# STUDIA
## UNIVERSITATIS BABEŞ-BOLYAI

## MATHEMATICA

1

1990



CLUJ-NAPOCA

Anul XXXV 1990

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STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXXV, 1, 1990

# CONTINUOUS SELECTIONS FOR MULTIFUNCTIONS AND THE PICARD PROBLEM FOR MULTIVALUED EQUATIONS

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Dedicated to Professor Á. Pál on his $60^{th}$ anniversary

*Received: May 18, 1989*
*AMS subject classification: 26E25*

**REZUMAT. — Selecţii continue pentru multifuncţii şi problema lui Picard pentru ecuaţii multivoce.** În lucrare se demonstrează o teoremă de existenţă a unei selecţii continue pentru o multifuncţie $F$ definită pe o submulţime compactă a lui $\mathbf{R}^{n+2}$. Se presupune că $F$ este o aplicaţie continuă ale cărei valori sînt submulţimi compacte, nevide, nu neapărat convexe. Ca o consecinţă, este obţinut un rezultat privind existenţa soluţiei problemei lui Picard pentru ecuaţia multivocă $\partial^2 z/\partial x\partial y \in F(x, y, z)$.

1. **Introduction.** In this paper we prove an existence theorem of a continuous selection for a multifunction $F$ defined in a compact subset of $\mathbf{R}^{n+2}$ and taking compact nonempty values, not necessarily convex. The theorem establishes the existence of a continuous selection for each of the functions $(x, y) \to F(x, y, z(x, y))$, with respect to a given family $\{z(x, y)\}$ of continuous functions $(x, y) \to z(x, y)$. This result is stronger that Theorem 1 [5]. It is analogous of [1].

As corollary, we obtain the existence of a solution for the Picard problem, associated with the multivalued hyperbolic equation $\frac{\partial^2 z}{\partial x\, \partial y} \in F(x, y, z)$, [4].

2. **Preliminary results [5].** Let be the multifunction $F: D \times B \to \text{comp}X$, where $D = [0, a] \times [0, b]$, $B$ is the closed ball centered in origin of $\mathbf{R}^n$ and with radius $c = M_1 + Mab$, $M_1$ given by (2.3), $M$ given by (2.4), $X$ is the closed ball centered in origin of $\mathbf{R}^n$ and with radius $M$. Obviously, $X$ is a compact space for the metric $d$ induced on $X$ by the norm of $\mathbf{R}^n$. Let $H$ be the Hausdorff—Pompeiu metric [3] on comp$X$ induced by $d$. Then comp$X$ is a compact metric space for $H$.

Let $\mathcal{C}(D; \mathbf{R}^n)$ be the Banach space of continuous functions from $D$ into $\mathbf{R}^n$ and $\mathcal{L}^1(D; \mathbf{R}^n)$ the Banach space of equivalence classes of Lebesgue integrable functions on $D$ and valued in $\mathbf{R}^n$.

Let the following hypotheses be satisfied:

($H^0$) The curve $\gamma: x = \psi(y)$, $0 \leqslant y \leqslant b$, is defined by the function $\psi \in C^1([0, b]; \mathbf{R})$, satisfying the conditions

$$\psi(0) = 0,\ 0 \leqslant \psi(y) \leqslant a,\ 0 \leqslant y \leqslant b. \tag{2.1}$$

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($H_1$) The functions $P \in AC([0, a]; \mathbf{R}^n)$, $Q \in AC([0, b]; \mathbf{R}^n)$, where $AC([\alpha_1, \alpha_2]; \mathbf{R}^n)$ is the space of absolutely continuous functions $f: [\alpha_1, \alpha_2] \to \mathbf{R}^n$, normed by

$$||f|| = \sup_{t \in [\alpha_1, \alpha_2]} ||f(t)|| + \int_{\alpha_1}^{\alpha_2} ||f'(t)|| dt,$$

satisfy the condition $P(0) = Q(0)$.

($H_2$) The function $\alpha : D \to \mathbf{R}^n$ defined by

$$\alpha(x, y) = P(x) + Q(y) - P(\psi(y)), \ (x, y) \in D, \tag{2.2}$$

is bounded and therefore, there is $M_1 > 0$ such that

$$||\alpha(x, y)|| \leqslant M_1, \ (x, y) \in D. \tag{2.3}$$

It follows that $\alpha$ is absolutely continuous function in the Carathéodory sense [2, §565—§568], $\alpha \in C^*(D; \mathbf{R}^n)$.

Let $K$ be the set of absolutely continuous functions $z : D \to \mathbf{R}^n$, $z \in C^*(D; \mathbf{R}^n)$, [2, §565—§568] satisfying the conditions (2.3), (2.4), (2.5), where

$$\left\|\frac{\partial^2 z(x, y)}{\partial x \, \partial y}\right\| \leqslant M, \ \text{a.e.}(x, y) \in D, \tag{2.4}$$

and

$$\begin{cases} z(x, 0) = P(x), \ 0 \leqslant x \leqslant a, \\ z(\psi(y), y) = Q(y), \ 0 \leqslant y \leqslant b. \end{cases} \tag{2.5}$$

PROPOSITION 1. *The set $K$ is a nonempty convex and compact subset of the Banach space $\mathcal{C}(D; \mathbf{R}^n)$.*

*Proof.* The relation $z \in K$ implies $z \in \mathcal{C}(D; \mathbf{R}^n)$. We observe that $\frac{\partial^2 z}{\partial x \, \partial y}$ exists a.e. $(x, y) \in D$, as $z \in C^*(D; \mathbf{R}^n)$ [2, §565—§568].

Let $M(x, y)$ be any point of $D$. Consider the parallel to $x$-axis, that intersects the curve $\gamma$ in the point $N(\psi(y), y)$. Let $M_0(x, 0)$ and $N_0(\psi(y), 0)$ be the ortogonal projections of $M$ and $N$ on the $x$-axis. Denote $D_0(x, y)$ the rectangle $M \, N \, M_0 \, N_0$, given by

$$D_0(x, y) = \{(u, v) \,|\, \psi(y) \leqslant u \leqslant x, \ 0 \leqslant v \leqslant y\}.$$

Integrating $\frac{\partial^2 z(x, y)}{\partial x \, \partial y}$ over $D_0(x, y)$, one obtains

$$\iint_{D_0(x, y)} \frac{\partial^2 z(u, v)}{\partial u \, \partial v} du dv = \int_0^y dv \int_{\psi(y)}^x \frac{\partial^2 z(u, v)}{\partial u \, \partial v} du = \int_0^y dv \frac{\partial z}{\partial v}(u, v)\Big|_{u=\psi(y)}^{u=x} =$$

$$= \int_0^y \frac{\partial z}{\partial v}(x, v) dv - \int_0^y \frac{\partial z}{\partial v}(\psi(y), v) dv = z(x, y) - z(x, 0) - z(\psi(y), y) + z(\psi(y), 0) =$$

$$= z(x, y) - P(x) - Q(y) + P(\psi(y)), \ (x, y) \in D.$$

Using (2.2) it follows

$$z(x, y) = P(x) + Q(y) - P(\psi(y)) + \iint\limits_{D_0(x,y)} \frac{\partial^2 z(u, v)}{\partial u\, \partial v}\, dudv = \alpha(x, y) +$$

$$+ \iint\limits_{D_0(x, y)} \frac{\partial^2 z(u, v)}{\partial u\, \partial v}\, dudv = P(x) + Q(y) - P(\psi(y)) + \int\limits_0^y dv \int\limits_{\psi(y)}^x \frac{\partial^2 z(u, v)}{\partial u\, \partial v}\, du, \ (x, y) \in D. \tag{2.6}$$

The compactness of $K$ follows using (2.6) and the Arzelà—Ascoli Theorem and its convexity is obvious.

*Remark.* The relation $z \in K$ implies $(x, y, z(x, y)) \in D \times B$ for every $(x, y) \in D$. Because each $z \in K$ generates a multifunction $(x, y) \to F(x, y, z(x, y))$ from $D$ into comp$X$, we shall denote this mapping by $G(z) : D \to \text{comp}X$, where

$$G(z)(x, y) = F(x, y, z(x, y)),\ (x, y) \in D. \tag{2.7}$$

3. **Continuous selections. The continuous case refined.** We prove the following result, analogous to Lemma 3 [1].

LEMMA. *Let $A : D \to \text{comp}X$ a continuous multifunction and $v : D \to \mathbf{R}^n$ a piecewise constant mapping such that $d(v(x, y), A(x, y)) < \rho$ for every $(x, y) \in D$. Then, for every $\varepsilon > 0$, there exists a piecewise constant mapping $w : D \to \mathbf{R}^n$ such that $d(v(x, y), w(x, y)) < \rho$ and $d(w(x, y), A(x, y)) < \varepsilon$ for every $(x, y) \in D$.*

*Proof.* Indeed, given $\varepsilon > 0$, we can choose a partition $(D_{ij})$ $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$ of $J = [0, a[\times[0, b[$ consisting of intervals $D_{ij} = [x_{i-1}, x_i[\times[y_{j-1}, y_j[$, such that $v|D_{ij} = z_{ij}$ and $H(A(x, y), A(x', y')) < \varepsilon$ for any $(x, y)$, $(x', y')$ in $D_{ij}$. Then, for each $(i, j)$, there exists a point $\xi_{ij} \in A(x_{i-1}, y_{j-1})$ such that $d(v(x_{i-1}, y_{j-1}), \xi_{ij}) < \rho$ and $d(\xi_{ij}, A(x, y)) < \varepsilon$ for every $(x, y) \in D_{ij}$. We define the mapping $w : D \to \mathbf{R}^n$ as follows: $w/D_{ij} = \xi_{ij}$ for each $(i, j)$, $w(a, y) = \lim\limits_{x \to a^-} w(x, y)$, $w(x, b) = \lim\limits_{y \to b^-} w(x, y)$. The mapping $w$ has the required properties. Obviously, if $(x, y) \in J$, then $(x, y) \in D_{ij}$ for an unique $D_{ij}$, such that $w(x, y) = \xi_{ij}$ and $v(x, y) = z_{ij} = v(x_{i-1}, y_{j-1})$, and consequently $d(v(x, y), w(x, y)) = d(v(x_{i-1}, y_{j-1}), \xi_{ij}) < \rho$ and $d(w(x, y), A(x, y)) = d(\xi_{ij}, A(x, y)) < \varepsilon$. By continuity, these inequalities are also true and for $x = a$, $y = b$.

THEOREM. *Let $F : D \times B \to \text{comp}X$ be a continuous multifunction. Then there exists a continuous mapping $g : K \to \mathfrak{L}^1(D ; \mathbf{R}^n)$ such that, for every $z \in K$, $g(z)$ is a regulated mapping in $D$ and $g(z)(x, y) \in G(z)(x, y)$ for every $(x, y) \in D$.*

*Proof.* We shall construct, for every $n \geqslant 1$, a continuous mapping $g^n : K \to \mathfrak{L}^1(D ; \mathbf{R}^n)$ such that, for every $z \in K$, $g^n(z)$ is a piecewise constant mapping of $D$ into $X$ which satisfies, at every $(x, y) \in D$,

$$d(g^n(z)(x, y),\ G(z)(x. y)) < 2^{-n}, \tag{3.1}$$

$$||g^{n+1}(z)(x, y) - g^n(z)(x, y)|| < 2^{-n-1}. \tag{3.2}$$

It follows that for every $z \in K$, the sequence $(g^n(z))$ converges uniformly in $D$ to a mapping $g(z)$ of $D$ into $X$ that is regulated in $D$ and satisfies $g(z)(x, y) \in G(z)(x, y)$ at every $(x, y) \in D$. Indeed, since the convergence is uniform in $D$

and each $g^n$ is continuous in $K$, $g$ will be a continuous mapping of $K$ into $\mathfrak{L}^1(D;\mathbf{R}^n)$ and this will prove the statement. The construction will be made in two stages. First choose a decreasing null sequence $(\Delta_n)$ of positive constants such that, for every $n \geqslant 1$,

$$H(F(x, y, z), F(\xi, \eta, \xi)) < 2^{-n-3} \tag{3.3}$$

for any points $(x, y, z)$, $(\xi, \eta, \zeta)$ in $D \times B$ with $||(x, y) - (\xi, \eta)|| < \Delta_n$, $||z - \xi|| < \Delta_n$. This is possible, because $F$ is uniformly continuous in $D$. We select, for every $n \geqslant 1$, a finite open covering $(U_k^n) 1 \leqslant k \leqslant N(n)$ of the compact space $K$ such that

$$\operatorname{diam} U_k^n < \Delta_n, \quad 1 \leqslant k \leqslant N(n).$$

Let $(p_k^n) 1 \leqslant k \leqslant N(n)$ be a continuous partition of unity subordinate to $(U_k^n) 1 \leqslant k \leqslant N(n)$. We denote $N(n) = N_1(n) N_2(n)$ and $p_k^n(z) = q_i^n(z) r_j^n(z)$, $i = \overline{1, N^1(n)}$, $j = \overline{1, N_2(n)}$. The functions $p_{ij}^n : K \to \mathbf{R}$, satisfy the properties:

a) $0 \leqslant p_{ij}^n(z) \leqslant 1$ for $z \in K$, $i = \overline{1, N_1(n)}$, $j = \overline{1, N_2(n)}$,

b) $p_{ij}^n(z) = 0$, if $z \notin U_{ij}^n$, $i = \overline{1, N_1(n)}$, $j = \overline{1, N_2(n)}$,

c) $\displaystyle\sum_{i=1}^{N_1(n)} \sum_{j=1}^{N_2(n)} p_{ij}^n(z) = \sum_{i=1}^{N_1(n)} \sum_{j=1}^{N_2(n)} q_i^n(z) r_j^n(z) = 1$, for $z \in K$.

We denote

$$W_k^n = \{z \in U_k^n / p_k^n(z) > 0\}, \quad 1 \leqslant k \leqslant N(n).$$

Then, for every $n \geqslant 1$ and every vector index $l = (l_1, l_2, \ldots, l_n)$ such that $1 \leqslant l_\nu \leqslant N(\nu)$, $\bigcap_{\nu=1}^{n} W_{l_\nu}^\nu \neq \emptyset$,

there exists a piecewise constant mapping $v_l^n : D \to X$ and a point $z_l^n \in \bigcap_{\nu=1}^{n} W_{l_\nu}^\nu$ such, that, at every $(x, y) \in D$,

$$d(v_l^n(x, y), G(z_l^n)(x, y)) < 2^{-n-1}. \tag{3.4}$$

This assertion is obviously true for $n = 1$. Suppose that it is true for $n = 1, 2, \ldots, p$. If $l = (l_1, l_2, \ldots, l_n)$ is such that (3.4) holds for $n = p$, we can use Lemma and construct, for every integer $s$ such that

$$1 \leqslant s \leqslant N(p+1), \quad \bigcap_{\nu=1}^{p} W_{l_\nu}^\nu \cap W_s^{p+1} \neq \emptyset,$$

a piecewise constant mapping $v_{(l,s)}^{p+1} : D \to X$ which satisfies, at every $(x, y) \in D$,

$$d(v_{(l,s)}^{p+1}(x, y), G(z_l^p)(x, y)) < 2^{-p-3}, \tag{3.5}$$

$$||v_{(l,s)}^{p+1}(x, y) - v_l^p(x, y)|| < 2^{-p-1}. \tag{3.6}$$

*Remark.* For any $n$-vector index $l = (l_1, l_2, \ldots, l_n)$ and integer $s$, we denote by $(l, s)$ the $(n+1)$-vector index $(l_1, l_2, \ldots, l_n, s)$.

Thus, if we fix a point

$$z^{p)+1}_{(l,s)} \in \bigcap_{\nu=1}^{p} W^{\nu}_{l_\nu} \cap W^{p+1}_{s},$$

we deduce for every $(x, y) \in D$,

$$d(v^{p+1}_{(l,s)}(x, y), G(z^{p+1}_{(l,s)}(x, y)) \leqslant d(v^{p+1}_{(l,s)}(x, y), G(z^{p}_{l})(x, y)) +$$
$$+ H(G(z^{p}_{l})(x, y), G(z^{p+1}_{(l,s)}(x, y)) < 2^{-p-3} + 2^{-p-3} = 2^{-(p+1)-1}.$$

Hence, the assertion is true for $n = p + 1$ and consequently, by induction, for every $n \geqslant 1$.

We next define, for every $z \in K$, a sequence of finite partitions of the interval $J$ as follows; given $z \in K$, we successively construct, for every $n \geqslant 1$ and every $2n$-vector index $l = (l_1, l_2, \ldots, l_n)$, $l = l^1 \times l^2$, $l^1 = (l^1_1, l^1_2, \ldots, l^1_n)$, $l^2 = (l^2_1, l^2_2, \ldots, l^2_n)$ $1 \leqslant l^1_\nu \leqslant N_1(\nu)$, $1 \leqslant l^2_\nu \leqslant N_2(\nu)$, $1 \leqslant \nu \leqslant n$, an interval $J^n_l(z) \subset J$ such that

$$J = \bigcup_{\substack{1 \leqslant i \leqslant N_1(1)\\ 1 \leqslant j \leqslant N_2(1)}} J^1_{ij}(z) \tag{3.7}$$

and

$$J^n_l(z) = \bigcup_{1 \leqslant s \leqslant N(n+1)} J^{n+1}_{(l,s)}(z), \quad n \geqslant 1. \tag{3.8}$$

Indeed, let

$$\begin{cases} x^1_0(z) = 0 \\ x^1_i(z) = x^1_{i-1}(z) + a q^1_i(z) \sum\limits_{j=1}^{N_2(1)} r^1_j(z), \; i = \overline{1, N_1(1)}, \end{cases}$$

and

$$\begin{cases} y^1_0(z) = 0 \\ y^1_j(z) = y^1_{j-1}(z) + b r^1_j(z) \sum\limits_{i=1}^{N_1(1)} q^1_i(z), \; j = \overline{1, N_2(1)}. \end{cases}$$

We denote

$$J^1_{ij}(z) = [x^1_{i-1}(z), x^1_i(z)[ \times [y^1_{j-1}(z), y^1_j(z)[$$

for each $i = \overline{1, N_1(1)}$, $j = \overline{1, N_2(1)}$. Then, obviously, $J^1_{ij}(z)$ is nonempty if and only if $z \in W^1_{ij}$, but (3,7) holds whatever $z \in K$ because $(p^1_{ij})1 \leqslant i \leqslant N_1(1)$, $1 \leqslant j \leqslant N_2(1)$ ia s partition of unity. More generally, if $l = (l_1, l_2, \ldots, l_n)$ is an $2n$-vector index with $1 \leqslant l_\nu \leqslant N(\nu)$, $l = l^1 \times l^2$, $l^1 = (l^1_1, l^1_2, \ldots, l^1_n)$, $l^2 = (l^2_1, l^2_2, \ldots, l^2_n)$, $1 \leqslant l^1_\nu \leqslant N_1(\nu)$, $1 \leqslant l^2_\nu \leqslant N_2(\nu)$, $1 \leqslant \nu \leqslant n$, for which $J^n_l(z)$ has been constructed let

$$\begin{cases} x^n_{(l^1,0)}(z) = x^n_{(l^1_1, l^1_2, \ldots, l^1_n - 1)}(z), \\ x^{n+1}_{(l^1,s^1)}(z) = x^{n+1}_{(l^1,s^1-1)}(z) + \left(a \prod\limits_{\nu=1}^{n} q_{l_{\nu^1}}(z) \sum\limits_{j=1}^{N_2(n)} r_j(z)\right) q^{n+1}_{s^1}(z) \end{cases}$$

and

$$\begin{cases} y^{n+1}_{(l^1,0)}(z) = y^{n}_{(l^2_1, l^2_2, \ldots, l^2_n - 1)}(z), \\ y^{n+1}_{(l^1,s^1)}(z) = y^{n+1}_{(l^1,s^1-1)}(z) + \left(b \prod_{\nu=1}^{n} r_{l^2_\nu}(z) \sum_{i=1}^{N_2(n)} q\ (z)\right) r^{n+1}_{s^2}(z). \end{cases}$$

and set

$$J^{n+1}_{(l,s)}(z) = [x^{n+1}_{(l^1,s^1-1)}, x^{n+1}_{(l^1,s^1)}(z)[\times[y^{n+1}_{(l^2,s^2-1)}(z), y^{n+1}_{(l^2,s^2)}(z)[$$

where $s = s^1 \times s^2$, for each $s = \overline{1, N(n+1)}$ ($s^1 = \overline{1, N_1(n+1)}$, $s^2 = \overline{1, N_2(n+1)}$). Then $J^{n+1}_{(l^1,s^1)}(z)$ is nonempty if and only if $z \in \bigcap_{\nu=1}^{n} W^{\nu}_{l_\nu} \bigcap W^{n+1}$, and, in particular, $z \in \bigcap_{\nu=1}^{n} W^{\nu}_{l_\nu}$ implies that (3.8) holds nontrivially. We observe, that in this case, diam $J^n_l(z) = ab \prod_{\nu=1}^{n} p^{\nu}_{l_\nu}(z) > 0$. However, whatever $z \in K$, we have by construction that

$$J = \bigcup \{J^{\nu}_l(z) : l = (l_1, l_2, \ldots, l_n),\ 1 \leqslant \nu \leqslant N(\nu),\ 1 \leqslant \nu \leqslant n\}. \qquad (3.9)$$

We define, for every $n \geqslant 1$, the required mapping $g^n$ of $K$ into $\mathfrak{L}^1(D ; \mathbf{R}^n)$ In view of (3.9) we can do this simply by prescribing, for every $z \in K$, the restriction of $g^n(z)$ to each of the intervals $J^n_l(z)$. For every $z \in K$, we define

$$g^1(z)/J = \sum_{\nu=1}^{N(1)} \chi[J^1_s(z)]v^1_s \qquad (3.10)$$

where $\chi$ is the characteristic function and set, for every $n \geqslant 1$, and every $2n$ vector index $l = (l_1, l_2, \ldots, l_n)$, $l = l^1 \times l^2$, $l^1 = (l^1_1, l^1_2, \ldots, l^1_n)$, $l^2 = (l^2_1, l^2_2, \ldots l^2_n)$, $1 \leqslant l_\nu \leqslant N(\nu)$, $1 \leqslant l^1_\nu \leqslant N_1(\nu)$, $1 \leqslant l^2_\nu \leqslant N_2(\nu)$, $1 \leqslant \nu \leqslant n$,

$$g^{n+1}(z)/J^n_l(z) = \sum_{\nu=1}^{N(n+1)} \chi[J^{n+1}_{(l,s)}(z)]v^{n+1}_{(l,s)}. \qquad (3.11)$$

This uniquely defines, for every, $n \geqslant 1$, $g^n(z)$ as a piecewise constant mapping in $J$, and hence we can extend $g^n(z)$ to $D = \overline{J}$ by setting

$$\begin{cases} g^n(z)(a, y) = \lim\limits_{x \to a^-} g^n(z)(x, y), \\ g^n(z)(x, b) = \lim\limits_{y \to b^-} g^n(z)(x, y). \end{cases} \qquad (3.12)$$

Obviously, for every $n \geqslant 1$, $g^n$ is a mapping of $K$ into $\mathfrak{L}^1(D ; \mathbf{R}^n)$. This construction implies, similarly with [5, Proposition 2] that each $g^n$ is continuous in $K$. Thus, only the inequalities (3.1) and (3.2) remain to be verified.

Let $z \in K$ be given and fix $(x, y) \in J$. Then, for every $n \geqslant 1$, there exists one and only one $2n$-vector index $l = (l_1, l_2, \ldots, l_n)$, $l = l^1 \times l^2$, such tha

$(x, y) \in J_l^n(z)$, This implies that, in particular, $z \in \bigcap_{\nu=1}^{n} W_{l_\nu}^{\nu}$ and consequently, by (3.11),

$$d(g^n(z)(x, y), G(z)(x, y)) = d(v_l^n(x, y), G(z)(x, y)) \leqslant d(v_l^n(x, y), G(z_l^n)(x, y)) + $$
$$+ H(G(z_l^n)(x, y), G(z)(x, y)) < 2^{-n-1} + 2^{-n-3} < 2^{-n}.$$

Moreover, if $(x, y) \in J_l^n$ then $(x, y) \in J_{(l,s)}^{n+1}$ for one and only one index $s$ with $1 \leqslant s \leqslant N(n+1)$, so that also $z \in \bigcap_{\nu=1}^{n} W_{l_\nu}^{\nu} \bigcap W_s^{n+1}$.
Hence, we deduce from (3.4) and (3.6) that

$$||g^{n+1}(z)(x, y) - g^n(z)(x, y)|| = ||v_{(l,s)}^{n+1}(x, y) - v_l^n(x,y)|| < 2^{-n-1}.$$

Thus, the inequalities (3.1), (3.2) hold at every $(x, y) \in D$. Obviously, by (3.12) and continuity, then remain valid at $x = a$, $y = b$. This completes the proof.

4. **Multivalued equations with partial derivatives.** Let us consider the multivalued equation

$$\frac{\partial^2 z}{\partial x \partial y} \in F(x, y, z), \ (x, y) \in D, \ z \in B, \tag{4.1}$$

where $F : D \times B \to \operatorname{comp}X$.

The Picard problem associated with (4.1) is defined in [4] and consists in finding of absolutely continuous function [2, §565—§568], $z \in C^*(D; \mathbf{R}^n)$, which satisfies (4.1) a.e. $(x, y) \in D$, and (2.5). As corollary of theorem of selection we state the following existence result.

THEOREM. *Let be satisfied the hypotheses* $(H_0)$, $(H_1)$, $(H_2)$, $(H_3)$, *where*:

$(H_3)$ $\quad F : D \times B \to \operatorname{comp}X$ *is a continuous multifunction.*

*Then, there exists a regulated mapping* $\lambda : D \to \mathbf{R}^n$ such *that the mapping* $z : D \to \mathbf{R}^n$, *given by*

$$\bar{z}(x, y) = P(x) + Q(y) - P(\psi(y)) + \iint_{D_0(x,y)} \lambda(u, v)dudv = \alpha(x, y) +$$
$$+ \iint_{D_0(x,y)} \lambda(u, v)dudv = P(x) + Q(y) - P(\psi(y)) + \int_0^y dv \int_{\psi(y)}^x \lambda(u, v)du,$$
$$(x, y) \in D, \tag{4.2}$$

*is a solution of the Picard problem* (4.1) + (2.5).

*Proof.* Let be $\lambda(x, y) + g(\bar{z})(x, y)$, $\bar{z} \in K$, $\lambda : D \to \mathbf{R}^n$, where $g$ exists by the theorem of selection. From $(H_1)$ the function $\bar{z}$ given by (4.2) is absolutely continuous function in the Carathéodory sense [2, §565—§568] $\bar{z} \in C^*(D; \mathbf{R}^n)$.

From (4.2) it follows $\frac{\partial^2 \bar{z}}{\partial x \partial y}(x, y) = \lambda(x, y) = g(\bar{z})(x, y) \in G(\bar{z})(x, y) = F(x, y, \bar{z}(x, y))$ a.e. $(x, y) \in D$ and $\bar{z}(x, 0) = P(x)$, $0 \leqslant x \leqslant a$, $\bar{z}(\psi(y), y) = Q(y)$, $0 \leqslant y \leqslant b$. Hence $\bar{z}$ is a solution of the Picard problem (4.1) + (2.5).

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STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXXV, 1, 1990

# ON THE ESTIMATE OF THE POINTWISE APPROXIMATION OF FUNCTIONS BY LINEAR POSITIVE FUNCTIONALS

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Dedicated to Professor A. Pal on his 60th anniversary

*Received: November 14, 1989*
*AMS subject classification: 41A36, 41A25*

**REZUMAT. — Asupra estimării punctuale a aproximării funcţiilor prin funcţionale liniare şi pozitive.** Fie $F: V \to R$ o funcţională liniară şi pozitivă cu proprietatea $F(e_j) = x^j$, $j = 0,1$, unde $V$ este un subspaţiu liniar de funcţii reale definite pe un interval $I$, $x \in I$ este fixat, iar $e_j$ sînt funcţiile $e_j(t) = tj$. In lucrare se dă o evaluarea generală pentru $|F(f) - f(x)|$ cînd $f \in V$, cu ajutorul unui modul de continuitate de ordinul doi generalizat. Evaluări concrete se dau apoi pentru modulul uzual de continuitate de ordinul doi. De asemenea se dau aplicaţii la aproximarea prin operatori liniari şi pozitivi.

0. **Introduction.** In the present paper, based on a new method we improve and generalize the estimates that we have obtained in [10] and [12] (see also [13]). In fact, the unified method that we apply here results by combining these methods. In order to enlarge the generality, we present this estimate with the aid of a generalized modulus of continuity of the second order and in terms of functionals, although the applications that we have in view are for the usual second order modulus of continuity and for the poinwise estimate of the approximation by linear positive operators that perserve linear functions. For such operators our estimate improves the general estimate given in [6]. In the same time, since our estimate requires not the continuity of the functions, nor the compactness of their domains, it is more general than the estimate in [6]. However, in other sens the second is more general.

1. **Main results.** Let $I$ be an arbitrary fixed interval of the real axis. We denote by $\mathcal{F}(I)$ the linear space of the real valued functions defined on I and by $\mathcal{F}_b(I)$ the subspace of $\mathcal{F}(I)$ of those functions that are bounded on each compact subinterval of $I$. For $j = 0, 1, 2, \ldots$ we denote by $e_j \in \mathcal{F}_b(I)$ the functions $e_j(t) = t^j (t \in I)$.

For any $f \in F(I)$ and any points of $I: t_1 < y < t_2$ we define:

$$\Delta(f, t_1, t_2, y) = \frac{t_2 - y}{t_2 - t_1} f(t_1) + \frac{y - t_1}{t_2 - t_1} f(t_2) - f(y). \tag{1.1}$$

In the following definition we shall consider a class of mappings that have similar properties as the second order modulus of continuity. For this reason we shall conventionaly call them general moduli of continuity of the second order.

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DEFINITION 1.1. Let $V$ be a subspace of $\mathcal{F}_b(I)$ that contains the functions $e_0$ and $e_1$. We say that a function $\omega : V \times (0, \infty) \to [0. \infty)$ is a *general modulus of continuity of the second order on* $V$ if the following conditions hold:

i) $\omega(f, h) \leqslant \omega(f, h_1)$, ($f \in V$ and $0 < h \leqslant h_1$) (1.2)

ii) $\omega(f + p, h) = \omega(f, h)$, ($f \in V$, $h > 0$, and $p = ae_0 + be_1$, where $a \in \mathbf{R}$, $b \in \mathbf{R}$).

*Examples of general moduli of continuity of the second order.*

A) The usual second order modulus of continuity, defined by:

$$\omega_2(f, h) = \sup\{|f(y + \rho) - 2f(y) + f(y - \rho)|,\ y \pm \rho \in I,\ 0 < \rho \leqslant h\}, \tag{1.3}$$
$$(f \in \mathcal{F}_b(I) \text{ and } h > 0).$$

B) A modified modulus of continuity of the second order, defined in [12] by:

$$\omega_2^{\bullet}(f, h) = \sup\{|\Delta(f, t_1, t_2, y)|;\ t_1, t_2 \in I,\ t_2 - h \leqslant t_1 < y < t_2\} \tag{1.4}$$
$$(f \in \mathcal{F}_b(I) \text{ and } h > 0).$$

C) More general moduli can be obtained in the following mode. Let $V_1$ be a linear subspace of $\mathcal{F}_b([0, 1])$ and let $L : V_1 \to \mathcal{F}_b([0,1]$ be a linear positive operator such that $L(e_j, y) = e_j(y)$, ($j = 0,1$ and $y \in [0, 1]$). For any $\alpha, \beta \in I$, $\alpha < \beta$ we denote by $q_{\alpha,\beta}$ the polynomial defined by $q_{\alpha,\beta}(t) = (\beta - \alpha)t + \alpha$. Let $V \subset \mathcal{F}_b(I)$ be a linear subspace with the property that for any $f \in V$ and any points $\alpha, \beta \in I$, $\alpha < \beta$ we have $f_{\alpha,\beta} \in V_1$, where we denote $f_{\alpha,\beta} = (f|_{[\alpha,\beta]}) \circ q_{\alpha,\beta}$. For any $\lambda \in (0, 1)$ and any $k > 0$ we denote:

$$\omega_{L,\lambda,k}(f, h) = \sup\{k\,|L(f_{\alpha,\beta},\lambda) - f_{\alpha,\beta}(\lambda)|;\ \alpha, \beta \in I,\ \beta - h \leqslant \alpha < \beta\}, \tag{1.5}$$
$$(f \in V \text{ and } h > 0).$$

$$\omega_{L,k}(f, h) = \sup_{\lambda \in (0,1)} \omega_{L,\lambda,k}(f, h),\ (f \in V \text{ and } h > 0). \tag{1.6}$$

If $\mathfrak{L}$ is a family of such linear positive operators $L : V_1 \to \mathcal{F}_b([0, 1])$ then we define:

$$\omega_{\mathfrak{L}}(f, h) = \sup_{L \in \mathfrak{L}} \omega_{L,1}(f, h),\ (f \in V \text{ and } h > 0) \tag{1.7}$$

We note that $\omega_2(f, h) = \omega_{B_1, \frac{1}{2}, 2}(f, 2h)$ and $\omega_2^{\bullet}(f, h) = \omega_{B_1, 1}(f, h)$, where $B_1$ is the Bernstein polynomial of degree 1: $B_1(f, \lambda) = (1 - \lambda)f(0) + \lambda f(1)$.

D) Other examples are:

$$\omega_1(f', h) = \sup\{|f'(y) - f'(t)|;\ y, t \in I,\ |y - t| \leqslant h\}, \tag{1.8}$$
$$(f \in C_1(I) \text{ and } h > 0),$$

as well as the least concave majorant $\tilde{\omega}_1(f', \cdot)$ of $\omega_1(f', \cdot)$, defined in [9] by:

$$\tilde{\omega}_1(f', h) = \sup\left\{\sum_{i=1}^{n} \lambda_i\omega_1(f', h_i)\,;\; n \geqslant 1,\; \sum_{i=1}^{n} \lambda_i = 1,\right. \tag{1.9}$$

$$\left.\sum_{i=1}^{n} \lambda_i h_i = h,\; \lambda_i \geqslant 0\right\},\quad (f \in C_1(I),\; h > 0).$$

DEFINITION 1.2. Let $\omega$ be a general modulus of continuity of the second order as in Definition 1.1, on the linear subspace $V \subset \mathcal{F}_b(I)$ and let the function $\psi : [0, \infty) \to [0, \infty)$. We say that $\omega$ satisfies the *condition* $(A(\psi))$ *on* $V$ if we have:

$$|\Delta(f, t_1, t_2 y)| \leqslant \left[\frac{t_2 - y}{t_2 - t_1}\,\psi\left(\left|\frac{t_1 - h}{h}\right|\right) + \frac{y - t_1}{t_2 - t_1}\,\psi\left(\left|\frac{t_2 - y}{h}\right|\right)\right]\omega(f, h) \tag{1.10}$$

$$(f \in V,\; h > 0 \text{ and } t_1 < y < t_2,\; t_1, t_2 \in I).$$

LEMMA 1.1. *Let $\omega$ be a general modulus of continuity of the second order on the linear subspace $V \subset \mathcal{F}_b(I)$ and let $\psi : [0, \infty) \to [0, \infty)$ be a function such that $\psi(t) \geqslant 1$ $(t > 0)$. We assume that for every two points of $I : a < b$ and for every function $g \in V$ such that $g(a) = 0 = g(b)$, if we denote $h = \frac{b-a}{2}$ we have the following relations:*

i) $|g(t)| \leqslant \omega(g, h), \quad (t \in [a, b])$

ii) *If* $y \in [a, a + h]$ *then* $$|g(t) - g(y)| \leqslant \psi\left(\left|\frac{t - y}{h}\right|\right)\omega(g, h) \tag{1.11}$$

$(t \in I \cap (b, \infty))$ *and if* $y \in [a + h, b]$ *then the same inequality holds for* $(t \in I \cap (-\infty, a))$.

*Then $\omega$ verifies the condition $(A(\psi))$ on $V$.*

*Proof.* Let $f \in V$ and let the real number $h > 0$. Let $t_1 < y < t_2$ three points of $I$. If $t_2 - t_1 \leqslant 2h$ let $a = t_1$, $b = t_2$ and $\rho = (t_2 - t_1)/2$. Also, let $p$ be the polynomial of degree one defined such that the function $g = f + p$ verifies the condition $g(a) = 0 = g(b)$. Then from (1.11)$-i$) with $\rho$ instead of $h$ we obtain:

$$|\Delta(f, t_1, t_2, y)| = |\Delta(g, t_1, t_2, y)| = |-g(y)| \leqslant \omega(g, \rho) \leqslant \omega(g, h) =$$

$$= \omega(f, h) \leqslant \left[\frac{t_2 - y}{t_2 - t_1}\psi\left(\left|\frac{t_1 - y}{h}\right|\right) + \frac{y - t_1}{t_2 - t_1}\,\psi\left(\left|\frac{t_2 - y}{h}\right|\right)\right]\cdot\omega(f, h).$$

Now let the case $t_2 - t_1 > 2h$. Then at least one of the conditions $t_2 - y > h$ and $y - t_1 > h$ holds. We only consider the case $t_2 - y > h$, since the proof in the other case is analogous. We denote $a = \max\{t_1, y - h\}$ and $b = a + 2h$. Let $p$ be the polynomial of degree one defined such that the function $g = f +$

$+p$ verifies the condition $g(a) = 0 = g(b)$. Because $y \in [a, a+h]$ and $t_2 \in I \cap (b, \infty)$ we can apply (1.11)—ii). Hence $|g(t_2) - g(y)| \leqslant \psi\left(\left|\frac{t_2 - y}{h}\right|\right) \cdot \omega(g, h)$.

If $y - t_1 > h$ we have $y = a + h \in [a + h, b]$ and $t_1 \in I \cap (-\infty, a)$. From (1.11)—ii) it results $|g(t_1) - g(y)| \leqslant \psi\left(\left|\frac{t_1 - y}{h}\right|\right) \cdot \omega(g, h)$. If $y - t_1 \leqslant h$ we have $a = t_1$ and since $|g(t_1) - g(y)| = |g(y)|$ the above inequality is also true.

From the relations already proved we have:

$$|\Delta(f, t_1, t_2, y)| = |\Delta(g, t_1, t_2, y)| \leqslant \frac{t_2 - y}{t_2 - t_1} \cdot |g(t_1) - g(y)| + \frac{y - t_1}{t_2 - t_1} \cdot$$

$$\cdot |g(t_2) - g(y)| \leqslant \left[\frac{t_2 - y}{t_2 - t_1} \cdot \psi\left(\left|\frac{t_1 - y}{h}\right|\right) + \frac{y - t_1}{t_2 - t_1} \cdot \psi\left(\left|\frac{t_2 - y}{h}\right|\right)\right] \cdot \omega(f, h)$$

In what follows $x$ will be a fixed point of the interval $I$. We denote by $\eta_x \in \mathcal{F}_b(I)$ the function defined by:

$$\eta_x(t) = \begin{cases} 0, & t = x \\ 1, & t \in I,\ t \neq x. \end{cases} \tag{1.12}$$

If $f \in \mathcal{F}(I)$ we denote by $\delta_x^+ f$ and $\delta_x^- f$ the functions defined by:

$$(\delta_x^{\pm} f)(t) = \begin{cases} f(t) - f(x), & t \in I,\ t \gtrless x \\ 0 & t \in I,\ t \leqq\geqq x. \end{cases} \tag{1.13}$$

It results the following representation: $f = \delta_x^+ f + \delta_x^- f + f(x) e_0$.

DEFINITION 1.3. Let $V$ be a linear subspace of $\mathcal{F}_b(I)$ and let $F : V \to \mathbf{R}$ be a linear positive functional. We say that the functional $F$ verifies the *condition* $(B(x))$, where, $x \in I$ if the following conditions are accomplished:

i) $e_j \in V$, $(j = 0,1)$

ii) $\eta_x \in V$. (1.14)

iii) If $f \in V$ then $|f| \in V$

iv) If $f \in V$ then $\delta_x^+ f \in V$ and $\delta_x^- f \in V$,

and

$$F(e_j) = x^j,\ (j = 0,1). \tag{1.15}$$

For a linear positive functional $F : V \subset \mathcal{F}_b(I) \to \mathbf{R}$ that verifies the condition $(B(x))$ we denote

$$M_x(F) = \frac{1}{2} F(|e_1 - x e_0|). \tag{1.16}$$

LEMMA 1.2. [12] *For a linear positive functional* $F : V \subset \mathcal{F}_b(I) \to \mathbf{R}$ *that satisfies the condition* $(B(x))$ *we have*:

$$F(|\delta_x^+ e_1|) = F(|\delta_x^- e_1|) = M_x(F). \tag{1.17}$$

*Proof.* Clearly $|\delta_x^\pm e_1| \in V$. We have $e_1 - xe_0 = \delta_x^+ e_1 + \delta_x^- e_1$ and from the condition $(B(x))$ we have $0 = F(e_1 - xe_0) = F(\delta_x^+ e_1) + F(\delta_x^- e_1)$. Then $F(|\delta_x^+ e_1|) = F(\delta_x^+ e_1) = -F(\delta_x^- e_1) = F(-\delta_x^- e_1) = F(|\delta_x^- e_1|)$. Also we have $|e_1 - xe_0| = |\delta_x^+ e_1| + |\delta_x^- e_1|$.

LEMMA 1.3. [12] *Let* $F : V \subset \mathcal{F}_b(I) \to \mathbf{R}$ *be a linear positive functional that satisfies the condition* $(B(x))$ *and such that* $M_x(F) \neq 0$. *Then the following representation holds*:

$$F(f) - f(x) = F_{t_1}(F_{t_2}(\varphi_{f,x}(t_1, t_2))), \ (f \in V), \tag{1.18}$$

*where* $\varphi_{f,x} : I \times I \to \mathbf{R}$ *is defined by*:

$$\varphi_{f,x}(t_1, t_2) = \frac{1}{M_x(F)}\{|\delta_x^+ e_1|(t_2)(\delta_x^- f)(t_1) + |\delta_x^- e_1|(t_1)(\delta_x^+ f)(t_2)\}, \tag{1.19}$$

*and the notation* $F_{t_j(g(t_1,\dots,t_n))}$ *means the value of the functional* $F$ *applied to the partial function* $t_i \to g(t_1, \dots, t_n)$ *when* $t_j = const$ $(i \neq j)$.

*Proof.* From (1.14) it follows that the partial functions $t_2 \to \varphi_{f,x}(t_1, t_2)$ for all fixed $t_1$ are in $V$. Next, the function $t_1 \to F_{t_2}(\varphi_{f,x}(t_1, t_2))$ belongs to $V$ since $F_{t_2}(\varphi_{f,x}(t_1, t_2)) = (\delta_x^- f)(t_1) + \frac{1}{M_x(F)} F(\delta_x^+ f) \cdot |\delta_x^- e_1|(t_1)$. Finally $F_{t_1}(F_{t_2}(\varphi_{f,x}(t_1, t_2))) = F(\delta_x^- f) + F(\delta_x^+ f) = F(f - f(x)e_0) = F(f) - f(x)$.

*Remark. 1.1.* For applications, the most important case of functionals as in Definition 1.3 can be obtained in the following mode. Let $\mu$ be a regular positive Borel measure on $I$ such that $\int_I e_j d\mu = x^j$, $(j = 0,1)$. Let $V = \mathfrak{L}_\mu(I) \cap \mathcal{F}_b(I)$ and let the functional $F : V \to \mathrm{R}$ be defined by:

$$F(f) = \int_I f d\mu, \ (f \in V). \tag{1.20}$$

If $I$ is compact, $x \in I$ and $F : C(I) \to \mathbf{R}$ is a linear positive functional such that $F(e_j) = x^j$ then $F$ is of the form (1.20). We can consider that the functional $F$ is prolonged on the whole space $V = \mathfrak{L}_\mu(I) \cap \mathcal{F}_b(I)$ and thus $F$ verifies the condition $(B(x))$.

For the functionals (1.20) the relation (1.17) becomes:

$$\int_{I^-} |e_1 - xe_0| d\mu = \int_{I^+} |e_1 - xe_0| d\mu = \frac{1}{2} \int_I |e_1 - xe_0| d\mu, \tag{1.17'}$$

where $I^- = I \cap (-\infty, x)$ and $I^+ = I \cap (x, \infty)$. Also (1.18) becomes:

$$F(f) - f(x) = \int_{I^-} d\mu(t_1) \int_{I^+} \frac{t_2 - t_1}{M_x(F)} \Delta(f, t_1, t_2, x) d\mu(t_2). \qquad (1.18')$$

Moreover it is shown in [12] that in this case the following equivalence holds:

$$(F(f) = f(x), \ (f \in V)) \text{ iff } (M_x(F) = 0). \qquad (1.21)$$

From (1.21) it results that $F(f) = f(x)$, $(f \in V)$ if $x$ is a end of the interval $I$.

*Remark 1.2.* More particular functionals (1.20) can be defined in the following mode. Let $A$ be a finite or countable set of indices and let the families $\{x_i \in I, \ i \in A\}$ and $\{c_i \geqslant 0, \ i \in A\}$ be such that $\sum_{i \in A} c_i = 1$ and $\sum_{i \in A} c_i \cdot x_i = x$. We consider $V = \{f \in \mathcal{F}_b(I), \ \sum_{i \in A} c_i | f(x_i) | < \infty\}$ and the functional $F : V \to \mathbf{R}$ defined by:

$$F(f) = \sum_{i \in A} c_i \cdot f(x_i), \ (f \in V). \qquad (1.22)$$

The main result of this section is the following Theorem:

THEOREM 1.1 *Let $F : V \subset \mathcal{F}_b(I) \to \mathbf{R}$ be a linear positive functional that satisfies the condition $(B(x))$, when $x \in \operatorname{Int} I$, and let $\omega$ be a general modulus of continuity of the second order on $V$ that satisfies the condition $(A(\psi))$. We suppose that $h > 0$ is a real number and $\psi \circ \left| \frac{e_1 - x e_0}{h} \right| \in V$. Then we have:*

$$|F(f) - f(x)| \leqslant F\left(\psi \circ \left| \frac{e_1 - x e_0}{h} \right| \right) \cdot \omega(f, h), \ (f \in V). \qquad (1.23)$$

*Proof.* We first consider the case $M_x(F) \neq 0$. we can write:

$$\varphi_{f,x}(t_1, t_2) = \begin{cases} \frac{t_2 - t_1}{M_x(F)} \cdot \Delta(f, t_1, t_2, x), \text{ if } t_1 < x < t_2 \\ 0 \quad , \text{ if } t_1 \geqslant x \text{ or } t_2 \leqslant x. \end{cases}$$

From the condition $(A(\psi))$ we have for $t_1 < x < t_2$:

$$|\Delta(f, t_1, t_2, x)| \leqslant \left[ \frac{t_2 - x}{t_2 - t_1} \cdot \psi\left( \left| \frac{t_1 - x}{h} \right| \right) + \frac{x - t_1}{t_2 - t_1} \cdot \psi\left( \left| \frac{t_2 - x}{h} \right| \right) \right] \cdot \omega(f, h) =$$

$$= \left[ \psi(0) + \Delta\left( \psi \circ \left| \frac{e_1 - x e_0}{h} \right|, t_1, t_2, x \right) \right] \omega(f, h) = \Delta(\sigma_{x,h,\psi}, t_1, t_2, x) \cdot \omega(f, h),$$

where we have denoted:

$$\sigma_{x,h,\psi} = \psi(0) \cdot \eta_x + \psi \circ \left| \frac{e_1 - x e_0}{h} \right| \in V.$$

Therefore we have: $|\varphi_{f,x}(t_1, t_2)| \leqslant (\varphi_{(\sigma_{x,h,\psi}),x}(t_1, t_2)) \cdot \omega(f, h)$.

From Lemma 1.3 and the condition $(B(x))$ we have:

$$|F(f) - f(x)| = |F_{t_1}(F_{t_2}(\varphi_{f,x}(t_1, t_2)))| \leqslant F_{t_1}(F_{t_2}(|\varphi_{f,x}(t_1, t_2)|)) \leqslant$$
$$\leqslant F_{t_1}(F_{t_2}((\varphi_{(\sigma_{x,h,\psi}),x})(t_1, t_2))\,\omega(f, h))) = [F(\sigma_{x,h,\psi}) - (\sigma_{x,h,\psi})(x)]\omega(f, h) =$$
$$= \left[\psi(0) \cdot F(\eta_x) + F\left(\psi_0 \left|\frac{e_1 - xe_0}{h}\right|\right) - \psi(0)\right]\omega(f, h) \leqslant F\left(\psi \circ \left|\frac{e_1 - xe_0}{h}\right|\right)\omega(f, h).$$

We consider now the general case. Since $x \in \operatorname{Int} I$ we choose two points of $I$: $a$ and $b$ such that $a < x < b$, and for any $\lambda \in (0, 1)$ we consider the functional $G_\lambda : V \to \mathrm{R}$ defined by:

$$G_\lambda(f) = \lambda \cdot F(f) + (1 - \lambda)\left[\frac{b - x}{b - a} \cdot f(a) + \frac{x - a}{b - a} \cdot f(b)\right], \quad (f \in V).$$

Since $G_\lambda$ satisfies the condition $(B(x))$ and $M_x(G_\lambda) > 0$, from above it follows:

$$|G_\lambda(f) - f(x)| \leqslant G_\lambda\left(\psi \circ \left|\frac{e_1 - xe_0}{h}\right|\right)\omega(f, h).$$

We have:

$$|F(f) - f(x)| = \left|\frac{1}{\lambda}(G_\lambda(f) - f(x)) + \frac{1 - \lambda}{\lambda}\Delta(f, a, b, x)\right| \leqslant$$
$$\leqslant \frac{1}{\lambda} G_\lambda\left(\psi \circ \left|\frac{e_1 - xe_0}{h}\right|\right)\omega(f, h) + \frac{1 - \lambda}{\lambda}|\Delta(f, a, b, x)|.$$

If we consider $f$ fixed and $\lambda$ tends to 1 we obtain (1.23).

2. **Estimates for the usual second order modulus of continuity** $\omega_2$. For any real number $a$ we denote by $]a[$ the greatest integer number that is less than $a$.

*Remark. 2.1.* In [12] it is proved that the modified modulus $\omega_2^*(., 2h)$ (see (1.4)) satisfies the condition $(A(\theta))$, where the function $\theta : [0, \infty) \to [0, \infty)$ is defined by $\theta(t) = (1 + ]t[)^2$, $(t \geqslant 0)$. By taking into account Lemma 2.1 from below we can infer that the modulus $\omega_2$ also satisfies the condition $(A(\theta))$. In this section we shall obtain other estimate that improves this one.

LEMMA 2.1. [12] *For every $f \in \mathcal{F}_b(I)$ and every real number $h > 0$ we have:*

$$\omega_2^*(f, 2h) \leqslant \omega_2(f, h). \tag{2.1}$$

*Proof.* Let $t_1, t_2 \in I$, $t_1 < y < t_2$ and $t_2 - t_1 \leqslant 2h$. We consider the polynomial $p$ of degree one defined such that the function $g = f + p$ to satisfy the condition $g(t_1) = 0 = g(t_2)$. We have $\Delta(f, t_1, t_2, y) = \Delta(g, t_1, t_2, y) = -g(y)$ and $\omega_2(f, h) = \omega_2(g, h)$.

Let $\varepsilon > 0$ be arbitrary choosen. Since $g$ is bounded on $[t_1, t_2]$ there is a point $u_\varepsilon \in (t_1, t_2)$ such that:

$$|g(u_\varepsilon)| > \sup\{|g(t)|\ ;\ t \in [t_1, t_2]\} - \varepsilon.$$

We only consider the case $u_\varepsilon \geq \frac{t_1 + t_2}{2}$ and $g(u_\varepsilon) > 0$, since the other one can be reduced to this. Then $2u_\varepsilon - t_2 \in [t_1, t_2]$ and $\omega_2(g, h) \geq |g(t_2) - 2g(u_\varepsilon) + g(2u_\varepsilon - t_2)| \geq -g(t_2) + 2g(u_\varepsilon) - g(2u_\varepsilon - t_2) \geq g(u_\varepsilon) - \varepsilon \geq |g(y)| - 2\varepsilon = |\Delta(g, t_1, t_2, y)| - 2\varepsilon$.

Since $t_1$, $t_2$, $y$ and $\varepsilon$ are arbitrary choosen Lemma is proved.

LEMMA 2.2. *Let $a < b$ be two points of $I$. Denote by $h = \frac{b-a}{2}$ and let $x \in [a, a+h]$. Let $g \in \mathcal{F}_b(I)$ such that $g(a) = 0 = g(b)$. Then the following inequalities hold:*

i) $|g(t)| \leq \omega_2(g, h), \quad (t \in [a, b])$

ii) $|g(t)| \leq \frac{4}{3}\,\omega_2(g, h), \quad \left(t \in \left(b, b + \frac{h}{2}\right] \cap I\right)$

iii) $|g(t)| \leq 2\omega_2(g, h), \quad (t \in (b, b+h] \cap I)$

iv) $|g(t) - g(t-h)| \leq 2\omega_2(g, h), \quad (t \in (b, b+h)] \cap I)$

v) $|g(t) - g(x)| \leq 2\omega_2(g, h), \quad \left(t \in \left(b, b + \frac{h}{2}\right] \cap I,\right.$ (2.2)

$\left. x \in \left[a + \frac{h}{2},\ a + h\right] \text{ and } t \leq 2x - a + h\right)$

vi) $|g(t) - g(x)| \leq 5\omega_2(g, h), \quad \left(t \in \left(b + h,\ b + \frac{3}{2}h\right] \cap I,\right.$

$\left. x \in \left[a + \frac{h}{2},\ a + h\right] \text{ and } t \leq 2x - a + 2h\right)$

vii) $|g(t) - g(x)| \leq \left(\frac{1}{2}k^2 + \frac{3}{2}k + 1\right)\omega_2(g, h), \quad (t \in (b, \infty) \cap I)$

*where* $k = 1 + ]\,|b - t|/h\,[$.

*Proof.* i) If $t \in (a, b)$ we have $g(t) = -\Delta(g, a, b, t)$ and we can apply Lemma 2.1.

ii) We have $4b - 3t \geq t - 2h$ and $4b - 3t \in [a, b]$. Hence from i) it results $|g(4b - 3t)| \leq \omega_2(g, h)$ and then:

$$|g(t)| = \left|\frac{1}{3}(g(t) - 2g(2b - t) + g(4b - 3t)) - \frac{1}{3}g(4b - 3t) + \right.$$

$$\left. + \frac{2}{3}(g(t) - 2g(b) + g(2b - t))\right| \leq \left(\frac{1}{3} + \frac{1}{3} + \frac{2}{3}\right)\omega_2(g, h) = \frac{4}{3}\,\omega_2(g, h).$$

iii) By using i) we have $|g(2b - t)| \leq \omega_2(g, h)$ since $2b - t \in [a, b]$ and then $|g(t)| \leq |g(t) - 2g(b) + g(2b - t)| + |g(2b - t)| \leq 2\omega_2(g, h)$.

iv) We have $t - 2h \in [a, b]$ and hence $|g(t - 2h)| \leqslant \omega_2(g, h)$. By using relations i) and iii) already proved, we have $|g(t) - g(t - h)| = \left| \frac{1}{2} (g(t) - 2g(t - h) + g(t - 2h)) + \frac{1}{2} g(t) - \frac{1}{2} g(t - 2h) \right| \leqslant \left(\frac{1}{2} + 1 + \frac{1}{2}\right) \omega_2(g, h) = 2\omega_2(g, h)$.

v) From the conditions of the hypothesis it results $2x - a \leqslant b$, $4x - t - 2a = (2x - a) + (2x - t - a) \geqslant 2x - a - h \geqslant a$, and $4x - t - 2a \leqslant 4x - b - 2a \leqslant b$. Hence from i) we have $|g(4x - t - 2a)| \leqslant \omega_2(g, h)$. Next we deduce: $|g(4x - t - 2a)| = |(g(t) - 2g(2x - a) + g(4x - t - 2a)) + 2g(2x - a) - g(t)| \geqslant |g(t) - 2g(2x - a)| - \omega_2(g, h) = |(g(t) - 2(g(2x - a) - 2g(x) + g(a)) - 4g(x)| - \omega_2(g, h) \geqslant |g(t) - 4g(x)| - 3\omega_2(g, h)$.

Hence $|g(t) - 4g(x)| \leqslant 4\omega_2(g, h)$.

From i) and ii) we have $|g(x)| \leqslant \omega_2(g, h)$ and respectively $|g(t)| \leqslant \frac{4}{3}\omega_2(g, h)$. If $g(x) \cdot g(t) < 0$ we have $|g(t) - 4g(x)| = |g(t)| + 4 \cdot |g(x)|$ and $|g(t) - g(x)| = |g(t)| + |g(x)|$. By denoting $p = |g(t)|$ and $q = |g(x)|$ and by taking into account that:

$$\max \left\{p + q : p \geqslant 0,\ q \geqslant 0,\ 0 \leqslant p \leqslant \frac{4}{3},\ p + 4q \leqslant 4\right\} = 2,$$

we obtain relation v). If $g(x) \cdot g(t) \geqslant 0$ then we have: $|g(t) - g(x)| \leqslant \frac{4}{3} \omega_2(g, h) \leqslant 2\omega_2(g, h)$.

vi) By using relations v) and iv) we have $|g(t - h) - g(x)| \leqslant 2\omega_2(g, h)$ and respectively $|g(t - h) - g(t - 2h)| \leqslant 2\omega_2(g, h)$. Then $|g(t) - g(x)| = |(g(t) - 2g(t - h) + g(t - 2h)) + (g(t - h) - g(x)) + (g(t - h) - g(t - 2h))| \leqslant (1 + 2 + 2)\, \omega_2(g, h) \leqslant 5\omega_2(g, h)$.

vii) Denote $y_j = b + j \cdot \frac{t - b}{k}$ for $j = 0, k$. We have:

$$g(t) = g(y_k) = \sum_{j=1}^{k-1} j \cdot (g(y_{k-j+1}) - 2g(y_{k-j}) + g(y_{k-j-1})) + k \cdot g(y_1) + (1 - k) \cdot g(y_0).$$

We have $g(y_0) = g(b) = 0$ and $y_1 \in (b, b + h]$ and from iii) we have $|g(y_1)| \leqslant 2\omega_2(g, h)$. We have also $|y_{j+1} - y_j| \leqslant h$. Then:

$$|g(t)| \leqslant \left(\left(\sum_{j=1}^{k-1} j\right) + 2k\right) \omega_2(g, h) = \left(\frac{1}{2} k^2 + \frac{3}{2} k\right) \omega_2(g, h).$$

Finally by using i) we have $|g(t) - g(x)| \leqslant |g(t)| + |g(x)| \leqslant \left(\frac{1}{2} k^2 + \frac{3}{2} k + \frac{1}{2} + 1\right) \omega_2(g, h)$.

LEMMA 2.3. *The modulus* $\omega_2$ *satisfies the condition* $(A(\theta_1))$ *on* $\mathcal{F}_b(I)$ *where* $\theta_1 : [0, \infty) \to [0, \infty)$ *is defined by:*

$$\theta_1(t) = \begin{cases} 0, & (t = 0) \\ 1, & (t \in (0, 1]) \\ 2, & \left(t \in \left(1, \frac{3}{2}\right]\right) \\ 3, & \left(t \in \left(\frac{3}{2}, 2\right]\right) \\ 5, & \left(t \in \left(2, \frac{5}{2}\right]\right) \\ \frac{1}{2}\,(]t[)^2 + \frac{3}{2}\,]t[+ 1, & \left(t \in \left(\frac{5}{2}, \infty\right)\right) \end{cases} \tag{2.3}$$

*Proof.* In order to apply Lemma 1.1 let $a < b$ two points of $I$, let $g \in \mathcal{F}_b(I)$ be such that $g(a) = 0 = g(b)$ and let us denote $h = \frac{b-a}{2}$. Relation (1.11)—i) results from (2.2)—i). Consider now a point $y \in [a, a+h]$, and let $t \in (b, \infty) \cap I$.

If $\left|\frac{t-y}{h}\right| \leq \frac{3}{2}$ then certainly $y \in \left[a + \frac{h}{2}, a+h\right]$, $t \in \left(b, b + \frac{h}{2}\right]$ and $t \leq 2y - a + h$. Then from (2.2) —v) we have:

$$|g(t) - g(y)| \leq 2\omega_2(g, h) = \theta_1\left(\left|\frac{t-y}{h}\right|\right)\omega_2(g, h).$$

If $\left|\frac{t-y}{h}\right| \in \left(\frac{3}{2}, 2\right]$ then $t \in (b, b+h] \cap I$. Then from (2.2)—i) and iii) we have: $|g(t) - g(y)| \leq |g(t)| + |g(y)| \leq 3\omega_2(g, h) = \theta_1\left(\left|\frac{t-y}{h}\right|\right)\omega_2(g, h)$.

If $\left|\frac{t-y}{h}\right| \in \left(2, \frac{5}{2}\right]$ then $y \in \left[a + \frac{h}{2}, a+h\right]$, $t \in \left(b+h, b + \frac{3}{2}h\right]$ and $t \leq 2y - a + 2h$. Then from (2.2)—vi) it results:

$$|g(t) - g(y)| \leq 5\omega_2(g, h) = \theta_1\left(\left|\frac{t-y}{h}\right|\right)\omega_2(g, h).$$

Finally, if $\left|\frac{t-y}{h}\right| > \frac{5}{2}$ we take into account that in (2.2)—vii)

$$k = 1 + \left]\left|\frac{t-b}{h}\right|\right[ \leq \left]\left|\frac{t-y}{h}\right|\right[ \text{ and hence } |g(t) - g(y)| \leq$$

$$\leq \left(\frac{1}{2}k^2 + \frac{3}{2}k + 1\right)\omega_2(g, h) \leq \theta_1\left(\left|\frac{t-y}{h}\right|\right)\omega_2(g, h).$$

If $y \in [a+h, b]$ then we take the interval $-I = \{-t;\ t \in I\}$, $a^* = -b$, $b^* = -a$ and we define $g^* \in \mathcal{F}_b(-I)$ by $g^*(t) = g(-t)$, $(t \in -I)$. We have

$-y \in [a^*, a^* + h)]$. Then from above we deduce for $t \in (-\infty, a) \cap I$ : $|g(t) - g(y)| = |g^*(-t) - g^*(-y)| \leqslant \theta_1\left(\left|\frac{-t+y}{h}\right|\right) \omega_2(g^*, h) = \theta_1\left(\left|\frac{t-y}{h}\right|\right) \omega_2(g, h)$.
Thus the condition (1.11)—ii) is completly proved. Consequently we can apply Lemma 1.1.

COROLLARY 2.1. *For every $j = 2,3,4$ the modulus $\omega_2$ satisfies the condition $(A(\theta_j))$ on $\mathcal{F}_b(I)$, where $\theta_j$: $[0, \infty) \to [0, \infty)$ are defined by*

$$\theta_2(t) = \theta_2^s(t) = 1 + t^s, \qquad (t \geqslant 0), \tag{2.4}$$

*where s is a real number such that $s \geqslant 2$.*

$$\theta_3(t) = (1 + ]t[)^2, \ (t \geqslant 0) \tag{2.5}$$

$$\theta_4(t) = 1 + \frac{1}{4} t^4, \ (t \in [0, 1]) \ \textit{and,}$$

$$\theta_4(t) = 1 + \frac{3}{4} t + \frac{1}{4} t^4, \ (t \in (1, \infty)). \tag{2.6}$$

*Proof.* Corrollary 2.1 results from Lemma 2.3 and from the inequalities: $\theta_j(t) \geqslant \theta_1(t)$, $(t \geqslant 0)$ for $j = 2, 3, 4$.

Indeed, for $\theta_2$ we have the following cases: if $t \in [0, 1]$ then $\theta_2(t) \geqslant 1$, if $t \in \left(1, \frac{3}{2}\right]$ then $\theta_2(t) > 2$, if $t \in \left(\frac{3}{2}, 2\right]$ then $\theta_2(t) > \frac{13}{4} > 3$, if $t \in \left(2, \frac{5}{2}\right]$ then $\theta_2(t) > 5$, if $t \in \left(\frac{5}{2}, 3\right]$ then $\theta_2(t) > \frac{29}{4} > 6$ and if $t > 3$ then $\theta_2(t) \geqslant 1 + t^2 \geqslant \frac{1}{2} t^2 + \frac{3}{2} t + 1 \geqslant \theta_1(t)$.

For $\theta_3$ we have the following cases: $\theta_3(0) = 0$, if $t \in (0, 1]$ then $\theta_3(t) = 1$, if $t \in (1, 2]$ then $\theta_3(t) = 4$, if $t \in (2, 3]$ then $\theta_2(t) = 9$ and if $t > 3$ then also $\theta_3(t) > \theta_1(t)$.

For $\theta_4$ we have the following cases: if $t \in [0, 1]$ then $\theta_4(t) \geqslant 1$, if $t \in \left(1, \frac{3}{2}\right]$ then $\theta_4(t) > 2$, if $t \in \left(\frac{3}{2}, 2\right]$ then $\theta_4(t) > 1 + \frac{3}{4} + \frac{81}{64} > 3$, if $t > 2$ then $\theta_4(t) > 1 + \frac{3}{4} ]t[ + \frac{1}{4} (]t[)^4 \geqslant \theta_1(t)$, since $u^3 - 2u - 3 \geqslant 0$ for $u \geqslant 2$.

THEOREM 2.1. *Let $F : V \subset \mathcal{F}_b(I) \to \mathbf{R}$ be a linear positive functional that satisfies the condition $(B(x))$, where $x \in$ Int $I$. Let the real number $h > 0$. Then for every $j = 1, 2, 3, 4$ if $\theta_j \circ \left|\frac{e_1 - xe_0}{h}\right| \in V$ we have*

$$|F(f) - f(x) \leqslant F\left(\theta_j\left(\left|\frac{e_1 - xe_0}{h}\right|\right)\right)\omega_2(f, h), \ (f \in V). \tag{2.7}$$

*Proof.* The relations (2.7) result directly from Theorem 1.1 and Lemma 2.3 (for $j = 1$) and Corrollary 2.1 (for $j = 2, 3, 4$).

COROLLARY 2.2. *Let $F : V \subset \mathcal{F}_b(I) \to \mathbf{R}$ be a linear positive functional that satisfies the condition $(B(x))$, where $x \in$ Int $I$. We suppose that $e_2 \in V$. Then:*

$$|F(f) - f(x)| \leqslant (1 + h^{-2} \cdot F((e_1 - xe_0)^2))\omega_2(f, h), \ (f \in V, \ h > 0). \tag{2.8}$$

$$|F(f) - f(x)| \leqslant 2\omega_2\left(f, \ (F((e_1 - xe_0)^2))^{\frac{1}{2}}\right), \ (f \in V). \tag{2.9}$$

*Remark 2.2.* i) Relation (2.8) improves the estimates in [6] and [10]. ii) Relation (2.7) for $j = 3$ is given in [12] but with another proof. (see Remark 2.1).
iii) Relation (2.9) improves the estimate in [5].
iv) The estimate (2.7) for $j = 4$ is specially constructed for the Bernstein polynomial.

3. **Applications to linear positive operators.** By using the estimates in the previous section we can obtain pointwise estimates for the linear positive operators that perserve linear functions.

A. *THE OPERATORS OF S. N. BERNSTEIN.* For any $n \in \mathbf{N}$, $n \geq 1$ the polynomial operator of S. N. Bernstein $B_n : \mathcal{F}_b([0, 1]) \to \mathcal{P}_n$ is defined by the formula:

$$B_n(f, x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \cdot p_{nk}(x), \text{ where} \tag{3.1}$$

$$p_{nk} = \binom{n}{k} \cdot x^k(1 - x)^{n-k}, \quad x \in [0, 1] \text{ and } f \in \mathcal{F}_b([0, 1])$$

THEOREM 3.1. *We have:*

$$| B_n(f, x) - f(x) | \leq 1,25\ \omega_2\left(f, \left(\frac{4x(1-x)}{n}\right)^{\frac{1}{2}}\right), \tag{3.2}$$

$$(n \in \mathbf{N},\ n \geq 1,\ x \in [0, 1],\ f \in \mathcal{F}_b([0, 1])).$$

*Proof.* If $x = 0$ or $x = 1$ the relation (3.2) is obvious. Let $x \in (0, 1)$. Then (3.2) results from (2.8) if we take $h = \left(\frac{4x(1-x)}{n}\right)^{\frac{1}{2}}$, and from the relation: $B_n((e_1 - xe_0)^2, x) = \frac{x(1-x)}{n}$.

*Remark 3.1.* The value 1,25 of the constant in (3.2) improves the constant equal to 3.25 given in [6].

THEOREM 3.2. *We have:*

$$1 \leq \sup_{\substack{n \in N \\ n \geq 1}} \sup_{\substack{f \in \mathcal{F}_b([0,1]) \\ f \neq \text{linear}}} \frac{\| B_b(f) - f \|}{\omega_2\left(f, n^{-\frac{1}{2}}\right)} \leq 1,115, \tag{3.3}$$

where $\| \cdot \|$ is the sup-norm.

*Proof.* In order to obtain the right inequality in (3.3) it is enough to estimate the difference $|B_n(f, x) - f(x)$ for $x \in (0, 1))$. We apply (2.7) for $j = 4$ to the functional $f \to B_n(f, x)$, $h = n^{-\frac{1}{2}}$ and $f \in \mathcal{F}_b([0, 1])_n$: $|B_n(f, x) - f(x)| \leq$
$\leq \left[1 + \frac{3}{4}\sum_k{}' n^{\frac{1}{2}}\left|\frac{k}{n} - x\right| \cdot p_{nk}(x) + \frac{n^2}{4} B_n((e_1 - xe_0)^4, x)\right]\omega_2\left(f, n^{-\frac{1}{2}}\right)$, where $\sum_k{}'$

denotes the sum taken over those indices $k$ for which $\left|\frac{k}{n} - x\right| > n^{-\frac{1}{2}}$.
In [14] and [15] it is proved the following inequality

$$n^{\frac{1}{2}} \sum_k{}' \left|\frac{k}{n} - x\right| \cdot p_{n,k}(x) \leqslant \varkappa - 1, \tag{3.4}$$

where $\varkappa = \frac{4306 + 837\sqrt{6}}{5832} \leqslant 1{,}09$ is the Sikkema's constant.

By denoting $T_{ns} = \sum_{k=0}^{n} (k - nx)^r p_{n,k}(x)$, from the relation:

$T_{n\,s+1}(x) = x(1 - x)\,[T'_{n\,s}(x) + ns\,T_{n\,s-1}(x)]$, $(s \geqslant 1,\ n \geqslant 1)$ that is proved in [8] and by taking into account $T_{n\,0}(x) = 1$, $T_{n\,1}(x) = 0$ we obtain:

$$T_{n\,4}(x) = x(1 - x)\,[(3n^2 - 6n)\,x(1 - x) + n] \cdot (n \geqslant 1).$$

We have $T_{n\,4}(x) \leqslant T_{n\,4}\left(\frac{1}{2}\right) = \frac{3}{16}n^2 - \frac{1}{8}n \leqslant \frac{3}{16}n^2$ Therefore:

$$|B_n(f, x) - f(x)| \leqslant \left(1 + \frac{3}{4} \cdot 0{,}09 + \frac{3}{64}\right) \omega_2(f, n^{-\frac{1}{2}}) = (1,1143\ldots)\omega_2(f, n^{-\frac{1}{2}}).$$

For the left inequality let us consider an arbitrary real number $\varepsilon\ 0 > \varepsilon < 1$ and let the function $f_\varepsilon \in F_b([0, 1])$ defined by:
$f_\varepsilon(t) = t/\varepsilon$ fot $0 \leqslant t \leqslant \varepsilon$, and $f_\varepsilon(t) = (1 - t)/(1 - \varepsilon)$ for $\varepsilon < t \leqslant 1$.

Let $n = 1$. Then: $\omega_2\left(f_\varepsilon, n^{-\frac{1}{2}}\right) = \omega_2(f_\varepsilon, 1) = |f_\varepsilon(0) - 2f_\varepsilon(\varepsilon) + f_\varepsilon(2\varepsilon)| = 1/(1-\varepsilon)$, and $B_1(f_\varepsilon, \varepsilon) - f_\varepsilon(\varepsilon) = -1$. Hence.

$$\frac{\|B_1(f_\varepsilon) - f_\varepsilon\|}{\omega_2(f_\varepsilon, 1)} \geqslant 1 - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary choosen the desired inequality is proved.

*Remark 3.2.* The value of the upper bound in (3.3) improves the value 1,43 given in [12]. In [2] and [6] it is given the value 3,25.

*B. THE OPERATORS OF SZASZ–MIRAKJAN.* For any $n \in \mathbf{N}$, $n \geqslant 1$ let $S_n : \mathcal{F}_b([0, \infty)) \to C[0, \infty)$ be the operator Szasz-Mirakjan given by the formula:

$$S_n(f, x) = \exp(-nx) \sum_{k=0}^{\infty} (nx)^k (k!)^{-1} f\left(\frac{k}{n}\right), \text{ for } x \in [0, \infty) \tag{3.5}$$

and $f \in V = \{f \in \mathcal{F}_b([0, \infty)), f$ such that the serie in (3.5) is absolutely con vergent for any $x \in [0, \infty)\}$.

From (2.8) by taking into account that $S_n((e_1 - xe_0)^2, x) = x/n$, and $S_n(f, 0) = f(0)$ we have:

THEOREM 3.3 *We have:*

$$|S_n(f, x) - f(x)| \leqslant (1 + x)\omega_2(f, n^{-\frac{1}{2}}), \tag{3.6}$$
$$(n \in \mathbf{N},\ n \geqslant 1,\ f \in V,\ \textit{and}\ x \in [0, \infty)).$$

A. *THE OPERATORS OF D. D. STANCU — PARTICULAR CASE.*

For $n \in \mathbf{N}$, $n \geqslant 1$ and the real number $\alpha \geqslant 0$ let the operator $L_n^\alpha : \mathcal{F}_b([0, 1]) \to \mathfrak{L}_n$ defined by:

$$L_n^\alpha(f, x) = \sum_{k=0}^{n} \binom{n}{k} \cdot \frac{\prod_{j=0}^{k-1} (x + j\alpha) \prod_{j=0}^{n-k-1} (1 - x + j\alpha)}{\prod_{j=0}^{n-1} (1 + j\alpha)} \cdot f\left(\frac{k}{n}\right) \tag{3.7}$$

for $f \in \mathcal{F}_b([0, 1])$ and $x \in [0, 1]$.

Using (2.8), since $L_n^\alpha((e_1 - xe_0)^2, x) = x(1 - x) \dfrac{1 + n\alpha}{n(1 + \alpha)}$, we have:

THEOREM 3.4. *For any $\alpha \geqslant 0$ and $n \in \mathbf{N}$, $n \geqslant 1$ we have:*

$$|L_n^\alpha(f,x) - f(x)| \leqslant 1{,}25\ \omega_2\left(f, \left(\frac{1 + n\alpha}{n(1 + \alpha)}\right)^{\frac{1}{2}}\right), \tag{3.8}$$

$$(f \in \mathcal{F}_b([0, 1]),\ x \in [0, 1]).$$

*Remark 3.3.* The constant 1,25 in (3.8) improves the constant equal to 3,25 given in [7].

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STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXXV, 1, 1990

# ON FEEBLY CONTINUOUS FUNCTIONS

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Dedicated to Professor Á. Pal on his 60th anniversary

Received: April 27, 1989
AMS subject classification: 54C05, 54C10

**REZUMAT — Asupra funcţiilor slab continue.** În lucrare se obţin unele descompuneri ale continuităţii slabe şi condiţii suficiente pentru ca o funcţie să fie slab continuă (continuă).

1. **Introduction.** In [4] Levine defines a set $A$ in a topological space $X$ to be semi-open if there exists an open set $U$ such that $U \subset A \subset \mathrm{Cl}(U)$, where $\mathrm{Cl}(U)$ denotes the closure of $U$. A set $A$ is semi-closed if its complement is semi-open. The intersection of all the semi-closed sets containing a set $A$ is the semi-closure of $A$, denoted by $\mathrm{sCl}(A)$. In a topological space $X$ a set $A$ is feebly open [6] if there exists an open set $U$ such that $U \subset A \subset \mathrm{sCl}(U)$. A set is feebly-closed if its complement is feebly-open. The intersection of all the feebly-closed sets containing a set $A$ in a topological space is the feebly-closure of $A$, denoted by $\mathrm{fCl}(A)$.

A set $A$ in a topological space $X$ is said to be $\alpha$-set [11] (preopen set [9]) if $A \subset \mathrm{Int}\,(\mathrm{Cl}(\mathrm{Int}(A)))$ $(A \subset \mathrm{Int}(\mathrm{Cl}(A)))$. It is known [3] that $A$ is $\alpha$-set if and only if $A$ is feebly-open.

In [4] Levine introduced the concept of semi-continuous functions. Neubrunnová [10] showed that semi-continuity is equivalent to quasi continuity due to Marcus [7]. On the other hand, Levine [5] introduced the concept of weakly continuous functions. In 1973, Popa and Stan [17] introduced the concept of weakly quasicontinuous functions. Weak quasi continuity is implied by both quasi continuity and weak continuity which are independent of each other.

It is shown in [14] that weak continuity is equivalent to semi-weak continuity in the sens of Costovici [1]. Recently, Mashhour et. al. [8] have defined and investigated a new class of functions called $\alpha$-continuous functions. These functions have been investigated by Noiri [13], [16]. In [6] Maheshwari and Jain introduced the concept of feebly continuous functions. These functions have been investigated by Lee and Chae [2] and the present author [19]. By [3] follows that feebly continuity is equivalent to $\alpha$-continuity. Recently, Noiri, [15] has introduced the notion of weakly $\alpha$-continuous func-

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tions (or weakly feebly continuous functions) and studied several properties weakly $\alpha$-continuous functions.

Ih this paper, we obtain some decompositions of feebly continuity and me sufficient conditions for a function to be feebly continuous (continuo

2. **Definitions.** DEFINITION 1. A function $f: X \to Y$ is said to be *fee continuous* [6] (reps. *precontinuous* [9]) if for every open set $V$ of $Y$, $f^{-1}(V)$ feebly-open (resp. preopen) in $X$.

DEFINITION 2. A function $f: X \to Y$ is said to be *weakly feeblycontinu* [15] (resp. *weakly continuous* [5]) if for each $x \in X$ and each open set $V$ conta ning $f(x)$, there exists a feebly-open (resp. open) set $U$ containing $x$ such th $f(U) \subset \mathrm{Cl}(V)$.

DEFINITION 3. A function $f: X \to Y$ is said to be *quasi continuous* (resp. *weakly quasi continuous* [17]) at $x \in X$ if for every open set $U$ containi $x$ and every open set $V$ containing $f(x)$, there exists a non-empty open set such that $G \subset U$ and $f(G) \subset V$, (resp. $f(G) \subset \mathrm{Cl}(V)$).

If $f$ is quasi continuous (resp. weakly quasi continuous) at every point $X$, then it is called quasi continuous (resp. weakly quasi continuous).

DEFINITION 4. A function $f: X \to Y$ is said to be *weakly almost continuo* [20] if for each open set $V$ containing $f(x)$ ,there exists a preopen set $U$ conta ning $x$ such that $f(U) \subset \mathrm{Cl}(V)$.

Weakly almost continuity is implied by both weak continuity and preco tinuity which are independent of each other.

DEFINITION 5. A point $x$ of a topological space $X$ is said to be *$\theta$-adhere* of a subset $A \subset X$ if $A \cap \mathrm{Cl}(V) \neq \emptyset$ for every open set $V$ containing $x$. The of all $\theta$-adherents points of $A$ is called the *$\theta$-closure* of $A$ and is denoted by $\mathrm{Cl}_\theta(A$ If $A = \mathrm{Cl}_\theta A$, then $A$ is called *$\theta$-closed*. The complement of a $\theta$-closed set is c led *$\theta$-open*. It is shown in [21] that $\mathrm{Cl}(A) = \mathrm{Cl}_\theta(A)$ for every open set $A$ a $\mathrm{Cl}_\theta(A)$ is closed for every subsets $A$ of $X$.

By [8], [15] and [20] we have the followins diagram

| | | | | |
|---|---|---|---|---|
| quasi continuous | ← | feebly continuous | → | precontinuous |
| ↓ | | ↓ | | ↓ |
| weakly | | weakly | | weakly |
| quasi continuous | ← | feebly continuous | → | almost continuous |

3. **Main results.** In [18, Theorem 1] it is proved that a precontinuous a quasi continuous function is weakly continuous. In [8] Mashhour et obtained the result that every precontinuous and quasi continuous function feebly continuous. In [13, Theorem 3.2] Noiri proved the following theore

THEOREM 1. *A function $f: X \to Y$ is feebly continuous if and only if precontinuous and quasicontinuous.*

As an improved of these results, we have the following two theorems:

THEOREM 2. *A function $f: X \to Y$ is feebly continuous if and only if precontinuous and weakly quasi continuous.*

*Proof.* Let $G$ be any open set of $Y$ and $x \in X$ such that $f(x) \in G$. As $f$ weakly quasi continuous by [14, Theorem 4.1] there is a semi-open set $U_1 \subset$ containing $x$ such that $f(U_1) \subset \mathrm{Cl}(V)$. As $f$ is precontinuous by [9, Theorem there is a preopen set $U_2 \subset X$ containing $x$ such that $f(U_2) \subset V$. By [13, Lemm

3.1] $U = U_1 \cap U_2$ is feebly-open, $x \in U$ and $f(U) \subset V$. By [8, Theorem 1.1] $f$ is feebly continuous. Conversely, if $f: X \to Y$ is feebly continuous, by Theorem 1, $f$ is precontinuous and quasi continuous, hence weakly quasi continuous.

THEOREM 3. *A function $f: X \to Y$ is feebly continuous if and only if $f$ is weakly almost continuous and quasi continuous.*

*Proof.* It is similar to the proof of Theorem 2.

The following theorem is proved in [5]:

THEOREM 4. *A function $f: X \to Y$ is continuous if and only if $f$ is weakly continuous and $f^{-1}$ (Fr(G)) is closed in $X$ for every open set $G \subset Y$.*

For the feebly continuous functions we have the following two theorems.

THEOREM 5. *A function $f: X \to Y$ is feebly continuous if and only if $f$ is weakly quasi continuous and $f^{-1}$ (Fr(G)) is preclosed in $X$ for every open set $G \subset Y$.*

*Proof.* If $f$ is feebly continuous, then $f$ is precontinuous [8] and by [9, Theorem 1] the inverse image under mapping $f$ of each closed set of $Y$ is preclosed in $X$, thus $f^{-1}$ (Fr(G)) is preclosed in $X$. If $f$ is feebly continuous then $f$ is quasi continuous [8], hence weakly quasi continuous.

Conversely, let $G$ be any open set of $Y$ and $x \in X$ such that $f(x) \in G$. The function $f$ being weakly quasi continuous by [14, Theorem 1] there is a semi-open set $V \subset X$ containing $x$ such that $f(V) \subset \mathrm{Cl}(G)$. Let us consider the set $U = V - f^{-1}(\mathrm{Fr}(G)) = V \cap (X - f^{-1}(\mathrm{Fr}(G))$. As $f^{-1}$ (Fr(G)) is preclosed in $X$ $X - f^{-1}$ (Fr(G)) is preopen. By [13, Lemma 3.1]. $U$ is feebly open. As $x \in V$ and $f(x) \in G$ it follows that $x \in U$. Let $y \in U$. Then $y \in V$ and $y \bar{\in} f^{-1}(\mathrm{Fr}(G))$, thus $f(y) \in \mathrm{Cl}(G)$ and $f(y) \bar{\in} \mathrm{Fr}(G)$, thus $f(y) \in G$. As $U$ is feebly open and contains $x$, it follows by [8, Theorem 1] that $f$ is feebly continuous.

THEOREM 6. *A function $f: X \to Y$ is continuous if and only if $f$ is weakly almost continuous and $f^{-1}(\mathrm{Fr}(G))$ is semi-closed in $X$ for every open set $G \subset Y$.*

*Proof.* It is similar to the proof of Theorem 5.

TEHOREM 7. *Let $Y$ be a regular space. Then the following conditions are equivalent for a function $g: X \to Y$:*

(a) *$g$ is feebly continuous.*

(b) *$g^{-1}$ $(\mathrm{Cl}_\theta(B))$ is feebly closed in $X$ for every subset $B$ of $Y$.*

(c) *$g$ is weakly feebly continuous.*

(d) *$g^{-1}$ $(V)$ is feebly closed in $X$ for every $\theta$-closed set $F$ of $Y$.*

(e) *$g^{-1}$ $(V)$ is feebly open for every $\theta$-open set $V$ of $Y$.*

(f) *$g$ is continuous.*

*Proof.* (a) $\to$ (b): Since $\mathrm{Cl}_\theta(B)$ is closed in $Y$ for every subset $B$ of $Y$, $g^{-1}$ $(\mathrm{Cl}_\theta(B))$ is feebly closed by [8, Theorem 1.1].

(b) $\Rightarrow$ (c): Let $B$ be any subset of $Y$. Then we have

$$\mathrm{fCl}(g^{-1}(B)) \subset \mathrm{fCl}(g^{-1}(\mathrm{Cl}_\theta(B))) = g^{-1}(\mathrm{Cl}_\theta(B)).$$

Therefore, $g$ is weakly feebly continuous by ([15, Lemma 2.2].

(c) $\Rightarrow$ (d): Let $F$ be any $\theta$-closed set of $Y$. By [15, Lemma 2.2] we have $\mathrm{fCl}(g^{-1}(F)) \subset g^{-1}(\mathrm{Cl}_\theta(F)) = g^{-1}(F)$. Therefore, $g^{-1}(F)$ is feebly-closed in $X$.

(d) $\Rightarrow$ (e): Let $V$ be any $\theta$-open set of $Y$. By hypothesis $g^{-1}(Y - V) = X - g^{-1}(V)$ is feebly-closed in $X$ and hence $g^{-1}(V)$ is feebly-open in $X$.

(e) $\Rightarrow$ (a): Since $Y$ is regular, $\mathrm{Cl}_\theta(B) = \mathrm{Cl}(B)$ for every subset $B$ of $Y$ and hence open set is $\theta$-open. Therefore, $g$ is feebly continuous.

(a) $\Leftrightarrow$ (f): Follows from [8, Remark].

*Remark 1.* In Theorem 7 the following implications hold even if the assu tions that $Y$ is regular is dropt: (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Leftrightarrow$ (e).

DEFINITION 6. A topological space $X$ is said to be *rim-compact* if e point of $X$ has a base of neighbourhoods with compact frontiers.

THEOREM 8. *If $Y$ is a rim-compact space and $f: X \to Y$ is weakly fe continuous function with the closed graph, then $f$ is feebly continuous.*

*Proof.* Let $x \in X$ and $V$ be any open set of $Y$ containing $f(x)$. Since Y rim-comact, there exist an open set $W$ such that $f(x) \in W \subset V$ and the fro $\mathrm{Fr}(W)$ is compact. It is obvious that $f(x) \bar{\in} \mathrm{Fr}(W)$. Thus for each $y \in \mathrm{Fr}($ we have $(x, y) \bar{\in} G(f)$. Since $G(f)$ is closed, there exists open sets $U_y(x) \subset X$ $V(y) \subset Y$ containing $x$ and $y$, respectively, such that $f(U_y(x)) \cap V(y) =$ The family $\{V(y) : y \in \mathrm{Fr}(W)\}$ is a cover of $\mathrm{Fr}(W)$ by open sets of $Y$. Since Fr is compact, there exist a finite number of points $y_1, y_2, \ldots, y_n$ in $\mathrm{Fr}(W)$ such t $\mathrm{Fr}(W) \subset \mathrm{U}\{V(y_i) : 1 \leqslant i \leqslant n\}$. Since $f$ is weakly feebly continuous, there ex a feebly-open set $U_0$ containing $x$ such that $f(U_0) \subset \mathrm{Cl}(W)$. Put $U = U_0$ $\cap \{\cap U_{y_i}(x) : 1 \leqslant i \leqslant n\}$. Then by [16, Lemma 3.3] $U$ is feebly-open $(U) \cap (Y - W) = \emptyset$. This shows that $f(U) \subset V$ and by [8, Theorem 1. is feebly cotinuous.

THEOREM 9. *If $Y$ is rim-compact Hausdorff and $f$ is weakly feebly co nuous, then $f$ is continuous.*

*Proof.* By [12, Theorem 4], $Y$ is regular and it follows from Theoren that $f$ is continuous.

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STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXXV, 1, 1990

# FINITE DIMENSIONAL VECTOR CONTRACTIONS AND THEIR FIXE POINTS

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Dedicated to Professor A. Pal. on his 60th anniversar

*Received: May 10, 1989*
*AMS subject calssification: 54H25*

**REZUMAT. — Contracţii generalizate şi puncte fixe.** În lucrare sînt obţinute mai multe teoreme de punct fix pentru aplicaţii $T: X \to X$, contractive în raport cu o metrică generalizată $d: X^2 \to R^n$.

**0. Introduction.** The well known Banach's fixed point theorem has bee extended in many directions until now. One of the most interesting of them consists in taking the metric $d$ of the ambient space $X$ with values in $\mathbf{R}^n_+$ and t impose upon the considered self-mapping $T$ of $X$ a contractivity condition lik

$$d(Tx, Ty) \leqslant A(d(x, y)), \quad x, y \in X. \qquad (K_1$$

where $A: \mathbf{R}^n \to \mathbf{R}^n$ is a (vector) *increasing* operator satisfying certain regularit assumptions. In particular, when $A$ is *linear* that is

$$A = (a_{ij}), \text{ with } a_{ij} \geqslant 0, \ 1 \leqslant i, j \leqslant n,$$

a basic result of this type has been established in 1964 by Perov [15] for th case of $A$ being an *a-matrix*, and in 1973 by Matkowski [13] for $A$ satis fying a *normality* condition (see the terminology of Section 1). Concerning th relationships between these notions, the answer — precised in the above mentio ned section — is that an $a$-matrix is necessarily normal and viceversa or, equi valently, that Perov's fixed point result is identical with Matrowski's. This im plicitly means that all „vector" type fixed point results based on such (linear techniques are immediately reductible to their "scalar" counterparts: for exam ple, the main statement in Balakrishna Reddy and Subrahmany am [2] is identical (from this viewpoint) with that obtained by Delean and Marinescu [20], the Czerwik's theorem in nothing but a varian of Reich's [16] and, finally, that the contractor type fixed points result esta blished in Balakrishna Reddy and Subrahmanyam [3] is re ductible to the Altman's one [1, ch. I, §5], as well shall prove in Section 2 The nonlinar case will be also considered under this perspective, in Section 3 where afixed point result extending in a strict way the one obtained by Kw pisz [12] is being formulated; the reduction to the Banach's fixed poin principle is then discussed for the obtained statement, in the spirit of Bessa ga's metrization theorem [4]. Some further considerations about these questio will be made elsewhere.

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**1. Normal and asymptotic type matrices.** Let $\mathbf{R}^n$ denote the usual vector mensional space, $\mathbf{R}^n_+$ the standard *positive cone* in $\mathbf{R}^n$, and $\leqslant$ the induced *ring*. Also, let $(\mathrm{R}^\circ_+)^n$ denote the interior of $\mathbf{R}^n$ and $<$ the *strict* (non-reflexive) *ring* induced by it, in the sense $x = (\zeta_1, \ldots, \zeta_n) < y = (\eta_1, \ldots, \eta_n)$ provided $\zeta_i < \eta_i$, $1 \leqslant i \leqslant n$. We shall cate by $L(\mathrm{R}^n)$ the (linear) space of all (real) $n \times n$ matrices $A = (a_{ij})$ and $\mathbf{R}^n$) the positive cone of $L(\mathbf{R}^n)$ consisting of all matrices $A = (a_{ij})$ with $a_{ij} \geqslant$ , $1 \leqslant i, j \leqslant n$. For each $A \in L\ (\mathbf{R}^n)$, let us put

$$\nu(A) = \inf\{\lambda \geqslant 0\,;\ Az \leqslant \lambda z,\ \text{for some } z < 0\}$$

call the considered matrix, *normal*, when $\nu(A) < 1$, or, equivalently, when system of inequalities

$$a_{i1}\zeta_1 + \ldots + a_{in}\zeta_n < \zeta_i,\ 1 \leqslant i \leqslant n, \tag{S}$$

a solution $z = (\zeta_1, \ldots, \zeta_n) < 0$, as it can be readily seen. Concerning the lem of characterizing this class of matrices, the following result obtained Matkowski [13] must be taken into consideration. Denote

$$a^{(1)}_{ij} = 1 - a_{ij},\ i = j \tag{N$_1$}$$

$$= a_{ij}\quad ,i \neq j,\ 1 \leqslant i, j \leqslant n$$

inductively (for $1 \leqslant k \leqslant n - 1$)

$$a^{(k+1)}_{ij} = a^{(k)}_{kk}a^{(k)}_{ij} - a^{(k)}_{ik}a^{(k)}_{ij},\ i = j \tag{N$_k$}$$

$$= a^{(k)}_{kk}a^{(k)}_{ij} + a^{(k)}_{ik}a^{(k)}_{kj},\ i \neq j,\ k + 1 \leqslant i, j \leqslant n.$$

THEOREM 1. *The matrix* $A \in L_+\ (\mathrm{R}^n)$ *is normal, if and only if*

$$a^{(i)}_{ii} > 0,\ 1 \leqslant i \leqslant n. \tag{C$_1$}$$

*Proof.* As already noted, the argument may be found in Matkowski's pa- however, for the sake of completeness, we shall supply a proof which dif- in part, from the original one.

*Necessity.* Assume (S) has a solution $z\ (\zeta_1 \ldots, \zeta_n) > 0$, that is

$$\left\{\begin{array}{l} a^{(1)}_{11}\zeta_1 - a^{(1)}_{12}\zeta_2 - a^{(1)}_{13}\zeta_3 - \ldots - a^{(1)}_{1n}\zeta_n > 0 \\ -\ a^{(1)}_{21}\zeta_1 + a^{(1)}_{22}\zeta_2 - a^{(1)}_{23}\zeta_3 - \ldots - a^{(1)}_{2n}\zeta_n > 0 \\ -\ a^{(1)}_{31}\zeta_1 - a^{(1)}_{32}\zeta_2 + a^{(1)}_{33}\zeta_3 - \ldots - a^{(1)}_{3n}\zeta_n > 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ -\ a^{(1)}_{n1}\zeta_1 - a^{(1)}_{n2}\zeta_2 - a^{(1)}_{n3}\zeta_3 - \ldots + a^{(1)}_{nn}\zeta_n > 0 \end{array}\right. \tag{S$^1_1$}$$

view of

$$a^{(1)}_{ij} \geqslant 0,\ i \leqslant i, j \leqslant n,\ i \neq j,$$

must have

$$a^{(1)}_{11}, \ldots, a^{(1)}_{nn} > 0\,;$$

e, in particular, $(\mathrm{C}_1)$ is fulfilled for $i = 1$. Further, let us multiply the first inequality of $(S_1)$ by the factor $a_{i1}^{(1)}/a_{11}^{(1)} \geqslant 0$ and add it to the $i$-th relation of the same system for $i = 2, 3, \ldots, n$; one gets (if we take into account the notations $(N_2)$ plus $a_{11}^{(1)} > 0$)

$$\begin{cases} a_{11}^{(1)}\zeta_1 - a_{12}^{(1)}\zeta_2 - a_{13}^{(1)}\zeta_3 - \ldots - a_{1n}^{(1)}\zeta_n > 0 \\ \qquad a_{22}^{(2)}\zeta_2 - a_{23}^{(2)}\zeta_3 - \ldots - a_{2n}^{(2)}\zeta_n > 0 \\ \quad - a_{32}^{(2)}\zeta_2 + a_{33}^{(2)}\zeta_3 - \ldots - a_{3n}^{(2)}\zeta_n > 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \quad - a_{n2}^{(2)}\zeta_2 - a_{n3}^{(2)}\zeta_3 - \ldots + a_{nn}^{(2)}\zeta_n > 0 \end{cases} \tag{$S_2$}$$

Since

$$a_{ij}^{(2)} \geqslant 0,\ 2 \leqslant i, j \leqslant n,\ i \neq j,$$

we necessarily have

$$a_{22}^{(2)}, \ldots, a_{nn}^{(2)} > 0$$

that is, $(C_1)$ is fulfilled for $i = 1, 2$. Now, if we multiply the second inequality of $(S_2)$ by the factor $a_{i2}^{(2)}/a_{22}^{(2)} \geqslant 0$ and add it to the $i$-th relation of the same system for $i = 3, \ldots, n$, one obtains that $(C_1)$ will be fulfilled with $i = 1, 2, 3$; continuing in this way, it is clear that, after $n$ steps, $(C_1)$ will be entirely satisfied.

*Sufficiency*. Let us admit that $(C_1)$ holds; our goal is to find a solution $z = (\zeta_1, \ldots, \zeta_n)$ for (S) with $\zeta_i > 0$, $1 \leqslant i \leqslant n$. To do this, let us start with the system

$$\begin{cases} a_{11}^{(1)}\xi_1 - a_{12}^{(1)}\xi_2 - a_{13}^{(1)}\xi_3 - \ldots - a_{1n}^{(1)}\xi_n = \sigma_1 \\ -a_{21}^{(1)}\xi_1 + a_{22}^{(1)}\xi_2 - a_{23}^{(1)}\xi_3 - \ldots - a_{2n}^{(1)}\xi_n = \sigma_2 \\ -a_{31}^{(1)}\xi_1 - a_{32}^{(1)}\xi_2 + a_{33}^{(1)}\xi_3 - \ldots - a_{3n}^{(1)}\xi_n = \sigma_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ -a_{n1}^{(1)}\xi_1 - a_{n2}^{(1)}\xi_2 - a_{n3}^{(1)}\xi_3 - \ldots + a_{nn}^{(1)}\xi_n = \sigma_n \end{cases} \tag{$S_1'$}$$

where $y = (\sigma_1, \ldots, \sigma_n) > 0$ is arbitrary fixed. Denote

$$\sigma_i^{(1)} = \sigma_i,\ 1 \leqslant i \leqslant n \tag{$N_1'$}$$

and, inductively (for $1 \leqslant k \leqslant n - 1$)

$$\sigma_i^{(k+1)} = a_{kk}^{(k)}\sigma_i^{(k)} + a_{ik}^{(k)}\sigma_k^{(k)},\ k + 1 \leqslant i \leqslant n. \tag{$N_k$}$$

Let us apply to $(S_1')$ the same transformations as in $(S_1)$; one gets (by these notations)

$$\begin{cases} a_{11}^{(1)}\xi_1 - a_{12}^{(1)}\xi_2 - a_{1\,13}^{(1)}\xi_3 - \ldots - a_{1n}^{(1)1}\xi_n = \sigma_1^{(1)} \\ \qquad a_{22}^{(2)}\xi_2 - a_{23}^{(2)}\xi_3 - \ldots - a_{2n}^{(2)2}\xi_n = \sigma_2^{(2)} \\ \quad - a_{23}^{(2)}\xi_2 + a_{33}^{(2)}\xi_3 - \ldots - a_{3n}^{(2)2}\xi_n = \sigma_3^{(2)} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \quad - a_{n2}^{(2)}\xi_2 - a_{n3}^{(2)}\xi_3 - \ldots + a_{nn}^{(2)}\xi_n = \sigma_n^{(2)} \end{cases} \tag{$S_2'$}$$

where, in addition,

$$\sigma_i^{(2)} > 0,\ 2 \leqslant i \leqslant n,$$

in view of

$$a_{ij}^{(1)} \geqslant 0,\ i \leqslant i,\ j \leqslant n,\ i \neq j.$$

If we apply to this new system the same transformations as in $(S_2)$ and, further, iterate these upon the obtained system $(S_3')$, etc., we arrive at the following diagonal form of $(S_1')$

$$\left\{\begin{array}{l} a_{11}^{(1)}\xi_1 - a_{12}^{(1)}\xi_2 - a_{12}^{(1)}\xi_3 - \ldots - a_{1n}^{(1)}\xi_n = \sigma_n^{(1)} \\ \qquad a_{22}^{(2)}\xi_2 - a_{23}^{(2)}\xi_3 - \ldots - a_{2n}^{(2)}\xi_n = \sigma_2^{(2)} \\ \qquad\qquad a_{33}^{(3)}\xi_3 - \ldots - a_{3n}^{(3)}\xi_n = \sigma_3^{(3)} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \qquad\qquad\qquad a_{nn}^{(n)}\xi_n = \sigma_n^{(n)} \end{array}\right. \qquad (S_n')$$

In view of $(C_1)$ plus the above positivity properties, it is clear that the unique solution $z = (\zeta_1, \ldots, \zeta_n)$ of $(S_n')$ must satisfy $\zeta_i > 0$, $1 \leqslant i \leqslant n$; this, combined with the equivalence between $(S_1')$ and $(S_n')$, ends the argument. q.e.d.

A useful variant of Matkowski's condition $(C_1)$ may now be depicted as follows. Letting $I$ denote the unitary matrix in $L(\mathbf{R}^n)$, indicate by $\Delta_1, \ldots, \Delta_n$ the successive "diagonal" minors of $I - A$ ,that is

$$\Delta_1 = 1 - a_{11},\ \Delta_2 = \det\begin{pmatrix} 1 - a_{11} & - a_{12} \\ - a_{21} & 1 - a_{22}\end{pmatrix},\ \ldots,\ \Delta_n = \det(I - A).$$

By the transformations we used in passing from $(S_1)$ to $(S_2)$ and from this to $(S_3)$, etc., one gets at once

$$\Delta_i = a_{ii}^{(i)},\ 1 \leqslant i \leqslant n,$$

so that, $(C_1$ may be (formally) written as

$$\Delta_i > 0,\ 1 \leqslant i \leqslant n. \qquad (C_2)$$

After Perov's terminology [15], a matrix $A \in L_+(\mathbf{R}^n)$ satisfying $(C_2)$ will be termed an *a-matrix*. We therefore proved that the notions of normal matrix and $a$-matrix are identical (over $L_+(\mathbf{R}^n)$).

For the applications we have in mind, further characterizations of this class of matrices are necessary. To this and, let $||\cdot||$ denote one of the usual *norms* in $\mathbf{R}^n$ (e.g., the euclidean one) as well as its compatible *matrix norm* introduced as

$$||A|| = \sup\{||Ax||;\ ||x|| \leqslant 1\{,\ A \in L(\mathrm{R}_n).$$

By convention, a matrix $A \in L(\mathbf{R}^n)$ will be said to be *asymptotic* if $A^p \to 0$

$$A^p \to 0 \text{ as } p \to \infty \text{ or, equivalently if}$$

$$A^p x \to 0 \text{ as } p \to \infty, \text{ for each } x \in \mathbf{R}^n.$$

The following simple result will be in effect for us.

LEMMA 1. *The matrix $A \in L_+(\mathbf{R}^n)$ is asymptotic if and only if $\sum_{p \in \mathbf{N}} A^p$ con ges in $(L(\mathbf{R}^n), ||\cdot||)$, the sum of this series being (the matrix) $(I - A)^{-1}$ (he $I - A$ is invertible in $L(\mathbf{R}^n)$ and its inverse belongs to $L_+(\mathbf{R}^n)$).*

*Proof.* Let the matrix $A$ be asymptotic. If $x \in \mathbf{R}^n$ satisfies $(I - A)x$ : then, the immediate consequence of such an assumption (by repeatedly apply $A$ to the equivalent equality)

$$x = A^p x, \text{ for all } p \in \mathbf{N}.$$

gives us $x = 0$ (if we take the limit as $p \to \infty$) proving that $(I - A)^{-1}$ ex as an element of $L(\mathbf{R}^n)$. Moreover, in view of

$$I - A^p = (I - A)(I + A + \ldots + A^{p-1}),\ p \geqslant 1,$$

one gets (again by a limit process)

$$I = (I - A)(I + A + A^2 + \ldots),$$

which ends the proof. q.e.d.

Before answering the question of which relationships exist between the c of matrices we just introduced and the preceding ones, let us give a useful re ming result about normal matrices.

LEMMA 2. *Let $A \in L_+(\mathbf{R}^n)$ be a normal matrix. Then, an equivalent n $||\cdot||^0$ in $\mathbf{R}^n$ and a number $\lambda$ in $(0, 1)$ exist with the properties*

a) $||Ax||^0 \leqslant \lambda ||x||^0,\ x \in \mathbf{R}^n_+$

b) $0 \leqslant x < y$ *implies* $||x||^0 < ||y||^0$.

*Proof.* By the hypothesis about $A$, we have promised a vector $z = \ldots, \zeta_n) < 0$ and a number $\lambda \in (\nu(A), 1)$ with $Az \leqslant \lambda z$. Let us introduce norm $||\cdot||^0$ in $\mathbf{R}^n$ by the convention

$$||x||^0 = \max \{\xi_i|/\xi_i;\ 1 \leqslant i \leqslant n\},\ x = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n.$$

(As a matter of fact, an equivalent norm exists generated by a scalar proc and satisfying (a) + (b) above; see in this direction Perov's paper we alre quoted. But, for our purposes, it will suffice having a non-smooth norm of kind). By the obvious relation

$$x \leqslant (||x||_0)z, \text{ for all } x \in \mathbf{R}^n_+,$$

one gets (if we take into account the choice of $z$)

$$Ax \leqslant (||x||_0)Az \leqslant \chi||x||^0 \cdot z,\ x \in \mathbf{R}^n_+$$

wherefrom, (a) results at once. Since (b) is almost immediate, we omit the tails. It remains only to prove that $||\cdot||^0$ is equivalent with, e.g., the euclid norm $||\cdot||$ in $\mathbf{R}^n$, But this follows easily by the relation (deduced from (D

$$||x||(\zeta_1^2 + \ldots + \zeta_n^2)^{-1/2} \leqslant ||x||^0 \leqslant ||x|| \cdot \max(\zeta_i^{-1};\ 1 \leqslant ui \leqslant n),\ n \in$$

and this completes the argument. q.e.d.

We are now in position to give a complete answer to the above posed blem.

THEOREM 2. *The notions of normal matrix and asymptotic matrix identical over $L$ $(\mathbf{R}^n)$.*

*Proof.* Let $A \in L_+(\mathbf{R}^n)$ be normal. By Lemma 2, we found an equiva norm $||\cdot||^0$ on $\mathbf{R}^n$ with the properties (a) + (b). From the former, it is clear that

$$A^p x \to 0 \text{ as } p \to \infty, \text{ for all } x \in \mathbf{R}^n_+,$$

which (by the properties of the cone $\mathbf{R}^n_+$) is equivalent with the asymptotic property. Conversely, assume $A \in L_+(\mathbf{R}^n)$ is asymptotic. Letting $x > 0$ be arbitrary fixed, put

$$z = \sum_{p \geq 0} A^p x \text{ (evidently, } z > 0).$$

As $Az = \sum_{p \geq 1} A^p x$, we necessarily have $z = x + Az$ which, combined with the choice of $x$, gives $Az < z$. The proof is complete. q.e.d.

We cannot close these developments without giving another characterization of asymptotic (or normal) matrices in terms of *spectral radius*; this fact — of marginal importance for the next section — is, however, sufficiently interesting for itself to be added here. Let $A \in L(\mathbf{R}^n)$ be a given matrix. Under the natural immersion of $\mathbf{R}^n$ in $\mathbf{C}^n$, let us call the number $\lambda \in \mathbf{C}$ an *eigenvalue* of $A$, provided $Az = \lambda z$, for some $z \in \mathbf{C}^n$ different from zero (called in this case an *eigenvector* of $A$). The number

$$\rho(A) = \sup\{|\lambda|;\ \lambda = \text{eigenvalue of } A\}$$

will be referred to as the *spectral radius* of $A$.

LEMMA 3. *The matrix $A \in L_+(\mathbf{R}^n)$ is asymptotic if and only if $\rho(A) < 1$.*

*Proof.* Suppose $A$ is asymptotic. For each eigenvalue, $\lambda$, of $A$, let $z \in \mathbf{C}_n$ be any eigenvector of $A$ corresponding to it. We therefore have $Az = \lambda z$ and this gives

$$A^p z = \lambda^p z, \text{ for all } p \in \mathbf{N}.$$

By the choice of $A$, plus $z \neq 0$, we must have $\lambda^p \to 0$ as $p \to \infty$, which cannot happen unless $|\lambda| < 1$. Hence $\rho(A) < 1$. Conversely, assume that the matrix $A = (a_{ij})$ in $L_+(\mathbf{R}^n)$ satisfies $\rho(A) < 1$, and put $A_\delta = a_{ij}{}^{(\varepsilon)})$, $\varepsilon > 0$, where

$$a_{ij}{}^{(\varepsilon)} = a_{ij} + \varepsilon,\ 1 \leqslant i, j \leqslant n.$$

We have $\rho(A_\varepsilon) < 1$, when $\varepsilon > 0$ is small enough (one may follow), to prove this, a direct argument based on the obvious fact

$$\det (A_\varepsilon) \to \det (A) \text{ when } \varepsilon \to 0+).$$

Now, $A_\varepsilon$ being a matrix over $\mathbf{R}^0_+$ (in the sense

$$a^{(\varepsilon)}_{ij} > 0,\ 1 \leqslant i, j \leqslant n).$$

for each $\varepsilon > 0$ we have, by the Perron—Frobenius theorem (see, e.g., Bushell [6] for a fixed point argument involving Hilbert's projective metric) that $A_\varepsilon$ has a positive eigenvalue $\mu = \mu(\varepsilon) > 0$ (which, in view of $\rho(A_\varepsilon) < 1$, must satisfy $\mu < 1$) as well as an eigenvector $z > 0$. Combining these informations, one gets

$$Az \leqslant A_\varepsilon z = \mu z < z.$$

Hence, $A$ is normal. This, along with Theorem 2, completes the argument. q.e.d.

The technical interest of this proof consists in avoiding the use of the norn Jordan forms (cf. R u s [17, ch. IV, §1]). For the standard argument we re to G a n t m a c h e r [11, ch. XIII, §3].

2. **Mappings of linear contractive type.** Let $X$ be an abstract set. In the f lowing, the notion of $\mathbf{R}^n$*—valued metric* on $X$ will be used to designate any fu tion $d : X^2 \to \mathbf{R}^n_+$ satisfying the usual sufficiency, symmetry and transitiv properties (the last one with $\leqslant$ standing for the ordering induced by $\mathbf{R}^n_+$). T *convergence* property of a sequence $(x_p)_{p\in\mathbf{N}}$ in $X$ towards a limit $x \in X$ bei introduced as $d(x_p, x) \to 0$ for $p \to \infty$, we have that each convergent seque is necessarily *d-Cauchy* (that is, $d(x_p, x_q) \to 0$ as $p, q \to \infty$) but the conve is not general valid; the ambient space $X$ will be said to be *d—complete* wh each $d$—Cauchy sequences converges. Finally, given the self-mapping $T$ of $X$, c it *A —contractive* (for $A \in L\ (\mathbf{R}^n)$) when ($K_1$) is being satisfied. We are interested the sequel to determine under what specific assumptions about $A$ it is true th the considered self-mapping has fixed points. In this direction, a basic answer concentrated in

THEOREM 3. *Suppose $X$ is $d$—complete and $T: X \to X$ is an* A—*co tractive mapping with $A \in L_+(\mathbf{R}^n)$ being normal (or, equivalently, asymptot Then*

a) *$T$ has a unique fixed point, $z \in X$*

b) *for each $x \in X$, the sequence $(T^px)_{p\in\mathbf{N}}$ converges to this fixed point un an error evluation expressed as*

$$d(T^px, z) \leqslant (I - A)^{-1}A^pd(x, \quad Tx),\ p \in \mathbf{N}. \qquad ($$

*Proof*. The standard one may be found, e.g., in P e r o v [15]. We sh give here an alternative argument based on ordering principles. For each $x \in X$, one has, by ($K_1$)

$$d(T^px,\ T^{p+1}x) \leqslant A^pd(x, Tx),\ p \in \mathbf{N}$$

and therefore, by Lemma 1, the function

$$\varphi(x) = \sum_{p\in\mathbf{N}} d(T^px,\ T^{p+1}x), \quad x \in X$$

is well defined and continuous from $X$ to $\mathbf{R}^n$. Of course, by this definition

$$d(x, Tx) = \rho(x) - \rho(Tx), \text{ for all } x \in X.$$

Now, if we define an ordering on $X$ by

$$x \leqslant y \text{ if and only if } d(x, y) \leqslant \rho(x) - \rho(y)$$

it is clear that each ascending (modulo $\leqslant$) sequence $(x_p)_{p\in\mathbf{N}}$ in $X$ is a $d$—Cau one, bounded from above. This, combined with a maximality result of the au [19] gives us that, for each $x \in X$, a maximal (modulo $\leqslant$) point $z \in X$ e with $x \leqslant z$. We claim this is our desired point. Indeed, as $z \leqslant Tz$, we ne sarily have $z$ is a fixed point of $T$. Moreover, noting that the structure ($X$ has maximal element (since $T$ has at most one fixed point) we actually h

$$T^px \leqslant z, \text{ for all } p \in \mathbf{N}$$

which, combined with $\varphi(z) = 0$ yields the desired conclusion. q.e.d.

As implicitly results from Lemma 2, the above statement is nothing but an equivalent formulation of the Banach fixed point principle; see also the remark in the above quoted Perov's paper. Indeed, letting $e : X^2 \to \mathbf{R}_+$ be the metric on $X$ defined as

$$e(x, y) = ||d(x, y)||^0, \; x, y \in X, \tag{D'}$$

it suffices to note that $X$ is e—complete and the contractivity condition $(K_1)$ implies

$$e(Tx, Ty) \leqslant \lambda\, e(x, y), \; x, y \in X,$$

wherefrom the assertion follows. The meta-conclusion we may derive from this could be formulated as: each contractivity argument involving $\mathbf{R}n$ — valued metrics and normal/asymptotic matrices may be translated into a contractivity argument involving ordinary metrics and subunitary positive numbers. The following examples will clarify this assertion.

*Example 1.* Let $X_i$, $1 \leqslant i \leqslant n$, be Hausdorff uniform spaces whose topologies are generated by the pseudometric families $(d_{w(i)}\,;\; w(i) \in \Gamma_i)$, $1 \leqslant i \leqslant n$, respectively and, putting $X = X_1 \times \ldots \times X_n$, let the operators $T_i : X \to X_i$, $1 \leqslant i \leqslant n$, be such that: for each $n$-uple $w = (w(1), \ldots, w(n))$ in $\Gamma = \Gamma_1 \times \times \ldots \times \Gamma_n$ there exists a normal matrix $A(w) = (a_{ij}^{(w)})$ in $L\ (\mathbf{R}^n)$, with

$$d_{w(i)}(T_i(x), T_i(y)) \leqslant \sum_j a_{ij}^{(w)} d_{w(j)}(x_j, y_j), \; 1 \leqslant i \leqslant n, \tag{K$_2$}$$

$$\textit{for each couple } x = (x_1, \ldots, x_n),\; y = (y_1, \ldots, y_n) \in X$$

Denoting

$$d_w(x, y) = \max\{d_{w(i)}(x_i, y_i)/\zeta_i(w)\,;\; 1 \leqslant i \leqslant n\}$$

$$(x = (x_1, \ldots, x_n),\; y = (y_1, \ldots, y_n) \in X)$$

where $z(w) = (\zeta_1(w), \ldots, \zeta_n(w)) > 0$ is that introduced by the normality condition, the family $(d_w\,;\; w \in \Gamma)$ defines a Hausdorff uniform structure over $X$, sequentially complete if all uniform structures on $X_i$, $1 \leqslant i \leqslant n$, are sequentially complete. We also put $T = (T_1, \ldots, T_n)$ (in the sense

$$T(x) = T_1(x), \ldots, T_n(x)),\; x \in X).$$

Then, the above inequality $(K_2)$ gives

$$d_{w(i)}(T_i(x),\; T_i(y)) \leqslant \Big(\sum_j a_{ij}^{(w)} \zeta_j(w)\Big) d_w(x, y) \geqslant \lambda(w)\zeta_i(w) d_w(x, y),\; 1 \leqslant i \leqslant n,$$

that is

$$d_w(Tx, Ty) \leqslant \chi(w) d_w(x, y),\; x, y \in X,$$

where $\chi(w) \in (0, 1)$ is again introduced by the normality condition imposed upon $A(w)$. Now, by the uniform version of the Banach contraction principle (see, e.g., Deleanu and Marinescu [10]) we have promised a fixed point for $T$; in other words, the main result in Balakrishna Reddy and Surahmanyam [2] can be completely reduced to these known statements.

*Example 2.* Let $(X_i, d_i)$, $1 \leqslant i \leqslant n$, be complete metric spaces and, indicating by $K(X_i)$ the class of all (nonempty) closed parts of $X_i$, $1 \leqslant i \leqslant n$, assume the operators $T_i : X = X_1 \times \ldots \times X_n \to K(X_i)$, $1 \leqslant i \leqslant n$, are such that a couple of matrices $B = (b_{ij})$, $C = (c_{ij})$ in $L_+(\mathbf{R}^n)$ with $A = B + C$ normal, and a number $\mu \in (0,1 - \nu(A))$ may be found with the property

$$H_i(T_i(x),\ T_i(y)) \leqslant \sum_j b_{ij} d_j(x_j,\ y_j) + \sum_j c_{ij}\,\mathrm{dist}_j(x_j,\ T_j(x)) +$$

$$\mu\,\mathrm{dist}_i(y_i,\ T_i(y)),\ 1 \leqslant i \leqslant n,\ \textit{for any pair}\ x = (x_1, \ldots, x_n) \qquad (\mathrm{K}_3)$$
$$y = (y_1, \ldots, y_n)\ \textit{of (arbitrary) points in}\ X$$

(Here, $H_i(\ldots)$, $\mathrm{dist}_i(\ldots)$ are the usual Hausdorff pseudometric and, respectively, the usual distance function in $(X_i, d_i)$, $1 \leqslant i \leqslant n$). Let $\lambda \in (\nu(A), 1 - \mu)$ and $z = (\zeta_1, \ldots, \zeta_n) > 0$ be such that $Az \leqslant \lambda z$ (possibly, by the definition of $\nu(A)$). Introducing a metric structure on $X$ by the convention

$$d(x, y) = \max\{d_i(x_i, y_i)/\zeta_i;\ 1 \leqslant i \leqslant n\}$$
$$(x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n) \in X)$$

and noting that, for each $n$-uple of pairs $Y_i, Z_i \in K(X_i)$, $1 \leqslant i \leqslant n$, and each point $x = (x_1, \ldots, x_n) \in X$ we have

$$H(Y, Z) = \max\{H_i(Y_i, Z_i)/\zeta_i;\ 1 \leqslant i \leqslant n\}$$
$$\mathrm{dist}\ (x, Y) = \max\{\mathrm{dist}_i(x_i,\ Y_i)/\zeta_i;\ 1 \leqslant i \leqslant n\}$$

where

$$Y = Y_1 \times \ldots \times Y_n,\ Z = Z_1 \times \ldots \times Z_n\ \text{(of course, } Y, Z \in K(X))$$

the above relations give (the notation $T = (T_1, \ldots, T_n)$ having the "multi-valued" meaning of the preceding one), for $1 \leqslant i \leqslant n$,

$$H_i(T_i(x),\ T_i(y)) \leqslant \sum_j b_{ij}\zeta_j\, d(x, y) + \sum_j c_{ij}\zeta_j\,\mathrm{dist}(x,\ Tx) +$$

$$+ \mu\zeta_i\,\mathrm{dist}(y,\ Ty) \leqslant (\lambda + \mu)\zeta_i \max\{d(x, y),\ \mathrm{dist}(x, Tx),\ \mathrm{dist}(y, Ty)\}$$

that is

$$H(Tx,\ Ty) \leqslant (\lambda + \mu)\max\{d(x, y),\ \mathrm{dist}(x, Tx),\ \mathrm{dist}(y, Ty),\ x, y \in X.$$

Therefore, all conditions in Reich's theorem [16] being fulfilled, we derive that a fixed point for $T$ must exist in $X$; this is exactly the main result in Czerwik [8] where a more technical proof based on a successive approximation method has been used.

*Example 3.* Let $X_i$, $Y_i$, $1 \leqslant i \leqslant n$, be Banach spaces and, putting $X = X_1 \times \ldots \times X_n$, let $X_*$ be a subset of $X$. Assume the operators $T_i : X_* \to Y_i$, $1 \leqslant i \leqslant n$, closed in the usual sense, are such that a normal matrix $A = (a_{ij})$ in $L_+(\mathbf{R}^n)$ and a number $\beta > 0$ exist with the properties: for each $x = (x_1, \ldots, x_n) \in X_*$ there may be determined bounded linear operators $\Gamma_i(x_i) \in L(Y_i, X_i)$, $1 \leqslant i \leqslant n$, in such a way that

$$\begin{cases} (x_1 + \Gamma_1(x_1)y_1, \ldots, x_n + \Gamma_n(x_n)y_n) \in X_* \quad \textit{and} \\ ||T_i(x_1 + \Gamma_1(x_1)y_1, \ldots, x_n + \Gamma_n(x_n)y_n) - T_i(x_1, \ldots, x_n) - y_i|| \leqslant \\ \sum_i a_{ij}||y_n||,\ 1 \leqslant i \leqslant n,\ \textit{for each}\ y = (y_1, \ldots, y_n) \in Y = Y_1 \times \ldots \times Y_n \end{cases} \qquad (\mathrm{K}_4)$$

$$||\Gamma_i(x_i)|| \leqslant \beta,\ 1 \leqslant i \leqslant n. \tag{K_5}$$

Letting $\lambda \in (\nu(A), 1)$ and $z = (\zeta_1, \ldots, \zeta_n) > 0$ ve such that $Az \leqslant \lambda z$, denote

$$\Gamma(x)y = (\zeta_1\Gamma_1(x_1)y_1, \ldots, \zeta_n\Gamma(x_n)y_n)$$

(for any couple $x = (x_1, \ldots, x_n) \in X_x$, $y = (y_1, \ldots, y_n) \in Y$)

and by $T$ the operator from $X_x$ to $Y$ defined as

$$T(x) = (\zeta_1 T_1(x), \ldots, \zeta_n T_n(x)),\ x \in X_x.$$

If we now write ($K_4$) for $(\zeta_1 y_1, \ldots, \zeta_n y_n) \in Y$ and take into account these conventions, one gets

$$||T(x + \Gamma(x)y) - T(x) - y|| \leqslant \lambda\ ||y||,\ y \in Y,$$

(here, $||\cdot||$ stands for the supremum norm in both $X$ and $Y$) as well as (by the relation contained in ($K_5$))

$$||\Gamma(x)|| \leqslant \gamma,\ \text{with } \gamma > 0 \text{ independent of } x \in X_x$$

Consequently, the contractor type Altman's theorem [1, ch. I, §5] is applicable; by that result, we deduce $T(x) = 0$ has a solution in $X_x$. In other words, the statement in Balakrishna Redd andy Subrahmanyam [3] is nothing but a variant of this "onedimensional" existence result.

The list of these examples may be continued with, e.g., the fixed point statements involving Krasnoselskij/Urysohn operators or the Altman type coincidence theorem appearing, in respectively, the first and second reference of the above quoted authors, but these seem to be not too representative; some further considerations about them will be done in a future paper.

3. **Some nonlinear versions.** Let $(X, d)$ be a complete ($\mathbf{R}^n$—valued) metric space and $T: X \to X$ an $A$—contractive self-mapping with $A: \mathbf{R}^n_+ \to \mathbf{R}^n_+$ (vector) *increasing* ($u \leqslant v$ implies $Au \leqslant Av$). As explicitly noted in the above section, a linearity assumption about $A$ (in the sense $A \in L_+(\mathbf{R}^n)$) makes the corresponding fixed point result involving $T$, reductible to Banach's. It remains now to study the nonlinear case (modulo $A$). Essentially, any fixed point statement of this type is again reductible to Banach's, in view of the Bessaga metrization theorem (cf. Deimling [9, ch. V, §17] and the references therein). But, this reduction process, obtained through a Zorn maximality argument, cannot be considered as effective; this "immaterial" dependence makes these statements be much more interesting than their linear counterparts. We shall start our discussion with the following result of this type obtained by Kwapisz [12].

THEOREM 4. *Let the self-mapping $T$ of $X$ be $A$-contractive (in the sense of*($K_1$)) *where the increasing operator $A$ fulfils*

$$\begin{cases} \text{for each } w \in \mathbf{R}^n_+, \text{ there exists } M(w) = \text{the maximal solution} \\ \text{in } \mathbf{R}^n_+ \text{ of } u = Au + w \end{cases} \tag{K_6}$$

$$u = Au \text{ for the only case } u = 0 \ (\text{that is, } M(0) = 0) \tag{K_7}$$

$$\begin{cases} (u_p)_{p \in N} \text{ decresing in } \mathbf{R}^n_+ \text{ and } u_p \to u \in \mathbf{R}^n_+ \text{ as } p \to \infty \\ \text{imply } Au_p \to Au \text{ as } p \to \infty. \end{cases} \tag{K_8}$$

*Then, conclusions* (a) + (b) *of Theorem 3 (minus the error evaluation form* (E)) *are valid.*

The maximal solution argument — originally developed by W a z e w s [20] — used in the above theorem is, of course, interesting from a techni viewpoint. But, a closed analysis shows it cannot cover (for $n = 1$) the stand result in this direction due to M a t k o w s k i [13]. In fact, the operator $A : \mathbf{R}^n_+ \to \mathbf{R}^n_+$ defined as (for some $z > 0$)

$A(u) = u - z$, when $u \geqslant z$

$= 0$, in the opposite situation

does not satisfy $(K_6)$ for $w > z$ but it obviously has all the properties invol in Matkowski's theorem. So, the question arises of whether or not an appropri substitution of these conditions by another ones may be performed in suc way that a covering property of this type be valid. The answer is affirmat and the conditions in question (containing in a strict sense $(K_6) - (K_8)$, as t above counterexample shows) are $(K_9) + (K_{10})$ below. In other words, the f lowing extended version of Theorem 4 may be formulated.

THEOREM 5. *Let the self-mapping $T$ of $X$ be $A$-contractive (in the sense* $(K_1)$) *where the (vector) increasing operator $A$ satisfies*

*the subset $S(A)$ of all $u \in \mathbf{R}^n_+$ with $Au < u$ is not empty!* (I

$A^p w \to 0$ *as* $p \to \infty$, *for each* $w \in \mathbf{R}^n_+$. (K

*Then, $T$ has a unique fixed point, $z \in X$, which is the limit of any sequence of su cessive approximations starting from an arbitrary point of $X$.*

*Proof.* Letting $x \in X$ be such a point, denote

$x_p = T^p x,\ p \in \mathbf{N}.$

We have, by $(K_1)$,

$d(x_p, x_{p+1}) \leqslant A^p(d(x_0, x_1)),\ p \in \mathbf{N},$

so that, by $(K_{10})$, $d(x_p, x_{p+1}) \to 0$ as $p \to \infty$. Let $u \in S(A)$ be arbitrary fix There exists, by the conclusion we just derived, a rank $p = p(u) \in \mathbf{N}$ w $d(x_p, x_{p+1}) \leqslant u - Au \leqslant u$. Without loss one may assume $p = 0$ (since, oth wise, we substitute $x = x_0$ by $x_p$ in these reasonings). We have successively ( $(K_1)$)

$d(x_0, x_1) \leqslant u \Rightarrow d(x_1, x_2) \leqslant Au \Rightarrow d(x_0, x_2) \leqslant$
$d(x_0, x_1) + d(x_1, x_2 \leqslant u - Au + Au = u,$
$d(x_0, x_2) \leqslant u \Rightarrow d(x_1, x_3) \leqslant Au \Rightarrow d(x_0, x_3) \leqslant$
$d(x_0, x_1) + d(x_1, x_3) \leqslant u - Au + Au = u,$ etc.,

and this gives

$d(x_0, x_q) \leqslant u,$ for all $q \in \mathbf{N}$.

Hence, again by $(K_1)$,

$d(x_p, x_{p+q}) \leqslant A^p u,\ p, q \in \mathbf{N},$

which tells us (via$(K_{10})$) that $(x_p)_{p \in N}$ is a $d$-Cauchy sequence. As $X$ is comple $x_p \to z$ as p $\to \infty$, for some $z \in X$; we claim $z$ is our desired point. In fact, lett $q \to \infty$ in the above relation yields (by the triangle inequality)

$d(x_p, z) \leqslant A^p u,$ for all $p \in \mathbf{N}$.

This (by $(K_1)$) again gives

$$d(x_{p+1}, Tz) \leqslant A^{p+1}u, \ p \in \mathbf{N}$$

and therefore (combining with $(K_{10})$) $x_p \to Tz$ as $p \to \infty$. By the uniqueness of the limit in $(X, d)$ we must then have $z = Tz$. Moreover, let $z^*$ be another fixed point of $T$ and put $w = d(z, z^*)$; one has

$$d(z, z^*) = d(T^p z, T^p z^*) \leqslant A^p w, \ p \in \mathbf{N},$$

wherefrom (by $(K_{10})$ again) $z = z^*$. The proof is complete. q.e.d.

In particular, for $n = 1$ (when (see [14]) $(K^9)$ is reductible to $(K_{10})$) this result is identical to the above quoted Matkowski's (cf. also Turinici [18]). On the other hand, when $A$ is linear, $(K^9)$ plus $(K_{10})$) are fulfilled in the normal case; so, Theorem 3 is a particular version of the above statement. As we already said, a reduction of Theorem 6 to Banach's contraction principle is (theoretically) possible, but very little can be said about the effectiveness of this procedure, in many situations (except the ones characterized by relations like

$$A(\tau z) \leqslant \lambda \tau z, \ \tau > 0 \ \text{(for some } \lambda \in (0, 1) \text{ and } z > 0)$$

when, by the construction of the associated metric $e : X^2 \to \mathbf{R}_+$ we indicated in (D') (see the preceding section) this objective is attainable). The situation is complicated by the fact that, under a weaker form of $(K_{10})$, namely

$$A^p u \to 0 \ \textit{as} \ p \to \infty, \ \textit{for each} \ u \in S(A) \qquad (K'_{10})$$

a fixed point of $T$ is to be reached, provided

$$d(x, Tx) \leqslant u, \ \textit{for some} \ x \in X, \ u \in S(A) \qquad (K_{11})$$

is being accepted and, moreover, two fixed points $z$, $z^*$ of $T$ are identical, whenever

$$d(z, z^*) \leqslant u, \ \text{for some} \ u \in S(A)$$

(the proof being almost evident, we omit the details). In other words, by these changes in Theorem 5, the fixed point of the ambient mapping is not unique, in general, and this makes Bessaga's reduction theorem be without object in such a case. We note in the same context that a sufficient condition for $(K'_{10})$ to be valid is the couple $(K_7) + (K_8)$ and also, that $(K'_{10})$ reduces to (hence is equivalent with) condition $(K_{10})$ provided $S(A)$ is cofinal in $\mathbf{R}^n_+$ (for each $v \in \mathbf{R}^n_+$ there exists $u \in S(A)$ with $v \leqslant u$).

The notion of contractive (in the sense of $(K_1)$) self-mapping may be also deemed in the larger context of the linear spaces endowed with a topological convergence structure and an ordering one, induced by a cone; some results in this direction have been obtained by Bohl [5, ch. IV, §4] and Collatz [7, ch. II, §11] in the case of $A$ being a linear and, respectively, nonlinear operator on that space leaving *invariant* the considered cone. These, however, do not cover ours, as it can be directly seen.

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STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXXV, 1, 1990

# SUITES D'APPLICATIONS MULTIVOQUES QUI SATISFONT À CERTAINES CONDITIONS DE CONTRACTIVITÉ

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À professeur A. Pal pour son 60e anniversaire

*Manuscrit reçu le 4 Septembre, 1989*
*AMS subject classification: 54H25, 54C60*

**REZUMAT. — Şiruri de multifuncţii care satisfac anumite condiţii de contractivitate.** În prezenta notă ne propunem să demonstrăm trei teoreme de punct fix comun pentru şiruri de aplicaţii multivoce definite pe spaţii metrice complete care satisfac inegalităţi contractive de tip (1) sau (5).

**1.** Dans cette note nous allons démontrer trois théorèmes de points fixes communs pour des suites d'applications multivoques définies sur des espaces métriques complets qui satisfont aux inégalités contractives de type (1) ou (5), en partant des résultats ressemblants obtenus par B. Fisher [2], H. Kaneko [5], T. Kubiak [4], I. A. Rus [7] et K. L. Singh, J. H. N. Whitfield [8], Nicoleta Negoescu [9] pour d'autres types de conditions de contractivité.

**2.** Soient $(X, d)$ un espace métrique, $A$ et $B$ deux ensembles nonvides de $X$ ex $x$ un élément fixé de $X$. Alors on définit:

$$D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}, \ D(x, A) = \inf\{d(x, A) : a \in A\},$$
$$H(A, B) = \max\{\sup_{a\in A} D(a, B), \sup_{b\in B} D(b, A)\}, \ H(x, A) = \sup\{d(x, a) : a \in A\},$$
$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}.$$

Aussi on définit les suivantes classes d'ensembles:

$$BN(X) = \{A : A \subset X, A \neq \emptyset \text{ et } A \text{ borné}\};\ CL(X) = \{A : A \subset X, A \neq \emptyset \text{ et } A \text{ fermé}\},$$

$$CB(X) = BN(X) \cap CL(X),\ Cpt(X) = \{A : A \subset X,\ A \neq \emptyset \text{ et } A \text{ compact}\}.$$

*Observations.* La fonction $D$ est continue (v. [3]).

Evidemment on a: $D(x, A) \leqslant \delta(x, A)$ et $\delta(A, B) \geqslant H(A, B)$.

La fonction $H$ est une métrique sur $CB(X)$ (et sur $Cpt(X)$) appellée la métrique de Hausdorff [v. [1]).

THEOREME 1. *Soient $(X, d)$ un espace métrique complet et $(S_n)$, $(T_n)$ deux suites d'applications multivoques de $X$ dans $CB(X)$. Supposons qu'il existe une constante $h$, $0 \leqslant h \leqslant 1$, telle que pour chaque $m, n \in \mathbf{N}^*$ et pour tous $x, y \in X$ on a:*

$$H^2(S_m x, T_n y) \leqslant h^2 \max\{d^2(x, y), D(x, S_m x)D(y, T_n y), D(x, T_n y)D(y, S_m x)\}. \quad (1)$$

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*Alors les suites d'applications multivoques $(S_n)$ et $(T_n)$ ont un point fixe commun, c'est-à-dire il existe un point $x \in X$ tel que $x \in S_m x \cap T_n x$ pour chaque $m, n \in \mathbf{N}^*$.*

*Démonstration.* Nous supposons d'abord que $h = 0$. Soient $x_0 \in X$ et $x_1 \in S_1 x_0$. Alors pour tous $n \in \mathbf{N}^*$, $D(x_1, T_n x_1) \leqslant H(S_1 x_0, T_n x_1) = 0$ donc $x_1 \in T_n x_1$. Aussi, pour tous $m \in \mathbf{N}^*$, $D(x_1, S_m x_1) \leqslant H(T_1 x_1, S_m x_1 = 0$ donc $x_1 \in S_m x_1 \cap T_n x_1$.

Nous supposons maintenant $h \neq 0$. Soient $x_0 \in x_1 \in S_1 x_0$, $x_2 \in T_1 x_1$. Nous construisons la suite $(x_n)$ oú $x_{2n-1} \in S_n x_{2n-2}$, $x_{2n} \in T_n x_{2n-1}$ sont tels que :

$$d(x_{2n-1}, x_{2n}) \leqslant \frac{1}{\sqrt{h}} H(S_n x_{2n-2}, T_n x_{2n-2}) \text{ et } \quad (2)$$

$$d(x_{2n}, x_{2n+1}) \leqslant \frac{1}{\sqrt{h}} H(S_{n+1} x_{2n}, T_n x_{2n-1}).$$

On suppose d'abord qu'il existe $n \in \mathbf{N}$ tel* que $x_n = x_{n+1}$.

Si $n$ est pair nous avons $x_{2n} \in S_{n+1}$. Alors, pour chaque $m$, il suit :

$$D^2(x_{2n}, T_m x_{2n}) \leqslant H^2(S_{n+1} x_{2n}, T_m x_{2n}) \leqslant h^2 \max\{d^2(x_{22n}, x_{2n}), D(x_{2n}, S_{n+1} x_{2n}) \cdot D(x_{2n}, T_m x_{2n}), \cdot D(x_{2n}, T_m x_{2n}),$$

$$D(x_{2n}, S_{n+1} x_{2n}) D(x_{2n}, T_m x_{2n})\} = 0 \text{ car } d(x_{2n}, x_{2n}) = 0, D(x_{2n}, S_{n+1} x_{2n}) = 0.$$

Donc $x_{2n} \in T_m x_{2n}$ pour chaque $m$.

De la même manière, pour chaque $m$, nous avons (en employant le fait que $x_{2n} \in T_{n+1} x_{2n}$) :

$$D^2(x_{2n}, S_m x_{2n}) \leqslant H^2(S_m x_{2n}, T_{n+1} x_{2n}) \leqslant h^2 \max\{d^2(x_{2n}, x_{2n}), D(x_{2n}, S_m x_{2n}) \cdot D(x_{2n}, T_{n+1} x_{2n}).$$

$$D(x_{2n}, T_{n+1} x_{2n}) D(x_{2n}, S_m x_{2n})\} = 0 \text{ car } d(x_{2n}, x_{2n}) = 0, D(x_{2n}, T_{n+1} x_2) = 0$$

Donc $D(x_{2n}, S_m x_{2n}) = 0$ et $x_{2n} \in S_m x_{2n}$, $\forall m \in \mathbf{N}^*$.

Donc dans le cas $x_n = x_{n+1}$, $n$ pair, nous avons montré que $x_2$ este un point fixe commun pour $S_p$ et $T_q$, p, $q \in \mathbf{N}^*$.

On obtient un résultat analogue si $n$ est impair.

Dans le cas $x_n \neq x_{n+1}$ pour chaque $n \in N^*$ nous montrerons que la suite $(x_n)$ est une suite de Cauchy. Nous obtenons, en employant les inégalités (1) et (2) :

$$d^2(x_{2n}, x_{2n+1}) \leqslant \left(\frac{1}{\sqrt{h}}\right)^2 H^2(S_{n+1} x_{2n}, T_n x_{2n-1}) \leqslant \frac{1}{h} h^2 \max\{d^2(x_{2n-1}, x_{2n}),$$

$$D(x_{2n}, S_{n+1} x_{2n}) D(x_{2n-1}, T_n x_{2n-1})\}.$$

$D(x_{2n}, T_n x_{2n-1}) D(x_{2n-1}, S_{n+1} x_{2n},$ donc nous a ons obtenu que :

$$d^2(x_{2n}, x_{2n+1}) \leqslant h \max\{d^2(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1}) d(x_{2n-1}, x_{2n})\}.$$

Il suit une des deux possibilités:

a) ou $\max\{d^2(x_{2n-1}. x_{2n}), d(x_{2n}, x_{2n+1})d(x_{2n-1}, x_{2n})\} = d^2(x_{2n-1}, x_{2n})$ et alors $d^2(x_{2n}, x_{2n+1}) \leqslant hd^2(x_{2n-1}, x_{2n})$ et $d(x_{2n}, x_{2n+1}) \leqslant \sqrt{hd}\,(x_{2n-1}, x_{2n})$,

b) ou $\max\{d^2(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1})d(x_{2n-1}, x_{2n})\} = d(x_{2n}, x_{2n+1})$
$d(x_{2n-1}, x_{2n})$
et alors $d^2(x_{2n}, x_{2n+1}) \leqslant hd(x_{2n}, x_{2n+1})d(x_{2n-1}, x_{2n})$ ou $d(x_{2n}, x_{2n+1}) \leqslant$
$\leqslant hd(x_{2n-1}, x_{2n})$

(car $x_{2n} \neq x_{2n+1}$ et $d(x_{2n}, x_{2n+1}) < 0$).

Mais $\max\{\sqrt{h}, h : 0 < h < 1\} = \sqrt{h}$ et donc:

$$d(x_{2n}, x_{2n-2}) \leqslant \sqrt{h}d(x_{2n+1}, x_{2n}), \ \forall n \in \mathbf{N}^*. \tag{3'}$$

De la même manière on peut montrer que:

$$d(x_{n2-1}, x_{2n}) \leqslant \sqrt{h}\, d(x_{2n-1}, x_{2n}). \tag{3''}$$

En répétant le raisonnement qui nous a conduit aux inégalités (3′) et (3″) nous obtenons: $d(x_{2n}, x_{2n+1}) \leqslant h^n d(x^0, x_1)$ et $d(x_{2n+1}, x_{2n+2}) \leqslant h^n d(x_1, x_2)$.

En notant par $r^0 = \max\{d(x^0, x_1), d(x_1, x_2)\}$, nous avons pour $m > n$:

$$d(x_n, x_m) \leqslant \sum_{i=0}^{m-n-1} d(x_{n+i}, x_{n+i+1}) \leqslant \sum_{i=0}^{m-n-1} h^{n+i}r_0 \leqslant h^n r^0(1 - h)$$

et $d(x_n, x_m) \to \infty$ pour $n \to \infty$. Donc $(x_n)$ est une suite de Cauchy.

Mais $(x_n) \subset X$ et $(X, d)$ est un espace métrique complet et il suit que $(x_n)$ est une suite convergente à un point $x \in X$ ($\lim\limits_{n\to\infty} x_n = x \in X$).

Alors: $D^2(x_{2m-2}, T_n x) \leqslant H^2(S_m x_{2m-2}, T_n x) \leqslant h^1 \max\{d^2(x_{2m-2}, x),$
$D(x_{2m-2}, S_m x_{2m-2})D(x, T_n x), \ D(x_{2m-2}, T_n x)D(x, S_m x_{2m-2})\}$ ou
$D^2(x_{2m-2}, T_n x) \leqslant h^2 \max\{d^2(x_{2m-2}, x), \ d(x_{2m-2}, x_{2m-1})D(x, T_n x),$
$D(x_{2m-2}, T_n x) \cdot d(x, x_{2m-1})\}$.

Pour $m \to \infty$ nous avons: $D^2(x, T_n x) \leqslant h^2 \max\{d^2(x, x), d(x, x)D(x, T_n x), D(x, T_n x)d(x, x)\} = 0$, ce que implique $D(x, T_n x) = 0$ et donc $x \in T_n x$ pour chaque $n$.

De la même manière on a $x \in S_m x$ pour chaque $m$, et la démonstration est finie.

Du théoreme 1 on a la suivante conséquence obtenue par nous dans [9], théorème 3.

THEOREME 2. *Soient $(X, d)$ un espace métrique complet et $S$, $T : X \to CB(X)$ deux applications multivoques qui satisfont à l'inégalité* (1) *pour $S_m = S$ et $T_n = T$. Alors $S$ et $T$ ont in point fixe commun $x \in X$.*

Aussi le théoréme 2 a lieu si $S = T : X \rightarrow BC(X)$.

Le théoreme 1 est vrai aussi si $(S_n)$ et $(T_n)$ sont deux suites d'opérateurs $S_n, T : X_n. X, \ n \in \mathbf{N}^*$.

Du théoréme 2 nous obtenons un théoréme analoque en remplaçant les applications multivoques $S, T : X \rightarrow CB(X)$ par deux opérateurs $S, T : X \rightarrow X$. Ce résultat est illustré par le suivant:

*Exemple.* Soit $X = \{1, 2, 3\}$. Nous définissons sur $X$ un métrique $d$ par: $d(1, 2) = 2, \ d(2, 3) = \frac{9}{4}, \ d(1, 3) = \frac{5}{4}$.

Soient $S, T : X \rightarrow X$ définis par $S1 = S2 = S3 = 1$ et $T1 = T3 = 1, T2 = 3$.

Alors il existe $\frac{5}{8} \leqslant h \leqslant 1$ tel que l'inégalité:

$d^2(Sx, Ty) \leqslant h^2 \max\{d^2(x, y), \ d(x, Sx)d(y, Ty), \ d(x, Ty)d(y, Sx)\}$ est satisfaite.

Alors il existe le point $x = 1$ tel que $1 = S1 = T1$

THEOREME 3. *Soient $(X, d)$ un espace métrique complet et $(S_n)$, $(T_n)$ des suites d'applications de $X$ dans $CB(X)$ qui sont convergentes ponctuellement á $S$ et $T$ respectivement. Nous supposons que $(S_n)$ et $(T_n)$ satisfont á la condition : il existe $0 \leqslant h < 1$ tel que:*

$$H^2(S_nx, T_ny) \leqslant h^2 \max\{d^2(x, y), D(x, S_nx)D(y, T_ny), S(x, T_ny)D(y, S_nx)\},$$
$$\forall n \in \mathbf{N}^*, \ \forall x, y \in X. \qquad (4)$$

Alors $S$ et $T$ ont un point fixe commun $u \in X$.

*Démonstration.* Nous montrons d'abord que pour tous $x, y \in X$ il suit:

$$|D(y, S_nx) - D(y, Sx)| \leqslant H(S_nx, Sx). \qquad (4')$$

En effet, soient $a \in S_nx$ et $b \in Sx$. Alors $d(y, a) \leqslant d(y, b) + d(b, a)$ et $d(y, a) \leqslant D(y, Sx) + D(a, Sx)$, donc $D(y, S_nx) \leqslant D(y, Sx) + H(Sx, S_nx)$.

De la même maniére, nous obtenons $D(y, Sx) \leqslant D(y, S_nx) + H(S_nx, Sx)$ et alors l'inégalité (4′) este vraie et donc on a (4′) $(D(y, S_nx) - S(y, Sx))^2 \leqslant H^2(S_nx, Sx)$.

On peut obtenir une inégalité analogue pour $T_n$ et $T$.

En employant l'inégalité (4″) et le fait que $H$ est continue, il suit que $S$ et $T$ satisfont aux hypothéses du théoréme 1 et donc $S, T$ ont un point fixe commun dans $X$.

THEOREME 4. *Soient $(X, d)$ un espace métrique complet, $(S_n)$, $(T_n)$ deux suites d'applications multivoques de $X$ dans $BN(X)$. Supposons qu'il existe une constante $h$, $0 \leqslant h < 1$, telle que pour tous $m, n \in \mathbf{N}^*$ et $x, y \in X$, nous avons:*

$$\delta^2(S_mx, T_ny) \leqslant h^2 \max\{d^2(x, y), H(x, S_mx)H(y, T_ny), D(x, T y)D(y, S_mx)\} \qquad (5)$$

Alors $(S_n)$ et $(T_n)$ ont un point fixe commun, c'est-á-dire il existe un point $u \in X$ tel que $u \in S_nu$ et $u \in T_nu$ pour tous $n \in \mathbf{N}^*$. Davantage, $S_m$ et $T_n$ ont un point fixe commun unique et $S_nu = T_mu = \{u\}$, $\forall m, n \in \mathbf{N}^*$.

*Démonstration.* Nous définissons une paire de suites d'applications $f_n, g_n:$ $: X \to X$ telles que : pour $x, y \in X$ soient $f_m x$, $g_n y$ des points dans $S_m x$ et $T_n y$ respectivement qui satisfont aux inégalités :

$$d(x, f_m x) \geqslant \sqrt{h} H(x, S_m x) \text{ et}$$

$$d(y, g\, y) \geqslant \sqrt{h} H(y, T\, y).$$

Alors, pour chaque $x, y \in X$ et $m, n \in \mathbf{N}^*$ nous obtenons :

$$d^2(f_m x, g_n y) \leqslant \{^2(S_m x, T_n y) \leqslant h \max\{hd^2(x, y), (\sqrt{h})^2 H(x, S_m x) H(y, T_n y),$$

$$(\sqrt{h})^2 D(x, T_n y) D(y, S_m x)\} \leqslant h \max\{d^2(x, y), d(x, f_m x) d(y, g_n y),$$

$$d(x, g_n) d(y, f_m x)\}.$$

Donc $(f_n)$ et $(g_n)$ satisfont aux hypothéses du théoréme 1 (pour des opérateurs) et donc il existe un point fixe commun de $(f_n)$ et $(g_n)$, $u \in X$.

Si nous supposons qu'ils existent deux points fixes communs $u$ et $v$ de $(f_n$ et $(g_n)$ nous obtenons :

$d^2(u, v) = d^2(f_m u, g_n v) \leqslant h \max\{d^2(u, v), 0, d^2(u, v)\} = hd^2(u, v)$. Donc $d(u, v) = 0$ c'est-á-dire $u = v$ et $u$ est un point fixe commun unique de $(f_n)$ et $(g_n)$.

Èvidemment $u \in S_m u$ et $u \in T_n u$ et donc nous avons :

$\delta^2(S_m u, T_n u) \leqslant h^2 \max\{0, H(u, S_m u) H(u, T_n u), 0\}$ pour chaque $m, n \in \mathbf{N}^*$ et

$H(u, S_m u) = 0$, $H(u, T_n u) = 0$, donc $S_m u = T_n u = \{u\}$ pour $\forall m, n \in \mathbf{N}^*$.

On voit que $u \in X$ est un point fixe commun de $(S_n)$ et $(T_n)$ si et seulement si $u$ est un point fixe commun de $f(_n)$ et $g(_n)$. Cela implique l'unicité du point fixe $u \in X$ et le fait que $S_m u = T_n u = \{u\}$.

*Observations* Le théoreme 4 est vrai aussi pour une paire d'applications multivoques, $S, T : X \to BN(X)$.

Le théoreme 4 a lieu et dans le cas $S = T : X \to BN(X)$.

Nous donnons maintenant un exemple d'application multivoque qui ne satisfait pas aux conditions de cette conséquence du théoréme 4, mais qui satisfait aux conditions du théoréme 2.

*Exemple.* Soit $X = [0, 1]$ avec le métrique usuelle d et $T : X \to BN(X)$, $Tx = \left[0, \frac{1}{2}\right]$, $\forall x \in [0, 1]$.

Soient $0 < x < y < \frac{1}{2}$, alors $\delta(Tx, Ty) = \frac{1}{2}$ mais $H(Tx, Ty) = 0$ et alors il n'existe pas un $0 \leqslant h < 1$ qui satisfait á l'inégalité (5), mais l'inégalité (1) pour une seule application est satisfaite.

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STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXXV, 1, 1990

# A SPLINE APPROXIMATION OF AN ARBITRARY ORDER FOR THE SOLUTION OF SYSTEM OF SECOND ORDER DIFFERENTIAL EQUATIONS

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Received: April 26, 1989
AMS subject classification: 65L05, 56D07

**REZUMAT. — Aproximare spline de ordin arbitrar a soluţiei unor sisteme de ecuaţii diferenţiale de ordinul al doilea.** În lucrare este construită o funcţie spline de aproximare a soluţiei sistemului $y'' = f_1(x, y, z)$, $z'' = f_2(x, y, z)$, cu condiţiile $y(x_0) = y_0$, $y'(x_0) = y_0'$, $z(x_0) = z_0$ and $z'(x_0) = z_0'$. Funcţiile spline folosite nu sînt în mod necesar polinomiale. Metoda folosită este cu un singur pas avînd ordinul de aproximare $0(h^{\alpha+2m})$ în $y^{(i)}$ şi $z^{(i)}$, $i=0, 1, 2, 0<\alpha<1$, dacă $f_1, f_2 \in C([0, 1] \times R^2)$.

1. **Introduction.** The aim of this paper is to construct a spline function approximation method for solving the nonlinear system of ordinary differential equations $y''= f_1(x, y, z)$, $z''=f_2(x, y, z)$ with $y(x_0) = y_0$, $y'(x_0) = y_0'$, $z(x_0) = z_0$ and $z'(x_0)= z_0'$. The spline functions are not necessarily polynomial splines. It is shown that the method is a one-step method $0(h^{\alpha+2m})$ in $y^{(i)}$ and $z^{(i)}$ where $i = 0\ (1)\ 2$, $0< \alpha \leqslant 1$ assuming $f_1, f_2 \in C([0, 1] \times \mathbf{R}^2)$. Here $m$ is the number of the iteration processes describing the spline functions defined in the method.

2. **Description on the method.** Consider the nonlinear system of ordinary differential equations

$$y'' = f_1(x, y, z), \qquad y(x_0) = y_0,\ y'(x_0) = y_0',$$
$$z'' = f_2(x, y, z) \qquad z(x_0) = z_0,\ z'(x_0) = z_0',$$

where $f_1, f_2 \in C([0, 1] \times \mathbf{R}^2)$.

Let $\Delta : 0 = x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots < x_n = 1$ be a partition of the interval $[0,1]$ and let $h = x_{k+1} - x_k$ for $k = 0(1)\ n - 1$.

Choosing an arbitrary positive integer $m$, then for any $x \in [x_k, x_{k+1}]$ we define the spline functions approximating $y(x)$, $z(x)$ by menas of the two functions $S_\Delta(x)$, $\bar{S}_\Delta(x)$ which are defined as follows:

$$S_\Delta(x) \equiv S_k^{[m]}(x) = S_{k-1}^{[m]}(x_k) + S_{k-1}^{'[m]}(x_k)(x - x_k) + \\ + \int_{x_k}^{x}\int_{x_k}^{t} f_1(u, S_k^{[m-1]}(u), \bar{S}_k^{[m-1]}(u))\, du\, dt \tag{2.1}$$

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$$\bar{S}_\Delta(x) \equiv \bar{S}_k^{[m]}(x) = \bar{S}_{k-1}^{[m]}(x) + S_{k-1}^{\prime[m]}(x_k)(x - x_k) +$$

$$= \int_{x_k}^{x}\int_{x_k}^{t} f_2(u, S_k^{[m-1]}(u), \bar{S}_k^{[m-1]}(u))du\, dt \quad (2.2) \qquad (2.2)$$

where $S_{-1}^{[m]}(x_0) = y_0$ and $\bar{S}_{-1}^{[m]}(x_0) = y_0'$. In equations (2.1) and (2.2) we use the following iteration process, where $S_{-1}^{\prime[m]}(x_0) = y_0'$ and $\bar{S}_{-1}^{\prime[m]}(x_0) = z_0'$

For $x \in [x_k\ x_{k+1}]$ $k = 0\,(1)\,n-1$, and $j = 1(1)m$,

$$S_k^{[0]}(x) = S_{k-1}^{[m]}(x_k) + S_{k-1}^{\prime[m]}(x_k)(x - x_k) + 1/2 f_1(x_k, S_{k-1}^{[m]}(x_k), \bar{S}_{k-1}^{[m]}(x_k))(x - x_k)^2$$

$$\bar{S}_k^{[0]}(x) = \bar{S}_{k-1}^{[m]}(x_k) + \bar{S}_{k-1}^{\prime[m]}(x_k)(x - x_k) + 1/2 f_2(x_k, S_{k-1}^{[m]}(x_k), \bar{S}_{k-1}^{[m]}(x_k))(x - x_k)^2$$

$$S_k^{[j]}(x) = S_{k-1}^{[m]}(x_k) + S_{k-1}^{\prime[m]}(x_k)(x - x_k) +$$

$$+ \int_{x_k}^{x}\int_{x_k}^{t_{m-j+1}} f_1(u_m - j + 1, S_k^{[j-1]}(u_m - j + 1), \bar{S}_k^{[j-1]}(u_m - j + 1)) \cdot du_{m-j+1} dt_{m-j+1}$$

$$\bar{S}_k^{[j]}(x) = \bar{S}_{k-1}^{[m]}(x_k) + \bar{S}_{k-1}^{\prime[m]}(x_k)(x - s_k) +$$

$$+ \int_{x_k}^{x}\int_{x_k}^{t_{m-j+1}} \bar{f}_2(u_{m-j+1}, S_k^{[j-1]}(u_{m-j+1}), \bar{S}_k^{[j-1]}(u_{m-j+1})) \cdot S u_{m-j+1} dt_{m-j+1}.$$

By construction, it is clear that both $S_\Delta(x)$ and $\bar{S}_\Delta(x) \in C^1([0,1])$.

3. **Error estimation and convergence.** In order to give error estimates for the method, we need to write down the exact solutions for the system under consideration using Taylor's expansion in the following form:

$$y(x) \equiv y^{[m]}(x) = y_k + y_k'(x - x_k) + \int_{x_k}^{x}\int_{x_k}^{t_1} f_1(u_1, y^{[m-1]}(u_1) z^{[m-1]}(u_1))\, du_1 dt_1 \quad (3.1) \qquad (3.1)$$

$$y'(x) \equiv y'^{[m]}(x) = y_k' + \int_{x_k}^{x} f_1(u_1, y^{[m-1]}(u_1), z^{[m-1]}(u_1))du_1 \quad (3.2) \qquad (3.2)$$

$$z(x) \equiv z^{[m]}(x) = z_k + z_k'(x - x_k) + \int_{x_k}^{x}\int_{x_k}^{t_1} f_2(u_1, y^{[m-1]}(u_1), z^{[m-1]}(t_1))du_1 dt_1 \quad (3.3) \qquad (3.3)$$

$$z'(x) \equiv z'^{[m]}(x) = z_k' + \int_{x_k}^{x} f_2(u_1, y^{[m-1]}(u_1), z^{[m-1]}(u_1))du_1. \quad (3.4) \qquad (3.4)$$

Note that, we have used the following notations: $y_k = y(x_k)$, $z_k = z(x_k)$, $'_k = y'(x_k)$, and $z'_k = z'(x_k)$. Also, the following iterations are defined:

or $x_k \leqslant u_m \leqslant t_m \leqslant u_{m-1} \leqslant t_{m-1} \leqslant \ldots \leqslant u_{m-j+1} \leqslant t_{m-j+1} \leqslant \ldots \leqslant u_1 \leqslant$

$\leqslant t_1 \leqslant x \leqslant x_{k+1}$, $j = 1(1)m$, we have

$$y^{[0]}(x) = y_k + y'_k(x - x_k) + 1/2y''(\xi_k)(x - x_k)^2,\ \xi_k \in (x_k, x_{k+1}),$$

$$z^{[0]}(x) = z_k + z'_k(x - x_k) + 1/2z''(\eta_k)(x - x_k)^2,\ \eta_k \in (x_k, x_{k+1}),$$

$$y^{[j]}(x) = y_k + y'_k(x - x_k) + \int_{x_k}^{x}\int_{x_k}^{t_{m-j+1}} f_1(u_{m-j+1},$$

$$y^{[j-1]}(u_{m-j+1}), z^{[j-1]}(u_{m-j+1}))du_{m-j+1}dt_{m-j+1}$$

$$z^{[j]}(x) = z_k + z'_k(x - x_k) + \int_{x_k}^{x}\int_{x_k}^{t_{m-j+1}} f_2(u_{m-j+1}, y^{[j-1]}(u_{m-j+1}),$$

$$z^{[j-1]}(u_{m-j+1}))du_{m-j+1}dt_{m-j+1}.$$

oreover, we use the following notations:

$$\begin{cases} e(x) = |y(x) - S_\Delta(x)|,\ e'(x) = |y'(x) - S'_\Delta(x)|, \\ \bar{e}(x) = |z(x) - \bar{S}_\Delta(x)|,\ \bar{e}'(x) = |z'(x) - \bar{S}'_\Delta(x)|, \\ e_k = |y_k - S_\Delta(x_k)|,\ \bar{e}_k = |z_k - \bar{S}_\Delta(x_k)|, \\ e'_k = |y'_k - S'_\Delta(x_k)|,\ \bar{e}'_k = |z'_k - \bar{S}'_\Delta(x_k)|. \end{cases} \tag{3.6}$$

We are now ready to give error estimates for the method.

LEMMA 3.1. *Let $e(x)$, $\bar{e}(x)$, $e'(x)$, $\bar{e}'(x)$, $e_k$, $\bar{e}_k$, $e'_k$, and $\bar{e}'_k$ be defined as in* (3.6) *d assume that $f_i$ satisfies the Lipschitz condition; that is, there exists a constant such that:*

*$_i(x, y_1, z_1) - f_i(x, y_2, z_2)| \leqslant L_i\{|y_1 - y_2| + |z_1 - z_2|\}$ for all $(x, y_1, z_1)$ and $, y_2, z_2)$ in the domain of definition of $f_i$ for $i = 1, 2$; then there exist constants ich make the folowing inequelities hold true:*

$$e(x) \leqslant e_k(1 + C_0h) + \bar{e}_kC_0h + e'_kC_1h + \bar{e}'_kC_2h + C_3h^{2m+2}\omega(h), \tag{3.7}$$

$$\bar{e}(x) \leqslant e_k\bar{C}_0h + \bar{e}_k(1 + C_0h) + e'_k\bar{C}_1h + \bar{e}'_k\bar{C}_2h + \bar{C}_3h^{2m+2}\omega(h). \tag{3.8}$$

$$e'(x) \leqslant e_kC'_0h + \bar{e}_kC'_0h + e'_k(1 + C'_1h) + \bar{e}'_kC'_1h + C'_3h^{2m+1}\omega(h), \tag{3.9}$$

$$\bar{e}'(x) \leqslant e_k\bar{C}'_0h + \bar{e}_k\bar{C}'_0h + \bar{e}'_kC'_1h + e'_k(1 + \bar{C}'_1h) + \bar{C}'_3h^{2m+1}\omega(h), \tag{3.10}$$

*ere* $\omega(h) = \max(\omega(y'', h), \omega(z'', h))$ *and* $\omega(f, h) = \sup\limits_{|x_1 - x_2| < h}|f(x_1) - f(x_2)|$.

*Proof*. Since, $e(x) = |y(x) - S_\Delta(x)| = |y^{[m]}(x) - S_k^{[m]}(x)|$, then by and (3.1) we get:

$$e(x) \leqslant |y_k - S_{k-1}^{[m]}| + |y_k' - S_{k-1}'^{[m]}(x_k)|\ |x - x_k| +$$

$$+ \int_{x_k}^{x}\int_{x_k}^{t_1} |f_1(u_1, y^{[m-1]}(u_1), z^{[m-1]}(u_1)) - f_1(u_1, S_k^{[m-1]}(u_1), \bar{S}_k^{[m-1]}(u_1))\ |du_1 dt_1 \leqslant$$

$$+ e_k' h + L_1 \int_{k_k}^{x}\int_{x_k}^{t_1} \{|y^{[m-1]}(u_1) - S_k^{[m-1]}(u_1)| + |z^{[m-1]}(u_1) - \bar{S}_k^{[m-1}(u_1)|\} du_1$$

we have used the fact that $S_k^{[j]}(x_k) = S_{k-1}^{[m]}(x_k)$ for all $j = 1(1)m$.

Let $I_i = |y^{[m-i]}(u_i) - S_k^{[m-i]}(u_i)|$ and

$J_i = |z^{[m-i]}(u_i) - \bar{S}_k^{[m-i]}(u_i)|$ for $i = 1(1)m$, then one gets:

$$I_i \leqslant |y_k - S_k^{[m]}(x_k)| + |y_k' - S_k'^{[m]}(x_k)|\ |u_i - x_k| +$$

$$+ L_1 \int_{x_k}^{u_i}\int_{x_k}^{t_i+1} \{|y^{[m-i-1]}(u_{i+1}) - S^{[m-i-1]}(u_{i-1})| +$$

$$+ |z^{[m-i-1]}(u_{i+1}) - \bar{S}_k^{[m-i-1]}(u_{i+1})|\} du_{i+1} dt_{i+1}$$

$$\leqslant e_k + e_k'|u_i - x_k| + L_1 \int_{x_k}^{u_i}\int_{x_k}^{t_{i+1}} \{I_{i+1} + J_{i+1}\} du_{i+1} dt_{i+1}.$$

Similarly, one can show that

$$J_i \leqslant \bar{e}_k + \bar{e}_k'|u_i - x_k| + L_2 \int_{x_k}^{u_i}\int_{x_k}^{t_i+1} |I_{i+1} + J_{i+1}\ |du_{i+1} dt_{i+1}.$$

Therefor,

$$I_i + J_i \leqslant (e_k + \bar{e}_k) + (e_k' + \bar{e}_k')|u_i - x_k| +$$

$$+ (L_1 + L_2) \int_{x_k}^{u_i}\int_{x_k}^{t_{i+1}} |I_{i+1} + J_{i+1}| du_{i+1} dt_{i+1}.$$

Thus, we get

$$e(x) \leqslant e_k + e_k' h + L_1 \int_{x_k}^{x}\int_{x_k}^{t_1} |I_1 + J_1| du_1 dt_1$$

$$\leqslant e_k + e'_k h + L_1 \int_{x_k}^{x}\int_{x_k}^{t_1} \{(e_k + \bar{e}_k) + (e'_k + \bar{e}'_k)|u_1 - x_k| +$$

$$+ (L_1 + L_2)\int_{x_k}^{u_1}\int_{x_k}^{t_2} [I_2 + J_2]du_2 dt_2\} du_1 dt_1$$

$$\leqslant e_k + e'_k h + L_1\left[(e_k + \bar{e}_k)\frac{h^2}{2!} + (e'_k + \bar{e}'_k)\frac{h^3}{3!}\right] +$$

$$+ L_1(L_1 + L_2)\int_{x_k}^{x}\int_{x_k}^{t_1}\int_{x_k}^{u_1}\int_{x_k}^{t_2} [I_2 + J_2]du_2 dt_2 du_1 dt_1$$

$$\leqslant e_k + e'_k h + L_1(e_k + \bar{e}_k)\left[\frac{h^2}{2!} + (L_1 + L_2)\frac{h^4}{4!}\right] +$$

$$+ L_1(e'_k + \bar{e}'_k)\left[\frac{h^3}{3!} + (L_1 + L_2)\frac{h^5}{5!}\right] +$$

$$+ L_1(L_1 + L_2)^2 \int_{x_k}^{x}\int_{x_k}^{t_1}\int_{x_k}^{u_1}\int_{x_k}^{t_2}\int_{x_k}^{u_2}\int_{x_k}^{t_3} [I_3 + J_3]du_3 dt_3 du_2 dt_2 du_1 dt_1$$

$$\ldots \qquad \ldots \qquad \ldots$$

As a result, we get

$$e(x) \leqslant e_k + e'_k h + L_1(e_k + \bar{e}_k)\left[\frac{h^2}{2!} + L_1 + L_2)\frac{h^4}{4!} + (L_1 + L_2)^2 \cdot\right.$$

$$\left. + \frac{h^6}{6} + \ldots + (L_1 + L_1)^{m-2}\frac{h^{2(m-1)}}{(2(m-1))!}\right] + L_1(e'_k + \bar{e}'_k)\left[\frac{h^3}{3!} +\right.$$

$$\left. + (L_1 + L_2)\frac{h^2}{5!} + (L_1 + L_2)^2\frac{h^7}{7!} + \ldots + (L_1 + L_2)^{m-2}\frac{h^{2m-1}}{(2m-1)!}\right] +$$

$$+ L_1(L_1 + L_2)^{m-1}\int_{x_k}^{x}\int_{x_k}^{t_1}\int_{x_k}^{u_1}\int_{x_k}^{t_2}\cdots\int_{x_k}^{u_{m-1}}\int_{x_k}^{t_m} [I_m + J_m]\cdot du_m dt_m \ldots du_1 dt_1.$$

But, $I_m = |y^{[0]}(u_m) - S_k^{[0]}(u_m)|$, then

$$I_m \leqslant |y_k - S_{k-1}^{[m]}(x_k)| + |y'_k - S'^{[m]}_{k-1}(x_k)|\ |u_m - x_k| + 1/2[\,|y''(\xi_k) -$$

$$- y''(x_k)|\,](u_m - x_k)^2 + 1/2|f_1(x_k, S_k^{[m]}(x_k)\bar{S}_k^{[m]}(x_k)) -$$

$$- f_1(x_k, y_k, z_k)| \cdot (u_m - x_k)^2$$

$$\leqslant e_k + e_k' |u_m - x_k| + 1/2\omega(y'', h)(u_m - x_k)^2 + 1/2L_1[|y_k - S_k^{[m]}(x_k)| + |z_k - \bar{S}_k^{[m]}(x_k)|](u_m - \ _k)^2$$

$$\leqslant e_k + e_k' |u_m - x_k| + 1/2\omega(y'', h)(u_m - x_k)^2 + 1/2\, L_1(e_k + \bar{e}_k)(u_m - x_k)$$

Similarly, one can show that

$$J_m \leqslant \bar{e}_k + \bar{e}_k' |u_m - x_k| + 1/2\omega(z'', h)(u_m - x_k)^2 + 1/2L_2(e_k + \bar{e}_k) \cdot (u_m - x$$

Consequently, we get

$$I_m + J_m \leqslant (e_k + \bar{e}_k) + (e_k' + \bar{e}_k') |u_m - x_k| + \omega(h)(u_m - x_k)^2 + 1/2(L_1 + L_2)(e_k + \bar{e}_k)(u_m - x_k)^2.$$

Finally, we end up with the following:

$$e(x) \leqslant e_k + e_k'h + L_1(e_k + \bar{e}_k)\left[\frac{h^2}{2!} + (L_1 + L_2)\frac{h^4}{4!} + (L_1 + L_2)^2 \cdot \frac{h^6}{6!} + \ldots + (L_1 + L_2)^{m-2}\frac{h^{2(m-1)}}{(2(m-1))!} + (L_1 + L_2)^{(m-1)}\frac{h^{2m}}{(2m)!} + (L_1 + L_2)^m \frac{h^{2m+2}}{(2m+2)!}\right] + L_1(e_k' + \bar{e}_k')\left[\frac{h^3}{3!} + (L_1 + L_2)\cdot\frac{h^5}{5!} + (L_1 + L_2)^2\frac{h^7}{7!} + \ldots + (L_1 + L_2)^{m-2}\frac{h^{2m-1}}{(2m-1)!} + (L_1 + L_2)^{m-1}\frac{h^{2m+1}}{(2m+1)!}\right] + L_1(L_1 + L_2)^{m-1}\frac{h^{2m+2}}{(2m+2)!}\omega(h)$$

$$\leqslant e_k + e_k'h + L_1h(e_k + \bar{e}_k)[1 + (L_1 + L_2) + (L_1 + L_2)^2 + \ldots + (L_1 + L_2)^{m-2} + (L_1 + L_2)^{m-1} + (L_1 + L_2)^m] + L_1h(e_k' + \bar{e}_k') \cdot [1 + (L_1 + L_2) + (L_1 + L_2)^2 + \ldots + (L_1 + L_2)^{m-2} + (L_1 + L_2)^{m-1}] + L_1(L_1 + L_2)^{m+1}\frac{h^{2m+2}}{(2m+2)!}\omega(h) \cdot$$

$$\leqslant e_k + e_k'h + L_1h(e_k + \bar{e}_k)\frac{[(L_1 + L_2)^{m+1} - 1]}{(L_1 + L_2) - 1} + L_1h(e_k' + \bar{e}_k')\cdot \frac{[(L_1 + L_2)^m - 1]}{(L_1 + L_2) - 1} + L_1(L_1 + L_2)^{m-1}\frac{h^{2m+2}}{(2m+2)!}\omega(h)$$

Let $C_0 = L_1\frac{[(L_1 + L_2)^{m+1} - 1]}{[(L_1 + L_2) - 1]}$ $C_2 = L_1\frac{(L_1 + L_2)^{m-1}}{(L_1 + L_2) - 1}$, and $C_3 = L_1\frac{(L_1 + L_2)^{m-}}{(2m+2)!}$

$$e(x) \leqslant e_k(1 + C_0h) + \bar{e}_kC_0h + e_k'C_1h + C_2\bar{e}_k'h + C_3h^{2m+2}\omega(h),$$

where $C_1 = 1 + C_2$. This proves the first part of the lemma.

Using the previous procedure (2.2), and (3.3) one can show the following estimate:

$$\bar{e}(x) \leqslant \bar{e}_k + \bar{e}_k' h + L_2 h(e_k + \bar{e}_k) \frac{[(L_1 + L_2)^{m+1} - 1]}{(L_1 + L_2) - 1} + L_2 h(e_k' + \bar{e}_k') \cdot$$

$$- \frac{(L_1 + L_2)^m - 1}{(L_1 + L_2) - 1} + L_2(L_1 + L_2)^{m-1} \frac{h^{2m+2}}{(2m+2)!} \omega(h).$$

Let $\bar{C}_0 = L_2 \dfrac{(L_1 + L_2)^{m+1} - 1}{(L_1 + L_2) - 1} \cdot \bar{C}_1 = L_2 \dfrac{(L_1 + L_2)^m - 1}{(L_1 + L_2) - 1}$, and

$\bar{C}_3 = L_2 \dfrac{(L_1 + L_2)^{m-1}}{(2m+2)!}$, then one gets

$$\bar{e}(x) \leqslant e_k \bar{C}_0 h + \bar{e}_k(1 + \bar{C}_0 h) + e_k' \bar{C}_1 h + \bar{e}_k' \bar{C}_2 h + \bar{C}_3 h^{2m+2}. \ \omega(h),$$

where $\bar{C}_2 = 1 + \bar{C}_1$. This proves the second part of the lemma.

Next, an estimate for e'(x) is given using (2.1) and (3.2):

$$e'(x) \leqslant |y_k' - S_{k-1}'^{[m]}(x_k)| + \int_{x_k}^{x} |f_1(u_1, y^{[m-1]}(u_1), z^{[m-1]}(u_1)) -$$

$$- f_1(u_1, S^{[m-1]}(u_1), \bar{S}_k^{[m-1]}(u))|du_1$$

$$\leqslant e_k' + L_1 \int_{x_k}^{x} [I_1 + J_1] du_1$$

$$\leqslant e_k' + L_1 \int_{x_k}^{x} \{(e_k + \bar{e}_k) + (e_k' + \bar{e}_k') \, | u_1 - x_k| + (L_1 + L_2) \cdot$$

$$\cdot \int_{x_k}^{u_1} \int_{x_k}^{t_2} [I_2 + J_2] \, du_2 dt_2\} du_1$$

$$\leqslant e_k' + L_1 \left[(e_k + \bar{e}_k)h + (e_k' + \bar{e}_k') \frac{h^2}{2!}\right] + L_1(L_1 + L_2) \cdot$$

$$\cdot \int_{x_k}^{x} \int_{x_k}^{v_1} \int_{x_k}^{t_2} [I_2 + J_2] \, du_2 dt_2 du_1$$

$$\leqslant e_k' + L_1(e_k + \bar{e}_k) \left[h + (L_1 + L_2) \frac{h^3}{3!}\right] + L_1(e_k' + \bar{e}_k') \left[\frac{h^2}{2!} + (L_1 + L_2) \frac{h^4}{4!}\right] +$$

$$+ L_1(L_1 + L_2)^2 \int_{x_k}^{x} \int_{x_k}^{u_1} \int_{x_k}^{t_2} \int_{x_k}^{x_3} \int_{x_k}^{t_3} [I_3 + J_3] \cdot du_3 dt_3 du_2 dt_2 du_1.$$

Hence, one gets

$$e'(x) \leqslant e'_k + L_1(e_k + \bar{e}_k)\Big[h + (L_1 + L_2)\frac{h^3}{3!} + (L_1 + L_2)^2\frac{h^5}{5!} + \ldots +$$

$$+ (L_1 + L_2)^{m-2}\frac{h^{2m-3}}{(2m-3)!}\Big] + L_1(e'_k + \bar{e}'_k)\Big[\frac{h^2}{2!} + (L_1 + L_2)\frac{h^4}{4!} +$$

$$+ (L_1 + L_2)^2\frac{h^6}{6!} + \ldots + (L_1 + L_2)^{m-2}\frac{h^{2m-2}}{(2m-2)!}\Big] + L_1 \cdot$$

$$\cdot (L_1 + L_2)^{m-1}\int_{x_k}^{x}\int_{x_k}^{u_1}\int_{x_k}^{t_2}\ldots\int_{x_k}^{u_{m-1}}\int_{x_k}^{t_m}[(e_k + \bar{e}_k) + (e'_k +$$

$$+ \bar{e}'_k)\,|u_m - x_k| + 1/2[\omega(y'', h) + \omega(z'', h)]\,(u_m - x_k)^2 +$$

$$+ 1/2(L_1 + L_2)\,(e_k + \bar{e}_k)\,(u_m - x_k)^2]\,du_m dt_m \ldots du_2 dt_2 du_1$$

$$e'(x) \leqslant e'_k + L_1(e_k + \bar{e}_k)\Big[h + (L_1 + L_2)\frac{h^3}{3!} + (L_1 + L_2)^2\frac{h^5}{5!} + \ldots +$$

$$+ (L_1 + L_2)^{m-2}\frac{h^{2m-3}}{(2m-3)!} + (L_1 + L_2)^{m-1}\frac{h^{2m-1}}{(2m-1)!} + 1/2(L_1 + L_2)^m \cdot$$

$$\cdot \frac{h^{2m+1}}{(2m+1)!}\Big] + L_1(e'_k + \bar{e}'_k)\Big[\frac{h^2}{2!} + (L_1 + L_2)\frac{h^4}{4!} + (L_1 + L_2)^2 \cdot$$

$$\cdot \frac{h^6}{6!} + \ldots + (L_1 + L_2)^{m-2}\frac{h^{2m-2}}{(2m-2)!} + (L_1 + L_2)^{m-1}\frac{h^{2m}}{(2m)}\Big] +$$

$$+ L_1(L_1 + L_2)^{m-1}\frac{h^{2m+1}}{(2m+1)!}\,\omega(h)$$

$$e'(x) \leqslant e'_k + L_1 h(e_k + \bar{e}_k)\,[1 + (L_1 + L_2) + (L_1 + L_2)^2 + \ldots +$$

$$+ (L_1 + L_2)^{m-2} + (L_1 + L_2)^{m-1} + (L_1 + L_2)^m] + L_1 h(e'_k + \bar{e}'_k)\,\cdot$$

$$\cdot\,[1 + (L_1 + L_2) + (L_1 + L_2)^2 + \ldots + (L_1 + L_2)^{m-2} +$$

$$+ (L_1 + L_2)^{m-1}] + L_1(L_1 + L_2)^{m-1}\frac{h^{2m+1}}{(2m+1)!}\,\omega(h)$$

$$e'(x) \leqslant e'_k + L_1 h(e_k + \bar{e}_k)\frac{(L_1 + L_2)^{m+1} - 1}{(L_1 + L_2) - 1} + L_1 h(e'_k + \bar{e}'_k)\,\cdot$$

$$\cdot\,\frac{(L_1 + L_2)^m - 1}{(L_1 + L_2) - 1} + L_1(L_1 + L_2)^{m-1}\frac{h^{2m+1}}{(2m+1)!}\,\omega(h)$$

Let $C'_0 = L_1\frac{(L_1 + L_2)^{m+1} - 1}{(L_1 + L_2) - 1}$, $C'_1 = L_1\frac{(L_1 + L_2)^m - 1}{(L_1 + L_2) - 1}$, and

$C'_3 = L_1\frac{(L_1 + L_2)^{m-1}}{(2m+1)!}$, then we get

$$e'(x) \leqslant e_k C_0' h + \bar{e}_k C_0 h + e_k'(1 + C_1' h) + \bar{e}_k' C_k' h + C_3' h^{2m+1} \omega(h).$$

Finally, it is now easy to show that

$$\bar{e}'(x) \leqslant \bar{e}_k' + L_2 h(e_k + \bar{e}_k)[1 + (L_1 + L_2) + (L_1 + L_2)^2 + \ldots + (L_1 + L_2)^{m-2} + (L_1 + L_2)^{m-1} + (L_1 + L_2)^m] + L_2 h(e_k' + \bar{e}_k') \cdot [1 + (L_1 + L_2) + (L_1 + L_2)^2 + \ldots + (L_1 + L_2)^{m-2} + (L_1 + L_2)^{m-1}] + L_2(L_1 + L_2)^{m-1} \frac{h^{2m+1}}{(2m+1)!} \omega(h)$$

$$\leqslant \bar{e}_k' + L_2 h(e_k + \bar{e}_k) \frac{(L_1 + L_2)^{m+1} - 1}{(L_1 + L_2) - 1} + L_2 h(e_k' + \bar{e}_k') \cdot \frac{(L_1 + L_2)^m - 1}{(L_1 + L_2) - 1} + L_2 (L_1 + L_2)^{m-1} \frac{h^{2m+1}}{(2m+1)!} \omega(h)$$

Let $C_0' = L_2 \frac{(L_1 + L_2)^{m+1} - 1}{(L_1 + L_2) - 1}$, $\bar{C}_1' = L_2 \frac{(L_1 + L_2)^m - 1}{(L_1 + L^7) - 1}$, and

$\bar{C}_3' = L_2 \frac{(L_1 + L_2)^{m-1}}{(2m+1)!}$, then we get

$$\bar{e}'(x) \leqslant e_k \bar{C}_0' h + \bar{e}_k \bar{C}_0' h + e_k' \bar{C}_1' h + \bar{e}_k'(1 + \bar{C}_1' h) + C_3' h^{2m+1} \omega(h),$$

this proves the last and the fourth part of the lemma.

Now, let $E(x) = [e(x)\ \bar{e}(x)\ \bar{e}'(x)\ \bar{e}'(x)]^T$

$$E_k = [e_k\ \bar{e}_k\ e_k'\ \bar{e}_k']^T$$

$$C = [C_3\ \bar{C}_3\ C_3'\ \bar{C}_3']$$

where T stands for the transpose. Note that, the initial conditions implies that $E_0 = [0\ 0\ 0\ 0]^T$, then from Lemma (3.1) we can write $E(x)$ in the following form:

$$E(x) \leqslant (I + Ah) E_k + Ch^{2m+1}\omega(h). \tag{3.11}$$

where $I$ is the identity matrix of order 4 and $A$ is the matrix defined as follows:

$$A = \begin{bmatrix} C_0 & C_0 & C_1 & C_2 \\ \bar{C}_0 & \bar{C}_0 & \bar{C}_1 & \bar{C}_2 \\ C_0' & C_0' & C_1' & C_1' \\ \bar{C}_0' & \bar{C}_0' & \bar{C}_1' & \bar{C}_1' \end{bmatrix}.$$

Let $||E(x)|| = ||E(x)||_\infty$, then (3.11) becomes

$$||E(x)|| \leqslant (1 + ||A||h)\ ||E_k|| + ||C||\ h^{2m+1}\omega(h)$$

the above inequality is true for all $x \in [x_k, x_{k+1}]$, $k = 0(1)n - 1$.

In particular, the following inequalities hold true:

$$(1 + \|A\|h)\,\|E_k\| \leqslant (1 + \|A\|h)^2\,\|E_{k-1}\| + (1 + \|A\|h)\,\|C\|h^{2m+1}\omega(h).$$

$$(1 + \|A\|h)^2\|E_{k-1}\| \leqslant (1 + \|A\|h)^3\,\|E_{k-2}\| + (1 + \|A\|h)^2\|C\|h^{2m+1}\omega(h).$$

$$(1 + \|A\|h)^3\,\|E_{k-2}\| \leqslant (1 + \|A\|h)^4\,\|E_{k-3}\| + (1 + \|A\|h)^3\|C\|h^{2m+1}\omega(h).$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$(1 + \|A\|h)^k\,\|E_1\| \leqslant (1 + \|A\|h)^{k+1}\|E_0\| + (1 + \|A\|h)^k\|C\|h^{2m+1}\omega(h).$$

Adding L.H.S. and R.H.S. we get

$$\|E(x)\| \leqslant h^{2m+1}\omega(h)\|C\|\sum_{j=0}^{k}(1 + \|A\|h)^j$$

$$< h^{2m+1}\omega(h)\ \|C\|\frac{|1 + \|A\|h|^{k+1} - 1}{1 + \|A\|h - 1}$$

$$= h^{2m}\ \omega(h)\frac{\|C\|}{\|A\|}\left\{\left[1 + \frac{\|A\|}{n}\right]^{k+1} - 1\right\}$$

$$= h^{2m}\omega(h)\frac{\|C\|}{\|A\|}\,[e^{\|A\|} - 1] \leqslant Bh^{2m}\omega(h).$$

$$\text{where } B = \frac{\|C\|}{\|A\|}\,[e^{\|A\|} - 1].$$

Using (2.1), (3.1) and (2.2), (3.3), it can be easily shown that

$$e''(x) = |y''(x) - S''_\Delta(x)| \leqslant C_1h^{2m}\ \omega(h)$$

and

$$\bar{e}''(x) = |z''(x) - \bar{S}''_\Delta(x)| \leqslant C_2h^{2m}\ \omega(h).$$

Thus, we have proved the main result of this paper.

THEOREM 3.2. *Let $y(x)$, $z(x)$ be the exact solution of the system of equations*

$$y''(x) = f_1(x, y, z),\ y(x_0) = y_0,\ y'(x_0) = y'_0$$

$$z''(x) = f_2(x, y, z),\ z(x_0) = z_0,\ z'(x_0) = z'_0.$$

*If $S_\Delta(x)$ and $\bar{S}_\Delta(x)$, defined by (2.1) and (2.2), are the approximate solutions and $f_1, f_2 \in C([0, 1]) \times \mathbf{R}^2)$, then for all $x \in [x_k, x_{k+1}]$. $k = 0(1)n - 1$, we have*

$$|y^{(i)} - S_\Delta^{(i)}(x)| \leqslant B_1h^{2m}\ \omega(h),\ i = 0(1)2$$

$$|z^{(i)} - \bar{S}_\Delta^{(i)}(x)| \leqslant B_2h^{2m}\ \omega(h),\ i = 0(1)2$$

*where $B_1$ and $B_2$ are constants independent of $h$.*

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STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXXV, 1, 1990

# A GLOBAL APPROXIMATION METHOD FOR SECOND ORD NONLINEAR BOUNDARY VALUE PROBLEMS

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*Received: September 4, 1989*
*AMS subject classification: 65L10, 65D07*

**REZUMAT. — O metodă de aproximare globală petru probleme la limită neliniare de ordinul al doilea.** Lucrarea propune o metodă globală de aproximare a soluţiei problemei bilocale neliniare $y'' + f(x, y) = 0$, $0 \leqslant x \leqslant 1$, $y(0) = \alpha$, $y(1) = \beta$, cu ajutorul funcţiilor spline cubice şi quintice. Metoda este o combinaţie între metoda colocaţiei spline şi o metodă discretă a multipaşilor. Ea are ordinul de convergenţă patru. Sînt date două exemple numerice care ilustrează aplicabilitatea metodei şi avantajele ei.

**1. Introduction.** We consider a numerical procedure based on spline fu to approximate the solution of nonlinear two-point boundary value prob ordinary differential equations. Though polynomials have long been the fu most widely used to approximate other functions, spline function ir circumstances is a more adaptable approximating function than a poly Besides, it provides a simple and powerful theory.

Generally, a spline approximation satisfies certain continuity con and some discretization equations. In this paper we shall consider a pr where the conditions of continuity are implemented explicitly. In view fact that boundary value problems involving differential equations o higher than two are not too common, we confine our attention to secon equations of the form

$$\frac{d^2y}{dx^2} + f(x, y) = 0,\ 0 \leqslant x \leqslant 1,\ y(0) = \alpha,\ y(1) = \beta,$$

where $\alpha$ and $\beta$ are constants. We assume that for

$$(x, y) \in S = \{0 \leqslant x \leqslant 1,\ -\infty < y < +\infty\},$$

$f(x, y)$ is continuous, $\partial f/\partial y$ exists and is continuous. Since the boundar problem (1.1) is likely to be nonlinear in nature, it may happen that w solution exists they are more than one. In this context, we recapitul work of L e e s (1966) where it is mentioned that whenever $u = \sup_S \partial f/\partial$ problem (1.1) possesses a unique solution. Differential equations of this particular, systems of such equations occur frequently, for example, in nical problems without dissipation.

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Cubic splines to obtain the solution of linear two-point boundary value problems have been used by Bickley (1968). Albasiny and Hoskins (1969) have discussed the application of cubic splines in the solution of second order linear problems and concluded that results which are accurate upto $O(h^4)$ can be obtained in a special case. Fyfe (1969) has used cubic splines to solve linear second order problems and examined deferred corrections, effects of nonuniform spacing and a mesh refinement procedure. Usmani (1980) has discussed a fourth order scheme for linear problems involving second order differential equations.

Recently, Chawla and Subramanian (1987) have suggested a fourth order method involving a cubic spline procedure coupled with fourth order Numerov scheme for the solution of nonlinear boundary value problem (1.1). In this paper, we suggest another fourth order solution procedure where a cubic spline coupled with a quintic spline scheme is used for the same problem. We have also made a systematic error analysis and established fourth order convergence of the present method. Numerical examples are supplemented to show the working of the method.

2. **Development of the numerical scheme.** We consider a uniform partition $\Delta$ of the interval $[0, 1]$ into $N$ subintervals by inserting the knots $x = jh$, $j = 0(1)\,N$, where step length $h = 1/N$, and $I_k = (x_{k-1}, x_k)$, $k = 1(1)\,N$. On the partition $\Delta$, a natural representation of a spline function of degree $m$ contains $(m+1)\,N$ parameters and hence we require as many relations. The continuity conditions provide $m(N-1)$ relations and $(N+1)$ relations are obtained through collocation. Hence, we need $m-1$ $(= N(m+1) - m(N-1) - (N+1))$ relations more for the complete determination of all the unknowns.

Let $S_j(x)$ be a quintic spline on the $j$ th interval $I_j$. To simplify the presentation, we use the abbreviations

$$S_j(x_k) = z_k, \text{ and } S_j''(x_k) = M_k$$

The following quintic spline relation can be derived from Ahlberg, Nilson and Walsh (1967), Albasiny and Hoskins (1971) as

$$z_{j-2} + 2z_{j-1} - 6z_j + 2z_{j+1} + z_{j+2} \tag{2.1}$$
$$= \frac{h^2}{20}(M_{j-2} + 26M_{j-1} + 66M_j + 26M_{j+1} + M_{j+2}),\ j = 2(1)\,N-2.$$

But, from differential equation and the boundary conditions of the problem (1.1), we obtain

$$M_i \approx y''(x_i) \approx -f(x_i, z_i),\ i = 1(1)\,N-1, \tag{2.2}$$

and

$$M_0 \approx y''(x_0) \approx -f(0, \alpha),\ M_N \approx y''(x_N) \approx -f(1, \beta). \tag{2.3}$$

Using relations (2.2) and (2.3) in (2.1), we get

$$-z_{j-2} - 2z_{j-1} + 6z_j - 2z_{j+1} - z_{j+2} \tag{2.4}$$
$$= \frac{h^2}{20}(f_{j-2} + 26f_{j-1} + 66f_j + 26f_{j+1} + f_{j+2}),\quad j = 2(1)\,N-2.$$

As the relation (2.4) gives $N-3$ equations in $N+1$ unknowns $z_i$, we need four relations more. This is in complete agreement with our earlier statement made in the beginning of this section. Since the boundary conditions give two relations determining $z_0 = \alpha$, and $z_N = \beta$, we need two more relations only. This can be achieved by using quartic splines in the neighbourhood of the two end points.

When a quartic spline is considered as the approximate solution, a relation similar to (2.4) may be obtained as

$$-z_{i-1} + 2z_i - z_{i+1} = \frac{h^2}{12}(f_{i-1} + 10f_i + f_{i+1}), \quad i = 1(1)\, N-1. \tag{2.5}$$

It may be mentioned that the relation (2.5) is equivalent to well-known fourth order Numerov scheme.

Now using the relation (2.5), we get two equations for $i = 1$ and 2. These two equations, after some algebraic simplifications, give a relation near the first boundary point $x_0 = a$ as

$$-4z_0 + 7z_1 - 2z_2 - z_3 = \frac{h^2}{12}(4f_0 + 41f_1 + 14f_2 + f_3). \tag{2.6}$$

In a similar fashion, we can derive a different relation near the second boundary point $x_N = b$ as

$$-z_{N-3} - 2z_{N-2} + 7z_{N-1} - 4z_N = \frac{h^2}{12}(f_{N-3} + 14f_{N-2} + 41f_{N-1} + 4f_N). \tag{2.7}$$

The equations (2.6) and 2.7) are the required two additional relations. Therefore the relations (2.6), (2.4) and (2.7) form a set of $(N-1)$ equations to determine the $N-1$ unknowns $z_j$, $j = 1(1)\, N-1$. As the function $f(x, y)$ is non-linear in $y$, some iterative procedure is necessary to solve the system. We consider the application of Newton's method to the above system.

With the nodal approximations $z_i$ to the true solution $y(x)$ known, the approximate second derivates $M_i$ are calculated from the relation (2.2). Using these values of $z_i$ and $M_i$, we construct a cubic spline interpolant $u(x)$ to the true solution $y(x)$ as

$$u(x) = \frac{(x_{i+1}-x)^3}{6h} M_i + \frac{(x-x_i)^3}{6h} M_{i+1} \tag{2.8}$$

$$+ \left(y_i - \frac{h^2 M_i}{6}\right)\left(\frac{x_{i+1}-x}{h}\right) + \left(y_{i+1} - \frac{h^2 M_{i+1}}{6}\right)\left(\frac{x-x_i}{h}\right), \quad x_i < x < x_{i+1}.$$

The above solution procedure may be summarized as follows:

Step 1. nodal approximations $z_i$ to the true solution $y(x)$ are computed using the quintic spline scheme,

Step 2. with the help of $z_i$ values, the approximate second derivatives $M_i$ are calculated from equation (2.2), and

Step 3. finally these values of $z_i$ and $M_i$ are used to construct a cubic spline interpolant $u(x)$ in the whole range $0 \leqslant x \leqslant 1$.

The method proposed here actually generates a global spline, not just its values at the nodes. That is, a continuous global approximation to the true solution $y(x)$ is produced.

In the next section, we devote our attention to error analysis and convergence of the method.

3. **Convergence.** In this section we show that the procedure described in the previous section has fourth order convergence. For the ease of analysis we introduce the vector and matrix notations. Let

$$Y = (y_1, \ldots, y_{N-1})^T,\ Z = (z_1, \ldots, z_{N-1})^T,\ F = (f_1, \ldots, f_{N-1})^T, \tag{3.1}$$

$$C = (4\alpha,\ \alpha,\ 0,\ \ldots, 0,\ \beta,\ 4\beta)^T,\ D = \left(\frac{1}{3}f_0,\ \frac{1}{20}f_0,\ 0, \ldots, 0,\ \frac{1}{20}f_N, \frac{1}{3}f_N\right)^T$$

be the vectors and the five-band matrices $J$ and $V$ are given by

$$J = \begin{bmatrix} 7 & -2 & -1 & & & & \\ -2 & 6 & -2 & -1 & & & \\ -1 & -2 & 6 & -2 & -1 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & -1 & -2 & 6 & -2 & -1 & \\ & & -1 & -2 & 6 & -2 & \\ & & & -1 & -2 & 7 & \end{bmatrix},\quad B = \frac{1}{60}\begin{bmatrix} 205 & 70 & 5 & & & & \\ 78 & 198 & 78 & 3 & & & \\ 3 & 78 & 198 & 78 & 3 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & 3 & 78 & 198 & 78 & 3 & \\ & & 3 & 78 & 198 & 78 & \\ & & & 5 & 70 & 205 & \end{bmatrix} \tag{3.2}$$

In a compact form, the system of equations (2.6), (2.4) and (2.7) can be written as

$$JZ - h^2[BF(Z) + D] = C. \tag{3.3}$$

For the exact solution $Y$, the equation (3.3) becomes

$$JY - h^2[BF(Y) + D] = C + T(h), \tag{3.4}$$

where the truncation error $T(h) = (t_1(h), \ldots, t_{N-1}(h))$ is given by

$$\begin{cases} t_1(h) = -\dfrac{h^6}{48}\, y^{(6)}(\xi_1),\ x_0 < \xi_1 < x_3, \\ t_i(h) = -\dfrac{h}{120}\, y^{(6)}(x_i) + 0(h^7),\ i = 2(1)N-2 \\ t_{N-1}(h) = -\dfrac{h^6}{48}\, y^{(6)}(\xi_{N-1}),\ x_{N-3} < \xi_{N-1} < x_N. \end{cases} \tag{3.5}$$

From the equations (3.3)—(3.4), we note that the difference $Y - Z$, say $E_d$, satisfies

$$(J - h^2BU)\,(Y - Z) = T(h) \tag{3.6}$$

where

$$F(Y) - F(Z) = U(Y - Z) \text{ (mean-value theorem)},$$

and $U = \{u_1, \ldots, u_{N-1}\}$ with $u_i$ being a certain value of $\partial f_i/\partial y_i$. Therefore equation (3.6) may be written as

$$E_d = (J - h^2 BU)^{-1}\, T(h). \qquad (3.7)$$

Now, from (3.5), the norm of the truncation error can be obtained as (see Usmani and Warsi (1980))

$$||T|| \leqslant \frac{1}{43}\, h^6 W^{(6)}, \text{ where } W^6 = \max_{0 \leqslant x \leqslant 1} |y^{(6)}(x)| \qquad (3.8)$$

(all norms are $\infty$ — norms unless otherwise stated).
We have to consider $||(J - h^2BU)^{-1}||$ to estimate the difference $Y - Z = E_d$ in equation (3.7).

Further analysis to find the above norm entails invoking some results of the classical theory of applied linear algebra, especially theorems concerning non-negative matrices, diagonal dominance, graph connectedness and monotonicity. Some important properties of the matrix $J$ which are useful in the present analysis are given in appendix A.

Following [12], we consider the norm $||(J - h^2BU)^{-1}||$ in two separate intervals, (i) $-\infty < u \leqslant 0$ and (ii) $0 < u < \pi^2$.

Case (i): $-\infty < u \leqslant 0$

In this interval, both the matrices $J$ and $(J - h^2BU)$ are clearly monotone and $J - h^2BU \geqslant J$. Therefore

$$||(J - h^2BU^{-1})|| \leqslant ||J^{-1}|| \leqslant \frac{1}{6}\left(\frac{1}{8h^2} + \frac{1}{2}\right) \text{(see appendix A)}. \qquad (3.9)$$

Case (ii): $-0 < u < \pi^2$.

Let $U = U^+ + U^-$, where $U^+ > 0$ and $U^- \leqslant 0$. So, we can write

$$J - h^2BU = M[I - h^2M^{-1}BU^+], \text{ where } M = J - h^2BU^-. \qquad (3.10)$$

As $B$ is a nonnegative matrix, it is easy to show that $M$ is monotone. Hence

$$M^{-1} \geqslant 0 \text{ and } ||M^{-1}|| \leqslant ||J^{-1}||. \qquad (3.11)$$

Following Henrici (1962), we state the following lemma for the norm of $[I - h^2M^{-1}BU^+]^{-1}$.

LEMMA 3.1. *For* $0 < u < 8$ *and* $||h^2M^{-1}BU^+|| \leqslant k < 1$, *the matrix* $[I - h^2M^{-1}BU^+]$ *exists and*

$$||[I - h^2M^{-1}BU^+]^{-1}|| < \frac{1}{1-k}, \textit{ for } h < H,$$

*where*

$$H < \sqrt{\frac{2}{u}\left(1 - \frac{u}{8}\right)}.$$

Since $||h^2M^{-1}BU^+|| \leqslant h^2u\left(\frac{1}{8h^2}+\frac{1}{2}\right) < 1$, for $h < H$, we get

$$||[I - h^2M^{-1}BU^+]^{-1}|| \leqslant \frac{1}{1 - h^2u\left(\frac{1}{8h^2}+\frac{1}{2}\right)}. \tag{3.12}$$

Therefore, from (3.10) we obtain

$$||(J - h^2BU)^{-1}|| \leqslant ||([I - h^2M^{-1}BU^+]^{-1}|| \cdot ||M^{-1}|| \tag{3.13}$$

$$\leqslant \frac{\frac{1}{8h^2}+\frac{1}{2}}{1 - h^2u\left(\frac{1}{8h^2}+\frac{1}{2}\right)},\ h < H.$$

The above results can now be stated as:

LEMMA 3.2. *For* $y \in C^6_{[0,1]}$ *and* $-\infty < u < 8$,

$$||E_d|| = ||Y - Z|| \leqslant \lambda h^4, \tag{3.14}$$

*where*

$$\lambda = \begin{cases} \frac{W^{(6)}}{2304}, & -\infty < u \leqslant 0, \\ \frac{W^{(6)}}{48(8-u)}, & 0 < u < 8. \end{cases}$$

Since a cubic spline is used as the global approximation to the solution $y(x)$, the norm of the corresponding error, say $E_c(x)$, can be calculated as (see H a l l, 1968)

$$||E_c(x)|| \leqslant \frac{5}{384} h^4W^{(4)}, \text{ where } W^{(4)} = \max_{0 \leqslant x \leqslant 1} |y^{(4)}(x)|. \tag{3.15}$$

Now, we are in a position to estimate the total error, $E(= E_c + E_d)$, of the method and we state it in a theorem as

THEOREM 3.1. *Let* $y \in C^6_{[0,1]}$ *and* $-\infty < u < 8$. *Then our method described in section* 2 *provides convergence of* $0(h^4)$ *for the problem* (1.1), *that is,*

$$||E|| \leqslant ||E_d|| + ||E_c|| \leqslant \psi h^4, \tag{3.16}$$

*where*

$$\psi = \lambda + \frac{5}{384} W^{(4)}.$$

Thus, the error in the present method is of order four. This fact is also verified by numerical illustrations.

4. **Solution of the difference equations.** Newton's method is discussed to solve the nonlinear system (3.3). From Kantorovich's result sufficient conditions are obtained which guarantees the convergence of Newton's method (see Henrici 1962).

For the nonlinear system (3.3), let

$$R(Z) = JZ - h^2[BF(Z) + D] - C, \tag{4.1}$$

and

$$Z^{(0)} = (Z_i^{(0)}, \ldots, Z_{N-1}^{(0)}) \tag{4.2}$$

be an initial approximation. Then, Newton's method for the system (3.3) is

$$R(Z^{(i)}) + R'(Z^{(i)})\, \Delta Z^{(i)} = 0 \tag{4.3}$$

whose solution is given by

$$\Delta Z^{(i)} = -[R'(Z^{(i)})]^{-1} R(Z^{(i)}), \ i = 0,\ 1,\ 2, \ldots \tag{4.4}$$

provided that the inverse of the matrix

$$A(Z) = J - h^2 BF'(Z) \tag{4.5}$$

exists for $Z = Z^{(i)}$, $i = 0, 1, 2, \ldots$
If the matrices $A(Z^{(\nu)})$, $\nu = 0, 1, 2, \ldots$ involved continue to be nonsingular, a sequence of successively better approximations $Z^{(\nu)}$ can be obtained by the algorithm

$$Z^{(\nu+1)} = Z^{(\nu)} + \Delta Z^{(i)}, \ \nu = 0,\ 1,\ 2, \ldots. \tag{4.6}$$

Suppose

$$\sup \frac{\partial f_i}{\partial z_i}\bigg|_{Z_i = Z_i^{(0)}} = u^{(0)}. \tag{4.7}$$

To obtain necessary bound for the inverse of the matrix $A(Z^{(0)})$ we consider the two cases: (i) $-\infty < u^{(0)} \leqslant 0$, and (ii) $0 < u^{(0)} < \pi^2$.
Case (i): $-\infty < u^{(0)} < 0$

Following arguments similar to those given in section 3, $A(Z^{(0)})$ is monotone and

$$||A(Z^{(0)})^{-1}|| \leqslant ||J^{-1}|| \leqslant \frac{1}{6}\left(\frac{1}{8h^2} + \frac{1}{2}\right) = B_0, \text{ say.} \tag{4.8}$$

Case (ii): $0 < u^{(0)} < \pi^2$.

Let $U = F'(Z^{(0)})$ and $U = U^+ + U^-$, where $U^+ > 0$ and $U^- \leqslant 0$. Suppose $M = J - h^2 BU^-$, then it is easy to prove that $M$ is irreducible and monotone. Let us consider the matrix $A(Z^{(0)})$. Now

$$A(Z^{(0)}) = M(I - h^2 M^{-1} BU^+). \tag{4.9}$$

As the product of two monotone matrices is a monotone matrix, $A(Z^{(0)})$ will be monotone provided $(I - h^2 M^{-1} BU^+)$ is monotone. Following Collatz

(1966; 378), the above condition becomes $||h^2M^{-1}BU^+|| < 1$, which after some simplifications becomes $h \leqslant H$, where

$$H < \sqrt{\frac{2}{u^{(0)}}\left(1 - \frac{u^{(0)}}{8}\right)}, \quad u^{(0)} < 8.$$

We state the above result in the following lemma.

LEMMA 4.1. *If* $0 < u^{(0)} < 8$, *then* $A(Z^{(0)})$ *is monotone for all* $h \leqslant H$, *provided*

$$H < \sqrt{\frac{2}{u^{(0)}}\left(1 - \frac{u^{(0)}}{8}\right)}. \tag{4.10}$$

So, we reach the conclusion that the inverse of the matrix $A(Z^{(0)})$ exists and is nonnegative.

Following arguments similar to those given in section 3, norm of the inverse of $A(Z^{(0)})$ may be obtained as

THEOREM 4.1. *For* $0 < u^{(0)} < 8$ *and for all* $h \leqslant H$, *we have*

$$||A(Z^{(0)})^{-1}|| < \frac{\omega}{8h^2(1 - u^{(0)}\omega/8)} = B^0, \text{ say,} \tag{4.11}$$

*where* $1 + 4h^2 = \omega$.

Now, if the initial approximation is properly choosen such that

$$JZ^{(0)} = C, \tag{4.12}$$

then

$$||R(Z^{(0)})|| \leqslant \sigma h^2, \tag{4.13}$$

where

$$||F(Z^{(0)})|| + \frac{1}{60}\max\left(|20f_0|, |3f_0|, |3f_N|, |20f_N|\right) \leqslant \sigma. \tag{4.14}$$

From (4.8), (4.11) and (4.13), we obtain

$$||A(Z^{(0)})^{-1}R(Z^{(0)})|| \leqslant B_0\sigma h^2 = \eta_0, \text{ say.} \tag{4.15}$$

Let $L_2(h_0) = \max |f^{yy}|$ over $0 \leqslant x \leqslant 1$ and for $y$ satisfying

$$\max_{1 \leqslant i \leqslant N-1} |Y - Z_i^{(0)}| \leqslant N(h_0)\eta_0, \text{ where } N(h_0) = (1 - \sqrt{1 - 2h^0})/h_0.$$

If $R = (r_1, \ldots, r_{N-1})$, then

$$\sum_{j,k=1}^{N-1} \left|\frac{\partial^2 r_i}{\partial z_j \partial z_k}\right| \leqslant 6h^2L_2(h^0) = K, \text{ say, for } 1 \leqslant i \leqslant N-1. \tag{4.16}$$

Now, the main theorem of Kantorovich's theorem which guarantees convergence of Newton's method is satisfied provided

$$h_0 = B_0\eta_0K \leqslant 1/2. \tag{4.17}$$

With the help of (4.8), (4.11), (4.15) and (4.16), we obtain the following result

THEOREM 4.2. *For the solution of the nonlinear system (3.3) by Newto method, let the initial approximation $Z^{(0)}$ satisfy* (4.12).
*If* $-\infty < u^{(0)} \leqslant 0$, *then Newton's method is convergent provided*

$$h_0 = \frac{\omega^2}{384} \sigma L_2(h_0) \leqslant 1/2 \tag{4.}$$

*If* $0 < u^{(0)} < 8$, *then Newton's method is convergent provided* $h \leqslant H^{(0)}$ *and*

$$h_0 = \frac{\omega^2}{64(1 - u^{(0)} \omega/8)^2} \sigma L_2(h^0) \leqslant 1/2 \tag{4.}$$

*In each case, the speed of convergence of the method is given by*

$$||Y - Z^{(i)}|| \leqslant \frac{1}{2^{i-1}} (2h_0)^{2^i-1} \eta_0. \tag{4.}$$

5. **Numerical illustrations.** In this section we present the computatio behaviour of the method formulated in section 2. Consider the two proble Example 1 [12]. $y'' + e^{-2y} = 0$, $0 \leqslant x \leqslant 1$, $y(0) = 0$, $y(1) = \ln 2$. with the e solution $y(x) = \ln(1 + x)$, and
Example 2. $y'' - \frac{1}{2} (1 + x + y)^3 = 0$, $y(0) = 0 = y(1)$. with exact solu $y(x) = \frac{2}{2 - x} - x - 1$.

The approximate values of the solution and its derivatives are obta using the method described in section 2. For the solution of the nonli system (3.3), it is easy to verify that $h_0 \leqslant 1/2$ for both the problems. Ta I and II contain values of maximum error in the solution at nodes and points. Table III contains various error terms in derivative approximation both the examples. The notation $||E_d||$ denotes $\max_{1 \leqslant i \leqslant N-1} |y_i - z_i|$ and $D$ denotes $\max |y^{(n)}(x_i + 0.5h) - u^{(n)}(x_i + 0.5h)|$, $i = 0(1)N - 1$, where $(n)$ the $n^{th}$ derivative with respect to $x$. For the sake of comparision we have computed the solution using the method given in [12] for both the exam and tabulated the maximum absolute errors at nodes and mid-points in colu $||E_N||$ and $e_N$ respectively. All computations included in this work are ca out on CYBER 180/840—A. The notation 6.04(—7) is used for 6.04 × 1

*Table 1*

**Errors for Example 1**

| $N$ | $\|\|E_d\|\|$ | $\|\|E_N\|\|$ | $e$ | $e_N$ |
|---|---|---|---|---|
| 8 | .60415 (—6) | .20165 (—5) | .15365 (—4) | .15786 (—4) |
| 16 | .29388 (—7) | .12867 (—6) | .10595 (—5) | .10839 (—5) |
| 32 | .24273 (—8) | .80668 (—8) | .69988 (—7) | .71032 (—7) |
| 64 | .16390 (—9) | .50476 (—9) | .45082 (—8) | .45466 (—8) |
| 128 | .10758 (—10) | .30520 (—10) | .28627 (—9) | .28756 (—9) |

*Table 2*

**Errors for Example 2**

| $N$ | $\|\|E_d\|\|$ | $\|\|E_N\|\|$ | $e$ | $e_N$ |
|---|---|---|---|---|
| 8 | .64841 (−5) | .16424 (−4) | .11700 (−3) | .12020 (−3) |
| 16 | .21645 (−6) | .10481 (−5) | .82514 (−5) | .84595 (−5) |
| 32 | .19245 (−7) | .66034 (−7) | .55176 (−6) | .56129 (−6) |
| 64 | .13277 (−8) | .41315 (−8) | .35786 (−7) | .36149 (−7) |
| 128 | .85362 (−10) | .25785 (−9) | .22810 (−8) | .22935 (−8) |

From the data presented in Tables I and II, we conclude that the theoretically established fourtjh order convergence is numerically verified. We also observe that alhough our method and the method given in [12] are both fourth order convergent, results are much better at the mesh points in our case.

*Table 3*

**Max. absolute errors in derivative approximation at mid-points**

| $N$ | For Example 1 $De$ | $D^2e$ | For Example 2 $De$ | $D^2e$ |
|---|---|---|---|---|
| 8 | .79437 (−5) | .92481 (−2) | .74526 (−4) | .69823 (−1) |
| 16 | .61595 (−6) | .25944 (−2) | .58842 (−5) | .20141 (−1) |
| 32 | .43842 (−7) | .68865 (−3) | .42402 (−6) | .54255 (−2) |
| 64 | .29470 (−8) | .17751 (−3) | .28781 (−7) | .14092 (−2) |
| 128 | .19248 (−9) | .45069 (−4) | .18820 (−8) | .35916 (−3) |

*APPENDIX A*

Here we present some interesting properties of the matrix $J$ which are useful in analysis of the method in section 2. $J$ is a symmetric, five-band matrix. It is also an irreducible matrix following a well-known result of graph theory, namely, a $n \times n$ complex matrix is irreducible if and only if its directed graph is strongly connected' (see Varga 1962). Besides, $J$ is diagonally dominant with strict dominance in first and last rows, and hence it is a irreducibly diagonally dominant matrix. So $J$ is monotone and $J^{-1}$ is nonnegative (see Henrici 1962). Furthermore, $J$ is also a Stieljes matrix (see Varga 1962). In addition, since it is irreducible, $J^{-1} > 0$.

The matrix $J$ can be written as the product of two symmetric tridiagonal matrices $P = (P_{ij})$ and $Q = (Q_{ij})$ where $P_{i,i} = 2$, $P_{i,i} \pm i = -1$, $Q_{i,i} = 4$, and $Q_{.,i} \pm i = 1$, that is $J = PQ$ and hence $J^{-1} = Q^{-1}P^{-1}$, where $J$ is given by (3.2).

A relation between the matrices $P$ and $Q$ can be established as $Q = 6I - P$. Therefore, $Q^{-1} =$ $= P^{-1}(6P^{-1} - I)^{-1}$, or $Q^{-1}P^{-1} = \frac{1}{6}(P^{-1} + Q^{-1})$. From Usmani (1980), the norm of $J^{-1}$ may be obtained as

$$||J^{-1}|| \leqslant \frac{1}{6} \left[||P^{-1}|| + ||Q^{-1}||\right] \leqslant \frac{1}{6}\left(\frac{1}{8h^2} + \frac{1}{2}\right).$$

**Acknowledgement.** We wish to express our sincere thanks to Prof. R. A. Usmani for his comments and suggestions. The second author acknowledges the financial support from grant 9/81(37)/85 — EMR — I of C.S.I.R. India

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STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXXV, 1, 1990

# STABILITY ANALYSIS FOR A NEW DIRECT INTEGRATION OPERATOR

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*Received: November 3, 1989*
*AMS subject classification: 65L20*

**REZUMAT. — Analiza stabilităţii pentru un nou operator de integrare directă.** Se studiază stabilitatea unui operator de integrare directă pentru ecuaţiile vibraţiilor liniare ale unui sistem cu $n$ grade de libertate.

1. **Introduction** A new direct integration operator has been introduced [3] for integrationof differential equations describing the non-linear dynamical response of a structure:

$$M\ddot{U} + g(\dot{U}) + f(U) = P(t), \tag{1}$$

or, particularly, the linear structure response, i.e.

$$M\ddot{U} + C\dot{U} + KU = P(t), \tag{2}$$

in which; $M$ = the mass matrix; $U = [u_1 \dots u_s]^T$ = the degree-of-freedom (deplascement) vector; $f$ and $g$ = non-linear stiffness and damping function, respectively; $C$ and $K$ = damping and stiffness matrix, respectively; and, a dot indicates differentiation with respect to time $t$.

The operator is defined by following formulae:

$$\begin{aligned}
U_1 &= U_0 + \dot{U}_0\Delta t + \ddot{U}_0(\Delta t)^2/2 + \dddot{U}_0(\Delta t)^3/6 + \beta(\Delta t)^3\Delta\dddot{U}_1 \\
\dot{U}_1 &= \dot{U}_0 + \ddot{U}_0\Delta t + \dddot{U}_0(\Delta t)^2/2 + \gamma(\Delta t)^2\Delta\dddot{U}_1 \\
\ddot{U}_1 &= \ddot{U}_0 + \dddot{U}_0\Delta t + \delta(\Delta t)\Delta\dddot{U}_1 \\
\dddot{U}_1 &= \dddot{U}_0 + 1\cdot\Delta\dddot{U}_1,
\end{aligned} \tag{3}$$

in which the subscript 0 and 1 denote function values in $t_0$ and $t_1 = t_0 + \Delta t$, respectively. The operator coefficients $\beta$, $\gamma$ and $\delta$ are given by:

$$\beta = \frac{(1-\theta)^{4-p}}{6p}, \quad \gamma = \frac{(1-\theta')^{3-p'}}{2p'}, \quad \delta = \frac{(1-\theta'')^{2-p''}}{p''}, \tag{4}$$

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in which $p$, $p'$ and $p''$ are positive integers arbitrarily choosen and $\theta$, $\theta'$ and $\theta'' \in \in (0, 1)$ and are associated with $p$, $p'$ and $p''$ respectively [3].
In the subsequent analysis we study the stability of the operator defined by Eqs. (3), when applied to the linear Equation (2). Because a change of the initial basis into the the basis formed by the eingenvectors of the problem $K\Phi_j = \omega^2 M \Phi_j$ will decuple the matricial Eq. (2) -see [2], [4], the operator stability will be analyzed for a single -degree-of-freedom (SDOF) system equation, i.e.

$$\ddot{u} + 2\xi\omega\dot{u} + \omega^2 u = p(t), \tag{5}$$

in which $\omega$ is the circular frequency, and $\xi$ is the damping ratio.

2. **Operator matrix.** The operator formulae Eqs. (3) written for a SDOF system take the following matrix form

$$X_1 = S_\theta X_\theta + R_\theta \Delta\dddot{u}_1, \tag{6}$$

in which:

$$X(t) = [u(t)\ \dot{u}(t)\ \ddot{u}(t)\ \dddot{u}(t)]^T, \tag{7a}$$

$$X_1(t) = X(t_1),\ X_\theta = X(t_\theta), \tag{7b}$$

$$S_\theta = \begin{bmatrix} 1 & \Delta t & \Delta t^2/2 & \Delta t^3/6 \\ 0 & 1 & \Delta t & \Delta t^2/2 \\ 0 & 0 & 1 & \Delta t \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

and

$$R_\theta = [\beta\Delta t^3\quad \gamma\Delta t^2\quad \delta\Delta t\quad 1]^T.$$

Eq. (5) can be put in the matrix form

$$A\, X(t) = p(t), \tag{9}$$

in which

$$A = [\omega^2\quad 2\zeta\omega\quad 1\quad 0]. \tag{10}$$

Writting Eq. (9) for $t = t_1$ as

$$A\, X = p_1 \tag{11}$$

in which $p_1 = p(t_1)$ and substituting Eq. (6) in Eq. (11) leads to

$$A\, S_\theta\, X_\theta + A R_\theta \Delta\dddot{u}_1 = p_1. \tag{12}$$

Let denote by

$$a = \frac{1}{\Delta t} A\ R_\theta = \beta x^2 + 2\xi\gamma x + \delta \tag{13a}$$

$$x = \omega\Delta t = 2\pi(\Delta t/T), \tag{14b}$$

in which $T = 2\pi/\omega$ is the system period.

Solving Eq. (12) with respect to $\Delta\dddot{u}_1$ yields

$$\Delta\dddot{u}_1 = -\frac{1}{a\Delta t} A\, S_0\, X_0 + \frac{p_1}{a\Delta t}. \tag{14}$$

Using Eq. (14) in Eq. (6), the latter one can be put in the form

$$X_1 = SX_0 + p_1 R \tag{15}$$

in which

$$R = \frac{1}{a\Delta t} R_0 = \frac{1}{a} [\beta\Delta t^2 \quad \gamma\Delta t \quad \delta 1/\Delta t]^T, \tag{16}$$

and the operator matrix $S$ is given by

$$S = S_0 - R\, A\, S_0 = (I - R\, A)S_0. \tag{17}$$

Explicitly, the matrix $S$ is given by:

$$S = \begin{bmatrix} 1-\beta c_1 & \Delta t(1-\beta c_2) & \Delta t^2(1/2-\beta c_3) & \Delta t^3(1/6-\beta c_4) \\ -\frac{1}{\Delta t}\gamma c_1 & 1-\gamma c_2 & \Delta t(1-\gamma c_3) & \Delta t_2(1/2-\gamma c_4) \\ -\frac{1}{\Delta t^2}\delta\, {}_1 & -\frac{1}{\Delta t}\delta c_2 & 1-\delta c_3 & \Delta t(1-\delta c_4) \\ -\frac{1}{\Delta t^3} c_1 & -\frac{1}{\Delta t^2} c_2 & -\frac{1}{\Delta t} c_3 & 1-c_4 \end{bmatrix} \tag{18}$$

in which:

$$\begin{aligned} c_1 &= x^2/a \\ c_2 &= (x^2 + 2\zeta x)/a \\ c_3 &= (x^2/2 + 2\zeta x + 1)/a \\ c_4 &= (x^2/6 + \zeta x + 1)/a, \end{aligned} \tag{19}$$

and $a$ is defined by Eq. (13).

Applying to matrix $S$ the similarity transformation defined by

$$S' = D^{-1} \cdot S \cdot D \tag{20}$$

in which

$$D = \text{diag}[d_{ii}] \quad ; d_{ii} = (\Delta t)^{4-i}, \quad i = \overline{1, 4} \tag{21}$$

resuts in

$$S' = \begin{bmatrix} 1-\beta c_1 & 1-\beta c_2 & 1/2-\beta c_3 & 1/6-\beta c_4 \\ -\gamma c_1 & 1-\gamma c_2 & 1-\gamma c_3 & 1/2-\gamma c_4 \\ -\delta c_1 & -\delta c_2 & 1-\delta c_3 & 1-\delta c_4 \\ -c_1 & -c_2 & -c_3 & 1-c_4 \end{bmatrix} \tag{22}$$

According to Eq. (20), matrix $S$ entries $s_{ij}$ can be generated by

$$s_{ij} = s'_{ij}(\Delta t)^{j-i}; \quad i, j = \overline{1, 4}, \tag{23}$$

in which $s'_{ij}$ are matrix $S'$ entries (Eq. (22)).

**3. Stability analysis.** Denoting in Eq. (15) $t_k$ and $t_{k+1}$ instead of $t_0$ and $t_1$ respectively, leads to the recurrence relation

$$X_{k+1} = S\,X_k + R\,p_k; \quad p_k = p(t_k) \tag{24}$$

Applying Eq. (24) succesively for $i = 0, 1, \ldots, n-1$, results in

$$X_n = S^n X_0 + (S^{n-1}p_1 + S^{n-2}p_2 + \ldots + Sp_{n-1} + I)R. \tag{25}$$

Defining the operator stability as the sensitivity of the solution $X_n = X(t_n)$ to small changes in initial conditions $X_0$, it can be seen that the latter term of Eq. (25) do not influence operator stability. Thus, the stability can be analyzed for the homogenous equation (5), i.e. $p(t) = 0, t \geqslant t_0$. In this case, Eq. (25) becomes

$$X_n = S^n X_0. \tag{26}$$

Let $J$ be the Jordan form of matrix $S$, and

$$S = T\,J\,T^{-1} \tag{27}$$

in which $T$ is the transformation matrix to Jordan form (see for instance [1]). Using Eq. (27) in Eq. (26), the latter one reads

$$X_n = (T\,J^n T^{-1})X_0, \tag{28}$$

from which it can be seen that the stability criterion will be the condition that the spectral radius $\rho(S)$ of matrix $S$, be bounded by 1:

$$\rho(S) \leqslant 1. \tag{29}$$

The foregoing conclusions and the criterion expressed by (29) are also pointed out in (2).

So, our task will be to find conditions which ensure that operator matrix $S$ have all eigenvalues of modulus less than or equal to unity. Matrix $S'$ will be employed instead of $S$, because having the same eigenvalues as $S$ and a sampler form.

The characteristic polynomial of matrix $S'$ will be found first, and then transformed in order to apply to it the Routh—Hurwitz criterion.

According to Eqs. (22) and (19), the characteristic polynomial of matrix $S'$ is

$$f(\lambda) = \det(S' - \lambda I) = \lambda P(z) \tag{30}$$

in which

$$z = 1 - \lambda,$$
$$P(z) = az^3 - bz^2 + cz - d, \tag{31}$$

and the coefficients of $P(z)$ are given by

$$
\begin{aligned}
a &= \beta x^2 + 1 + 2\xi x \\
b &= x^2(\gamma + \delta/2 + 1/6) + 1 + 2\xi x(\delta + 1/2) \\
c &= x^2(1 + \delta) + 2\xi x \\
d &= x^2.
\end{aligned}
\tag{32}
$$

From Eq. (30) follows the property expressed by

THEOREM 1. *For any choice of $\beta$, $\gamma$ and $\delta$, the operator matrix $S$ has an eigenvalue $\lambda = 0$.*

The three other eigenvalues are related to the roots of polynomial $P(z)$— Eqs. (31), (32), by $\lambda = 1 - z$.

As the transformation

$$w = \frac{\lambda - 1}{\lambda + 1}$$

maps the unit circle of the $\lambda$ plane into the left of $w$ plane - see [6], pp. 239, the corresponding transformation in $z - w$ variables is

$$z = \frac{2w}{w - 1}\cdot \tag{33}$$

Substituting Eq. (33) in Eq. (31) yields the polynomial

$$Q(w) = d_0 w^3 + d_1 w^2 + d_2 w + d_3, \tag{34}$$

in which:

$$
\begin{aligned}
d^0 &= 2\left[\beta - \frac{\gamma}{2} + \frac{1}{24} + \frac{\delta - 1/2}{x^2} + 2\frac{\zeta}{x}\left(\gamma - \frac{\delta}{2}\right)\right] \\
d_1 &= \gamma - \frac{\delta}{2} - \frac{1}{12} + \frac{1}{x^2} + \frac{2\zeta}{x}\left(\delta - \frac{1}{2}\right) \\
d_2 &= \frac{1}{2}\left[\left(\delta - \frac{1}{2}\right) + \frac{2\zeta}{x}\right] \\
d_3 &= \frac{1}{4}\cdot
\end{aligned}
\tag{35}
$$

The condition $|\lambda| \leqslant 1$ is equivalent to the condition $\mathrm{Re}(w) \leqslant 0$, in which $w$ are the roots of polynomial $Q(w)$ defined by Eq. (34) (35). It follows then,

THEOREM 2. *The operator defined by Eq. (3) is stable if and only if, the the coefficients $d_i$, $i = \overline{0, 3}$ of polynomial $Q(w)$ satisfie the following conditions:*

$$d_1 \geqslant 0,\ d_2 \geqslant 0, \tag{36a}$$

$$d_0 > 0, \tag{36b}$$

$$d_1 d_2 - d_0 d_3 \geqslant 0. \tag{36c}$$

*Proof.* Eqs. (36) are Routh—Hurwitz conditions applied to coefficients $d_i$ of polynomial $Q(w)$—see [6], [5].

For a system without damping, i.e. $\xi = 0$, coefficients $d_i$—Eq. (35) read

$$\begin{aligned} \gamma_0 &= 2\left(\beta - \frac{\gamma}{2} + \frac{1}{24} + \frac{\delta - 1/2}{x^2}\right) \\ \gamma_1 &= \gamma - \frac{\delta}{2} - \frac{1}{12} + \frac{1}{x^2} \\ d_2 &= \frac{1}{2}\,(\delta - 1/2) \\ \gamma_3 &= 1/4, \end{aligned} \tag{37}$$

and the conditions (36a)—(36b) are equivalent to the following ones:

$$1^\circ \ \ \delta \geqslant \frac{1}{2} \tag{38}$$

$$2^\circ \ \ \beta - \frac{\delta}{2} + \frac{1}{24} \geqslant 0 \text{ and } \forall \eta,\ x > 0 \tag{39a}$$

or

$$\beta - \frac{\delta}{2} + \frac{1}{24} < 0 \text{ and } x^2 < \frac{\delta - 1/2}{\dfrac{\delta}{2} - \beta - \dfrac{1}{24}} \tag{39b}$$

$$3^\circ \ \ \gamma - \frac{\delta}{2} - \frac{1}{12} \geqslant 0 \text{ and } \forall x,\ x > 0 \tag{40a}$$

or

$$\gamma - \frac{\delta}{2} - \frac{1}{12} < 0 \text{ and } x^2 < \frac{1}{\dfrac{\delta}{2} + \dfrac{1}{12} - \gamma} \tag{40b}$$

4° Condition (36c) can be expressed as

$$\beta < \frac{(\gamma + 1/6)^2}{2} \text{ and } \delta_1 < \delta < \delta_2 \tag{41}$$

in which

$$\delta_{1,2} = (\gamma + 1/6) \pm [(\gamma + 1/6)^2 - 2\beta]^{1/2}.$$

We have therefore,

THEOREM 3. *For un undamped system ($C = 0$ in Eq. (2) and $\zeta = 0$ in Eq. (5)) the operator Eqs. (3) are stable if and only if coefficients $\beta$, $\gamma$ and $\delta$ satisfie Eqs. (38), (39), (40) and (41).*

*Particularly, if Eqs. (38), (39a), (40a) and (41) are satisfied, the operator is unconditionally stable, i.e. it is stable for $\forall x,\ x > 0$.*

**4. Numerical examples.** Examples 1 and 2 refer to un undamped **system.**
*Example 1.*

$$\beta = 1/28, \ \gamma = 1/4, \ \delta = 1/1.5 \qquad (42)$$

Conditions (38) and (41) are satisfied; conditions (39b) and (40b) give $x^2 < 3 \cdot 5$, from wich it follows the stability limit (see Eq. (13b)):

$$\frac{\Delta t}{T} < \frac{\sqrt{3 \cdot 5}}{2\pi} = \cdot 29775 \qquad (43)$$

Indeed, the spectral radii computed directly from matrix $S'$(by QR iteration) were: $\Delta t/T = 2975 \ldots \rho = .9979$; $\Delta t/T = .2980 \ldots \rho = 1.00267$. The variation of the spectral radius $\rho$ can be followed in Fig. 1, in which the moduli of eigenvalues $\lambda_i$ are plotted against $\Delta t/T$: $\rho = \max_{i=\overline{1,3}} |\lambda_i|$; the minimal $\rho$—value is .85224 at $\Delta t/T = .28066$.

*Example 2.* The special case $\delta = 1/2$ (44)

In this case we have: $d_0 = 2(\beta - \gamma/2 - 1/24)$, $d_1 = \gamma - 1/3 + 1/x^2$, $d_2 = 0$ and $d_3 = 1/4$.

If $d_0 \neq 0$ we have not $\mathrm{Re}(w) < 0$: indeed, let be $w_1 \in R$, $w_2, w_3 \in C$ $(w_3 = \overline{w_2})$.

From $d_2 = 0$ it results: $w_1 \cdot 2 \ \mathrm{Re}\,(w_2) = -\ |w_2|^2$ and then, $w_1$ and $\mathrm{Re}(w_2)$ have opposite signs.

Consequently, if $\delta = 1/2$ we must have also

$$\beta - \frac{\gamma}{2} + \frac{1}{24} = 0 \qquad (45)$$

Thus, $d_0 = 0$, $d_2 = 0$ and the characteristic equation $Q(w) = 0$ reduces $t_0$

$$d_1 w^2 + 1/4 = 0;$$

if $d_1 < 0$ it follows that $\mathrm{Re}(w) > 0$, we must have then $d_1 > 0$, i.e.:

$$\gamma \geqslant \frac{1}{3} \ \text{and} \ \forall x, \ x > 0 \qquad (46a)$$

or

$$\gamma < \frac{1}{3} \ \text{and} \ x^2 < \frac{1}{1/3 - \gamma} \qquad (46b)$$

As a numerical example, let be:

$$\gamma = 1/24, \ \gamma = 1/16, \ \delta = 1/2. \qquad (47)$$

Conditions (44) and (45) are satisfied and the condition (46b) gives the stability limit

$$\frac{\Delta t}{T} < \frac{\sqrt{6}}{2\pi} = .38985.$$

Numerically: $\Delta t/T = .389 \ldots \rho = 1.000$; $\Delta t/T = .40 \ldots \rho = 1.4473$

**Note 1.** The choice in Eq. (47) corresponds to the choice of $p = 4$, $p' =$ = and $p'' = 2$ in Eq. (4). This is the only choice which eliminates the need of estimating $\theta$, $\theta'$ and $\theta''$.

*Example 3.* Damped system

a) $$\beta = 1/28,\ \gamma = 1/4,\ \delta = 1/1.5$$

Coefficients $d_i$ — Eq.(37)—are:

$$d_0 = \frac{1}{3}\left(\frac{1}{x^2} - \zeta\frac{1}{x} - \frac{2}{7}\right),$$

$$d_1 = \frac{1}{6}\left(6\frac{1}{x^2} + 2\zeta\frac{1}{x} - 1\right),$$

$$d_2 = \zeta\frac{1}{x} + \frac{1}{12},$$

$$d_3 = \frac{1}{4}$$

Let consider $\zeta = 0.1$: Eqs. (36a)—(36c) from Theorem 2, give $x < 1.704$, i.e. $\Delta t/T < .27199$.

b) $$\beta = 1/24,\ \gamma = 1/6,\ \delta = 1/2$$

In this case,

$$d_0 = \frac{2\zeta}{x}\left(\frac{-1}{12}\right) < 0$$

and the operator is unstable for $\forall\zeta$, $\zeta < 0$.

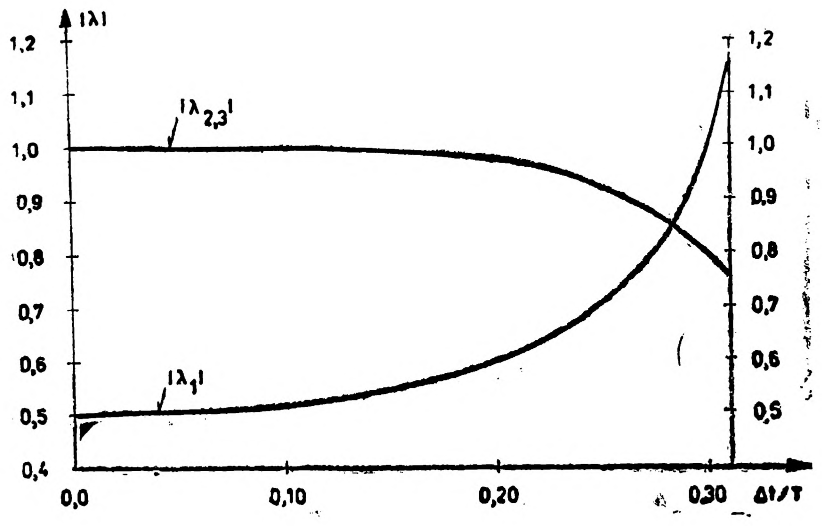


Fig. 1. Spectral radius of the operator $\beta = 1/28$, $\gamma = 1/4$, $\delta = 1/1.5$ ($\rho = \max_{i=\overline{1,3}} |\lambda_i|$).

**Note 2.** *Choosing operator coefficients and time step.* The operator defined by Eqs. (3) was derived under assumption $\ddot{u}$ = constant for $t_0 \in [t_0, t_0 + \Delta t]$ — – see [3]. In order to meet this assumption, the time-step-to-period-ratio $\Delta t/T$ have to be choosen much less than the stability limit found as it is done in the foregoing examples.

Several numerical test indicated coefficients $\beta = 1/28$, $\gamma = 1/4$ and $\delta = 1/1.5$, and a time step $\Delta t \leqslant T/50$, as one of the best choice meeting operator stability and accuracy, for both undamped and damped system, (for a multi-degrees-of-freedom system $T$ is the shortest system period).

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STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXXV, 1, 1990

# STATISTICAL ESTIMATE OF THE SAFETY COEFFICIENT AT VARIABLE LOADING THROUGH ASYMMETRICAL CYCLES USING PARABOLIC MODELLING

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Dedicated to Professor A. Pal on his 60th anniversary

*Received: September 22, 1989*
*AMS subject classification:*

**REZUMAT. — Evaluarea statistică a coeficientului de siguranţă la solicitări variabile prin cicluri asimetrice utilizînd modelarea parabolică.** În prezenta notă se stabileşte o relaţie îmbunătăţită pentru calculul coeficientului de siguranţă la solicitări variabile prin cicluri asimetrice.

Compared to the classical methods of approximation of the Haigh type diagram [1], the present paper aims at establishing an improved relation for the calculus of the safety coefficient at variable loadings through asymmetrical cycles.

Soderberg [1] approximates the Haigh type diagram through the AC straght line, Serensen [2] and Gh. Buzdugan [3], through the $ABC$ broken line, respectively the quarter of ellipse having the $OC$ and $OA$ semiaxis.

These methods approximate the diagram of resistances at weariness neglecting a part of the real field, (case [1] and [2]) or increasing this field [3].

The expressions of the safety coefficients obtained by Soderberg and Gh. Buzdugan, using the classical notations.

$$\psi = \sigma_v/\sigma_{-1}, \qquad \theta = \sigma_m/\sigma_c,$$

are

$$c_d = \frac{1}{\psi + \theta}, \tag{1}$$

$$c_e = \frac{1}{\sqrt{\psi^2 + \theta^2}}, \tag{2}$$

$M(\sigma_m, \sigma_v)$ representing the coordinates of a current point on the omothetic curve.

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The continuation suggests the approximation of the Haigh type diagram through a parabola, the condition that it pass through points $A$ and $C$ and approximate the $M_1(x_i, y_i)$, $i = 1 \ldots, n$ points obtained experimentally in the best way possible being imposed on it.

Coefficient $b$ of the parabola

$$y = -\frac{1}{\sigma_c^2}(b\sigma_c + \sigma_{-1})x^2 + bx + \sigma_{-1} \tag{3}$$

that passes through points $A$ and $C$ is determined through the method of the smallest squares from the condition,

$$\sum_{i=1}^{n}\left[b\left(x_i - \frac{x_i^2}{\sigma_c}\right)^2 - \frac{\sigma_{-1}}{\sigma_c^2}x_i^2 + \sigma_{-1} - y_i\right]^2 ]^2 = \text{minimum} \tag{4}$$

resulting the determination

$$b = -S/\left(\frac{1}{\sigma_c^2}\sum_{i=1}^{n}x_i^4 - \frac{2}{\sigma_o}\sum_{i=1}^{n}x_i^3 + \sum_{i=1}^{n}x_i^2\right) \tag{5}$$

where

$$S = \frac{\sigma_{-1}}{\sigma_c^3}x_i^3 - \frac{\sigma_{-1}}{\sigma_c^2}\sum_{i}x_i^3 - \frac{\sigma_{-1}}{\sigma_o}\sum_{i=1}^{n}x_i^2 + \frac{1}{\sigma_c}\sum_{i=1}^{n}x_i^2y_i + \sigma_{-1}\sum_{i=1}^{n}x_i - \sum_{i=1}^{n}x_iy_i \tag{6}$$

Considering parabola (3) as a limit cycle (safety coefficient equal to the unit) its omothetic parabola to the origin, the points of which represent loading cycles with the same safety coefficient $c > 1$, for the same type of efforts concentrator, it has the equation

$$y = -\frac{1}{\sigma_c^2}(b\sigma_c + \sigma_{-1}) \cdot c \cdot x^2 + bx + \frac{\sigma_{-1}}{c} \tag{7}$$

$b$ being determined by relation (5).

Considering that in this relation $(x, y)$ represent the coordinates of a current point $M(\sigma_m, \sigma_v)$ in the omothetic parabola, it receives the form

$$\sigma_v = -\frac{1}{\sigma_c^2}\, b\sigma_c + \sigma_{-1})c\sigma_m^2 + b\sigma_m + \frac{\sigma_{-1}}{c} \tag{8}$$

wherefrom, noting

$$B = b\sigma_m/\sigma_{-1} \tag{9}$$

expression

$$c = \frac{1}{\sqrt{\frac{1}{4}(\psi - B)^2 + \theta^2\left(1 + b\frac{\sigma_o}{\sigma_{-1}}\right)} + \frac{1}{2}(\psi - B)} \tag{10}$$

results from the safety coefficient at variable loaginds through asymmetrical cycles.

In the particular case in which $n = 1$ and point $M_1$ has the coor $\sigma_{0/2}$, $\sigma_{0/2}$ (the positive pulsatory cycle to be checked) the statistical esti no longer necessary, the parabola passing through this point, and the v $b$ is

$$b = -\frac{\frac{\sigma_{-1}}{\sigma_0^2}\left(\frac{\sigma_2^2}{2}\right) - \sigma_{-1} + \frac{\sigma_0}{2}}{\frac{1}{\sigma_c}\left(\frac{\sigma_0}{2}\right)^2 - \frac{\sigma_0}{2}}$$

Modelling through relation (3) with the value of coefficient $b$ given situaties the approximation curve in the immediate vicinity of the Hai diagram, leading thus to values close to the real situation.

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STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXXV, 1, 1990

# PERIODIC SOLUTIONS OF CERTAIN SYSTEMS OF FOURTH, FIFTH AND SIXTH ORDER DIFFERENTIAL EQUATIONS

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Received: July 25, 1990
AMS subject classification: 34C25

**ABSTRACT.** — In this paper, the sufficient conditions for the existence of periodic solutions of the certain fourth, fifth and sixth order nonlinear system of equations are given. Thus the $n$-dimensional analogues of the results given in [1] [2] and [6] are obtained.

**1. Introduction.** This work is concerned with the problem of existence of periodic solutions of real fourth, fifth and sixth order nonlinear system of equations of the forms.

$$X^{(4)} + A\dddot{X} + B(X, \dot{X}, \ddot{X}, \dddot{X})\ddot{X} + \frac{d}{dt}\operatorname{grad} C(X) + D(X) = P_1(t, X, \dot{X}, \ddot{X}, \dddot{X}) \tag{1.1}$$

$$X^{(5)} + EX^{(4)} + F(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)})\dddot{X} + \frac{d}{dt}\operatorname{grad} G(\dot{X}) + H(\dot{X}, \ddot{X})\dot{X} + K(X) = P_2(t, X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)}) \tag{1.2}$$

and

$$X^{(6)} + LX^{(5)} + MX^{(4)} + N(\ddot{X})\dddot{X} + \frac{d}{dt}\operatorname{grad} U(\dot{X}) + S(\dot{X}, \ddot{X})\dot{X} + T(X) = P_3(t, X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)}, X^{(5)}) \tag{1.3}$$

Here $X$, the unknown function of $t$, is an element of the real $n$-dimensional space $\mathbf{R}^n$ with components $(x_1, x_2, \ldots, x_n)$. $A$, $E$, $L$ and $M$ are constant $n \times n$ matrices. $B$, $F$, $H$, $N$ and $S$ are continuous $n \times n$ matrices depending on the arguments shown in (1.1)—(1.3). $C$, $G$ and $U: \mathbf{R}^n \to \mathbf{R}$ are functions of class $C^2$. The functions $D$, $K$ and $T: \mathbf{R}^n \to \mathbf{R}^n$ are of class $C$.

$$P_1: J \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n, P_2: J \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n, \mathbf{P}_3: J \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$$

are continuous in their arguments and $w$-periodic in $t$. Where $J$ being the infinite range $-\infty < t < \infty$.

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Our object is to establish the following results:

THEOREM 1. *Suppose that*

(i) *$A$ is symmetric;*

(ii) *there exists a constant $a_2 > 0$ such that*

$$||B(X, \dot{X}, \ddot{X}, \dddot{X})\ddot{X}|| \leqslant a_2 ||\ddot{X}|| \text{ for all } X, \dot{X}, \ddot{X}, \dddot{X} \in \mathbf{R}^n; \qquad (1.4)$$

(iii) *the matrix*

$$\left(a_4 - \frac{1}{4} a_2^2\right) I \qquad (1.5)$$

*is positive definite, where*

$$a_4 = \inf_{||x||^2 > 1} \left(\frac{<D(X), X>}{||X||^2}\right), \qquad (1.6)$$

*and $I$ is the $n \times n$ identity matrix;*

(iv) *there exist constants $\alpha_1 > 0$, $\beta_1 \geqslant 0$ such that*

$$||P_1]t, X, \dot{X}, \ddot{X}, \dddot{X})|| \leqslant \alpha_1 + \beta_1(||X|| + ||\dot{X}|| + ||\ddot{X}||) \qquad (1.7)$$

*for all $t$ and all $X, \dot{X}, \ddot{X}, \dddot{X} \in \mathbf{R}^n$.*

*Under these conditions, a constant $\varepsilon_0 > 0$ exists such that if $\beta_1 \leqslant \varepsilon_0$ then the equation (1.1) has at least one w-periodic solution.*

THEOREM 2. *Assume that*

(i) *$E$ and $H$ are symmetric;*

(ii) *there exists a function $\bar{K}: \mathbf{R}^n \to \mathbf{R}$ such that*

$$\frac{\partial \bar{K}}{\partial x_i} = K_i(X), \; i = 1, 2, \ldots, n; \qquad (1.8)$$

(iii) *there exists a constant $b_2 > 0$ such that*

$$||F(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)}) \dddot{X}|| \leqslant b_2 ||\dddot{X}|| \text{ for all } X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)} \in \mathbf{R}^n \qquad (1.9)$$

*where*

$$b_4 = \inf_{\dot{X}, \ddot{X}} \lambda_i(H(\dot{X}, \ddot{X})) \geqslant \frac{1}{4} b_2^2 \qquad (1.10)$$

*for all $\dot{X}, \ddot{X} \in \mathbf{R}^n$ and $\lambda_i(H(\dot{X}, \ddot{X}))$, $(i = 1, 2, \ldots, n)$ denote the eigenvalues of $H(\dot{X}, \ddot{X})$;*

(iv) *there exist constnats $\alpha_2 > 0$, $\beta_2 \geqslant 0$ such that*

$$||P_2(t, X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)})|| \leqslant \alpha_2 + \beta_2(||\dot{X}|| + ||\ddot{X}||) \qquad (1.11)$$

*for all t and all* $X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)} \in \mathbf{R}^n$;

(v) *the function K satisfies either*

$$\langle K(X), \operatorname{sgn} X \rangle \to +\infty \text{ as } ||X|| \to \infty, \tag{1.12}$$

*or*

$$\langle K(X), \operatorname{sgn} X \rangle \to -\infty \text{ as } ||X|| \to \infty, \tag{1.13}$$

*where* $<, >$ *denotes the usual inner product in* $\mathbf{R}^n$ *and* $\operatorname{sgn} X = (\operatorname{sgn} x_1, \ldots, \operatorname{sgn} x_n)$.

*Thus there exists a constant* $\varepsilon > 0$ *such that if* $\beta_2 \leqslant \varepsilon$, *the equation (1.2) has at least one w-periodic solution.*

THEOREM 3. *If*

(i) *L, M and S are symmetric ;*

(ii) *there exists a function* $\overline{T} : \mathbf{R}^n \to \mathbf{R}$ *such that*

$$\frac{\partial \overline{T}}{\partial x_i} = T_i(X), \ i = 1, 2, \ldots, n \tag{1.14}$$

(iii) *there exists a constant* $c_3 > 0$ *such that*

$$||N(\ddot{X})\dddot{X}|| \leqslant c_3 ||\dddot{X}|| \text{ for all } \ddot{X}, \dddot{X} \in \mathbf{R}^n; \tag{1.15}$$

(iv) *the matrix*

$$c_5 I - \frac{1}{4} c_3^2 L^{-1} \operatorname{sgn} L \tag{1.16}$$

*is positive definite; where*

$$c_5 = \inf \lambda_i(S(\dot{X}, \ddot{X})) \text{ or } -\sup \lambda_i(S(\dot{X}, \ddot{X})), \ i = 1, 2, \ldots, n \tag{1.17}$$

*according to the positive or negative definite of L and* $\lambda_i(S(\dot{X}, \ddot{X}))$ *denote the eigenvalues of* $S(\dot{X}, \ddot{X})$.

(v) *there exists a constant* $\alpha_3 > 0$ *such that*

$$||P_3(t, X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)}, X^{(5)})|| \leqslant \alpha_3 \tag{1.18}$$

*for all t and all* $X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)}, X^{(5)} \in \mathbf{R}^n$;

(vi) *the function T satisfies either*

$$< T(X), \operatorname{sgn} X > \to \infty \text{ as } ||X|| \to \infty \tag{1.19}$$

*or*

$$< T(X), \operatorname{sgn} X > \to -\infty \text{ as } ||X|| \to \infty. \tag{1.20}$$

*Then the equation* (1.3) *admits of at least one w-periodic solution.*

Theorems 1, 2 and 3 are $n$-dimensional analogues of the results obtained in [1], [2] and [6].

*Remark 1.* Theorem 2 can also be established for an equation of the form

$$X^{(5)} + EX^{(4)} + F(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)})\dddot{X} + \frac{d}{dt}\operatorname{grad} G(\dot{X}) + H_1(X)\dot{X} + K(X) = P_2(t, X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)}) \quad (1.21)$$

in which $E$, $F$, $G$, $K$ and $P_2$ are exactly as before. But the coefficient $H_1$ is symmetric continuous $n \times n$ matrix depending only on $X$ and satisfying the condition:

$$b_4 = \inf_X \lambda_i(H_1(X)) > \frac{1}{4} b_2^2 .$$

If we take $F(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)}) = F_1(\ddot{X})$ in (1.21), we obtain the equation given in [5].

*Remark 2.* The result related Theorem 3 can be established for the equation of form:

$$X^{(6)} + LX^{(5)} + MX^{(4)} + N(\ddot{X})\dddot{X} + \frac{d}{dt}\operatorname{grad} U(\dot{X}) + S_1(X)\dot{X} + T(X) = P_3(t, X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)}, X^{(5)}) \quad (1.22)$$

where $L$, $M$, $N$, $U$, $T$ and $P_3$ are exactly as before but the $S_1$ is symmetric continuous $n \times n$ matrix depending only on $X$ and satisfying the condition:

$$c_5 = \inf_X \lambda_i(S_1(X)) \text{ or } -\sup_X \lambda_i(S_1(X)).$$

*Remark 3.* Using the Theorem 3, E z e i l o ' s Theorem 2 [2] can be easly extended by replacing a continuous function of $X$, $g_3(x)$ say, in the place of the constant $d_3$.

**2. Some preliminaries.** The proof of all three theorems are based on the well-known Leray—Schauder fixed point technique, with the equations embedded in a suitable parameter-dependent equations. For Theorem 1, the parameter-dependent equation is

$$X^{(4)} + A\dddot{X} + \{(1 - \mu)a_2 + \mu B(X, \dot{X}, \ddot{X}, \dddot{X})\}\ddot{X} + \mu \frac{d}{dt} \operatorname{grad} C(X) + (1 - \mu)a_4 X + \mu D(X) = \mu P_1(t, X, \dot{X}, \ddot{X}, \dddot{X}) \quad (2.1)$$

while for Theorem 2 and Theorem 3, the paremeter-dependent equations are respectively

$$X^{(5)} + EX^{(4)} + \{(1 - \mu)b_2 + \mu F(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)})\}\dddot{X} + \mu \frac{d}{dt} \operatorname{grad} G(\dot{X}) + \{(1 - \mu)b_4 + \mu H(\dot{X}, \ddot{X})\}\dot{X} + (1 - \mu)b_5 X + \mu K(X) = \mu P_2(t, X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)}) \quad (2.2)$$

and

$$X^{(6)} + LX^{(5)} + MX^{(4)} + \{(1-\mu)c_3 + \mu N(\ddot{X})\}\dddot{X} + \mu \frac{d}{dt}\text{grad } U(\dot{X})$$

$$+ \{(1-\mu)c_5 \operatorname{sgn} L + \mu S(\dot{X}, \ddot{X})\}\dot{X} + (1-\mu)c_6 X + \mu T(X) =$$

$$= \mu P_3(t, X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)}, X^{(5)}). \quad (2.3)$$

Where, in all equations, the parameter $\mu$ satisfies $0 \leqslant \mu \leqslant 1$. The constants $b_5$ in (2.2) and $c_6$ in (2.3) are arbitrary but their signs will be positive or negative according as $K$ in (2.2) and $T$ in (2.3) are subject to (1.12) or (1.13) and (1.19) or (1.20) respectively.

Observe that for $\mu = 1$, (2.1) (2.2) and (2.3) reduce to the original equations (1.1), (1.2) and (1.3) respectively. For $\mu = 0$ (2.1) reduces to

$$X^{(4)} + A\dddot{X} + a_2\dot{X} + a_4 X = 0 \quad (2.4)$$

and (2.2) to

$$X^{(5)} + EX^{(4)} + b_2\ddot{X} + b_4\dot{X} + b_5 X = 0 \quad (2.5)$$

and also (2.3) to

$$X^{(6)} + LX^{(5)} + MX^{(4)} + c_3\dddot{X} + c_5 \operatorname{sgn} L\dot{X} + c_6 X = 0. \quad (2.6)$$

It is easy to see from hypothesis (iii) of Theorem 1, the equation (2.4) has no nontrivial $w$-periodic solutions. Also, if $b_5 \neq 0$ and $c_6 \neq 0$, by (1.10) and (1.16), the same results hold for the equations (2.5) and (2.6).

To prove Theorems it suffices [3] to verify the existence of priori bounds $\delta$, $\gamma$ and $\nu$ which are independent of $\mu (0 \leqslant \mu \leqslant 1)$ such that any $w$-periodic solutions of $X(t)$ will hold followings: For the equation (2.1)

$$||X(t)|| \leqslant \delta, \ ||\dot{X}(t)|| \leqslant \delta, \ ||\ddot{X}(t)|| \leqslant \delta, \ ||\dddot{X}(t)|| \leqslant \delta, \ \delta > 0; \quad (2.7)$$

for (2.2)

$$||X(t)|| \leqslant \gamma, \ ||\dot{X}(t)|| \leqslant \gamma, \ ||\ddot{X}(t)|| \leqslant \gamma, \ ||\dddot{X}(t)|| \leqslant \gamma,$$
$$||X^{(4)}(t)|| \leqslant \gamma, \ \gamma > 0; \quad (2.8)$$

and finally for (2.3)

$$||X(t)|| \leqslant \nu, \ ||\dot{X}(t)|| \leqslant \nu, \ ||\ddot{X}(t)|| \leqslant \nu, \ ||\dddot{X}(t)|| \leqslant \nu,$$
$$||X^{(4)}(t)|| \leqslant \nu, \ ||X^{(5)}(t)|| \leqslant \nu, \ \nu > 0 \quad (2.9)$$

must be satisfied for all $t \in [\tau, \tau + \omega]$ and arbitrary $\tau$.

**3. Outline of proof of theorems.** The technique for the verifications of (2.7), (2.8) and (2.9) for (2.1), (2.2) and (2.3) rsepectively are the same as that used in [1], [2], [6] and we shall therefore skip inessential details. The following result holds:

Let $X : [0, \omega] \to \mathbf{R}^n$ be an $w$-periodic function of class $C^4$, $C^5$ and $C^6$ for the equations (2.1) (2.2) and (2.3) respectively. Then for some $\tau > 0$,

$$\int_{\tau}^{\tau+\omega} ||X^{(j)}(t)||\,dt < \frac{1}{4}\,\omega^2\pi^{-2}\int_{\tau}^{\tau+\omega} ||X^{(j+1)}(t)||\,dt \qquad (3.1)$$

where $j = 1, 2, 3$ for $X(t) \in C^4$ $j = 1, 2, 3, 4$ for $X(t) \in C^5$ and $j = 1, 2, 3, 4,$ for $X(t) \in C^6$. For the proof of (3.1) see [4].

To verify (2.7) for (2.1) let $X = X(t) \in C^4$ be an $w$-periodic solution (2.1). Consider the function $V = V(X, \dot{X}, \ddot{X}, \dddot{X})$ defined by

$$V = <\dot{X},\ \ddot{X} + \frac{1}{2}A\dot{X}> - <X,\ \dddot{X} + A\ddot{X}> - \mu<X, \text{grad}\, C(X)> + $$
$$+ \mu C(X). \qquad (3.2)$$

A straightforward differentiation of (3.2), using (2.1), gives

$$\dot{V} = <\ddot{X},\ \ddot{X}> + <X,\ B^*\ddot{X}> + <X,\ D^*> - \mu<X,\ P_1> \qquad (3.3)$$

where

$$B^* = (1-\mu)a_2 + \mu B(X, \dot{X}, \ddot{X}, \dddot{X}),\ D^* = (1-\mu)a_4 X + \mu D(X).$$

Observe from hypothesis (ii) of Theorem 1 that

$$||B^*(X, \dot{X}, \ddot{X}, \dddot{X})\ddot{X}|| \leqslant a_2||\ddot{X}|| \qquad (3.4)$$

for all $X, \dot{X}, \ddot{X}, \dddot{X} \in \mathbf{R}^n$. Also from (1.6), it is clear that

$$<X,\ D^*(X)> \geqslant a_4||X||^2 \text{ for } ||X|| \geqslant 1.$$

Thus, for some constant $\delta_1 > 0$

$$<X,\ D^*(X)> \geqslant a_4||X||^2 - \delta_1 \qquad (3.5)$$

where $\delta_1$ is independent of $\mu$. Comning the estimates (3.5) and (3.4) with (3.3) we have that

$$\dot{V} \geqslant ||\ddot{X}||^2 - a_2||X||\,||\ddot{X}|| + a_4\,||X||^2 - \delta_1 - \mu <X,\ P_1>.$$

**From this point onwards, the arguments in [1] apply. Indeed by using (1.7) and (3.1) and proceeding as in [1] with $\beta_1$ chosen sufficiently small, it can b readly shown that**

$$\int_{0}^{\omega} ||X^{(j)}(t)||^2 dt \leqslant \delta_2,\ \mathrm{j} = 0, 1, 2.$$

**where $\delta_2 > 0$ is a constant. The first two inequalities in (2.7) now follow, jus as in [1]. By taking the inner product of (2.1) with $X^{(4)}$ and integrate from $t =$ $= 0$ to $t = w$ as in [1], the last two inequalities can be obtained obviously.**

To verify (2.8) for (2.2) consider the function $V = V(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)})$ defined, for any solution $X = X(t) \in C^{(5)}$ of (2.2), by

$$V = -<\dot{X}, X^{(4)} + E\dddot{X}> + <\ddot{X}, \dddot{X}> + \frac{1}{2}<E\ddot{X}, \ddot{X}> - \mu<\dot{X}, \text{grad } G(\dot{X})> \tag{3.6}$$

$$-\bar{K}^*(X) + \mu G(\dot{X})$$

just as in [6]. Where the function $\bar{K}^* : \mathbf{R}^n \to \mathbf{R}$ is given by

$$\frac{\partial \bar{K}^*(X)}{\partial x_i} = \mu K_i(X) + (1 - \mu) b_5 x_i, \ i = 1, 2, \ldots, n \tag{3.7}$$

Differentiating $V$ and using (2.2) and ( .7) gives

$$\dot{V} = <\dddot{X}, \dddot{X}> + <H^*\ddot{X}, \ddot{X}> + <F^*\dddot{X}, \dot{X}> - \mu<\dot{X}, P_2>$$

where

$$H^* = (1 - \mu) b_4 + \mu H(\dot{X}, \ddot{X})$$

and

$$F^* = (1 - \mu) b_2 + \mu F(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)}).$$

Note that in view of (1.9) and (1.10)

$$||F^*(X, \dot{X}, \ddot{X}, \dddot{X}, X,^{(4)})\dddot{X}|| \leqslant b_2||\dddot{X}||$$

and

$$<H^*(\dot{X}, \ddot{X})\dot{X}, \dot{X}> \geqslant b_4||\dot{X}||^2$$

for all $X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)} \in \mathbf{R}^n$.

The arguments in [6], with $P_2$ subject to (1.11), will show readily that, for some constants $\gamma_1 > 0$, $\gamma_2 > 0$

$$\dot{V} \geqslant \gamma_1(||\dddot{X}||^2 + ||\dot{X}||^2) - \frac{1}{2}\beta_2||\ddot{X}||^2 - \gamma_2,$$

where $\gamma_1$, $\gamma_2$ are independent of $\mu$. From the $w$-periodicity of $X$ and (3.1) if $\beta_2$ is sufficiently small it can be readily shown that

$$\int_0^{\omega} ||X^{(j)}(t)||^2 dt < \gamma_3, \ j = 1, 2, 3.$$

The first three inequalities in (2.8) now follow as in [6]. To obtain the last two inequalities take the inner product of (2.2) with $X^{(5)}$.

Finally for the verification of (2.9) for (2.3) let $X = X(t) \in C^6$ be an $w$-periodic solution of (2.3). Consider $V = V(X, \dot{X}, \ddot{X}, \dddot{X}, X,^{(4)}, X^{(5)})$ defined by

$$V = W \text{sgn} L \tag{3.8}$$

where

$$W = -<\dot{X}, X^{(5)} + LX^{(4)} + M\dddot{X}> + <\ddot{X}, X^{(4)} + L\dddot{X}> + \frac{1}{2}<M\ddot{X}, \ddot{X}>$$

$$-\frac{1}{2}<\dddot{X}, \dddot{X}> - \mu<\dot{X}, \text{grad } U(\dot{X})> + \mu U(\dot{X}) - T^*(X).$$

Here the function $\bar{T}^*: \mathbf{R}^n \to \mathbf{R}$ is given by

$$\frac{\partial \bar{T}^*(X)}{\partial x_i} = \mu T_i(X) + (1-\mu)c_6 x_i, \quad i = 1, 2, \ldots, n. \qquad (3.9)$$

On differentiating $V$ and using (2.3) and (3.9), we have

$$\dot{V} = (<L\dddot{X}, \dddot{X}> + <\dot{X}, N^*\dddot{X}> + <\dot{X}, S^*\dot{X}> - <\mu\dot{X}, P_3>\mathrm{sgn}L \qquad (3.10)$$

where

$$N^* = (1-\mu)c_3 + \mu N(\ddot{X})$$

and

$$S^* = (1-\mu)c_5 \mathrm{sgn}L + \mu S(\dot{X}, \ddot{X}).$$

By (1.15), (1.16) and (1.17) we obtain that

$$||N^*(\ddot{X})\dddot{X}|| < c_3||\dddot{X}|| \qquad (3.11)$$

and

$$<\dot{X}, S^*(\dot{X}, \ddot{X})\dot{X}> \geqslant c_5||\dot{X}|| \text{ or } <\dot{X}, S^*(\dot{X}, \ddot{X})\dot{X}> \leqslant -c_5||\dot{X}||^2 \qquad (3.12)$$

for all $\dot{X}, \ddot{X}, \dddot{X} \in \mathbf{R}^n$.
Combining the estimates (3.12), (3.11) with (3.10) and noting that the matrix $L\mathrm{sgn}L$ is positive definite, we have

$$\dot{V} \geqslant \nu_1||\dddot{X}||^2 - c_3||\dot{X}||\,||\dddot{X}|| + c_5||\dot{X}||^2 - \alpha_3||\dot{X}||$$

where $\nu_1 > 0$ is the least eigenvalue of $L\mathrm{sgn}L$.

After this point the arguments in [2] can be applied. From the $w$-periodicity of $X$ and (3.1) it would follow that

$$\int_0^\omega ||X^{(j)}(t)||dt < \nu_2, \quad j = 1, 2, 3.$$

Thus, the first three inequalities in (2.9) now follow as in [2]. By taking the inner product of (2.3) with $X^{(6)}$ and using by Schwarz's inequality the remaining last three inequalities can be obtained easily.

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# RECENZII

W. Bruns, U. Vetter, **Determinantal Rings**, *Lecture Notes in Math.*, 1327, Springer Verlag, Berlin Heidelberg 1988, 236 p.

Let $U$ be an $m \times n$ matrix over a ring $A$. For $t \leqslant \min(m, n)$, the ideal generated by the $t$-minors of $U$ is denoted $I_t(U)$.

Let $B$ be a commutative ring, and consider an $m \times n$ matrix $X = (X_{ij})$ whose entries are independent indeterminates over $B$. If $B[X]$ denotes the polinomial ring $B[X_{ij} : 1 \leqslant i \leqslant \leqslant m, 1 \leqslant j \leqslant n]$ and $I_t(X)$ is the ideal generated by the $t$-minors of $X$, let $R_t(X) = B[X]/I_t(X)$ the residue class rings. These rings are the most prominent members of a larger class of rings of type $B[X]/I$ called *determinantal rings*. Their defining ideals $I$ can be described as follows: given integers $1 \leqslant u_1 < \ldots < u_p \leqslant m$, $0 \leqslant r_1 < < \ldots < r_p < m$, and $1 \leqslant v_1 < \ldots < v_q \leqslant n$, $0 \leqslant s_1 < \ldots < s_q < n$, the ideal $I$ is generated by the $(r_i + 1)$ -minors of the first $u_i$ rows and the $(s_j + 1)$ -minors of the first $v_j$ columns, $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q$.

Over an algebraically closed field $B = K$ of coefficients one can associate a geometric object with the ring $R_t(X)$. Having chosen bases $d_1, \ldots, d_m$ in an $m$-dimensional vector space $V$ and $e_1, \ldots, e_n$ in an $n$-dimensional vector space $W$, one identifies Hom $K$ $(V, W)$ with the $mn$-dimensional affine space of $m \times n$ matrices, of which $K[X]$ is the coordinate ring. Let $V_k = \sum_{i=1}^{k} Kd_i$ and $W_k^* = \sum_{z-1}^{k} Ke_i^*$ with $e^*$ the dual basis. Then the ideal $I$ above defines the *determinantal variety* $\{f \in \mathrm{Hom}_K (V, W) / / rk\, f / V_{u_i} \leqslant r_i, rk\, f^* / W^*_{v_j} \leqslant s_j, 1 \leqslant i \leqslant p, 1 \leqslant j \leqslant q\}$.

The authors also treat simultaneously a second class of rings: the homogeneous coordinate rings of the Schubert varieties, called *Schubert cycles*.

Algebrically one can consider every determinantal ring as a dehomogenezation of a Schubert cycle. In geometric terms one passes from a (projective) Schubert variety to an affine determinantal variety by removing a hyperplane „at infinity".

Linear algebra over determinantal rings is also discussed.

Gr. CĂLUGĂREANU

S. Rempel and B.-W. Schulze, **Asymptotics for Elliptic Mixed Boundary Problems**, *Mathematical Research*, vol. 50, Akademie-Verlag, Berlin 1989, 418 p.

Microlocal analysis, including the theory of pseudo-differential operators ($\psi$DO) and the theory of Fourier integral operators (FIO) is a powerfull tool in the investigation of boundary value problems for linear partial differential operators. Combining methods from analysis (both real and complex), functional analysis, algebra, differential geometry and topology it lead to substantial progress and to the proofs of deep results concerning global properties of the solutions of these problems, such as, for instance, the famous Atiyah-Singer index theorem, Fredholm property etc.

$\psi$DO's are a class of operators including the linear partial differential operators as well as the simplest functions of them (e. g. the inverses of elliptic operators and their complex powers). Establishing a correspondence between these operators and some class of functions, called symbols, one extends the operational calculus, developed in the case of constant coefficients and based on Fourier transform, to the case of variable ones.

At it is well known the parametrix of an elliptic differential operator on a manifold without boundary is a $\psi$DO but, in the presence of the boundary, the parametrix may contain also other terms. A lucid and fairly complete presentation of this situation can be found in an other monograph by the same authors. „Index Theory of Elliptic Boundary Problems" Akademie-Verlag, Berlin 1982 (Russian translation Mir Editors, Moscow 1986).

The present book is dealing with the pseudo-differential calculus for boundary value problems with discontinuous (mixed) boundary conditions and geometric singularities of the boundary (manifolds with conical singularities and edges). In this case, due mainly to the presence of geometrical singularities, the corresponding algebras of operators with symbolic structures containing the parametrices are of high complexity and the theory is far from being in a final form (the authors mention many open problems in the Notes section of each chapter). The main tools used in this study is the Mellin transform, Mellin operators and Mellin symbols, following the ideas developed by the authors in two fundamental papers published in the Mathematische Nachrichten 111 (1983), 41–109, and 116 (1984), 269–314.

The book is divided into four chapters: 1. Operators on the half axis; 2. Continuous asymptotics and higher order operators; 3. Boundary value problems; 4. Mixed boundary value problems on manifolds with edges.

Chapter 1 begins with the study of the classical Mellin transform defined on $C_0^\infty\ (R_+)$ (space of infinite differentiable functions with compact support) by $Mu\,(z) = \int_0^\infty u(t)\cdot t^{z-1}\,dt$ and extended first to an isomorphism from $L^2(R_+)$ to $L^2(\mathrm{Re}\, z = 1/2)$ and then to a meromorphic function on $C$. The basic idea in applying Mellin symbols (a specified class of meromorphic functions on $C$) is to identify $\mathrm{Re}\, z = 1/2$ with the conormal direction to the boundary. This chapter contains also a detailed study of function spaces with discrete conormal singularity, because the functional analysis in this simplest case, of discrete asymptotics, contains all the basic elements of a more general theory which is developed in the second chapter. In fact troughout the book, the authors return frequently to the discrete case. Green operators, Mellin, operators, Mellin symbols are also considered.

In the second chapter the results obtained in the first one for discrete asymptotics are extended to continuous asymptotics by associating with certain given subsets $\Lambda$, $\Lambda'$ of $C$ some function spaces $\mathcal{E}_\Lambda$ and $C^\infty_{\Lambda,\Lambda'}$. The authors show, on an example, that this more complicated situation can effectively occur at it is treated by considering some analytic functionals defined on appropriate function spaces.

The third chapter is devoted to the study of pseudo-defferential boundary value problems without transmision property. Again some adequate function spaces on a cone and on a wedge are considered. Green operators with or without boundary symbols and Mellin operators are applied to study ellipticity and Fredholm property for these problems.

The last chapter of the book is dealing with mixed boundary value problems on manifolds with edges. In this case, beside the $R_+$ -calculus a $\psi$DO -calculus along the edge is also applied and, since the boundary conditions may change when crossing the edge, some extra-boundary conditions of Shapiro-Lopatinski type along the edge, have to be imposed Also, in parametrices one gets potentials needing for matrix valued operators in the sense of Boutet de Monvel's algebra or Vishik-Eskin's work.

Including many original results of authors, the book presents in an unified fundamental results from the theory of ps differential boundary value problems on folds with singularities. Although not ea read the book is clearly written and co a plenty of results and methods. We recom it warmly to all interested in partial differ equations and related areas.

S. CO

W. Tutschke, **Solution of Initial Problems in Classes of Generalized A Functions.** *Teubner-Texte zur Mathematik,* 110, Leipzig 1989, pp. 180.

The main goal of the book is the ap tion of scales of Banach spaces of gene analytic functions for solving initial valu blems for differential equations. To mak book self-contained, the author include the needed background functional-analyti terial in detail, such that the book can be as an introductory text by a beginner who to enter the domain. But the presentatio gresses rapidly up to recent results, mo them obtained by the research group „Pa komplexe Differentialgleichungen" of the I matica Department at Halle University, so the book will be of interest for the spec in the field too. (The author published an book on the same subject: Partielle kom Differentialgleichungen in einer und in me komplexen Variablen, Berlin 1977).

The first chapter of the book „I value problems in Banach spaces" begins a brief introduction to the calculus of B space-valued functions defined on an int of the real axis (differentiation and Rie integration). The chapter contains also an cation of the method of succesive approxim for solving the initial value problem:

$$\frac{du}{dt} = f(t, u),\ u(0) = u_0,$$

where $f\colon I \times B \to B$, $I$ is an interval in $B$ a Banach space. As an example, the of an infinite system of ordinary differ equations is reduced to (1).

The next chapter is devoted to the of scales of Banach spaces, which are fa of Banach spaces $B_s$ and linear injections $B_s \to B_{s'}$, $||I_{ss'}|| \le 1$ for all $s$, $s'$ in an interval $(0, s_0)$ and $s' < s$. As principal ex one considers scales of Banach spaces of morphic functions on some domains $G_s$ compactly contained in a given domain $G$ in the complex plane, such that $\cup G_s = G$. The developed theory allows the author to extend the method of successive approximations for solving initial value problems in scales of Banach spaces (Chapter 3 of the book).

Chapter 4 is concerned with the classical Cauchy-Kovalevskaya theorem for the complex equation

$$\frac{\partial^k w}{\partial t^k} = f(t, z, w, p), \tag{2}$$

where $f = (f_1, \ldots, f_m)$, $f_j$ holomorphic functions of $z$, $w$ and $p$, $z = (z_i) 1 \leqslant i \leqslant n$, $w = (w_j)$ $1 \leqslant j \leqslant m$, $w_j$ holomorphic functions of $z$ and

$$p = \left( \frac{\partial w_1}{\partial t}, \frac{\partial w_1}{\partial z_1}, \ldots, \frac{\partial^k w_n}{\partial t_n^k} \right).$$

By a famous result of H. Lewy (1957) there are infinitely many differentiable functions $f$ such that the differential equation

$$\frac{\partial w}{\partial t} = f\left(t, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right) \tag{3}$$

has no solution. H. Lewy's result was the starting point of a lot of papers looking for conditions ensuring the solvability of the equation (3). The Cauchy-Kovalevskaya theorem shows that this is the case if $f$ and $w$ are holomorphic functions, satisfying some boundedness and Lipschitz conditions together with their first derivatives. Some conditions are also imposed on the initial vectors.

Chapter 5 is dealing with the proof of Holmgren theorem on power series representation of the solution of a system of differential equations in the real case (i.e. real functions and real variables).

The study of generalized analytic functions is done in Chapter 6 „Basic properties of generalized analytic functions". These are solutions $w = w(z)$ to the differential equation

$$\frac{\partial w}{\partial \bar{z}} = a(z)w + b(z)\bar{w}, \tag{4}$$

where $a(z)$ and $b(z)$ are complex-valued continuous functions defined on a domain in the complex plane. Obviously that, for $a = b = 0$ one obtains the classical Cauchy-Riemann condition characterizing holomorphic functions.

The rest of the book — Chapters : 7 Initial value problems with generalized analytic initial functions : 8. Contraction-mapping principles in scales of Banach spaces : 9. Further existence theorems — are devoted to a systematic application of the method of scales of Banach spaces of generalized analytic functions for solving initial value problems. Among the topics treated here we mention : overdetermined first order systems, scales of pseudoholomorphic functions in L. Bers' sense, Euler's polygonal line method, Gronwall lemma etc.

Written by an eminent specialist in the field with substantial contributions to the subject the book is a valuable contribution to the theory of initial value problems with generalized analytic functions. Starting from the introductory notinos the book brings the reader to the frontiers of current research, stressing on the main ideas of the theory and of its connections with other domains of investigations.

We recommend it warmly to all interested in the applications of the functional-analytic methods to differential equations.

S. COBZAŞ

**Numerical Treatment of Differential Equations.** (Proceedings of the Fourth Seminar „NUMDIFF — 4" held in Halle, 1987), *Teubner — Texte zur Mathematik*, Band 104.

The papers contained in the proceedings are divided into three sections. The first one is : Results on Ordinary Differential Equations, Differential Algebraic Equations and Delay Equations. Some works lay stress on stiff differential equations and on extensions of Runge — Kutta methods to delay equations.

In the second part entitled : Results on Partial Differential Equations and Related Topics, some numerical techniques are discussed (finite differences, finite elements and method of lines). Two interesting works are devoted to convection — diffusion problems. Furthermore, some questions of numerical stability in nonlinear problems are considered.

In the last section : Applications in Science and Technology, the works deal with concrete questions of applicability of differential equations and differential algebraic equations to problems in science and technology. The authors use a large variety of techniques to obtain desired numerical results.

An increased number of works have a high mathematical level. Others treat very interesting applications : the equations of Prandtl's boundary layer, the modelling of nonnewtonian fluid flow, periodic phenomena in reaction diffusion systems, etc.

However, there is a good balance between theoretical aspects (numerical stability, error estimations in numerical methods, treatment of higher — index differential algebraic equation); the analysis of numerical methods for: stiff systems, two — point boundary value problem parabolic and hyperbolic equations and their application to concrete problems.

C. I. GHEORGHIU

**Seminar Analysis of Karl — Weierstrass — Institute 1986/87** Edited by B.—W. Schulze and H. Triebel, Teubner Texte zur Mathematik Band 106, Teubner Leipzig 1988, 332 p.

The volume is the continuation of a corresponding series published by the Karl—Weierstrass—Institute of Mathematics 1981—85 (the volume 1985/86 appeared as Band 96 of Teubner-Texte). This volume contains thirteen papers on partial differential equations, function spaces, global analysis and differential geometry with applications to mathematical physics More than half of the book is occupied by a paper by J. Eiccorn, Elliptic differential operators on noncompact manifolds, pp. 4—169, which is concerned with the spectral theory of certain self-adjoint differential operators over noncompact Riemann manifolds. The paper is only an introduction ot this reach field of investigation and, as the author asserts in the preface, an extended version is planned to appear later. The next paper is B.—W. Schultze, Elliptic complexes on manifolds with conical singularities, 170—223, where the theory of single, differential operators on manifolds with singularities, developed by the author jointly with S. Rempel, is extended to complexes. Other papers included in this volume are A. Juhl, On the Poisson transformation for differentua forms on hyperbolic spaces, 224—236; W. Hoffmann, On a trace formula for Hecke operators, 237—245; K.—D. Kirchberg, Some results concerning the Dirac operator on compact Kähler spin manifolds, 247—255; two interesting papers by Th. Schmidt on Infinite-dimensional supermanifolds, 256—268, and on Supergeometry and its application in physics, 269—286; R. Johnson, Recent results on weighted inequalities for the Fourier transform, 287—296; H. Triebel, Atomic representations of Fpsqspaces and Fourier integral operators, 297—305; B. Lange and M. Lorenz, Propagation of singularities for opera tors with double involutive characteristic 306—311; H.—G. Leopold, Pseudo diferential operators and function spaces of variable order of differentiation; W. Sickel, Superposition of functions in spaces of Besov—Triebel—Lizorkin type. The critical case $1 < s < n/p$, 319 326; H.—J. Schmeiser and W. Sickel, On [...] mation by Riesz and Abel—Poisson m[...] periodic Besov—Lizorkin—Triebel spaces[...] 332.

Written by eminent specialists in t[...] the papers included in this volume conta[...] tially new results obtained by the auth[...] will be of great interest for all working [...] domains of research.

S. C[...]

**Proceedings of the Second Inter[...] Symposium on Numerical Analysis, Prag[...]** Teubner—Texte zur Mathematik, Ba[...] Leipzig, 1988.

There have been 12 plenary lectu[...] 49 section lectures at this Symposium [...] to the following themes: Numerical A[...] Approximation Theory and Smoothing, [...] Element Methods (superconvergence), Bifu[...] Problems, Numerical Methods in ODE, N[...] cal Methods in PDE, Eigenvalue Proble[...] Computational Statistics.

Only 9 plenary lectures and 34 [...] lectures are included in this volume. A [...] these plenary lectures is given here fo[...] pleteness: Axelsson O., „A priori boun[...] discretization error estimates for parabol[...] blems", Douglas J. Jr., „Three models for [...] looding in a naturally fractured petroleu[...] voir", Feistauer M., Zenisek A., "Finite [...] variational crimes in nonlinear elliptic pro[...] Folta J., "Notes on the history of nu[...] analysis in its connections with Prague", G[...] S., "Survey of convergence criteria of [...] gonal power processes", Hackbusch W[...] new multi-grid method", Necas J., "Fin[...] ment approach to the transonic flow pr[...] Parter S., "Remarks on the solution of [...] systems of equations", Tichonov A.N[...] problems with inaccurate data".

The presentation of the titles a[...] authors of these papers represents a gu[...] of their high mathematical level. The[...] are up to date such as: variational form[...] for nonlinear elliptic problems (based es[...] on important results of Czerchoslovak sc[...] equations), stability and error estimates [...] bounds and discretization error) of [...] finite element approximations for time-de[...] convection-diffusion equations etc.

As to the section lecture; some [...] refer to the use of FEM for solving so[...] blems of fluid mechanis (ideal comp[...] fluid flow in a plane cascade, math[...]

modelling of urban air pollution, etc.) for the study of elasto-plastic bodies behaviour and for the study of dynamic behaviour of solids.

Some problems regarding spline function (shape preserving splines, spline approximation in the stabilization method for solving nonlinear boundary value problems, etc.).

Two of the papers refer to the Runge—Kutta methods (their modification for stiff problems and the Lyapunov matrix equation for these methods).

The other papers deal with mathematical aspects which are more or less related to the above mentioned topics.

We consider these proceedings to be useful for those intersted in numerical analysis and especially in its applications in ODE and PDE.

C. I. GHEORGHIU

Lothar Budach, Bernd Graw, Christoph Meinel, Stephan Waack, **Algebraic and Topological Properties of Finite Partially Ordered Sets,** Teubner Texte zur Mathematik Vol. 109, Teubner, Leipzig, 1988.

The book is a revised, expanded and completed version of the first author's lectures held at Humboldt-University during the academic year 1983/84. The original lectures have been considerably extended by new contributions in the area and by revised proofs of known results.

The use of methods which have been developed in the algebraic topology and commutative algebra allows a new prospective on combinatoriai problems. The authors' intention is to close the gap which appears in using topological methods in the theory of finite partially ordered sets (posets).

The common theme for the five chapters of the book are the principle of inclusion-exclusion and the theorem of Rota concerning Galois connections of posets.

Chapter 1 constitutes an introduction to the theory of partially ordered sets which comprises the original form of the theorem of Rota.

From Chapter 2 on the book uses a "diagram cohomology" introduced there. A Leray spectral sequences is developed and a homological proof (due to Baclawski) of Rota's theorem is presented.

Chapter 3 is devoted to the study of homotopy properties of posets. Using Quillen's theorem it is shown that Galois connections are nothing else than a very special case of homotopy equivalence which yields a new proof of Rota's theorem.

Chapter 4 introduces the Möbius algebra which is a kind of a combinatorial analogue of the Chow rings of algebraic geometry. Also analogue of the Riemann—Roch theorem of Grothendieck is proved which leads again to a new proof of Rota's theorem.

Chapter 5 gives a brief introduction to Cohen—Macaulayness and shellability as an important combinatorial property which implies the Cohen—Macaulayness.

Finally an appendix is given, containing applications of algebraic topological properties of posets in computation theory. By the authors' remark, this is the original motivation of the volume. The book is provided with a literature containing 60 items and a index of terms.

According its originality and style the volume can be highly recommended to all interested in the exciting field ot posets and their applications.

A. B. NÉMETH

Victor Guillemin, Shlomo Sternberg, **Symplectic Techniques in Physics,** Cambridge University Press, 1984, 468 pp., ISBN 0521 24 8663.

This is one of the most elaborated book on symplectic geometry written in an interdisciplinary spirit of the mathematics and theoretical physics. The Preface presents the subject starting from a hystorical and methodological point of view giving some general comments over the book, whose purpose is twofold: to provide an introduction in the matter and to expose the main results from a present-day approach. The content is the following:

I. *Introduction.* This first chapter is quite general, demanding only few mathematical prerequisites. Here are presented some physical aspects generating the symplectic techniques going from the Hamiltonian mechanics and the various theories of light. So, one discuss successively the relationships between the linear optic. the geometric optic and the wave optics, as well as the corresponding relations with the classical and the quantum mechanics.

II. *The Geometry of the moment map.* This is a highly mathematical chapter. There are presented the normal forms, the Darboux-Weinstein theorem on the local symplectomorphism of two symplectic manifolds and the equivariant version of the theorem relative to a compact group action, the moment map and some of its physical and mathematical aplications as the harmonic analysis of group representations. Convexity properties of the toral group actions as well as the principles of geometric quantisation are also presented.

III. *Motion in a Yang-Mills field and the principle of general covariance.* The chapter is devoted to the study of the particles in a Yang-Mills field. The symplectic structure of the cotangent bundle of the total space of the principal bundles is studied with application to the isotropic and co-isotropic embeddings, and to an symplectic analog of the induced representation.

IV. *Complete integrability.* Here, the group — theoretical method and the moment map are applied to study the complete integrability of different mechanical systems. Among the topics covered we mention the fibrations by tori, systems of Calogero type, Solitons and co-adjoint structures, the algebra of formal pseudodifferential operators, the higher-order calculus of variations in one variable.

V. *Contractions of symplectic homogeneous spaces.* Results on the cohomology of Lie algebras and on the contractions of homogeneous symplectic spaces, with applications to the Galilean and Poincaré elementary particles are given.

The work is concluded with an ample list of References and a general Index.

On the basis of their broad experience and scientifical work, the authors performed a valuable and original synthesis on the symplectic geometry. The symplectic techniques play a crucial role in the mathematical formulation of many problems from the classical and the modern physics. A subject of common interest to both mathematicians and teoretical physicists is treated systematicaly and exhaustively being an excellent text for graduate courses, or even for scientifical research.

M. ȚARINĂ

A. Di Nola, S. Sessa, W. Ped E. Sanchez, **Fuzzy-relation equatio their applications to knowledge engi** Kluver Academic Publishers, Dordrecht ton, London, 1989 (Theory and D Library, Series D: System Theory, ledge Engineering and Problem Solvin

The book is an in-depth study fuzzy relation equations and their applic The authors are outstanding speciali the domain and they provide us a c hensive and up-to-date account of the

The book is organized as follows:

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In chapter lo the approximate so of the systems of fuzzy relations equati studied.

Chapters 11—13 deal with the cations of fuzzy relation equations in ficial Intelligence. The treated topic uncertainty in knowledge-based system structions, validation and optimizati knowledge bases, imprecise reasonin the inference mechanisms in the expert

Chapter 14 presents the theory a design of fuzzy logic controllers.

Chapter 15 contains two bibliogr papers on fuzzy relation equations and on fuzzy relations.

An author index and a subject complete the book.

The book is a very lucid and c hensive tratement of fuzzy relation eq It will be a useful tool for researchers field. Researchers in different or relate as well as the students will be benefit f introduction to relevant literature.

The book is highly recommen all interested in fuzzy sets and relate The book is very important for the ment of the fuzzy set theory and its tions. Our thanks are due to the and to the Publisher.

D. DUMIT

Imprimeria „ARDEALUL" Cluj C-da nr. 4/1990

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