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ON A CLASSIFICATION OF COMPUTER BASED SYSTEMS

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REZUMAT. — **O clasificare a sistemelor bazate pe calculator.** Se încearcă o scurtă prezentare a unei clasificări bazate pe tendința de convergență a diverselor domenii ale informaticii.

1 **Introduction.** The present paper presents an outline of a larger work concerning a classification of computer based systems. Its basic idea is the classification of such systems based on their functionality (the actions they are able to perform) and its aim is to provide a basis for a unified method for defining data base, functional, logical and decisional systems. The classification assumes a uniform representation of knowledge for each type of system considered.

2. The Clasifiction

In the followings we shall consider a **computer based system (CS)** to be a system which can perform the aquisition, processing and producing of informations using a computer. The basic functions of such a system are:
– reception of informations,
– reperesentation of informations,
– processing of informations based on the representation,
– transmition of informations.

Although the reception and transmition of information are very important from the point of view of communication with the environment the basic activity of the above mentioned systems is the information processing based on the representation used. The information processing capabilities are going to be the basis of classification of CS-s in the followings.

At each moment a CS contains the representation of a part of the world called *its universe*. The universe may change in time. The changes depend on the CS's communication with its environment and on the processings it can perform. These processings are determined by the mechanisms incorporated in the CS. The mechanisms may be either processors of the computer used by the CS or procedures implemented on such processors. Depending on the incorporated mechanisms we classifiy CS-s in the following hierarchy:
– data base type systems,
– functional systems,

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— inferential systems,
— decision systems,

We consider this classification hierarchical in the sense that each type i by enriching the previous one by specific mechanisms.

— **Data base type systems.** These are CS-s which only include n for storing and retrieving informations. By *storing* we mean the updating the universe by adding or substituting informations. *Upda* process of searching the universe for informations matching certain (

This type of systems carry out the most elementary intellectua which consists of storing of informations and returning them with no tions.

The main problem with this type of systems is that of search (The field of data base management which covers the study and desig systems is rich in both theoretical and technical results.

— **Functional systems.** This type of systems assume the existenc universe of a special class of objects called *functional relations* (or simply f and include mechanisms for handling these objects. The specific acti cerning functions is their evaluation. ***Evaluation*** is the process by wl informations are produced starting from the existing ones in the unive on the functional dependencies defined by a function.

Since functions are defined on objects of the universe, evaluating mec take advantage of retrieving mechanisms. This is true even if the und universe is very small, say just a few variables of a program. The storing nisms are useful for updating the universe with intermediary or final r the evaluation.

Functional systems perform an intellectual activity of higher level c to data base type systems since they produce new informations not e present in the universe. The limitations of such systems are due to t that new informations obtainable are in a special (functional) relation w contents of the universe.

The most important requirement for such a system is the efficiency evaluating mechanisms.

— **Inferential systems.** Inferential systems are those which include nisms for producing new informations from those existing in the univers nonfunctional relations. These mechanisms use various *search strateg* obtaining these new informations. The use of storing and retrieving mech is motivated exactly by the same reasons as in the case of functional sy Since evaluating mechanisms are more efficient than search, inferential s take advantage of them in obtaining the so called computable inter informations.

The intellectual activity closest to that performed by these systems of reasoning. This is of higher level than the one performed by functior tems, since the new informations produced depend on the informations universe through more general relations than the functional ones.

This type of systems constitute the main object of study of the h part of artificial intelligence.

– **Decision systems.** All the systems of the above mentioned type perform activities according to goals either encapsulated in them or set by the user. Decision systems should be able to set their own goals depending on the informations in the universe and certain criterions of choice. After setting a goal this type of systems may use any of the mechanisms from the lower levels to reach it.

The intellectual activity this type of systems try to perform is that of choosing among various alternatives based on given criterions. An example of decision system may be the command system of a robot.

Based on a uniform representation of the universe including functions, relations and decision criteria and taking into account the above presented classification may be the basis for a unified definition and specification methodology for all types of CS-s may be developed. This may lead to a better integration of various fields of computer science and to a new instrument for designing the future computer generations.

3. Conlcusions

In this paper we tried to give a short presentation of a classification of computer based systems which is based on the obvious tendency of convergence of various fields of computer science. There is a lot of work to be done in this direction although the basis of this work is mainly contained in the results obtained in data base management, functional programming, logical systems and decision theory.

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A FUZZY TRAINING ALGORITHM

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REZUMAT. — Un algoritm fuzzy de instruire Se dă o metodă pentru instruirea unui clasificator fuzzy binar. Pentru instruire se utilizează o partiţie binară fuzzy a unei mulţimi de instruire.

1. Introduction. The aim of this note is to give a training procedure for a fuzzy binary classifier. For such a classifier the training classes are supposed to be fuzzy sets on a training set X.

Since many classes of real objects have not sharp boundries the fuzzy set may be successfully used to describe the classification structure of a data set (see [1], [3], [4]). We suppose that the cluster structure of the training set X is given by a binary fuzzy partition $\{A_1, A_2\}$ of X. This partition may be obtained for instance by using the Fuzzy Divisive Hierarchical Clustering algorithm ([3], [4], [6]).

Let $X \subset \mathbf{R}^d$ and f be a linear function on $\mathbf{R}^d$, $f(x) = v^T x$. A measure of separation of A_1 and A_2 by the hyperplane $f(x) = 0$ (or by the vector v) may be defined. It is high desirable to find a separating vector v for which the separation degree is as great as posisible.

In this paper we'll search for a hiperplane H such that for every class A_i the sum of distances (in this class) to H of the misclassified points be minimum We may admit that H obtained in this way is a good separation hyperplane. However we don't know if H is the best hypeplane. i.e. it maximizes the separation degree of the fuzzy classes A_1 and A_2.

Let B_1 and B_2 be the classical sets corresponding to A_1 and A_2. Even if B_1 and B_2 are not linearly separable it is possible to obtain a separation hyperplane for the fuzzy sets A_1 and A_2. Moreover we are able to measure how good is this hypeplane i.e. the separation degree realized by it.

2. Fuzzy partitions. Let X be a nonempty set considered as the universe of discourse. A fuzzy set on X is a function $A : X \to [0, 1]$. $A(x)$ is the membership degree of x to A. If A and B are fuzzy sets on X we may define

$$(A \cap B)(x) = T(A(x), B(x)), \quad \forall x \in X, \tag{1}$$

$$(A \cup B)(x) = S(A(x), B(x)), \quad \forall x \in X, \tag{2}$$

where T is a t-norm and S is its dual conorm (see [4], [5]).

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In this paper we consider the particular definitions

$$(A \cap B)(x) = T_\infty(A(x), B(x)) = \max(0, A(x) + B(x) - 1), \quad (3)$$

$$(A \cup B)(x) = S_\infty(A(x), B(x)) = \min(1, A(x) + B(x)). \quad (4)$$

Let C be a fuzzy set on X. If A_1 and A_2 are disjoint fuzzy sets and their union is C, then $\{A_1, A_2\}$ is called a *fuzzy partition* of C. It is easy to see that

$$\left.\begin{array}{l} A_1 \cap A_2 = \emptyset \\ A_1 \cup A_2 = C \end{array}\right\} \Leftrightarrow A_1(x) + A_2(x) = C(x), \quad \forall x \in X. \quad (5)$$

The equivalence (5) holds if and only if the set operations are defined by using a pair (T, S), with $T = T_\infty$ and $S = S_\infty$ (see [5]).

3. **Training with fuzzy sets.** Let $X = \{x^1, \ldots, x^p\}$, $x^j \in \mathbf{R}^d$ be a data set. Let us suppose that the cluster structure of X is given bu a binary fuzzy partition $\{A_1, A_2\}$ of X. We have thus

$$A_1(x) + A_2(x) = 1, \quad \forall x \in X.$$

In what follows we'll admit A_1 and A_2 are two training sets. These fuzzy training sets may be used to design a binary classifier. Therefore we'll search for a separating vector v. The corresponding decision rule will be:

x is assigned to class 1 if $v^T x > 0$;
x is assigned to class 2 if $v^T x < 0$.

A fuzzy classifier may also be considered. Let us suppose that we have a procedure to compute the membership degree of a new considered point x to the class A_i. If the class A_i has the prototype L^i, then we put:

$$A_1(x) = \frac{d(x, L^2)}{d(x, L^1) + d(x, L^2)}, \quad (6)$$

where $d(x, L^i)$ is the Euclidean distance of x to prototype L^i.

The decision rule for the fuzzy classifier may be written as:

x is assigned to class 1 if $A_1(x) > 0.5$ and $v^T x > 0$;
x is assigned to class 2 if $A_2(x) > 0.5$ and $v^T x < 0$.

Let $g : \mathbf{R}^d \to \mathbf{R}$ be a discriminant function

$$g(x) = w^T x + a, \quad (7)$$

where $x \in \mathbf{R}^d$, $a \in \mathbf{R}$. Let us denote by Y the space of the augmented vectors y, where

$$y = \begin{pmatrix} x \\ 1 \end{pmatrix}, \quad x \in X.$$

The fuzzy set A_i induces on Y a fuzzy set A_i^*:

$$A_i^*(y) = A_i(x), \quad y = \begin{pmatrix} x \\ 1 \end{pmatrix}.$$

what follows we'll write A_i instead of A_i^*. If we put

$$v = \begin{pmatrix} w \\ a \end{pmatrix},$$

discriminant function (7) becomes

$$g(y) = v^T y.$$

The non-separation degree of A_1 and A_2 with respect to g may be defined (see also [8]) as

$$M(A_1, A_2; g) = \inf\{k \in [0, 1] \mid [A_1(y) \leq k, \forall y \in g^{-1}(-\infty, 0] \text{ and } A_2(y) \leq k, \forall y \in g^{-1}[0, \infty)] \text{ or } [A_2(y) \leq k, \forall y \in g^{-1}(-\infty, 0] \text{ and } A_1(y) \leq k, \forall y \in g^{-1}[0, \infty)]\}. \quad (9)$$

The degree of separation of A_1 and A_2 by the hyperplane $g(y) = v^T y = 0$ may be defined as

$$D(A_1, A_2; g) = D(A_1, A_2; v) = 1 - M(A_1, A_2; g). \quad (10)$$

We admit that the fuzzy classes A_1 and A_2 are t-*linearly separable* if there exists a vector v for which $D(A_1, A_2, v) = t > 0$.
Ideally a vector v^* so that

$$D(A_1, A_2; v^*) = \max D(A_1, A_2; v),$$

is needed. We'll give a procedure to obtain a separating vector v. The procedure doesn't guarantee that the obtained vector is exactly v^*.

We'll eliminate the points of equal membership to A_1 and A_2. We considere the normalization of the sample vectors defined as

$$z = \begin{cases} y & \text{if } A_1(y) > 0.5 \\ -y & \text{if } A_2(y) > 0.5 \end{cases} \quad (11)$$

The fuzzy classes of the vectors z are also denoted by A_1, A_2.

Let H be the hyperplane of equation $v^T z = 0$. The set E_i of the samples of the class i misclassified by H is defined as

$$E_i = \{z \mid v^T z \leq 0 \text{ and } A_i(z) > 0.5\}. \quad (12)$$

Let d be a norm induced distance on $\mathbf{R}^d$. We denote by d_i the distance in the fuzzy set A_i induced by d. Distance with respect to A_i between a point z and the hyperplane H is (see [3] and [7]) given by

$$d_i(z, H) = A_i(z) d(z, H) = A_i(z) \mid -v^T z \mid. \quad (13)$$

We denote by $J_i(v)$ the sum of the distances in A_i between the points of E_i and H:

$$J_i(v) = \sum_{z \in E_i} d_i(z, H) = \sum_{z \in E_i} A_i(z)(-v^T z). \quad (14)$$

Consider a citerion function $J : \mathbf{R}^{d+1} \to \mathbf{R}$, where

$$J(v) = J_1(v) + J_2(v).$$

Therefore

$$J(v) = \sum_{i=1}^{2} \sum_{z \in E_i} A_i(z)(-v^T t).$$

On aim is to minimize J. In this respect a gradient descent procedure is Let v^k be the vector solution at the k-th step of the procedure. Then

$$v^{k+1} = v^k + a_k h^k,$$

where a_k is a positive number and h^k is the antigradient of J in v^k, i.e.

$$h^k = -\nabla J(v^k) = \sum_{i=1}^{2} \sum_{z \in E_i} A_i(z) z.$$

For our purpose it is sufficient to consider $a_k = c > 0$.

If the sample vectors are cyclically considered a sequence $(z^j)_{j \geq 1}$ is obt

$$z^{mp+j} = z^j, \quad m = 0, 1, 2, \ldots$$

The trainign procedure can thus be written as:

A. Choose an arbitrary vector $v \in \mathbf{R}^{d+1}$

B. Put $v^{k+1} = \begin{cases} v^k + cA_i(z)z^k, & \text{if } A_i(z^k) > 0.5 \text{ and } (v^k)^T z^k \leq 0 \\ v^k, & \text{otherwise.} \end{cases}$

This training procedure is a generalization of the well-known perc algorithm (see [2]). Therefore we may call it the Fuzzy Perceptron Algor

If v is the separating vector obtained by our procedure, the separation sure of the fuzzy classes A_1 and A_2 by $\bar{v}$ is given by $D(A_1, A_2; \bar{v})$.

In a further paper we'll study the convergence properties of the algo and some numerical examples will be given.

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A PRELIMINARY BIBLIOGRAPHY ON FUZZY CLUSTERING AND RELATED FIELDS

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REZUMAT. — O bibliografie preliminară asupra clasificării cu mulţimi fuzzy şi domenii conexe. Se indică principalele orientări în acest domeniu şi se dă o listă conţinînd 285 de lucrări.

A clustering algorithm is a procedure by which a collection of objects is partitioned into disjoint subsets or clusters. In the fuzzy clustering every data point belongs, with some degrees of membership to all clusters. A cluster of points corresponds to a fuzzy set on the data set X. The cluster structure of data is generally given by a fuzzy partition of X.

Fuzzy clustering should be useful in applications where the clusters touch, overlap, there exist bridges between clusters or there are isolated points.

The notion of fuzzy set was first introduced by Zadeh [281] and the use of fuzzy sets in clustering was first suggested by Bellman, Kalaba and Zadeh [14].

Negoita [203] has used a separation theorem of fuzzy sets to describe a cluster based information retrieval system.

Ruspini [224] has introduced a notion of a fuzzy partition to describe the cluster structure of a data set.

Gitman and Levine [132] have proposed an algorithm to detect the unimodal fuzzy sets. The obtained clusters are not fuzzy sets.

Dunn [115] has generalized the minimum-variance clustering procedure to a Fuzzy ISODATA clustering technique.

Bezdek [15] has generalized Dunn's approach to obtain an infinite family of algorithms known as the Fuzzy c-Means (**FCM**) algorithms.

FCM procedure has been generalized to obtain the cluster substructure of a fuzzy class by Dumitrescu [80]. A convenient notion of partition of a fuzzy class and a method to derive the criterion function have been used. Based on the Generalized FCM procedure a hierarchical clustering method has been proposed [101]. This method produces a fuzzy hierarchy and is a solution for the cluster validity problem.

Roubens [223] has considered a non-metric approach of the fuzzy clustering problem.

Backer [8] has proposed a clustering method that simultaneously generates a pair of optimal hard and induced fuzzy partition.

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Thomason [263] and Lee [180] have suggested to use of the languages to pattern recognition.

The use of fuzzy relations to clustering have been considered by F and Turner [126], Tamura et. al. [259], Dunn [118], Ye Bang [280], Bezdek and Harris [24].

In the following it's proposed a preliminary bibliography on fuzzy ring and related fields that we hope to be of a real utility to those wh interested in this field.

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A COMBINED ITERATIVE METHOD FOR SOLVING OPERATORIAL EQUATIONS IN FRECHET SPACES

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REZUMAT. — O metodă iterativă combinată pentru rezolvarea ecuaţiilor operatoriale în spaţii Fréchet. În lucrare sînt date rezultate similare cu cele din [2] şi [3], dar în spaţii Fréchet. Aceasta pentru că există spaţii cvasinormate care nu sînt normate şi există posibilitatea de a defini familii de cvasinorme folosind o normă.

1. In order to apply the methods of Newton-Kantorovici type for solving the operatorial equations

$$P(x) = \theta \tag{1}$$

where $P: X \to Y$, to each iterative step, is used the inverse of linear map: the Frechet derivative of the first order (see, the Newton-Kantorovici method) or the divided difference of the first order (see the chord method).

This difficulty can be eliminated taking both the sequence (x_n) of iteratives and the sequence (A_n) of linear operators, where $(A_n): Y \to X$, sequence which is convergent to the inverse linear operator which appears in the iterativ method.

S. Ul'm [1] gives a method for a simultaneous approximation both for the solution of (1) and for the inverse of the Fréchet derivative.

A. Diaconu and I. Păvăloiu ([2], [3]) are studying such a combined method, in Banach spaces, which generalizes the Ul'm-s method.

In this paper we are given such results in Fréchet spaces. This, not only from the reason that there exist quasinormed spaces which are not normate spaces [5], but also because there existes the possibility to define a family of quasinormes, using a norme, and to choose the family of parameters such that the convergence conditions of the corresponding iterative method to be satisfied.

2. Let be the iterative method [2]

$$x_{n+1} = x_n - A_n P(x_n)$$
$$A_{n+1} = A_n(2I - [x_n, x_{n+1}, P]A_n), \quad n = 0, 1 \ldots \tag{2}$$

obtained from the method of chord, where x_0 is an arbitrary selected point in the Fréchet spaces X, I the identity operator on Y.

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The operator $P: X \to Y$ is a continuous one and $A \subset (Y, X)^*$; $(Y, X)^*$ is the space of linear and continuous operator defined on Y values in X, an iterative operator associated of the operator P, i.e. an ope. for which the solution x^* of the equation (1) is a fixed point.

We denoted with $[x_n, x_{n+1}; P]$ the generalized divided differen. P on nodes (x_n, x_{n+1}) and by $\| \cdot \| : X \to R_+$, the quasinorme induced an invariant distance $d = X \times X \to R_+$, i.e. $d(x, y) = d(x - y, 0)$ and $\|x\| = d(x, 0)$.

In connection with the iterative method (2), we have the following

THEOREM. *If the following conditions are verified in the hall* $S = \{x \mid \|x - x\| \leqslant R\}$:

1°. *the operator P has an inversable divided difference of first order*

$$\|[x, y; P]^{-1}\| \leqslant B < +\infty, \ \forall x, y \in S$$

and has a divided difference of second order with

$$\|[x, y, z; P]\| \leqslant M < +\infty, \ \forall x, y, z \in S;$$

2°. *the A_0 is a bounded operator and* $\|A_0\| \leqslant 2B$;

3°. *the following inequality holds*

$$max\{4MB^2\|P(x_0)\|, (1/9)u\| I - [x_0, x_1; P]A_0\|\}^2 \leqslant cd$$

$$when \ u = \sqrt{4MB^2\|P(x_1\|}, \ d<1, \ R = \frac{cd}{2MB(1 - d^{p-1})}, \ c = 1/9$$

$$p = \frac{1 + \sqrt{5}}{2} - \varepsilon \ for \ any \ \varepsilon \in]0, 1[,$$

hen

(i) *the sequences (x_n) and (A_n) given by* (2) *are convergent*;

(ii) *the equation* (1) *has the solution $x^* \in S$ and*

$$\lim_{n \to \infty} x_n = x^*;$$

(iii) *If* $A^* = \lim\limits_{n \to \infty} A_n$ *then* $A^* = \lim\limits_{n \to \infty} [x_n, x_{n+1}; P]^{-1}$;

(iv) *the following inequalities are satisfied*:

$$\|x^* - x_n\| \leqslant \frac{dp^n(p-1)}{18MB(1 - d^{p^n(p-1)})}$$

and

$$\|A^* - A_n\| \leqslant \frac{2B}{9}\left[2d^{p^{n-1}} + \frac{3d^{p^n}}{1 - d^{p^n(p-1)}}\right], \ n = 0, 1, \ldots$$

Proof. We shall prove, by induction, the following relations:

a) $x_n \in S, \ n = 0, 1, \ldots$

b$_1$) $r_n = 4MB^2\|P(x_n\| \leqslant \theta_n d^{a_n} \leqslant (1/9)d^{p^n}; \ n = 0, 1, \ldots$

b$_2$) $d_n = \|I - [x_{n-1}, x_n; P]A_n\| \leqslant c_n d^{b_n} \leqslant (1/9)d^{p^{n-1}}; \ n = 0, 1, \ldots$

where (θ_n) and (c_n) are sequences generated by the relations

$$\theta_{n+1} = \theta_n^2 d^{a_n - a_{n-1}} + \theta_n \theta_{n-1} + \theta_n c_n d^{b_n - a_{n-1}}$$

$$c_{n+1} = (c_n d^{b_n - a_{n-1}} + \theta_n d^{a_n - a_{n-1}} + \theta_{n-1})^2$$

with $\theta_0 = \theta_1 = c_1 = 1/9$, (a_n), (b_n) are sequences generated by the relation

$$\begin{cases} b_{n+1} = 2a_{n-1} \\ a_{n+1} = a_n + a_{n-1} \end{cases}$$

with $a_0 = 1$, $a_1 = 2$, $b_1 = 1$;

c) $)|A_n|(\leqslant 2B$, $n = 1, 2, \ldots$.

Indeed, from the previous hypothesis we have, $x_0 \in S$.

Using 3° it results, for $n = 0$, $r_0 \leqslant (1/9)d$ and $)|A_0|(\leqslant 2B$. For $n = 1$, b_2 becomes $d_1 \leqslant (1/9)d$.

Next, we assume that the relations $a)-c)$ are true for $n = k > 0$, and we show that they are true for $n = k + 1$.

Using (2) we have

$$)|x_{k+1} - x_0|(\leqslant \sum_{i=0}^{k})|x_{i+1} - x_i|(\leqslant \sum_{i=0}^{k})|A_i|(\cdot)|P(x_i)|(\leqslant$$

$$\leqslant 2B \frac{d}{36MB^2}(1 + d^{p-1} + \ldots + d^{p^k-1}) \leqslant$$

$$\leqslant \frac{d}{18MB^2}(1 + d^{p-1} + \ldots + d^{k(p-1)}) < \frac{d}{18MB(1 - d^{p-1})} = R$$

because $k(p-1) \leqslant p^k - 1$, and so $x_{k+1} \in S$.

To prove the property b_1), based on Newton's interpolation formula, we have

$$)|P(x_{k+1})|(\leqslant)|P(x_{k+1}) - P(x_k) - [x_{k+1}, x_k; P](x_{k+1} - x_k)|(+$$
$$+)|P(x_k) + [x_{k+1}, x_k; P](x_{k+1} - x_k) \leqslant$$
$$\leqslant)|[x_{k+1}, x_k, x_{k-1}; P]|(\cdot)|x_{k+1}, x_k|(\cdot)|(x_{k+1} - x_{k-1}|(+$$
$$+)|P(x_k) + [x_{k-1}, x_k; P](x_{k+1}, - x_k|(,$$

and, by the induction hypothesis, we get

$$)|P(x_{k+1})|(\leqslant M) |A_k|(\cdot) |P(x_k)|(\cdot$$
$$\cdot ( )|A_k|(\cdot) |P(x_k)|(+) |A_{k-1}|(\cdot) |P(x_{k-1})|( ) +$$
$$+)|P(x_k)|(\cdot) |I - [x_{k-1}, x_k; P]A_k|). \tag{5}$$

Also

$$)|I - [x_k, x_{k+1}; P]A_{k+1}|(\leqslant) |I - 2[x_k, x_{k+1}; P]A_k +$$
$$+ ([x_k, x_{k+1}; P]A_k)^2 |(\leqslant)| I - [x_k, x_{k+1}; P]A_k|(^2 \leqslant$$
$$\leqslant\{) |I - [x_{k-1}, x_k; P]A_k|(+)| [x_{k+1}, x_k; P] -$$
$$- [x_k, x_{k+1}; P]|(\cdot) |A_k\}^2 \leqslant \{) |I - [x_k, x_{k-1}; P]A_k|(+$$
$$+)|A_k|(\cdot)| [x_{k-1}, x_k, x_{k+1}; P]|(\cdot () |x_{k+1} - x_k|(+)|x_k - x_{k-1}()\}^2. \tag{6}$$

Because $)|A_k|(\leqslant 2B$, in base of (5), (6) and with notation from b₂), we have

$$r_{k+1} \leqslant r_2^k + r_k \cdot r_{k-1} + r_k \cdot d_k$$
$$d_{k+1} \leqslant (d_k + r_k + r_{k-1})^2.$$

From these relations, like in [3], it follows

$$r_{k+1} \leqslant \theta_{k+1} d^{a_{k+1}} < (1/9) d^{p^{k+1}}$$
$$d_{k+1} \leqslant c_{k+1} d^{b_{k+1}} < (1/9) d^{p^k}$$

and so the b) properties are true for $n = k + 1$.

To prove the property c), we have

$$)|A_{k+1}|(=)\ |\ [x_k, x_{k+1}; P]^{-1} + A_{k+1} - [x_k, x_{k+1}; P]^{-1}|(\leqslant$$
$$\leqslant)\ |\ [x_k, x_{k+1}; P]^{-1}\ |((1+)\ |I - [x_k, x_{k+1}; P]A_{k+1}|(\ )\leqslant$$
$$\leqslant B(1 + d_{k+1}) < B(1 + (1/9)d^{p^k}) < 2B.$$

In base of a) — c), it results

$$)|x_{n+m} - x_n|(\leqslant \sum_{i=n}^{n+m-1})|x_{i+1} - x_i|(\leqslant \sum_{i=n}^{n+m-1})|A_i\ |(\cdot)\ |P(x_i)\ |(\leqslant$$
$$\leqslant \frac{d^{p^n}}{18MB}[i + d^{p^n} + \ldots + (d^{p^n(p-1)})^{m-1} + \ldots] < \frac{d^{p^n}}{18MB(1 - d^{p^n(n-1)})}.$$

Because $d < 1$, it results that the sequence (x_n) is a fundamental one, therefore it is convergent to $x^* \in S$.

From b₁) it results

$$\lim_{n\to\infty} P(x_n) = 0$$

and because P is continuous, we get that x^* is a solution of the equation.

Making, in (8) $m \to \infty$, we obtain the error of which x_n approximates solution x^*:

$$)|x^* - x_n|(\leqslant \frac{d^{p^n(p-1)}}{18MB(1 - d^{p^n(p-1)})}$$

where from it results, for $n = 0$

$$)|x^* - x_0|(\leqslant \frac{d^{p-1}}{18MB(1 - d^{p-1})} < R$$

so $x^* \in S$.

To establish the convergence of the sequence (A_n) we have, by (2),

$$) |A_{i+1} - A_i |(=) |A_i(2I - [x_i, x_{i+1}; P]A_i) - A_i| (\leqslant$$
$$\leqslant) |A_i| (\cdot) |I - [x_i, x_{i+1}; P]A_i|(\leqslant$$
$$\leqslant 2B (d_i + r_i + r_{i-1}) \leqslant 2B/9 (d^{p^i} + 2d^{p^{i-1}})$$

nd so

$$) |A_{m+n} - A_n |(\leqslant \sum_{i=n}^{n+m-1}) |A_{i+1} - A_i |(\leqslant$$
$$\leqslant 2B/9 \left(\sum_{i=n}^{n+m-1} d^{p^i} + 2 \sum_{i=n}^{n+m-1} d^{p^{i-1}} \right) =$$
$$= 2B/9 \left(2d^{p^{n-1}} + d^{p^{n+m-1}} + 3 \frac{d^{p^n}}{1 - d^{p^n(p-1)}} \right),$$

which denotes that the convergence $A(n) \subset (Y,X)^*$ is fundamental.
Because X,Y are Fréchet spaces, it results

$$\lim_{n\to\infty} A_n = A^*$$

and

$$) |A^* - A_n |(\leqslant 2B/9 \left(2d^{p^{n-1}} + \frac{3d^{p^{n-1}}}{1 - d^{p^n(p-1)}} \right).$$

We prove that the sequence $([x_n, x_{n+1}; P]^{-1})$ is convergent to the A^*, too
We have

$$) |A_n - [x_n, x_{n+1}; P]^{-1} | (\leqslant) | [x_n, x_{n+1}; P]^{-1}| (\cdot) |I - [x_n, x_{n+1}; P]A_n |(\leqslant$$
$$\leqslant 2B/9 (2d^{p^{n-1}} + d^{p^n}),$$

and so

$$) |A^* - [x_n, x_{n+1}; P]^{-1}| (\leqslant) |A^* - A_n| (+) |A_n - [x_n, x_{n+1}; P]^{-1}| (<$$
$$<) |A - A_n| (+2B/9 (2d^{p^{n-1}} + d^{p^n})$$

from where the statement results.
Hence, the theorem is proved.

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AN ALGORITHM FOR CONVEX DECOMPOSITIONS OF FUZZY PARTITIONS

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REZUMAT. — Un algoritm pentru descompunerea convexă a unei partiţii nuanţate. În lucrare se prezintă un nou algoritm care realizează descompunerea convexă a reprezentării matriceale a unei partiţii nuanţate. Algoritmul este comparat cu algoritmul Minimax (MM) al lui Bezdek şi Harris [1]. Se infirma astfel o conjectură din [1]. Se dovedeşte inconsistenţa Teoremei 3 din [1].

1. Introduction. Let $X = \{x_1, \ldots, x_p\}$ be a non-empty set and A_i, $i = 1, \ldots, n$, are fuzzy sets on X. It may be proved (see [2 – 5]) that $P = \{A_1, \ldots, A_n\}$ is a fuzzy partition of X if and only if the condition

$$\sum_{i=1}^{n} A_i(x) = 1, \tag{1}$$

holds for every $x \in X$.

Any fuzzy partition $\{A_1, \ldots, A_n\}$ of X may be represented by an $n \times p$ matrix. Denote by A this matrix. The elements of A are

$$a_{ij} = A_i(x^j),\ i = 1, \ldots, n\ ;\ j = 1, \ldots, p. \tag{2}$$

The sum of elements of every column in A is 1. By the convex decomposition of a fuzzy partition we'll understand the convex decomposition of the associated matrix A. The matrices considered in this paper are all matrices representing a finite fuzzy partition of X.

The problem of the convex decomposition of a fuzzy partition has been addressed in the papers [1], [3], [6]. In this paper we propose a new algorithm for the convex decomposition.

2. Non-degenerate convex decomposition. According to [1] we denote

$$P_n = \left\{U \in \{0,1\}^{n\times p} \mid \sum_{i=1}^{n} u_{ij} = 1,\ \forall j, \sum_{j=1}^{p} u_{ij} > 0,\ \forall i\right\}, \tag{3}$$

$$P_{no} = \left\{U \in \{0,1\}^{n\times p} \mid \sum_{i=1}^{n} u_{ij} = 1,\ \forall j\right\}, \tag{4}$$

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$$P_{fn} = \left\{U \in [0,1] \mid \sum_{i=1}^{n} u_{ij} = 1, \ \forall j, \sum^{p} u_{ij} > 0, \ \forall i\right\},$$

$$P_{fno} = \left\{U \in [0,1]^{n \times p} \mid \sum^{n} u_{ij} = 1, \ \forall j\right\},$$

$P_n(P_{no})$ is the space of the non-degenerate (degenerate) matrices repre-
hard or classical partition on X.

In [1] the following strict inclusions are given:

$$\mathbf{conv}\ P_n \subset P_{fn},$$

$$P_{fn} \subset \mathbf{conv}\ P_{no}.$$

In [1] is also claimed that „the additional property in P_{fn} ne
distinguish it as a member of conv P is not yet known".

In [3] a neccessary and sufficient condition for a matrix $A \in P_{fn}$ to
a convex decomposition in hard non-degenerate partitions has been st
This condition is given by the next

THEOREM 1 ([3]). Let $A \in P_{fn}$. $A \in \mathbf{conv}\ P_n$ if and only if the cond

$$\sum_{j=1}^{p} a_{ij} \geqslant 1,$$

holds for every $i = 1, \ldots, n$.

Proof. The necessity is easy to prove (see [3]). For sufficiency, a cons-
tive proof has been given in [6]. This proof supplies an algorithm for
convex decomposition of a fuzzy partition.

Remark. A fuzzy partition $P = \{A_1, \ldots, A_n\}$ of X admits a con
decomposition in hard non-degenerate partitions if and only if

$$\sum_{j=1}^{p} A_i(x_j) \geqslant 1, \ i = 1, \ldots, n.$$

3. The MMM decomposition algorithm. Let A be the matrix representa
of a fuzzy partition $P = \{A_1, \ldots, A_n\}$. We are now able to propose a new algori
for the convex decomposition of A. This algorithm is very easy to progra
mate. Contrary to other decomposition algorithms ([1], [6]) it doesn't ex
citely use the notion of a path in the matrix. Some interesting properties of t
algorithm will be done.

In order to obtain the convex decomposition

$$A = \sum_{k} c_k U^k,$$

where $0 \leqslant c_k \leqslant 1$, $\sum_k c_k = 1$, and U^k is a boolean matrix representing a clas
partition of X, at the first step defines

$$c_1 = \min_{j=1,\ldots,p} \max_{i=1,\ldots,n} a_{ij}.$$

for every $j = 1, \ldots, p$ denote

$$I_{1,j} = \{l \mid \min_{i=1,\ldots,n} \{a_{ij} | a_{ij} \geqslant c_1\} = a_{lj}\}. \tag{13}$$

From the expresion of c_1 it is evident that

$$I_{i,j} \neq 0, \ \forall j = 1, \ldots, p. \tag{14}$$

If, for a fixed j, i_j is the unique element of $I_{1,j}$ we put

$$u^1_{i_j j} = 1 \tag{15}$$

and

$$u^1_{ij} = 0, \text{ for every } i \neq i_j. \tag{16}$$

If card $I_{1,j} > 1$ then i_j is an arbitrary element from $I_{1,j}$.

We obtained a matrix $U^1 = (u^1_{ij})$. In the convex decomposition this matrix has the coefficient c_1. The process repeats for the matrix

$$R = A - c_1 U^1. \tag{17}$$

and continues iteratively until $R = 0$.

At the k-th step we have

$$c_k = \min_{j=1,\ldots,p} \max_{i=1,\ldots,n} r_{ij} \tag{18}$$

and

$$I_{k,j} = \{l \mid \min_i \{r_{ij} \mid r_{ij} \geqslant c_k\} = r_{lj}\}. \tag{19}$$

The boolean matrix obtained at the k-th step is U_k. For every $j = 1, \ldots, p$ we choose a single $i_j \in I_{k,j}$ and put

$$u^k_{i_j j} = 1, \ u^k_{ij} = 0, \text{ for } i \neq i_j. \tag{20}$$

The convex decomposition algortihm we presented may be called the Mini-Minimax (**MMM**) algorithm. This algorithm may be descrided as follows:

4. MMM Convex Decomposition Algorithm.

S1. Put $k := 1$, $R := A$.

S2. Compute

$$c_k = \min_{j=1,\ldots,p} \max_{i=1,\ldots,n} r_{ij}$$

S3. For $j = 1, \ldots, p$ compute

$$I_{k,j} = \{l \mid \min \{r_{ij} | r_{ij} \geqslant c_k\} = a_{lj}\}$$

S4. Compute the matrix U^k. For every $j = 1, \ldots, p$ choose $i_j \in I_{k,j}$ and put

$$u^k_{i_j j} = 1, \ u^k_{ij} = 0, \text{ for } i \neq i_j$$

S5. Up-date the matrix R, $R := R - c_k U^k$.

S6. If R is the zero matrix, then stop.

Otherwise put $k := k + 1$ and go to S2.

Let us now prove that the **MMM** algorithm is correct. For every $j = 1, \ldots, p$ we have

$$1 = \sum_{i=1}^{n} a_{ij} = \sum_{i=1}^{n} \sum_{k} c_k u_{ij}^k$$
$$= \sum_{k} \sum_{i=1}^{n} c_k u_{ij}^k.$$

Therefore

$$\sum_{k} c_k \sum_{i=1}^{n} u_{ij}^k = 1$$

Since for every $j = 1, \ldots, p$

$$\sum_{i=1}^{n} u_{ij}^k = 1$$

we obtain

$$\sum_{k} c_k = 1.$$

The correctness of the algorithm is then proved.

Remark. The algorithm doesn't guarantee that the obtained hard part are all non-degenerate.

The number of coefficients in a convex decomoposition may be calle *length* of this decomposition.

5. **Properties of the MMM-decompositions.** In this section we'll co the results obtained by the **MMM** algorithm and the results of the **MM** rithm of Bezdek and Harris [1] This comparaison underlines some inter properties of the **MMM**-decomposition.

Let us remember that in the **MM** algorithm at every step k a $((i_1,1),)\ldots,)(i_p,p))$ in the matrix is considered. The pair (i_j, j) belongs to parth if

$$r_{i_j \hat{j}} = \max_{i} r_{ij}.$$

If more than one index i_j occurs one of these is chosen. A matrix obtained where

$$u_{ij}^k = \begin{cases} 1, & \text{if } i = i_j \\ 0, & \text{otherwise} \end{cases}$$

coefficient c_k of U^k is the same as for the **MMM** decomposition algorithm the matrices generally differ.

With respect to the **MM**-decomposition the Theorem 3 from [1] states the coefficient vector $c = (c_1, \ldots, c_s)$ of all **MM**-decomposition of a matrix is lexicographically larger than the coefficient vector $d = (d_1, \ldots, d_q)$ of any er convex decomposition of A.

In [1] is also claimed the following

Conjecture. If $\sum_{k=1}^{s} c_k U^k$ is any **MM**-decomposition of a matrix A and W_q is any other decomposition of A, then $s \leqslant q$.

Example 1. Consider the matrix

$$A = \begin{pmatrix} 0.6 & 0.4 & 0.2 & 0.1 \\ 0 & 0.3 & 0.5 & 0.5 \\ 0.4 & 0.3 & 0.3 & 0.4 \end{pmatrix} \tag{27}$$

There exist 48 **MM**-decompositions of A. These decompositions have the

$$A = 0.4 \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + 0.3 U_2 + 0.1 U_3 + 0.1 U_4 + 0.1 U_5, \tag{28}$$

these decompositions are degenerate.

The **MMM** algorithm gives only two decompositions — a non-degenerate a degenerate one. The non-degenerate decomposition is

$$A = 0.4 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} + 0.3 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + 0.2 \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} + 0.1 \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{29}$$

the degenerate decomposition is given by

$$A = 0.4 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} + 0.3 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} + 0.2 \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + 0.1 \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{30}$$

The coefficient vector of every **MM**-decomposition of matrix A considered he example is

$$c = (0.4,\ 0.3,\ 0.1,\ 0.1,\ 0.1), \tag{31}$$

the coefficient vector of the **MMM**-decomposition is

$$d = (0.4,\ 0.3,\ 0.2,\ 0.1). \tag{32}$$

The vector d is lexicrographically larger than c. The result stated by rem 3 from [1] is therefore incorrect.

Since the length od every **MM**-decomposition of A is longer than the of the **MMM**-decomposition it follows that the Conjecture of Bezdek and fails.

Example 2. For the matrix

$$B = \begin{pmatrix} 0.6 & 0.3 & 0.4 & 0.4 \\ 0 & 0.4 & 0.1 & 0.5 \\ 0.4 & 0.3 & 0.5 & 0.1 \end{pmatrix}$$

there exist 35 **MM**-decompositions. These decompositions are eighter form

$$B = 0.4\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + 0.3\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + 0.1v_3 + 0.1v_4 + 0.1v_5$$

ot they may be written as

$$B = 0.4\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + 0.3\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} + 0.1W_3 + 0.1W_4 + 0.1W_5$$

Therefore every possible **MM**-decomposition of B is degenerate.

The possible **MMM**-decompositions of B are again two. Every decomposition is non-degenerative. These decompositions are:

$$B = 0.4\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + 0.3\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + 0.2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} + 0.1\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

and

$$B = 0.4\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + 0.3\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} + 0.2\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + 0.1\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

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ON THE SHAPE OF BEZIER SURFACES,

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...subject classification: 65D05, 68V05

REZUMAT. — Asupra formei suprafeţei Bezier. În lucrare se generalizează unele rezultate obţinute anterior in [4] prin introducerea in ecuaţia (5) a parametrilor reali nenegativi p şi q. În figurile 1—12 s-a urmărit influenţa acestora precum şi a parametrilor naturali r şi s asupra formei suprafeţei. În ultima parte am considerat ecuaţia vectorială (11) corespunzătoare unei suprafeţe de tip Bezier cu proprietăţi la limită.

A Bezier surface is represented by the vectorial equation

$$B(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} b_{m,i}(u) b_{n,j}(v) P_{ij}, \tag{1}$$

...ere:

$$b_{p,k}(t) = \binom{p}{k} t^k \cdot (1-t)^{p-k}, \ t \in [0, 1]. \tag{2}$$

Therefore, the shape of a Bezier surface is determined by the points P_{ij} (i $= 0,1, \ldots, m$; $j = 0,1, \ldots, n$) and the functions $b_{m,i}(u)$ and $b_{n,j}(v)$, $u, v \in$ $[0,1]$; $i = 0,1, \ldots, m$; $j = 0,1, \ldots, n$. This surface lies only on the ...nts P_{oo}, P_{mo}, P_{on} and P_{mn}. If the points P_{ij} ($i = 0,1, \ldots, m$; $j = 0,1, \ldots$ $, n$) are fixed, then the shape of Bezier surface is dependent only by the ...is functions $b_{p,k}$.

In paper [4] the functions $b_{p,k}$ was replaced by the functions $w_{p,k,r}$, where:

$$w_{p,k,r}(t) = \begin{cases} \binom{p-r}{k} t^k (1-t)^{p-r-k+1}, & \text{if } 0 \leqslant k < r, \\ \binom{p-r}{k} t^k (1-t)^{p-r-k+1} + \binom{p-r}{k-r} t^{k-r+1} (1-t)^{p-r}, & \text{if } r \leqslant k \leqslant p-r \\ \binom{p-r}{k-r} t^{k-r+1} (1-t)^{p-k}, & \text{if } p-r < k \leqslant p \end{cases} \tag{3}$$

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r being a non-negativ integer, such $2r < p$. The vectorial equation cording to this case is:

$$S_{r,s}(u, v) = \sum_{i=0}^{m-r} \sum_{j=0}^{n-s} b_{m-r,i}(u) b_{n-s,j}(v) \, [P_{ij} + u(P_{r+i,j} - P_{ij}) +$$
$$+ v(P_{i,s+j} - P_{ij}) + uv(P_{r+i,s+j} - P_{i,s+j} - P_{r+i,j} + P_{ij})],$$

for $(u, v) \in [0, 1] \times [0, 1]$.

Can be observed that for $(r, s) \in \{0, 1\} \times \{0, 1\}$ the equation (4) is of the form (1).

Also, in [4] we have given the formulas for the derivates $B^{(p}$ $B^{(p,o)}(1, v)$, $B^{(p,o)}(u, o)$, $B^{(o,q)}(u, 1)$, $B^{(p,q)}(o, o)$, $B^{(p,q)}(1, o)$, $B^{(p,q)}(1, o)$, $B^{(p}$ and similar for the derivates of function $S_{r,s}$.

These formulas show us the possibility to control the shape of the surface corresponding to function $S_{r,s}(u, v)$, for the fixed points P_{ij} ($i = 0, \ldots, m$; $j = 0, 1, \ldots, n$), by choosing conveniently the natural numbers

Another way to control the shapes of a Bezier surface is to in two new real non-negative parameters p and q as follows:

$$S_{r,s}(u, v) = \sum_{i=0}^{m-r} \sum_{j=0}^{n-s} b_{m-r,i}(u) b_{n-s,j}(v) \, [P_{ij} + u^p(P_{r+i} - P_{ij}) + \tag{5}$$
$$+ v^p(P_{i,s+j} - P_{ij}) + u^p v^q(P_{r+i\,s+j} - P_{i,s+j} - P_{r+i,j} + P_{i,j})],$$

for $u, v \in [0, 1]$.

The dependence of the shapes of Bezier surface by the parameters and q is illustred in figures 1—12 corresponding to the following set of P_{ij} ($i = 0, 1, \ldots, 5$; $j = 0, 1, \ldots, 5$).

i \ j	0	1	2	3	4	5
0	(0,0,2)	(0,2,1)	(0,3,3)	(0,5,4)	(0,9,3)	(0,6,2)
1	(2,0,2)	(3,2,2)	(2,3,3)	(1,4,4)	(1,5,3)	(2,5,3)
2	(3,0,2)	(4,2,0)	(4,3,0)	(4,5,2)	(3,6,2)	(3,7,1)
3	(5,0,2)	(5,2,3)	(6,4,1)	(6,6,1)	(6,8,5)	(4,7,3)
4	(6,0,2)	(6,1,5)	(6,3,0)	(7,5,0)	(5,4,3)	(6,6,3)
5	(8,0,2)	(6,2,2)	(6,4,0)	(9,4,2)	(9,5,4)	(7,7,4)

The surfaces are seen, by an observer, from the point which does wnes $x0z$ and $x0y$ the angles $\alpha = 30°$, respectively $\beta = 15°$.

From (5) are deduced the following special cases:

$$S_{r,s}^{\infty\infty}(u, v) = \sum_{i=0}^{m-r} \sum_{j=0}^{n-s} b_{m-r,i}(u) b_{n-s,j}(v) \, P_{ij},$$

$$S_{r,s}^{0,0}(u, v) = \sum_{i=0}^{m-r} \sum_{j=0}^{n-s} b_{m-r,i}(u) b_{n-s,j}(v) \, P_{r+i,s+j},$$

$$S_{r,s}^{0,\infty}(u, v) = \sum_{i=0}^{m-r} \sum_{j=0}^{n-s} b_{m-r,i}(u) b_{n-s,j}(v)\ P_{r+i,j}. \tag{9}$$

$$S_{r,s}^{\infty,0}(u, v) = \sum_{i=0}^{m-r} \sum_{j=0}^{n-s} b_{m-r,i}(u) b_{n-s,j}(v)\ P_{i,s+j}, \tag{10}$$

Finally, we consider the vectorial equation:

$$G(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} g_{m,i}(u) g_{n,j}(v)\ P_{ij}\ ;\ u, v \in [0, \infty) \tag{11}$$

re:

$$g_{p,k}(t) = \binom{p}{k} t^k (1 + t)^{-p}, \quad t \in [0, \infty),$$

a Bleimann-Butzer-Hahn basis.

This equation determines a surface of Bezier type which lies only on point ,and:

$$\lim_{u\to\infty} G(u, o) = P_{m,o}, \qquad \lim_{v\to\infty} G(o, v) = P_{o,n}, \qquad \lim_{\substack{u\to\infty \\ v\to\infty}} G(u, v) = P_{mn}.$$

Also, we have for the derivatives:

$$G^{(p,q)}(o, o) = B^{(p,q)}(o, o),$$

$$\lim_{u\to\infty} G^{(p,q)}(u, o) = B^{(p,q)}(1, 0),$$

$$\lim_{v\to\infty} G^{(p,q)}(o, v) = B^{(p,q)}(0, 1),$$

$$\lim_{\substack{u\to\infty \\ v\to\infty}} G^{(p,q)}(u, v) = B^{(p,q)}(1, 1).$$

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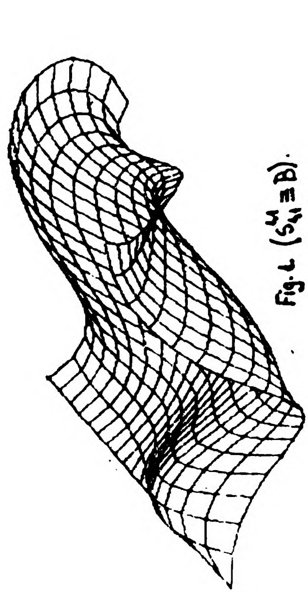

Fig. 1. ($S^{44}_{4,1} \equiv B$).

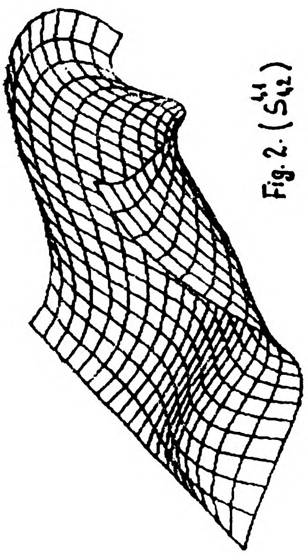

Fig. 2. ($S^{44}_{4,2}$)

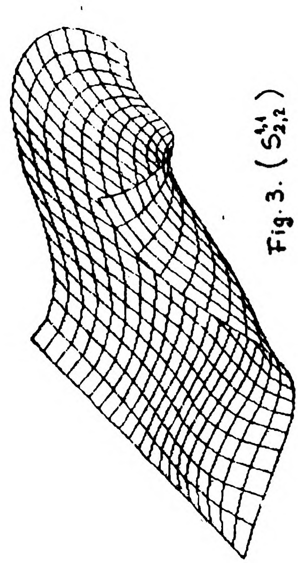

Fig. 3. ($S^{44}_{2,2}$)

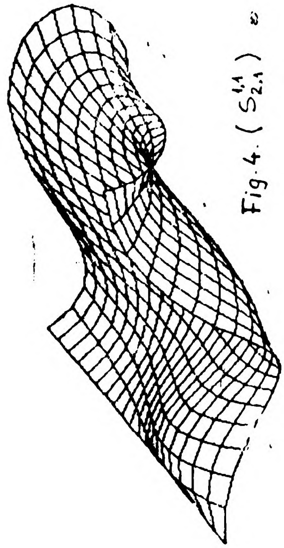

Fig. 4. ($S^{44}_{2,1}$)

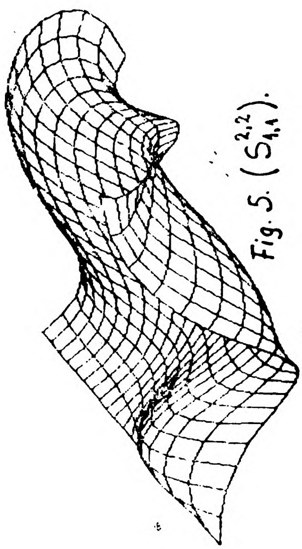

Fig. 5. ($S^{2,2}_{1,1}$).

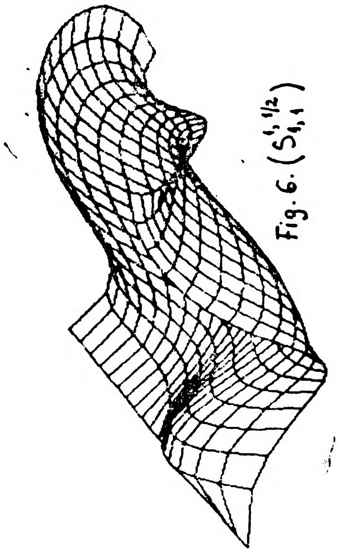

Fig. 6. ($S^{1,1/2}_{1,1}$)

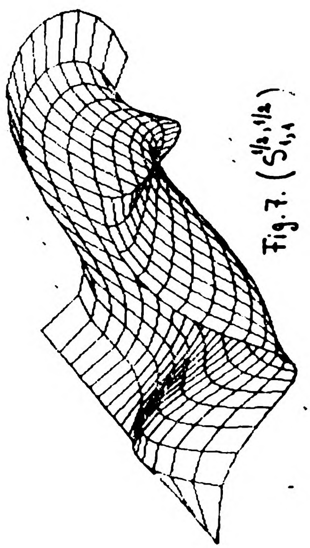

Fig. 7. ($S^{1/2,1/2}_{1,1}$)

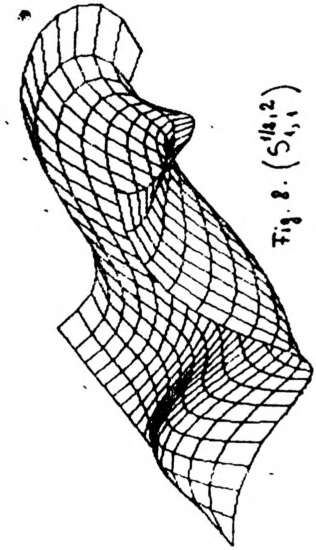

Fig. 8. ($S^{1/2,2}_{1,1}$)

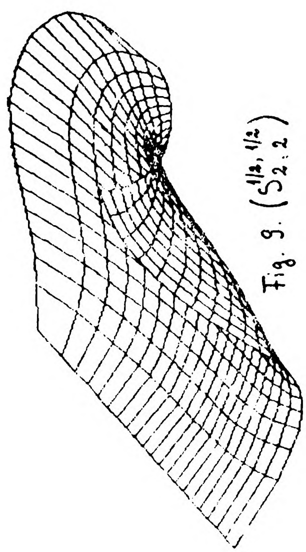

Fig. 9. ($S_{2,2}^{1/2,1/2}$)

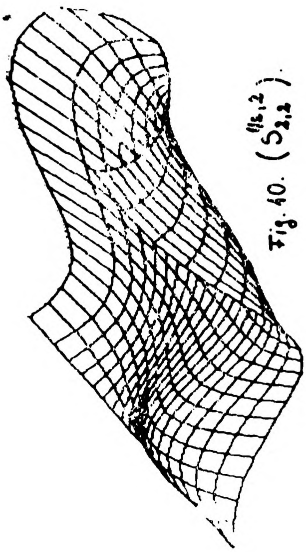

Fig. 10. ($S_{2,2}^{1/2,2}$)

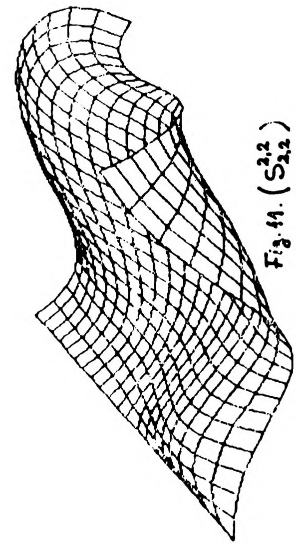

Fig. 11. ($S_{2,2}^{2,2}$)

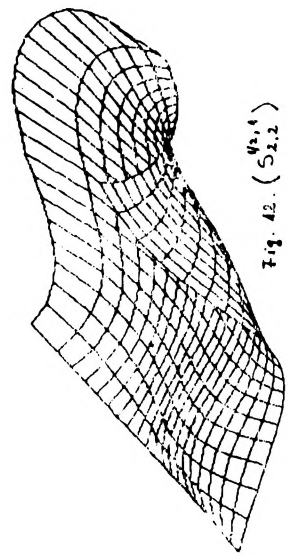

Fig. 12. ($S_{2,2}^{1/2,1}$)

STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXXV, 3, 1990

THE PRINCIPLE OF MAJORANT AND THE METHOD ANALOGOUS TO THE CHEBYSHEV'S METHOD FOR SOLVING THE OPERATORRIAL EQUATIONS WHICH DEPEND ON PARAMETERS

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REZUMAT. — Principiul majorantei şi metoda analoagă metodei lui Cebîşev pentru rezolvarea ecuaţiilor operatoriale ce depind de un parametru. În lucrare se prezintă principiul majorantei şi o metodă analoagă metodei lui Cebîşev, pentru rezolvarea ecuaţiilor operatoriale care depind de un parametru.

1. Let be the equation

$$P(x, a) = 0 \tag{1}$$

where $P: X \times M \to X$ is a nonlinear continuous operator, X is a Fréchet space and M is a quasinormed space, $\theta \in X$ being the null element.

Suppose that the equation (1) is majorized [2] by the equation

$$Q(z, b) = 0 \tag{1'}$$

where $Q: D \subset \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $D = [z_o, z'] \times [b_o, b']$, is a real continuous function of two real variables.

In order to obtain the solution $x^*(a) \in X$ of the equation (1) we use the iterative method

$$x_{n+1}(a) = x_n(a) - \Lambda_{n,a} P(x_n, a) - \Lambda_{n,a} [x_n, x_{n-1}, x_{n-2}; P^{(a)}] \Lambda_{n-1,a} P(x_{n-1}, a) \bar{\Lambda}_{n,a} P(x_n, a) \tag{2}$$

where

$$\Lambda_{n,a} = [x_n, x_{n-1}; P^{(a)}]^{-1}$$
$$\bar{\Lambda}_{n,a} = [x_n, x_{n-2}; P^{(a)}]^{-1}$$

the inverse of the first partial divided difference of the operator P in the points (x_n, x_{n-1}), respectively (x_n, x_{n-2}), supposing that a is constant.

To find the solution $z^*(b)$ of the majorant equation, we consider the iteration

$$z_{n+1}(b) = z_n(b) - \frac{Q(z_n, b)}{[z_n, z_{n-1}; Q^{(b)}]} \cdot \left(1 + \frac{[z_n, z_{n-1}, z_{n-2}; Q^{(b)}] Q(z_{n-1}, b)}{[z_n, z_{n-2}; Q^{(b)}][z_{n-1}, z_{n-2}; Q^{(b)}]}\right) \tag{2'}$$

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where $[z', z''; Q^{(b)}]$ and $[z', z'', z'''; Q^{(b)}]$, the first and second partial difference of Q are denoted.

In the paper [3], in the case of equation

$$P(x) = \theta$$

majorized by the real equation

$$Q(z) = 0$$

the following theorem was proved:

THEOREM A. *If the following properties are valide for the initial ... mations $x_{-2}, x_{-1}, x_0 \in S \subset X$, respectively z_{-2}, z_{-1}, z_0* from $[z_0, z']$

1′. *There is the operator $\Lambda = -[x^{(1)}, x^{(2)}; P]^{-1}$ and the quasinorm [4]) satisfied the relation*

$$)\,|\Lambda|\,(\leq -\frac{1}{[z^{(1)}, z^{(2)}; Q]} < B, \ \forall x^{(i)} \in S$$

$(i = 1, 2)$, *where S is defined by*

$$)\,|x - x_0|\,(\leq z' - z_0, \text{ and })\,|x_s - x_0|\,(\leq z_0 - z_s, \ s = -2, -1;$$

2′. $)\,|P(x_i)|\,(\leq Q(z_i), \ i = -2, -1, 0;$

3′. $)\,|x^{(1)}, x^{(2)}, x^{(3)}; P|\,(\leq [z^{(1)}, z^{(2)}, z^{(3)}; Q]$

$$x^{(i)} \in S, \ z^{(i)} \in [z_0, z'], \ i = \overline{1, 3};$$

4′. $)\,|x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}; P]\,(\leq [z^{(1)}, z^{(2)}, z^{(3)}, z^{(4)}; Q]$

$$x^{(i)} \in S, \ z^{(i)} \in [z_0, z'(, \ i = \overline{1, 4};$$

then the equation (1″) *has a solution $x^* \in S$, solution which is the lim... sequence generated by the method analogous to the Chebyshev method, the ... gence order being given by the unequality*

$$)\,|x^* - x_0|\,(\leq z^* - z_0$$

z^* being the unique solution of equation (2″), *solution given by the metho... tangent hyperbolas.*

2. The Theorem A will be used on proving the existence of the s... of the operator equations depending by one parameter, like the equati...

For the partial divided difference, we will use the notation:

$$[x^{(1)}, x^{(2)} \mid a^{(1)}, a^{(2)}; p^{(a|x)}] = [a^{(1)}, a^{(2)}; [x^{(1)}, x^{(2)}; p^{(a)}]^{(x)}]$$

$$[x^{(1)}, x^{(2)}, x^{(3)}; p^{(a^2)}] = [x^{(1)}, x^{(2)}; [x^{(2)}, x^{(3)}; p^{(a)}]^{(a)}]$$

$$[x^{(1)}, x^{(2)}, x^{(3)} \mid a^{(1)}, a^{(2)}; p^{(a^2|x)}] = [a^{(1)}, a^{(2)}; [x^{(1)}, x^{(2)}, x^{(3)}; p^{(a^2)}]^{(x)}].$$

We prove the following

THEOREM 1. *If, for the initial approximations $x_0, x_{-1}, x_{-2} \in X$, respectively z_0, z_{-1}, z_{-2} $[z, z']$ the following conditions are satisfied:*

1°. *The operator $\Lambda_{0,a_0} = -[x_0, x_{-1}; P^{(a_0)}]^{-1}$ there exists and*

$$) | \Lambda_{0,a_0} | (\leqslant -\frac{1}{[z_0, z_{-1}; Q^{(b)}]} = B_{0,b_0};$$

2°. $) | P(x_i, a_0) | (\leqslant Q(z_i, b_0),\ i = -2, -1, 0;$

3°. $) | [x^{(1)}, x^{(2)}, x^{(3)}; p^{(a_0^2)}] | (\leqslant [z^{(1)}, z^{(2)}, z^{(3)}; Q^{(b_0^2)}]$

$) | [x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}; p^{(a_0^3)}] | (\leqslant [z^{(1)}, z^{(2)}, z^{(3)}, z^{(4)}; Q^{(b_0^3)}]$

$\forall x^{(i)} \in S$, *where* S *is given by* $) | x - x_0 | (\leqslant z - z_0 \leqslant z' - z_0,\ i = \overline{1, 4};$

4°. $) | [a^{(1)}, a^{(2)}; p^{(x_0)}] | (\leqslant [b^{(1)}, {}^{(2)}; Q^{(z_0)}]\ \ \forall a^{(i)} \in \sigma$

σ *beig given by* $) | a - a_0 | (\leqslant b - b_0 \leqslant b' - b_0,\ i = \overline{1, 2}$

5°. $) | [x^{(1)}, x^{(2)}, x^{(3)} | a^{(1)}, a^{(2)}; p^{(a_0^2 | x_0)}] | (\leqslant$

$\leqslant [z^{(1)}, z^{(2)}, z^{(3)} | b^{(1)}, b^{(2)}; Q^{(b_0^2 | z_0)}].$

$\forall x^{(i)} \in S,\ a^{(j)} \in \sigma,\ i = \overline{1, 3};\quad j = 1, 2;$

6°. $) | x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)} | a^{(1)}, a^{(2)}; p^{(a_0^3 | x_0)})] | (\leqslant$

$\leqslant [z^{(1)}, z^{(2)}, z^{(3)}, z^{(4)} | b^{(1)}, b^{(2)}; Q^{(b_0^3 | z_0)}].$

$\forall x^{(i)} \in S,\ i = \overline{1, 4}$ *and* $a^{(i)} \in \sigma,\ i = 1, 2$

then, from the existence the solution $z^*(b) \in [z_0, z']$ (for any $b \in [b_0, b']$ of equation (1′) which is the limit of the sequence generated by (2′), it results the existence of the solution $x^*(a)$ of the equation (1), for any $a \in \sigma$, solution which is the limit of the sequence generated by the iterative method (1′), the convergence order being given by the inequality

$$) | x^*(a) - x_0 | (\leqslant z^*(b) - z_0. \tag{3}$$

Proof. The conditions 1°—3° of the theorem can be applied in the case of equations independent of parameters, i.e. for equation such as $P(x, a_0) = 0$ and $Q(z, b_0) = 0$, a_0 and b_0 being fixed. For these equations the existence of solution $x^*(a_0)$ results from the Theorem A.

We prove that these conditions of Theorem A are satisfied for any $a \in \sigma$ and $b \in [b_0, b']$.

a) Let us consider the operator

$$I + \Lambda_{0,a_0}[x_0, x_{-1}; p^{(a)}] =$$
$$= \Lambda_{0,a_0}([x_0, x_{-1}; p^{(a)}] - [x_0, x_{-1}; p^{(a_0)}]) =$$
$$= \Lambda_{0,a_0}[x_0, x_{-1} | a, a_0; p^{(a|x)}] (a - a_0).$$

Taking into account the condition 5° of Theorem A, we can write

$$\| I + \Lambda_{0,a_0}[x_0, x_{-1}; p^{(a)}] \| \leqslant B_{0,b_0}[z_0, z_{-1} \mid b, b_0; Q^{(b\mid x)}](b - b_0) =$$
$$= B_{0,b_0}([z_0, z_{-1}; Q^{(b)}] - [z_0, z_{-1}; Q^{(b_0)}]) =$$
$$= 1 + \frac{[z_0, z_{-1}; Q^{(b)}]}{[z_0, z_{-1}; Q^{(b_0)}]} = q.$$

By hypothesis, we have $[z_0, z_{-1}; Q^{(b)}] < 0$ and from the existence solution $z^*(b) \in [z_0, z']$, $\forall\, b \in [b_0, b']$ it results

$$[z_0, z_{-1}; Q^{(b)}] < 0.$$

If this is false, from 4° and 2° results $Q(z, b) \geqslant Q(z_0, b) \geqslant Q(z_0, b_0)$ and so the equation has not solution in $[z_0, z']$.

It results then $q < 1$, and from Banach's theorem it follows the existence of the operator

$$H^{-1} = [I - (I + \Lambda_{0,a_0}[x_0, x_{-1}; p^{(a)}])^{-1} =$$
$$= -\Lambda_{0,a_0}[x_0, x_{-1}; P^{(a)}]^{-1}.$$

Then it results the existence of

$$H^{-1}\Lambda_{0,a_0} = [-\Lambda_{0,a_0}[x_0, x_{-1}; P^{(a)}]^{-1}]^{-1}\Lambda_{0,a_0} =$$
$$= -[x_0, x_{-1}; P^{(a)}]^{-1} \quad \Lambda_{0,a}$$

for which, in base of 1°, we have

$$\| \Lambda_{0,a} \| = \frac{1}{1-q} B_{0,b_0} \quad \cdots \frac{1}{[z_0, z_{-1}; Q^{(b)}]} = B_{(0,a)}$$

so the condition 1′ of Theorem A is verified.

b) To prove that in the conditions of Theorem 1, the condition 2′ theorem A is verified, we consider the inequality

$$\| P(x_0, a) \| = \| P(x_0, a) + P(x_0, a_0) - P(x_0, a_0) \|$$

which, using the condition 2°, may be written

$$\| P(x_0, a) \| \leqslant \| P(x_0, a_0) \| + \| P(x_0, a) - P(x_0, a_0) \| =$$
$$= Q(z_0, b_0) + \| [a, a_0; P^{(x_0)}](a - a_0) \|.$$

On the base of 4°, we have

$$\| P(x_0, a) \| \leqslant Q(z_0, b_0) + [b, b_0; Q^{(z_0)}](b - b_0) =$$
$$= Q(z_0, b_0) + Q(z_0, b) - Q(z_0, b_0) = Q(z_0, b)$$

In the same way we can obtain

$$\| P(x_i, a) \| \leqslant Q(z_i, b), \quad i = -2, -1.$$

c) We consider the relation

$$[x^{(1)}, x^{(2)}, x^{(3)}; P^{(a^2)}] = [x^{(1)}, x^{(2)}, x^{(3)}; P^{(a_0^2)}] +$$
$$+ [x^{(1)}, x^{(2)}, x^{(3)}; P^{(a^2)}] - [x^{(1)}, x^{(2)}, x^{(3)}; P^{(a_0^2)}] =$$
$$= [x^{(1)}, x^{(2)}, x^{(3)}; P^{(a_0^2)}] + [x^{(1)}, x^{(2)}, x^{(3)} | a, a_0; P^{(a^2|x)}] (a - a_0)$$

ich, using the conditions 3° and 5°, becomse

$$) | [x^{(1)}, x^{(2)}, x^{(3)}; P^{(a^2)}] | (\leqslant$$
$$\leqslant [z^{(1)}, z^{(2)}, z^{(3)}; Q^{(b_0^2)}] + [z^{(1)}, z^{(2)}, z^{(3)} | b, b_0; Q^{(b^2|z)}] (b - b_0) =$$
$$= [z^{(1)}, z^{(2)}, z^{(3)}; Q^{(b_0^2)}] + [z^{(1)}, z^{(2)}, z^{(3)}; Q^{(b^2)}] -$$
$$- [z^{(1)}, z^{(2)}, z^{(3)}; Q^{(b_0^2)}] = [z^{(1)}, z^{(2)}, z^{(3)}; Q^{(b^2)}].$$

d) Considering the relation

$$[x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}; P^{(a^3)}] =$$
$$= [x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}; P^{(a_0^3)}] + [x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}; P^{(a^3)}] -$$
$$- [x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}; P^{(a_0^3)}] = [x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}; P^{(a_0^3)}] +$$
$$+ [x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}]; a, a_0; P^{(a^2|x)}] (a - a_0)$$

the condition 1° and 6°, it follows

$$) | [x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}; P^{(a)}] | (\leqslant [z^{(1)}, z^{(2)}, z^{(3)}, z^{(4)}; Q^{(b^3)}] +$$
$$+ [z^{(1)}, z^{(2)}, z^{(3)}, z^{(4)} | b, b_0; Q^{(b^2|z)}] (b - b_0) =$$
$$= [z^{(1)}, z^{(2)}, z^{(3)}, z^{(4)}; Q^{(z_0^3)}] + [z^{(1)}, z^{(2)}, z^{(3)}, z^{(4)}; Q^{(b^3)}] -$$
$$- [z^{(1)}, z^{(2)}, z^{(3)}, z^{(4)}; Q^{(b_0^3)}] = [z^{(1)}, z^{(2)}, z^{(3)}, x^{(4)}; Q^{(b^3)}]$$

The hypothesis of Theorem A being true, it results the conclusion of Theorem 1.

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COMPUTING THE d-COMPLEXITY OF WORDS BY FIBONACCI-LIKE SEQUENCES

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REZUMAT. — Calculul d-complexității cuvintelor prin șiruri aproape-Fibonacci. Articolul studiază complexitatea cuvintelor, care se definește ca numărul subcuvintelor distincte ale unui cuvînt. Ca măsură a complexității se folosește d-complexitatea (introdusă în [1]).

1. Introduction. The complexity of words is of great importance not only in the computer science, but also in other domains. We give here some results in the computation of the d-complexity of words using Fibonacci-like sequences. The definitions and notations used in this papers are from [1].

Let X be an alphabet, and X^k the set of all words of the length k over X.

DEFINITION. 1. Let d, k and s be positive integers, $p = x_1x_2 \ldots x_k \in X^k$. A *d-subword of p* is defined as $q = x_{i_1}x_{i_2} \ldots x_{i_s}$ where

$$i_1 \geqslant 1$$
$$0 \leqslant i_{j+1} - i_j \leqslant d \quad \text{for} \quad j = 1, 2, \ldots, s-1$$
$$i_s \leqslant k.$$

DEFINITION 2. For $p \in X^k$ the *d-complexity* $K_d(p)$ is the number of all different d-subwords of p.

Example. Let X be the English alphabet and $p = beer$. In this word there are three 1-subwords of length 1 (b, e, r), three 1-subwords of length 2 (be, ee, er), two 1-subwords of length 3 (bee, eer), and a single 1-subword of length 4 ($beer$). Then $K_1(p) = 3 + 3 + 2 + 1 = 9$. The above 1-subwords are 2-subwords too, and there exists a single new 2-subwords (ber), then $K_2(p) = 10$. To compute the 3-complexity of p, let us find the 3-subwords which aren't 2- or 1-subwords. There is only ine: br. Then $K_3(p) = 11$. Because of length of p, $K_d(p) = 11$ for all $d \geqslant 3$.

Notation. In the case of words of length k, consisting of different symbols, the d-complexity will be denoted by $N(k, d)$.

DEFINITION 3. If p is a word, consisting of different symbols and d a positive integer, then $a_{i,d}(p)$ will denote the number of *d-subwords of p which terminate in position* i.

The followings are true (from [1]):

a. For any $k \geqslant 1$ and $p \in X^k$ hold

$$k \leqslant K_1(p) \leqslant \frac{k(k+1)}{2} \tag{1}$$

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b. For $n \geqslant 2$, $k \geqslant 1$, $d \geqslant 1$ and $p \in X^k$

$$k \leqslant K_d(p) \leqslant 2^k - 1 \qquad (2)$$

c. If $k \geqslant 1$, $p \in X^k$, consisting of different symbols, then

$$a_{i,d}(p) = 1 + a_{i-1,d}(d) + a_{i-2,d}(d) + \ldots + a_{i-d,d}(p) \qquad (3)$$

for $i = 1, 2, \ldots, k$.

Another relations are given in [2].

2. **Computing the d-complexity of words.** The d-complexity of a word with different symbols can be obtain by the formula:

$$N(k, d) = \sum_{i=1}^{k} a_{i,d}(p) \qquad (4)$$

where p is any word of k different symbols.

Because of (3) we can write

$$a_{i,d} + \frac{1}{d-1} = \left(a_{i-1,d} + \frac{1}{d-1}\right) + \ldots + \left(a_{i-d,d} + \frac{1}{d-1}\right)$$

for $d > 1$.

Let be

$$c_{i,d} = (d-1)\, a_{i,d} + 1 \qquad (5)$$

Then

$$c_{i,d} = c_{i-1,d} + c_{i-2,d} + \ldots + c_{i-d,d}$$

and the sequence $c_{i,d}$ is one of a Fibonacci-type.

For any d we have $a_{1,d} = 1$, and from this $c_{i,d} = d$ results. Therefore the numbers $c_{i,d}$ are defined by the following recurrence equation:

$$c_{n,d} = c_{n-1,d} + c_{n-2,d} + \ldots + c_{n-d,d} \quad \text{for } n > 0$$
$$c_{n,d} = 1 \quad \text{for } n \leqslant 0$$

It is easy to prove that this numbers can be generated by the following generating function:

$$F_d(z) = \sum_{n \geqslant 0} c_{n,d} z^n = \frac{1 + (d-2)z - z^2 - \ldots - z^d}{1 - 2z + z^{d+1}}$$

Because of (5), the d-complexity $N(k, d)$ can be expressed with the numbers $c_{i,d}$ by the following formula:

$$N(k, d) = \frac{1}{d-1}\left(\sum_{i=1}^{k} c_{i,d} - k\right) \quad \text{for } d > 1,$$

and $N(k, 1) = \frac{k(k+1)}{2}$

From this the following formula results:

$$N(k, d) = N(k - 1, d) + \frac{1}{d - 1}(c_{k,d} - 1),$$

d if the $c_{k,d}$ are known, the $N(k, d)$ may be calculated in $O(1)$ time.

Let F_n (with $F_0 = 0$, $F_1 = 1$) be denoted the Fibonacci numbers. Then, have $c_{n,2} = F_{n+2}$

d

$$N(k, 2) = \sum_{i=1}^{k} F_{i+2} - k = F_{k+4} - k - 3$$

Taking into account the formula of F_n, we have

$$N(k, 2) = \left\lfloor \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^{k+4} + \frac{1}{2} \right\rfloor - k - 3 =$$

$$= \left\lfloor \frac{7\sqrt{5} + 15}{10} \left(\frac{1 + \sqrt{5}}{2} \right)^{k} + \frac{1}{2} \right\rfloor - k - 3$$

This can be written as

$$[3.0652475\ (1.6180339)^k + 0.5] - k - 3$$

The table 1 lists the $N(k, d)$ values for $k \leqslant 10$ and $d \leqslant 10$.

The following proposition gives the value of $N(k, d)$ in almost all cases.

PROPOSITION. *For $k \geqslant 2d - 2$ we have*

$$N(k, k - d) = 2^k - (d - 2) \cdot 2^{d-1} - 2. \tag{6}$$

Proof. Let $k \geqslant 2d - 2$. Then $N(k, k - d - 1)$ may be computed as follows. Between the $N(k, k - d)$ subwords there are exactly $d \cdot 2^{d-1}$ in which $i_{j+1} - i_j = k - d$ for some j (see definition 1), because of the d cases of the choosed positions with distance $k - d$, and 2^{d-1} cases of choosing the other letters. (The distance between the other letters must be less than $k - d$, then $k - d + 1 \geqslant d - 1$ or $k \geqslant 2d - 2$). Then

$$N(k, k - d - 1) = N(k, k - d) - d \cdot 2^{d-1}$$

For $d = 1, 2, 3, \ldots$ we have

$$N(k, k - 2) = N(k, k - 1) - 1$$

$$N(k, k - 3) = N(k, k - 2) - 2 \cdot 2^1$$

$$N(k, k - 4) = N(k, k - 3) - 3 \cdot 2^2$$

. . . .

$$N(k, k - d) = N(k, k - d + 1) + (d - 1) \cdot 2^{d-2}$$

Adding the obvious relation

$$N(k, k - 1) = 2^k - 1$$

and summing all these, we obtain

$$\begin{aligned} N(k, k-d) &= 2^k - 1 - (1 + 2 \cdot 2^1 + 3 \cdot 2^2 + \ldots + (d-1) \cdot 2^{d-2}) \\ &= 2^k - 1 - (d-2) \cdot 2^{d-1} - 1 \\ &= 2^k - (d-2) \cdot 2^{d-1} - 2 \text{ qu.e.d.} \end{aligned}$$

(The sum in the bracket may be obtained by computing $f'(2)$ in two way $f(x) = 1 + x + \ldots + x^{d-1}$).

A more conveniable form of this formula is

$$N(k, d) = 2^k - (k - d - 2) \cdot 2^{k-d-1} - 2 \text{ for } \frac{k-2}{2} \leqslant d \leqslant k-1$$

But, the following may be easily obtained, also:

$$N(k, d) = N(k, d-1) + (k-d) \cdot 2^{k-d-1} \text{ for } \frac{k-2}{2} \leqslant d \leqslant k-1$$

Another way to compute the $N(k, d)$ is that which compute the numbe sequences of lenght k of zeros and ones, with no more than $d-1$ adja zeros. One 1 in such a sequence represents the presence of a letter of a w in a given d-subword, but one 0 the absence of the corresponding letter. $b_{k,d}$ denote the number of sequences of zeros and ones, of length k, in which first and the last position has 1, and adjacent zeros may be atmost $d-1$

Then may be proved that

$$\begin{aligned} b_{k,d} &= b_{k-1,d} + b_{k-2,d} + \ldots + b_{k-d,d} \text{ for } k > 1 \\ b_{1,d} &= 1 \\ b_{k,d} &= 0 \text{ for all } k \leqslant 0 \end{aligned}$$

or

$$b_{k,d} = 2 \cdot b_{k-1,d} - b_{k-1-d,d}$$

This second formula may be proved easily. There are $b_{k-1,d}$ dsequence of len $k-1$ with desired property. Adding one 1 or 0 in the e.g. $(k-1)^{th}$ posit in each sequence, we obtain $2 \cdot b_{k-1,d}$ sequences, but between these b_{k-1} have d adjacent zeros. This sequences $b_{k,d}$ are of Fibonacci type.

Adding zeros on the left and/or right to these sequences, we can obtain number $N(k, d)$, as number of all these sequences. Thus

$$N(k, d) = b_{k,d} + 2 \cdot b_{k-1,d} + 3 \cdot b_{k-2,d} + \ldots + k \cdot b_{1,d}$$

Table

Values of N(k, d)

k \ d	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	3	3	3	3	3	3	3	3	3	3
3	6	7	7	7	7	7	7	7	7	7
4	10	14	15	15	15	15	15	15	15	15
5	15	26	30	31	31	31	31	31	31	31
6	21	46	58	62	63	63	63	63	63	63
7	28	79	110	122	126	127	127	127	127	127
8	36	133	206	238	250	254	255	255	255	255
9	45	221	383	462	494	506	510	511	511	511
10	55	364	709	894	974	1006	1018	1022	1023	1023

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METHOD FOR IMPROVING THE RESULTS OF CERTAIN CLUSTERING PROCEDURES

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REZUMAT. — Metodă de îmbunătăţire a rezultatelor unor algoritmi de clasificare. În acest articol se propune o metodă de clasificare în două etape care reuneşte avantajele metodelor bazate pe teoria grafelor, respectiv pe reprezentarea clusterilor prin prototipuri. În primele două părţi se formulează o problemă de clasificare şi se arată că, în anumite condiţii, metoda înlănţuirii simple furnizează partiţia cerută. Apoi se enunţă un alt algoritm care generalizează algoritmul lui Prim de determinare a arborelui de acoperire minimal, arătîndu-se că generează aceeaşi partiţie. În încheiere se compară complexităţile celor doi algoritmi şi se descrie metoda de clasificare în două etape urmată de un exemplu numeric.

1. **Introduction and notations.** Standard clustering algorithms based on the representation of clusters by prototypes and the minimization of a certain functional, such as k-Means or Fuzzy c-Means, often misclassify points situated at the border of the clusters. On the other hand, graph-theoretical methods usually produce accurate classifications but have high complexities. In this paper, we propose a two phase clustering procedure which gathers the advantages of the two types of methods mentioned above.

Let us consider a finite set of objects $O = \{o_1, o_2, \ldots, o_m\}$ and a set of centers $C = \{c_1, c_2, \ldots, c_n\}$. Put $X = O \cup C$ and let $\rho : X \times X \to \mathbf{R}_+$ be a dissimilarity coefficient. The triple (O, C, ρ) will be called a configuration. We formulate the following problem: classify the objects from O, associating them with the centers from C according to the dissimilarity coefficient ρ (we are not concerned now about what the centers represent and how they were obtained). This means we must find a partition $\mathfrak{P}$ of X so that:

$$\text{Card}\ \mathfrak{P} = n \tag{1a}$$

$$\{c_i, c_j\} \not\subset \mathfrak{P}, \ (\forall)\ 1 \leqslant i, j \leqslant n,\ i \neq j,\ P \in \mathfrak{P} \tag{1b}$$

In this paper we discuss some classification algorithms for the problem stated above under a certain condition, which simplifies the results, the general case being treated in a further paper. This condition, which will be precisely formulated in the next paragraph, is a rather natural condition; roughly speaking, we demand that the centers are not arbitrary points in the configuration of X; but authentic ,,prototypes'' for n clusters; therefore we call this a homogeneity condition for a certain configuration of X.

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Under the homogeneity assumption, the single linkage method generates a partition of X with n clusters, each cluster containing a center. This means that selecting the edges from $X \times X$ in the increasing order of the dissimilarity between their extremities, paths between two centers may not appear. The complexity of this algorithm is $O(m^3)$. Another clustering algorithm, complexity $O(m^2)$, is then presented as a generalization of Prim's algorithm (see Prim 1957) ; in comparison with the single linkage algorithm, this one reduces the set in which the least dissimilarity edge is searched for and n dendrograms are also constructed. In order to prove that these two algorithms generate the same partition of X, we characterize this partition in terms of an ultrametric on X.

We now introduce some notations. Let $\Pi = \bigcup_{k \geq 2} X^k$ the set of paths in X, $\Pi(U, V) = \{p \in \Pi \mid p = (x_0, x_1, \ldots, x_k),\ x_0 \in U,\ x_k \in V,\ k \in N^*\}$ the set of paths between U and V; for $p = (x_0, x_1, \ldots, x_k) \in \Pi$ we note with $l(p) = \sup\{\rho(x_{i-1}, x_i) \mid i = 1, 2, \ldots, k\}$ and with δ the following ultrametric:

$$\delta(x, y) = \inf\ \{l(p) \mid p \in \Pi(x, y)\}.$$

For $U, V \subseteq X$, we put $\delta(U, V) = \inf\{\delta(x, y) \mid x \in U,\ y \in V\}$. According to Bezdek and Harris (1978), this ultrametric corresponds to the minmax transitive closure of the fuzzy relation associated with δ. Let $\alpha \in \mathbf{R}_+$ and $R_\alpha \subseteq X \times X$ be defined as follows:

$$xR_\alpha y \Leftrightarrow \delta(x, y) < \alpha.$$

Then R_α is an equivalence on $X \times X$ and the set of partitions $\{X/R_\alpha \mid \alpha \in \mathbf{R}_+\}$ coincides with the set of partitions generated by the single linkage method and with the set of balls $\{\{B(x, \alpha) \mid x \in X\} \mid \alpha \in \mathbf{R}_+\}$. For details, see Zahn (1971), Duda and Hart (1973).

2. **The Single Linkage Method under the Homogeneity Assumption.** In this paragraph, we formulate the homogeneity condition for a certain configuration and study its implications upon the single linkage method.

DEFINITION 1: *A configuration satisfies the homogeneity condition (is homogeneous) if the following relation holds:*

$$sup\{\delta(x, C) \mid x \in O)\} < inf\{\delta(c_i, c_j) \mid 1 \leq i,\ j \leq n,\ i \neq j\} \tag{2}$$

In other words, we demand that point-to-center dissimilarities are less than intercenter dissimilarities. The homogeneity condition can easily be formulated in relation with the single linkage method; it means that the single linkage method „recognizes" the centers, keeping them in separate clusters during the classification of the objects from O.

PROPOSITION 1: *A configuration is homogeneous if there exists $\alpha \in \mathbf{R}_+$ so that the partition X/R_α satisfies conditions (1a) and (1b).*

Proof. Supposing that (2) holds, we choose $\alpha = \inf\{\delta(c_i, c_j) \mid 1 \leq i, j \leq n, i \neq j\}$. On the one hand $\delta(c_i, c_j) \geq \alpha$ for every $i \neq j$ implies $(c_i, c_j) \notin R_\alpha$ and Card $(X/R_\alpha) \geq n$; on the other hand $\delta(x, C) < \alpha$ for every $x \in O$ implies Card $(X/R_\alpha) \leq n$ and hence (1a) and (1b) are deduced.

Conversely, the existence of α, with X/R_α satisfying (1b) implies $\delta(c_i, c_j) \geqslant \geqslant \alpha$ for every $i \neq j$. Since Card $(X/R_\alpha) = n$ and each center is in a separate cluster, for every $x \in O$ we deduce the existence of a center $c_x \notin C$ so tha $\delta(x, c_x) < \alpha$ Q.E.D.

The following proposition indicates that any configuration may be converted into a homogeneous one if we eliminate from O all the objects which do not satisfy the homogeneity condition.

PROPOSITION 2: *Let (O, C, ρ) be a configuration and $O' = \{x \in O \mid \delta(x, C) < < \inf\{\delta(c_i, c_j) \mid 1 \leqslant i, j \leqslant n, i \neq j\}\}$, $X' = O' \cup C$, $\rho' = \rho \mid X' \times X'$. Then (O', C, ρ') is a homogeneous configuration.*

Proof: Obviously we have $\delta'(x, y) \geqslant \delta(x, y)$ for every $x, y \in X'$. For $x \in \in O'$ we shall prove that $\delta'(x, C) = \delta(x, C)$. Let $c_x \in C$ and $p \in \Pi(x, c_x)$ so that $l(p) = \delta(x, C)$. Suppose $\delta'(x, c_x) > \delta(x, c_x)$, which means that p contains a node $y \in O \setminus O'$. We have:

$$\inf\{\delta(c_i, c_j) \mid i \neq j\} > \delta(x, C) = \delta(x, c_x) = \max\{\delta(x, y), \delta(y, c_x)\} \geqslant$$
$$\geqslant \delta(y, c_x) \geqslant \delta(y, C) \geqslant \inf\{\delta(c_i, c_j) \mid i \neq j\}.$$

This is a contradiction which proves $\delta'(x, C) = \delta(x, C)$. Hence:

$$\sup\{\delta'(x, C) \mid x \in O'\} = \sup\{\delta(x, C) \mid x \in O'\} <$$
$$< \inf\{\delta(c_i, c_j) \mid i \neq j\} \leqslant \inf\{\delta'(c_i, c_j) \mid i \neq j\}.$$

This means (O', C, ρ') is homogeneous Q.E.D.

From now on we suppose that the homogeneity condition holds. Let $\{X_1, X_2, \dots, X_n\}$ $(c_i \in X_i,\ i = 1, 2, \dots, n)$, be the partition of X satisfying (1a) and (1b) generated by the single linkage method. The following proposition gives two properties of this partition, the first one being a characterization of it.

PROPOSITION 3: *For $i, j \in \{1, 2, \dots, n\}$, we have:*

a) $\delta(x, c_i) < \delta(x, c_j)$, if $i \neq j$, $x \in X_i$;

b) $\delta(x, c_i) = \delta(x, c_j) = \delta(x, C) \Rightarrow i = j$.

Proof: a) Let $\alpha \in \mathbf{R}_+$ so that $X/R_\alpha = \{X_1, X_2, \dots, X_n\}$. Then $x \in X_i$ and $x \notin X_j$ imply:

$$\delta(x, c_i) < \alpha \leqslant \delta(x, c_j)$$

b) There exists k so that $x \in X_k$. Then $\delta(x, C) \leqslant \delta(x, c_k) < \alpha \Rightarrow \delta(x, c_i) = = \delta(x, c_j) < \alpha \Rightarrow x \in X_i,\ x \in X_j \Rightarrow i = j = k$.

3. **Generalized Prim's Algorithm.** In this paragraph we give a new graph-theoretical algorithm equivalent with the single linkage algorithm for homogeneous configurations. The algorithm, which is a generalization of Prim's algorithm (see Prim 1957), is described below:

Step 1. Let $k := 1$; $N_k := O$; $X_i^k := \{c_i\}$, $i = 1, 2, \dots, n$;

Step 2. Let $L_k = N_k \times (X \setminus N_k)$;

Search for $(x_k, y_k) \in L_k : \rho(x_k, y_k) = \min\{\rho(x, y) \mid (x, y) \in L_k\}$

Step 3. For $i := 1$ to n do

$$\text{if } y_k \in X_i^k \cup \{c_i\} \text{ then let } X_i^{k+1} := X_i^k \cup \{x_k\}$$

$$\text{else let } X_i^{k+1} := X_i^k ;$$

Step 4. Let $N_{k+1} := N_k \setminus \{x_k\}$; $k := k + 1$;

Step 5. If $N_k \neq \emptyset$ go to 2 else print $(X_i^k, i = 1, 2, \ldots, n)$ stop.

The algorithm finishes after exactly m iterations. The output partiti X is given by $X_i = X_i^{m+1}$, $i = 1, 2, \ldots, n$. The set N_k contains the ob not yet classified before the iteration k. The set L_k in which the least c milarity edge is searched for contains fewer elements than in the single lin method. Although the least dissimilarity edge at a certain iteration may be unique, the output partition is. At every iteration, exactly one object i O is associated with a center ; unlike this algorithm, the single linkage met has iterations which unify two classes of objects without associating th with a center, while other iterations associate a whole class with a center. I $n = 1$ we obtain Prim's algorithm.

THEOREM *If $\{X_1, \ldots, X_n\}$ is an output partition of the algorithm above, we have :*

$$(\forall)\ i, j \in \{1, 2, \ldots, n\},\ i \neq j,\ (\forall)\ x \in X_i :\ \delta(x, c_i) < \delta(x, c_j)$$

Proof : We shall prove this theorem by induction with respect to k, the nu ber of the iteration. For $k = 1$ the assertion of the theorem is evident. Supp it now valid for a certain k and $y_k \in X_i^k$. We have to prove that $\delta(x_k, c_i)$ $< \delta(x_k, c_j)$, $(\forall)\ j \neq i$. Indeed, if $j \neq i$, using $\delta(y_k, c_i) < \delta(y_k, c_j)$, we obtain

$$\delta(x_k, c_i) \leqslant \max\ \{\delta(x_k, y_k), \delta(y_k, c_i)\} \leqslant \max\ \delta\{(x_k, y_k), \delta(y_k, c_j)\} \leqslant$$

$$\leqslant \max\{\ \delta(x_k, y_k), \delta(y_k, x_k), \delta(x_k, c_j)\} = \max\ \{\delta(x_k, y_k), \delta(x_k, c_j)\} \leqslant$$

$$\leqslant \max\ \{\rho(x_k, y_k), \delta(x_k, c_j)\} = \delta(x_k, c_j).$$

The last equality follows from the existence of an edge from L_k belonging to th minimal path between x_k and c_j and from the minimality of (x_k, y_k) in L_k. Hen $\delta(x_k, c_i) \leqslant \delta(x_k, c_j)$, $(\forall)\ j \neq i \Rightarrow \delta(x_k, c_j) = \delta(x_k, C)$. If there would exist j $\neq i$ so that $\delta(x_k, c_i) = \delta(x_k, c_j)$, according to proposition 3b) follows $i = j$, co tradiction. This proves $\delta(x_k, c_i) < \delta(x_k, c_j)$, $(\forall)\ j \neq i$. Q.E.D.

The theorem guarantees the uniqueness of the output partition. From th theorem and proposition 3a) follows the equivalence of the generalized Prim algorithm and the single linkage algorithm for homogeneous configuration

4. **The Complexity of the Algorithms.** In order to implement the alg ithms presented in the previous paragraphs it is necessary to describe them a rather different manner. We shall denote by $\rho(A, B) = \min\limits_{\substack{x \in A \\ y \in B}} \rho(x, y)$.

Algorithm 1′ (single linkage)

Step 1. Let $O_i := \{o_i\}$, $i = 1, 2, \ldots, m$; $C_j := \{c_j\}$, $j = 1, 2, \ldots, n$;
$I := \{1, 2, \ldots, m\}$.

Step 2. Search for $i_0, j_0 : \rho(O_{i_0}, C_{j_0}) = \min\{\rho(O_i, C_j) \mid i \in I, j = 1, 2 \ldots, n\}$

Step 3. Search for $i_1 > j_1 : \rho(O_{i_1}, O_{j_1}) = \min\{\rho(O_i, O_j) \mid i, j \in I, i > j\}$.

Step 4. If $\rho(O_{i_0}, C_{j_0}) < \rho(O_{i_1}, O_{j_1})$ then $C_{j_0} := C_{j_0} \cup O_{i_0}$; $I := I \setminus \{i_0\}$
else $O_{j_1} := O_{i_1} \cup O_{j_1}$; $I := I \setminus \{i_1\}$.

Step 5. If $I \neq \emptyset$ then go to 2
else print $(C_j,\ j = 1, 2, \ldots, n)$ stop.

Algorithm 2′ (generalized Prim's algorithm)

Step 1. Let $O_i := \{o_i\},\ i = 1, 2, \ldots, m$; $C_j := \{c_j\},\ j = 1, 2, \ldots, n$;
$I := \{1, 2, \ldots, m\}$.

Step 2. Search for $i_0, j_0 : \rho(O_{i_0}, C_{j_0}) = \min\{\rho(O_i, C_j) \mid i \in I,\ j = 1, 2, \ldots, n\}$

Step 3. Let $C_{j_0} := C_{j_0} \cup O_{i_0}$; $I := I \setminus \{i_0\}$

Step 4. If $I \neq \emptyset$ then go to 2
else print $(C_j,\ j = 1, 2, \ldots, n)$ stop.

At the end of algorithm 1′ and 2′, sets C_j will contain the objects of O associated with c_j, $j = 1, 2, \ldots, n$. The following iterative relation for the estimation of the interset dissimilarity is used:

$$\rho(A_1 \cup A_2, B) = \min\{\rho(A_1, B),\ \rho(A_2, B)\}.$$

We shall now evaluate the complexities of the two algorithms as a function of the complexity of the comparison operation: CP $(<)$. The complexity of the minimal dissimilarity estimation at the iteration $m - k + 1$ is:

$$\left[kn + \frac{k(k-1)}{2} - 1\right]\mathrm{CP}\ (<), \quad \text{in algorithm } 1',$$

$$(kn - 1)\ \mathrm{CP}\ (<), \quad \text{in algorithm } 2'.$$

The complexity of the estimation of the dissimilarity between the unified class and the others at the iteration $m - k + 1$ is:

$(k - 1)$CP $(<)$ in algorithm 1′, when O_{i_0} and C_{j_0} are unified

$(n + k - 2)$CP $(<)$, in algorithm 1′ when O_{i_1} and O_{j_1} are unified

$(k - 1)$CP $(<)$, in algorithm 2′.

Adding these complexities for $k = 1, 2, \ldots, m$, we obtain:

$$\left(\frac{1}{6}m^3 + \frac{n+1}{2}m^2 + \frac{3n-10}{6}m\right)\mathrm{CP}(<) \leqslant \mathrm{CP}(1') \leqslant$$

$$\leqslant \left(\frac{1}{6}m^3 + \frac{n+1}{2}m^2 + \frac{9n-16}{6}m - n\right)\mathrm{CP}\ (<)$$

$$\mathrm{CP}(2') = \left(\frac{n+1}{2}m^2 + \frac{n-3}{2}m\right)\mathrm{CP}(<).$$

For n fixed, we have $\mathrm{CP}(1') = O(m^3)$, $\mathrm{CP}(2') = O(m^2)$.

5. A Two Phase Clustering Procedure. The single linkage and the generalized Prim's algorithms we studied in this paper are hierarchical ascending clustering algorithms. The homogeneity condition which is supposed valid is in fact a compatibility condition between a previous procedure which determined the centers and these algorithms.

We now present a two phase clustering procedure which combines two types of clustering methods: graph-theoretical ones and methods based on the minimization of a certain functional. In the first phase, we perform for instance the Fuzzy c-Means algorithm (see Dunn 1973, Bezdek 1981) or a fuzzy hierarchical algorithm (see Dumitrescu 1988); then we defuzzify the fuzzy partition obtained associating each object with the cluster in which it has the highest membership degree. For each hard cluster we consider the objects in it correctly classified if their membership degree in the corresponding fuzzy cluster is higher than a certain threshold. The other objects will be reclassified by one of the algorithms presented above (obviously we shall prefer the generalized Prim's algorithm). Denoting by C_i the set of objects correctly classified in the hard cluster i, we take as centers the sets C_i; the dissimilarity $\rho(x, C_i)$ means point-set dissimilarity, i.e. the least dissimilarity between x and the objects in C_i.

This two phase clustering procedure gathers the advantages of the methods it consists of: the accuracy of the classification provided by graph-theoretical methods and the capacity of the Fuzzy c-Means algorithm to rapidly detect compact well-separated clusters. At the same time, some of the disadvantages of the two types of methods are eliminated: the incapacity of the Fuzzy c-Means algorithm to correctly classify the objects situated at the border of the clusters, especially in the case of unequal or nonconvex clusters (this being the major disadvantage which suggested the idea of this procedure) and the high complexity order of graph-theoretical methods (on the one hand the complexity of the generalized Prim's algorithm is one order lower than the complexity of the single linkage algorithm and on the other hand only a few objects from the total number of objects to be classified are reclassified by a graph-theoretical algorithm).

Here is now a numerical example. We considered 120 points in the two dimensional Euclidian space. The hard partition obtained by defuzzification from the fuzzy partition given by Fuzzy c-Means is presented in figure 1. Points having the membership degree in a fuzzy cluster higher than 67% from the highest membership degree in that cluster are considered correctly classified; on the figure they were encircled and represent the centers of the clusters. It is easy to see that the homogeneity condition holds. The other points were reclassified by the generalized Prim's algorithm, the results being presented in figure 2. These results are quite satisfactory. Let us notice that 17 from 34 points reclassified were wrongly classified by Fuzzy c-Means.

6. Cluster Analysis on Real Data. In this paragraph, the two phase clustering procedure presented above is used to analyze data concerning physico-geographical conditions and hydroenergetical potentials for a set of 123 hydrographical basins located in the North of Romania. Our purpose is to determine the influence of the local physico-geographical conditions upon the theoretical

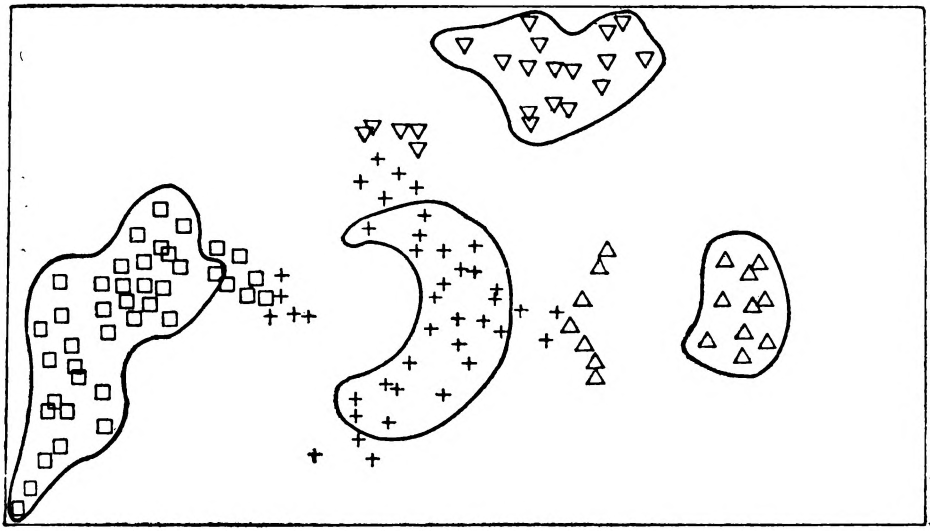

Fig. 1. Classification with FCM.

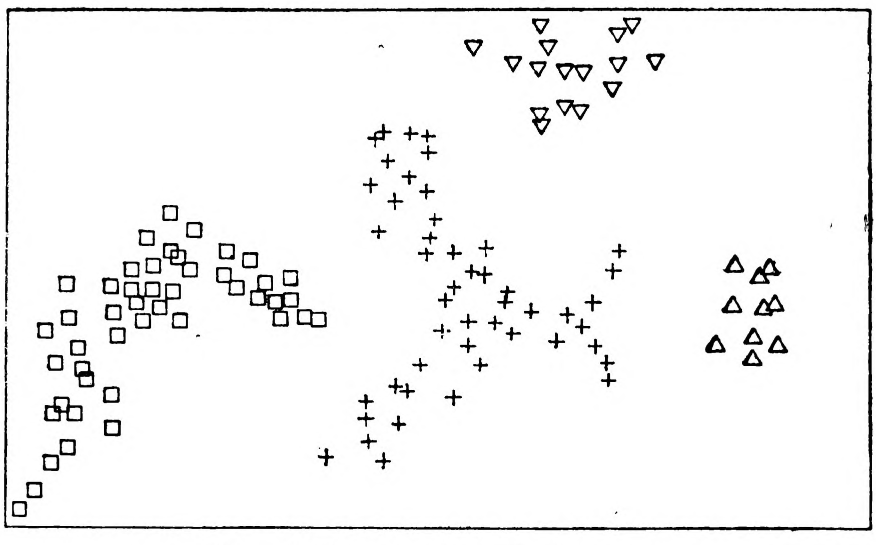

Fig. 2. Reclassification with algorithm 2.

hydroenergetical potential. 15 variables were studied; they were associated by factor analysis with 4 factors as follows:

1. Theoretic hydroenergetical potentials: a) rainfall potential; b) drainage potential; c) linear potential;

2. Morphometrical elements: a) surface of the basin; b) absolute altit c) relative altitude; d) density of the drainage; e) relief energy; f) slope of basin; g) slope of the river.

3. Average coefficient of erosion resistence.

4. Hydroclimatical elements: a) average anual rainfall; b) average d nage; c) drainage coefficient; d) drainage variation coefficient.

Cluster analysis used the factorial scores. A hierarchical classification cedure with fuzzy sets determined 4 fuzzy clusters for a certain threshold Dumitrescu 1988), clusters which were then defuzzyfied. Studying the com sition of the clusters on succesive levels, we notice the increase of the te rial homogeneity of the physico-geographical conditions from the upper to lower levels. The threshold in the hierarchical classification was chosen acc ding to the clusters homogeneity degree requiered by further statistical ana sis. The results obtained by hierarchical classification were improved usin procedure implementing the generalized Prim's algorithm. The scatter containing a bifactorial representation of the clusters lobtained by reclassif tion of certain samples is given in figure 3. The geographical location of clusters is shown in figure 4. The improvement of the classification me in this case, an increase in interpretability. Here is now the interpreta of the results:

Zone I is characterized by the highest altitudes, crystalline, erosion re tent formations, high relief energy and high values of the hydroclimatical e ments.

Zone II is superposed on neogene eruptive and also on crystalline of low altitudes; it is characterized by low absolute altitudes, high relative altitu and the highest values of the hydroclimatical elements.

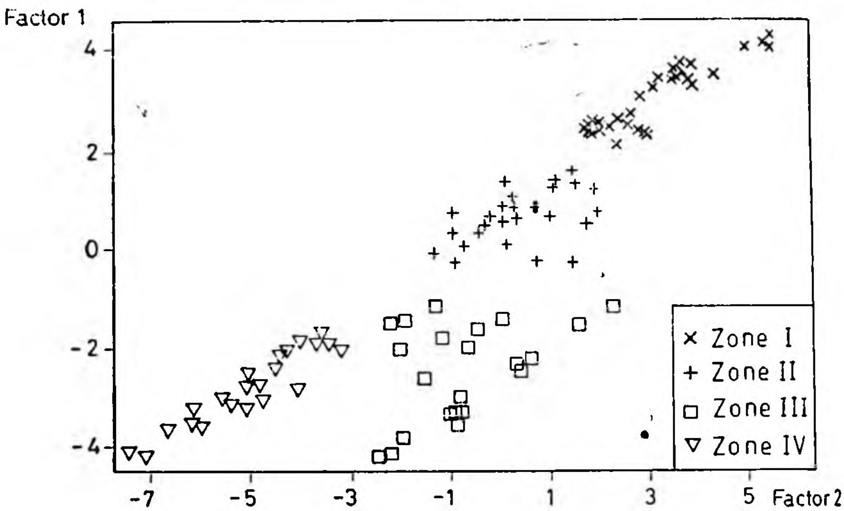

Fig. 3. Bifactorial representation of data.

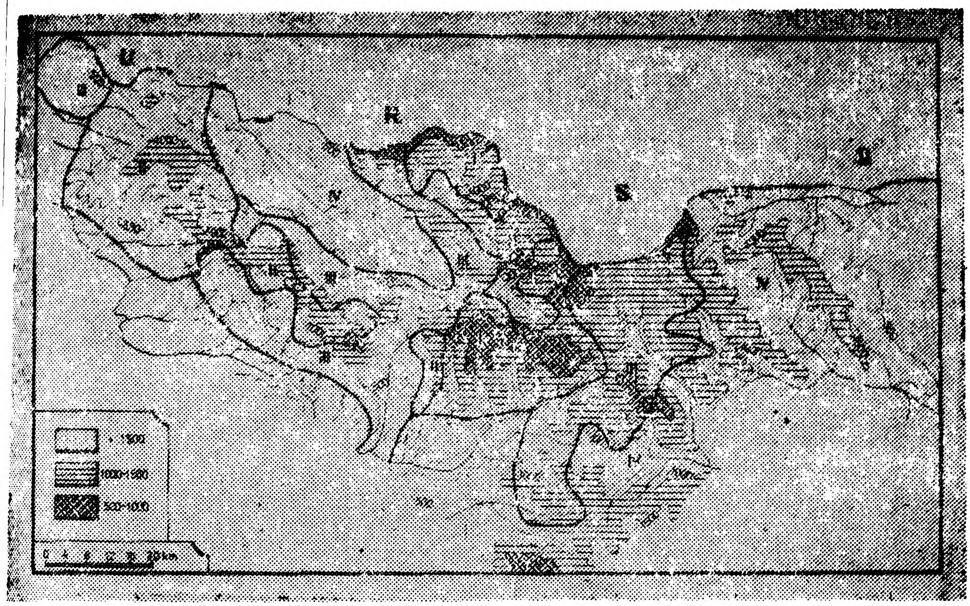

Fig. 4. Geographical location of the clusters.

Zone III represents the border more or less extended of the first two zones from spatial and also from hydroclimatical point of view. It is superposed on volcanic sedimentary and on the lower altitude crystalline.

Zone IV consists of basins situated in two distinct geographical areas, similar from geological and hydroclimatical point of view. It represents the depressionary area developed on cuaternary and neogene sedimentary formations and the paleogene and neogene pleated sedimentary formations from the eastern region. Both areas are sheltered from the main direction of atmospherical circulation by mountain chains. All variables have the lowest values in this zone.

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N THE MEASURES OF INFORMATION OF ORDER α ASSOCIATED WITH SOME PROBABILITY DISTRIBUTIONS

ION MIHOC*

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S subject classification: 60E05

REZUMAT. — Asupra informaţiei de ordinul α asociată unor distribuţii de probabilitate. În această lucrare se prezintă anumite probleme de bază legate de funcţia generatoare de informaţie corespunzătoare măsurii cantităţii de informaţie de ordinul α. De asemenea, lucrarea mai cuprinde o caracterizare informaţională a distribuţiei gama generalizată precum şi caracterizări informaţionale pentru distribuţiile de probabilitate ce aparţin clasei distribuţiei gama generalizată.

1. **The information generating function of order** α. Let X be an absolu-ly continuous random variable, that is, a random variable heving a probabi-ty density function f(x) and which is defined on the probability space, in he sense of Kolmogorov (Ω, K, P).

DEFINITION 1. [3] The *measure of the amount of information associated to he absolutely continuous random variable* X (or the entropy of X) respectively, he amount of information contained by the probability space (Ω, K, P) is efined by the following expresion

$$H(X) = - \int_a^b f(x) \cdot \log_2 f(x) dx, \tag{1.1}$$

rovided that the integral of the right of (1.1) exists) where $[a, b]$ is the domain f definition of the probability density dfunction f(x).

Remark 1. If (Ω, K, P) is a probability space generated by an random xperiment A and $A_1, A_2, \ldots, A$ are the possible autcomes of random ex-riment A, then

$$H(X) = H(A) = H(\mathfrak{A}) = - \sum_{i=1}^{n} p_i \cdot \log_2 p_i \tag{1.2}$$

here

$$p_i = P(A_i) > 0, \; i = \overline{1, n}; \; \sum_{i=1}^{n} p_i = 1, \; \mathfrak{A} = (p_1, \ldots, p_n), \tag{1.2a}$$

presents the amount of information furnished by the random experiment

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The quantity $H(A)$ defined by (1.2) is interpreted either as a measure of entropy (i.e. of uncertainty) or as a measure of information. Both interpretations are justified. As a matter of fact the difference between these two interpretations consists only in that whether we imagine ourselves in a moment before carrying out an experiment whose n possible results have the probabilities $p_1, p_2, \ldots, p_n$ (in which case $H(A)$ measures our uncertainty concerning the result of the experiment) or we imagine ourselves in a moment after the experiment has been carried out (in which case $H(A)$ measures the amount of information we got from the experiment). Also, we can speak as well that the quantity (1.2) represents the amount of information contained by the random variable X generated by the random experiment A.

Rényi [5] introduced a new measure of the amount of information associated to a random variable X.

DEFINITION 2. *The measure of the amount of information of order α associated to a random variable X,* has the form

$$I_\alpha(X) = I_2(x;\alpha) = \frac{1}{1-\alpha}\log_2\left(\int_R [f(x)]^\alpha dx\right), \quad \alpha > 0, \ \alpha \neq 1 \tag{1.3}$$

if X is a continuous random variable, respectively, the form

$$I_\alpha(X) = I_\alpha(\mathfrak{X}) = I_2(x;\alpha) = \frac{1}{1-\alpha}\log_2\left(\sum_{i=1}^{n} p_i^\alpha\right), \quad \alpha < 0, \ \alpha \neq 1, \tag{1.4}$$

if X is a discrete random variable.

It is interesting to note that on the basis of the relation (1.4) we have

$$\lim_{\alpha\to 1} I_\alpha(\mathfrak{X}) = H(\mathfrak{X}) = -\sum_{i=1}^{n} p_i \cdot \log_2 p_i, \tag{1.5}$$

i.e. we get back the ordinary Shannon entropy when α tends to 1. This means that Shannon's entropy belongs to the family of information measures of order α described in (1.4) and corresponds to $\alpha = 1$.

The information quantities of order α defined above are generally known as Rényi's information of order α.

Also one can see from (1.4) that

$$0 \leqslant I_\alpha(\mathfrak{X}) \leqslant \log_2 n, \tag{1.6}$$

and $I_\alpha(\mathfrak{X}) = 0$ can hold if and only if the distribution is degenerated ($\mathfrak{X} = (0, 0, \ldots, 1)$, for instance) while $I_\alpha(\mathfrak{X}) = \log_2 n$ holds if and only if $p_1 = p_2 = \ldots = p_n = \frac{1}{n}$. It is also easily seen that $I_\alpha(\mathfrak{X})$ is a monoton decreasing function of α.

If in (1.4) we assume that $\alpha = 2$, then we obtain

$$\mathrm{I}_2(\mathfrak{X}) = -\log_2\left(\sum_{i=1}^{n} p_i^2\right) = -\log_2 \mathrm{E}_n(\mathfrak{X}), \tag{1.7}$$

here

$$E_n(\mathfrak{X}) = \sum_{i=1}^{n} p_i^2 \tag{1.8}$$

just the Onicescu information energy [1] introduced in the information theory r an analogy to the kinetic energy from mechanics.

From (1.7) it follows that

$$E_n(\mathfrak{X}) = 2^{-I_2(\mathfrak{X})}. \tag{1.9}$$

DEFINITION 3. The information generating function associated to the Rényi formation of order α has the following form

$$T_X(u\,;\,\alpha) = \left(\sum_{i=1}^{n} p_i^{\alpha}\right)^{\frac{ku}{1-\alpha}}, \quad \alpha > 0,\ \alpha \neq 1,\ u \in R, \tag{1.10}$$

X is a discrete random variable, respectively, the form

$$T_X(u\,;\,\alpha) = \left(\int_R [f(x)]^{\alpha} dx\right)^{\frac{ku}{1-\alpha}}, \quad \alpha > 0,\ \alpha \neq 1,\ u \in R, \tag{1.11}$$

X is a continuous random variable, where

$$k = \log_2 e. \tag{1.12}$$

THEOREM 1. *The information generating function (1.11) has the following fundamental property*

$$T'_X(0\,;\,\alpha) = \frac{1}{1-\alpha} \log_2 \left(\int_R [f(x)]^{\alpha} dx\right) = I_{\alpha}(X). \tag{1.13}$$

Proof. According to the relation (1.11) we obtain

$$\frac{d}{du}\,[T_X(u\,;\,\alpha)] = T'_X(u\,;\,\alpha) = \frac{k}{1-\alpha}\left(\int_R [f(x)]^{\alpha} dx\right)^{\frac{ku}{1-\alpha}} \cdot \log e \left(\int_R [f(x)]^{\alpha} dx\right),$$

respectively, the relation

$$T'_X(O\,;\,\alpha) = I_{\alpha}(X) = \frac{k}{1-\alpha} \log e \left(\int_R [f(x)]^{\alpha} dx\right),$$

if we have in view that

$$\log_2 e \cdot \log e A = \log_2 A,\ k = \log_2 e. \tag{1.14}$$

2. **On measure of the amount of information of order α associated to the generalized gamma distribution.** Let X be a continuous random variable defined on (Ω, K, P).

DEFINITION 4. [4] We say that the random variable X *follows a generalized gamma distribution* if its probability density has the form

$$f(x) = f(x; a, b, r) = \frac{a}{\Gamma\left(\frac{r}{a}\right) \cdot b^{r/a}} \cdot x^{r-1} \cdot e^{-\frac{x^a}{b}}, \quad x > 0, \tag{2.1}$$

when

$$b > 0, \quad \frac{r}{a} > 0. \tag{2.2}$$

TEOREM 2. *If X is a absolutly continuous random variable which follows a generalized gamma distribution, then the information generating function of order α, $T_X(u; \alpha)$, respectively, the measure of the amount of information $I_2(x; \alpha)$ associates will be*

$$T_X(u; \alpha) = \left\{ \frac{b^{\frac{1}{a}}}{a \cdot \alpha^{\frac{\alpha(r-1)+1}{a(1-\alpha)}}} \cdot \frac{\left[\Gamma\left(\frac{\alpha(r-1)+1}{a}\right)\right]^{\frac{1}{1-\alpha}}}{\left(\Gamma\left(\frac{r}{a}\right)\right)^{\frac{\alpha}{1-\alpha}}} \right\}^{ku}, \quad u \in R, \tag{2.3}$$

respectively,

$$I_2(x; \alpha) = I_2(x; \alpha, a, b, r) = \tag{2.4}$$

$$= \frac{1}{1-\alpha} \log_2 \left\{ \frac{b^{\frac{1-\alpha}{a}}}{a^{1-\alpha} \cdot \alpha^{\frac{\alpha(r-1)+1}{a}}} \cdot \frac{\Gamma\left(\frac{\alpha(r-1)+1}{a}\right)}{\left(\Gamma\left(\frac{r}{a}\right)\right)^{\alpha}} \right\},$$

wheer

$$b > 0, \quad \frac{r}{a} > 0, \quad \alpha > 0, \quad \alpha \neq 1. \tag{2.5}$$

Proof. The integral form of the information generating function of order α associated to the random variable X, with the probability density function (2.1) will be

$$T_X(u; \alpha) = \left(\frac{a}{\Gamma\left(\frac{r}{a}\right) \cdot b^{\frac{r}{a}}}\right)^{\frac{\alpha \cdot ku}{1-\alpha}} \cdot \left(\int_0^{\infty} x^{\alpha(r-1)} \cdot e^{-\frac{\alpha \cdot x^a}{b}} dx\right)^{\frac{ku}{1-\alpha}}, \tag{2.6}$$

where

$$u \in R, \quad k = \log_2 e, \quad b > 0, \quad \frac{r}{a} > 0, \tag{2.7}$$

d

$$\Gamma(s) = \int_0^\infty x^{s-1} \cdot e^{-x} dx, \quad s > 0 \tag{2.8}$$

the Euler integral of the second kind or gamma function.

If we compute the derivative of the function $T\ (u\ ;\ \alpha)$ we obtain

$$T'_X(u\ ;\ \alpha) = \frac{d}{du}\ [T_X(u\ ;\ \alpha)] = \tag{2.9}$$

$$= \frac{k}{1-\alpha} \cdot \left(\frac{a}{\Gamma\left(\frac{r}{a}\right) \cdot b^{\frac{r}{a}}} \right)^{\frac{\alpha \cdot ku}{1-\alpha}} \cdot \left(\int_0^\infty x^{\alpha(r-1)} \cdot e^{-\frac{\alpha \cdot x^a}{b}} dx \right)^{\frac{ku}{1-\alpha}} \cdot$$

$$\cdot \left\{ \log_e \left(\int_0^\infty x^{\alpha(r-1)} \cdot e^{\frac{-\alpha x^a}{b}} dx \right) + \alpha \cdot \log_e \left(\frac{a}{\Gamma\left(\frac{r}{a}\right) \cdot b^{\frac{r}{a}}} \right) \right\},$$

, hence, using the property (1.13) we obtain

$$T'_X(0\ ;\ \alpha) = \frac{k}{1-\alpha} \left\{ \log_e \left(\int_0^\infty x^{\alpha(r-1)} \cdot e^{-\frac{\alpha\ x^a}{b}} dx \right) + \right.$$

$$\left. + \alpha \cdot \log_e \left(\frac{a}{\Gamma\left(\frac{r}{a}\right) \cdot b^{\frac{r}{a}}} \right) \right\}. \tag{2.10}$$

According to the relations (1.14) from (2.10) we obtain just the integral of the amount of information of order α, namely

$$I_2(x\ ;\ \alpha,\ a,\ b,\ r) = \frac{1}{1-\alpha} \log_2 \left[\left(\frac{a}{\Gamma\left(\frac{r}{a}\right) \cdot b^{\frac{r}{a}}} \right)^\alpha \int_0^\infty \left(x^{r-1} \cdot e^{-\frac{x^a}{b}} \right)^\alpha dx \right]. \tag{2.11}$$

Now, if we make the change of variables

$$t = \frac{\alpha}{b} \cdot x^a, \tag{2.12}$$

the information generating function (2.6) can be expressed as follows

$$T_X(u\ ;\ \alpha) = \left\{ \frac{b^{\frac{1}{a}}}{a \cdot \alpha^{\frac{\alpha(r-1)+1}{a(1-\alpha)}}} \cdot \frac{\left[\Gamma\left(\frac{\alpha(r-1)+1}{a} \right) \right]^{\frac{1}{1-\alpha}}}{\left(\Gamma\left(\frac{r}{a} \right) \right)^{\frac{\alpha}{1-\alpha}}} \right\}^{ku} \tag{2.13}$$

e $\frac{r}{v} > 0$, $b > 0$, $\alpha > 0$, $\alpha \neq 1$, $u \in R$.

From (2.13) we obtain

$$T'_X(u\,;\,\alpha) = \left\{ \frac{b^{\frac{1}{a}}}{a \cdot \alpha^{\frac{\alpha(r-1)+1}{a(1-\alpha)}}} \cdot \frac{\left[\Gamma\left(\frac{\alpha(r-1)+1}{a}\right)\right]^{\frac{1}{1-\alpha}}}{\left(\Gamma\left(\frac{r}{a}\right)\right)^{\frac{\alpha}{1-\alpha}}} \right\}^{ku} \cdot$$

$$\cdot k \log e \left\{ \frac{b^{\frac{1}{a}}}{a \cdot \alpha^{\frac{\alpha(r-1)+1}{a(1-\alpha)}}} \cdot \frac{\left[\Gamma\left(\frac{\alpha(r-1)+1}{a}\right)\right]^{\frac{1}{1-\alpha}}}{\left(\Gamma\left(\frac{r}{a}\right)\right)^{\frac{r}{a}}} \right\} \qquad (2.14)$$

and, whence if we have in view the relations (1.14) and, respectively, the fundamental property of the information generating function of order α, (1.13) we obtain

$$I_2(x\,;\,\alpha, ,a, b, r) = T'_X(0\,;\,\alpha) = \qquad (2.15)$$

$$= \frac{1}{1-\alpha} \log_2 \left\{ \frac{b^{\frac{1-\alpha}{a}}}{a^{1-\alpha} \cdot \alpha^{\frac{\alpha(r-1)+1}{a}}} \cdot \frac{\Gamma\left(\frac{\alpha(r-1)+1}{a}\right)}{\left(\Gamma\left(\frac{r}{a}\right)\right)^{\alpha}} \right\},$$

e.i. just the measure of order α of amount of information associated to the random variable X which follows the generalized gamma distribution. This completes the proof.

3. **Particular cases.** Let X be a continuous random variable which follows the generalized gamma distribution, that is, its probability density function has the form (2.1).

According to Theorem 2, for $T_X(u\,;\,\alpha)$ and $I_2(x\,;\,\alpha, a, b, r)$ we obtained the forms (2.3) and (2.4).

In the next we will present some particular cases. These particular cases are probability distributions which belong to the class of generalized gamma distribution.

3.1. **Gamma distribution.** If $a = 1$, $b = 1$ and $r = p > 0$, then the continuous random variable X follows the gamma distribution and we have

$$f(x\,;\,1, 1, p) = f(x\,;\,p) = \frac{1}{\Gamma(p)} \cdot x^{p-1} \cdot e^{-x}, \quad x > 0, \ p > 0, \qquad (3.1.1)$$

$$T_X(u\,;\,\alpha) = \left\{ \frac{1}{\alpha^{\frac{\alpha(p-1)+1}{1-\alpha}}} \cdot \frac{[\Gamma(\alpha(p-1)+1)]^{\frac{1}{1-\alpha}}}{(\Gamma(p))^{\frac{\alpha}{1-\alpha}}} \right\}^{ku}, \quad u \in R, \qquad (3.1.2)$$

$$I_2(x\,;\,\alpha, p) = \frac{1}{1-\alpha} \cdot \log_2 \left\{ \frac{\Gamma[\alpha(p-1)+1]}{\alpha^{\alpha(p-1)+1} \cdot (\Gamma(p))^{\alpha}} \right\} \qquad (3.1.3)$$

where $\alpha > 0$, $\alpha \neq 1$, $p > 0$.

3.2. **Exponential distribution.** If $a = 1$, $b > 0$, $r = 1$, then the continuous random variable X follows the exponential distribution. The probability density function, the information generating function and the amount of information corresponding to this probability distribution will be the followings

$$f(x;b) = \frac{1}{b}\cdot e^{-\frac{1}{b}x}, \quad x > 0,\ b > 0, \tag{3.2.1}$$

$$T_X(u;\alpha) = \left(\frac{b}{\alpha^{\frac{1}{1-\alpha}}}\right)^{ku}, \quad u \in R,\ k = \log_2 e, \tag{3.2.2}$$

$$I_2(x;\alpha,b) = \frac{1}{1-\alpha}\log_2\left(\frac{b}{\alpha^{\frac{1}{1-\alpha}}}\right), \quad \alpha > 0,\ \alpha \neq 1, \tag{3.2.3}$$

3.3. **Erlang distribution.** If $a = 1$, $r = m$, $m \in N$, $b > 0$. then the continuous random variable X has the Erlang distribution and we obtain

$$f(x;b,m) = \frac{1}{b^m\cdot(m-1)!}\cdot x^{m-1}\cdot e^{-\frac{x}{b}}, \quad x > 0, \tag{3.3.1}$$

$$T_X(u;\alpha) = \left\{\frac{b}{\alpha^{\frac{\alpha(m-1)+1}{1-\alpha}}}\cdot\frac{\left[\Gamma\left(\frac{\alpha(m-1)+1}{1-\alpha}\right)\right]^{\frac{1}{1-\alpha}}}{((m-1)!)^{\frac{\alpha}{1-\alpha}}}\right\}^{ku}, \quad u \in R \tag{3.3.2}$$

respectively

$$I_2(x;\alpha,b,m) = \frac{1}{1-\alpha}\cdot\log_2\left\{\frac{b^{1-\alpha}}{\alpha^{\alpha(m-1)+1}}\cdot\frac{\Gamma[\alpha(m-1)+1]}{((m-1)!)^{\alpha}}\right\} \tag{3.3.3}$$

when $\alpha > 0$, $\alpha \neq 1$, $k = \log_2 e$.

3.4. **Chi-square (χ^2) distribution.** If $a = 1$, $b = 2\sigma^2$ and $r = \frac{s}{2}$, then we obtain a random variable X which follows the chi-square distribution and we have

$$f(x;b,s) = f(x;\sigma,s) = \frac{1}{(2\sigma^2)^{\frac{s}{2}}\cdot\Gamma\left(\frac{s}{2}\right)}\cdot x^{\frac{s}{2}-1}\cdot e^{-\frac{x}{2\sigma^2}}, \quad x > 0 \tag{3.4.1}$$

$$T_X(u;\alpha) = \left\{\frac{2\sigma^2}{\alpha^{\frac{\alpha\left(\frac{s}{2}-1\right)+1}{1-\alpha}}}\cdot\frac{\left[\Gamma\left(\alpha\left(\frac{s}{2}-1\right)+1\right)\right]^{\frac{1}{1-\alpha}}}{\left(\Gamma\left(\frac{s}{2}\right)\right)^{\frac{\alpha}{1-\alpha}}}\right\}^{ku}, \quad u \in R \tag{3.4.2}$$

$$I_2(x;\alpha,b,s)=\frac{1}{1-\alpha}\log_2\left\{\frac{(2\sigma^2)^{1-\alpha}}{\alpha^{\alpha\left(\frac{s}{2}-1\right)+1}}\cdot\frac{\Gamma\left[\alpha\left(\frac{s}{2}-1\right)+1\right]}{\left(\Gamma\left(\frac{s}{2}\right)\right)^{\alpha}}\right\},\qquad (3.4.3)$$

where $\alpha > 0$, $\alpha \neq 1$, $\sigma > 0$.

3.5. **Rayleight's distribution.** If $a = r = 2$, $b > 0$, the n the continuous random variable X has the Reyleigh's distribution and its probability density function has the form

$$f(x;b)=\frac{2x}{b}\cdot e^{-\frac{x^2}{b}},\ x>0,\ b>0.\qquad (3.5.1)$$

Also we have

$$T_X(u;\alpha)=\left\{\frac{\sqrt{b}}{2\cdot\alpha^{\frac{\alpha+1}{2(1-\alpha)}}}\cdot\left[\Gamma\left(\frac{\alpha+1}{2}\right)\right]^{\frac{1}{1-\alpha}}\right\}^{ku},\ u\in R,\qquad (3.5.2)$$

$$I_2(x;\alpha;b)=\frac{1}{1-\alpha}\cdot\log_2\left\{\frac{b^{\frac{1-\alpha}{2}}}{2^{1-\alpha}\cdot\alpha^{\frac{\alpha+1}{2}}}\cdot\Gamma\left(\frac{\alpha+1}{2}\right)\right\}\qquad (3.5.3)$$

where $\alpha > 0$, $\alpha \neq 1$, $b > 0$.

3.6. **Maxwell's distribution.** If $a = 2$, $r = 3$, $b > 0$, we obtain

$$f(x;b)=\frac{4x^2}{\sqrt{\pi\cdot b^3}}\cdot e^{-x^2/b},\ x>0,\ b>0,\qquad (3.6.1)$$

$$T_X(u;\alpha)=\left\{\frac{\sqrt{b}}{2\cdot b^{\frac{2\alpha+1}{2(1-\alpha)}}}\cdot\frac{\left[\Gamma\left(\frac{(2\alpha+1)}{2}\right)\right]^{\frac{1}{1-\alpha}}}{\left(\frac{1}{2}\sqrt{\pi}\right)^{\frac{\alpha}{1-\alpha}}}\right\}^{ku},\ u\in R\qquad (3.6.2)$$

$$I_2(x;\alpha,b)=\frac{1}{1-\alpha}\cdot\log_2\left\{\frac{b^{\frac{1-\alpha}{2}}}{2^{1-\alpha}\cdot\alpha^{\frac{2\alpha+1}{2}}}\cdot\frac{\Gamma\left(\frac{(2\alpha+1)}{2}\right)}{\left(\frac{\sqrt{\pi}}{2}\right)^{\alpha}}\right\}\qquad (3.6.3)$$

when $\alpha > 0$, $\alpha \neq 1$, $b > 0$.

3.7. **Cut normal distribution.** If $a = 2$, $r = 1$, $b = 2\sigma^2$, $\sigma > 0$, then the continuous random variable X has the normal distribution cut leftsidedly in the point $x = 0$. For a thus random variable we have

$$f(x;\sigma)=\frac{2}{\sqrt{2\pi}\cdot\sigma}\cdot e^{-\frac{x^2}{2\sigma^2}},\ x>0,\ \sigma>0,\qquad (3.7.1)$$

$$T_X(u\,;\,\alpha,\,\sigma) = \left(\frac{\sigma\cdot\sqrt{\pi}}{\sqrt{2}\cdot\alpha^{\frac{1}{2(1-\alpha)}}}\right)^{ku};\ u\in R, \tag{3.7.2}$$

$$I_2(x\,;\,\alpha) = \frac{1}{1-\alpha}\cdot\log_2\left[\frac{1}{\sqrt{\alpha}}\left(\sqrt{\frac{2}{\pi}}\right)^{\alpha-1}\right], \tag{3.7.3}$$

hen $\alpha > 0$, $\alpha \neq 1$.

Remark 2. The Theorem 2, together with the results of the last section ow that the generalized gamma distribution has a important interest, namely, e informational characterizations of the particular cases mentioned above ay be reduced to the informational characterization of the generalized gamma stribution defined by the probability density function (2.1).

4. **The relation between the measure of the amount of information of order and Shannon's information measure.** Let X be an absolut continuous dom variable which follows the generalized gamma distribution, that is, its bability density function has the form (2.1).

The Shannon's information measure which correspondsg to this random iable will be

$$H_2(X) = H(X) = k\cdot H_e(X),\quad k = \log_2 e, \tag{4.1}$$

re

$$H_e(X) = -\int_0^\infty f(x\,;\,a,\,b,\,r)\log_e f(x\,;\,a,\,b,\,r)dx, \tag{4.2}$$

$$H_2(X) = H(X) = -\int_0^\infty f(x\,;\,a,\,b,\,r)\cdot\log_2 f(x\,;\,a,\,b,\,r)dx. \tag{4.3}$$

The measure of the amount of information of order α corresponding to same random variable has the form

$$I_2\,(x\,;\,\alpha,\,b,\,a,\,r) = \frac{1}{1-\alpha}\log_2\left\{\frac{b^{\frac{1-\alpha}{a}}}{a^{1-\alpha}\cdot\alpha^{\frac{\alpha(r-1)+1}{a}}}\cdot\frac{\Gamma\left(\frac{\alpha(r-1)+1}{a}\right)}{\left(\Gamma\left(\frac{r}{a}\right)\right)^{\alpha}}\right\}, \tag{4.4}$$

re $b > 0$, $\frac{r}{a} > 0$, $\alpha > 0$, $\alpha \neq 1$.

THEOREM 3. *Between the measures* $I_2\,(x\,;\,\alpha,\,a,\,b,\,r)$ and $H_2(X)$ exist the wing relation

$$\lim_{\alpha\to1} I_2(x\,;\,\alpha,\,a,\,b,\,r) = H_2(X) = k\cdot H_e(X), \tag{4.5}$$

where

$$H_e(X) = \frac{1}{a}\left\{r + \ln\left[b\left(\frac{\Gamma\left(\frac{r}{a}\right)}{a}\right)^a\right] - (r-1)\, D\ln\Gamma\left(\frac{r}{a}\right)\right. \quad (4.6)$$

and $D \ln \Gamma(t) = \psi(t)$ *is digamma function.*

Proof. Using the following notation

$$A(\alpha) = \frac{b^{\frac{1-\alpha}{a}}}{a^{1-\alpha}\cdot\alpha^{\frac{\alpha(r-1)+1}{a}}}\cdot\frac{\Gamma\left(\frac{\alpha(r-1)+1}{a}\right)}{\left(\Gamma\left(\frac{r}{a}\right)\right)^{\alpha}}, \quad (4.7)$$

we obtain a new form for the measure of the amount of information $I_2(x;\alpha,a,b,r)$ namely.

$$I_2(x;\alpha,a,b,r) = \frac{1}{1-\alpha}\cdot\log_2 A(\alpha). \quad (4.8)$$

By passing to the limit $\alpha\to 1$, (4.8) follows imediately

$$\lim_{\alpha\to 1} I_2(x;\alpha,a,b,r) = \frac{1}{\log_e 2}\cdot\lim_{\alpha\to 1}\frac{A'(\alpha)}{A(\alpha)}, \quad (4.9)$$

if we have in view that

$$A(1) = 1. \quad (4.10)$$

Let us introduce the following notations

$$A_1(\alpha) = b^{\frac{1-\alpha}{a}}\cdot\Gamma\left(\frac{\alpha(r-1)+1}{a}\right) \quad (4.11)$$

$$A_2(\alpha) = a^{1-\alpha}\cdot\alpha^{\frac{\alpha(r-1)+1}{a}}\cdot\left(\Gamma\left(\frac{r}{a}\right)\right)^{\alpha}. \quad (4.12)$$

Differentiating (4.11) we get

$$A_1'(\alpha) = -\frac{1}{a}\log_e b\cdot b^{\frac{1-\alpha}{a}}\cdot\Gamma\left(\frac{\alpha(r-1)+1}{a}\right) + \frac{r-1}{a}\cdot b^{\frac{1-\alpha}{a}}\cdot\Gamma_\beta'(\beta) \quad (4.13)$$

where

$$\beta = \frac{\alpha(r-1)+1}{a} \quad (4.13a)$$

$$\Gamma(p) = \int_0^\infty t^{p-1}\cdot e^{-t}\,dt, \quad p > 0, \quad (4.13b)$$

$$\Gamma_\beta'(\beta) = \int_0^\infty t^{\frac{\alpha(r-1)+1}{a}}\cdot\log_e t\cdot e^{-t}\,dt. \quad (4.13c)$$

Also, differentiating (4.12) we get

$$A_2'(\alpha) = a^{1-\alpha} \cdot \alpha^{\frac{\alpha(r-1)+1}{a}} \cdot \left(\Gamma\left(\frac{r}{a}\right)\right)^{\alpha} \left\{\log_e \Gamma\left(\frac{r}{a}\right) + \right. \tag{4.14}$$

$$\left. + \frac{r-1}{a} \cdot \log_e \alpha + \frac{\alpha(r-1)+1}{a} - \log_e a\right\}.$$

Putting (4.13) and (4.14) in the right-hand of the relation

$$A'(\alpha) = \frac{A_1'(\alpha) \cdot A_2(\alpha) - A_1(\alpha) \cdot A_2'(\alpha)}{[A_2(\alpha)]^{-2}} \tag{4.15}$$

finally get

$$A'(\alpha) = A(\alpha)\left\{\frac{1}{a}\,[(r-1) \cdot D \ln \Gamma(\beta) - \log_e (b \cdot e \cdot \alpha^{r-1}) - \right. \tag{4.16}$$

$$\left. - \alpha\,(r-1)] + \log_e \Gamma \frac{a}{\left(\frac{r}{a}\right)}\right\}$$

here

$$D \ln \Gamma(\beta) = \frac{\Gamma'(\beta)}{\Gamma(\beta)}, \quad \beta = \frac{\alpha(r-1)+1}{a}. \tag{4.16a}$$

From this last relation (4.16) we get

$$(\alpha)\Big|_{\alpha=1} = A'(1) = \frac{1}{a}\left\{(r-1) \cdot D \ln \Gamma\left(\frac{r}{a}\right) - r - \log_e\left[b \left(\frac{\Gamma\left(\frac{r}{a}\right)}{a}\right)^{a}\right]\right\}, \tag{4.17}$$

we have in view that

$$D \ln \Gamma(\beta)\,|_{\alpha=1} = D \ln \Gamma\left(\frac{r}{a}\right). \tag{4.17a}$$

We now turn to the expression (4.9) and taking into account (4.17b) we

$$\lim_{\alpha\to 1} I_2(x\,;\,\alpha,\,a,\,b,\,r) = \frac{1}{\log_e 2} \cdot \frac{1}{a}\left\{r + \log_e\left[b \cdot \left(\frac{\Gamma\left(\frac{r}{a}\right)}{a}\right)^{a}\right] - \right. \tag{4.18}$$

$$\left. - (r-1) \cdot D \ln \Gamma\left(\frac{r}{a}\right)\right\},$$

ause $A(1) = 1$.

Making use of [2]

$$H_e(X) = \frac{1}{a}\left\{ r + \log_e \left[b \cdot \left(\frac{\Gamma\left(\frac{r}{a}\right)}{a} \right)^{a} \right] - (r-1) \cdot D \ln \Gamma\left(\frac{r}{a}\right) \right\},$$

we find that

$$\lim_{\alpha \to 1} I_2(x\,;\,\alpha, a, b, r) = H_2(X) = k \cdot H_e(X), \quad k = \log_2 e.$$

This completes the proof.

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OF FINDING THE SHORTEST PATHS OF A DIGRAPH

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REZUMAT. — Asupra determinării drumurilor minime ale unui digraf. În această lucrare se propune un program FORTRAN 77 pentru determinarea tuturor drumurilor de lungime minimă între oricare două noduri specificate ale unui digraf pozitiv-capacitat.

Introduction. For a given arc-weighted digraph $D = (V, E)$, having $(|V| = n)$, the set of nodes (vertices), E the set of arcs (edges), and $C = (c_{ij})$, $i, j = 1,2, \cdots, n$ the matrix of the arc costs, the *shortest path problem* [1-3] is the problem of finding all the shortest paths (the paths for which the sum of weights is minimum) from a specified starting node NOD1 $\in V$ to specified ending node NOD2 $\in V$, provided that at least such a path exists (for a review of the extant algorithms for this problem see [1]).

What is often required in practice is not simply the shortest but also the second, third, etc. shortest paths in a graph. With this information, one would then decide on the best path to choose, using also criteria which are either difficult to incorporate directly into the algorithms or which are subjective in nature.

Moreover, the second, third, etc. shortest paths can be used in a sensitivity analysis of the shortest path problem. In this paper, we shall give a FORTRAN 77 program, for the solution of the above mentioned problem, for the case where all c_{ij} are nonnegative, this case occuring often enough in practice (e.g., see [1—3]), to warrant the description of a special algorithm.

Description of the method. We shall assume that matrix C does not satisfy the triangulary condition, i.e., c_{ij} is not less than $c_{ik} + c_{kj}$ for all i, j and k, otherwise the shortest path between v_i and v_j is always the single arc (v_i, v_j) and the problem becomes nonexistent. In particular, if an arc (v_i, v_j) does not exist in G, then ist cost will be assumed to have been set to ∞ (e.g., see [1, 2]).

Solution of the problem are independent of physical interpretation of the weights. The unweighted digraph is merely a special case, interpreted as a weighted digraph with every weight equal to 1. The shortest paths given by the program will be then the paths of fewest arcs.

Let $C^* = (c^*_{ij})$, $i, j = 1, 2, \ldots, n$ be, such that c^*_{ij} is equal to the weight of shortest path from v_i to v_j (if such a path exists) and to ∞ if there exists no path between v_i and v_j.

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Let v_i and v_j be two distinct nodes of D, for which $c^*_{ij} \neq \infty$, i.e., there
at least a path from v_i to v_j. Obviously, (v_k, v_j) is the last arc of a sh
path from v_i to v_j if and only if

$$c^*_{ij} = c^*_{ik} + c_{kj}.$$

Thus, *if* the indices k for which (1) holds are $k_1, k_2, \ldots, k_t$, then
$s = 1, 2, \ldots, t$ are the last arcs of the shortest paths from v_i to v_j. For
one of these indices, we repeat the procedure, i.e., we find the last arcs
form (w, v_{k_s}), belonging to the shortest paths from v_i to v_{k_s}, by taking
the role of v_i, and so on.

If (1) does not hold for any $k \neq i, j$, then the arc (v_i, v_j) is the single sh
path from v_i to v_j.

In this way, all the shortest paths from v_i to v_j can be generated (the me
described above, is summarized in the FORTRAN 77 program **SPD** (Sh
Paths of a Digraph). In the sequel, we have:

N = the nodes' number of the digraph,
DD = the matrix C,
DM = the matrix C^*,
INF = our „infinity" (∞), which is a very large number, say 10

The subroutine MDM computes the matrix DM, from DD, according to
algorithm described in [2].

Example. For the digraph with matrix

C =

I \ J	1	2	3	4	5	6
1	0	1	4	∞	∞	∞
2	∞	0	2	5	8	∞
3	∞	∞	0	2	5	∞
4	∞	∞	∞	0	3	5
5	∞	∞	∞	∞	0	1
6	∞	∞	∞	∞	∞	0

the program SPD has found the following shortest paths from node 1 to no

(1, 2, 3, 5, 6).
(1, 2, 3, 4, 5, 6,),

of weight equal to 9.

Conclusions. Considerable experimentations, on a PDP-11 computer,
conducted to investigate the efficiency (the efficiency of the program ca
measured in terms of the computational times required to obtain the solu
of the proposed method. Based on this computational experience, it ma
said that the above program can be successfully used for large-sized digr
generating all the shortest paths in a reasonably computer memory and co
tational tim

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```
C                     ****************
C                     *              *
C                     * PROGRAM - SPD *
C                     *              *
C                     ****************
C
C
C       -----------------------------------------------
C       !                                             !
C       !  This program finds all the shortest paths  !
C       ! between any two specified nodes,NOD1 and    !
C       ! NOD2,of an arc-weighted digraph with non -  !
C       ! negative arc costs.                         !
C       !                                             !
C       !                   ****                      !
C       !                                             !
C       !         Written by Dr. Danut Marcu          !
C       !                                             !
C       !               Winter 1989                   !
C       !                                             !
C       -----------------------------------------------
C
C
C       DRIVER - PROGRAM
C       ----------------
C
C
        REAL INF
        DIMENSION DD(6,6),DM(6,6),MS(6,6),IK(6),NS(6),NSL(6)
C
C       ....DATA ENTRY....
C
        ACCEPT *,N
        ACCEPT *,INF
        ACCEPT *,NOD1
        ACCEPT *,NOD2
        DO 1 I=1,N
1       ACCEPT *,(DD(I,J),J=1,N)
C
C       ....PRINT THE WEIGHT OF SHORTEST PATHS....
C
        CALL MDM(N,DD,DM,INF)
        PRINT *,DM(NOD1,NOD2)
C
C       ....PRINT THE SHORTEST PATHS FROM NOD1 TO NOD2....
C
        CALL PMS(N,DD,DM,MS,IK,NOD1,NOD2)
        CALL PSP(N,MS,NS,NSL,NOD1,NOD2)
        STOP
        END
```

```
      SUBROUTINE PSP(N,MS,NS,NSL,NOD1,NOD2)
      DIMENSION MS(N,N),NS(N),NSL(N)
      IT=1
      NS(1)=NOD2
13    NNS=2
      NJK=MS(NOD2,IT)
4     N1=NNS-1
      DO 1 L=1,N1
      IF(NS(L).EQ.NJK) GO TO 2
1     CONTINUE
      NS(NNS)=NJK
      K=1
12    IF(MS(NJK,K).EQ.0) GO TO 3
10    NJK=MS(NJK,K)
      NNS=NNS+1
      GO TO 4
3     IF(NS(NNS).NE.NOD1) GO TO 9
      DO 5 L=1,NNS
5     NSL(L)=NS(NNS-L+1)
      PRINT 6,(NSL(L),L=1,NNS)
6     FORMAT(//,1X,6I4)
9     NNS=NNS-1
      IF(NNS.EQ.1) GO TO 7
      NJK=NS(NNS)
      L=1
11    IF(MS(NJK,L).NE.NS(NNS+1)) GO TO 8
      K=L+1
      IF(MS(NJK,K).EQ.0) GO TO 3
      GO TO 10
8     L=L+1
      GO TO 11
2     NNS=N1
      NJK=NS(NNS)
      K=K+1
      GO TO 12
7     IT=IT+1
      IF(MS(NOD2,IT).NE.0) GO TO 13
      RETURN
      END
```

```
      SUBROUTINE PMS(N,DD,DM,MS,IK,NOD1,NOD2)
      DIMENSION DD(N,N),DM(N,N),MS(N,N),IK(N)
      DO 1 L=1,N
      DO 1 K=1,N
1     MS(L,K)=0
      DO 2 L=1,N
2     IK(L)=0
      K=0
      DO 3 L=1,N
      IF(L.EQ.NOD1) GO TO 3
      IF(L.EQ.NOD2) GO TO 3
      IF(DM(NOD1,NOD2).NE.DM(NOD1,L)+DD(L,NOD2)) GO TO 3
      K=K+1
      MS(NOD2,K)=L
3     CONTINUE
      IF(K.EQ.0) GO TO 4
7     DO 5 L=1,N
      IF(IK(L).EQ.1) GO TO 5
      IF(MS(L,1).EQ.0) GO TO 5
      K=1
9     IF(MS(L,K).NE.0) GO TO 6
      IK(L)=1
      GO TO 7
6     IT=0
      DO 8 NJK=1,N
      IF(NJK.EQ.MS(L,K)) GO TO 8
      IF(DM(NOD1,MS(L,K)).NE.DM(NOD1,NJK)+DD(NJK,MS(L,K))) GO TO 8
      IT=IT+1
      MS(MS(L,K),IT)=NJK
8     CONTINUE
      K=K+1
      GO TO 9
5     CONTINUE
10    RETURN
4     MS(NOD2,1)=NOD1
      GO TO 10
      END
```

```
      SUBROUTINE MDM(N,DD,DM,INF)
      REAL INF
      DIMENSION DD(N,N),DM(N,N)
      DO 1 L=1,N
      DO 1 K=1,N
      DM(L,K)=DD(L,K)
      DO 2 L=1,N
      DD(L,L)=0
      DO 3 J=1,N
      DO 3 I=1,N
      IF(DM(I,J).EQ.INF) GO TO 3
      DO 3 K=1,N
      IF(DM(I,J)+DM(J,K).GE.DM(I,K)) GO TO 3
      DM(I,K)=DM(I,J)+DM(J,K)
      CONTINUE
      DO 4 L=1,N
      DM(L,L)=0
      RETURN
      END
```

BINARY TREES, AN EULER'S PROBLEM AND FINITE SEQUENCES OF NUMBERS

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S subject classification: 68P05, 05C05, 51D20

REZUMAT. — Arbori binari, o problemă a lui Euler şi secvenţe finite de numere. În acest articol se studiază relaţia care există între arbori binari, o problemă a lui Euler privind împărţirea unui poligon convex în triunghiuri prin diagonale ce nu se intersectează în interiorul poligonului, şi o mulţime finită de secvenţe de numere. Se deduce faptul că o corespondenţă biunivocă se poate stabili între următoarele mulţimi: A_n = mulţimea arborilor binari cu n noduri, T_{n+2} = mulţimea posibilităţilor de a împărţi un poligon convex cu $n+2$ laturi în triunghiuri prin diagonale ce nu se intersectează în interiorul poligonului, şi mulţimea $S_n = \{(s_1, s_2, \ldots, s_{2n+1}) \mid s_1 = s_{2n+1} = 1;\ |s_i - s_{i+1}| = 1$, pentru $i = 1, 2, \ldots, 2n;\ s_j \in N^*, j = 2, 3, \ldots, 2n\}$.

Different properties and applications of the binary trees are studied in [2], where one shows that the number of elements of A_n is:

$$\frac{1}{2n+1} \cdot \binom{2n+1}{n} \tag{1}$$

In [1] there is studied and solved an Euler's problem, which requests to determine the number of possibilities to divide a convex polygon with n sides into triangles by diagonals which do not cross each other inside the polygon. Denoting by T_n the set of these possibilities, the number of elements of T_{n+2} is also given by (1).

In [3] we have studied the set:

$$S_n = \{(s_1, s_2, \ldots, s_{2n+1}) \mid s_1 = s_{2n+1} = 1;\ |s_i - s_{i+1}| = 1,$$
$$i = 1, 2, \ldots, 2n;\ s_j \in N^*,\ j = 2, 3, \ldots, 2n\},$$

while in [4] we used this set to analyse an extremal problem concerning the property of consecutive retrieval.

THEOREM 1 [3]. *The number of elements of S_n is given by* (1).

Proof. We shall construct a binary tree labelled as follows:
- the root (lying on the level 1) has the label 1;
- a node with the label i has the two subtrees constructed observing the same rule, the left-subtree has the root labelled with $i-1$, while the root of the right-subtree is labelled with $i+1$;

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— the subtree having the root labelled with 0 is replaced by the em[illegible] subtree.

The number of elements from S_n is equal to the number of paths (in t[illegible] labelled binary tree) joining the root to vertices lying on the level $2n-1$ a[illegible] having the label equal to 1.

If b_{ij} is the number of nodes on the level i which have the label equal t[illegible] j, then:

$$b_{11} = 1;$$
$$b_{ij} = 0 \text{ for } i < j \text{ or } i*j = 0;$$
$$b_{ij} = b_{i-1,\,j-1} + b_{i-1,\,j+1} \text{ for } i \geqslant 1,\ j \geqslant 1,\ i \geqslant j.$$

From the construction manner of this labelled binary tree one obser[illegible] that on an even level we have only nodes with even labels, while on an od[illegible] level we have only nodes with odd labels.

The value precised in the theorem is equal to the value $b_{2n+1,1}$. By [illegible] we construct the numbers (c_{ij}) as follows:

$$c_{i1} = c_{i2} = 0;\ c_{ij} = 0 \text{ for } j > i;$$
$$c_{i,j+2} = b_{2i,2(i-j+1)} \text{ for } 1 \leqslant j \leqslant i; \qquad (3)$$

namely only the values b_{ij} in which the first index is even are considered an[illegible] only the nonzero elements are taken from such a level, but in reverse orde[illegible]

From (2) and (3) we obtain:

$$c_{i1} = c_{i2} = 0,\ i \geqslant 1;\ c_{13} = 1;\ c_{ij} = 0, \text{ for } j > 3;$$
$$c_{ij} = c_{i-1,\,j-2} + 2\cdot c_{i-1,\,j-1} + c_{i-1,j}, \text{ for } 3 \leqslant j \leqslant i+2;$$
$$c_{ij} = 0, \text{ for } j > i+2.$$

By (2) and (3) we obtain: $b_{2n+1,1} = b_{2n,2} = c_{n,n+2}$.

We shall consider further down the set of numbers (d_{ij}):

$$d_{i1} = d_{i2} = 0 \text{ for } i = 1, 2, \ldots;$$
$$d_{13} = 1;\ d_{1j} = 0 \text{ for } j > 3; \qquad (4)$$
$$d_{ij} = d_{i-1,\,j-2} + 2d_{i-1,\,j-1} + d_{i-1,j} \text{ for } i \geqslant 1 \text{ and } j \geqslant 3.$$

One shows by induction with respect to i that for $i \leqslant j \leqslant i+3$ we hav[illegible]

$$c_{ij} = d_{ij} - d_{i,j-2}.$$

Consider the generating function:

$$d(x, y) = \sum_{i,j \geqslant 1} d_{ij}\, x^i y^j.$$

One obtains from (4) that this function fulfils the relationship:

$$x\cdot d(x, y) + 2xy\cdot d(x, y) + xy^2\cdot d(x, y) = d(x, y) - xy^3,$$

ce:

$$d(x, y) = xy^3/\{1 - x(1+y)^2\} = xy^3 \sum_{i \geq 0} \{x(1+y)^2\}^i =$$

$$= xy^3 \sum_{i \geq 0} x^i \cdot \sum_{j=0}^{2i} \binom{2i}{j} y^j = \sum_{i,j \geq 0} \binom{2i}{j} x^{i+1} y^{j+3} = \sum_{i,j=1} \binom{2i-2}{j-3} x^i y^j.$$

From (6) we obtain:

$$d_{ij} = \binom{2i-2}{j-3}, \tag{7}$$

ile (5) and (7) yield:

$$b_{2n+1,1} = c_{1,n+2} = \binom{2n-2}{n-1} - \binom{2n-2}{n-3} = \frac{1}{2n+1} \binom{2n+1}{n},$$

erefore the same value as in (1).

We shall give further down a new proof for theorem 1. Consider the points (i, j), $i, j \in \boldsymbol{N}$, $1 \leqslant i, j \leqslant n+1$. Let d be a path between the vertices $(1, 1)$ and $P(n+1, n+1)$, which fulfils the following two conditions:
) one can pass from a point $P(i, j)$ into $P(i+1, j)$ or $P(i, j+1)$;
) one cannot pass over the main diagonal (only points $P(i, j)$ with $i \geqslant j$ are ed).

Observe that the path d has 2n edges, hence the vertices are:

$$d = [P_1, P_2, \ldots, P_k, \ldots, P_{2n+1}],$$

ile $P_1 = P(1, 1)$, $P_{2n+1} = P(n+1, n+1)$. If the vertex $P(i, j)$ is on the th position in the path d, then $k = i + j - 1$. Corresponding to the path d construct the sequence $s = (s_1, s_2, \ldots, s_{2n+1})$ as follows:

$s_1 = 1$;

if $P(i, j)$ is on the k-th position in the path d, then:

$s_{i+1} = s_i + 1$ if on the $(k+1)$-th position in d is $P(i+1, j)$;

$s_{i+1} = s_i - 1$ if on the $(k+1)$-th position in d is $P(i, j+1)$.

One notices the fact that $s \in S_n$ and the number of paths d is equal to e number of elements from the set S_n.

Let b_{ij} be the number of paths between $P(1, 1)$ and $P(i, j)$ which fulfil above specified conditions (a) and (b). Then:

$$\begin{aligned} &b_{i1} = 1 \text{ for } i \geqslant 1; \\ &b_{ii} = b_{i,i-1} \text{ for } i = 2, 3, \ldots, n+1; \\ &b_{ij} = b_{i-1,j} + b_{i,j-1} \text{ for } 1 < j < i \leqslant n+1. \end{aligned} \tag{8}$$

The last relationship (8) can be extended to the whole net of points (i, j), $i, j \geqslant 1$, if we use the following initial values:

$$b_{1,2} = 0 \text{ and } b_{1,j} = 2 - j \text{ for } j > 2.$$

In order to determine more easily the values b_{ij}, we shall start from following initial values:

$$b_{-1,0} = 0\,;\ b_{0,0} = b_{1,0} = 1\,;\ b_{i,0} = 0 \text{ for } i \geqslant 2\,;$$
$$b_{-1,1} = b_{-1,2} = -1\,;\ b_{-1,j} = 0 \text{ for } j \geqslant 3\,;$$

and use the recurrence relationship:

$$b_{ij} = b_{i-1,j} + b_{i,j-1} \text{ for } i \geqslant 0 \text{ and } j \geqslant 1.$$

With the relationship (10) and initial values (9), the values b_{ij} obt from (8) do not change.

Consider the generating function:

$$B(x, y) = \sum_{i \geqslant 0, j \geqslant 1} b_{ij} x^i y^j.$$

Using (9) and (10), we obtain successively:

$$B(x, y) = \sum_{i \geqslant 0, j \geqslant 1} (b_{i-1,j} + b_{i,j-1}) x^i y^j =$$
$$= x \cdot \sum_{i \geqslant 0, j \geqslant 1} b_{i-1,j}\, x^{i-1} y^j + y \cdot \sum_{i \geqslant 1, j \geqslant 1} b_{i,j-1}\, x^i y^{j-1} =$$
$$= x\left(\sum_{j \geqslant 1} b_{-1,j}\, x^{-1} y^j + \sum_{i \geqslant 1, j \geqslant 1} b_{i-1,j}\, x^{i-1} y^j\right) +$$
$$+ y\left(\sum_{i \geqslant 0} b_{i,0}\, x^i + \sum_{i \geqslant 0, j \geqslant 2} b_{i,j-1}\, x^i y^{j-1}\right) =$$
$$= -y - y^2 + x \cdot \sum_{i \geqslant 0, j \geqslant 1} b_{i,j} x^i y^j + y + xy + y \cdot \sum_{i \geqslant 0, j \geqslant 1} b_{i,j}\, x^i y^j =$$
$$= xy - y^2 + (x + y) \cdot \sum_{i \geqslant 0, j \geqslant 1} b_{i,j}\, x^i y^j = xy - y^2 + (x + y) B(x,$$

One obtains the following expression for the generating function (1

$$B(x, y) = \frac{xy - y^2}{1 - (x + y)},$$

which is successively transformed as follows:

$$B(x, y) = (xy - y^2) \cdot \sum_{i \geqslant 0} (x + y)^i = (xy - y^2) \cdot \sum_{i \geqslant 0} \sum_{j=0}^{i} \binom{i}{j} x^j y^{i-j} =$$
$$= (xy - y^2) \cdot \sum_{i,j \geqslant 0} \binom{i + j}{i} x^i y^j = \sum_{i,j \geqslant 0} \binom{i + j}{i} x^{i+1} y^{j+1} -$$
$$- \sum_{i,j \geqslant 0} \binom{i + j}{i} x^i y^{j+2} = \sum_{i,j \geqslant 1} \binom{i + j - 2}{i - 1} x^i y^j - \sum_{i \geqslant 0, j \geqslant 2} \binom{i + j - 2}{i} x^i$$
$$= \sum_{i \geqslant 1, j \geqslant 2} \left[\binom{i + j - 2}{i - 1} - \binom{i + j - 2}{j}\right] x^i y^j + y \sum_{i \geqslant 1} x^i - \sum_{j \geqslant 2} y^j =$$
$$= \sum_{i \geqslant 1, j \geqslant 2} (i + j - 1)/i \cdot \binom{i + j - 2}{i - 1} x^i y^j + y \sum_{i \geqslant 1} x^i - \sum_{j \geqslant 2} y^j.$$

The values b_{ij} can be determined by identifying the coefficients from the last expression obtained for $B(x, y)$ with those from (11). For $b_{n+1,n+1}$ we obtain:

$$b_{n+1,n+1} = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{2n+1}\binom{2n+1}{n},$$

therefore the value specified by Theorem 1.

DEFINITION The sequence $s = (s_1, s_2, \ldots, s_{2n+1})$ is called symmetrical sequence if $s_i = s_{2n+2-i}$, $i = 1, 2, \ldots, n+1$.

THEOREM 2. *The number of symmetrical sequences of length $2n+1$ is:*

$$\binom{n}{[(n+1)/2]}.$$

Proof. Consider the binary tree constructed at the first proof of theorem **1.** From this construction mode it results that an empty subtree can appear only on an even level (the root is on the level 1) and is the subtree of a node with the label 1. The number of nodes with the label 1 on the level $2k+1$ is provided by theorem 1 as being:

$$a_{k+1} = \frac{1}{2k+1}\cdot\binom{2k+1}{k}. \tag{12}$$

The number of symmetrical sequences of length $2n+1$ is equal to the number of sequences $(s_1, s_2, \ldots, s_{n+1})$, $s \in N^*$ for $j = 1, \ldots, n+1$, $s_1 = 1$ and $|s_i - s_{i+1}| = 1$ for $i = 1, \ldots, n$. This number of sequences is equal to the number of nodes on the level $n+1$ of the binary tree. If b_k is the number of nodes on the level k, then we obtain:

$$\begin{aligned} &b_0 = 1/2;\ b_{2k+1} = 2b_{2k} \text{ for } k = 1, 2, \ldots \\ &b_{2k} = 2b_{2k-1} - a_k \text{ for } k = 1, 2, \ldots \end{aligned} \tag{13}$$

since on the level $2k$ there exist a_k empty subtrees.

From (13) we obtain:

$$b_{2k+2} = 4b_{2k} - a_k. \tag{14}$$

Consider the following generating function:

$$g(x) = \sum_{k\geq 0} b_{2k}x^k. \tag{15}$$

Using (14) we obtain successively:

$$4g(x) - \sum_{k\geq 0} a_k x^k = \sum_{k\geq 0}(4b_{2k} - a_k)x^k = \sum_{k\geq 0} b_{2k+2}x^k =$$

$$= \frac{1}{x}\sum_{k\geq 0} b_{2k+2}x^{k+1} = \frac{1}{x}\left(\sum_{k\geq 0} b_{2k}x^k - b_0\right) = \frac{1}{x}\cdot(g(x) - 1/2).$$

It is shown in [2] that for the values a_k we have:

$$\sum_{k\geq 0} a_k x = \frac{1}{2k}(1 - \sqrt{1-4x}).$$

Then:

$$4g(x) - \frac{1}{2x}(1 - \sqrt{1-4x}) = \frac{1}{x}(g(x) - 1/2).$$

hence:

$$g(x) = \frac{1}{2\sqrt{1-4x}} = \frac{1}{2}\sum_{k\geq 0}\binom{-1/2}{k}(-4x)^k =$$

$$= \frac{1}{2}\sum_{k\geq 0}(-1)^k \frac{(2k)!}{4^k \cdot k! \cdot k!}(-4x)^k = \frac{1}{2}\sum_{k\geq 0}\binom{2k}{k}\cdot x^k.$$

From (13), (14) and the last above relationship, one obtains:

$$b_{2k} = \frac{1}{2}\binom{2k}{k}), \; b_{2k+1} = \binom{2k}{k}.$$

If $n + 1 = 2k$, then:

$$b_{n+1} = b_{2k} = \frac{1}{2}\binom{2k}{k} = \binom{2k-1}{k} = \binom{n}{[(n+1)/2]}.$$

If $n + 1 = 2k + 1$, then:

$$b_{n+1} = b_{2k+1} = \binom{2k}{k} = \binom{n}{[n/2]} = \binom{n}{[(n+1)/2]},$$

since n is even in this case.

These relationships show that theorem 2 is true.

The fact that the sets A_n, T_{n+2}, S_n have the same number of eleme suggested us the idea to construct bijection correspondences between the coup of sets (A_n, S_n) and (T_{n+2}, S_n): from this, a bijection correspondence betw A_n and T_{n+2} can also be deduced. These bijection correspondence point o the fact that the elements from A_n and T_{n+2} can be represented (encoded) elements from S_n.

In [2] one gives the following recursive definition for the traversal binary tree in preorder:

a) traverse the root;
b) traverse the left subtree in preorder;
c) traverse the right subtree in preorder.

Let $A \in A_n$ be a binary tree with n nodes. Since every vertex has tw subtrees, one deduces the fact that are $2n$ subtrees in A.

LEMMA 1 [2]. *If A is a binary tree with n nodes, then, out of the $2n$ subtr $n + 1$ are empty subtrees.*

The *proof* is given in [2].

Let $B = (B_1, B_2, \ldots, B_{2n})$ the order in which the $2n$ subtrees appear in traversal of the binary tree A in preorder. Starting from the binary tree A, ence from the set of subtrees B, we can construct a vector binary values $= (V_1, V_2, \ldots, V_{2n})$ according to the following rule, for $i = 1, 2, \ldots, n$:

$$V_i = \begin{cases} 1, \text{ if } B_i \text{ is nonempty subtree,} \\ 0, \text{ in otherwise;} \end{cases} \tag{16}$$

Using the vector V, a vector $s = (s_1, s_2, \ldots, s_{2n+1})$ can be constructed as llows:

$$s_1 = 1;\ s_2 = 2;$$

$$s_{i+1} = \begin{cases} s_{i+1} + 1, \text{ if } V_i = 1; \\ s_{i+1} - 1, \text{ if } V_i = 0; \text{ for } i = 1, 2, \ldots, 2n - 1. \end{cases} \tag{17}$$

Example. For $n = 3$ we shall represent corresponding to the five binary rees, the vectors V and s constructed according to (16) and (17):

$$V_1 = (1, 1, 0, 0, 0, 0);\ s_1 = (1, 2, 3, 4, 3, 2, 1);$$
$$V_2 = (1, 0, 1, 0, 0, 0);\ s_2 = (1, 2, 3, 2, 3, 2, 1);$$
$$V_3 = (1, 0, 0, 1, 0, 0);\ s_3 = (1, 2, 3, 2, 1, 2, 1);$$
$$V_4 = (0, 1, 1, 0, 0, 0);\ s_4 = (1, 2, 1, 2, 3, 2, 1);$$
$$V_5 = (0, 1, 0, 1, 0, 0);\ s_5 = (1, 2, 1, 2, 1, 2, 1);$$

Using the vector $V = (V_1, V_2, \ldots, V_{2n})$, we determine a vector $W = (W_1, W_2, \ldots, W_{2n})$ according to the following rule:

$$W_i = 1 \text{ if } V_i = 1; \text{ and } -1 \text{ if } V_i = 0; \text{ for } i = 1, 2, \ldots, 2n \tag{18}$$

With the vector W, the vector s constructed according to (16) and (17) can be determined more easily as follows:

$$s_1 = 1;\ s_2 = 2;\ s_{i+2} = s_{i+1} + W_i, \text{ for } i = 1, 2, \ldots, 2n - 1. \tag{19}$$

LEMMA 2. (a) $|W_1 + W_2 + \ldots + W_k \geqslant -1$ *for* $k = 1, \ldots, 2n - 1$;

(b) $|W_1 + W_2 + \ldots + W_{2n-1} = -1$.

Proof. For the n vertices of the binary tree, there will exist $n - 1$ nonempty subtrees and $n + 1$ empty subtrees (according to Lemma 1). From this fact one deduces that $n - 1$ values from vector W are equal to 1, while $n + 1$ values are equal to -1. Since $W_{2n} = -1$, it results that the relationship (b) is true.

The relationship (a) will be proved by induction with respect to n.

For $n = 1$, the binary tree has a single node, hence $W = (-1, -1)$. In this case the relationship is true. Suppose that it is true for all binary trees with at most n nodes, and consider a binary tree with n nodes ($n > 1$). The binary tree consists of a root and two subtrees: B_1 and B_2. If $B_1 = \emptyset$, then $B_2 = \emptyset$, hence $W_1 = -1$, $W_2 = 1$. In this case the relationship (a) is true for

$k = 1$ and $k = 2$, and, in addition $W_1 + W_2 = 0$. The vector $(W_3, W_4,$ $\ldots, W_{2n})$ corresponds to the binary tree B_2, with $n - 1$ nodes, for which the relationship is true (according to the hypothesis of the induction).

If $B_1 \neq \emptyset$, let m be the number of nodes from B_1. Since $m < n$, it res that for B_1 the relationship (a) is true. Since $B_1 \neq \emptyset$ we obtain that W_1 = and $(W_2, W_3, \ldots, W_{m+1})$ corresponds to the tree B_1, hence $W_1 + (W_2 + W_3 + \ldots + W_{m+1}) = 0$. So, the relationship (a) is true in this case, too.

The vector V with the values 0 and 1 (or the vector $s \in S_n$) can be to the representation of the binary tree A. We shall describe further do the algorithm which determines the vector V starting from the binary tree. For this purpose, suppose that the binary tree is stored with the vectors **LLINK** and **RLINK** (according to [2]) (the binary tree is stored only under the for of the two vectors since the informations associated to the vertices are n needed). The link towards an empty subtree will be precised by a zero va in the two vectors.

The algorithm uses a stack **ST**. Suppose that the root of the tree is stor at the address **R**.

Algorithm 1:

```
i : = 0 ; {i is index for the vector V}
k : = 1 ; {k is index for the stack ST}
ST[1] : = R ;
P : = LLINK[R] ; {P runs through the nodes of the tree}
Cont : = true ; {Cont is a boolean variable for the cycle of the algorithm}
while Cont do
  if P=0 then
    begin i : =i+1 ; V[i] : =0 ;
      if k=0 then Cont : =false
       else begin P : =ST[k] ; k : =k-1 ; P : =RLINK[P] end
    end
   else begin i : =i+1 ; V[i] : = 1 ;
      k : =k+1 ; ST[k] : =P ; P : =LLINK[P]
     end ;
```

One deduce from this algorithm the fact that the vector V was determined according to (16). If one wishes the determination of the vector s, then the relationship (17) will be used.

The algorithm which follows will determine the binary tree corresponding to the vector $V = (V_1, V_2, \ldots, V_{2n})$. It uses the significances of the elements from V, precised by the relationship (16). In this algorithm **ST** represents stack. The binary tree will be represented by means of the vectors **LLINK** and **RLINK**, while the root be stored at the address 1.

Algorithm 2:

```
i : =1 ; {i is address of the node from the binary tree to be constructed}
k : =1 ; {k is index for the stack ST}
ST[1] : =1 ; {the first node is put into the stack}
t : =1 ; {t=1 if LLINK will be completed for the node i, and t=0 if RLINK
         will be completed}
```

```
for j:=1 to 2n do
if t=1 then
 if V[j]=1 then
  begin LLINK[i]:=i+1; i:=i+1; k:=k+1; ST[k]:=i end
 else begin LLINK[i]:=0; t:=0; i:=ST[k]; k:=k-1 end
else if V[j]=1 then
  begin RLINK[i]:=i+1; i:=i+1; k:=k+1 ST[k]:=i; t:=1 end
 else begin RLINK[i]:=0; i:=ST[k]; k:=k-1 end;
```

Since W and V were constructed starting from the vector $s \in S_n$, for every $k = 1, \ldots, 2n$ we have: $W_1 + W_2 + \ldots + W_k \geqslant -1$, therefore each time we extract an element from the stack (by i : = ST[k]; k : = k − 1) this fact is possible.

THEOREM 3. *Both the algorithm 1 and the algorithm 2 establish a bijection correspondence between the sets A_n and S_n.*

Proof. The vector s can also be constructed by means of the relationship (16), (18) and (19). From (19) and Lemma 2 it results that the vector s belongs to S_n. From the fact that the traversal of a binary tree in preorder is unique, and from the determination mode of the vector V, it results that for a binary tree A we can determine a unique vector $s \in S_n$. This vector is determined by means of the algorithm 1 and relationship (16).

Conversely, considering a vector $s = (s_1, s_2, \ldots, s_{2n-1}, s_{2n})$ from S_n, we can determine a vector $W = (W_1, W_2, \ldots, W_{2n-1}, W_{2n})$ as follows:

$$W_i = s_{i+2} - s_{i+1} \text{ for } i = 1, 2, \ldots, 2n - 1;$$
$$W_{2n} = -1.$$

From the vector W we determine the vector $= (V_1, V_2, \ldots, V_{2n})$ as follows:

$$V_i = 1 \text{ if } W_i = 1, \text{ and } 0 \text{ if } W_i = -1; \text{ for } i = 1, \ldots, 2n.$$

By means of the algorithm 2 one constructs uniquely a binary tree, using the vector V.

Let $P = P_1 P_2 \ldots P_{n+2}$ be a convex polygon with $n + 2$ vertices and let $T \in T_{n+2}$ be a partition of the polygon P into triangles by diagonals which do not cross each other inside the polygon. The elements t_k from T are triangles, hence $t_k = (P_{a(k)}, P_{b(k)}, P_{c(k)})$, where $P_{a(k)}$, $P_{b(k)}$, $P_{c(k)}$ are vertices of the polygon P. Suppose that:

$$\begin{gathered} a(k) < b(k) < c(k), \text{ for } k = 1, 2, \ldots, n; \\ c(i) \leqslant c(j), \text{ for } 1 \leqslant i < j \leqslant n; \\ \text{if } c(i) = c(j), \text{ then } a(i) < a(j), \text{ for } 1 \leqslant i < j \leqslant n. \end{gathered} \tag{20}$$

Using (20), we obtain a unique indexation of the elements $\{t_1, \ldots, t_n\}$ from T.

LEMMA 3. *Let $c' = (c'_1, c'_2, \ldots, c'_n)$ and $c'' = (c''_1, c''_2, \ldots, c''_n)$ be the vectors associated to the partitions $T', T'' \in T_{n+2}$, for which the conditions (20) are fulfilled. If $T' \neq T''$, then $c' \neq c''$ (hence the only vector c is sufficient in order to precise a partition).*

Proof. The lemma will be proved by induction with respect to n. $n = 2$, the two possible partitions are: $T' = \{(P_1, P_2, P_3); (P_1, P_3, P_4)\}$; $T'' = \{(P_1, P_2, P_4); (P_2, P_3, P_4)\}$, one obtains the vectors $c' = (3, 4)$ and $c'' = (4, 4)$, which are different.

Suppose that the lemma is true for polygons with at most $n + 1$ vertices and we shall prove the lemma for polygons with $n + 2$ vertices. Let $T' = (t'_1, \ldots, t'_n)$ and $T'' = (t''_1, \ldots, t''_n)$ be two different partitions from T_{n+2} and let c' and c'' be the vectors associated to these two partitions.

If $t'_1 = t''_1$ (hence $c'_1 = c''_1$), then from $T' \neq T''$ we obtain that $(t'_2, \ldots, t'_n) \neq (t''_2, \ldots, t''_n)$. From the hypothesis of the induction we obtain that $(c'_2, \ldots, c'_n) \neq (c''_2, \ldots, c''_n)$, hence $c' \neq c''$.

If $t' = (P_{i1}, P_{j1}, P_{k1}) \neq t'' = (P_{i2}, P_{j2}, P_{k2})$, then the following two cases must be analysed:

(a) $k1 = c'_1 \neq k2 = c''_1$, therefore $c' \neq c''$;

(b) $k1 = c'_1 = k2 = c''_1$. Denote $k = k1$, $i = i1$, $j = i2$. Since $t'_1 \neq t''_1$, we obtain that $i \neq j$, and suppose that $i < j$. From (20) and from the fact that P_iP_k is a diagonal and cannot cross other edges of the triangles belonging to the partition T', we obtain that t'_1 is of the form (P_i, P_{i+1}, P_k). Analogously, we obtain that t''_1 is of the form (P_j, P_{j+1}, P_k). From the same conditions and from the fact that $i < j < k$, one obtains:

$$t'_2 = (P_{i+1}, P_{i+2}, P_k);\ t''_2 = (P_{j+1}, P_{j+2}, P_k);\ t'_3 = (P_{i+2}, P_{i+3}, P_k);$$

$$t''_3 = (P_{j+2}, P_{j+3}, P_k);\ \ldots;\ t'_{k-j-1} = (P_{k+i-j-2}, P_{k+i-j-1}, P_k);$$

$$t''_{k-j-1} = (P_{k-2}, P_{k-1}, P_k);\ t'_{k-j} = (P_{k+i-j-1}, P_{k+i-j}, P_k);$$

$t''_{k-j} = (P_{k1}, P_{k2}, P_{k3})$, with $k3 \neq k$, hence $c''_{k-j} \neq c'_{k-j}$. It follows that $c' \neq c''$.

We shall describe further down an algorithm which determines a vector $s = (s_1, s_2, \ldots, s_{2n+1}) \in S_n$ starting from $T \in T_{n+2}$. Suppose that the partition T is precised by the vector $c = (c_1, \ldots, c_n)$.

Algorithm 3:

```
s[1]:=1; s[2]:=2; j:=2; c[0]:=3;
for k:=1 to n do
  begin {a} d[k]:=c[k]-c[k-1];
        {b} if d[k]>0 then
              for i:=1 to d[k] do
                begin j:=j+1; s[j]:=s[j-1]+1 end;
        {c} j:=j+1; s[j]:=s[j-1)-1
  end;
```

The algorithm which follows determines a partition $T \in T_{n+2}$ for the polygon $P = P_1P_2 \ldots P_{n+2}$ starting from a vector $s \in S_n$. The partition will be precised by the vertices $(P_{a(k)}, P_{b(k)}, P_{c(k)})$, $k = 1, \ldots, n$, of the triangles which constitute the partition. The algorithm uses a stack **ST.**

Algorithm 4:

```
k:=2; {k is index for the stack ST}
ST[1]: =1; ST[2]: =2; {the indices of the first two vertices of the poligon P
                       are introduced into the stack}
j:=2; {j is index for the vertices of the polygon P}
m:=0; {m is index for the elements of the partitions T}
for i: =2 to 2n+1 do
 if s[i]>s[i-1] then
  begin k: =k+1; j: =j+1; ST[k]: =j end
 else begin m: =m+1
    a[m]: =ST[k-2]; b[m]: =ST[k-1]; c[m]: =ST[k];
    ST[k-1]: =ST[k]; k: =k-1
     {the last but one element of the stack is removed}
 end;
```

THEOREM 4. *Both the algorithm 3 and the algorithm 4 establish a bijection correspondence between the sets S_n and T_{n+2}.*

Proof. Let $t_1, t_2, \ldots, t_k$ be the first k triangles from T. These ones are formed with the vertices belonging to the set:

$$M = \bigcup_{i=1}^{k} \{P_{a(i)}, P_{b(i)}, P_{c(i)}\},$$

in which the greatest index of the vertices from M is $c(k)$. In a set with $c(k)$ vertices we can form at most $c(k) - 2$ triangles, hence $k \leqslant c(k) - 2$, or $c(k) \geqslant k + 2$.

The vertex P_{n+2} belongs to at least one triangle, and from (20) it results that the triangle t_n contains the vertex P_{n+2}, hence $c_n = n + 2$. The two assignements of the cycle from the point {b} of the algorithm 3 are performed $n - 1$ times since:

$$d_1 + d_2 + \ldots + d_n = c_n - c_0 = n + 2 - 3 = n - 1.$$

In the algorithm 3 there are calculed $2n + 1$ values for the vector s, since two values are calculated at the beginning of the algorithm, $n - 1$ values at the point {b} of the cycle with respect to k, and n values at the point {c} of this cycle.

For a certain value of k, at the point {c} of the algorithm 3 one calculates a value s_j, and:

$$s_j = s_{j-1} - 1 = s_2 + (d_1 + d_2 + \ldots + d_k) - k =$$
$$= 2 + c[k] - c[0] - k = c[k] - k - 1 \geqslant k + 2 - k - 1 \geqslant 1.$$

Therefore $s_j \geqslant 1$ for every value $j = 1, 2, \ldots, 2n + 1$.

From the above reasons we deduce that for a $T \in T_{n+2}$ one obtains a sequence $s \in S_n$.

For a certain value of i from the algorithm 4, there will exist in the stack $1 + s_{i-1}$ vertices. For every i with $s_i < s_{i-1}$ there will exist in the stack at least three vertices (from $1 < s_i < s_{i-1}$ we obtain that $1 + s_{i-1} \geqslant 1 + 2 = 3$)

therefore one may construct a triangle. From the construction manner of sequence $s \in S_n$, it results that there exist n values of i for which $s_i <$ hence n triangles are constructed, while for a sequences s a single partition is constructed. If s', $s'' \in S$, $s' \neq s''$, then let i be the first value for w the corresponding possitions into the two sequences differ, therefore $s'_i \neq$ From the construction manner of the elements from S_n follows that $s'_i < s'_{i-1}$ $s''_i > s''_{i-1}$, or conversely: $s'_i > s'_{i-1}$ and $s''_i > s''_{i-1}$. We deduce from this that the partitions T' and T'' constructed by means of the sequences s' s'' are different.

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THE SIMPLEX ALGORITHM IS HOWEVER BETTER THAN KHACHIYAN'S ALGORITHM

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REZUMAT. — Algoritmul simplex este, totuşi, mai bun decît algoritmul lui Khachiyan. Lucrarea prezintă comparativ rezultatele obţinute, pe un calculator I 102 F, pentru diferite instanţe ale problemei de optimizare liniară atît cu algoritmul simplex cît şi cu o variantă îmbunătăţită a algoritmului lui Khachiyan. Varianta algoritmului simplex este cea dată de Zukhovitskii şi Avdeyeva [3]. Cu toate că algoritmul simplex efectuează un număr mare de iteraţii, acest algoritm are timp de lucru mai scurt. De asemeni, s-a arătat că varianta iniţială [2, 4] a algoritmului lui Khachiyan conduce la depăşiri de reprezentare a numerelor în calculator pentru dimensiuni relativ mici (10—12).

Algorithm for Linear Optimization Problem. In order to compare the ...k times for the solution of some linear optimization problem instances, we ...orated programs for the simplex algorithm and for two algorithms which ...e Khachiyan's algorithm as basic iteration, the last two ones using respectively the sliding objective method and the bisection method [2].

As it is known, the linear optimization problem can be stated in different ...ivalent forms. We considered that this problem is given in the most general ...n:

$$\max \{c^T x \mid Ax < (=)\ b\}. \quad (1)$$

...refore the programs allow the data input for any instance, without requi... certain preliminary transformations of this one. Nevertheless, one must ...ly construct a linear optimization problem in the canonical form:

$$\max \{-q^T y + Q \mid Dy \leqslant p,\ y \geqslant 0\} \quad (2)$$

...se transformations are performed by expressing the free variables (for which ... condition $x_i \geqslant 0$ is not required) as functions of the other variables, and ...n removing the equalitytype constraints (null rows).

The simplex algorithm [3] performs modified Jordan steps (MJS) in the ...plex table corresponding to the problem (2), which has the form:

	$-y_1$	$-y_2$	$\dots$	$-y_n$	1
y_{n+1}	$d_{n+1,1}$	$d_{n+1,2}$	$\dots$	$d_{n+1,n}$	p_{n+1}
$\vdots$	$\dots$	$\dots$	$\dots$	$\dots$	$\vdots$
y_m	d_{m1}	d_{m2}	$\dots$	d_{mn}	p_m
f	q_1	q_2	$\dots$	q_n	Q

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The description of the simplex algorithm is the following one:

step 1: (remove the independent variables)
— if there are free variables, then perform MJS with pivot elements from these columns and (if possible) from the null rows
— if Jordan step can no longer be performed and there still free independent variables, then either the optimum value does not depend of these ones (if in the row of f there appear zero on all corresponding columns), or the problem has no finit optimum (otherwise).

step 2: (remove the null rows)
— perform **MJS** with pivot elements from the null rows by removing the pivot columns from the simplex table.

step 3: (determine an admissible solutions)
— if there exists $p_r < 0$ with $n + 1 \leqslant r \leqslant m$, then
  if rthere exists $d_{rj} < 0$, then
    determine i_o which fulfils (3)
    perform a **MJS** with the pivot element d_{i_0j}
    go to step 3
  else stop, the problem is unsolvable
else go to step 4.

step 4: (determine an optimum solution)
— if there exists $q_j < 0$, then
  if there exists $d_{ij} > 0$, then
    determine i which fulfils (3)
    perform a **MJS** with the pivot element d_{i_0j}
    go to step 4
  else stop, the problem has not finite optimum
else stop, the optimum solution is obtined for
$y_1 = y_2 = \ldots = y_n = 0$ and $y_i = p_i$, $i = n + 1, \ldots, m$.

$$\frac{p_{i_0}}{d_{i_0}} = \min\left\{\frac{p_i}{d_{ij}} \,\middle|\, \frac{p_i}{d_{ij}} > 0,\ n + 1 \leqslant i \leqslant m\right\}$$

As basic operation in the next algorithms for solving the linear optimization problem we used the Khachiyan's algorithm variant given in [5]. The variant was proposed by Konig and Pallaschke and completed by the author with a procedure for the initial ellipsoid determination. This subalgorithm will be denote by K, while I_a denote the active restriction set.

The algorithm which solves problem (1) by using the silding objective method is:

step 1: (determine an admissible solution)
— determine with K an admissible solution x_0 and an ellipsoid $E_0 = (x_0, A_0)$ containing the optimum solution
— let $d_{m+1} = \sqrt{a_{m+1}^T A_0 a_{m+1}}$ and $t = o$.

step 2: (stopping test)

— let $y = x_t + \theta s$, where $s \in \{0, -a_{m+1}, -A_t a_{m+1}\}$,
and $\theta = \min\left\{\dfrac{b_i - a_i^T x_t}{a_i^T s} \,\middle|\, a_i^T s > 0,\ i \in I_a\right\}$.

— let $z = A_t a_{m+1}/d_{m+1}$, $f_i = a_i^T z$, $i \in I_a$
$e_i' = a_i^T y - b_i$, $i \in I_a$, $L = \{i \in I_a \mid f_i < 0\}$
$\xi = \min\{e_i'/f_i \mid i \in L\}$,

— compute η as in step 4 of K (using the values computed at this step),

— if $\eta - \xi < \varepsilon$, then stop and chose y as optimum point.

step 3: (chose of the new system)

— compute $b_{m+1} = a_{m+1}^T y$, $\xi = -\theta a_{m+1}^T s/d_{m+1}$

— let $k = m + 1$,

— solve $a_i x \leqslant b_i$, $i \in I_a U \{m + 1\}$, with K starting with step 3 of this one,

— go to step 2.

We programed for this algorithm the variants 1 (corresponding to the ise $s = 0$), 2 (for $s = -a_{m+1}$) and 3 (for $s = -A_t a_{m+1}$).

The algorithm which solves the linear optimization problem by means the bisection method and uses the algorithm K consists of:

step 1: (determine an admissible solution)

— determine with K an admissible solution x_0 and an ellipsoid $E_0 = (x_0, A_0)$ which contains the optimum solution,

— let $d_{m+1} = \sqrt{a_{m+1}^T A_0 a_{m+1}}$.

step 2: (problem choise in compatibility case)

— let $z = A_0 a_{m+1}/d_{m+1}$, $f_i = a_i^T z$, $i \in I_a U \{m + 1\}$,
$L = \{i \in I_a \mid f_i < 0\}$, $\xi = \min\{e_i/f_i \mid i \in L\}$,

— compute η as in step 4 of K,

— if $\eta - \xi < \varepsilon$, then stop and chose x_0 as optimum point
else chose $b_{m+1} = a_{m+1}^T x_0 - \dfrac{2\xi + \eta}{3} d_{m+1}$,
save data (x_0, A_0, e_i, d_i) for old ellipsoid,
let $b_{m+1} = b_{m+1}$, $\xi = \xi$, $\eta' = \eta$.

step 3: (problem solution for the new bisection)

— solve $a_i x \leqslant b_i$, $i \in I_a U \{m + 1\}$, with K,

— save the indices of the removed hyperplanes,

— if the system is compatible, then go to 2 with the last ellipsoid denote (x_0, A_0),
else go to step 4,

— if during calculation card $(I_a) \leqslant n$, then go to step 5.

step 4: (problem choise in incompatibility case)

— load again the lastly saved data (x_0, A_0, e_i, d_i)

— let $\eta = b'_{m+1}$, $\xi = \xi'$ and $b_{m+1} = a^T_{m+1} x_0 - \frac{2\xi + \eta}{3} d_{m+1}$,

— if $\eta - \xi < \varepsilon$, then stop and chose x_0 as optimum point else $b'_{m+1} = b_{m+1}$ and go to step 3.

step 5: (end of step 3)

— if card $(I_a) = n$, then solve $a_i x = b_i$, $i \in I_a$, and if the solution x fulfils all constraints removed at the last s[illegible] then x is optimum point, stop, else or if card $(I_a) < n$, go to step 4.

Test Problems. As test instances we chosen [1]:

$$c^T = (h^{n-1}, h^{n-2}/p, \dots, 1/p^{n-1}),$$

$$b^T = (1, p, p^2, \dots, p^{n-2}),$$

$$A = \begin{vmatrix} 1 & 0 & \dots & 0 \\ 2hp & 1 & \dots & 0 \\ 2h^2p^2 & 2hp & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 2h^{n-1}p^{n-1} & 2h^{n-2}p^{n-2} & \dots & 1 \end{vmatrix}$$

which are „wrong" examples for the simplex algorithm, i.e. for some va[illegible] of p and h this algorithm needs $2^n - 1$ iterations to reach the optimum s[illegible] tion $x^T = (0, 0, \dots, p^{n-1})$. The presented tests were performeted for the chois[illegible]

a) $p = 2$, $h = 0.625$,

b) $p = 2.5$, $h = 0.5$,

c) $p = 5$, $h = 0.25$.

The obtained results are listed in the table:

n	Simplex algorithm		Objective sliding method 1		2		3		Bisection method	
	Steps	Time	Pr.	Time	Pr.	Time	Pr.	Time	Pr.	Time
0	1		2		3		4		5	
a) p = 2; h = 0.625										
4	7	0.02	9	10.56	5	9.42	4	9.20	3	8.84
5	11	0,04	12	11.40	7	9.74	4	9.88	3	10.58
6	15	0.06	16	14.90	11	12.42	7	10.98	3	14.14
7	21	0.12	22	19.26	13	17.44	13	17.40	4	23.06
8	26	0.18	27	24.76	20	21.96	16	20.24	3	21.80
9	31	0.22	28	31.48	21	29.64	15	28.20	3	28.82
10	34	0.30	40	51.20	27	43.64	23	42.20	3	33.28
11	42	0.40	52	73.92	34	68.34	23	66.84	3	46.82
12	46	0.52	61	96.48	43	91.38	40	95.10	3	43.72

Table continued

0	1		2		3		4		5	
p = 2.5 ; h = 0.5										
4	15	0.04	11	10.62	7	9.34	6	8.10	3	8.62
5	31	0.10	13	10.48	11	10.12	8	9.94	3	9.94
6	63	0.22	22	14.20	15	13.70	12	14.48	3	12.12
7	96	0.42	24	16.84	18	16.44	16	19.12	4	23.04
8	98	0.52	36	29.44	28	26.68	21	29.06	3	20.64
9	102	0.64	48	42.26	35	40.96	24	36.96	3	26.36
10	110	0.78	51	58.82	37	53.12	33	53.54	3	27.80
11	240	2.00	63	92.68	48	86.34	41	87.86	3	34.98
12	243	2.36	74	114.28	63	109.12	43	110.38	3	32.16
p = 5 ; h = 0.25										
4	15	0.04	16	10.44	11	8.88	9	8.64	4	8.90
5	31	0.10	24	11.62	18	10.26	11	10.42	3	8.92
6	63	0.24	37	17.26	30	14.22	21	15.66	3	11.28
7	127	0.58	51	24.12	38	20.98	22	21.32	3	10.96
8	255	1.40	61	34.82	53	33.22	38	33.68	28	68.40
9	511	3.24	78	52.84	63	53.14	41	49.72		
10	1023	7.56	80	88.98			48	86.80		
11	2047	17.44			82	117.21	62	107.14		
12	4095	41.06	82	102.72						

In this table, the computing times are expressed in seconds, while for the rithms which use Khachiyan's algorithm there was given the number problems (Pr.) (system of inequations) to be solved until the solution of considered problem was reached, and not the total number of cuts (iterans).

Concluding, we can specify that the simplex algorithm, although exponential, es results in a time shorter than the algorithms based on Khachiyan's rithm. Such last algorithms lead sometimes to the impossibility of solving problems (see, e.g. Table), due to overflows of numer representation in puter.

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