z_{p-1} - z_p}\right) \right\},$$

$$\tilde{L}_p(z) = \frac{1}{2\pi i} \left\{ \ln \frac{z - z_{p+1}}{z - z_{p-1}} + F_{1/1-\mu}\left(\frac{z - z_p}{z_{p-1} - z_p}\right) - F_{1/1-\mu}\left(\frac{z - z_p}{z_{p+1} - z_p}\right) \right\},$$

$$\tilde{L}_{p+1}(z) = \frac{1}{2\pi i} \left\{ \frac{z - z_{p+2}}{z_{p+1} - z_{p+2}} \ln \frac{z - z_{p+2}}{z - z_{p+1}} - 1 + F_{1/1-\mu}\left(\frac{z - z_p}{z_{p+1} - z_p}\right) \right\}$$

where $F(z) = \int_0^1 \frac{t}{t - \alpha}\, dt$, while for the others $\tilde{L}_j(z)$ $(j \neq p,\ p - 1,\ p + 1)$ t already established expressions are still valid.

Concerning the coefficients $L_{kj} = L_k(z_j)$, for $k \neq j$ they could directly calculated from the expression of $L_j(z)$ by using the equality $\lim_{z \to z_p} (z - z_p) \ln (z - z_p) = 0$ in the case of $k = j - 1$ or $k = j + 1$. For $k = j$ as we have

$$\tilde{L}_j(z) = \frac{1}{2\pi i} \left\{ \frac{z - z_j}{z_j - z_{j-1}} \ln \frac{z - z_j}{z - z_{j-1}} + \frac{z - z_j}{z_j - z_{j+1}} \ln \frac{z - z_{j+1}}{z - z_j} + \ln \frac{z - z_{j+1}}{z - z_{j-1}} \right\}$$

we get immediately

$$L_{jj} = \frac{1}{2\pi i} \ln \left(\frac{z_j - z_{j+1}}{z_j - z_{j-1}} \right)$$

where it takes the same principal determination for logarithm.

Using then the general calculus already performed for $\tilde{L}_j(z)$ and L_{jk}, $w(z_k) - w_B(z_k) = u_k - iv_k$ and $L_{jk} = M_{kj} + iN_{kj}$ we are led again to the re algebraic homogenous system

$$u_k = \sum_{j=1}^{n} M_{kj} u_j + \sum_{j=1}^{n} N_{kj} v_j$$

$$v_k = -\sum_{j=1}^{n} M_{kj} v_j + \sum_{j=1}^{n} N_{kj} u_j.$$

system which will be completed in this case by the complex equation

$$\sum_{j=1}^{n} w_j \int_C L_j(\zeta) d\zeta = \Gamma \text{ or, equaivalent with}$$

$$\sum_{j=1}^{n} u_j \operatorname{Re} \int_C L_j(\zeta) d\zeta + v_j \operatorname{Im} \int_C L_j(\zeta)\, d\zeta = \Gamma,$$

$$\sum_{j=1}^{n} u_j \operatorname{Im} \int_C L_j(\zeta) d\zeta = \sum_{j=1}^{n} v_j \operatorname{Re} \int_C L_j(\zeta)\, d\zeta.$$

These last two real equations allow to determine an unique solution of the above homogenous system which includes also the data on C. This unique solution once introduced in the integral representation of the problem (i.e in our case the Cauchy formula) leads to the complete determination of the complex velocity in every point of the domain to the flow.

Regarding the singularities $\{z_r\}_{r=\overline{1,q}}$ of the fluid flow admitting that they are vortices (and so $\Gamma_k \neq 0$, $k=\overline{1,q}$) the absence of external forces implies the fulfilling of a so called „freedom condition'' for them i.e.

$$\frac{dz_r}{dt} + 1 + im + i\omega z_r = \lim_{z \to z_r} \left[w(z) + \frac{i\Gamma_r}{z - z_r} \right], \; r = \overline{1, q}.$$

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THEOREMS OF UNIQUE FIXED POINT FOR PAIRS OF EXPANSION MAPPINGS

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REZUMAT. — Teoreme de punct fix pentru perechi de funcţii expansive. În această notă sînt demonstrate trei teoreme de punct fix pentru perechi de funcţii expansive, teoreme care generalizează rezultatele din [2]—[5].
Rezultatul principal este următorul:

Fie (X, d) un spaţiu metric complet şi $f, g: (X, d) \to (X, d)$ două funcţii surjective satisfăcînd inegalitatea (1) pentru orice x, y din X, unde Ψ satisface proprietăţile (B) şi (B^*) cu $h > 1$. Dacă proprietăţile (C^*) şi (U) au loc atunci f şi g au un punct fix comun unic.

In [1]—[5] some fixed point theorems of expansion mappings are proved. In this note using a combination of methods used in [2]—[4] other fixed point theorems for pair of expansion mappings are proved, which generalize some results from [2]—[5].

Let $\mathbf{R}_+$ be the set of all non negative real numbers and $\psi: \mathbf{R}_+^3 \to \mathbf{R}_+$ be a real-valued function.

DEFINITION 1. A real valued function $\psi: \mathbf{R}_+^3 \to \mathbf{R}_+$ satisfies *property* (B) if for every $u, v \in \mathbf{R}_+$ such that $u \geqslant \psi(v, u, v)$, then $u \geqslant hv$ where $\psi(1, 1, 1) = h \geqslant 1$ [1].

DEFINITION 2. A real valued function $\psi: \mathbf{R}_+^3 \to \mathbf{R}_+$ satisfies *property* (B^*) if for every $u, v \in \mathbf{R}_+$ such that $u \geqslant \psi(v, v, u)$, then $u \geqslant hv$, where $\psi(1, 1, 1) = h \geqslant 1$.

DEFINITION 3. A real valued function $\psi: \mathbf{R}_+^3 \to \mathbf{R}_+$ satisfies *property* (C^*) if is continuous.

DEFINITION 4. A real valued function $\psi: \mathbf{R}_+^3 \to \mathbf{R}_+$ satisfies *property* (U) if $\psi(u, 0, 0) > u$, *for all* $u > 0$ [3].

THEOREM 1. *Let (X, d) be a metric space and f, g be two self mappings of X, satisfying the inequality*

$$d(fx, gx) \geqslant \psi(d(x, y), d(x, fx), d(y, gy)) \tag{1}$$

or all x, y in X, where ψ satisfies property (U). If f and g have a fixed point z, then z is a unique common fixed point for f and g.

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Proof. Suppose that f and g have a second common fixed point $z' \neq z$. Then $d(z, z') = d(fz, gz') \geqslant \psi(d(z, z'), d(z, fz), d(z', gz')) = \psi(d(z, z'), 0, 0) > d(z, z')$, which implies $z = z'$. Contradiction.

COROLLARY 1 (Theorem 1 ; [3]). *Let (X, d) be a metric space and f a self mapping of X, satisfying the inequality*

$$d(fx, fy) \geqslant \psi(d(x, y), d(x, fx), d(y, fy)) \tag{2}$$

for all x, y in X with $x \neq y$, where ψ satisfies property (U). If f has a fixed point z, then z is a unique fixed point for f.

THEOREM 2. *Let (X, d) be a complete metric space and f, g two surjective self mappings of X satisfying the inequality (1) for x, y in X, where ψ satisfies properties (B) and (B^*) with $h > 1$. If properties (C^*) and (U) hold then f and g have a unique fixed point.*

Proof. Let $x_0 \in X$. Since f is surjective there is a point $x_1 \in f^{-1}(x_0)$. Since g is surjective there is a point $x_2 \in g^{-1}(x_1)$. Continuing in that manner one obtains a sequence $\{x_n\}$ with $x_{2n+1} \in f^{-1}(x_{2n})$ and $x_{2n+2} \in g^{-1}(x_{2n+1})$. Suppose $x_{2n+1} = x_{2n}$ for some n.

Since $d(x_{2n+1}, x_{2n}) = d(gx_{2n+2}, fx_{2n+1})$ by (1) we have:

$$d(x_{2n+1}, x_{2n}) - d(gx_{2n+2}, fx_{2n+1}) \geqslant \psi(d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, fx_{2n+1}),$$
$$d(x_{2n+2}, gx_{2n+2})) = \psi(d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n}), d(x_{2n+2}, x_{2n+1}))$$

which implies by property (B)

$$0 \geqslant hd(x_{2n+1}, x_{2n+2})$$

which implies $x_{2n+1} = x_{2n+2}$ The condition $x_{2n+1} = x_{2n}$ implies that x_{2n} is a fixed point of f since $x_{2n+2} = x_{2n+1}$, x_{2n} is a fixed point of g. Similarly $x_{2n+2} = x_{2n+1}$ leads to x_{2n+1} being a common fixed point of f and g. Assume $x_n = x_{n+1}$ for each n. From (1)

$$d(x_{2n}, x_{2n+1}) = d(fx_{2n+1}, gx_{2n+2}) \geqslant \psi(d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n}),$$
$$d(x_{2n+2}, x_{2n+1}))$$

which implies by property (B^*)

$$d(x_{2n}, x_{2n+1}) \geqslant hd(x_{2n+1}, x_{2n})$$

and

$$d(x_{2n+1}, x_{2n}) \leqslant \frac{1}{n} d(x_{2n+1}, x_{2n}) \leqslant \ldots \leqslant \left(\frac{1}{n}\right)^{2n} d(x_1, x_0)$$

then by a routine calculation one can show that $\{x_n\}$ is a Cauchy sequence and since X is complete there is a $x \in X$ so

$$\lim x_n = x.$$

Let $y \in f^{-1}(x)$. Then we have

$$d(x_{2n+2}, x) = d(gx_{2n+2}, fy) \geqslant \psi(d(x_{2n+2}, y), d(y, fy), d(x_{2n+2}, gx_{2n+2})) =$$
$$= \psi(d(x_{2n+1}, fy), d(y, fy), d(x_{2n+2}, x_{2n+1}))$$

Letting n tend to infinity we have by continuity of ψ

$$0 \geqslant \psi(d(x, y), d(y, x), 0)$$

which implies by properties (B*)

$$0 \geqslant \psi(d(x, y), d(x, y), 0) \geqslant hd(x, y)$$

and thus $y = x$, then $x = f(x)$.

Let $z \in g^{-1}(x)$. Then we have

$$d(x_{2n+1}, x) = d(x_{2n+1}, gz) \geqslant \psi(d(x_{2n+1}, z), d(x_{2n+1}, fx_{2n+1}), d(z, gz)) =$$
$$= \psi(d(x_{2n+1}, z)), d(x_{2n+1}, x_{2n}), d(z, x))$$

Letting n tend to infinity we have by continuity of ψ:

$$0 \geqslant \psi(d(x, z), 0, d(x, z))$$

which implies by properties (B)

$$0 \geqslant \psi(d(x, z), 0, d(x, z)) \geqslant hd(x, z)$$

and thus $z = x$, then $x = z = g(x)$.

So x is a common fixed point of f and g. By property (U) and Theorem 1, x is a unique common fixed point of f and g.

COROLLARY 2. (Theorem 2; [3]). *Let (X, d) be a complete metric space and $f: (X, d) \to (X, d)$ a surjective mapping satisfying the inequality (2) for all x, y in X with $x \neq y$ where ψ satisfies properties (B) with $h > 1$. If properties (C) and (U) hold, then f has a unique fixed point.*

THEOREM 3. *Let (X, d) be a complete metric space and $f, g: (X, d) \to (X, d)$ two surjective mappings. If there are non negative real numbers a, b, c with $b < 1$, $c < 1$, and $a + b + c > 1$ such that*

$$d^k(fx, gy) \geqslant ad^k(x, y) + bd^k(x, fx) + cd^k(y, gy) \tag{3}$$

where $k \geqslant 1$ for each x, y in X, then f and g have a common fixed point. If $a > 1$ then the common fixed point is unique.

Proof. Let

$$\psi(d(x, y), d(x, fx), d(y, gy)) = [ad^k(x, y) + bd^k(x, fx) + cd^k(y, gy)]^{\frac{1}{k}}$$

Then we have

$$u \geqslant (av^k + bu^k + cv^k)^{\frac{1}{k}}$$

$$u^k \geqslant av^k + bu^k + cv^k$$

$$u^k(1 - b) \geqslant (a + b)v^k$$

$$u \geqslant v\left(\frac{a + c}{1 - b}\right)^{\frac{1}{k}} \geqslant v(a + b + c)^{\frac{1}{k}} = hv$$

Similarly we have

$$u \geqslant (av^k + bv^k + cu^k)^{\frac{1}{k}}$$

$$u^k \geqslant av^k + bv^k + cu^k$$

$$u \geqslant v\left(\frac{a + b}{1 - c}\right)^{\frac{1}{k}} \geqslant v(a + b + c)^{\frac{1}{k}} = hv$$

On other hand, if $a > 1$ we have $\psi(u, 0, 0) = au > u$. By Theorem 2 follows that f and g have a unique common fixed point.

COROLLARY 3. *Let (X, d) be a complete metric space and $f: (X, d) \to (X, d)$ a surjective mapping. If there are non negative real numbers a, b, c with $b < 1$ and $a + b + c > 1$ such that*

$$d^k(fx, fy) \geqslant ad^k(x, y) + bd^k(x, fx) + cd^k(y, fy) \tag{4}$$

for each x, y in X, with $x \neq y$, then f has a fixed point. Further if $c > 1$ then the fixed point is unique.

Remark. For $k = 2$ we have Theorem 1 of [2].

For $k = 1$ we have Theorem 2 of [5].

COROLLARY 4 (Theorem 1, [4]). *Let f, g be surjective self mappings of a complete metric space (X, d). Suppose there is a constant $a > 1$ such that*

$$d(fx, gx) \geqslant ad(x, y) \tag{5}$$

for each x, y in X. Then f and g have a unique common fixed point.

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BILINEAR OPERATORS AND APPROXIMATION SPACES

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REZUMAT. — Operatori biliniari şi spaţii de aproximare. Se consideră o generalizare a spaţiilor de aproximare X_q^α [5], [7], [3], care în cazul spaţiului $\mathcal{L}(E, F)$ coincide cu idealul Lorentz-Marcinkiewicz $\mathcal{L}_{\varphi, q}(E, F)$ [1]. Se studiază o categorie de operatori biliniari pe aceste spaţii şi din proprietăţile lor se deduce stabilitatea la produsul tensiorial al idealelor $\mathcal{L}_{\varphi, q}$, pentru unele funcţii φ şi $0 < q < \infty$ [11].

0. **Introduction.** Let be $(X, \|\ \|_X)$ a quasi-normed abelian group and $(G_n)_{n=0}^{\infty}$ a sequence of subsets of X such that

$$G_0 = \{0\}, \tag{1}$$

$$G_n \subset G_{n+1};\ n = 0, 1, 2, \ldots, \tag{2}$$

$$G_n \pm G_m \subset G_{n+m},\ n, m = 0, 1, 2, \ldots. \tag{3}$$

Given any $f \in X$, put

$$E_n(f) = \inf\{\|f - g\|_X : g \in G_{n-1}\},\ n = 1, 2, \ldots$$

In this way we associate to each $f \in X$ the decreasing sequence of non-negative numbers $(E_n(f))$.

Let $0 < \alpha < \infty$ and $0 < q < \infty$. The classical approximation space X_q^α consists of all $f \in X$ which have a finite quasi — norm

$$\|f\|_{X_q^\alpha} = [\Sigma (n^\alpha E_n(f))^q \cdot n^{-1}]^{\frac{1}{q}}.$$

These spaces have been investigated by many authors [5], [7], [3].

If $X = \mathfrak{L}(E, F)$ — the space of all linear and bounded operators $T: E \to F$ where E and F are Banach spaces and $G_n = \{R \in \mathfrak{L}(E, F) : \text{rank } R \leqslant n\}$ $n = 0, 1, 2, \ldots$, we obtain $E_n(f) = a_n(T) = \inf\{\|T - R\| : R \in G_{n-1}\}$ [6], [1], [10].

Then $X_q^\alpha = \mathfrak{L}_{\frac{1}{\alpha}, q}(E, F)$, where $\mathfrak{L}_{p, q}(E, F)$ are the Lorentz operator ideals [1], [6], [10] $(\mathfrak{L}_{p, q}(E, F) = \{T : \Sigma(n^{\frac{1}{p}} a_n(T))^q n^{-1} < \infty\})$.

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In this paper we consider a generalization of the spaces X_q^α and some bilinear operators on these spaces are studied. In the final we deduce the tensor stability of some Lorentz–Marcinkiewicz operator ideals. (This final result is also obtained in [11]). The limit case of X_q^α (with $\alpha = 0$) is investigated in [3].

1. **The spaces $\mathbf{X}_q^\varphi$.** We denote by $\mathfrak{B}$ the set of all continuous functions $\varphi : (0, \infty) \to (0, \infty)$ with $\varphi(1) = 1$ and such that

$$\sup_{s>0} \frac{\varphi(st)}{\varphi(s)} < \infty \text{ for every } t > 0.$$

For $0 < q < \infty$, the approximation space X_q^φ consists of all $f \in X$ which have a finite quasi-norm:

$$||f||_{X_q^\varphi} = [\Sigma(\varphi(n)E_n(f))^q n^{-1}]^{\frac{1}{q}}, \text{ where } \varphi \in \mathfrak{B}.$$

For $\varphi(t) = t^\alpha$ $(\alpha \in (0, \infty))$ we obtain the spaces X_q^α or X_q [3] if $\alpha = 0$.

Remark. If $X = \mathfrak{L}(E, F)$, the space X_q^φ coincides with the Lorentz–Marcinkiewicz operator ideal $\mathfrak{L}_{\varphi, q}(E, F)$ [1], [11] and more, if $\varphi(t) = t^{\frac{1}{p}}(1 + |\log t|)^\gamma$, we obtain the Lorentz–Zygmund ideals $\mathfrak{L}_{p, q, \gamma}(E, F)$ [1], [2].

Next we describe the behaviour of some bilinear operators under X_q^φ – spaces.

Take three quasi-normed abelian groups X, Y, Z and let (G_n), (F_n), (H_n) be sequences of subsets in X, Y, Z respectively, satisfying conditions (1) to (3). Let $B : X \times Y \to Z$ be an operator. We assume that there is $M < \infty$ such that for all f, f_0, $f_1 \in X$ and all g, g_0, $g_1 \in Y$ the following holds:

(a) $B(f_0 \pm f_1, g) = B(f_0, g) \pm B(f_1, g)$
(b) $B(f, g_0 \pm g_1) = B(f, g_0) \pm B(f, g_1)$
(c) $||B(f, g)||_Z \leqslant M ||f||_X \cdot ||g||_Y$
(d) $B(G_m, F_n) \subset H_{mn}$ for $m, n = 0, 1, 2, \ldots$

In the paper [3] is proved that the operator $B : X_q \times Y_q \to Z_q$ is bounded for $0 < q < \infty$. Here we prove this fact for the case of X_q^φ spaces.

Let $\bar{\mathfrak{B}}$ be the subclass of $\mathfrak{B}$ containing the functions φ for which there exists a constant c such that

$$\varphi^*(n) = \max \{(\varphi(t)) : t \in \{n^2, n^2 + 1, \ldots, (n + 1)^2\}\} \leqslant c \cdot \varphi(n)$$

for all natural numbers n; $(c \in (0, \infty))$.

Using a similar argument to one in [8] we prove the following:

LEMMA 1.1. *Let X, Y, Z and B as above. Then there is a constant $c < \infty$ independent of f, g and n such that*

$$E_{n^2}(B(f, g)) \leqslant c[E_n(f)\|g\|_Y + E_n(g)\cdot\|f\|_X],\ n = 1,2, \ldots$$

Proof. Let c_1, c_3 be the constans of the quasi-triangle inequality in X and Z respectively. Given any $\varepsilon > 0$, find

$f_0 \in G_{n-1}$ with $\|f - f_0\| < E_n(f) + \varepsilon$. and $g_0 \in F_{n-1}$ with $\|g - g_0\|_Y < E_n(g) + \varepsilon$.

By (d), $B(f_0, g_0) \in H_{n^2-1}$ and hence, using (a), (b), (c), we get

$$E_{n^2}(B(f, g)) \leqslant \|B(f, g) - B(f_0, g_0)\|_Z = \|B(f - f_0, g) - B(f_0, g - g_0)\|_Z \leqslant c_3M[\|f - f_0\|_X\cdot\|g\|_Y + \|f_0 - f + f\|_X\cdot\|g - g_0\|_Y]$$

$$\leqslant c_3M[(E_n(f) + \varepsilon)\|g\|_Y + c_1(E_n(f) + \varepsilon + \|f\|_X)(E_n(g) + \varepsilon)] \leqslant$$

$$\leqslant c_3M[(E_n(f) + \varepsilon)\|g\|_Y + c_1(2\|f\|_X + \varepsilon)(E_n(g) + \varepsilon)]\ .$$

Since ε is arbitrary it results the relation, with $c = 2Mc_1c_3$. From this result we obtain

THEOREM 1.2. *For all $\varphi \in \bar{\mathfrak{B}}$ and $0 < q < \infty$ the operator $B : X_q^\varphi \times Y_q^\varphi \to Z_q^\varphi$ is bounded.*

Proof. Since the sequence $(E_n(f))$ is decreasing we can write

$$[\Sigma(\varphi(n)E_n(B(f, g))^q n^{-1}]^{\frac{1}{q}} \leqslant [\Sigma(2n + 1)(\varphi^*(n)E_{n^2}(B(f, g))^q n^{-2}]^{\frac{1}{q}} \leqslant$$

$$\leqslant 3^{\frac{1}{q}}[\Sigma(c\cdot\varphi(n)E_{n^2}(B(f, g)))^q\cdot n^{-1}]^{\frac{1}{q}}$$

$$\leqslant c(q)[\Sigma(\varphi(n)(E_n(f)\|g\|_Y + E_n(g)\|f\|_X))^q n^{-1}]^{\frac{1}{q}} \leqslant$$

$$\leqslant \bar{c}(q)[(\Sigma(\varphi(n)E_n(f))^q n^{-1})^{\frac{1}{q}}\|g\|_Y + (\Sigma(\varphi(n)E_n(g)\|f\|_X)^q n^{-1})^{\frac{1}{q}}] < \infty.$$

For $f \in X_q^\varphi$, $\|f\|_X = E_1(f) \leqslant \|f\|_{X_q^\varphi} < \infty$.

If $X_q^\varphi = \mathfrak{L}_{\varphi,q}(E, F) = \{T \in \mathfrak{L}(E, F) : (\Sigma(\varphi(n)a_n(T)^q n^{-1})^{\frac{1}{q}} < \infty\}$, [1], is the Lorentz—Marcinkiewicz operator ideal and the bilinear operator is the tensor product operator $T_1 \hat{\otimes}_\tau T_2$ [2], [4], [8], where τ is a tensor norm, we obtain the following:

COROLLARY 1.3. *For all $\varphi \in \bar{\mathfrak{B}}$ and $0 < q < \infty$ the operator ideal $\mathfrak{L}_{\varphi,q}$ is tensor product stable.*

This result is obtained in other way in [11].

Remarks. If $\varphi(t) = (1 + |\log t|)^\gamma$, $-\infty < \gamma < \infty$ we have the Lorentz–Zygmund ideals $\mathfrak{L}_{\infty, q, \gamma}(E, F)$ [2]. From the corollary 1.3 it results that $\mathfrak{L}_{\infty, q, \gamma}$ is tensor product stable for $0 < q < \infty$. In [2] is proved, in other way, the stability for $0 < q \leqslant 1$. For $\gamma = 0$, in [3], is proved the stability for $0 < q < \infty$, but this result is known [9] since

$\mathfrak{L}_{\infty, q} = \{T : \Sigma a_n^q(T) n^{-1} < \infty\}$ coincides with the ideal $\mathfrak{L}_q^* = \{T : \Sigma a_{2^n}^q(T) < \infty\}$.

If $X = [\mathfrak{A}, A]$ is an operator ideal [6], [10] and $G_n = \{R \in \mathfrak{A} : \text{rank } R \leqslant n\}$, $n = 0, 1, 2, \ldots,$ then $E_n(T) = a_n^A(T) = \inf\{A(T - R) : R \in G_{n-1}\}$ and $X_q^\varphi = \{T \in \mathfrak{A} : \Sigma(\varphi(n) a_n^A(T))^q n^{-1} < \infty\} = \mathfrak{L}_{\varphi, q}^A(E, F)$ is a new operator ideal which is tensor product stable, if $\varphi \in \bar{\mathfrak{B}}$ and $A(T_1 \hat{\otimes}_\tau T_2) \leqslant c \cdot A(T_1) \cdot A(T_2)$.

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DRAG PERTURBATIONS IN THE NODAL PERIOD FOR CIRCULAR SATELLITE ORBITS

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ABSTRACT. — The initially circular motion of an artificial satellite in the terrestrial atmosphere is studied from the point of view of the difference which appears between the nodal period and the corresponding Keplerian one, as an effect of the aerodynamic drag. Using the Total Density thermospheric model to describe the upper atmospheric density distribution, an analytic expression for the mentioned difference is established. Some particular cases are considered.

1. **Introduction.** The difference which arises between the nodal period of an artificial satellite and the corresponding Keplerian period as an effect of the aerodynamic drag constitutes a problem discussed by rather few authors (*e.g.* [1, 3, 4, 7, 8]). Analytic formulae for this difference were established for an upper atmospheric density distribution given by the simple law:

$$\rho = \rho_p \exp(-(r - r_p)/H), \tag{1}$$

where ρ = atmospheric density, H = density scale height, r = geocentric radius vector of the satellite, while the index „p" refer to values corresponding to the perigee.

In this paper we shall estimate analytically the difference caused by the aerodynamic drag between the nodal period, defined as:

$$T_\Omega = \int_0^{2\pi} (dt/du)\, du. \tag{2}$$

where u = argument of latitude, and the corresponding Keplerian period T_0 imposing the following conditions:

(i) the initial orbit of the satellite is circular;

(ii) the atmospheric density distribution is described by the Total Density (TD) thermospheric model.

2. **Basic equations.** Consider the notion of an artificial satellite under the influence of a perturbing factor of unspecified nature, but depending on a small parameter σ, and let S, T, W respectively be the radial, transversal,

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and binormal components of the perturbing acceleration. Let us describe the motion by means of the Newton—Euler system written in the form [4]:

$$\begin{aligned}
dp/du &= 2(Z/\mu)r^3T,\\
dq/du &= (Z/\mu)(r^3kBCW/(pD) + r^2T(r(q + A)/p + A) + r^2BS),\\
dk/du &= (Z/\mu)(-r^3qBCW/(pD) + r^2T(r(k + B)/p + B) - r^2AS),\\
d\Omega/du &= (Z/\mu)r^3BW/(pD),\\
di/du &= (Z/\mu)r^3AW/p,\\
dt/du &= Zr^2(\mu p)^{-1/2},
\end{aligned} \tag{3}$$

where p = semi-latus rectum, $q = e\cos\omega$, $k = e\sin\omega$ (e = eccentricity, ω = = argument of perigee), Ω = longitude of the ascending node, i = inclination, $A = \cos u$, $B = \sin u$, $C = \cos i$, $D = \sin i$, μ = Earth's gravitational parameter, and:

$$Z = (1 - r^2C\dot{\Omega}/(\mu p)^{1/2})^{-1}. \tag{4}$$

Let $z_j = z_j(u)$, $j = \overline{1, 5}$, be the elements of the set $\{p, q, k, \Omega, i\}$, and consider, as usually, that their variations over one revolution are small, such that they can be taken as constant (and equal to $z_{j_0} = z_j(u_0)$, where $u_0 = u(t_0)$) in the right-hand side of the equations (3), and these ones can be separately considered. The orbital elements can be written as $z_j = z_{j_0} + \Delta z_j$, the variations Δz_j of z_j in the interval $[u_0, u]$ being determined from:

$$\Delta z_j = \int_{u_0}^{u} (dz_j/du)\, du, \; j = \overline{1, 5}, \tag{5}$$

where the integrals can be estimated from (3) by successive approximations, with $Z \approx 1$, limiting the process to the first order approximation.

Replacing now (4) into the last equation (3), and expanding this one in binomial series, we obtain to first order in σ:

$$dt/du = r^2(\mu p)^{-1/2} + r^4C\dot{\Omega}/(\mu p) = f(z_j;\ \sigma;\ u),\ j = \overline{1, 5}. \tag{6}$$

The expansion of the function f in Taylor series on the hypersurface $H^* = H^*(z_{j_0}, \sigma = 0; u)$, $j = \overline{1, 5}$, with respect to the small quantities Δz_j and σ yields:

$$f = f_0 + \sum_{j=1}^{5} (\partial f/\partial z_j)_0 \Delta z_j + (\partial f/\partial \sigma)_0 \sigma + \ldots, \tag{7}$$

where the index „0" refers to $u = u_0$.

Lastly, consider the orbit equation in polar coordinates:

$$r = p/(1 + e\cos v) = p/(1 + Aq + Bk), \tag{8}$$

where v = true anomaly, and replace (8) into (6). Now, calculating the partial derivatives required by (7), neglecting the terms of the form $(\partial(r^4 C\dot{\Omega}/(\mu p))/ /\partial z_j)\,\Delta z_j$, $j = \overline{1, 5}$, which contain σ^2, and substituting the resulted $f = dl/du$ into (2), we can write:

$$T_\Omega = T_0 + \Delta T = T_0 + \sum_{j=1}^{4} I_j, \tag{9}$$

where:

$$T_0 = f_0 = p_0^{3/2}\mu^{-1/2}(1 + Aq_0 + Bk_0)^{-2}, \tag{10}$$

and ΔT is just the difference we search for. Its terms are:

$$\begin{aligned}
I_1 &= \frac{3}{2}\, p_0^{1/2}\mu^{-1/2}\int_0^{2\pi}(1 + Aq_0 + Bk_0)^{-2}\Delta p\, du,\\
I_2 &= -2p_0^{3/2}\mu^{-1/2}\int_0^{2\pi}(1 + Aq_0 + Bk_0)^{-3}A\Delta q\, du,\\
I_3 &= -2p_0^{3/2}\mu^{-1/2}\int_0^{2\pi}(1 + Aq_0 + Bk_0)^{-3}B\Delta k\, du,\\
I_4 &= \int_0^{2\pi}(\partial(r^4 C\dot{\Omega}/(\mu p))/\partial\sigma)\sigma\, du.
\end{aligned} \tag{11}$$

These formulae are general from two viewpoints: the nature of the perturbing factor is unspecified, and the orbit may have any subunitary eccentricity. The method was firstly applied in [12] for quasi-circular orbits and for a gravitational perturbing factor.

3. Circular case. Let us impose the condition (i). In this case ($e_0 = 0$ hence $q_0 = k_0 = 0$), (8) becomes:

$$r_0 = p_0, \tag{12}$$

while (11) acquire the form:

$$\begin{aligned}
I_1 &= \frac{3}{2}\, p_0^{1/2}\mu^{-1/2}\int_0^{2\pi}\Delta p\, du,\\
I_2 &= -2p_0^{3/2}\mu^{-1/2}\int_0^{2\pi} A\,\Delta q\, du,\\
I_3 &= -2p_0^{3/2}\mu^{-1/2}\int_0^{2\pi} B\,\Delta k\, du,\\
I_4 &= p_0^{3}\mu^{-1}C_0\int_0^{2\pi}(\partial\dot{\Omega}/\partial\sigma)\sigma\, du.
\end{aligned} \tag{13}$$

For simplicity reasons, in the following sections we shall no longer use the index „0" to mark the initial ($u = u_0$) values of z_j; $j = \overline{1, 5}$, and of functions of these ones (unless necessary, and then this fact will be specified). Every other unspecified index „0" appearing in our calculations is a simple notation and does not refer to $u = u_0$. In fact, every quantity which does not depend on u (explicity, or through A and B) will be considered constant (over one revolution) and equal to its value at $u = u_0$.

4. **Perturbing acceleration.** Consider now that the perturbing acceleration is due to the aerodynamic drag. Its components will be (*e.g.* [11]):

$$S = -\rho\delta v_{rel} v_r,$$
$$T = -\rho\delta v_{rel}(v_n - rCw), \tag{14}$$
$$W = -\rho\delta v_{rel} rDwA,$$

where δ = drag parameter of the satellite, w = constant angular velocity of rotation of the atmosphere with respect to the Earth's axis, v_{rel} = satellite speed with respect to the air flow, v_r, v_n = radial and transversal components of the satellite velocity with respect to the Earth's centre. The velocities are [6]

$$v_{rel} = (\mu/p)^{1/2}(1 + 2Aq + 2Bk + q^2 + k^2)^{1/2},$$
$$v_r = (\mu/p)^{1/2}(Bq - Ak), \tag{15}$$
$$v_n = (\mu/p)^{1/2}(1 + Aq + Bk)$$

With (15) and (12), and taking into account the last sentence of Section 3, the expressions (14) become:

$$S = 0,$$
$$T = \rho\delta(\mu/p)^{1/2}(Cw - p^{-3/2}\mu^{1/2}), \tag{16}$$
$$W = -\rho\delta(\mu p)^{1/2}DwA.$$

5. **Equations for the orbital elements.** By virtue of the considerations made in Section 2, we shall separately have in view the first five equations (3) Since at the analytic calculation of the integrals (5) we take $Z \approx 1$, we shall write these equations omitting in advance the factor Z. So, by (16) and (12), the mentioned equations acquire the form:

$$dp/du = 2pb(x + y)\rho,$$
$$dq/du = 2b(x + y)A\rho,$$
$$dk/du = 2b(x + y)B\rho, \tag{17}$$
$$d\Omega/du = -b(x/C)AB\rho,$$
$$di/du = -b(Dx/C)A^2\rho,$$

where we used the abbreviations:

$$b = p^{5/2}\mu^{-1/2}\delta, \quad x = Cw, \quad y = -p^{-3/2}\mu^{1/2} = -2\pi/T_0. \tag{18}$$

6. **Expression of the density.** In order to write the equations (17) into a suitable form for performing the integrals (5), we still have to express the density as function only of u (through A, B). For this purpose, according to the condition (ii), we used the TD thermospheric model, whose variants TD 86 [9] and TD 88 [10] differ by certain quantities (k_{nj}, B_j, g_2; see below), but use the same analytic formulae. This model expresses the density as:

$$\rho = X_0 \sum_{n=1}^{7} h_n g_n, \tag{19}$$

where:

$$X_0 = 10^{-8}(1 + a_1(F_x - F_b))(a_2 + f_m)(1 + a_3(K_p - 3)) \tag{20}$$

features the general dependence of the density on the solar and geomagnetic activity (F_x = radio solar flux on 10.7 cm for the previous day, F_b = radio solar flux on 10.7 cm averaged for three solar rotations, K_p = three-hourly planetary geomagnetic index, $f_m = (F_b - 60)/160$). The height-dependence of the density is described by:

$$h_n = \sum_{j=0}^{3} k'_{nj} \exp((120 - h)B_j), \tag{21}$$

where $k'_{nj} = 10^8 k_{nj}$ (k_{nj} = numerical constants tabulated in the TD model) are at most of the order of unity and were introduced by us in order to assign to X_0 the part of the small parameter σ (see Section 2); h = height (km); $B_0 = 0$, $B_j = (40\, j)^{-1}$ in TD 86 model, $B_j = (29\, j)^{-1}$ in TD 88 model ($j = \overline{1, 3}$).

Lastly, the terms g_n (allowing respectively for: mean density, individual dependence on the mean radio solar flux, North-South asymmetry, annual semiannual, diurnal and semidiurnal variation) are:

$$\begin{aligned} &g_1 = 1,\ g_2 = f_m + a_4 \text{ (TD 86) or } g_2 = f_m/2 + a_4 \text{ (TD 88)},\\ &g_3 = \sin(d - p_3)\sin\varphi,\ g_4 = (a_5 f_m + 1)\sin(d - p_4),\\ &g_5 = (a_6 f_m + 1)\sin(2(d - p_5)),\\ &g_6 = (a_7 f_m + 1)\sin(t - p_6)\cos\varphi,\\ &g\ = (a_8 f_m + 1)\sin(2(t - p_7))\cos^2\varphi, \end{aligned} \tag{22}$$

where d = day count in the year, φ = latitude, t = local time (in hours); the constants a_i ($i = \overline{1, 8}$) and the phases p_n ($n = \overline{3, 7}$) are tabulated in the TD model.

The height is given by:

$$h = r - R(1 - \varepsilon \sin^2\varphi), \tag{23}$$

where R = mean equatorial terrestrial radius, ε = Earth's oblateness. With $\sin\varphi = DB$ and with (12), we derive the following expansion:

$$\exp((120 - h)B_j) = A_j(1 + \varepsilon R D^2 B_j A^2),\ j = \overline{0, 3}, \tag{24}$$

where $A_j = \exp(B_j(120 - p + R - \varepsilon RD^2)$. Replacing (24) into (21), and denoting:

$$K_{no} = \sum_{j=0}^{3} k'_{nj} A_j, \quad K_{n1} = \sum_{j=0}^{3} k'_{nj} \varepsilon RD^2 A_j B_j, \tag{25}$$

the terms h_n acquire the general form:

$$h_n = K_{n0} + K_{n1} A^2, \; n = \overline{1, 7}. \tag{26}$$

Examining now the terms g_n, we remark from (22) that only g_3, g_6 and g_7 change during one revolution of the satellite. From $\sin \varphi = DB$ we have immediately:

$$g_3 = \sin(d - p_3) DB, \tag{27}$$

while from $\sin(\alpha - \Omega) = CB/\cos \varphi$ and $\cos(\alpha - \Omega) = A/\cos \varphi$, where α is the right ascension of the satellite, and introducing the notations $L = \Omega - \alpha_\odot - p_6 + \pi$, $L' = 2(\Omega - \alpha_\odot - p_7 + \pi)$, where $\alpha_\odot$ is the Sun's right ascension, we obtain for g_6 and g_7 [2, 5]:

$$g_6 = (a_7 f_m + 1)(CB \cos L + A \sin L), \tag{28}$$

$$g_7 = (a_8 f_m + 1)(2CAB \cos L' + (1 + C^2) \sin L' - C^2 \sin L'). \tag{29}$$

Finally, using (27)—(29), and denoting:

$$\begin{aligned} G_1 &= D \sin(d - p_3), & G_6 &= -(a_8 f_m + 1) C^2 \sin L', \\ G_4 &= (a_7 f_m + 1) \sin L, & G_7 &= (a_8 f_m + 1)(1 + C^2) \sin L', \\ G_5 &= (a_7 f_m + 1) C \cos L, & G_8 &= 2(a_8 f_m + 1) C \cos L', \end{aligned} \tag{30}$$

we rewrite (22) as:

$$\begin{aligned} & g_1 = 1, \; g_2 = G_0, \; g_3 = G_1 B, \; g_4 = G_2, \; g_5 = G_3, \\ & g_6 = G_4 A + G_5 B, \; g_7 = G_6 + G_7 A^2 + G_8 AB, \end{aligned} \tag{31}$$

all coefficients G_j, $j = \overline{0, 8}$, being constant over one revolution.

We replace now (26) and (31) into (19), and, using the abbreviating notations:

$$\begin{aligned} M_0 &= K_{10} + K_{20} G_0 + K_{40} G_2 + K_{50} G_3 + K_{70} G_6, & N_0 &= K_{30} G_1 + K_{60} G_5, \\ M_1 &= K_{60} G_4, & N_1 &= K_{70} G_8, \\ M_2 &= K_{11} + K_{21} G_0 + K_{41} G_2 + K_{51} G_3 + K_{71} G_6 + K_{70} G_7, & N_2 &= K_{31} G_1 + K_{61} G_5, \\ M_3 &= K_{61} G_4, & N_3 &= K_{71} G_8, \\ M_4 &= K_{71} G_7, & N_4 &= 0, \end{aligned} \tag{32}$$

the density appears as a function only of w (through A and B):

$$\rho = X_0 \sum_{j=0}^{4} (M_j A^j + N_j A^j B) \tag{33}$$

7. **Variations of the orbital elements.** Replacing (33) into (17) and introducing artificially $N_{-2} = N_{-1} = N_5 = M_5 = 0$, we obtain:

$$
\begin{aligned}
dp/du &= 2X_0 pb(x+y) \sum_{j=0}^{4} (M_j A^j + N_j A^j B), \\
dq/du &= 2X_0 b(x+y) \sum_{j=0}^{4} (M_j A^{j+1} + N_j A^{j+1} B), \\
dk/du &= 2X_0 b(x+y) \sum_{j=0}^{5} ((N_j - N_{j-2})A^j + M_j A^j B), \\
d\Omega/du &= -X_0 b(x/C) \sum_{j=0}^{5} ((N_j - N_{j-2})A^{j+1} + M_j A^{j+1} B), \\
di/du &= -X_0 b(Dx/C) \sum_{j=0}^{4} (M_j A^{j+2} + N_j A^{j+2} B).
\end{aligned} \tag{34}
$$

Examining (13), one remarks that the integrals (5) must be performed only for p, q and k. Selecting therefore the first three equations (34), performing the integrals (5), and adopting the intermediate notations:

$$
\begin{aligned}
&P_{10} = 0, && P_{20} = 0, && P_{30} = 0, \\
&P_{11} = -N_0, && P_{21} = 0, && P_{31} = -M_0, \\
&P_{12} = -N_1/2, && P_{22} = -N_0/2, && P_{32} = -M_1/2, \\
&P_{13} = -N_2/3, && P_{23} = -N_1/3, && P_{33} = -M_2/3, \\
&P_{14} = -N_3/4, && P_{24} = -N_2/4, && P_{34} = -M_3/4, \\
&P_{15} = 0, && P_{25} = -N_3/5, && P_{35} = -M_4/5,
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
&P'_{10} = M_1 + 2M_3/3, && P'_{20} = M_0 + 2M_2/3 + 8M_4/15, && P'_{30} = N_1/3 + 2N_3/15, \\
&P'_{11} = M_2/2 + 3M_4/8, && P'_{21} = M_1/2 + 3M_3/8, && P'_{31} = -N_0/2 + N_2/8, \\
&P'_{12} = M_3/3, && P'_{22} = M_2/3 + 4M_4/15, && P'_{32} = -N_1/3 + N_3/15, \\
&P'_{13} = M_4/4, && P'_{23} = M_3/4, && P'_{33} = -N_2/4, \\
&P'_{14} = 0, && P'_{24} = M_4/5, && P'_{34} = -N_3/5, \\
&P'_{15} = 0, && P'_{25} = 0, && P'_{35} = 0;
\end{aligned} \tag{36}
$$

$$P''_1 = M_0 + M_2/2 + 3M_4/8, \quad P''_2 = M_1/2 + 3M_3/8, \quad P''_3 = N_0/2 + N_2/8; \tag{37}$$

$$F_i^0 = -\left(\sum_{j=0}^{5} (P_{ij} A^j + P'_{ij} A^j B) + P''_i u\right)_{u=u_0}, \quad i = \overline{1,3}, \tag{38}$$

the variations we search for acquire the very compact form:

$$\Delta z_i = 2X_0 c_i b(x + y) \left(\sum_{j=0}^{5} (P_{ij}A^j + P'_{ij}A^jB) + P''_i u + F_i^0\right), \quad i = \overline{1, 3}, \tag{39}$$

with $z_1 = p$, $z_2 = q$, $z_3 = k$, $c_1 = p$ (constant $= p_0$), $c_2 = c_3 = 1$.

8. **Results.** To obtain I_1, I_2, I_3, we substituted (39) into (13) and performed the integrations. Then we used (35)—(37) to express the results in terms of M_j, N_j, and introduced provisionally the abreviation:

$$Y = \pi X_0 p^{3/2}\mu^{-1/2} b/16. \tag{40}$$

As to I_4, we used the last equation (3) and the fourth equation (34), then we performed the last integral (13), assigning to X_0 the part of the small parameter σ, and used (40). The results are:

$$\begin{aligned} I_1 &= Y(x + y)(-24N_1 - 9N_3 + 12\pi(8M_0 + 4M_2 + 3M_4) + 96F_1^0), \\ I_2 &= Y(x + y)(16N_1 + 8N_3), \\ I_3 &= Y(x + y)(64N_0 - 16N_1 + 16N_2 - 8N_3), \\ I_4 &= Yx(-4N_1 - 2N_3). \end{aligned} \tag{41}$$

Lastly, performing $\Delta T = I_1 + I_2 + I_3 + I_4$, introducing (40) and (18) into the result, and using the abbreviating notations:

$$\begin{aligned} H &= (64N_0 - 28N_1 + 16N_2 - 11N_3 + 12\pi(8M_0 + 4M_2 + 3M_4) + \\ &+ 96F_1^0)/16, \\ J &= (64N_0 - 24N_1 + 16N_2 - 9N_3 + 12\pi(8M_0 + 4M_2 + 3M_4) + \\ &+ 96F_1^0)/16, \end{aligned} \tag{42}$$

we obtain:

$$\Delta T = \pi X_0 p^4 \mu^{-1} \delta(HCw - 2\pi J/T_0). \tag{43}$$

9. **Particular cases.** Suppose that the atmospheric rotation is neglected ($w = 0$). In this case, taking also into account (18), the difference (43) reduces to:

$$\Delta T = -\pi X_0 p^{5/2} \mu^{-1/2} \delta J. \tag{44}$$

Observe that the same formula (44) is obtained from (43) if $w \neq 0$, but $C=0$, In physical terms, for polar circular orbits, the difference ΔT due to the aerodynamic drag does not depend on the atmospheric rotation.

Suppose, for another particular case, that the initial values of the orbital elements refer to the ascending node ($u_0 = 0$). Imposing this condition to (38) then using (35), and introducing the results into (42), these last expressions become:

$$H^0 = (160N_0 + 20N_1 + 48N_2 + 13N_3 + 12\pi(8M_0 + 4M_2 + 3M_4))/16,$$
$$J^0 = (160N_0 + 24N_1 + 48N_2 + 15N_3 + 12\pi(8M_0 + 4M_2 + 3M_4))/16, \quad (45)$$

and (43) acquires the form:

$$\Delta T = \pi X_0 p^4 \mu^{-1} \delta (H^0 Cw - 2\pi J^0/T_0). \quad (46)$$

Obviously, the same restrictions (negligible atmospheric rotation, or polar orbit) can be imposed to (46). The result will be (44) in which J^0 replaces J.

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IN MEMORIAM

PROFESSOR FRANCISC RADÓ

(1921–1990)

FRANCISC RADÓ was born on May 21, 1921 in Timişoara (Romania), where he got his primary and secondary education at jewish schools. He inherited his mathematical talent from his father Arthur Radó, a railway engineer with great interest in mathematics, who has had some original contributions in algebra. The father disclosed for his son the beauties of mathematics, conducted his steps in this direction and had long discussions with him on mathematical problems. In 1939 F. RADÓ succeeded to enter the Engineering School in Bucharest, occupying the third position at the admission exam among thousands of candidates.

After one year of study he was forbidden to continue his studies. It followed one idle year and three years of labour camp. The conditions were hard, nevertheless he didn't interrupt to study mathematics, usually hidden behind heaps of excavated earth.

In 1944, in the new romanian regime he was accepted as a student again; switched to study mathematics and got his degree in January 1946 at the University of Cluj. Until the end of 1949 he taught mathematics in Timişoara, first at the Jewish Lyceum, where he once was a school boy, and then, after the nationalization of schools in 1948, he joined a state secondary school. He was also appointed as assistant professor at the newly founded Pedagogical Institut in Timişoara during one academic year.

In January 1950 he was nominated at the Bolyai University in Cluj. In 1959, the two universities in Cluj (with teaching languages romanian and hungarian, respectively) were united to from the Babeş-Bolyai University, where RADÓ was active until his retirement in 1985 (from 1969 as full professor). Between 1951 and 1968 he also held a second position as research scientist at the Academic Institut for Computation in Cluj. He taught during the 1969–70 academic year at the University of Waterloo (Canada) as visiting professor on the invitation of Professor J. Aczél.

His primary interest was algebraic foundations of geometry. Prior to this he studied nomographic representability related to the theory of webs and functional equations (his PhD thesis in 1959 belongs to this domain), then the embedding conditions of a semi-web into a Reidemeister web. He continued with noninjective collineations of two Desarguesian projective planes, which led him to ring geometry. Under W. Benz's influence he became very interested in characterizing isometries of metric vector spaces with as weak assumptions as possible. He also obtained remarkable results in mathematical programming.

At the university he taught geometry, algebra, theory of numbers and theory of complex functions. He published several courses in geometry and exercise books. Wrote jointly with B. Orbán the book „Geometry under modern view point" in hungarian language, showing how various euclidean, affine, projective and reflexion geometries and their algebraic counterparts are interweaved. In a review on this book Professor F. Kárteszi wrote: „Such an ample treatment of the subject promised in the title didn't appear till now in hungarian. Moreover, such a thorough examination of the relations among the presented structures didn't occur in any other language either" (Az Eötvös Loránd Tudományegyetem Természettudományi Karának Szakmódszertani Közleményi XIV/1, Budapest, 1981). For his scientific and teaching activity RADÓ was awarded prizes by the Romanian Ministry of Education in 1960, 1967 and 1981.

In 1952 he married Martha Gidáli, one of his students. During the years to come they worked together on aspects of modernizing the teaching of geometry and had an essential contribution in writing new geometry manuals for the secondary school. Martha succeeded in ensuring favourable conditions to thorough scientific work. Their two daughters took up careers in computer science.

Professor RADÓ is among the editors of „Journal of Geometry" (rigth from the start) and of „Aequationes Mathematicae". More than 20 years he acted as reviewer for „Mathematical Reviews" and „Zentralblatt für Mathematik".

He gave talks at Universities in Bucharest, Iassy, Timişoara, Budapest, Debrecen, Giessen, Bochum, Waterloo, Montreal, Hamilton, Buffalo, Frankfurt, Pavia, Hamburg, Hannover, Braunschweig, München, Kiel, Duisburg, Oldenburg, Toronto, Pittsburgh, New York (Courant Institut), Brockport, New Brunswick, Tel Aviv and Haifa. In July 1982 he spend one month at the University of Hamburg being sponsored by W. Blaschke Fund, on the recommendation of Prof. Benz. In 1985, on the invitation of Prof. Aczél he acted as a visiting research scientist at the University of Waterloo for two months. He presented from his original researches on a series of mathematical meetings and symposiums, but was prohibited to attend many others due to the ever increasing restrictions imposed by the Ceauşescu government. Moreover, he couldn't go to the 1972 meeting on Higher Geometry, held in Oberwolfach, in spite of the fact that he was one of the organizers.

Some of Professor RADÓ's scientific results:

His studies in nomography comprise: i) criteria for various nomographic representability in terms of functional equations without differentiability assumptions (1959); ii) projective transformation of alignment nomograms in order to obtain minimal error. He showed that for a nomogram in a projective family to be optimal it is necessary and sufficient that the points of maximal error on each scale fulfil a specific geometric condition (1964, 1969). Studies in this direction were continued by B. Orbán, V. Groze, A. Vasiu.

He generalized Malcev's theorem concerning the conditions for a semigroup to be injectively embeddable into a group. Giving to this problem a geometric form he established necessary and sufficient conditions for a semi-web to be injectively embeddable in a regular web (satisfying Reidemeister's condition). This is not only a more general result, but the rather sophisticated Malcev's chains appear in a simpler and natural manner (1965).

Let S be a non-singular incidence structure such that to any point A and any line L there exists a point P and a line D with $A \in D$, $P \in D$, $P \in L$. Then S can be injectively imbedded in a projective plane with given general closure condition C if and only if all „projective conclusions" of C hold in S (1969).

A fundamental theorem of projective geometry states that any injective collineation of two projective spaces over fields K and K' respectively, is induced by a semi-linear mapping of the underlying vector spaces, related to certain monomorphism $s: K \to K'$. In 1956 W. Klingenberg has extended this theorem for non-injective collineations of spaces of finite dimension. The role of s is taken by a place (valuation morphism); he worked with coordinates. RADÓ has generalized Klingenberg's theorem to projective spaces of arbitrary dimension, by suitable adjusting Artin's method, instead of using coordinates (1969). This method was aferwards applied in many situations, especially for ring geometries by K. Mathiak, J. Brandstetter, P. Hartmann, F. Machala, H. Havlicek and others. Related to these ideas is also RADÓ's following result.

Let P and P' be Desarguesian projective planes and C a subset of P which contains three) non-concurrent lines and a point not incident with them. Then any (non necessarily injective collineation of C into P' can be uniquely extended to a collineation of P into P' (1970). Severa refinements and generalizations of this theorem are due e.g. to B. Orbán and J. L. Zemmer; D. S

Carter and A. Vogt have given a thorough study on collinearity-preserving mappings between Desarguesian planes.

RADÓ also gave an example of a valuation of a skew-field such that the corresponding valuation ring is non invariant. Thus, in Schilling's definition the invariance of the valuation ring is not a consequence of the other axioms (1970).

In a joint paper by Aczél, Djoković, Kannappan and RADÓ, the contribution of RADÓ is the following: Let G, H be groups, S a subsemigroup of G such that $G = S \cdot S^{-1}$ and let $f: S \to H$ be a morphism. The f can be extended to a morphism of G into H in a unique way (1971).

In the usual definition of skew-field RADÓ has replaced the axioms of the existence of unit and inverses by a weaker requirement, which can be stated as follows: Let $(F, +, \cdot)$ be a distributive nearring, $F^* = F\setminus\{0\}$, $M = \cap \{xF \cup Fx \mid x \in F^*\}$ and $N = \{x \in F \mid 0 \notin xF^*\}$. Then, $(F, +, \cdot)$ is a skew-field if and only if $M \cap N \neq 0$.

W. Leissner has characterized by geometric axioms the plane geometry over a weakly 1-finite ring (i.e. a ring with 1 such that $ab = 1 \Rightarrow ba = 1$). RADÓ gave an extension to arbitrary rings with identity. The geometric axiom concerning the existence and unicity of the line incident with two non-neighbouring points A and B is now replaced by the requirement that the intersection of all lines incident with A and B be a line. Leissner's other axioms remain essentially the same (1980). Corresponding results for the multidimensional case were given by W. Leissner, R. Severin and K. Wolf.

In the last years great effort have been made to characterize isometries (and semi-isometries) of different metric spaces under weak assumptions. Remind two theorems: A. D. Alexandrov's (1950) and F. S. Beckman—D. A. Quarles' (1953). The first states that a *bijection* of the Minkowski spacetime M_n $(n \geq 3)$ into itself, which preserves the zero pseudo-euclidean distances in *both directions* is necessarily linear (up to translation). By the second, for given $r > 0$, *any* mapping f of the Euclidean space E_n $(n \geq 2)$ *into* itself such that $|PQ| = r \Rightarrow |P^fQ^f| = r$ must be a motion (i.e. linear up to translation). Unlike Beckman-Quarles theorem, that of Alexandrov does not hold in the two-dimensional case, but a bijective mapping $f: M_2 \to M_2$ which leaves invariant one pseudo-euclidean distance $r \neq 0$, in both directions, is a Lorentz transformation (i. e. linear up to translation), as was proved by W. Benz in 1977. RADÓ generalized this result going over from the real numbers to an arbitrary field K with char $K \neq 2$ or 3, i.e. he proved the result above for an Artinean plane (non singular, non-anisotropic metric vector plane over K) (1980). The main idea was to draw in the proof Hua's theorem concerning the characterization of field monomorphisms. Then W. Benz generalized further in the sense that f can be *any* mapping of the Artinean plan into itself preserving the distances 1 *in one direction;* then f must be semi-linear (up to translation), provided the field K satisfies one of a list of assumptions (taking 1 istead of $r \neq 0$ is really no restriction). The following important step was made by RADÓ in 1983 by reducing Benz's assumptions only to char $K \neq 2$, 3 or 5 or $K = GF(5^m)$, $m > 1$, Among RADÓ's other results in this area is worthwise to mention: Let V be a metric vector space over the field $GF(p^q)$, $p > 2$, $3 \leq \dim V = n < \infty$; given a mapping $f: V \to V$ such that $|PQ| = 1 \Rightarrow |P^fQ^f| = 1$, then f is semi-linear (up to translation), provided $n \not\equiv 0, -1, -2 \pmod{p}$ or the discriminant of V satisfies a certain condition. The proof is based on the condition for a regular simplex to exist in a Galois space, which is of interest for its own sake, as well.

He formulated in 1963 the „branch and bound" method to solve the disjuntive programming problem, independently from the other authors who gave similar algorithms. In his review on RADÓ's paper (published in romanian), E. Balas writes in the International Abstracts in Operations Research 7, No. 1, 1967: „The basic principle of the branch and bound technique devised by Land and Doing for the integer programming problem and adapted by Little, Murty, Sweeney and Karel for solving the travaling salesman problem, is rediscovered here independently, stated in a more general form and used to solve a very important problem: linear programs with logical constrains.. It is to be noted that this work precedes the important paper by P. Bertier and B. Roy on the S.E.P.".

An extension theorem for Pexider's equation, in a general setting is proved in a joint paper with J. A. Baker, which is then used to generalize aggregatoin theorems for allocation problems by Aczél, Ng and Wagner (1987).

Among the various problems of optimally locating a new service facility with respect to the locations of n existing facilities, the Steiner problem arises when one seeks to minimize the sum of the weighted distances from the new facility to the existing ones (if $n = 3$ and weights $= 1$, it is the 300 years old Fermat problem). E. Weiszfeld proposed an iterative algorithm for solving the Steiner problem, but not until the 1970's, however, did H. Kuhn and L. M. Ostresh demonstrate that this algorithm always solves the problem. In 1958 W. Miele proposed an extension of the Weifeld algorithm to the Euclidean multifacility location problem (with several new facilities). RADÓ has provided, in an arbitrary dimensional Euclidean space, for a slightly modified version of Miele's algorithm Kuhn-Ostresh type theorems for convergence and optimality, both under mild assumptions (1988).

PROF. F. RADO's PUBLICATIONS

A. Research papers

1. *Observații asupra unui sistem liniar infinit* (*Remarks on an infinite linear system;* romanian). Bull. şt. secţ. şt. mat. fiz., 5 (1953), 285–292.
2. (With G. Călugăreanu) *Asupra unei probleme de propagare a căldurii* (*On a problem of heat propagation;* romaniam). Bul. şt. secţ. şt. mat. fiz., 6 (1954), 17–30.
3. (With L. Bal). *Două teoreme referitoare la separarea variabilelor în nomografie*. Comunic. Acad. R.P.R., 5 (1955), 285–290 *(Two theorems concerning the separation of variables in nomography)*.
4. (With L. Bal) *Separarea variabilelor în nomografie (The separation of variables in nomography)*. Comunic. Acad. R.P.R., 5 (1955), 303–305.
5. *Condiții de dependență liniară pertru trei funcții continue (Linear dependence conditions for three continuous functions)*. Studii şi cercet. Cluj, ser. I, 6 (1955), 51–63.
6. (With E. Gergely, L. Bal and G. Ionescu) *Despre nomogramele romboidale. (On rhomboidal nomograms)*. Lucrările consf. de geometrie diferenţială, Timişoara, 361–366 (1955).
7. *Certaines propriétés des intégrales des équations différentielles linéares*. Lucrările celui de al IV-lea Congres al matematicienilor români, 1956, 162–163.
8. *Az algebrai egyenletek numerikus megoldásáról (On the numerical solution of the algebraic equations;* hungarian). Bul. Univ. Babeş şi Bolyai, ediţia 1, maghiară, 2 (1957), 13–24.
9. *Cea mai bună transformare proiectivă a scărilor la nomograme cu puncte aliniate (The best projective transformation of the scales of alignment nomograms)*. Studii şi cercet. de matematică, Cluj, 8 (1957), 161–168.
10. *Despre contactele nomogramelor cu transparent (On the contacts of the nomograms with transparents)*. Studii şi cercet. de matem., Cluj, 8 (1957), 319–329.
11. *Ecuații funcționale în legătură cu nomografia (Functional ecuations in connection with nomography)*. Studii şi cercet. de matematică Cluj, 9 (1958), 249–320.
12. *Équations fonctionelles caractérisant les nomogrammes avec trois échelles rectilignes*. Mathematica, 1 (24), fasc. 1 (1959), 143–166.
13. *Sur quelques équations fonctionnelles avec plusieurs fonctions a deux variables*, ibid, 1 (24), fasc. 2. (1959) 321–339.
14. (With J. Aczél and G. Pickert) *Nomogramme, Gewebe und Quasigruppen*. Mathematica, 2 (25), (1960), 5–24.
15. *Despre problema programării liniare (On the problem of linear programming)*. Studii şi cercet. de mat. Cluj, Fasc. anexă, 11 (1960), 167–177.
16. (With E. Munteanu) *Calculul şarjelor celor mai economice la cuptoarele de topit fontă (The calculation of the most economic cha ges of the furnaces to melt cast iron)*. Studii şi cercet. de mat. Cluj, Fasc. anexă, 11 (1960), 149–158.

17. *Generalizarea ţesuturilor spaţiale pentru structuri algebrice (Generalization of spatial webs related to algebraic structures)*. Studia Univ. Babeş-Bolyai, 1960, ser. I, fasc. 1, 41–55.
18. *Eine Bedingung für die Regularität der Gewebe*. Mathematica, 2 (25) (1960), 325–334.
19. *Le calcul aproximatif des extrèmes des fonctions*. Mathematica, 3 (26), (1961), 171–177.
20. *Caractérisation de l'ensemble des intégrales des équations différentielles linéaires homogènes a coefficients constantes d'ordre donné*. Mathematica 4 (27), (1962), 131–143.
21. *Characteristica spojnicovych nomogramu rodu nula funcnimi rovnicemi (The characterization of alignment nomograms of genus zero by functional equations)* (czech language). Nomografické Metody, Praha, 1962, 86–91.
22. *Nomograme, ţesuturi şi cuasigrupuri (Nomograms, webs and quasigroups)*. Studii şi cercet. de mat., fasc. anexă 13 (1962) 305–319.
23. (With L. Németi) *Ein Wartezeitproblem in der Produktion*. Mathematica 5 (28) (1963), 65–95.
24. *Programare liniară cu condiţii logice (Linear programming with logical conditions)*. Comunicările Acad. R.P.R., 13, nr. 12 (1963), 1039–1042.
25. *Un algorithme pour résoudre certains problèmes de programmation mathématique*. Mathematica 6 (29) (1964), 105–116.
26. *Ţesuturi ternare regulate (Regular ternary webs)*. Studia Univ. Babeş-Bolyai, ser. 1, fasc. 1, 1964, 39–60.
27. (With M. Hosszú) *Über die Klasse von ternären Quasigruppen*. Acta math. Acad. sci. Hung., 15 (1964), 29–36.
28. (With V. Groze and B. Orbán) *Propriétés extrémales dans une classe de fonctions et application à la transformation des nomogrammes*. Mathematica 6 (29), (1964), 307–326.
29. *Einbettung eines Halbgewebes in ein reguläres Gewebe und eines Halbgruppoids in eine Gruppe*. Mathem. Zeitschrift, 89 (1965), 395–410.
30. *Über die beste projektive Transformation der geradlinigen Leiter*. Zeitschr. f. angewandte Mathematik und Mechanik, 45 (1965), 356–359.
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32. (With V. Peteanu) *Structures algébriques rattachées aux problèmes d'ordonnancement*. Lucrările colocv. de teoria aproximativă a funcţiilor, Cluj, 1967, 1–35.
33. *Sur le problème de flôt à coût minimum*. Studia Univ. Babeş-Bolyai, ser. Math. Phys., fasc. 1, 1968, 67–79.
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35. *Einbettung von Inzidenzstrukturen in projektive Ebenen mit vorgegebenen Schliessungssätzen*. Archiv der Mathematik, 20 (1969), 311–319.
36. *Darstellung nicht bijektiver Kollineationen eines projektiven Raumes durch verallgemeinerte semilineare Abbildungen*. Math. Zeitschr., 110 (1969), 153–170.
37. *Non-Injective Collineations on Some Sets in Desarguesian Projective Planes and Extension of Non-Commutative Valuations*. Aequationes Math. 4 (1970), 307–321.
38. *Congruence Preserving Isomoprhisms of the Translation Group Associated with a Translation Plane*. Canad. Journal of Math., 23 (1971), 214–221.
39. *A Branch and Cut Method for Solving Mixed One-Zero Programs*. Publ. de l'Univ. de Montréal, Fac. de Sciences, 22 (1970), 1–18.
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44. (With J. Aczél) *Involutory Functions Induced by Abelian Groups*. Studia Univ. Babeş-Bolyai, ser. Math. Mech., 1974, 32–40.
45. *A Generalization of Valued Vector Spaces. Applications to Homomorphisms of Projective Spaces and to Hjelmslev Geometries*. Preprint, Cluj-Napoca, 1975, 29 pages.
46. *Remark on a Functional Equation*. Aequationes Math., 14 (1976), 228–230.
47. (With R. Miron) *On the Definition of a Module and a Vector Space by independent Axioms*. Mathematica, 18 (41) (1976), 179–186).

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50. *Affine Geometries and Affine Barbilian Structures*. Lucrările Col. Geom. şi top., 22–24 sept. 1978, Cluj-Napoca, 27–45.
51. *On the Definition of Skew Fields*. Archiv der Mathematik, 32 (1979), 441–444.
52. *On the Characterization of Affine Isometries*. Resultate der Mathematik, 2 (1979), 171–185.
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54. *Affine Barbilian Structures*. Journ. of Geom., 14 (1980), 75–102.
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57. *A Characterization of the Semi-Isometries of a Minkowski Plane over a Field K*. Journ. of Geom., 21 (1983), 164–183.
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59. *Teoreme de tip Beckman-Quarles în plane Minkowski generalizate peste un cîmp. (Beckman-Quarles Type Theorems in Generalized Minkowski Planes over Fields)*. Lucrările colocv. naţ. de geom. şi top., Piatra Neamţ, iunie 1983, Iaşi 1984, 15–24.
60. *On the Characterization of the Semi-Isometries of a Galois Space*. Lucrările colocv. naţ. de geom. şi top., Timişoara, 2–7 iulie 1984, 237–240.
61. *Mappings of the Galois Planes Preserving the Unit Euclidean Distance*. Aequationes math., 29 (1985), 1–6.
62. (With I. Muntean) *Un model al geometriei din manualul pentru clasa a IX-a*. Lucrările seminarului de „Didactica Matematicii", Cluj-Napoca, 1985, 67–82 *(A model for the geometry presented in the textbook for grade IX)*.
63. *On mappings of the Galois space*. Israel Journal of Math., 53 (1986), 217–230.
64. *On the Combinatorial Types of Polytopes over an Ordered Skew Field*. Research Seminars, Faculty of Mathematics, Cluj-Napoca, no. 10 (1986), 3–8.
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66. *On the Semi-Isometries of Metric Vector Spaces*. Lucrările colocv. naţ. de geom. şi top Tîrgovişte, 12–14 aprilie 1986; Bucureşti, 1988, 239–244.
67. (With J. A. Baker) *Pexider's Equation and Aggregation of Allocations*. Aequationes Math., 32 (1987), 227–239.
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B. Lecture Notes, Textbooks, Books

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2. (With L. Bal) *Lecţii de nomografie (Lectures in nomography)*, Ed. Tehnică, Bucureşti, 1956.
3. (With V. Cseke and E. Kiss) *Feladatgyűjtemény középiskolai matematikai körök számára) (Exercise book for mathematical circles in secondary school)*. Ed. Tehnică, Bucureşti, vol. 1, 1957; vol. 2, 1959.
4. (With P. Szilágyi) *Analitikus mértan példatár (Exercise Book for Analytical Geometry)*, Lithographed, Cluj, 1958.
5. *Curs de geometrie (A Course in Geometry)*, Univ. Babeş-Bolyai, Cluj, 1972.
6. *Lecţii de programare liniară (Lectures in Linear Programming)*. Univ. Babeş-Bolyai, Cluj-Napoca, 1975.
7. (With G. Galbură) *Geometrie (Geometry)*. Ed. Didactică şi Pedagogică, Bucureşti, 1979.
8. (With B. Orbán, V. Groze and A. Vasiu) *Culegere de probleme de geometrie (Exercise Book for Geometry)*. Univ. Babeş-Bolyai, 1979.
9. (With B. Orbán) *A geometria mai szemmel (Geometry under modern view point)*, Ed. Dacia, Cluj-Napoca, 1981.
10. *Sistemul axiomatic al lui Bachmann (Bachmann's Axiomatic System)*. Chapter IX of the book „The Foundations of Arithmetic and Geometry" by R. Miron and D. Brînzei). Ed. Academiei R.S.R., Bucureşti, 1983, 218–233.

11. *Matematikai kislexikon (Little Mathematical Enciclopaedia;* (hungarian) in common with Gy. Maurer, B. Orbán, P. Szilágyi and M. Vincze, Ed. Kriterion, Cluj-Napoca, 1983.
12. *Geometria pentru perfecţionarea profesorilor (Geometry to improve the teachers' knowledge)* in common with A. C. Albu, V. Obădeanu, I. P. Popescu and D. Smaranda; editors R. Miron and D. I. Papuc. Ed. Didactică şi Pedagogică, Bucureşti, 1983.

C. **Informing Writings**

1. *A Bolyai Emlékkönyv matematikai vonatkozásai (The Mathematical Aspects of the Bolyai Remembrance Book;* hungarian). Utunk, X. évf., 5 sz., 1954 jan. 29.
2. *Algebrai és geometriai strukturák kapcsolata (Connenctions between Algebraic and Geometrical Structures;* hungarian), Korunk 1973-as Evkönyve, 257—271.
3. *Vektorterek (Vector Spaces;* hungarian). Matematikai Lapok, 1977, No. 10, 370—378.
4. *Geometria absolută (Absolute Geometry;* romanian), Lucrările Şimp. Bolyai János, organizat cu prilejul împlinirii a 175 de ani de la naştere, Cluj-Napoca, 15.12.1977. Facultatea de Matematică, Cluj-Napoca, 1979, 99—111.
5. *Cîteva momente din istoria geometriei (Some moments from Hystory of Geometry;* romanian). Lucrările Seminarului de „Didactica Matematicii, 4, Cluj-Napoca, 1988, 213—230.

D. **Papers on Prof. F. RADÓ**

V. Groze, M. Ţarină, A. Vasiu — *The Life and Work of Professor F. RADÓ*, Research Seminars, Seminar on Geometry, Faculty of Mathematics, "Babeş-Bolyai" University Cluj-Napoca, Preprint Nr. 2, 1991.

RECENZII

Mathematical Models & Methods in Applied Sciences (M³AS) (Editors N. Bellomo and F. Brezzi), World Scientific Publishing Company, Singapore, New Jersey, London, Hong Kong, Volume 1, Number 1, March 1991, 123p., ISSN :0218-2025.

,,This journal is schedulet to be published quarterly. The purpose of this journal is to provide a medium of exchange for scientists engaged in applied sciences (physics, mathematical physics, natural and technological sciences) where there exists a non-trivial interplay between mathematics, mathematical modelling of real systems and mathematical and computer methods oriented towards the qualitative and quantitative analysis of real physical systems.

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R. PRECUP

CRONICA

I. Publicații ale seminariilor de cercetare ale catedrelor de Matematică (seria de preprinturi):

Preprint 1—1989, Seminar on Functional Analysis and Numerical Methods (edited by I. Păvăloiu);
Preprint 2—1989, Seminar on Mechanics (edited by P. Brădeanu);
Preprint 3—1989, Seminar on Differential Equations (edited by I. A. Rus);
Preprint 4—1989, Seminar on Geometry (edited by M. Țarină);
Preprint 5—1989, Seminar on Celestral Mechanics and Space Research (edited by A. Pál);
Preprint 6—1989, Itinerant Seminar on Functional Equations, Approximation and Convexity (edited by E. Popoviciu);
Preprint 7—1989, Seminar on Mathematical Analysis (edited by I. Muntean);
Preprint 8—1989, Seminar on Optimization Theory (edited by I. Kolumbán);
Preprint 9—1989, Seminar on Computer Science (edited by S. Groze);
Preprint 10—1989, Seminar on Complexity (edited by Gh. Coman).

II. Manifestări științifice organizate de Facultatea de Matematică în 1989:

1. Ședințele de comunicări lunare ale catedrelor de matematică;
2. Seminarul itinerant de ecuații funcționale, aproximare și convexitate (18–20 mai 1989);
3. Conferința interdisciplinară de Astronomie, Astrofizică, Științele universului, Științele pămîntului (9–10 iunie 1989).

IMPRIMERIA „ARDEALUL" Cluj-Napoca, com. 34/1992.

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