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## UNIVERSITATIS BABEŞ-BOLYAI

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# GENERALIZED INTEGERS AND BONSE'S THEOREM

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**REZUMAT. — Intregi generalizaţi şi teorema lui Bonse.** În această lucrare se extinde teorema lui Bonse la cazul intregilor generalizaţi.

1. **Introduction.** Bonse ([6]) proved that the number 30 is the largest in the set of all natural numbers $a$ such that if $r \in \mathbf{Z}$ (the set of integers), $1 < r < a$ and $(r, a) = 1$, then $r$ is a prime number. In a recent paper ([4]) Hiroshi Iwata extended Bonse's theorem. To state Iwata's result we introduce the following notations:

$$\mathbf{N} = \{1, 2, 3, 4, \ldots\}, \text{ the set of all natural numbers;}$$
$$P = \{2, 3, 5, 7, \ldots\}, \text{ the set of all (positive) primes;}$$

$p_n$: the n-th prime in $P$;

$P_k = \{p_{j_1} p_{j_2} \ldots p_{j_i} : j_1 \leqslant j_2 \leqslant \ldots \leqslant j_i,\ 0 \leqslant i \leqslant k\}$, the set of all products of $i$ prime numbers with $0 \leqslant i \leqslant k$ for $k \in \mathbf{N}$, where $i = 0$ means $P_k \ni 1$;

$$M_k = \{a \in \mathbf{N} : r \in \mathbf{N},\ r < a,\ (r, a) = 1 \Rightarrow r \in P_k\}.$$

With these notations, Bonse's theorem is described as $\sup M_1 = 30$, and Iwata's result reads: $\sup M_k < \infty$ for all $k \in \mathbf{N}$.

Recently, Solomon W. Golomb ([2]) posed the problem to determine the largest odd integer $n$ such that every odd integer $j$ with $1 < j < n$ and $(j, n) = 1$ is a prime. As an answer to this problem I found that 105 is the largest odd integer with the required property. The proof of this fact will be given in section 2.

Golomb's problem suggests to treat the phenomena in a more abstract setting, namely in the domain of the so-called generalized integers. These will be described in section 3, the main result of the paper (Theorem 4.1) and the auxilliary results being placed in section 4.

2. **The solution of Golomb's problem.** We begin by seeing how we can find the numbers with the required property. Starting from 9, every such integer $n$ must be divisible by 3, because if it does not, then $n$ will be relatively prime to 9, and 9 is not prime. Analogously, starting from 25, every such integer must be divisible by 5. But it is also divisible by 3, so that it must

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be divisible by 3·5. In this way we obtain the following table:

| | | | |
|---|---|---|---|
| Starting from | 9, $n$ must be divisible by | $3 =$ | 3 |
| Starting from | 25, $n$ must be divisible by | $3\cdot 5 =$ | 15 |
| Starting from | 49, $n$ must be divisible by | $3\cdot 5\cdot 7 =$ | 105 |
| Starting from | 121, $n$ must be divisible by | $3\cdot 5\cdot 7\cdot 11 =$ | 1155 |
| . . . . . . . | . . . . . . . . . . | . . . . | . . |

Between 9 and 25 the only possible values for n are 9, 15, 21; between 25 and 49 only 45; between 49 and 121 only 105. Between 121 and 169 we have no possibility because 1155 is greater than 169. It can be seen that if we continue in this way, i.e. if $17^2$ is less than $3\cdot 5\cdot 7\cdot 11\cdot 13$, $19^2$ is less than $3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17$, and so on, then 105 will be the greatest odd integer $n$ with the required property. We have to prove that $p^2_{m+1} < p_2 p_3 \ldots p_m$, for $m \geqslant 5$. This inequality can be easily proved by induction on $m$, also using the inequality $p_{m+1} < 2p_m$, which was first proved by Chebyshev.

3. **The generalized integers.** The original definition and the name of „generalized primes" were given by Arne Beurling ([1]) in 1937. In this paper they will be defined as follows:

Suppose, given a sequence of real numbers (called generalized primes) such that

$$1 < q_1 < q_2 < q_3 < \ldots .$$

Form the set of all possible $q$ — products, i.e., products $q_1^{\alpha_1} q_2^{\alpha_2} \ldots$, where $\alpha_1, \alpha_2, \ldots$ are integers $\geqslant 0$ of which all but a finite number are 0. Call these numbers generalized integers and suppose that no two generalized integers are equal if their $\alpha$'s are different. Then arrange the sequence of generalized integers as an increasing sequence:

$$1 = b_1 < b_2 < b_3 < \ldots .$$

Let $\mathfrak{Q} = \{q_n : n \in \mathbf{N}\}$ and $\mathfrak{N} = \{b_n : n \in \mathbf{N}\}$.

Thus the generalized primes need not be natural primes, nor even integers. From the definition, the basic properties of the generalized integers are that they can be multiplied and ordered, that is, counted, but not added. However, division of one generalized integer by another is easily defined as follows: we say $d \mid b_n$, if $\exists c$ so that $dc = b_n$ and both $d$ and $c$ belong to $\mathfrak{N}$. From these definitions, it follows that greatest common divisor, multiplicative functions, Möbius function, Euler $\varphi$ — function, unitary divisors, etc., for the generalized integers can be defined.

In what follows, we assume that the generalized integers satisfy the following condition:

(i) the total number $N(x)$ of numbers $b \in \mathfrak{N}$ with $b \leqslant x$ is finite, for each real $x > 0$.

It is not difficult to verify that the condition (i) is equivalent to: (i)′ the total number $\pi(x)$ of numbers $q \in \mathfrak{Q}$ with $q \leqslant x$ is finite, for each real $x > 0$.

To extend Bonse's theorem and Iwata's result we need an assumption on the size of $N(x)$. Throughout of the paper we assume that the generalized integers satisfy the following basic asymptotic axiom:

AXIOM A. There exists a constant $0 \leqslant t < 1$, such that

$$N(x) = x + 0(x^t) \text{ as } x \to \infty.$$

Obviously, this axiom is satisfied by the positive integers. For,

$$N(x) = [x] = x + 0(1).$$

As a first consequence of the axiom A we have:

3.1. PROPOSITION. *The number of generalized primes is infinite.*

*Proof.* Suppose there are only a finite number of generalized primes, say $q_1, q_2, \ldots, q_k$. Then any $b_n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_k^{\alpha_k}$. Also, if $b_n \leqslant x$ then $q_i^{\alpha_i} \leqslant x$ for $i = 1, 2, \ldots, k$. Hence

$$\alpha_i \leqslant \frac{\log x}{\log q_i} \leqslant \frac{\log x}{\log q_1}.$$

Denote by $[x]$ the number of positive integers $\leqslant x$. Since $\alpha_i$ must be integral, the number of possible products formed from $q_1, q_2, \ldots, q_k$ and $\leqslant x$ is less than or equal to

$$\left(\left[\frac{\log x}{\log q_1}\right] + 1\right)^k.$$

Hence

$$N(x) \leqslant \left(\left[\frac{\log x}{\log q_1}\right] + 1\right)^k = 0(\log^k x),$$

which contradicts axiom A.

4. **The extension of Bonse's theorem.** Let introduce the following notations:

$$\mathfrak{A}_k = \{q_{j_1} q_{j_2} \ldots q_{j_i} : j_1 \leqslant j_2 \leqslant \ldots \leqslant j_i, \ 0 \leqslant i \leqslant k\},$$

the set of all products of $i$ generalized primes with $0 \leqslant i \leqslant k$ for $k \in \mathbf{N}$, where $i = 0$ means $\mathfrak{A}_k \ni 1$,

$$\mathfrak{M}_k = \{b \in \mathfrak{N} : s \in \mathfrak{N}, \ s < b, \ (s, b) = 1 \Rightarrow s \in \mathfrak{A}_k\}.$$

With these notations, the main result of the paper reads:

4.1. THEOREM. $\sup \mathfrak{M}_k < \infty$, *for every* $k \in \mathbf{N}$.

For our purpose we need several auxilliary results. The first of these is the prime number theorem for generalized integers, the proof of which may be found in [5].

4.2. PROPOSITION. *As* $x \to \infty$,

$$\pi(x) \sim \frac{x}{\log x}.$$

4.3. COROLLARY. $\lim\limits_{n \to \infty} q_n/(n \log n) = 1.$

4.4. COROLLARY. $\lim\limits_{n \to \infty} q_{n+1}/q_n = 1.$

4.5. COROLLARY. *For a fixed natural number $i$,*

$$\lim_{n\to\infty} q_{n+1}/q_{n-i} = 1.$$

4.6. PROPOSITION. *For a fixed natural number $k$, we have*

$$\lim_{n\to\infty} q_{n+1}^{k+1}(q_1q_2 \dots q_n)^{-1} = 0.$$

*Proof.* If $n$ is large enough, $0 < q_{n+1}^{k+1}(q_1q_2\dots q_n)^{-1} \leqslant q^k_{n+1}(q_nq_{n-1}\dots q_{n-k-1})^{-1} = q_{n-k-1}^{-1}(q_{n+1}/q_{n-k})(q_{n+1}/q_{n-k+1})\dots(q_{n+1}/q_n) \to 0\cdot 1\cdot 1\cdot\dots\cdot 1 = 0$ (as $n\to\infty$)

4.7. COROLLARY. *For a given $k \in \mathbf{N}$, we can find $n(k) \in \mathbf{N}$ such that $q_1q_2 \dots q_n > q_{n+1}^{k+1}$ for all $n \geqslant n(k)$.*

4.8. PROPOSITION. *If $b \in \mathfrak{M}_k$ and $b > q_n^{k+1}$ then $q_1q_2 \dots q_n \mid b$.*

*Proof.* If $b \in \mathfrak{M}_k$, $b > q_n^{k+1}$ and $\exists i \in \{1, 2, \dots, n\}$ such that $q_i \nmid b$, then $(b, q_i^{k+1}) = 1$ and $q_i^{k+1} \leqslant q_n^{k+1} < b$, which contradicts to the fact that $b \in \mathfrak{M}_k$.

4.9. PROPOSITION. *If $b \in \mathfrak{M}_k$ then $b \leqslant q_{n(k)}^{k+1}$, where $n(k)$ is a fixed number in corollary 4.7 depending only on $k$.*

*Proof.* Assume $b \in \mathfrak{M}_k$ and $q_{n+1}^{k+1} \geqslant b > q_n^{k+1}$ for some $n \geqslant n(k)$, then by proposition 4.8, $q_1q_2 \dots q_n \mid b$, so we have $b \geqslant q_1q_2 \dots q_n$. Moreover, by corollary 4.7, $q_1q_2 \dots q_n > q_{n+1}^{k+1}$, hence $b > q_{n+1}^{k+1}$, which is a contradiction.

*Proof of the theorem* 4.1. Proposition 4.9 means that $\mathfrak{M}_k$ is a finite set. Thus $\sup \mathfrak{M}_k < \infty$ and our theorem has been proved.

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# ON THE COMPOSITION OF SOME ARITHMETIC FUNCTIONS

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**REZUMAT. — Asupra compunerii unor funcţii aritmetice.** În această lucrare sînt studiate ecuaţii ataşate funcţiilor compuse $f(g(n))$; unde $f, g: \mathbf{N} \to \mathbf{R}$ sînt funcţiile aritmetice $\sigma$, $\varphi$, $\sigma^*$, $\psi$, reprezentind: suma divizorilor, indicatorul lui Euler, suma divizorilor unitari şi funcţia lui Dedekind.

The aim of this paper is the initial study of some equations for the composite functions $f(g(n))$, where $f, g: \mathbf{N} \to \mathbf{R}$ are the arithmetical functions $\sigma$, $\varphi$, $\sigma^*$, $\psi$, representing the sum of divisors, Euler's totient, the sum of unitary divisors, and Dedekind's function, respectively.

**Introduction.** Let $\sigma(n)$ denote the sum of divisors of the natural number $n$. It is well-known that a natural number $n$ is called perfect if $\sigma(n) = 2n$. Euclid and Euler ([5], [6]) have determined all the even perfect numbers, by showing that they are of the form: $n = 2^k(2^{k+1} - 1)$, where $2^{k+1} - 1$ is a prime ($k \geqslant 1$). It is not known if there are an infinity of such primes (i.e. of the form $2^a - 1$, the ,,Mersenne primes"), so it is not known if the number of even perfect numbers is infinite. On the other hand no odd perfect number is known. This problem seems to be one of the most difficult problems in Number theory ([5]). D. Suryanarayana [14] defines the notion of superperfect number, i.e. number $n$ with the property $\sigma(\sigma(n)) = 2n$ and he and H.—J. Kanold [8], [14] obtain the general form of even superperfect numbers. Kanold [8] proves that all odd superperfect numbers are perfect squares, but we do not know if there exists at leat one number of this type. More generally, D. Bode [2] defines $m$-perfect numbers as numbers $n$ for which $\underbrace{\sigma(\sigma \ldots (\sigma(n)) \ldots)}_{m} = 2n$, and shows that for $m \geqslant 3$ there are no even $m$-perfect numbers. If $\sigma(n) = 2n - 1$, $n$ has been called almost perfect ([5], [7]). Powers of 2 are almost perfect, it is not known if any other numbers are. If $\sigma(n) = 2n + 1$, $n$ has been called quasi-perfect ([3], [5]). In the light of the above notations, if $\sigma(\sigma(n)) = 2n + 1$, it would be consistent to call $n$ quasi-superperfect and by analogy, almost superperfect if $\sigma(\sigma(n)) = 2n - 1$. However, we use a unified notation of Ch. Wall [15], who defined $f$-perfect numbers by $f(n) = 2n$. Thus a superperfect number will be a $\sigma \circ \sigma$-perfect number, etc.

In what follows we denote by $\sigma^*(n)$ the sum of unitary divisors of $n$, i.e. those divisors $d \mid n$ with the property $(d, n/d) = 1$. If $\sigma^*(n) = 2n$, then $n$

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is called unitary perfect ([11], [12]). The Dedekind $\psi$ function is defined by

$$\psi(n) = n \prod_{p \mid n} \left(1 + \frac{1}{p}\right), \quad \psi(1) = 1 \tag{1}$$

in analogy with the well-known representation of Euler's totient

$$\varphi(n) = n \prod_{p \mid n} \left(1 - \frac{1}{p}\right), \quad \psi(1) = 1 \tag{2}$$

where $p$ runs over all prime divisors of $n$. It is well-known also that if $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ is the canonical representation of $n$, then we have ([11], [12]):

$$\sigma^*(n) = \prod_{i=1}^{r} (p_i^{\alpha_i} + 1), \quad \sigma^*(1) = 1 \tag{3}$$

$$\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{\alpha_i + 1} - 1}{p_i - 1}, \quad \sigma(1) = 1 \tag{4}$$

1. $\psi \circ \psi$ — **Perfect numbers and related problems.** In this section we determine all the $\psi \circ \psi$ — perfect numbers, i.e. we solve the equation

$$\psi(\psi(n)) = 2n \tag{5}$$

We need two lemmas.

LEMMA 1. $\psi(ab) \geqslant a \cdot \psi(b)$ *for all* $a, b = 1, 2, 3, \dots$ (6)

*Proof.* Let $a = \Pi p^\alpha \cdot \Pi q^\beta$, $b = \Pi p^{\alpha'} \cdot \Pi t^\gamma$, $(p, q) = 1$, $(p, t) = 1$ be the prime factorizations of $a$ and $b$ (for simplicity we do not use indices). Then $\frac{\psi(ab)}{\psi(b)} = a \cdot \Pi\left(1 + \frac{1}{q}\right) \geqslant a$, with equality only for all $\beta = 0$, that is if for each prime $p \mid a$ we have also $p \mid b$ (Particularly if $a \mid b$)

LEMMA 2. *If* $a \mid b$, *then* $\psi(a) \mid \psi(b)$. (7)

*Proof.* Let $a = \Pi p^\alpha$, $b = \Pi p^{\alpha+\alpha'} \cdot \Pi q^\beta$ with $(p, q) = 1$ $(\alpha, \alpha', \beta \geqslant 0)$. Then

$$\frac{\psi(b)}{\psi(a)} = \Pi p^{\alpha'} \cdot \Pi q^\beta \cdot \left(1 + \frac{1}{q}\right) = \Pi p^{\alpha'} \cdot \Pi(q^\beta + q^{\beta-1}) \in N$$

THEOREM 1. *The only solution of equation* (5) *is* $n = 3$.

*Proof.* First let $n$ be even, so $n = 2^k \cdot m$ with $k \geqslant 1$, $m \geqslant 1$, $(2, m) = 1$. Using the multiplicativity of $\psi$, i.e. $\psi(ab) = \psi(a)\ \psi(b)$ for all $(a, b) = 1$ (an easy consequence of (1)), and relation (6), we obtain successively: $\psi(\psi(n)) = \psi(3 \cdot 2^{k-1} \cdot \psi(m)) \geqslant \psi(m) \cdot \psi(3.2^{k-1}) \geqslant m \cdot 3 \cdot 2^k = 3n$, by the obvious relations $\psi(m) \geqslant m$ (equality only for $m = 1$) and $\psi(3.2^{k-1}) = 3.2^k$. Thus we have proved that

$$\psi(\psi(n)) \geqslant 3n, \ n \text{ even.} \tag{8}$$

which clearly implies that (5) has no even solutions. Now let $n = \Pi p^\alpha$ (with $p \mid n$ odd primes) be an odd solution of (5). Then $\psi(n) = \Pi p^{\alpha-1} \cdot \Pi(p + 1)$ is divisible by $2^r$ (since $2 \mid (p+1)$), where $r = \omega(n)$ is the number of distinct prime factors of $n$. Lemma 2 yields $2^{r-1} \cdot 3 = \psi(2^r) \mid \psi(\psi(n)) = 2n = 2\Pi p^\alpha$ (*), so if $r \geq 3$, this is impossible. For $r = 1$ we get $3 \mid \Pi p^\alpha = p^\alpha$, thus $n = 3^\alpha$. For $\alpha > 1$ we have $\psi(3^\alpha) = 3^{\alpha-1} \cdot 4$ and $\psi(\psi(3^\alpha)) = 3^{\alpha-1} \cdot 8 \neq 2 \cdot 3^\alpha$, while for $\alpha = 1$, we have equality. Thus $n = 3$ is a solution. We now have to study the case $r = 2$, when (*) shows that $3 \mid p^\alpha q^\beta$, i.e. $n = 3^\alpha \cdot q^\beta$, $(3, q) = 1$. If $\alpha > 1$ or $\beta > 1$ we evidently get $3q \mid \psi(n)$, so by (7) we obtain $4(q+1) \mid \psi(\psi(n)) = 2 \cdot 3^\alpha q^\beta$, a contradiction. But we cannot have also $\alpha = 1$, $\beta = 1$ because then we would obtain $\psi(n) = \psi(3)\ \psi(q) = 4(q+1) = 8m$ $(m \in \mathbb{N})$ and $\psi(8) = 12 \nmid 6q$ ($q$ being odd)

*Remark.* The proof of Theorem 1 (relation (8)) shows that all even solutions of $\psi(\psi(n)) = 3n$ are of the form $n = 2^k$

LEMMA 3. *If* $a \mid b$, *then* $\frac{\psi(a)}{a} \leq \frac{\psi(b)}{b}$. (9)

*Proof.* Let $b = qa$; then in view of (6) one can write: $\psi(b) \geq q\psi(a) = \frac{b}{a}\psi(a)$, yielding (9).

Now, in connection with D. Bode's result, we prove:

THEOREM 2. *For* $m \geq 3$, *there are no* $\underbrace{\psi \circ \psi \circ \ldots \circ \psi}_{m}$ — *perfect numbers.*

*Proof.* Firstly we observe that for all $n \geq 2$ we have:

$$\psi(\psi(n)) \geq \frac{3}{2}\psi(n) \tag{10}$$

Indeed, this is trivial for $n = 2$, while for $n \geq 3$ remark that $\psi(n)$ is alway even (this follows from $\psi(n) = \Pi(p^\alpha + p^{\alpha-1})$). Then $2 \mid \psi(n)$ and (9) enable us to write $\frac{\psi(2)}{2} \leq \frac{\psi(\psi(n))}{\psi(n)}$, which is exactly (10).

Now $\psi(\psi(\psi(n))) \geq \frac{3}{2}\psi(\psi(n)) \geq \frac{3}{2} \cdot \frac{3}{2} \cdot \psi(n) \geq \frac{9}{4}(n+1) > 2n$ by (10) and $\psi(n) \geq n + 1$ for $n \geq 2$. An easy inductive argument proves the theorem for all $m \geq 3$.

*Remark.* Equality occurs in (10) when $\psi(n) = 2^m$ (powers of 2). From (1 it is not difficult to deduce that $n$ must be a product of distinct Mersenne primes.

2. $\psi \circ \sigma$ — **Perfect numbers.**

THEOREM 3. *All even solutions of the equation*

$$\psi(\sigma(n)) = 2n \tag{11}$$

*have the form*: $n = 2^k$, *with* $2^{k+1} - 1 =$ *prime.*
$n = 3$ *is a solution of* (11). *If* $n$ *is an odd solution of* (11), *then*: (a) $n \not\equiv -1$ (mod 3), (b) $n \not\equiv 7$ (mod 12), (c) $n \not\equiv -4$ (mod 21), $n \not\equiv 10$ (mod 21). *If* $n$ *is a solution of* (11), *then* $2^\alpha \cdot 3^\beta \nmid \psi(n)$ *for all* $\alpha, \beta \geq 1$.

*Proof.* As in the proof of Theorem 1 $n = 2^k \cdot m$, ($m$ odd) is an even number. The multiplicativity of $\sigma$ and equality $\sigma(2^k) = 2^{k+1} - 1$ together with (6) imply $\psi(\sigma(2^k m)) \geqslant \sigma(m)\ \psi(2^{k+1}-1) \geqslant m \cdot 2^{k+1}$ with equality only for $m = 1$ and $2^{k+1} - 1 =$ prime. (Indeed, $\psi(n) = n + 1$ iff $n$ is a prime, which follows at once from (1)). Thus we have $\psi(\sigma(n)) \geqslant 2n$ for all even $n$, with equality for $n = 2^k$, where $2^{k+1} - 1$ is prime. This proves the first part of the theorem.

For the second part we first remark that

$$3 \mid \sigma(3k-1), \text{ for all } k \geqslant 1 \tag{12}$$

Indeed, let $d \mid (3k-1)$. Then the complementer divisor $\frac{3k-1}{d}$ has the form $3s - 1$ if $d \equiv 1 \pmod 3$ and the form $3s + 1$ if $d \equiv -1 \pmod 3$. In all cases $d + \frac{3k-1}{d} \equiv 0 \pmod 3$, implying (12). Now from (12), using (7), we can derive that $4 = \psi(3) \mid \psi(\sigma(3k-1)) = 2(3k-1)$ if $n = 3k - 1$ would be a solution of (11). This is clearly impossible, proving (a). In order to prove (b) and (c) use the known fact ([6]) that $\sigma(n)$ is odd only if $n = a^2$ or $n = 2a^2$ ($a \geqslant 1$ natural number). Thus if $n \neq a^2, 2a^2$, then $2 \mid \sigma(n)$, so by (7) $3 \mid n$ for a such solution of (11). First we select $n = 4m + 3$, with $m = 3k + 1$. Since $a^2 = 0$ or $1 \pmod 4$, and $2a^2 = 0$, or $2 \pmod 4$, we have $4m + 3 \neq a^2, 2a^2$. For $m = 3k + 1$ we obtain an obvious contradiction. Thus we cannot have $n = 4m + 3 = 12k + 7$. For (c) we can use the same method based on the numbers $7m + 3 \neq a^2, 2a^2$.

For the last proposition in the theorem, suppose $a \mid \sigma(n)$. Then (9) leads to $\psi(\sigma(n)) \geqslant \sigma(n)\ \psi(a)/a$, so for a solution $n$ of (11) and for $a = 2^\alpha \cdot 3^\beta$ we obtain (with $\psi(a) = 2a$ for these a) that $\sigma(n) \leqslant n$, i.e. $n = 1$, a contradiction.

*Remarks.* 1) Ch. Wall [15] has found all the $\psi$ – perfect numbers, by proving that $\psi(a) = 2a$ iff $a = 2^\alpha \cdot 3^\beta$ with $\alpha \geqslant 1$, $\beta \geqslant 1$.

2) The method of proof of Theorem 1 immediately gives that the equation

$$\frac{\psi(\sigma(n))}{2n} = \frac{m+1}{m} \tag{13}$$

for $n$ even, $n \neq 2^a$ (power of 2), where $m =$ greatest odd divisor of $n$; has the solutions $n = 2^k \cdot m$, with $m$ and $2^{k+1} - 1$ both primes.

3) It is easy to see that $\sigma(m) \geqslant \psi(m)$ for all $m$, so $\sigma(\sigma(n)) \geqslant \psi(\sigma(n)) \geqslant 2n$ for even $n$, with equality only for $n = 2^k$ where $2^{k+1} - 1 =$ prime. By this way we can reobtain Suryanarayana's [14] and Kanold's [8] result on the even superperfect numbers.

### 3. $\sigma \circ \psi$ – Perfect numbers.

LEMMA 4. *$\sigma(ab) \geqslant a\sigma(b)$ for all $a, b \geqslant 1$* (14)

*Proof.* As in the proof of (6), let $a = \prod p^\alpha \prod q^\beta$, $b = \prod p^{\alpha'} \prod t^\gamma$. Then

$$\frac{\sigma(ab)}{\sigma(b)} = \prod\left(\frac{p^{\alpha+\alpha'+1}-1}{p^{\alpha'+1}-1}\right)\prod\left(\frac{q^{\beta+1}-1}{q-1}\right).$$

The simple inequalities $(p^{\alpha+\alpha'+1}-1)/(p^{\alpha'+1}-1) \geqslant p^\alpha$ and $(q^{\beta+1}-1)/(q-1) \geqslant q^\beta$, ($\alpha, \alpha', \beta \geqslant 0$) imply (14).

THEOREM 4. *The only even solution of the equation*

$$\sigma(\psi(n)) = 2n \tag{15}$$

*is* $n = 2$. *If* $n$ *is an odd solution of* (15) *and* $p$ *is an odd prime having the property* $\sigma(p+1) > 2p$, *then* $p \nmid n$. *Particularly* $3, 5, 7, 11, 17, 19, \ldots \nmid n$.

*Proof.* Let $n = 2^k m$ (as usual) be an even number. Then $\sigma(\psi(n)) = \sigma(3 . 2^{k-1} \cdot \psi(m)) \geqslant \psi(m)\sigma(3.2^{k-1}) \geqslant m \cdot 4(2^k - 1)$ by (14) and $\psi(m) \geqslant m$. Now $4(2^k - 1) \geqslant 2^{k+1}$ for all $k \geqslant 1$, with equality for $k = 1$, thus the first part of the theorem is established. If $n$ is an odd number, suppose $p \mid n$. Write $n = p^\alpha \cdot N$, where $(p, N) = 1$. Here $\psi(n) = p^{\alpha-1} \cdot (p+1)\,\psi(N)$ and by Lemma 4 we get $\sigma(\psi(n)) \geqslant \psi(N)\sigma(p^{\alpha-1} \cdot (p+1)) \geqslant N \dfrac{p^\alpha - 1}{p - 1}\sigma(p+1)$. The assumed property and the inequality $(p^\alpha - 1)/(p - 1) \geqslant p^{\alpha-1}$ for all $\alpha \geqslant 1$, prove at once that $\sigma(\psi(n)) > 2n$. The inequality $\sigma(p+1) > 2p$ ($p$ odd prime) is not satisfied for $p = 13, 37, \ldots$ in which case the above argument is not sufficiently strong. We conjecture that (15) never has an odd solution.

*Remark.* By the method used in the first part we conclude that $\sigma(\psi(n))/2n \geqslant (m+1)/m$ if $n$ is even, $n \neq 2^a$, and $m$ denotes the greatest odd divisor of $n$.

**4. Almost and quasi $\psi \circ f$ — Perfect Numbers.** The equations

$$\psi(f(n)) = 2n \pm 1 \tag{16}$$

are very easy to study, because as we have remarked in the proof of Theorem 2, $\psi(m)$ is always even if $m \geqslant 3$. Thus in (16) we can have solutions only for $f(n) = 1$ or $f(n) = 2$. Without any difficulty we obtain: $\psi(\psi(n)) = 2n - 1 \Leftrightarrow n = 1, 2$; $\psi(\psi(n)) \neq 2n + 1$; $\psi(\varphi(n)) = 2n - 1 \Leftrightarrow n = 1$; $\psi(\varphi(n)) \neq 2n + 1$; $\psi(\sigma(n)) = 2n - 1 \Leftrightarrow n = 1$; $\psi(\sigma(n)) \neq 2n + 1$; etc.

**5. Almost and quasi $\sigma^* \circ \sigma^*$ — Perfect numbers.**

LEMMA 5. $\sigma^*(n) = 2^a$ *(power of* 2) *iff* $n = p_1 \ldots p_r$, *where* $p_i$ $(i = 1, r)$ *are distinct Mersenne primes.*

*Proof.* Relation (3) shows that $\sigma^*(n) = 2^m$ iff $p^\alpha + 1 = 2^a$ for all prime divisors $p$ of $n$. We shall prove that this is not the case if $\alpha > 1$. Firstly, let $p$ be of the form $p = 4k + 1$. Then $(4k+1)^\alpha + 1 = 4M + 2 = 2(2M + 1) \neq 2^a$. Twofold, let $p = 4k - 1$. Then $(4k-1)^\alpha + 1 = 4N + 2$ if $\alpha$ is even, so we must have $\alpha$ odd, $\alpha = 2s + 1$. It is easy to see by Newton's binomial formula that $(4k-1)^{2s+1} + 1 = (4k)^{2s+1} - C^1_{2s+1}(4k)^{2s} + \ldots + (2s + 1)(4k) = 4k \cdot A$, where $A$ is odd, $A > 1$. Evidently, $4k \cdot A$ cannot be a power of 2.
For $\alpha = 1$, $p + 1 = 2^a$ iff $p = 2^a - 1 =$ Mersenne prime and the lemma is proved.

THEOREM 5. $\qquad \sigma^*(\sigma^*(n)) \neq 2n \pm 1 \quad (n \neq 1.\ 3).$ (17)

*Proof.* $\sigma^*(m)$ is odd only if $m$=power of 2, so $\sigma^*(n)=$[illegible]d by Lemma 5, $n = p_1 \ldots p_r$ — *a* product of distinct Mersenne primes. But $\sigma^*$[illegible] $= 2^m + 1 = 2p_1 \ldots p_r \pm 1 \Leftrightarrow p_1 \ldots p_r = 2^{\ldots}$ [illegible] $\ldots p_r = 2^{m-1} + 1$
The first equality cannot be [illegible] the same is valid for the second one for $r \geqslant 2$. If $r = 1$, then [illegible] is acceptable. Now let $r > 2$, e.g.

$r = 4$ for simplicity and consider $(2^a - 1)(2^b - 1)(2^c - 1)(2^d - 1) = 2^{a+b+c+d} - \Sigma 2^{a+b+c} - \Sigma 2^{a+b} - \Sigma 2^a + 1 \neq 2^{m-1} + 1$ $(a < b < c < d)$ because on the left side we have $2^a \cdot A$, $A$ odd, $A > 1$, which is not a power of 2.

6. **On the equation** $\sigma(\sigma^*(n)) = n + 2$.

THEOREM 6. *The even solutions of the equation*

$$\sigma(\sigma^*(n)) = n + 2 \tag{18}$$

*are of the form* $n = 2^{2^s}$, *where* $2^{2^s} + 1$ *is a Fermat prime. There are no odd solutions.*

*Proof.* Let $n = 2^k \cdot m$, $m$ odd. Then $\sigma^*(n) = (2^k + 1)\sigma^*(m)$ and by (14) we can deduce that $\sigma(\sigma^*(n)) \geq \sigma^*(m)\sigma(2^k + 1) \geq m(2^k + 2) \geq m \cdot 2^k + 2 = n + 2$. Here we have equality when $m = 1$ and $2^k + 1$ is prime, i.e. is a Fermat prime. Then, as it is known, $k$ must be a power of 2, $k = 2^s$. Now let $n$ be odd, firstly $n = p^\alpha$ ($p$ odd prime). Then $\sigma(p^\alpha + 1) = p^\alpha + 2$ cannot hold, since $p^\alpha + 1$ is composite number ($\sigma(n) = n + 1$ iff $n$ = prime, this easily follows e.g. from (4)). On the other hand, if $n$ has at least two prime factors, let $n = ab$, with $(a, b) = 1$, $a > 1$, $b > 1$. Then $\sigma(\sigma^*(n)) = \sigma(\sigma^*(a)\sigma^*(b)) \geq \sigma^*(a)\sigma(\sigma^*(b)) \geq (a + 1)(\sigma^*(b) + 1) \geq (a + 1)(b + 2) > ab + 2$. Therefore (18) is not valid for odd $n$ and this finishes the proof.

7. $\psi \circ \varphi$ **— Perfect numbers.**

The next result follows by the same lines as Lemma 1:

LEMMA 6. $\psi(ab) \leq \psi(a)\ \psi(b)$, $a, b = 1, 2, 3, \ldots$. (19)

THEOREM 7. *The equation*

$$\psi(\varphi(n)) = 2n \tag{20}$$

*has not odd solutions. If $n$ is even and if we suppose $\psi(\varphi(m)) < 2m$ (i.e. $m$ is $\psi\circ\varphi$ — deficient), where $m$ denotes the greatest odd divisor of $n$, then $\psi(\varphi(n)) < 2n$ (i.e. $n$ is $\psi \circ \varphi$ — deficient).*

*Proof.* Let $n$ be an odd solution and suppose that it has $r$ distinct odd prime factors. We then clearly have $2^r \mid \varphi(n)$, so by Lemma 2, $2^{r-1} \cdot 3 \mid \psi(\varphi(n)) = 2n$. This is not true for $r \geq 3$. Ler $r = 1$; when $n = p^\alpha$, and in fact $n = 3^\alpha$. An easy computation shows that this situation is not possible. Let $r = 2$, i.e. $n = 3^\alpha \cdot p^\beta$ with $3 \nmid p$. Now $\varphi(n) = 2 \cdot 3^{\alpha-1} \cdot p^{\beta-1} \cdot (p - 1)$ and for $\alpha > 1$ or $\beta > 1$, $3 \mid \varphi(n)$ or $p \mid \varphi(n)$, leading to $\psi(3) = 4 \mid 2 \cdot 3^\alpha \cdot p^\beta$, $\psi(p) = (p + 1) \mid 2 \cdot 3^\alpha \cdot p^\beta$. Thus $\alpha = \beta = 1$, implying $\varphi(n) = 2(p - 1)$ and $\psi(p - 1) = 2p$. But $2 \mid (p - 1)$, so $3 \mid \psi(p - 1) = 2p$ only when $p = 3$. We can verify that $p = 3$ is not solution. If $n = 2^k \cdot m$ ($m$ odd), with $\psi(\varphi(m)) < 2m$, then applying Lemma 6 one has $\psi(\varphi(n)) \leq \psi(2^{k-1})\ \psi(\varphi(m)) < 3 \cdot 2^{k-1} \cdot m < 2^{k+1} \cdot m = 2n$.

*Remark.* The inequality $\psi(\varphi(m)) < 2m$ ($m$ odd) is not valid always, in fact it can be proved that $\limsup \psi(\varphi(m))/m = \infty$. On the other hand there are some plausible arguments to conjecture here that $\psi(\varphi(m)) \geq m$ for all odd $m$. For some results related to this problem we quote [1], [10]. We notice that in view of $\sigma(a) \geq \psi(a)$, this conjecture is stronger than the conjecture of Makowski and Schinzel [9]: $\sigma(\varphi(m)) \geq m$.

Finally, we deal with.

**8. On the equation $\varphi(\varphi(n)) = \frac{n}{4}$.**

The following lemma may be proved similarly with Lemma 4:

LEMMA 7. $\varphi(ab) \leqslant a\varphi(b)$ *for all* $a,\ b \geqslant 1$. (21)

THEOREM 8. *All solutions of the equation*

$$\varphi(\varphi(n)) = \frac{n}{4} \tag{22}$$

*are of the form* $n = 2^k$ $(k \geqslant 2)$.

*Proof.* Obviously $n$ must be even: $n = 2^k \cdot m$, $m$ odd. Then by (21) and $\varphi(m) \leqslant m$, one obtains successively: $\varphi(\varphi(n)) \leqslant \varphi(m) \cdot \varphi(2^{k-1}) \leqslant m \cdot 2^{k-2} = \frac{n}{4}$. Since $\varphi(m) = m$ only when $m = 1$, the theorem is proved.

THEOREM 9. *Let $p$ be a fixed odd prime and denote $A = \{n : p^2 \mid n\}$. Then the equation*

$$\varphi(\varphi(n)) = \frac{n}{2} - \frac{n}{p} + \frac{n}{2p^2} \tag{23}$$

*has solutions in $A$ only when $p$ is a Fermat prime, and all solutions may be written in the form $n = p^k$, $k \geqslant 2$.*

*Proof.* Let $n = p^k \cdot N$, where $k \geqslant 2$, $(p, N) = 1$. Then $\varphi(n) = p^{k-1}(p-1)\varphi(N)$ and by (21) $\varphi(\varphi(n)) \leqslant \varphi(N)\ \varphi(p^{k-1}(p-1)) \leqslant N \cdot \varphi(p^{k-1})\varphi(p-1) \leqslant Np^{k-2}(p-1) \cdot \frac{p-1}{2} = \frac{n}{2} - \frac{n}{p} + \frac{n}{2p^2}$.

Here we have used that $\varphi(p-1) \leqslant (p-1)/2$, which is a simple consequence of (21) (by writing $p - 1 = 2(p-1)/2$). There is equality only for $N = 1$ and $p - 1 = 2^a$, so $p = 2^a + 1 =$ Fermat prime, and $n = p^k$, $k \geqslant 2$.

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# A REFINEMENT OF JENSEN INEQUALITY AND APPLICATIONS

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**REZUMAT. — O rafinare a inegalităţii lui Jensen şi aplicaţii.** În lucrare se stabileşte o rafinare a inegalităţii lui Jensen şi se dau citeva particularizări în legătură cu anumite rezultate clasice pentru numere reale. Apoi se aplică inegalitatea lui Jensen pentru a se obţine citeva generalizări în spaţii normate ale unor inegalităţi din [3].

**1.** Further on, $X$ will be a linear space over the real or complex number field and $C$ will be a convex set in $X$. Then the following refinement of Jensen's inequality is valid.

THEOREM 1.1. *Let $f: C \to \mathbf{R}$ be a convex (concave) real function on $C$, $x_i \in C$, $p_i \geq 0$ and $P_n := \sum_{i=1}^{n} p_i > 0$ $(1 \leq i \leq n)$. Then the following inequality holds:*

$$f\left(\frac{\sum_{i=1}^{n} p_i x_i}{P_n}\right) \leq (\geq) \frac{\sum_{i_1,\ldots,i_{k+1}=1}^{n} p_{i_1} \cdots p_{i_{k+1}} f\left(\frac{x_{i_1} + \ldots + x_{i_{k+1}}}{k+1}\right)}{P_n^{k+1}}$$
$$\leq (\geq) \frac{\sum_{i_1,\ldots,i_k=1}^{n} p_{i_1} \cdots p_{i_k} f\left(\frac{x_{i_1} + \ldots + x_{i_k}}{k}\right)}{P_n^{k}} \leq (\geq) \ldots \leq (\geq) \frac{\sum_{i=1}^{n} p_i f(x_i)}{P_n} \tag{1.1}$$

*where $k$ is a positive integer such that $1 \leq k \leq n-1$.*

*Proof.* The first inequality follows by Jensen's inequality:

$$f\left(\frac{\sum_{i_1,\ldots,i_{k+1}=1}^{n} p_{i_1} \cdots p_{i_{k+1}} \left(\frac{x_{i_1} + \ldots + x_{i_{k+1}}}{k+1}\right)}{\sum_{i_1,\ldots,i_{k+1}=1}^{n} p_{i_1} \cdots p_{i_{k+1}}}\right) \leq (\geq)$$
$$\leq (\geq) \frac{\sum_{i_1,\ldots,i_{k+1}=1}^{n} p_{i_1} \cdots p_{i_{k+1}} f\left(\frac{x_{i_1} + \ldots + x_{i_{k+1}}}{k+1}\right)}{\sum_{i_1,\ldots,i_{k+1}=1}^{n} p_{i_1} \cdots p_{i_{k+1}}}$$

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since
$$\sum_{i_1,\ldots,i_{k+1}=1}^{n} p_{i_1}\ldots p_{i_{k+1}}\left(\frac{x_{i_1}+\ldots+x_{i_{k+1}}}{k+1}\right)=\sum_{i=1}^{n} p_i x_i \text{ and}$$
$$\sum_{i_1,\ldots,i_{k+1}=1}^{n} p_{i_1}\ldots p_{i_{k+1}} = P_n^{k+1}.$$

Now, putting
$$y_1=\frac{x_{i_1}+\ldots+x_{i_k}}{k},\ y_2=\frac{x_{i_2}+\ldots+x_{i_{k+1}}}{k},\ \ldots,\ y_{k+1}=\frac{x_{i_{k+1}}+\ldots+x_{i_{k-1}}}{k},$$
by Jensen's inequality:
$$f\left(\frac{y_1+\ldots+y_{k+1}}{k+1}\right)\leqslant(\geqslant)\frac{f(y_1)+\ldots+f(y_{k+1})}{k+1}$$
we deduce that:
$$f\left(\frac{\sum_{i=1}^{k+1} x_i}{k+1}\right)\leqslant(\geqslant)\frac{f\left(\frac{x_{i_1}+\ldots x_{i_k}}{k}\right)+\ldots+f\left(\frac{x_{i_{k+1}}+\ldots+x_{i_{k-1}}}{k}\right)}{k+1}$$

Multiplying with the nonnegative real numbers $p_{i_1},\ldots,p_{i_{k+1}}$ and summing after $i_1,\ldots,i_{k+1}$ we obtain
$$\sum_{i_1,\ldots,i_{k+1}=1}^{n} p_{i_1}\ldots p_{i_{k+1}} f\left(\frac{x_{i_1}+\ldots+x_{i_{k+1}}}{k+1}\right)\leqslant(\geqslant)P_n\sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\ldots p_{i_k}\times$$
$$\times f\left(\frac{x_{i_1}+\ldots+x_{i_k}}{k}\right)$$
from where results the second inequality in (1.1).

The theorem is proven.

COROLLARY 1.2. *If $f: C\to R$ is a convexe (concave) function on $C$ and $x_i\in C$ $(1\leqslant i\leqslant n)$ then:*
$$f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)\leqslant(\geqslant)\frac{\sum_{i_1,\ldots,i_{k+1}=1}^{n} f\left(\frac{x_{i_1}+\ldots+x_{i_{k+1}}}{k+1}\right)}{n^{k+1}}\leqslant(\geqslant)$$
$$\leqslant(\geqslant)\frac{\sum_{i_1,\ldots,i_k=1}^{n} f\left(\frac{x_{i_1}+\ldots+x_{i_k}}{k}\right)}{n^k}\leqslant(\geqslant)\ldots\leqslant(\geqslant)\frac{\sum_{i=1}^{n} f(x_i)}{n} \tag{1.2}$$
*where* $1\leqslant k\leqslant n-1$.

The following result gives a refinement of the polygonal inequality in normed linear spaces.

CONSEQUENCE 1.3. *Let* $(X, \|\cdot\|)$ *be a normed linear space and* $x_i \in X$ $(1 \leq i \leq n)$. *Then the following inequality is true:*

$$\left\|\sum_{i=1}^{n} x_i\right\| \leq \frac{\sum\limits_{i_1,\ldots,i_{k+1}=1}^{n}\left\|\frac{x_{i_1}+\cdots+x_{i_{k+1}}}{k+1}\right\|}{n^k} \leq$$
$$\leq \frac{\sum\limits_{i_1,\ldots,i_k=1}^{n}\left\|\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right\|}{n^{k-1}} \leq \ldots \leq \sum_{i=1}^{n}\|x_i\| \tag{1.3}$$

*where* $1 \leq k \leq n-1$.

Now we shall give some applications of the previous theorem

APPLICATIONS 1.4. a. Let $x_i$, $p_i \geq 0$ such that $P_n > 0$. Then

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \geq \left[\prod_{i_1,\ldots,i_{k+1}=1}^{n}\left(\frac{x_{i_1}+\cdots+x_{i_{k+1}}}{k+1}\right)^{p_{i_1}\cdots p_{i_{k+1}}}\right]^{1/P_n^{k+1}} \geq$$
$$\geq \left[\prod_{i_1,\ldots,i_k=1}^{n}\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right)^{p_{i_1}\cdots p_{i_k}}\right]^{1/P_n^{k}} \geq \ldots \geq \left(\prod_{i=1}^{n} x_i^{p_i}\right)^{1/P_n} \tag{1.4}$$

The proof follows by Theorem 1.1 for $f\colon (0,\infty)\to\mathbf{R}$, $f(x) = \ln x$.

b. Let $x_i$, $p_i \geq 0$ $(1 \leq i \leq n)$ such that $P_n > 0$ and $p > 1$. Then

$$\left(\frac{\sum\limits_{i=1}^{n} p_i x_i}{P_n}\right)^p \leq \frac{\sum\limits_{i_1,\ldots,i_{k+1}=1}^{n} p_{i_1}\cdots p_{i_{k+1}}\left(\frac{x_{i_1}+\cdots+x_{i_{k+1}}}{k+1}\right)^p}{P_n^{k+1}} \leq$$
$$\leq \frac{\sum\limits_{i_1,\ldots,i_k=1}^{n} p_{i_1}\cdots p_{i_k}\left(\frac{x_{i_1}+\cdots+x_{i_{k+1}}}{k}\right)^p}{P_n^{k}} \leq \ldots \leq \frac{\sum\limits_{i=1}^{n} p_i x_i^p}{P_n}. \tag{1.5}$$

The proof follows by Theorem 1.1 for $f\colon \mathbf{R}_+\to\mathbf{R}_+$, $f(x) = x^p$.

2. Further on, we shall apply Jensen's inequality to obtain generalizations in normed linear spaces of some results established in [3].

Throught in the sequel $(X, \|\cdot\|)$ will be a real or complex normed linear space.

LEMMA 2.1. *Let* $f\colon \mathbf{R}_+\to\mathbf{R}_+$ *be a nondecreasing convex function. Then for every* $x_i \in X$, $p_i \geq 0$ $(1 \leq i \leq n)$ *such that* $P_n > 0$, *the following inequality is valid:*

$$f\left(\frac{1}{P_n}\left\|\sum_{i=1}^{n} p_i x_i\right\|\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(\|x_i\|). \tag{2.1}$$

The proof is a simple consequence of Jensen's inequality.

We remark that, from (2.1) we deduce easy Theorem 2 and Remark 2 of [2]. The special case of this fact is the following inequality:

$$\left\|\sum_{i=1}^{n} z_i\right\|^r \leqslant \left(\sum_{i=1}^{n} q_i^{1/(1-r)}\right)^{r-1} \sum_{i=1}^{n} q_i \|z_i\|^r \tag{2.2}$$

where $q_i \geqslant 0$, $z_i \in X$ and $1 \leqslant r \leqslant 2$.

From this result, substituiting $q_i$ by $1/p_i$ $(1 \leqslant i \leqslant n)$ and by

$$\left(\sum_{i=1}^{n} p_i^{1/(r-1)}\right)^{r-1} \leqslant \sum_{i=1}^{n} p_i \qquad (1 \leqslant r \leqslant 2) \tag{2.3}$$

we obtain easy

$$\frac{\left\|\sum\limits_{i=1}^{n} z_i\right\|^r}{\sum\limits_{i=1}^{n} p_i} \leqslant \sum_{i=1}^{n} \frac{\|z_i\|^r}{p_i} \qquad (1 \leqslant r \leqslant 2) \tag{2.4}$$

where $p_i > 0$ $(1 \leqslant i \leqslant n)$, what gives a generalization for (see [3])

$$\frac{|z_1 + z_2|^r}{u+v} \leqslant \frac{|z_1|^r}{u} + \frac{|z_2|^r}{v} \tag{2.5}$$

$$uv(u+v) > 0, \qquad z_1, z_2 \in C, \qquad 1 \leqslant r \leqslant 2,$$

in the particular case when $u$ and $v$ are positive real numbers.

LEMMA 2.2. *Let* $f: \mathbf{R}_+ \to \mathbf{R}_+$ *be a nondecreasing convex function,* $p_1 > 0$, $p_i \leqslant 0$ $(2 \leqslant i \leqslant n)$ *and* $P_n > 0$. *Then*

$$f\left(\frac{1}{P_n}\left\|\sum_{i=1}^{n} p_i x_i\right\|\right) \geqslant \frac{1}{P_n}\sum_{i=1}^{n} p_i f(\|x_i\|). \tag{2.6}$$

The proof follows by the previous lemma substituiting $p_1$ by $P_n$, $p_i$ by $-p_i$ $(2 \leqslant i \leqslant n)$, $x_1$ by $\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i$ and $x_i$ by $x_i$ $(2 \leqslant i \leqslant n)$.

Putting in Lemma 2.2 $f: \mathbf{R}_+ \to \mathbf{R}_+$, $f(x) = x^r$ $(1 \leqslant r \leqslant 2)$ and $x_i = z_i/p$ $(z_i \in X,\ 1 \leqslant i \leqslant n)$, from (2.6) we obtain

$$\left\|\sum_{i=1}^{n} z_i\right\|^r \geqslant P_n^{r-1} \sum_{i=1}^{n} p_i |p_i|^{-r} \|z_i\|^r. \tag{2.7}$$

If we change $p_i|p_i|^{-r}$ by $q_i$, it results:

$$\left\|\sum_{i=1}^{n} z_i\right\|^r \geqslant \left(\sum_{i=1}^{n} q_i |q_i|^{r/(1-r)}\right)^{r-1} \sum_{i=1}^{n} q_i \|z_i\|^r \tag{2.8}$$

where

$$0 < q_1 \leqslant \left(\sum_{i=2}^{n} |q_i|^{1/(1-r)}\right)^{1/r} \text{ and } q_i \leqslant 0 \ (2 \leqslant i \leqslant n).$$

Now, putting in (2.8) $q_i = p_i^{-1}$ $(1 \leqslant i \leqslant n)$ and using the following inequality established in [4]:

$$\left(p_1^{1/(r-1)} - \sum_{i=2}^{n} |p_i|^{1/(r-1)}\right) \geqslant p_1 - \sum_{i=2}^{n} |p_i| = \sum_{i=1}^{n} p_i \tag{2.9}$$

we deduce

$$\frac{\left\|\sum_{i=1}^{n} z_i\right\|^r}{\sum_{i=1}^{n} p_i} \geqslant \sum_{i=1}^{n} \frac{\|z_i\|^r}{p_i} \qquad (1 \leqslant r \leqslant 2) \tag{2.10}$$

where $p_1 > 0$, $p_i \leqslant 0$, $P_n > 0$ $(2 \leqslant i \leqslant n)$.

The inequality (2.4) and (2.10) for $n = 2$ give

$$\frac{\|z_1 + z_2\|^r}{u + v} \leqslant \frac{\|z_1\|^r}{u} + \frac{\|z_2\|^r}{v} \tag{2.11}$$

if $uv(u + v) > 0$

and

$$\frac{\|z_1 + z_2\|^r}{u + v} \geqslant \frac{\|z_1\|^r}{u} + \frac{\|z_2\|^r}{v} \tag{2.12}$$

if $uv(u + v) < 0$

where $z_1, z_2 \in X$ and $1 \leqslant r \leqslant 2$, which extend in normed linear spaces some inequalities for the complex numbers established in [3].

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# NEW DINI THEOREMS FOR SEQUENCES WHICH SATISFY GENERALIZED ALEXANDROV CONDITIONS

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**REZUMAT. — Noi teoreme Dini pentru şiruri care satisfac condiţii generalizate ale lui Alexandrov.** Impunind asupra unui şir de funcţii continue condiţii de tip Alexandrov mai generale decit in [1—2], se obţin citeva generalizări ale rezultatelor de tip Dini din lucrările mai sus citate. Apoi, ţinind cont că aceste condiţii Alexandrov sînt de tip sumă şi că logaritmul transformă produsele în sume, se obţine un rezultat Dini pentru şiruri de funcţii continue care verifică o condiţie Alexandrov de tip produs.

1. **Introduction.** Considering for a sequence of real continuous functions a generalized Alexandrov condition, some extensions of the Dini theorems proved in [1], was obtained in [2].

The purpose of this note is twofold. Firstly, imposing to the sequence of continuous functions a more general Alexandrov condition as in [2], we extend the theorem 2.1 of [2].

Then, taking into account that generalized Alexandrov conditions are of sum type and that the logarithm transforms the products in sums, a Dini result for sequences of continuous functions which satisfy a generalized Alexandrov condition of product type is obtained.

2. **Main Results.** Given a nonvoid set $T$, an integer $k \geqslant 0$, $k+1$ positive numbers $c_0, \ldots, c_k$ and $k+1$ continuous functions $H_i : \mathbf{R}_+ \to \mathbf{R}_+$ with the properties: $H_i(0) = 0$ for all $i \in \{0, \ldots, k\}$, existing $i_0$ such that $H_{i_0}$ is strictly monotonically increasing on $\mathbf{R}_+$, we say that a sequence of functions $x_n : T \to \mathbf{R}$ satisfies a generalized Alexandrov condition relative to a function $x : T \to \mathbf{R}$, through the functions $H_i$, $i = \overline{0, k}$, if

$$\sum_{i=0}^{k} c_i \cdot H_i(|x(t) - x_{n+1+k-i}(t)|) \leqslant \sum_{i=0}^{k} c_i \cdot H_i(|x(t) - x_{n+k-i}(t)|), \tag{1}$$

for all $t \in T$ and all $n \in N$.

THEOREM 2.1. *Let $T$ be a nonvoid countably compact topological space and $(x_n)_{n\in N}$ be a sequence of continuous functions $x_n : T \to \mathbf{R}$ which converges pointwise on $T$ to a continuous function $x : T \to \mathbf{R}$. If $(x_n)_n$ satisfies a generalized Alexandrov condition relative to $x$ and through the functions $H_i$, $i = \overline{0, k}$, then $(x_n)_n$ converges uniformly on $T$ to $x$.*

*Proof.* Let $\varepsilon > 0$ and let $y_n : T \to \mathbf{R}$ be the function defined by $y_n(t) = \sum_{i=0}^{k} c_i \cdot H_i(|x(t) - x_{n+k-i}(t)|)$, $t \in T$, $n \in N$. Since $y_n$ are continuous, the

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sets $T_n = \{t \in T ; y_n(t) < \varepsilon\}$ are open. From (1) we have $T = \bigcup\{T_n : n \in N\}$. Indeed, given a $t$ in $T$, the convergence $x_n(t) \to x(t)$ and the continuity of $H_i$, $i = \overline{0, k}$, produces a number $n \in N$ such that

$$H_i(|x(t) - x_{n+k-i}(t)|) < \frac{\varepsilon}{c_i(k+1)},$$

for all $i = \overline{0, k}$, whence

$$y_n(t) < \sum_{i=0}^{k} c_i \cdot \frac{\varepsilon}{c_i \cdot (k+1)} = \varepsilon,$$

so that $t \in T_n$.

By the countable compactness of $T$, there exists an $n_0 \in N$ such that $T = T_1 \cup \ldots \cup T_{n_0} = T_{n_0}$, hence using (1) and the inequality $\sum_{i=0}^{k} c_i \cdot H_i(|x(t) - x_{n_0+k-i}(t)|) < \varepsilon$ for all $t \in T_{n_0}$, one has

$$\sum_{i=0}^{k} c_i \cdot H_i(|x(t) - x_{n+k-i}(t)|) \leqslant \sum_{i=0}^{k} c_i H_i(|x(t) - x_{n_0+k-i}(t)|) < \varepsilon,$$

for all $t \in T_{n_0}$ and all $n \geqslant n_0$. From here, it follows

$$H_{i_0}(|x(t) - x_{n+k-i_0}(t)|) < \frac{\varepsilon}{c_{i_0}}, \tag{2}$$

for all $t \in T$ and all $n \geqslant n_0$.
Taking into account $H_{i_0}$ is continuous function, strictly monotonically increasing and $H_{i_0}(0) = 0$, evidently the inverse function $H_{i_0}^{-1}$ is continuous, strictly monotonically increasing and $H_{i_0}^{-1}(0) = 0$. From (2), we obtain

$$H_{i_0}^{-1}(H_{i_0}(|x(t) - x_{n+k-i_0}(t)|)) = |x(t) - x_{n+k-i_0}(t)| < H_{i_0}^{-1}\left(\frac{\varepsilon}{c_{i_0}}\right), \tag{3}$$

for all $t \in T$ and all $n \geqslant n_0$.

Now, because $H_{i_0}^{-1}$ is continuous in the point 0 and $H_{i_0}^{-1}(0) = 0$ it immediately follows that simultaneous by $\varepsilon$, $H_{i_0}^{-1}\left(\frac{\varepsilon}{c_{i_0}}\right)$ becomes arbitrarly small, whence taking into account (3), the uniform convergence of $(x_n)$ on $T$ to $x$ is proved.

*Remarks.* 1). For $H_i(t) = t^{r_i}$, $t \in \mathbf{R}_+$, $r_i > 0$ $i = \overline{0, k}$, we arrive to the theorem 2.1 of [2].

2). Let us choose in (1)

$k = 2$, $T = [0, 1]$, $c_0 = 2$, $c_1 = 1$, $H_0(t) = t$ and $H_1(t) = \exp(t) - 1$, $t \in [0, 1]$

(where $\exp(t) = e^t$). The above theorem becomes:
"if the sequence of continuous functions $x_n : [0, 1] \to \mathbf{R}$ converges pointwise on $[0, 1]$ to 0 and satisfies: $0 \leqslant x_n(t)$,

$$x_{n+2}(t) \leqslant x_{n+1}(t) + \frac{1}{2}[\exp(x_n(t)) - \exp(x_{n+1}(t))], \quad t \in [0, 1], \tag{4}$$

$n \in N$, then $(x_n)$ converges uniformly on $[0, 1]$ to 0."

Let us observe that if $0 \leqslant x_n(t)$ is a monotonically decreasing sequence, then (4) is satisfied.

3. **A generalized Alexandrov condition of product type.** In [2] (corollary 2.3) the following result is proved:

"Let $T$ be a nonvoid countably compact topological space and $(x_n)_n$ be a sequence of continuous functions $x_n : T \to \mathbf{R}$, which converges pointwise on $T$ to 0. If $0 \leqslant x_n(t)$ and

$$x_{n+2}(t) \leqslant \frac{1}{2} \cdot [x_n(t) + x_{n+1}(t)], \tag{5}$$

for all $t \in T$ and all $n \in N$, then $(x_n)$ converges uniformly on $T$ to 0".

*Remark.* (5) condition is of sum type.

In what follows we will prove a Dini result which uses a generalized Alexandrov condition of product type.

THEOREM 3.1. *Let $T$ be a nonvoid countably compact topological space and $(y_n)_n$ be a sequence of continuous functions $y_n : T \to \mathbf{R}$ which converges pointwise on $T$ to a continuous function $y : T \to \mathbf{R}$.*
*If $0 < y(t) \leqslant y_n(t)$ for all $t \in T$ and all $n \in N$ and*

$$y_{n+2}(t) \leqslant [y_{n+1}(t) \cdot y_n(t)]^{1/2} \tag{6}$$

*for all $t \in T$ and all $n \in N$, then $(y_n)_n$ converges uniformly on $T$ to $y$.*

*Proof.* If we denote $x_n(t) = y_n(t)/y(t)$, evidently $x_n \in C(T)$, $x_n(t) \geqslant 1$ for all $t \in T$ and all $n \in N$, $x_n(t) \xrightarrow{n} 1$ pointwise on $T$. The sequence $(x_n)_n$ also satisfies (6), which is evidently equivalent with

$$\ln[x_{n+2}(t)] \leqslant \frac{1}{2} (\ln[x_{n+1}(t)] + \ln[x_n(t)]), \; t \in T, \; n \in N. \tag{7}$$

But taking $k = 1$, $c_0 = 2$, $c_1 = 1$, $H_0(t) = H_1(t) = \ln(1 + t)$, $t \in \mathbf{R}_+$, $x(t) = 1$ for all $t \in T$, $x_n(t) \geqslant 1$ for all $t \in T$ and all $n \in N$ in (1), we evidently obtain (7). Hence, by theorem 2.1 $(x_n)_n$ converges uniformly on $T$ to 1, which can be write

$$[y_n(t) - y(t)]/[y(t)] \xrightarrow{n} 0,$$

uniformly on $T$. But

$$|y_n(t) - y(t)| = |y(t)| \cdot [|y_n(t) - y(t)|/|y(t)|] \leqslant M \cdot$$

$\cdot [|y_n(t) - y(t)|/|y(t)|]$ (where $|y(t)| \leqslant M$ for all $t \in T$) which immediately implies $y_n \to y$ uniformly on $T$.

*Remarks.* 1). Let us observe that if $(y_n)_n$ is monotonically decreasing and satisfies $0 < y_n(t)$ for all $t \in T$ and all $n \in N$ then it also satisfies (6). Conversely, it is easy to show that the sequence $y_n(t) = e^{x_n(t)}$, $t \in [0, 1]$, $n \in N$, where $x_n(t)$ is defined as in [2], Remark 2.4, is not monotonous but satisfies (6).

2). For other recent extensions regarding the Dini results see [3], [4] and [5] p. 45.

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# ON CERTAIN UNIVALENCE CRITERIA

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**REZUMAT.** — Asupra unor anumite criterii de univalenţă. Folosindu-se o metodă pentru Φ — funcţii datorată lui St. Ruscheweyh [8] se obţin noi condiţii suficiente de univalenţă.

1. **Introduction.** Let $A$ be the class of functions $f$ which are analytic in the unit disk $U = \{z \in C;\ |z| < 1\}$, with $f(0) = 0$ and $f'(0) = 1$.

DEFINITION 1. ([3]). A function $f(z) \in A$ is called *Φ-like* in $U$ if only if:

$$\text{Re}\,\frac{zf'(z)}{\Phi(f(z))} > 0,\quad z \in U, \tag{1}$$

where $\Phi(w)$ is analytic in $f(U)$, $\Phi(0) = 0$, $\text{Re}\ \Phi'(0) > 0$.

DEFINITION 2. A function $f(z) \in A$ is called *spiral-like* if and only if there exists real number $\gamma$ such that

$$\text{Re}\ \exp\,(i\gamma)\,\frac{zf'(z)}{f(z)} > 0,\quad z \in U \tag{2}$$

Spiral-like functions are known to be univalent in $U$.

DEFINITION 3. ([4]). Let $\alpha \geqslant 0$ and $f(z) \in A$, $f(z)\,f'(z) \neq 0$ in $0 < |z| < 1$. Defining

$$J(\alpha, f(z)) = (1 - \alpha)\,\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf''(z)}{f'(z)} + 1\right) \tag{3}$$

if

$$\text{Re}\ J(\alpha,\ f(z)) > 0 \text{ for } z \in U, \tag{4}$$

then $f(z)$ is said to be an *α-convex* function.

DEFINITION 4. Let $f(z)$, $g(z)$ be two analytic functions in $U$. We say that $f(z)$ is *subordinate* to $g(z)$, written $f(z) \prec g(z)$, if there exists a function $\varphi(z)$ analytic in $U$ which satisfies $\varphi(0) = 0$, $|\varphi(z)| < 1$ and

$$f(z) = g(\varphi(z)),\quad |z| < 1. \tag{5}$$

L. Brickmann [3], using methods from the theory of differential equations, proved that every Φ-like function is univalent in $U$ and, in turn, that every analytic normalized univalent function in $U$ is Φ-like for a certain Φ.

St. Ruscheweyh [8] gives an elementary function — theoretic proof of the univalence of the Φ-like functions.

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In this note, by using the same method, we obtain certain sufficient conditions for univalence.

2. **Preliminaries.** F. C. Avhadiev and L. A. Aksentiev [1] have proved the following theorem:

THEOREM A. *Let $f(z)$, $g(z) \in A$. If:*

$$\left|\frac{zg''(z)}{g'(z)}\right| \leqslant \frac{1}{1-|z|^2}, \quad \forall z \in U \tag{6}$$

*and*

$$\operatorname{Log} f'(z) \prec \operatorname{Log} g'(z), \ \operatorname{Log} f'(0) = \operatorname{Log} g'(0) = 0, \tag{7}$$

*then the function $f(z)$ is univalent in $U$.*

There are known the next univalence criteria:

THEOREM B. (Becker)([2]). *Let $f(z) \in A$. If:*

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leqslant 1, \quad \forall z \in U, \tag{8}$$

*then the function $f(z)$ is univalent in $U$.*

THEOREM C. (Nehari) ([6]). *If $f(z) \in A$ and:*

$$|\{f;\ z\}| \leqslant \frac{2}{(1-|z|^2)^2}, \quad \forall z \in U, \tag{9}$$

*where $\{f;\ z\}$ is the Schwartzian derivative of $f(z)$:*

$$\{f;\ z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2, \tag{10}$$

*then the function $f(z)$ is univalent in $U$.*

In [6] was obtained the next univalence criterion:

THEOREM D. *Let $f(z) \in A$ and $\alpha \in C$. If* $\operatorname{Re}\alpha > 0$ *and:*

$$(1-|z|^{2\operatorname{Re}\alpha})\left|\frac{zf''(z)}{f'(z)}\right| \leqslant \operatorname{Re}\alpha, \quad \forall z \in U. \tag{11}$$

*then the function*

$$H_\alpha(z) = \left[\alpha\int_0^z u^{\alpha-1} f'(u)\, du\right]^{1/\alpha} \tag{12}$$

*is analytic and univalent in $U$.*

We will need the following lemma to prove our results:

LEMMA 1. *Let $f(z) \in A$ and let $\Phi(w)$ be an analytic function in $f(U)$, $\Phi(0) = 0$, $\Phi'(0) = c$,* $\operatorname{Re} c > 0$, $\frac{\Phi(w)}{w} \neq 0$ *for all $w \in f(U)$.*

*The differential equation*

$$\frac{zf'(z)}{\Phi(f(z))} = \frac{1}{c}\ \frac{zG'(f(z))\,f'(z)}{G(f(z))} \tag{13}$$

*has the analytic solution:*

$$G(w) = w \exp\left[\int_0^w \left(\frac{c}{\Phi(u)} - \frac{1}{u}\right) du\right], \tag{14}$$

$w = f(z)$.

*Proof.* Let $f(z) = w$ in (13). Then:

$$\frac{G'(w)}{G(w)} = \frac{c}{\Phi(w)} \text{ and } \frac{G'(w)}{G(w)} - \frac{1}{w} = \frac{c}{\Phi(w)} - \frac{1}{w}.$$

Integrating from 0 to $w$ we obtain:

$$G(w) = w \exp\left[\int_0^w \left(\frac{c}{\Phi(u)} - \frac{1}{u}\right) du\right].$$

**3. New criteria of univalence.**

THEOREM 1 [7]. *Let $f(z) \in A$. If $f(z)$ is a $\Phi$-like function then $f(z)$ is univalent in $U$.*

*Proof.* By assumption $\Phi(w) = cw + \ldots$ is analytic in $f(U)$, $\operatorname{Re} c > 0$ and $\operatorname{Re} \frac{zf'(z)}{\Phi(f(z))} > 0$. Let $F(z) = G(f(z))$ and $G(w)$ from (14).

From (13) we have:

$$\frac{zf'(z)}{\Phi(f(z))} = \frac{1}{c} \frac{zF'(z)}{F(z)}.$$

Because $\operatorname{Re} c > 0$ there exists a real number $\gamma$, $|\gamma| < \frac{\pi}{2}$ such that $c = |c| e^{i\gamma}$.

Then we have:

$$\frac{zf'(z)}{\Phi(f(z))} = |c|^{-1} e^{-i\gamma} \frac{zF'(z)}{F(z)}$$

By (1) we have:

$$\operatorname{Re} e^{-i\gamma} \frac{zF'(z)}{F(z)} > 0.$$

We have too $F(0) = F'(0) - 1 = 0$. Then $F(z)$ is a $\gamma$-spiral-like, that is a univalent function. Because $F(z) = G(f(z))$ and $G(w)$ is analytic in $f(U)$ we obtain that $f(z)$ is univalent in $U$.

THEOREM 2. *Let $f(z) \in A$, $f(z) \cdot f'(z) \neq 0$ in $0 < |z| < 1$, let $\Phi(w)$ be an analytic function in $f(U)$, $\Phi(0) = 0$, $\Phi'(0) = c$, $\operatorname{Re} c > 0$, $\frac{\Phi(w)}{w} \neq 0$ in $f(U)$, let $\alpha$ be a real fixed number and let:*

$$K(\alpha, f(z), \Phi(f(z))) = (1 - \alpha)\, c \frac{zf'(z)}{\Phi(f(z))} + \alpha\left(1 + \frac{zf''(z)}{f'(z)} + zf'(z) \frac{c - \Phi'(f(z))}{\Phi(f(z))}\right) \tag{15}$$

*If*

$$\operatorname{Re} K(\alpha, f(z), \Phi(f(z))) > 0, \quad \forall z \in U, \tag{16}$$

*then $f(z)$ is univalent in $U$.*

*Proof*. We remark that the function

$$F(z) = [f(z)] \exp\left[\int_0^{f(z)} \left(\frac{c}{\Phi(u)} - \frac{1}{u}\right) du\right] \tag{17}$$

is analytic in $U$ and

$$\frac{zF'(z)}{F(z)} = c\,\frac{zf'(z)}{\Phi(f(z))}. \tag{18}$$

Because $f(z)\,f'(z) \neq 0$ in $0 < |z| < 1$, from (17) and (18) we have $F(z) \cdot F'(z) \neq \neq 0$ in $0 < |z| < 1$. Combining (3), (15) and (18) we obtain:

$$J(\alpha,\ F(z)) = K(\alpha,\ f(z),\ \Phi(f(z)))$$

From (16) we have Re $J(\alpha,\ F(z)) > 0$ for all $z \in U$. Therefore $F(z)$ is an $\alpha$ — — convex function, hence a univalent function in $U$. Since $F(z) = G(f(z))$, where $G(w)$ is the analytic function defined by (14) we obtain that $f(z)$ is univalent in $U$.

*Remark* 1). Using Theorem 2 we can obtain new classes of univalent functions in $U$. In the case $\Phi(w) = cw$, $K(\alpha,\ f(z),\ cf(z)) = J(\alpha,\ f(z))$ and (16) becomes Re $J(\alpha,\ f(z)) > 0$, which defines the class of $\alpha$ — convex functions.

THEOREM 3. *Let $f(z)$, $F(z)$, $H(z) \in A$, and let $\Phi(w)$ be an analytic function in $f(U)$, $\Phi(0) = 0$, $\Phi'(0) = c$, Re $c > 0$, $\frac{\Phi(w)}{w} \neq 0$ for all $w \in f(U)$.*

*If*

$$\left|\frac{zH''(z)}{H'(z)}\right| \leqslant \frac{1}{1 - |z|^2},\quad \forall z \in U, \tag{19}$$

$$\operatorname{Log} F'(z) \prec \operatorname{Log} H'(z),\ \operatorname{Log} F'(0) = \operatorname{Log} H'(0) = 0, \tag{20}$$

*and $F(z) = G(f(z))$, where $G(w)$ is given by* (14), *then $f(z)$ is univalent in $U$.*

*Proof*. From (19), (20) and Theorem A we obtain that $F(z)$ ia univalent in $U$. Because we have $F(z) = G(f(z))$, where $G(w)$ is the analytic function defined by (14) it follows that $f(z)$ is univalent in $U$.

*Remark* 2). If $\Phi(w) = cw$, then $G(w) = w$ and Theorem 3 becomes Theorem A.

THEOREM 4. *Let $f(z) \in A$ and let $\Phi(w)$ be an analytic function* in $f(U)$, *$\Phi(0) = 0$, $\Phi'(0) = c$, Re $c > 0$, $\frac{\Phi(w)}{w} \neq 0$ for all $w \in f(U)$.*

*If*

$$(1 - |z|^2)\left|\frac{zf''(z)}{f'(z)} + z\,f'(z)\,\frac{c - \Phi'(f(z))}{\Phi(f(z))}\right| \leqslant 1,\quad \forall z \in U, \tag{21}$$

*then $f(z)$ is univalent in $U$.*

*Proof*. Let $F(z)$ be the function from (17). Then

$$\frac{zF''(z)}{F'(z)} = \frac{zf''(z)}{f'(z)} + z\,f'(z)\,\frac{c - \Phi'(f(z))}{\Phi(f(z))} \tag{22}$$

Combining (21) and (22) we obtain:

$$(1 - |z|^2)\left|\frac{zF''(z)}{F'(z)}\right| \leqslant 1, \quad \forall z \in U. \tag{23}$$

From (23) and Theorem B it follows that $F(z)$ is univalent in $U$. Since $F(z) = G(f(z))$, where $G(w)$ is the analytic function defined by (14) we obtain that $f(z)$ is univalent in $U$.

*Remark* 3). In the case $\Phi(w) = cw$ from Theorem 4 it follows Theorem B.

THEOREM 5. *Let* $f(z) \in A$ *and let* $\Phi(w)$ *be an analytic function* in $f(U)$, $\Phi(0) = 0$, $\Phi'(0) = c$, Re $c > 0$, $\frac{\Phi(w)}{w} \neq 0$ *for all* $w \in (U)$.

*If*

$$\left|\{f; z\} - f'^2(z)\frac{c^2 - \Phi'^2(f(z)) + 2\Phi''(f(z))\,\Phi(f(z))}{2\Phi^2(f(z))}\right| \leqslant \frac{2}{(1 - |z|^2)^2}, \quad \forall z \in U. \tag{24}$$

*where* $\{f;\ z\}$ *is the Schwartzian derivative of* $f(z)$, *then the function* $f(z)$ *is univalent in* $U$.

*Proof.* Let $F(z)$ be the function defined by (17). Then:

$$\frac{F''(z)}{F'(z)} = \frac{f''(z)}{f'(z)} + f'(z)\frac{c - \Phi'(f(z))}{\Phi(f(z))} \tag{25}$$

Combining (24) and (25) we obtain:

$$|\{F;\ z\}| \leqslant \frac{2}{(1 - |z|^2)^2}, \quad \forall z \in U. \tag{26}$$

From (26) and Theorem C it follows that $F(z)$ is univalent in $U$. Since $F(z) = G(f(z))$, where $G(w)$ is the analytic function defined by (14) we obtain that $f(z)$ is univalent in $U$.

*Remark* 4). In the case $\Phi(w) = cw$ from Theorem 5 it results Theorem C.

THEOREM 6. *Let* $f(z) \in A$, *let* $\alpha \in C$, Re $\alpha > 0$, *and let* $\Phi(w)$ *be an analytic function in* $f(U)$, $\Phi(0) = 0$, $\Phi'(0) = c$, Re $c > 0$, $\frac{\Phi(w)}{w} \neq 0$ *for all* $z \in U$.

*If*

$$(1 - |z|^{2\mathrm{Re}\,\alpha})\left|\frac{zf''(z)}{f'(z)} + zf'(z)\frac{c - \Phi'(f(z))}{\Phi(f(z))}\right| \leqslant \mathrm{Re}\ \alpha, \quad \forall z \in U, \tag{27}$$

*then the function*

$$\left[\alpha\int_0^z u^{\alpha-1} f'(z)\exp H(u)\,\frac{cf(z)}{\Phi(f(z))}\,du\right]^{1/\alpha}, \tag{28}$$

*where*

$$H(z) = \left[\int_0^{f(z)}\left(\frac{c}{\Phi(u)} - \frac{1}{u}\right)du\right] \tag{29}$$

*is analytic and univalent in* $U$.

*Proof.* Let $F(z)$ be the function defined by (17). Then we obtain (22). Combining (22) and (27) we have:

$$(1-|z|^{2\mathrm{Re}\,\alpha})\left|\frac{zF''(z)}{F'(z)}\right| \leqslant \mathrm{Re}\,\alpha, \quad \forall z \in U. \tag{30}$$

From (30) and Theorem D we obtain that the function

$$\left[\int_0^z u^{\alpha-1} F'(u)\, du\right]^{1/\alpha} \tag{31}$$

is analytic and univalent in $U$.
By using the expression of $F(z)$ from (17) we obtain

$$F'(z) = f'(z)\cdot \exp\left[\int_0^{f(z)} \left(\frac{c}{\Phi(u)} - \frac{1}{u}\right) du\right] \frac{c\, f(z)}{\Phi(f(z))},$$

and the conclusion of theorem immediately follows.

*Remark* 5). If we take $\Phi(w) = cw$ in Theorem 6, then we obtain Theorem D.

*Remark* 6). Results similar to those included in this note have been obtained in [5], for $c = 1$.

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# UNIVALENCE CONDITIONS OBTAINED BY DIFFERENTIAL SUBORDINATIONS

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REZUMAT. — **Condiţii de univalenţă obţinute prin subordonări diferenţiale.** În această lucrare sînt stabilite subordonări diferenţiale care conduc la Φ-funcţii, deci la funcţii univalente.

**1. Introduction.** Let $H$ be the space of functions regular in the unit disc $U = \{z \in \mathbf{C} / |z| < 1\}$, let $A = \{f \in H / f(0) = f'(0) - 1 = 0\}$ and let $A_0 = \{f \in A / f(z) \neq 0,\ z \in U \setminus \{0\}\}$.

Let $g$ and $G$ be functions in $H$. We say that $g$ is subordinate to $G$, written $g \prec G$ or $g(z) \prec G(z)$, if $G$ is univalent, $g(0) = G(0)$ and $g(U) \subset G(U)$.

In this article we obtain certain sufficient conditions for univalence using the univalence property of Φ-like functions with respect to $G$.

We will need the following definitions and lemmas to prove our results.

DEFINITION 1. [6] Let $G$ be a convex conformal mapping of $U$, $G(0) = 1$. Let $\Phi$ be analytic in a domain containing $f(U)$, $\Phi(0) = \Phi'(0) - 1 = 0$, $\Phi(w) \neq 0$ in $f(U) - \{0\}$. A function $f \in A$ is called *Φ-like with respect to G*, if and only if

$$\frac{zf'(z)}{\Phi(f(z))} \prec G(z). \tag{1}$$

If we let $G(z) = \frac{1+z}{1-z}$ in this definition, we obtain the definition of Φ-like function due to Brickman [10]. In this case condition (1) reduces to

$$\operatorname{Re} \frac{zf'(z)}{\Phi(f(z))} > 0,\quad z \in U. \tag{2}$$

Let $f$ be Φ-like with respect to $G$, where $0 \notin G(U)$. Then $f$ is univalent in $U$ ([6], Corollary 3).

In the particular cases $\Phi(w) = w$ and $\Phi(w) = e^{-i\gamma} w$, $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ we obtain the starlike functions with respect to $G$, the γ-spirallike functions with respect to $G$ and respectively the starlike an the γ-spirallike functions.

The conditions $G(0) = 1$ and $\Phi'(0) - 1 = 0$ of definition 1. can be replaced by $G(0) = c$, $\Phi'(0) - \frac{1}{c} = 0$ where $c$ is complex with $\operatorname{Re} c > 0$.

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DEFINITION 2. [2] The univalent function $q$ is said to be a ***dominant of the differential subordination***

$$\psi(p(z),\ zp'(z),\ z^2p''(z)) \prec h(z), \tag{3}$$

if $p \prec q$ for all $p$ satisfying (3).

If $\tilde{q}$ is a dominant of (3) and $\tilde{q} \prec q$ for all other dominants $q$ of (3), then $\tilde{q}$ is said to be the best dominant.

The best dominant, if it exists, is unique up to a rotation of $U$.

LEMMA 1. [3] *Let $q$ be univalent in $U$ and let $\theta$ and $\Phi$ be analytic in a domain $D$ containing $q(U)$ with $\Phi(w) \neq 0$ when $w \in q(U)$.*
*Set $Q(z) = zq'(z)\Phi(q(z))$, $h(z) = \theta(q(z))$ and suppose that:*

(i) *$Q$ is starlike (univalent) in $U$, and*

(ii) $\operatorname{Re} \dfrac{zh'(z)}{Q(z)} = \operatorname{Re}\left[\dfrac{\theta'(q(z))}{\Phi(q(z))} + \dfrac{zQ'(z)}{Q(z)}\right] > 0,\ z \in U.$

*If $p$ is analytic in $U$ with $p(0) = q(0)$, $p(U) \subset D$ and*

$$\theta(p(z)) + zp'(z)\,\Phi(p(z)) \prec \theta(q(z)) + zp'(z)\,\Phi(q(z)) = h(z), \tag{4}$$

*then $p \prec q$ and $q$ is the best dominant of* (4).

## 2. Main results.

THEOREM 1. *Let $f \in A_0$ and let $\Phi$ be analytic in a domain $D$ containing $f(U)$, $\Phi(0) = 0$, $\Phi'(0) = \dfrac{1}{c}$, $\operatorname{Re} c > 0$, $\Phi(w) \neq 0$ in $f(U)\setminus\{0\}$. Let $\alpha$ be a complex number and $q$ be convex (univalent) function in $U$ with $q(0) = c$ and*

$$\operatorname{Re}[q(z)/\alpha] > 0 \text{ in } U. \text{ If } Q(z) = \alpha\frac{zq'(z)}{q(z)}$$

*is starlike (univalent) in $U$ and if*

$$J_\Phi(\alpha, f(z)) = \frac{zf'(z)}{\Phi(f(z))} + \alpha\left\{1 + \frac{zf''(z)}{f'(z)} - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}\right\} \prec q(z) + \alpha\frac{zq'(z)}{q(z)} = G(z) \tag{5}$$

*then $\dfrac{zf'(z)}{\Phi(f(z))} \prec q(z)$ and $q(z)$ is the best dominant of* (5).

*Proof.* If we let $p(z) = \dfrac{zf'(z)}{\Phi(f(z))}$, $p$ is analytic in $U$ with $p(0) = c = q(0)$ and (5) can be rewritten as

$$p(z) + \alpha\frac{zp'(z)}{p(z)} \prec q(z) + \alpha\frac{zq'(z)}{q(z)} = G(z) \tag{6}$$

Since

$$\operatorname{Re}\frac{zG'(z)}{Q(z)} = \operatorname{Re}\left[\frac{q(z)}{\alpha} + \frac{zQ'(z)}{Q(z)}\right] > 0$$

the conclusion of the theorem will follow from lemma 1.

Let $k(z) = \frac{z}{(1-z)^2}$. If we let $q(z) = \frac{zk'(z)}{k(z)} = \frac{1+z}{1-z}$ and $\alpha > 0$ in theorem 1., $q$ is convex, $q(0) = 1$ and maps the unit disc $U$ onto the half plane Re $w > 0$ and

$$G(z) = \frac{1+z}{1-z} + \frac{2\alpha z}{1-z^2} = J(\alpha, k(z))$$

We note that $G(U)$ is the complex plane slit along the half-lines Re $w = 0$, Im $w \geqslant \sqrt{\alpha(\alpha+2)}$ and Re $w = 0$, Im $w \leqslant -\sqrt{\alpha(\alpha+2)}$. By using Theorem 1. we obtain the following result.

COROLLARY 1. *Let $f \in A_0$, $\Phi$ be analytic in a domain containing $f(U)$, $\Phi(0) = 0$, $\Phi'(0) = 1$, $\Phi(w) \neq 0$ in $f(U)\setminus\{0\}$ and $\alpha > 0$. If*

$$J_\Phi(\alpha, f(z)) \prec J(\alpha, k(z))$$

*then $f$ is $\Phi$-like, i.e.*

$$J_\Phi(\alpha, f(z)) \prec J(\alpha, k(z)) \Rightarrow J_\Phi(0, f(z)) \prec J(0, k(z)).$$

In particular, we easily obtain

$$f \in A_0 \text{ and } |J_\Phi(\alpha, f(z)) - 1| < 1 + \alpha \Rightarrow \text{Re}\, \frac{zf'(z)}{\Phi(f(z))} > 0,\ z \in U \tag{7}$$

and

$$f \in A_0, |\text{Im}\, J_\Phi(\alpha, f(z))| < \sqrt{\alpha(\alpha+2)} \Rightarrow \text{Re}\, \frac{zf'(z)}{\Phi(f(z))} > 0,\quad z \in U \tag{8}$$

where $\Phi$ is given in Corollary 1.

For $\alpha = 1$, from Corollary 1. we deduce:

COROLLARY 2. *Let $f \in A_0$ and $\Phi$ be analytic in a domain containing $f(U)$, $\Phi(0) = 0$, $\Phi'(0) = 1$, $\Phi(w) \neq 0$ in $f(U)\setminus\{0\}$. If*

$$1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{\Phi(f(z))} - \frac{z[\Phi(f(z))]'}{\Phi(f(z))} \prec \frac{1+z}{1-z} + \frac{2z}{1-z^2}$$

*then $f$ is $\Phi$-like.*

In particular, we have by (7):

$$\left|\frac{f''(z)}{f'(z)} + \frac{f'(z)}{\Phi(f(z))} - \frac{[\Phi(f(z))]'}{\Phi(f(z))}\right| < 2 \Rightarrow \text{Re}\, \frac{zf'(z)}{\Phi(f(z))} > 0,\ z \in U.$$

and by (8):

$$\left|\text{Im}\left\{\frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{\Phi(f(z))} - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}\right\}\right| < \sqrt{3} \Rightarrow \text{Re}\, \frac{zf'(z)}{\Phi(f(z))} > 0,\ z \in U.$$

If we take $\Phi(w) = e^{-i\gamma} w$, $-\frac{\pi}{2} < \gamma < \frac{\pi}{2}$, in theorem 1. and in corollary 1 and 2 we obtain the following sufficient condition of spirallikeness.

COROLLARY 3. *Let $f \in A$, $\alpha$ be a complex and $\gamma$ be a real number, $|\gamma| < \frac{\pi}{2}$ and $q$ be convex (univalent) in $U$ with $q(0) = e^{i\gamma}$ and* Re $[q(z)/\alpha] > 0$,

$z \in U$. *If* $Q(z) = \alpha \frac{zq'(z)}{q(z)}$ *is starlike (univalent) in* $U$ *and if*

$$J_\gamma(\alpha, f(z)) = (e^{i\gamma} - \alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec q(z) + \alpha\frac{zq'(z)}{q(z)} = G(z) \tag{9}$$

*then* $f$ *is* $\gamma$*-spirallike with respect to* $q$.

If we let $q(z) = \frac{e^{i\gamma} + e^{-i\gamma}z}{1-z}$ with $\gamma$ real and $|\gamma| < \frac{\pi}{2}$ in corollary 3, $q$ is convex, $q(0) = e^{i\gamma}$, and maps the unit disc $U$ onto the half plane $\mathrm{Re}\, w > 0$

$$G(z) = \frac{e^{i\gamma} + e^{-i\gamma}z}{1-z} + \alpha\left(\frac{e^{-i\gamma}z}{e^{i\gamma} + e^{-i\gamma}z} + \frac{z}{1-z}\right) = J_\gamma(\alpha, k_\gamma(z))$$

where $k(z) = \frac{z}{(1-z)^{1+e^{-2i\gamma}}}$ is the Koebe $\gamma$-spirallike function. $G(U)$ is the complex plane slit along the half-lines $\mathrm{Re}\, w = 0$,

$$\mathrm{Im}\, w \geqslant (\sqrt{\alpha(2\cos\gamma + \alpha)} - \alpha\sin\gamma)/\cos\gamma$$

and

$$\mathrm{Re}\, w = 0, \mathrm{Im}\, w \leqslant -(\sqrt{\alpha(2\cos\gamma + \alpha)} + \alpha\sin\gamma)/\cos\gamma.$$

By using corollary 3. we obtain the following result:

COROLLARY 4. *Let* $f \in A$, $\alpha > 0$ *and* $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. *Then*

$$J_\gamma(\alpha, f(z)) \prec J_\gamma(\alpha, k_\gamma(z)) \Rightarrow g \in SP_\gamma, \tag{10}$$

*i.e.*

$$J_\gamma(\alpha, f(z)) \prec J_\gamma(\alpha, k_\gamma(z)) \Rightarrow J_\gamma(0, f(z)) \prec J_\gamma(0, k_\gamma(z)).$$

In particular

$$f \in A,\ \alpha > 0,\ |J_\gamma(\alpha, f(z)) - (1 - i\alpha\,\mathrm{tg}\,\gamma)| < 1 + \frac{\alpha}{\cos\gamma},$$

$$z \in U \Rightarrow f \in SP_\gamma,\ \gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

and

$$f \in A,\ \alpha > 0,\ |\mathrm{Im}\ J_\gamma(\alpha, f(z)) + \alpha\,\mathrm{tg}\,\gamma| < \frac{\sqrt{\alpha(2\cos\gamma + \alpha)}}{\cos\gamma},$$

$$z \in U \Rightarrow f \in SP_\gamma,\ \gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

For $\gamma = 0$ these conditions are starlike conditions and was obtained by P. T. Mocanu in [5].

THEOREM 2. *Let* $f \in A_0$, *let* $\Phi$ *be analytic in a domain containing* $f(U)$, $\Phi(0) = 0$, $\Phi'(0) = \frac{1}{c}$, $\mathrm{Re}\, c > 0$ *and* $\Phi(w) \neq 0$ *in* $f(U)\setminus\{0\}$ *and let* $\alpha$ *be a real*

*number*
$$\operatorname{Re}\left\{\frac{zf'(z)}{\Phi(f(z))}+\alpha\left[1+\frac{zf''(z)}{f'(z)}-\frac{z[\Phi(f(z))]'}{\Phi(f(z))}\right]\right\}>0,\ z\in U, \tag{11}$$

*then* $\operatorname{Re}\frac{zf'(z)}{\Phi(f(z))}>0,\ z\in U,$ *i.e.*

$$\operatorname{Re} J_\Phi(\alpha, f(z))>0,\ z\in U\Rightarrow \operatorname{Re} J_\Phi(0, f(z))>0,\ z\in U.$$

*Proof.* If we let $p(z)=\frac{zf'(z)}{\Phi(f(z))}$, $p$ is analytic in $U$ with $p(0)=c$, $\operatorname{Re} c>0$ and (11) can be rewritten as

$$\operatorname{Re}\left[p(z)+\alpha\frac{zp'(z)}{p(z)}\right]>0 \tag{12}$$

which implies $\operatorname{Re} p(z)=\operatorname{Re}\frac{zf'(z)}{\Phi(f(z))}>0$ in $U$.

In the particular cases $\Phi(w)=w$ this theorem was proved by S. S. Miller, P. T. Mocanu and M. O. Reade in [4].

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# MAXIMUM PRINCIPLES FOR SOME DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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**REZUMAT. — Principii de maxim pentru ecuaţii diferenţiale cu argument modificat.** În prezenta lucrare se stabilesc principii de maxim pentru ecuaţii diferenţiale cu argument modificat, după care se dă o aplicaţie referitoare la mulţimea valorilor proprii ale unui probleme de valori proprii pentru o ecuaţie diferenţială cu argument modificat.

1. **Introduction.** Let us consider the following second order differentia operator with deviating arguments

$$L(y)(x) := y''(x) + p(x)y'(x) + q(x)y(x) + \sum_{i=1}^{m} q_i(x)y(g_i(x)),$$

for all $x \in [a, b]$; where $p, q, q_i, g_i \in C[a, b]$, $i = \overline{1, m}$. Let $a_1, b_1 \in \mathbf{R}$ be such that $a_1 \leqslant a$, $b \leqslant b_1$, $a_1 \leqslant g_i(x) \leqslant b_1$, for all $x \in [a, b]$.

The object of this paper is to establish maximum principles and minimum principles for the solutions of the following differential inequalities with peviating arguments:

$$L(y) \geqslant 0 \tag{1}$$

$$L(y) > 0 \tag{2}$$

$$L(y) = 0 \tag{3}$$

We follow terminologies and notations from [6] and [7].

2. **Maximum and minimum principles.** We have

THEOREM 1. (see [6]). *Let $y$ be a solution of* (1). *If*

$$q_i(x) \geqslant 0 \text{ and } q(x) + \sum_{i=1}^{m} q_i(x) < 0 \text{ for all } x \in \,]a, b[ \tag{4}$$

*then $y$ satisfies the maximum principle.*

THEOREM 2. (see [6]). *Let $y$ be a solution of* (3). *If $L$ satisfies* (4), *then $y$ satisfies the maximum principle and the minimum principle.*

THEOREM 3. (see [6]). *Let $y$ be a solution of* (2). *If*

$$q_i(x) \geqslant 0 \text{ and } q(x) + \sum_{i=1}^{m} q_i(x) \leqslant 0 \text{ for all } x \in \,]a, b[ \tag{5}$$

*then $y$ satisfies the maximum principle.*

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Now, we have

THEOREM 4. *Let $y$ be a solution of* (3). *We assume that*

$$q_i(x) \geqslant 0 \quad and \quad q(x) + \sum_{i=1}^{m} q_i(x) \leqslant 0 \quad for \ all \quad x \in ]a, b[ \tag{6}$$

*If there exists $M \geqslant 0$ such that $y(x) \leqslant M$, for all $x \in [a, b]$ and there exists $c \in ]a, b[$ such that $y(c) = M$, then $y(x) = M$ for all $x \in [a, b]$.*

*Proof.* Suppose the contrary: there exists $d \in ]a, b[$ such that $y(d) < M$. We shall prove that this assumption leads us to a contradiction.

(i) The case $d > c$. Let us consider the function

$$z(x) = e^{\alpha(x-c)} - 1$$

with $\alpha > 0$ to be chosen suitable. We have

$$z(x) < 0, \text{ for all } x \in ]a, c[,$$
$$z(c) = 0,$$
$$z(x) > 0, \text{ for all } x \in ]c, b[,$$

and

$$L(z)(x) = [\alpha^2 + \alpha p(x) + q(x)(1 - e^{-\alpha(x-c)})] \, e^{\alpha(x-c)} + \sum_{i=1}^{m} q_i(x) \, e^{\alpha(g_i(x)-c)} > 0$$

for $\alpha$ sufficiently large, from (6).

Let $w(x) = y(x) + \varepsilon z(x)$, where $0 < \varepsilon < \dfrac{M - y(d)}{z(d)}$.

We have

$$w(x) < M, \text{ for all } x \in ]a, c[,$$
$$w(d) = y(d) + \varepsilon z(d) < y(d) + M - y(a) = M,$$

because $z(d) > 0$, and $w(c) = M$.

Therefore we have a maximum larger then $M$ in the interior of $]a, d[$.

But

$$L(w)(x) = L(y)(x) + \varepsilon L(z)(x) = \varepsilon L(z)(x) > 0,$$

for $\alpha$ chosen before, and

$$q_i(x) \geqslant 0, \ q(x) + \sum_{i=1}^{m} q_i(x) \leqslant 0 \text{ on } ]a, d[ \subseteq ]a, b[.$$

This reprezents a contradiction with Theorem 3, so Theorem 4 is proved in the case (i).

(ii) The case $d < c$. We make a similary argument for the function $z(x) = e^{-\alpha(x-c)} - 1$.

The following result generalises the maximum principle and the minimum principle.

THEOREM 5. *Let $y$ be a solution of* (3), *where $L$ satisfies $q_i(x) \geqslant 0$, for all $x \in ]a, b[$, $i = \overline{1, m}$. If there exists a function $w \in C[a_1, b_1] \cap C^2[a, b]$ with*

$$w(x) > 0, \text{ for all } x \in [a, b] \tag{7}$$

$$L(w)(x) < 0, \text{ for all } x \in ]a, b[ \tag{8}$$

*then the function $y/w$ satisfies the maximum and the minimum principle.*

*Proof.* Let $v = \frac{y}{w}$. We have

$$L(y)(x) = L(vw)(x) = w(x)v''(x) + (2w'(x) + p(x)w(x))v'(x) + (w''(x) +$$

$$+ p(x)w'(x) + q(x)w(x))v(x) + \sum_{i=1}^{m} q_i(x)w(g_i(x))\, v(g_i(x)) = 0.$$

Dividing by $w > 0$ we obtain

$$v''(x) + 2\left(\frac{w'(x)}{w(x)} + p(x)\right)v'(x) + \left(\frac{w''(x)}{w(x)} + p(x)\frac{w'(x)}{w(x)} + q(x)\right)v(x) +$$

$$+ \sum_{i=1}^{m} q_i(x)\frac{w(g_i(x))}{w(x)} v(g_i(x)) = 0.$$

But $q_i(x) \geqslant 0$, for all $x \in ]a, b[$, $i = \overline{1, m}$, and because of (7), (8) we may apply Theorem 2. Hence the function $v = \frac{y}{w}$ satisfies the maximum and the minimum principle.

3. **Eigenvalue problem.** Let us consider the following eigenvalue problem corresponding to a boundary value problem for a differential equation with deviating arguments

$$y''(x) + p(x)y'(x) + [q(x) + \lambda q_2(x)]\, y(x) + [q_1(x) + \lambda q_3(x)]\, y(g(x)) = 0 \tag{9}$$

for all $x \in [a, b]$

$$\begin{aligned} y(x) &= 0, \quad x \in [a_1, a], \\ y(x) &= 0, \quad x \in [b, b_1]. \end{aligned} \tag{10}$$

where $p, q, q_1, q_2, q_3, g \in C[a, b]$.

DEFINITION 1. Any real number for which there exists an untrivial solution of (9) + (10) in called *eigenvalue* of the problem (9) + (10).

Assume that there exists a positive number $\eta$ such that $q_2(x) \geqslant \eta > 0$ and $q_3(x) \geqslant \eta > 0$ for all $x \in [a, b]$.
We have

THEOREM 6. *In these conditions, any eigenvalue of the problem* (9) + (10) *is contained in*

$$\left(-\infty, \max_{a \leqslant x \leqslant b}\left\{-\frac{q_1(x)}{q_3(x)}\right\}\right) \cup \left(\min_{a \leqslant x \leqslant b}\left\{-\frac{q(x) + q_1(x)}{q_2(x) + q_3(x)}\right\}, +\infty\right).$$

*Proof.* If an eigenvalue $\lambda$ satisfies

$$\lambda \geqslant \max_{a \leqslant x \leqslant b} \left\{-\frac{q_1(x)}{q_3(x)}\right\} \text{ and } \lambda \leqslant \min_{a \leqslant x \leqslant} \left\{-\frac{q(x) + q_1(x)}{q_2(x) + q_3(x)}\right\},$$

that is

$q_1(x) + \lambda q_3(x) \geqslant 0$ and $q(x) + q_1(x) + \lambda[q_2(x) + q_3(x)] \leqslant 0$ for all $x \in [a, b]$, we are in the conditions of Theorem 2, so any solution $y$ of (9) + (10) satisfies the maximum and the minimum principle. That means that $y$ takes his positive maximums and his negative minimums in $[a_1, a] \cup [b, b_1]$. Hence $y = 0$.

It results that such a $\lambda$ cannot be an eigenvalue for (9) + (10).

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# CONTINUOUS SELECTIONS FOR MULTIFUNCTIONS SATISFYING THE CARATHEODORY TYPE CONDITIONS. THE PICARD PROBLEM ASSOCIATED TO A MULTIVALUED EQUATION

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**REZUMAT. — Selecţii continue pentru multifuncţii care satisfac condiţii de tip Carathéodory. Problema lui Picard asociată unei ecuaţii multivoce.** Se demonstrează o teoremă de existenţă a unei selecţii continue pentru fiecare din aplicaţiile $(x, y) \to F(x, y, z(x, y))$, unde $F$ este o multifuncţie definită pe o mulţime compactă din $\mathbf{R}^{n+2}$ cu valori-mulţimi compacte, neconvexe din $\mathbf{R}^n$, relativ la o familie dată de funcţii continue $(x, y) \to z(x, y)$, $(x, y) \in D = [0, a] \times [0, b]$, $F$ satisfăcind condiţii de tip Carathéodory. Folosind acest rezultat se demonstrează o teoremă de existenţă a unei soluţii absolut continue pentru problema lui Picard asociată ecuaţiei multivoce $\frac{\partial^2 z}{\partial x \partial y} \in F(x, y, z)$.

1. **Introduction.** By analogy to [1], in this note one proves an existence theorem for a continuous selection for each of the functions $(x, y) \to F(x, y, z(x, y))$, where $F$ is a multifunction defined on a compact set in $\mathbf{R}^{n+2}$ and valued in the set of nonconvex compact sets in $\mathbf{R}^n$, relatively to a given family of continuous functions $(x, y) \to z(x, y)$, $F$ satisfying the Carathéodory type conditions. The main result (Theorem 2), is an extension of Theorem 1 [2], where $F$ is a continuous multifunction. The results in [1] remain true in this case. As consequence, one obtains an existence theorem of an absolutely continuous solution for the Picard problem associated to the multi-valued equation $\frac{\partial^2 z}{\partial x \partial y} \in F(x, y, z)$, [3], [4].

2. **Preliminaries.** We recall the main notations and results given in [2]. Let be the multifunction $F : D \times B \to \text{comp } X$, $D = [0, a] \times [0, b]$, $B \subset \mathbf{R}^n$ is the closed ball centered in origin and of radius $c = M_1 + Mab$, $M_1$ given by (3), $M$ given by (4), and $X \subset \mathbf{R}^n$ is the closed ball centered in origin and of radius $M$. Obviously, $X$ is a compact space in the metric $d$ induced on $X$ by the norm of $\mathbf{R}^n$. The family of nonempty compact sets in $X$, denoted comp$X$, is a compact metric space for the Hausdorff metric $H$ on comp$X$, induced by the metric $d$.

Let $\mathcal{C}(D; \mathbf{R}^n)$ the Banach space of continuous functions from $D$ to $R^n$ and $\mathcal{L}^1(D; \mathbf{R}^n)$ be the Banach space of equivalence classes of Lebesgue integrable functions defined on $D$ and valued in $\mathbf{R}^n$.

Let the following hypotheses be satisfied:

$(H_0)$ The curve $\gamma : x = \psi(y)$, $0 \leqslant y \leqslant b$, is defined by the function $\psi \in C^1([0, b]; \mathbf{R})$, satisfying the conditions

$$\psi(0) = 0, \ 0 \leqslant \psi(y) \leqslant a, \ 0 \leqslant y \leqslant b. \tag{1}$$

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$(H_1)$ The functions $P \in AC([0, a]; \mathbf{R}^n)$, $Q \in AC([0, b]; \mathbf{R}^n)$, where $AC([\alpha_1, \alpha_2]; \mathbf{R}^n)$ is the space of absolutely continuous functions $f: [\alpha_1, \alpha_2] \to \mathbf{R}^n$, endowed with the norm

$$\|f\| = \sup_{t \in [\alpha_1, \alpha_2]} \|f(t)\| + \int_{\alpha_1}^{\alpha_2} \|f'(t)\| \, dt,$$

satisfy the condition $P(0) = Q(0)$.

$(H_2)$ The function $\alpha: D \to \mathbf{R}^n$ defined by

$$\alpha(x, y) = P(x) + Q(y) - P(\psi(y)), \ (x, y) \in D, \tag{2}$$

is bounded and therefore, there is $M_1 > 0$ such that

$$\|\alpha(x, y)\| \leq M_1, \ (x, y) \in D. \tag{3}$$

It follows that $\alpha$ is absolutely continuous on $D$; $\alpha \in C^*(D; \mathbf{R}^n)$, [5, §565—§568].

Let $K$ be the set of absolutely continuous functions $z: D \to \mathbf{R}^n$, $z \in C^*(D; \mathbf{R}^n)$, satisfying (3), (4), (5), where

$$\left\| \frac{\partial^2 z(x, y)}{\partial x \partial y} \right\| \leq M, \text{ a.e. } (x, y) \in D, \tag{4}$$

and

$$\begin{cases} z(x, 0) = P(x), \ 0 \leq x \leq a, \\ z(\psi(y), y) = Q(y), \ 0 \leq y \leq b. \end{cases} \tag{5}$$

PROPOSITION 1. *The set $K$ is a nonempty, convex and compact subset of* $\mathcal{C}(D; \mathbf{R}^n)$.

Denote $D_0(x, y)$ the rectangle given by

$$D_0(x, y) = \{(u, v) \mid \psi(y) \leq u \leq x, \ 0 \leq v \leq y\} \subset D; \ (x, y) \in D.$$

Integrating $\frac{\partial^2 z(x, y)}{\partial x \partial y}$ over $D_0(x, y)$, which exists a.e. $(x, y) \in D$, as $z \in C^*(D; \mathbf{R}^n)$, [5], and using (2) one obtains

$$z(x, y) = P(x) + Q(y) - P(\psi(y)) + \int_0^y dv \int_{\psi(y)}^x \frac{\partial^2 z(u, v)}{\partial u \partial v} du =$$

$$= \alpha(x, y) + \iint_{D_0(x, y)} \frac{\partial^2 z(u, v)}{\partial u \partial v} du dv, \ (x, y) \in D. \tag{6}$$

*Remark.* The relation $z \in K$ implies $(x, y, z(x, y)) \in D \times B$ for each $(x, y) \in D$. Therefore, each $z \in K$ generates a map $(x, y) \to F(x, y, z(x, y))$ of $D$ into comp$X$, denoted $G(z)$. This function $G$ is associated to the multifunction $F$ by the relation

$$G(z)(x, y) = F(x, y, z(x, y)), \ (x, y) \in D. \tag{7}$$

The main theorem in [2] on the existence of continuous selection is the following:

THEOREM 1. *If* $F: D \times B \to \operatorname{comp} X$ *is a continuous multifunction, then there exists a continuous function* $g: K \to \mathfrak{L}^1(D; \mathbf{R}^n)$ *such that, for every* $z \in K$, $g(z)(x, y) \in G(z)(x, y)$, *a.e.* $(x, y) \in D$.

**3. Continuous selections. The measurable case.** The Theorem 1 can be extended to multifunctions defined on $D \times B$ and valued in $\operatorname{comp} X$, and satisfying the Carathéodory type conditions, similar to [1].

THEOREM 2. *Suppose* $F: D \times B \to \operatorname{comp} X$ *satisfies the following hypotheses:*

*a) for each* $z \in B$, $(x, y) \to F(x, y, z)$ *is measurable in* $D$,

*b) for each* $(x, y) \in D$, $z \to F(x, y, z)$ *is continuous in* $B$.

*Then, there exists a continuous function* $g: K \to \mathfrak{L}^1(D; \mathbf{R}^n)$ *such that, for each* $z \in K$, $g(z)(x, y) \in G(z)(x, y)$, *a.e.* $(x, y) \in D$.

*Proof.* One uses the proof of Theorem 1, which rests upon the following two properties of the multifunction $F: D \times B \to \operatorname{comp} X$, which are direct consequences of the assumed continuity:

1° the family $\{F(x, y, .)\}, (x, y) \in D$ of maps of $B$ into $\operatorname{comp} X$ is uniformly equicontinuous;

2° for each $z \in \mathcal{C}(D; \mathbf{R}^n)$ the map $G(z)(x, y) \to F(x, y, z(x, y))$ is measurable in $D$.

We shall show that the hypotheses of Theorem 2 induce in $F$ essentially the same properties. It is a consequence of the following two lemmas.

The Lemma 1 is a similar result of the theorems of Scorza—Dragoni type for multifunctions, [1], [6], [7].

LEMMA 1. *Suppose* $F: D \times B \to \operatorname{comp} X$ *satisfies the hypotheses of Theorem 2. Then, for each* $\varepsilon > 0$, *there exists a closed set* $E \subset D$ *with* $\mu(D - E) \leqslant \varepsilon$, *such that the family* $\{F(x, y, .)\}$, $(x, y) \in E$, *of maps of* $B$ *into* $\operatorname{comp} X$ *is uniformly equicontinuous.*

*Proof.* Let be the functions $\delta_n: D \to \mathbf{R}$, $n \geqslant 1$, defined by

$$\delta_n(x, y) = \sup\left\{\rho > 0 \,;\, \| u - v \| < \rho \Rightarrow H(F(x, y, u), F(x, y, v)) < \frac{1}{n},\ u, v \in B\right\}. \quad (8)$$

To verify the conclusion of this lemma, one shows that for each $\varepsilon > 0$, there exists a closed set $E \subset D$, with $\mu(D - E) \leqslant \varepsilon$, such that the restriction $\delta_n/E$ is continuous.

Let $A = \{z_m\}_{m \geqslant 1}$ be a dense subset of $B$. The hypothesis *a)* implies the existence of a closed set $L \subset D$ with $\mu(D - L) \leqslant \frac{\varepsilon}{2}$ such that the restriction to $L$ of every function $(x, y) \to F(x, y, z_m)$ is continuous. Then, each restriction $\delta_n/L$ is upper semicontinuous, hence measurable.

Indeed, suppose, by reductio ad absurdum, that for (fixed) $n \geqslant 1$ there exists a point $(x_0, y_0) \in L$, a constant $\nu > 0$, and a sequence $\{(x_k, y_k)\}_{k \geqslant 1}$ convergent to $(x_0, y_0)$ in $D$, such that

$$\delta_n(x_k, y_k) \geqslant \delta_n(x_0, y_0) + \nu, \text{ for all } k \geqslant 1. \quad (9)$$

Let $v_1$, $v_2$ be points in $B$ with

$$\delta_n(x_0, y_0) \leqslant \|v_1 - v_2\| \leqslant \delta_n(x_0, y_0) + \frac{3}{4}\nu, \tag{10}$$

such that, for some $\Delta > 0$,

$$H(F(x_0, y_0, v_1), F(x_0, y_0, v_2)) = \frac{1}{n} + \Delta. \tag{11}$$

Then, there exists points $z_{m_1}$, $z_{m_2}$ in $A$ and a point $(x_k, y_k) \in L$ such that

$$\|v_1 - z_{m_1}\| < \frac{\nu}{8}, \ \|v_2 - z_{m_2}\| < \frac{\nu}{8}. \tag{12}$$

Also, by hypothesis *a*) we have

$$\begin{cases} H(F(x_0, y_0, v_1), F(x_0, y_0, z_{m_1})) < \frac{\Delta}{8} \\ H(F(x_0, y_0, v_2), F(x_0, y_0, z_{m_2})) < \frac{\Delta}{8}, \end{cases} \tag{13}$$

and

$$\begin{cases} H(F(x_0, y_0, z_{m_1}), F(x_k, y_k, z_{m_1})) < \frac{\Delta}{8} \\ H(F(x_0, y_0, z_{m_2}), F(x_k, y_k, z_{m_2})) < \frac{\Delta}{8}. \end{cases} \tag{14}$$

According to (9), (10), (12) one obtains

$$\|z_{m_1} - z_{m_2}\| \leqslant \|z_{m_1} - v_1\| + \|v_1 - v_2\| + \|v_2 - z_{m_2}\| < \frac{\nu}{4} + \\ + \|v_1 - v_2\| \leqslant \delta_n(x_0, y_0) + \nu \leqslant \delta_n(x_k, y_k). \tag{15}$$

The relations (8) and (15) imply

$$H(F(x_k, y_k, z_{m_1}), F(x_k, y_k, z_{m_2})) < \frac{1}{n}. \tag{16}$$

Then, from (13), (14), (16) one obtains

$$H(F(x_0, y_0, v_1), F(x_0, y_0, v_2)) \leqslant H(F(x_0, y_0, v_1), F(x_0, y_0, z_{m_1})) + \\ + H(F(x_0, y_0, z_{m_1}), F(x_k, y_k, z_{m_1})) + H(F(x_k, y_k, z_{m_1}), F(x_k, y_k, z_{m_2})) + \\ + H(F(x_k, y_k, z_{m_2}), F(x_0, y_0, z_{m_2})) + H(F(x_0, y_0, z_{m_2}), F(x_0, y_0, v_2)) < \\ < H(F(x_k, y_k, z_{m_1}), F(x_k, y_k, z_{m_2})) + \frac{1}{2}\Delta < \frac{1}{n} + \frac{1}{2}\Delta, \tag{17}$$

which is in contradiction to (11).

It follows that there exists a closed set $E \subset L$ with $\mu(E - L) \leqslant \frac{\varepsilon}{2}$ such that each restriction $\delta_n/E$ is continuous. Obviously, by construction, $\mu(D - E) \leqslant \varepsilon$.

LEMMA 2. *Suppose* $F : D \times B \to \text{comp}X$ *satisfies the hypotheses of Theorem 2. Then, for each* $z \in \mathcal{C}(D; \mathbf{R}^n)$, $G(z) : (x, y) \to F(x, y, z(x, y))$ *is measurable in* $D$.

*Proof.* Let $z \in \mathcal{C}(D; \mathbf{R}^n)$ be given, and let $\{z_n\}_{n\geqslant 1}$ be a sequence of piecewise constant maps in $D$ that converges to $z$ uniformly in $D$. We need only show that, for every $\varepsilon > 0$, there exists a closed set $E_0 \subset D$ with $\mu(D - E_0) \leqslant \leqslant \varepsilon$ such that $G(z)_{/E_0}$ is continuous.

Taking into account Lemma 1, there is a closed set $E \cup D$ with $\mu(D - E) \leqslant \frac{\varepsilon}{2}$ such that the sequence $\{G(z_n)\}_{n\geqslant 1}$ converges to $G(z)$ uniformly in $E$. Since each $G(z_n)$ is measurable in $D$ and hence in $E$, there exists a closed set $E_0 \subset E$ with $\mu(E - E_0) \leqslant \frac{\varepsilon}{2}$ such that each restriction $G(z_n)/E_0$ is continuous. This implies that $G(z)/E_0$ is continuous.

To prove the Theorem 2, one shows an increasing sequence $\{E_n\}_{n\geqslant 1}$ of closed sets $E_n \subset D$ with $\mu(D - E_n) \leqslant 2^{-n}$ such that each restriction $F/E_n \times B$ is continuous and one defines for each $n \geqslant 1$,

$$F^n(x, y, z) = \begin{cases} F(x, y, z), & (x, y, z) \in E_n \times B \\ 0, & (x, y, z) \in D - E_n \times B. \end{cases}$$

Then, as a consequence of Theorem 1, there exists, for every $n \geqslant 1$ a continuous map $g^n : K \to \mathfrak{L}^1(D; \mathbf{R}^n)$ such that for each $z \in K$, $g^n(z)(x, y) \in \in F(x, y, z(x, y))$ a.e. $(x, y) \in E_n$.

Let $A_1 = E_1$ and $A_{n+1} = E_{n+1} - E_n$ for every $n \geqslant 1$ so that $E_n = \bigcup_{k=1}^{n} A_k$ and define $g(z)$, for each $z \in K$, by setting

$$g(z)/A_n = g^n(z) \text{ and } g(z)/D - \bigcup_{n=1}^{\infty} A_n = 0.$$

Obviously, $g$ maps $K$ into $\mathfrak{L}^1(D; \mathbf{R}^n)$ and, for each $z \in K$, $g(z)(x, y) \in F(x, y, z(x, y))$ a.e. $(x, y) \in D$. Also, $g : K \to \mathfrak{L}^1(D; \mathbf{R}^n)$ is continuous since, for any $z \in K$ and $w \in K$

$$\int_D \| g(z)(x, y) - g(w)(x, y) \| \, dxdy = \int_{D-E_n} \| g(z)(x, y) - g(w)(x, y) \| \, dxdy +$$

$$+ \int_{E_n} \| g(z)(x, y) - g(w)(x, y) \| \, dxdy \leqslant 2^{-n+1} M +$$

$$+ \sum_{k=1}^{n} \int_{A_k} \| g^k(z)(x, y) - g^k(w)(x, y) \| \, dxdy$$

whatever $n \geqslant 1$ and each $g^k$ is continuous, by construction.

**4. The Picard problem associated to a multi-valued equation.** Consider the multi-valued equation

$$\frac{\partial^2 z}{\partial x \partial y} \in F(x, y, z), \quad (x, y) \in D, \quad z \in B, \tag{18}$$

where $F: D \times B \to \text{comp}X$.

The Picard problem associated to the equation (18) is defined in [3], [4] and consist in the determination of an absolutely continuous function satisfying (18) and (5) in the hypotheses $(H_0)$, $(H_1)$, $(H_2)$.

As in [2] we establish the following theorem:

THEOREM 3. *Suppose $F: D \times B \to \text{comp}X$ satisfies the hypotheses a), b) of the Theorem 2. If the hypotheses $(H_0)$, $(H_1)$, $(H_2)$ are fulfilled, the Picard problem (18) + (5) has at least an absolutely continuous solution $\bar{z}: D \to \mathbf{R}^n$, $\bar{z} \in C^*(D; \mathbf{R}^n)$.*

The proof uses the Theorem 2 and is similar to that given for Theorem 2 [2].

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# ON A CLASS OF OPERATORS OF APPROXIMATION OF FUNCTIONS OF TWO VARIABLES

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**REZUMAT. — Asupra unei clase de operatori de aproximare a funcţiilor de două variabile.** Folosind calculul umbral şi şirurile polinomiale de tip binomial, corespunzătoare la doi delta operatori $P$ şi $Q$, se construieşte un operator $L^{P,Q}_{m,n}$, util în aproximarea funcţiilor continue de două variabile. Acest operator, ataşat delta operatorilor $P$ şi $Q$, reprezintă o extindere bidimensională a unui operator introdus recent de către L. Lupaş şi A. Lupaş [2]. Considerînd cazul cînd aceşti operatori sînt de tip pozitiv, se studiază convergenţa şirului $(L^{P,Q}_{m,n})$ către funcţia $f$ continuă pe patratul unitate $\Omega$ şi se evaluează ordinul de aproximare cu ajutorul modulului de continuitate. Se dau două exemple ilustrative.

**1.** In a recent paper [2] L. Lupaş and A. Lupaş have considered an approximation operator $L_m: C[0, 1] \to C[0, 1]$, defined — for any $m \in \mathbf{N}$, by the formula

$$(L_m f)(x) = \frac{1}{p_m(m)} \sum_{k=0}^{m} \binom{m}{k} p_k(mx)\, p_{m-k}(m - mx)\, f\left(\frac{k}{m}\right), \tag{1}$$

where $(p_m)$ represents a sequence of polynomials of binomial type, characterized by the fact that $p_m(x)$ is exactly of degree $m$ in $x$, for any $m \in \mathbf{N}$, and the following identities are satisfied

$$p_m(x + y) = \sum_{k=0}^{m} \binom{m}{k} p_k(x)\, p_{m-k}(y).$$

It should be mentioned that similar liniar operators of approximation have been considered earlier by G. Moldovan [4] and C. Manole [3].

Concerning the polynomials of binomial type, we notice that a detailed exposition of them has been presented by Gian—Carlo Rota and his collaborators in [5], where they have used the so-called "umbral" or "symbolic" calculus.

**2.** An important tool in the algebraic approach to the special polynomials occuring in combinatorics, theory of enumeration and in approximation theory is the so-called *delta operator*, usually denoted by a letter $Q$. This is a shift-invariant operator, that is it commutes with all shift operators: $QE^a = E^aQ$, $a \in \mathbf{R}$, $E^a$ being the shift operator, characterized by the fact that it translates the argument of a polynomial by $a$. If $\alpha$ is a constant, then we have $Q\alpha = 0$ and if $p$ is a polynomial of degree $m$, then $Qp$ is a polynomial of degree $m - 1$.

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The delta operators have analogous properties with the derivative operator $D$.

A polynomial sequence $(p_m)$ is called the *sequence of basic polynomials* for the delta operator $Q$ if: $p_0(x) = 1$, $p_m(0) = 0$ $(m > 0)$ and $(Qp_m)(x) = mp_{m-1}(x)$. In [5] there has been proved that every delta operator has a unique sequence of basic polynomials associated with it and that if $(p_m)$ is a basic sequence of some delta operator $Q$, then it is a sequence of polynomials of binomial type; conversely, if $(p_m)$ is a sequence of polynomials of binomial type, then it is a basic sequence for some delta operator.

Let us denote by $M$ the multiplication operator, i.e., $(Mp)(t) = tp(t)$. One defines the *Pincherle derivative* of a shift-invariant operator $T$ by the formula $T' = TM - MT$. One verifies immediately that $T'$ is also a shift-invariant operator. The following result is known: An operator $U$ is a delta operator if and only if $U = DV$ for some shift-invariant operator $V$, where the inverse operator $V^{-1}$ exists, $D$ being the derivative operator.

Let us denote by $e_k$ the function defined by $e_k(t) = t^k$ $(k = 0, 1, 2, \ldots)$. For computing a given delta operator, applied to a certain function, one can use a result from the paper [5]:

*If $(p_m)$ is a sequence of basic polynomials for a delta operator $U = DV$, then for any $m \in \mathbf{N}$ we have:*

$$p_m = U'V^{-m-1}e_m = V^{-m}e_m - (V^{-m})'e_{m-1} = e_1V^{-m}e_{m-1} = e_1(U')^{-1}p_{m-1}.$$

3. Let $(p_m)$ and $(q_n)$ be the basic sequences of polynomials corresponding to two delta operators $P$ and $Q$, respectively. We assume that $p_m(m) \neq 0$ and $q_n(n) \neq 0$ for any $m, n \in \mathbf{N}$.

In this paper we consider a bivariate extension of the operator $L_m$ defined at (1) and we investigate its approximation properties for continuous functions on the unit square $\Omega = [0,1]^2$.

To any function $f: \Omega \to \mathbf{R}$ we associate the linear operator $L^{P,Q}_{m,n}$ defined, for any $m, n \in \mathbf{N}$, by

$$(L^{P,Q}_{m,n}f)(x, y) = \frac{1}{A_{m,n}} \sum_{j=0}^{n} w_{m,n,k,j}(x, y) f\left(\frac{k}{m}, \frac{j}{n}\right), \tag{2}$$

where

$$w_{m,n,k,j}(x, y) = \binom{m}{k}\binom{n}{j} p_k(mx)\, q_j(ny)\, p_{m-k}(m - mx)\, q_{n-j}(n - ny) \tag{3}$$

and

$$A_{m,n} = p_m(m)\, q_n(n). \tag{4}$$

In view of investigating the approximation properties of these operators, we consider next the couple $(v^P_m, w^Q_n)$, where

$$v^P_m = 1 - \frac{m(m-1)}{p_m(m)} (P'^{-2} p_{m-2})(m)$$

$$w^Q_n = 1 - \frac{n(n-1)}{q_n(n)} (Q'^{-2} q_{n-2})(n).$$

We shall say that the couple of operators $(P, Q)$ belongs to the class $(V, W)$ is and only if we have: (i) $P$ and $Q$ are delta operators; (ii) $p'_m(0) \geqslant 0$ and $q_n(0) \geqslant 0$ for $m, n = 1, 2, \ldots$; (iii) $\lim v_m^P = \lim w_n^Q = 0$ as $m, n \to \infty$.

Assuming that $L_{m,n}^{P,Q}$, defined at (2) — (4), are positive operators, for investigating their convergence properties on the space of continuous functions $C(\Omega)$, we need first to find the values of these operators for the test functions $e_{r,s}$, where $e_{r,s}(x, y) = x^r y^s$ for $0 \leqslant r + s \leqslant 2$. If we take into account the theorem of Gian — Carlo Rota, mentioned above, we find immediately that:

$$L_{m,n}^{P,Q} e_{r,s} = e_{r,s} \ (r, s = 0, 1)$$

and

$$(L_{m,n}^{P,Q} e_{2,0})(x, y) = x^2 + x(1 - x)v_m^P,$$

$$(L_{m,n}^{P,Q} e_{0,2})(x, y) = y^2 + y(1 - y)w_n^Q.$$

By using a result given in [2], we can see that if $L_{m,n}^{P,Q}$ are positive operators then the sequences $(v_m^P)$, $(w_n^Q)$ are bounded and for any $m, n = 1, 2, \ldots$ we have: $0 \leqslant v_m^P \leqslant 1$, $0 \leqslant w_n^Q \leqslant 1$; on the other hand, if $(P, Q) \in (V, W$ then $L_{m,n}^{P,Q}$ are positive operators.

We are now in position to formulate the following

THEOREM 1. *If* $(P, Q) \in (V, W)$ *and* $f \in C(\Omega)$, *then we have*

$$\lim_{m,n \to \infty} L_{m,n}^{P,Q} f = f,$$

*uniformly on* $\Omega$.

For proving this result we can use the classical criterion of uniform convergence of Bohman—Korovkin, from the theory of linear positive operators, taking into consideration that

$$\lim_{m,n \to \infty} L_{m,n}^{P,Q} e_{r,s} = e_{r,s} (0 \leqslant r + s \leqslant 2),$$

uniformly on $\Omega$, because $e_{0,0}$, $e_{1,0}$, $e_{0,1}$, $e_{1,1}$ are fixed points of the operator $L_{m,n}^{P,Q}$ and from the definition of $(V, W)$ we have that $v_m^P \to 0$, $w_n^Q \to 0$ as $m, n \to \infty$.

4. In order to find quantitative estimates for the rapidity of convergence of the sequence $(L_{m,n}^{P,Q} f)$ to $f$, when $f \in C(\Omega)$, we shall use the modulus of continuity, defined by

$$\omega(f; \delta) = max\,|f(M'') - f(M')|,$$

when $M'(x', y')$ and $M''(x'', y'')$ are points from $\Omega$, and $d(M', M'') \leqslant \delta$ ($\delta \in \in \mathbf{R}_+$), with

$$d(M', M'') = \sqrt{(x'' - x')^2 + (y'' - y')^2}.$$

According to an inequality given in [1], in our case we have

$$||f - L_{m,n}^{P,Q} f|| \leqslant 2\omega(f; \mu_{m,n}),$$

where

$$\mu_{m,n} = \| L_{m,n}^{P,Q}((t-x)^2 + (\tau - y)^2;\ x, y) \|^{1/2},$$

with the notation

$$(L_{m,n}^{P,Q} f)(x, y) = L_{m,n}^{P,Q}(f;\ x,\ y).$$

If we take into account that

$$L_{m,n}^{P,Q}((t-x)^2;\ x, y) = x(1-x)v_m^P \leqslant \frac{1}{4} v_m^P,$$

$$L_{m,n}^{P,Q}((\tau-y)^2;\ x, y) = y(1-y)w_n^Q \leqslant \frac{1}{4} w_n^Q,$$

we obtain

$$\mu_{m,n} \leqslant \frac{1}{2}\sqrt{v_m^P + w_n^Q}.$$

Because $\omega(f;\ \lambda\delta) \leqslant (1+\lambda)\omega(f;\ \delta)$ $(\lambda > 0)$,
we arrive at a result included in

THEOREM 2. *If $(P, Q) \in (V, W)$ and $f \in C(\Omega)$, then the following Popoviciu-type inequality*

$$\|f - L_{m,n}^{P,Q} f\| \leqslant 3\omega(f; \sqrt{v_m^P + w_n^Q}) \tag{5}$$

*holds.*

*Illustrations.* (i) In the special case $P = Q = D$, one obtains the bidimensional-Bernstein operator $B_{m,n}$ for the unit square, when we have $v_n^D = \frac{1}{m}$, $w_n^D = \frac{1}{n}$ and the corresponding inequality is

$$\|f - B_{m,n} f\| \leqslant 3\omega\left(f;\ \sqrt{\frac{1}{m} + \frac{1}{n}}\right). \tag{6}$$

(ii) In the case when we choose the delta operators

$$P = \frac{1}{\alpha}(I - E^{-\alpha}),\ Q = \frac{1}{\beta}(I - E^{-\beta}) \qquad (\alpha, \beta \neq 0),$$

one arrives (see [3]) at the Stancu bidimensional linear positive operator [6]: $S_{m,n}^{\alpha,\beta}$, defined by

$$(S_{m,n}^{\alpha,\beta} f)(x, y) = \sum_{k=0}^{m} \sum_{j=0}^{n} p_{m,k}(x;\ \alpha)\, q_{n,j}(y;\ \beta)\, f\left(\frac{i}{m},\ \frac{j}{n}\right),$$

where

$$p_{m,k}(x;\ \alpha) = \binom{m}{k} \frac{x^{[k,\,-\alpha]}(1-x)^{[m-k,\,-\alpha]}}{1^{[m,\,-\alpha]}},$$

$$q_{n,j}(y;\ \beta) = \binom{n}{j} \frac{y^{[j,\,-\beta]}(1-y)^{[n-j,\,-\beta]}}{1^{[n,\,-\beta]}}.$$

Because in this case we have

$$v_m^P = \frac{1+\alpha m}{m+\alpha m}, \quad w_n^Q = \frac{1+\beta n}{n+\beta n},$$

we obtain the following inequality

$$\|f - S_{m,n}^{\alpha,\beta} f\| \leqslant 3\omega(f;\ \delta_{m,n}^{\alpha,\beta}), \tag{7}$$

where

$$\delta_{m,n}^{\alpha,\beta} = \left[\frac{1+\alpha m}{m+\alpha m} + \frac{1+\beta n}{n+\beta n}\right]^{1/2}.$$

A similar evaluation of the order of approximation has been given in [6], but using another distance $d$ between the points $M'$ and $M''$ of $\Omega$.

If we assume that $\alpha = \alpha(m) \to 0$ and $\beta = \beta(n) \to 0$ as $m, n \to \infty$, then from the inequality (7) there follows the uniform convergence of the sequence $(S_{m,n}^{\alpha,\beta} f)$ to the function $f \in C(\Omega)$. If $\alpha = \beta = 0$, then we have $S_{m,n}^{0,0} = B_{m,n}$ and (7) reduces to the inequality (6).

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# SOME CHARACTERIZATIONS OF INNER PRODUCT SPACES AND APPLICATIONS

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**REZUMAT. – Citeva caracterizări ale spaţiilor prehilbertiene şi aplicaţii.** În lucrare se stabilesc citeva caracterizări ale produselor scalare in clasa semi-produselor scalare de tip Lumer şi se aplică rezultatele obţinute la operatori definiţi pe spaţii liniare reale in dualitate.

1. **Introduction.** Let $X$ denote a linear space over the real field $\mathbf{R}$. In this paper we shall give some characterizations of inner product spaces in terms of semi-inner products in the sense of Lumer [3] by the use of some results established in [4] and [2]. Further, we shall give some applications for the operators on a dual system of real linear spaces.

1.1. DEFINITION. ([3], [1] pp. 389, [2]) A mapping $(,) : X \times X \to \mathbf{R}$ is called *semi-inner product in the sense of Lumer* or *L-semi-inner product*, for short, if the following conditions are satisfied:

(i) $(x + y, z) = (x, z) + (y, z),\ x, y, z \in X$;

(ii) $(\lambda x, y) = \lambda(x, y),\ \lambda \in \mathbf{R},\ x, y \in X$;

(iii) $(x, x) > 0$ if $x \neq 0$;

(iv) $|(x, y)|^2 \leqslant (x, x)(y, y),\ x, y \in X$;

(v) $(x, \lambda y) = \lambda(x, y),\ \lambda \in \mathbf{R},\ x, y \in X$.

We note that, the mapping $X \ni x \mapsto (x, x)^{1/2} \in \mathbf{R}_+$ is a norm on $X$ and every inner product is a $L$-semi-inner product on $X$.

For details concerning the properties of $L$-semi-inner products we send to [1], [2], [3]. Further, we list the properties of the functional (,) that will be used in the sequel.

1.2. THEOREM. ([1] pp. 389, [2]). *Let $(X, ||.||)$ be a normed linear space and (,) a L-semi-inner product which generates the norm $||.||$. Then $(X, ||\cdot||)$ is a smooth space iff (,) is continuous i.e.,*

$$\lim_{t \to 0} (y, x + ty) = (y, x),\ x, y \in X. \tag{C}$$

1.3. DEFINITION. ([5], [1] pp. 389, [2]). Let $(X, ||.||)$ be a real normed linear space and $f: X \to \mathbf{R}$, $f(x) := \frac{1}{2}\ ||x||^2,\ x \in X$. Then the mapping

$$(x, y)_T := \lim_{t \to 0+} [f(y + tx) - f(y)]/t,\ x, y \in X,$$

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is called *semi-inner product in the sense of Tapia* or *T-semi-inner product*, for short.

For details concerning the properties of T-semi-inner product, we send to [1], [2] or [5]. We only note that the following identity holds:

$$(x, y)_T = \tau(y, x)\,||x||, \quad x, y \in X,$$

where $\tau(y, x) := \lim_{t \to 0+} (||y + tx|| - ||y||)/t$ if $x, y \in X$. For the properties of the functional $\tau$ we send to [4].

The following lemma proved in [2[ is important in the sequel.

1.4. LEMMA. *Let $(X, ||.||)$ be a smooth normed linear space over the real field* **R** *and* (,) *the L-semi-inner product which generates the norm $||.||$. Then*

$$\text{(I)} \qquad (y, x)_T = (y, x) = \lim_{t \to 0} [(x, x + ty) - (x, x)]/t, \; x, y \in X.$$

Now, we state some well known results that we shall use in what follows:

1.5. THEOREM. ([4], Lemma 1.). *The normed space X of three or more dimensions is prehilbertian iff*

$$x \perp y \text{ implies } y \perp x \text{ for all pairs } x, y \text{ in } X, \; ||x|| = ||y|| = 1, \qquad (1.1.)$$

*where "$\perp$" denote Birkhoff's orthogonality in normed space $(X, ||.||)$.*

Consider the following condition:

$$\text{if } ||x|| = ||y|| = 1, \text{ then } \tau(x, y) = 0 \text{ implies } \tau(y, x) = 0. \qquad (1.2.)$$

Then we have

1.6. THEOREM. ([4], Lemma 2). *The following conditions are equivalent on X:*

(i) *Condition* (1.2.) *holds,*

(ii) *The space X is smooth and condition* (1.1.) *holds.*

1.7. *Remark.* Note that the restriction $||x|| = ||y|| = 1$ can be dropped in (1.1.) and — if we assume $y \neq 0$ — also in (1.2.).

The following result is a consequence of above theorems:

1.8. THEOREM. ([4], Lemma 3.). *If dim $(X) \geqslant 3$ and* (1.2.) *holds, then X is prehilbertian.*

If we want to drop the assumption $\dim(X) \geqslant 3$ in these characterizations, then it is necessary to strong the condition (1.2.).

1.9. THEOREM. ([4], Lemma 4.). *The space X is prehilbertian iff $\tau(x, y) = \tau(y, x)$ for all pairs $x, y$ in X, $||x|| = ||y|| = 1$.*

A slight improvement of Theorem 1.9. is given by

1.10. THEOREM. ([4], Proposition 1.) *The space X is prehilbertian iff $|\tau(x, y)| = |\tau(y, x)|$ for all pairs $x, y$ in X, $||x|| = ||y|| = 1$.*

Another result due to P. L. Papini ([4], Proposition 2.) is the following:

1.11. THEOREM. *Let $(X, ||.||)$ be a real normed linear space. Then the following conditions are equivalent:*

(i) *Condition* (1.1.) *holds;*

(ii) *If $||x|| = ||y|| = 1$, then $\tau(x, y) \geqslant 0$ implies $\tau(y, x) \geqslant 0$.*

Finally, we give

1.12. THEOREM. ([4], Theorem 4.) *The normed space $X$ of three or more dimensions is prehilbertian if and only if the following condition holds.*

$0 \leqslant \tau(x, y)\,\tau(y, x)$ *for every pairs* $x, y$ *in* $X$, $\|x\| = \|y\| = 1$.

2. **Theorems of characterization.** In this section we shall give some theorems of characterization of inner product spaces by the use of Lemma 1.4. and Theorems 1.5. — 1.12.

2.1. THEOREM. *Let $X$ be a real linear space and a mapping $(,) : X \times X \to \mathbf{R}$. Then the following sentences are equivalent:*

(I) $(,)$ *is an inner product on* $X$;

(II) $(,)$ *is a continuous L-semi-inner product which satisfies one of the following conditions:*

$$\lim_{t\to 0}\, [(x, x + ty) - (x, x)]/t = (x, y) \quad \textit{for all} \quad x, y \in X \tag{2.1.}$$

(see Theorem 3.2. of [2]), *or*

$$\lim_{t\to 0}\, [(x, x + ty) - 1]/t = \lim_{t\to 0}\, [(y, y + tx) - 1]/t \quad \textit{for all} \quad x, y \in X \quad \textit{with} \quad (x, x) = (y, y) = 1, \tag{2.2.}$$

*or*

$$\left|\lim_{t\to 0}\, [(x, x + ty) - 1]/t\right| = \lim_{t\to 0}\, [(y, y + tx) - 1]/t \quad \textit{for all} \quad x, y \in X \quad \textit{with} \quad (x, x) = (y, y) = 1. \tag{2.3.}$$

*Proof.* "(I) ⇒ (II)". It is evident by the properties of inner product. We omit the details.

"(II) ⇒ (I)". Firstly, we remark that by Lemma 1.4. and by identity (I), the conditions (2.1.), (2.2.) and (2.3.) are equivalent respectively with:

$$(y, x) = (x, y), \quad x, y \in X; \tag{2.1'.}$$

$$\tau(x, y) = \tau(y, x), \quad x, y \in X \text{ and } \|x\| = \|y\| = 1; \tag{2.2'.}$$

and

$$|\tau(x, y)| = |\tau(y, x)|, \quad x, y \in X \text{ and } \|x\| = \|y\| = 1; \tag{2.3'.}$$

By Theorem 1.9. and 1.10 of Introduction, we deduce that $(,)$ is on inner product on $X$.

The theorem is proved.

Now, we give the second theorem of characterization of inner products in terms of semi-inner products in the sense of Lumer.

2.2. THEOREM. *Let $X$ be a real linear space of three or more dimensions and the mapping $(,) : X \times X \to \mathbf{R}$. Then the following sentences are equivalent:*

(I) $(,)$ *is an inner product on* $X$;

(II) $(,)$ *is a continuous L-semi-inner product and one of the following conditions holds:*

$$(y, x) = 0 \quad \textit{implies} \quad (x, y) = 0 \quad \textit{if} \quad \|x\| = \|y\| = 1, \tag{2.4.}$$

or

$$\lim_{t\to 0}\ [(x, x+ty) - 1]/t = 0 \textit{ implies } \lim_{t\to 0}\ [(y, y+tx) - 1]/t = 0 \quad \textit{if } ||x|| = ||y|| = 1, \tag{2.4.'}$$

or

$$(y, x) \geqslant 0 \textit{ implies } (x, y) \geqslant 0 \textit{ if } ||x|| = ||y|| = 1, \tag{2.5.}$$

or

$$\lim_{t\to 0}\ [(x, x+ty) - 1]/t \geqslant 0 \textit{ implies } \lim_{t\to 0}\ [(y, y+tx) - 1]/t \geqslant 0 \quad \textit{if } ||x|| = ||y|| = 1, \tag{2.5.'}$$

or

$$(y, x)(x, y) \geqslant 0 \textit{ if } ||x|| = ||y|| = 1, \tag{2.6.}$$

or

$$\lim_{t\to 0} \{[(x, x+ty) - 1][(y, y+tx) - 1]\}/t^2 \geqslant 0 \textit{ if } ||x|| = ||y|| = 1. \tag{2.6.'}$$

*Proof.* "(I) $\Rightarrow$ (II)". It is evident by the properties of inner product.

"(II) $\Rightarrow$ (I)". We remark that, the conditions (2.4'.), (2.5'.) and (2.6'.) are equivalent (by Lemma 1.4.), with (2.4.), (2.5.) and (2.6.).

On the other hand, (2.4.), (2.5.) and (2.6.) are equivalent with

$$\tau(x, y) = 0 \text{ implies } \tau(y, x) = 0 \text{ if } ||x|| = ||y|| = 1, \tag{2.4''.}$$

$$\tau(x, y) \geqslant 0 \text{ implies } \tau(y, x) \geqslant 0 \text{ if } ||x|| = ||y|| = 1, \tag{2.5''.}$$

$$\tau(x, y)\tau(y, x) \geqslant 0 \text{ if } ||x|| = ||y|| = 1. \tag{2.6''.}$$

By the use of Theorem 1.8., Theorem 1.11. and Theorem 1.12., we deduce that (,) is an inner product on $X$, and the theorem is proved.

3. **Applications.** The main purposes of this section are to apply the characterization theorems obtained above to operators defined on a dual system of real linear spaces.

Let $(X, Y, \langle,\rangle)$ be a dual system of real linear spaces. Further, we shall consider an operator $A : X \to Y$ satisfying the following positivity condition:

(P) $\langle x, A(x)\rangle \geqslant 0$ if $x \in X$ and $\langle x, A(x)\rangle = 0$ implies $x = 0$.

In these assumptions, we can establish the next result.

3.1. THEOREM. *Let $(X, Y, \langle,\rangle)$ be a dual system of real linear spaces and $A : X \to Y$ an operator satisfying the condition* (P). *Then the following sentences are equivalent:*

(A) *$A$ is a linear operator and* $\langle x, Ay\rangle = \langle y, Ax\rangle$, $x, y \in X$;

(AA) *$A$ satisfies the conditions*

(i) $A(\alpha x) = \alpha A x$, $\alpha \in \mathbf{R}$, $x \in X$;

(ii) $|\langle x, Ay\rangle|^2 \leqslant \langle x, Ax\rangle\langle y, Ay\rangle$, $x, y \in X$;

(iii) $\lim_{t\to 0}\langle y, A(x+ty)\rangle = \langle y, Ax\rangle$, $x, y \in X$;

*and one of the following assumptions holds*:

$$\lim_{t\to 0}\langle x,[A(x+ty)-Ax]/t\rangle=\langle x, Ay\rangle,\ \ x,y\in X, \tag{3.1.}$$

*or*

$$\lim_{t\to 0}\langle x,[A(x+ty)-Ax]/t\rangle=\lim_{t\to 0}\langle y,[A(y+tx)-Ay]/t\rangle$$
$$\text{where } x,y\in X \text{ and } \langle x, Ax\rangle=\langle y, Ay\rangle=1, \tag{3.2.}$$

*or*

$$\lim_{t\to 0}\langle x,[A(x+ty)-Ax]/t\rangle=\lim_{t\to 0}\langle y,[A(y+tx)-Ay]/t\rangle$$
$$\text{where } x,y\in X \text{ and } \langle x, Ax\rangle=\langle y, Ay\rangle=1. \tag{3.3.}$$

*Proof.* "(A) $\Rightarrow$ (AA)". Putting $(x,y)_A:=\langle x, Ay\rangle$, then $(,)_A$ is an inner product on $X$ and the conditions (AA) is fulfiled.

"(AA) $\Rightarrow$ (A)". Putting $(x,y)_A:=\langle x, Ay\rangle$, $x,y\in X$, then $(,)_A$ is a continuous $L$-semi-inner product which satisfies the conditions (2.1.), (2.2.) or (2.3.). This means that, by Theorem 2.1., $(,)_A$ is an inner product on $X$ what implies the linearity of $A$ and the condition: $\langle x, Ay\rangle=\langle y, Ax\rangle$ for all $x,y\in X$.

The theorem is proved.

The second result is embodied in the next theorem.

3.2. THEOREM. *If* $\dim(X)\geq 3$, *then the condition* (A) *holds if and only if the relations* (i)—(iii) *are valid and one of the following assertions are true*:

$$\langle y, Ax\rangle=0 \text{ implies } \langle x, Ay\rangle=0 \text{ if } \langle x, Ax\rangle=\langle y, Ay\rangle=1, \tag{3.4.}$$

*or*

$$\lim_{t\to 0}\langle x,[A(x+ty)-Ax]/t\rangle=0 \text{ implies } \lim_{t\to 0}\langle y,[A(y+tx)-Ay]/t\rangle=0$$
$$\text{if } \langle x, Ax\rangle=\langle y, Ay\rangle=1, \tag{3.4'.}$$

*or*

$$\langle y, Ax\rangle\geq 0 \text{ implies } \langle x, Ay\rangle\geq 0 \text{ if } \langle x, Ax\rangle=\langle y, Ay\rangle=1, \tag{3.5.}$$

*or*

$$\lim_{t\to 0}\langle x,[A(x+ty)-Ax]/t\rangle\geq 0 \text{ implies } \lim_{t\to 0}\langle y,[A(y+tx)-Ay]/t\rangle\geq 0$$
$$\text{if } \langle x, Ax\rangle=\langle y, Ay\rangle=1, \tag{3.5'.}$$

*or*

$$\langle y, Ax\rangle\langle x, Ay\rangle\geq 0 \text{ if } \langle x, Ax\rangle=\langle y, Ay\rangle=1, \tag{3.6.}$$

*or*

$$\lim_{t\to 0}(\langle x,[A(x+ty)-Ax]/t\rangle\langle y,[A(y+tx)-Ay]/t\rangle)\geq 0$$
$$\text{if } \langle x, Ax\rangle=\langle y, Ay\rangle=1. \tag{3.6'.}$$

The proof follows by Theorem 2.2. and we omit the details.

3.3. *Remark*. If $(X, \|\cdot\|)$ is a real normed linear space and $A: X\to X^*$ is an operator satisfying the property:

(PN) $\quad Ax(x)\geq 0$ if $x\in X$ and $Ax(x)=0$ implies $x=0$, then the above theorems can be applyed for the dual mapping

$$\langle , \rangle : X \times X^* \to \mathbf{R}, \ \langle x, x^* \rangle = x^*(x), \ x \in X \text{ and } x^* \in X^*.$$

If $(X;(,))$ ia a real prehilbertian space and $A: X \to X$ is an operator satisfying the coresponding property:

PH) $\quad (Ax, x) \geqslant 0$ if $x \in X$ and $(Ax, x) = 0$ implies $x = 0$,

then the previous results remain valid too.

We omit the details.

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# ON LAGRANGIAN AND HAMILTON SPACES

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**REZUMAT. — Asupra Lagrangianului şi a spaţiilor lui Hamilton.** În prima parte a lucrării se definesc noţiunile de $G$ — aplicaţie, $I$ — aplicaţie şi $\overset{0}{I}$ — aplicaţie şi se stabilesc cîteva proprietăţi ale lor. În partea a doua aceste proprietăţi sînt extinse pentru spaţiile lui Hamilton.

The terminology and notation are essentially based on R. Miron [1] [2].

**1.** Let $M$ be a $C^\infty$ — differentiable, real, $n$-dimensional manifold and $(TM, \Pi, M)$ its tangent bundle. If $(U, \varphi(x) = x^i, i = 1, \ldots, n)$ is a locally chart on $M$, then $(\Pi^{-1}(U); x^i, y^i)$ is a locally chart on $TM$ (the coordinate of point $(x, y) \in T(M)$ are $(x^i, y^i)$). If $N$ is a non-linear connection on $TM$, locally given by $N^i_j(x, y)$, then $\left\{\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^k_i \frac{\partial}{\partial y^k}\right\}$ is a locally basis of a horizontal distribution $H$, on $TM$, determined by $N$, which is suplimentary to vertical distribution $V$ with locally basis $\{\partial/\partial y^i\}$. For the module of vector fields $\mathfrak{X}(TM)$ a locally basis is $\{\delta/\delta x^i, \partial/\partial y^i\}$ and the dual basis is $\{dx^i, \delta y^i\}$ [1]. Let $S = (x, y, y, (S^i))$ be a semispray an $M$, and $C$ a differentiable curve $C: I \to M$ which is a road relative to $S$, locally given by [5]:

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0, \tag{1.1.}$$

where $G^i = -\frac{1}{2} S^i$ and $N^i_j(x, y) = \partial G^i/\partial y^j$ ($S^i_{(x,y)}$ are differentiable of class $C^\infty$ on $\tau M = TM - 0$; $y \neq 0$). If $L_c S = S$, where $L_c$ is the Lie derivative with respect to $c = y^i \frac{\partial}{\partial y^i}$ (Liouville's vector field) then the system (1.1.) is by R. Miron [5]:

$$\frac{d^2x^i}{dt^2} + N^i_j\left(x, \frac{dx}{dt}\right) \frac{dx^j}{dt} = 0, \tag{1.2.}$$

and the integral curves of $S$ are the autoparalles of $N$.

Let $L^n = (M, L)$ be a Langrange space, where $L(x, y)$ is a real differentiable function on $TM$ (where the Finsler tensor field $g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j}$ is non degenerate) and the Finsler forms:

$$\omega = \omega_i \, dx^i; \quad \theta = g_{ij} \delta y^i \wedge dx^j, \tag{1.3.}$$

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where $\omega_i = \frac{1}{2}\frac{\partial L}{\partial y^i}$, which are globaly defined on $TM$. If $N^i_j = \overset{k}{N}{}^i_j(x, y)$ (the non linear connection of Kern) then we have, by V. Oproiu:

$$\overset{k}{\theta} = d\omega \tag{1.4}$$

where $d\omega$ is the exterior differentiable of $\omega$ ([1]).

If $N^i_j(x, y)$ is a fixed non linear connection and

$$L_{ij}(x, y) \overset{def}{=} g_{ia}\, N^a_j - g_{ja}\, N^a_i - \mathfrak{L}_{ij};\ \ \mathfrak{L}_{ij} = \frac{1}{2}\left(\frac{\partial^2 L}{\partial x^j\, \partial y^i} - \frac{\partial^2 L}{\partial x^i\, \partial y^j}\right) \tag{1.5}$$

then $L_{ij}$ is a Finsler tensor field and $L_{ij} = -L_{ji}$. Hence the system $L_{ij} = 0$ is invariant to transformation $(\Pi^{-1}, x^i, y^i) \to (\Pi^{-1}(\bar{U});\ \bar{x}^i, \bar{y}^i)$, $(x, y) \in U \cap \bar{U}$.
Denoting,

$$B_i = \frac{1}{2}\left(\frac{\partial^2 L}{\partial x^a\, \partial y^i} y^a - \frac{\partial L}{\partial x^i}\right);\ \ \overset{k}{G}{}^i = \frac{1}{2} g^{ia}\, B_a. \tag{1.6}$$

from (1.5) one obtains

$$g_{ia}\frac{\partial G^a}{\partial y^j} - g_{ja}\frac{\partial G^a}{\partial y} - \mathfrak{L}_{ij} = 0. \tag{1.7}$$

A solution of $L_{ij} = 0$ is the non linear connection of Kern (or the canonical non linear connection),

$$\overset{k}{N}{}^i_j(x, y) = \frac{\partial G^i}{\partial y^i}. \tag{1.8}$$

We have

THEOREM (1.1). *The set $N(L)$, of non-linear connection $\{N^i_j(x, y)\}$, which is determined by the Lagrangian $L(x, y)$ is given by*

$$N^i_j = \overset{k}{N}{}^i_j - A^i_j;\ g_{ia}A^a_j - g_{ja}A^a_i = 0. \tag{1.9}$$

We obtain a generalized form of Oproiu theorem. It is,

THEOREM (1.2). *We have $\theta = d\omega$ for every $N \in N(L)$.*

The variational principle to the Lagrangian $L(x(t), y(t))$, for the parametrized curves $C : x^i = x^i(t)$, on $M$, gives the Euler-Lagrange equations for the geodesics $C$ for Lagrange space $L^n$.

R. Miron [5] shows that $C$ is the integrale curve of the semispray $S$ with $S^i = -2\overset{k}{G}{}^i$. Hence the equations for the geodesic $C$, of Lagrange space, are given by

$$\frac{d^2x^i}{dt^2} + 2\overset{k}{G}{}^i\left(x, \frac{dx}{dt}\right) = 0. \tag{1.10}$$

DEFINITION (1.1). The set $N(L)$ is called the set of *intrinsec non-linear connections* of the Lagrange space $L^n = (M, L)$.

DEFINITION (1.2). If $L^n = (M, L)$ and $\widetilde{L}^n = (M, \widetilde{L})$ are two Lagrange spaces, then the map $G: L^n \to \widetilde{L}^n$ given by:

$$G: \{(x, y), L\} \to \{(x, y), \widetilde{L}\} \tag{1.11}$$

with,

$$\widetilde{L}(x, y) = aL(x, y) + \alpha_i(x) y^i, \tag{1.12}$$

where $a = ct$ and $\alpha = \alpha_i(x)dx^i \in \Lambda_1^0(M)$, is called a *G-map*.

DEFINITION (1.3). A map $O: L^n \to \widetilde{L}^n$ given by

$$O: \{(x, y), g\} \to \{(x, y), \widetilde{g} = ag\} \tag{1.13}$$

is called a *omotetic* map.

From (1.13) we have.

PROPOSITION (1.1). *A G-map*: $L^n \to \widetilde{L}^n$ *is an omotetic map.*

DEFINITION (1.4). If $G$ is a map: $L^n \to \widetilde{L}^n$ so that a integrale curve $C$, of semispray $S$, determined by $\overset{k}{N}$, is a integrale curve of semispray $\widetilde{S}$, determined by $\overset{k}{\widetilde{N}}$, then $\overset{\circ}{G}$ is called a *geodesic map*.

From (1.11) we have

THEOREM (1.3). *A G-map is a geodesic map* $\overset{\circ}{G}$ *iff*

$$\frac{\partial \alpha_i(x)}{\partial x^j} = \frac{\partial \alpha_j(x)}{\partial x^i}. \tag{1.14}$$

In this case the 1-form $\alpha = \alpha_i(x)\, dx^i$ is a closed 1-form and locally is exact: exist a differentiable function $f(x)$ so that

DEFINITION (1.5). A *I-map*: $L^n \to \widetilde{L}^n$ is called *g-isometrical* map, iff $\widetilde{g}(x, y) = g(x, y)$.

From (1.13) we have,

THEOREM (1.4). *A G-map is a I-map iff* $a = 1$.

If $\alpha_i(x)$ is arbitrary of $M$, and $L^n$ is a fixed Lagrange space, then the *class* of Lagrange space $\{(M, \widetilde{L})\}$, where

$$\widetilde{L}(x, y) = L(x, y) + \alpha_i(x)y^i, \tag{1.16}$$

*defines same metrical Finsler structure* $(M, g(x, y))$.

From (1.18) we have,

THEOREM (1.5). *A G-map preserve the class* $N(L)(N(\widetilde{L}) = N(L))$ *iff G is* $G^\circ$

DEFINITION (1.6). *A* $\overset{\circ}{I}$*-map*: $L^n \to \widetilde{L}^n$ is called a *iscmetrical map* if preserve

$$g\ (\widetilde{g}(x, y) = g(x, y)) \text{ and } N(L)(N(\widetilde{L}) = N(L)).$$

THEOREM (1.6). *A G-map is a* $\overset{\circ}{I}$*-map iff G is a* $\overset{\circ}{G}$*-map amd a I-map.*

In general, if $(M, L)$ is fixed in $\{M, \widetilde{L}\}$; where $\{M, \widetilde{L}\}$ is the set of

Lagrange spaces which define same metrical Finsler structure and $N \in N(L)$, then we have

$$\tilde{N}^i_j(x, y) = N^i_j(x, y) + B^i_j(x, y) \tag{1.17}$$

where $B(x, y)$ are the tensor fields solutions of system:

$$g_{ia} B^a_j - g_{ja} B^a_i = \frac{1}{2}\left(\frac{\partial \alpha_i}{\partial x^j} - \frac{\partial \alpha_j}{\partial x^i}\right). \tag{1.18}$$

**2. Hamilton spaces.** If $(\overset{*}{T}M, \overset{*}{\Pi}, M)$ is the cotangent bundle and $(x^i, p_i)$ are the coordonates of a point $u \in \overset{*}{\Pi}^{-1}(U) \subset \overset{*}{T}M$ then $\{\partial/\partial p_i\}$ is a local basis of the vertical distribution $V: u \in \overset{*}{T}M \to V_u \subset T_u \overset{*}{T}M$.

Let $N$ be a nonlinear connection on $\overset{*}{T}M$ which determines a subbundle $H\overset{*}{T}M$ of $T\overset{*}{T}M$ such that $T\overset{*}{T}M = H\overset{*}{T}M \oplus V\overset{*}{T}M$ is the Whitney sum (where $V\overset{*}{T}M$ is the vertical subbundle of $T\overset{*}{T}M$). If $N_{ki}(x, p)$ are the coefficients of the nonlinear connection, $N$, on $\overset{*}{\Pi}^{-1}(U)$, then $\left\{\frac{\overset{*}{\delta}}{\delta x^i} = \frac{\partial}{\partial x^i} + N_{ij}(x, p)\frac{\partial}{\partial p_j}\right\}$ is a local basis of the horizontal distribution $N: u \in \overset{*}{T}M \to N_u \subset T_u \overset{*}{T}M$ where $T_u \overset{*}{T}M = N_u \oplus V_u$.

There exists a nonlinear connection on $\overset{*}{T}M$, if $M$ is a paracompact differentiable manifold.

A local basis of $\mathfrak{X}(\overset{*}{T}M)$ is $\left(\frac{\overset{*}{\delta}}{\delta x^i}, \frac{\partial}{\partial p_i}\right)$ and is colled, by R. Miron, a local basis adapted.

For the $d$-tensor fields, on $M$, given by R. Miron

$$R_{ijk} = \frac{\overset{*}{\delta} N_{jk}}{\delta x^i} - \frac{\overset{*}{\delta} N_{ik}}{\delta x^j}; \quad \tau_{ij} = N_{ij} - N_{ji}, \tag{2.1}$$

one obtains:

$$\left[\frac{\overset{*}{\delta}}{\delta x^i}, \frac{\overset{*}{\delta}}{\delta x^j}\right] = R_{ijk}\frac{\partial}{\partial p_k}; \quad \left[\frac{\overset{*}{\delta}}{\delta x^i}, \frac{\partial}{\partial p_k}\right] = -\frac{\partial N_{ir}}{\partial p_k}\frac{\partial}{\partial p_r},$$

and $\left[\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_k}\right] = 0$. Evidently, $N$ is integrable iff $R_{ijk} = 0$. In [2] R. Miron consider the forms:

$$\tilde{p} = p_i\, dx^i; \quad \eta = \frac{1}{2}\tau_{ij}\, dx^i \wedge dx^j; \quad \theta = \overset{*}{\delta} p_i \wedge dx^i, \tag{2.2}$$

in the dual basis, $(dx^i, \overset{*}{\delta} p_i = dp_i - N_{ij} dx^j)$ of the basis $\left(\frac{\overset{*}{\delta}}{\delta x^i}, \frac{\partial}{\partial p_i}\right)$.

In [2], R. Miron define an Hamilton space by $H^n = (M, H)$ where $H: \overset{*}{T}M \to R$ is a $C^\infty$ — function and the Hessian $a^{ij}(x, p) = \frac{1}{2}\frac{\partial^2 H}{\partial p_i \partial p_j}$ is non-degenerate, $\forall(x, p) \in \overset{*}{T}M$. This concept is fecund in the theoretical physics.

The map $\varphi : (x, y) \in TM \to (x, p) \in \overset{*}{T}M$ where $p = \omega$ are given by (1.3), is a local diffeomorphism from $TM$ to $\overset{*}{T}M$. If $(x, y) \in TM$ is fixed, there exists an open $D \subset TM$, $(x, y) \in D$, and an open $\overset{*}{D} = \varphi(D) \subset \overset{*}{T}M$ and $\varphi^{-1}$, the inverse function of $\varphi/D$ is given by:

$$x^i = x^i,\ y^i = \Phi^i(x, p)\quad \forall(x, p) \in \overset{*}{D}\ (\Pi(D) = D_1 \subset M).$$

The restriction of $L^n$ to $D$ is $L^n_{/D}$. Let $H$ be the function, on $\overset{*}{D}$,

$$H(x, p) = -L(x, y) + 2p_i y^i;\ y^i = \Phi^i(x, p) \qquad (p = \omega) \tag{2.2}$$

and the Hamilton space $H_D = (M, H)$. The transformation (2.2) is the Legendre transformation: $L^n_{/D} \to H^n_{/\overset{*}{D}}$.

If $\mathfrak{T} = \frac{1}{2} H$, $\mathfrak{L} = \frac{1}{2} L$, then one obtains ([2])

$$\frac{\partial \mathfrak{T}}{\partial x^i} + \frac{\partial \mathfrak{L}}{\partial x^i} = 0; \qquad \frac{\partial \mathfrak{T}}{\partial p^i} = \Phi^i(x, p) \tag{2.3}$$

and from Theorem (8.5). R. Miron [2], one obtains

$$\frac{d^2 x^i}{dt^2} + 2\overset{k}{G^i}\left(x, \frac{dx}{dt}\right) = 0 \Leftrightarrow \frac{dx^i}{dt} - \frac{\partial \mathfrak{T}}{\partial p_i} = 0; \qquad \frac{dp_i}{dt} + \frac{\partial \mathfrak{T}}{\partial x^i} = 0 \tag{2.4}$$

The last equations are the Hamilton equations.

The nonlinear connection $\overset{c}{N}(x, p)$, $\overset{*}{T}M_{/\overset{*}{D}}$ given by

$$\overset{c}{N}_{ij}(x, p) = -g_{ja}\ \overset{k}{N^a_j}(x, y) + \frac{1}{2}\ \frac{\partial^2 L(x, y)}{\partial y^j\, \partial x^i};\ y^i = \Phi^i(x, p) \tag{2.5}$$

depends by $H$, only.

From (2.5) and (1.5) we have:

$$\overset{c}{N}_{ij} - \overset{c}{N}_{ji} = L_{ij} = 0 \tag{2.6}$$

since $N^i_j = \overset{k}{N^i_j}$. One obtains, in this mode:

THEOREM (2.1). (R. Miron [2]) $\cdot\ \overset{c}{\tau}_{ij} = 0$; $\underset{ijk}{S}\ \overset{c}{R}_{ijk} = 0$ *and* $\overset{c}{\theta}$ *is integrable, and*

$$\frac{d^2 x^i}{dt^2} + 2\overset{k}{G^i}\left(x, \frac{dx}{dt}\right) = 0 \Leftrightarrow \frac{dx^i}{dt} - \frac{\partial \mathfrak{T}}{\partial p_i} = 0; \qquad \frac{\overset{*c}{\delta} p_i}{\delta t} + \frac{\overset{*c}{\delta} \mathfrak{T}}{\delta x^i} = 0. \tag{2.7}$$

Evidently, $H$ is constant on the integrale curves of the Hamilton equations (2.7) since we have $dH = 0$ on this curves. A relation of type (1.4) is obtained by

THEOREM (2.2) $\overset{c}{\theta} = dp$.

Let $N(L)$ be the set of intrinsec nonlinear connection of $L^n_{ID}$ and

$$N_{ij}(x, p) = -g_{ja}N^a_i(x, y) + \frac{1}{2}\frac{\partial^2 L(x, y)}{\partial y^j \partial x^i};\ y^i = \Phi^i(x, p). \tag{2.8}$$

We have,

$$N_{ij}(x, p) - N_{ji}(x, p) = L_{ij} = 0 \tag{2.9}$$

$$N_{ij}(x, p) = \overset{c}{N}_{ij}(x, p) + A_{ij}(x, p) \tag{2.10}$$

where

$$A_{ij}(x, p) = A^a_i(x, y)g_{ja}(x, y);\ y^i = \Phi^i(x, p) \tag{2.11}$$

$$A_{ij}(x, p) - A_{ji}(x, p) = 0 \tag{2.12}$$

We obtain,

THEOREM (2.3). *The set $N(H)$, of intrinsec nonlinear connection of $H^{n}_{\overset{*}{D}}$, the dual space of $L^n_{ID}$, is given by* (2.10), (2.11) *and* (2.12).

THEOREM (2.4). *For every $N \in N(H)$ we have*

$$\theta = dp;\ \underset{ijk}{S} R_{ijk} = 0 \tag{2.13}$$

*and $\theta$ is integrable.*

THEOREM (2.5). *The form of Hamilton equations* (2.7) *is an invariant on $N(H)$,*

$$\frac{dx^i}{dt} - \frac{\partial \mathcal{H}}{\partial p_i} = 0;\quad \frac{\overset{*}{\delta} p_i}{dt} + \frac{\overset{*}{\delta}\mathcal{H}}{\delta x^i} = 0,\ \ \forall N \in N(H). \tag{2.14}$$

Let $\mathcal{F}$ be an 1-parametric family of support elements $(x, p) \in \overset{*}{T}M$, in the points $x \in M$, given by ([2])

$$x^i = x^i(t),\ p_i = p_i(t),\ t \in (a, b),\ \text{rank}\left\|\frac{dx^i}{dt}\right\| = 1 \tag{3.18}$$

Then $\mathcal{F}$ is called the geodesic 1-parametric family of the support elements relative to $N$, if its support elements are horizontal relative to $N$. It is denote by $\mathcal{F}_N$ and one obtains ([2])

$$\frac{\delta p_i}{dt} = 0. \tag{3.19}$$

Denoting $g_{ks}\frac{dx^s}{dt} = \mathcal{A}_k(x, y)$, $y^i = \Phi^i(x, p)$, from (2.12) we obtain,

THEOREM (2.6). *If $\mathcal{F}$ is $\overset{c}{\mathcal{F}}_N$ then $\mathcal{F}$ is $\mathcal{F}_N$ for any $N \in N(H)$ iff $A^s_i\mathcal{A}_s = 0$ and* $\det(A^s_i) = 0$ *on $\mathcal{F}$.*

Let $C$ be the geodesics of $L^n$ given by (2.7) and $\overset{\circ}{N}(L) \subset N(L)$ given by (1.9) (1.10) and det $(A_j^i) = 0$, on $C$. If $\overset{\circ}{N}(H) \subset N(H)$ is given by (2.10)–(2.12) and det $(A_j^i) = 0$, on $C$, then we have

$$\left(\frac{\overset{*c}{\delta} p_i}{dt} = 0 \Leftrightarrow \frac{\overset{*}{\delta} p_i}{dt} = 0\right) \Leftrightarrow A_i^a(x, y)\, \mathfrak{a}_a = 0\,; \qquad y^i = \Phi^i(x, p) \tag{2.13}$$

$$\left(\frac{\overset{*c}{\delta} \mathfrak{T}}{\delta x^i} = 0 \Leftrightarrow \frac{\overset{*}{\delta} \mathfrak{T}}{\delta x^i} = 0\right) \Leftrightarrow A_i^a\, \mathfrak{a}_a = 0 \tag{2.14}$$

where

$$\mathfrak{a}_a = g_{ka} \frac{dx^k}{dt}.$$

Let $\underset{1}{\overset{\circ}{N}} \subset \overset{\circ}{N}(H)$ be the set of nonlinear connections $N$ with the property $A_{ij} \frac{\partial \mathfrak{T}}{\partial p_j} = 0$, on $C$, We have

THEOREM (2.7). *We have*

$$\frac{\overset{*c}{\delta} \mathfrak{T}}{\delta x^i} = 0 \Leftrightarrow \frac{\overset{*}{\delta} \mathfrak{T}}{\delta x^i} = 0 \qquad \forall N \in \underset{1}{\overset{\circ}{N}}(H). \tag{2.15}$$

THEOREM (2.8). *Let $\mathfrak{F}$ be an 1-parametric family of support elements $(x, p)$ induced by the geodesic of $L^n$ on $(M, H_{/\overset{*}{D}}{}^N)$. If $\mathfrak{F}$ is $\mathfrak{F}^N$ then $\mathfrak{F}$ is $\overset{*}{\mathfrak{F}}{}^N$ for every $N \in \underset{1}{N}(H)$.*

From (1.16) we obtain,

$$\tilde{p}_i(x, y) = p_i(x, y) + \frac{1}{2}\, \alpha_i(x, y) \qquad \forall (x, y) \in TM, \tag{2.16}$$

where

$$p_i = \frac{1}{2} \frac{\partial L}{\partial y^i}, \qquad \tilde{p}_i = \frac{1}{2} \frac{\partial \tilde{L}}{\partial y^i}.$$

The map $\tilde{\varphi} : (x, y) \in TM \to (x, \tilde{p}) \in \overset{*}{T}M$ is a local diffeomorphism from $TM$ to $\overset{*}{T}M$. If $(x, y) \in TM$ is fixed, there exists an open $\tilde{D}^* = \tilde{\varphi}(\tilde{D})$ where $\tilde{D}$ is an open $\tilde{D} \subset TM$, $(x, y) \in \tilde{D}$, and $D_2 = \Pi(\tilde{D})$ is an open set in $M$, such that $\tilde{\varphi}_{/\tilde{D}}$ is inversable in the form $\tilde{\varphi}^{-1}$ given by

$$x^i = x^i, \; y^i = \tilde{\Phi}^i(x, \tilde{p}) \qquad \forall (x, \tilde{p}) \in \tilde{D}^* \tag{2.17}$$

Let $(M, \tilde{L}_{/\tilde{D}})$ be the restriction of $\tilde{L}^n = (M, \tilde{L})$ to $\tilde{D}$ and $\tilde{H}^n = (M, \tilde{H}_{/\tilde{D}^*})$ its dual Hamilton space, where $\tilde{H}$ is the Hamilton function, on $\tilde{D}^*$, given by

$$\tilde{\mathfrak{T}}(x, \tilde{p}) = -\tilde{\mathfrak{L}}(x, y) + \tilde{p}_i y^i, \qquad y^i = \tilde{\Phi}^i(x, \tilde{p}), \qquad (x, y) \in D \cap \tilde{D} \tag{2.18}$$

The map $G$ given by (1.16) induces the map $\Omega: (x, p) \to (x, \widetilde{p})$ where $\widetilde{p}$ is given by (2.16). This points are called the corresponding points. In this points we have $\widetilde{\Phi}^i(x, \widetilde{p}) = \Phi^i(x, p)$ since $\widetilde{p}_i = p_i + \frac{1}{2}\alpha_i$

One obtains:

THEOREM (2.9). *In the corresponding points of $H^n$ and $\widetilde{H}^n$ we have*

$$\widetilde{\mathfrak{T}}(x, \widetilde{p}) = \mathfrak{T}(x, p); \quad \frac{\partial\widetilde{\mathfrak{T}}}{\partial\widetilde{p}_i} = \frac{\partial\mathfrak{T}}{\partial p_i}; \quad \widetilde{a}^{ij} = a^{ij} \tag{2.19}$$

THEOREM (2.10). *If the map $G$ given by* (1.16) *is a $\overset{\circ}{I}$-map, in the corresponding points $(x, p) \xrightarrow{\Omega} (x, \widetilde{p})$ we have* (2.20).

THEOREM (2.11). *If $G$ is a $\overset{\circ}{I}$ map, in the corresponding points of $(M, H_{/D^\bullet})$ and $(M, \widetilde{H}_{/\widetilde{D}^\bullet})$ we have*

$$\overset{c}{\widetilde{N}}_{ki}(x, \widetilde{p}) = \overset{c}{N}_{ki}(x, p) + \frac{1}{2}\frac{\partial\alpha_i}{\partial x^k}; \qquad \widetilde{p}_i = p_i + \frac{1}{2}\alpha_i \tag{2.21}$$

$$\overset{\bullet c}{\widetilde{\delta}}\widetilde{p}_i = \overset{\bullet c}{\delta} p_i. \tag{2.22}$$

THEOREM (2.12). *Let $\mathfrak{F}$ be an 1-parametric family of support elements $(x, p)$ induced by the geodesics of $L^n_{/D}$ on $(M, H_{/D^\bullet})$ and $\widetilde{\mathfrak{F}}$ for $(M, \widetilde{H}_{/\widetilde{D}^\bullet})$. If $\mathfrak{F}$ is $\mathfrak{F}^{\overset{c}{N}}$ then $\widetilde{\mathfrak{F}}$ is $\mathfrak{F}^{\overset{c}{\widetilde{N}}}$ and conversellу.*

If $G$ is an $\overset{\circ}{I}$-map, since $\widetilde{g}(x, y) = g(x, y)$ and $\overset{k}{\widetilde{N}}{}^i_j(x, y) = \overset{k}{N}{}^i_j(x, y)$, we have:

THEOREM (2.13). *If $G$ is an $\overset{\circ}{I}$ map, then the set $N(\widetilde{H})$ of intrinsec nonlinea connections of $\widetilde{H}^n = (M, \widetilde{H}_{/\widetilde{D}^\bullet})$ is given by*

$$\widetilde{N}_{ij}(x, \widetilde{p}) = \overset{c}{\widetilde{N}}_{ij}(x, \widetilde{p}) + A_{ij}(x, \widetilde{p}) \tag{2.23}$$

*where,*

$$A_{ij}(x, \widetilde{p}) = A^a_i(x, y)\, g_{ja}(x, y); \; y^i = \widetilde{\Phi}^i(x, \widetilde{p}) \tag{2.24}$$

$$A_{ij} - A_{ji} = 0 \tag{2.25}$$

*since* $\widetilde{L}_{ij} = 0$.

THEOREM (2.13). *If $G$ is an $\overset{\circ}{I}$-map, then we have*

$$\widetilde{N}_{ij}(x, \widetilde{p}) = N_{ij}(x, \widetilde{p}) + \frac{1}{2}\frac{\partial\alpha_j(x)}{\partial x^i}, \qquad \forall N \in N(H),\ \widetilde{N} \in N(\widetilde{H}) \tag{2.26}$$

$$\frac{dx^i}{dt} - \frac{\partial\mathfrak{T}}{\partial p_i} = 0,\ \frac{\overset{\bullet}{\delta} p^i}{dt} + \frac{\overset{\bullet}{\delta}\mathfrak{T}}{dx^i} = 0 \Leftrightarrow \frac{dx^i}{dt} - \frac{\partial\widetilde{\mathfrak{T}}}{\partial\widetilde{p}^i} = 0;\ \frac{\overset{\bullet}{\widetilde{\delta}}\widetilde{\mathfrak{T}}}{\delta x^i} + \frac{\overset{\bullet}{\widetilde{\delta}}\widetilde{p}_i}{dt} = 0$$

*in the corresponding points of $(M, H_{/D^\bullet})$ and $(M, \widetilde{H}_{/\widetilde{D}^\bullet})$.*

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# ON A GENERAL FIXED POINT PRINCIPLE FOR $(\theta, \varphi)$-CONTRACTIONS

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**REZUMAT. — Asupra unui principiu general de punct fix pentru $(\theta, \varphi)$ — contracţii.** În (17) am dat un principiu general de punct fix pentru $(\theta, \varphi)$-contracţii. În prezenta lucrare se pun în evidenţă câteva consecinţe şi aplicaţii ale acestui principiu. În finalul lucrării se formulează două conjecturi.

**1. Introduction.** Let $X$ be a nonempty set, $P(X) := \{Y \subset X \mid Y \neq \Phi\}$ and $Y \in P(X)$. We denote by $\mathbf{M}(Y)$ the set of all mappings $f: Y \to Y$.

Definition 1 (see (17)). A triple $(X, S, M)$ is a *fixed point structure* if

(i) $S \subset P(X)$, $S \neq \Phi$

(ii) $M: P(X) \to \bigcup_{Y \in P(X)} \mathbf{M}(Y)$ $Y \mapsto M(Y) \subset \mathbf{M}(Y)$, is a mapping such that, if $Z \subset Y$, $Z \neq \Phi$, then $M(Z) \supset \{f|_Z : f \in M(Y)$ and $f(Z) \subset Z\}$,

(iii) Every $Y \in S$ has the fixed point property with respect to $M(Y)$

Definition 2 (see (17)). Let $X$ be a nonempty set, $Z \subset P(X)$, and $Z \neq \Phi$. A mapping $\theta: Z \to R_+$ has the *intersection property* if $Y_n \in Z$, $Y_{n+1} \subset Y_n$, $n \in N$ and $\lim_{n\to\infty} \theta\ (Y_n) = 0$ implis $\bigcap_{n\in N} Y \neq \Phi$

Definition 3 (see 17)). Let $(X, S, M)$ be a fixed point structure, $\theta: Z \to R_+$, $S \subset Z \subset P(X)$ and $\eta: P(X) \to P(X)$. The pair $(\theta, \eta)$ is *compatible* with $(X, S, M)$ if

(i) $\eta$ is a closure operator, $S \subset \eta(Z) \subset Z$, and $\theta(\eta(Y)) = \theta(Y)$ for all $Y \in Z$,

(ii) $\{Y \in Z \mid \theta(Y) = 0\} \cap \{Y \in Z \mid \eta(Y) = Y\} \subset S$.

Definition 4 (see (11)). A function $\varphi: R_+ \to R_+$ is a *comparison function* if

(i) $\varphi$ is monoton increasing,

(ii) $(\varphi^n(t))_{n\in N}$ converges to 0 for all $t \geq 0$.

Definition 5 (see (17)). Let $X$ be a nonempty set, $Y \subset X$, $Z \subset P(X)$, $Z \neq \varphi$, $\varphi: R_+ \to R_+$ and $\theta: Z \to R_+$. A mapping $f: Y \to X$ is a $(\theta, \varphi)$ — *contraction* if

(i) $A \in P(Y) \cap Z$ implies $f(A) \in Z$,

(ii) $\theta(f(A)) \leqslant \varphi(\theta(A))$, for all $A \in P(Y) \cap Z \cap I(f)$,

(iii) $\varphi$ is a comparison function

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Definition 6 (see (17)). A mapping $f: Y \to X$ is a *strong* $(\theta, \varphi)$*-contraction* if

(i) $A \in P(Y) \cap Z$ implies $f(A) \in Z$,

(ii) $\theta(f(A)) \leqslant \varphi(\theta(A))$, for all $A \in P(Y) \cap Z$,

(iii) $\varphi$ is a comparison function.

The following general fixed point principle is given in /17/:

THEOREM A. *Let* $(X, S, M)$ *be a fixed point structure and* $(\theta, \eta)(\theta: Z \to R_+$, $\eta: P(X) \to P(X))$ *a compatible pair with* $(X, S, M)$. *Let* $Y \in \eta(Z)$ *and* $f \in M(Y)$. *If*

(i) $\theta|_{\eta(Z)}$ *has the intersection property,*

(ii) $f$ *is a* $(\theta, \varphi)$*-contraction,*

*then*

(a) $F_f \neq \Phi$,

(b) *if* $F_f \in Z$, *then* $\theta(F_f) = 0$.

The aim of this paper is to present some consequences of this general result and to formulate two conjectures.

2. **$(\delta, \varphi)$-contractions.** Let $(X, d)$ be a metric space, $S = \{\{x\} \mid x \in X\}$, $M(Y) = \mathrm{M}(Y)$, $\theta = \delta$, $Z = P_b(X)$ and $\eta(A) = \bar{A}$. From the Theorem A we have

THEOREM 1 (see 2, 7, 8). *Let* $(X, d)$ *be a bounded complete metric space and* $f: X \to X$ *a* $(\delta, \varphi)$*-contraction. Then*

(a) $f$ *is a Picard mapping,*

(b) $f$ *is a Janos mapping.*

*Proof.* (a) From the Theorem A we have $F_f = \{x^*\}$. Let $x_0 \in X$ and $B_n := \{f^n(x_0), f^{n+1}(x_0), \ldots, x^*\}$. From $f(B_n) = B_{n+1} \subset B_n$, and $\delta(B_{n+1}) \leqslant \varphi(\delta(B_n))$, it follows that $\delta(B_n) \to 0$ as $n \to \infty$, i.e, $f^n(x_0) \to x^*$ as $n \to \infty$.

(b) $x^* \in \bigcap_{n \in N} f^n(X)$ and $\delta(f^n(X)) \to 0$ as $n \to \infty$.

From the Theorem 1 we have:

THEOREM 2. *Let* $(X, d)$ *be a complete metric space and* $f: X \to X$ *a mapping such that:*

(i) *there exists* $m \in N^*$ *such that* $f^m(X)$ *is bounded,*

(ii) $f^m$ *is a* $(\delta, \varphi)$*-contraction*

*Then:*

(a) $f$ *is a Picard mapping,*

(b) $f$ *is a Janos mapping.*

*Proof.* From the Theorem 1, $F_{f^m} = \{x^*\}$ and $f^{km}(x) \to x^*$ as $k \to \infty$. On the other hand $X \supset f(X) \supset \ldots \supset f^m(X) \supset f^{m+1}(x) \supset \ldots$, and $(f^{km}(X)) \to 0$ as $k \to \infty$. Thus $F_f = \{x^*\}$, $f^n(x) \to x^*$ as $n \to \infty$ and $\bigcap_{n \in N} f^n(X) = \{x^*\}$.

*Remark* 1. From the Theorem 1 and 2 we have some results given by Kannan, Reich, Rus, Amann, Avramescu, Boyd—Wong, Browder, Chatterjea, Ciric, Delbosco, Fischer, Fuchssteiner, Furi Hegedüs, Iseki, Kasahara, Rhoades, Tan, Taskovic, Walter, ... (see [2, 5, 6, 7, 8, 11, 17]).

3. **$(\alpha, \varphi)$-contractions.** Let $(X, \|.\|)$ be a Banach space.

Definition 7 (see [1, 5, 6, 9, 10]). A mapping $\alpha : P_b(X) \to R_+$ is called *a measure of noncompactness* iff

(i) $\alpha(A) = 0$ implies $\bar{A} \in P_{cp}(X)$,

(ii) $\alpha(\bar{A}) = \alpha(A)$, for all $A \in P_b(X)$,

(iii) $A \subset B$ implies $\alpha(A) \leqslant \alpha(B)$, for all $A, B \in P_b(X)$,

(iv) $\alpha(\text{co } A) = \alpha(A)$ for all $A \in P_b(X)$,

(v) if $A_n \in P_{b,cl}(X)$, $A_{n+1} \subset A_n$, $n \in N$, and $\alpha(A_n) \to 0$ as $n \to \infty$, then $\bigcap_{n \in N} A_n \neq \varphi$ and $\alpha(\bigcap_{n \in N} A_n) = 0$.

From the Theorem A we have:

THEOREM 3 (see 9). *Let $X$ be a Banach space, $\alpha$ a measure of noncompactness on $X$, $Y \subset X$ a closed convex bounded subset of $X$ and $f: Y \to Y$ a continuous $(\alpha, \varphi)$contraction. Then $F_f \neq \Phi$ and $\alpha(F_f) = 0$.*

*Proof.* Let $S = P_{cp,cv}(X)$, $M(Y) = C(Y, Y)$, $\theta = \alpha$ and $\eta(Y) = \overline{\text{co}}(Y)$. From the Theorem A, $F_f \neq \Phi$ and $\alpha(F_f) = 0$.

THEOREM 4 (see [3, 5, 6, 11]). *Let $X$ be a Banach space, $\alpha$ a measure of noncompactness on $X$, $Y \subset X$ a closed convex subset of $X$ and $f: Y \to Y$ a continuous $(\alpha, \varphi)$-contraction such that $f(Y)$ is bounded.*

*Then $F_f \neq \Phi$ and $\alpha(F_f) = 0$.*

*Proof.* We have $\overline{\text{co}}(f(Y)) \in I(f)$. The proof follows from the Theorem 3.

*Remark* 2. From the Theorem 3 and 4 we have some results given by Darbo, Banas-Goebel, Rus, Danes, Pasicki, ... (see 1, 5, 6, 9, ...).

4. **$(\beta, \varphi)$-contractions.** We bigin with

Definition 8 (see [9, 19]). Let $X$ be a Banach space. A mapping $\beta : P_b(X) \to R_+$ is a *measure of nonconvexity* on $X$ iff

(i) $\beta(A) = 0$ implies $\bar{A}$ is a convex set,

(ii) $\beta(\bar{A}) = \beta(A)$ for all $A \in P_b(X)$,

(iii) $\beta : (P_{b,cl}(X), H) \to R_+$ is continuous.

*Example* 1. $\beta = \beta_{EL}$; $\beta_{EL}(A) = H(A, \text{co}A)$

Let $(X, d, W)$ be a convex metric space. In what follow we suppose that if $x, y \in X$ and $\{x, y\}$ is a convex set, then $x = y$.

From the Theorem A we have

THEOREM 5 (see [15, 19]). *Let $X(, d, W)$ be a strictly convex metric space with property* (C), *$Y \in P_{b,cl}(X)$ and $f: Y \to Y$ a nonexpansive $(\beta_{EL}, \varphi)$-contraction. Then $F_f = \{x^*\}$.*

*Proof.* Let $S = P_{b,cl,cv}(X)$, $M(Y) := \{f: Y \to Y \mid f \text{ nonexpansive}\}$, $\eta(A) = \bar{A}$ *and* $\theta = \beta_{EL} : P_b(X) \to R_+$. By a theorem of Takahashi, $(X, S, M)$ is a fixed point structure. From the Theorem A we have $F_f \neq \Phi$ and $\beta_{EL}(F_f) = 0$. Thus we have $F_f = \{x^*\}$.

THEOREM 6. *Let $(X, d, W)$ be a strictly convex metric space with property (C), $Y \in P_{cl}(X)$ and $f^m : Y \to Y$ a nonexpansive $(\beta_{EL}, \varphi)$ — contraction such hat $f^m(X)$ is bounded (for some $m \in N^*$).*

*Then $F_f = \{x^*\}$.*

*Proof.* We have $f^m(Y) \in I(f^m)$. From the Theorem 5, $F_{f^m} = \{x^*\}$, i.e, $F_f = \{x^*\}$.

THEOREM 7. *Let $X$ be a Hilbert space, $Y \subset X$ a closed subset of $X$ and $f : Y \to Y$ a mapping such that for some $m \in N^*$:*

(i) *$f^m(Y)$ is bounded,*

(ii) *$f^m : Y \to Y$ is a nonexpansive $(\beta_{EL'}, \varphi)$-contraction.*

*Then $F_f = \{x^*\}$.*

*Proof.* The proof follows from the Theorem 6.

5. **$(\gamma, \varphi)$-contractions.** Definition 9 (see [13]). Let $X$ be a Banach space. A mapping $\gamma : P_b(X) \to R_+$ is called a *measure of non compactconvexity* if

(i) $\gamma(A) = 0$ implies $\bar{A} \in P_{cp,cv}(X)$,

(ii) $\gamma(\bar{A}) = \gamma(A)$ for all $A \in P_b(X)$,

(iii) if $A_n \in P_{b,cl}(X)$, $A_{n+1} \subset A_n$ and $\gamma(A_n) \to 0$ as $n \to \infty$, then $\bigcap_{n \in N} A_n \neq \Phi$ and $\gamma(\bigcap_{n \in N} A_f) = 0$.

From the Theorem A we have

THEOREM 8 (see [13]). *Let $X$ be a Banach space, $\gamma$ a measure of non compact-convexity on $X$, $Y \subset X$ a closed bounded subset of $X$ and $f : Y \to Y$ a continuous $(\gamma, \varphi)$ — contraction.*

*Then $F_f = \{x^*\}$.*

*Proof.* Let $S = P_{cp,cv}(X)$, $M(Y) = C(Y, Y)$, $\theta = \gamma : P_b(X) \to R_+$ and $\eta(A) = \bar{A}$. The proof follow from the Theorem A.

From the Theorem 8 we have

THEOREM 9. *Let $X$ be a Banach space, $\gamma$ a measure of non compact-convexity on $X$, $Y \subset X$ a closed subset of $X$ and $f : Y \to Y$ a mapping such that for some $m \in N^*$:*

(i) *$f^m(Y)$ is bounded,*

(ii) *$f^m : Y \to Y$ is a continuous $(\gamma, \varphi)$ — contraction.*

*Then $F_f = \{x^*\}$.*

6. **Nonself mappings.** Definition 10 (Brown (see [16, 18]). Let $X$ be a nenempty set and $Y \subset X$ a nonempty subset of $X$. A mapping $\rho : X \to Y$ is called a *retraction* of $X$ onto $Y$ if $\rho/_Y = 1_Y$.

Definition 11 (Brown see [16, 18]). A mapping $f : Y \to X$ is *retractible* onto $Y$ if there is a retraction $\rho : X \to Y$ such that $F_f = F_{\rho \circ f}$.

We have

THEOREM 10 (see [18]). *Let $(X, S, M)$ be a fixed point structure and $(\theta, \eta)$ $(\theta : Z \to R_+, \eta : P(X)$ a compatible pair with $(X, S, M)$. Let $Y \in \eta(Z)$, $f : Y \to X$*

*a mapping and* $\rho: X \to Y$ *a retraction. We suppose that*

(i) $\theta|_{\eta(z)}$ *is a mapping with the intersection property,*

(ii) $f$ *is a strong* $(\theta, \varphi)$*-contraction,*

(iii) $f$ *is retractible onto* $Y$ *by* $\rho$ *and* $\rho \circ f \in M(Y)$,

(iv) $\rho$ *is* $(\theta, l)$*-Lipschitz* $(l \in R_+)$,

(v) *the function* $l\varphi$ *is a comparison function.*

*Then* $F_f \neq \Phi$ *and if* $F_f \in Z$, *then* $\theta(F_f) = 0$.

*Proof.* The mapping $\rho \circ f: Y \to Y$ is a strong $(\theta, l\varphi)$-contraction. By the theorem A, $F_{\rho \circ f} \neq \Phi$. From the condition (iii) follows $F_{\rho \circ f} = F_f \neq \Phi$. Let $F_f \in Z$. From $f(F_f) = F_f$ and the condition (ii) we have $\theta(F_f) = 0$.

*Remark* 3. From the theorem 10 we have some results given by Leray—Schuder, Brown, Caristi, Petryshyn, Reich, Rus, Banas, Williamson, ... (see [1, 4, 5, 6, 14, 18],...).

7. **Asymptotic fixed point theorems.** At the end of this paper we formulate the following

*Conjecture* 1. Let $(X, S, M)$ be a fixed point structure and $(\theta, \eta)(\theta: Z \to R_-, \eta: P(X) \to P(X)$ a compatible pair with $(X, S, M)$. Let $f \in M(X)$ be such that.

(i) $\theta_{\eta(Z)}$ has the intersection property,

(ii) $f$ is a $(\theta, \varphi)$-contraction,

(iii) there exists $m \in N^*$ such that $f^m(X) \in Z$.

Then $F_f \neq \Phi$, and if $F_f \in Z$, then $\theta(F_f) = 0$.

*Conjecture* 2. Let $(X, S, M)$ be a fixed point structure and $(\theta, \eta)$ a compatible pair with $(X, S, M)$. Let $f \in M(X)$ be such that

(i) $\theta|_{\eta(Z)}$ has the intersection property,

(ii) there exists $m \in N^*$ such that $f^m(X) \in Z$, and $f^m$ is a $(\theta, \varphi)$-contraction.

Then $F_f \neq \Phi$.

These conjectures are in connection with the following lang-standing,

*Conjecture* 3 (see [3]). Let $X$ be a Banach space, $Y \to X$ a nonempty closed bounded convex set and $f: Y \to Y$ a continuous mapping such that $f^m$ is compact for som $m \in N^*$. Then $f$ has a fixed point.

*Remark* 4. For some partial solutions, of the conjecture 3, given by Bourgin, Browder, Frum—Ketkov, Nussbaum, Fournier, Granas, Steinlein, ... see [3, 11] and [17].

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# THE ERROR ANALYSIS IN INTERPOLATION BY COMPLEX SPLINE FUNCTIONS

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**REZUMAT. — Studiul erorii în interpolarea prin funcţii spline complexe.** Se evaluează norma funcţiei $F = \sigma - f$, unde $f$ este o funcţie analitică în domeniul $\Omega$ deschis din $\mathbf{R}^2$, cînd cunoaştem valorile $f_k = f(z_k)$, $k = 0, 1, \ldots, n+1$, pe nodurile $z_k$ ale unei diviziuni date pe o curbă închisă $\Gamma$, rectificabilă Jordan, conţinută în $\Omega$, iar $\sigma$ este o funcţie spline cubică complexă de interpolare a lui $f$. În plus $f$ şi $\sigma$ îndeplinesc condiţia lui Hölder.

1. **Introduction.** Let $\Gamma$ be a closed curve, Jordan rectifiable, that belong to the open domain $\Omega$ from $\mathbf{R}^2$. One considers the partition

$$\Delta_\Gamma : \{P_0, P_1, \ldots, P_n, P_{n+1};\ P = P_{n+1}\} \tag{1}$$

that divides the curve $\Gamma$ in the arcs $\Gamma_k$ from $P_{k-1}$ to $P_k$, $k = 1, \ldots, n+1$.

On denotes

$$h_k = z_k - z_{k-1},\quad k = 1, \ldots, n+1 \tag{2}$$

where $z_k = x_k + iy_k$; $x_k, y_k \in \mathbf{R}$ is the affix of

$$P_k,\quad k = 0, 1, \ldots, n+1 \quad (z_0 = z_{n+1})$$

Knowing the values $f_k = f(z_k)$, $k = 0, 1, \ldots, n+1$, of a given function $f$, that is analytic in $\Omega$, in the paper [13] it was constructed a complex cubic spline $\sigma$, of the form

$$\sigma(z) = \frac{M_k - M_{k-1}}{6h_k}(z - z_{k-1})^3 + \frac{M_{k-1}}{2}(z - z_{k-1})^2 + m_{k-1}(z - z_{k-1}) + f_{k-1},\ z \in \Gamma_k,$$
$$k = 1, 2, \ldots, n+1, \tag{3}$$

where $m_j = \sigma'(z_j)$ and $M_j = \sigma''(z_j)$, $j = 0, 1, \ldots, n+1$, while $h_j$, $j = 1, 2, \ldots, n+1$ is given by the formula (2), that interpolates the function $f$.

From the conditions

$$\begin{cases} \sigma(z_k) = f_k,\ k = 0, 1, \ldots, n+1, \ (f_{n+1} = f_0) \\ \sigma'(z_k) = m_k,\ k = 0, 1, 2, \ldots, n+1, \end{cases} \tag{4}$$

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that can be written in the form

$$\begin{cases} M_k' = 6\dfrac{f_k - f_{k-1}}{h_k^2} - 6\dfrac{m_{k-1}}{h_k} - 2M_{k-1} \\ m_k = 3\dfrac{f_k - f_{k-1}}{h_k} - 2m_{k-1} - \dfrac{M_{k-1}}{2}h_k, \quad k = 1, 2, \ldots, n+1 \end{cases} \tag{5}$$

with $m_0 = a$, $M_0 = b$; $a, b \in \mathbf{R}$, it follows that the spline function (3) exists and it is unique [13].

In the paper [13] it is also evaluated the value

$$|\sigma(z) - f(z)| \text{ for } z = \frac{z_{k-1} + z_k}{2}, \; z \in \Gamma_k, \; k = 1, \ldots, n+1$$

when $f$ satisfies the Hölder's condition on $\Gamma$ and it is obtained

$$|\sigma(z) - f(z)| \leqslant \frac{|h_k|}{8}\left[A|h_k|^{\mu-1}\left(1 + \frac{8}{2^\mu}\right) + \frac{1}{2}|M_{k-1}| \cdot |h_k| + 3|m_{k-1}|\right], \tag{6}$$

where $A$ is the Hölder's constant, $\mu \in (0, 1)$ is the Hölder's exponent and $M_{k-1}$, $m_{k-1}$, $k = 2, \ldots, n+1$, are obtained by (5) with initial data $M_0$, $m_0$.

**2.** In the present paper one estimates the norm of the error function $F$, $F = \sigma - f$, in the conditions that $\sigma, f \in H^\mu(\Gamma)$ — the Banach space of the functions $\varphi$ which satisfies the condition

$$|\varphi(t'') - \varphi(t')| \leqslant A|t'' - t'|^\mu, \quad \forall t', t'' \in \Gamma, \tag{7}$$

with the norm [15]

$$||\varphi||_{H^\mu} = ||\varphi||_\infty + M_\mu(\varphi) \tag{8}$$

where

$$M_\mu(\varphi) = \sup_{t', t'' \in \Gamma} \frac{|\varphi(t') - \varphi(t'')|}{|t' - t''|^\mu}. \tag{9}$$

Now, if we consider the parameter $s$ (the length of the arc), $s = s(t)$, $t \in \Gamma$, we have

$$s_k = s(z_k), \; k = 0, 1, \ldots, n+1.$$

Next, one denotes by

$$\lambda_k = s_k - s_{k-1}, \tag{10}$$

the measure of the arc $\Gamma_k$, $k = 1, \ldots, n+1$.

Taking into account the formulas (2) and (10), we have

$$|h_k| \leqslant \lambda_k, \; k = 1, 2, \ldots, n+1 \tag{11}$$

First, we estimate the value

$$\Delta_k F = |F(t') - F(t'')|, \; \forall t', t'' \in \Gamma_k \; (t' \neq t''), \tag{12}$$

where

$$F(t') = \sigma(t') - f(t'), \qquad F(t'') = \sigma(t'') - f(t'')$$

We have

$$F(t') - F(t'') = \sigma(t') - f(t') - \sigma(t'') + f(t'') = f(t'') - f(t') +$$
$$+ \frac{M_k - M_{k-1}}{6h_k} [(t' - z_{k-1})^3 - (t'' - z_{k-1})^3] +$$
$$+ \frac{M_{k-1}}{2} [(t' - z_{k-1})^2 - (t'' - z_{k-1})^2] +$$
$$+ m_{k-1}[(t' - z_{k-1}) - (t'' - z_{k-1})] + f_{k-1} - f_{k-1}, \qquad \forall t', t'' \in \Gamma_k.$$

So, one obtains

$$\Delta_k F \leqslant |f(t'') - f(t')| + \frac{1}{6}\left|\frac{M_k - M_{k-1}}{h_k}\right| \cdot |(t' - z_{k-1})^3 - (t'' - z_{k-1})^3| +$$
$$+ \frac{1}{2} |M_{k-1}| \cdot |(t' - z_{k-1})^2 - (t'' - z_{k-1})^2| +$$
$$+ |m_{k-1}| \cdot |t' - z_{k-1} - t'' + z_{k-1}| = |f(t'') - f(t')| +$$
$$+ \frac{1}{6}\left|\frac{M_k - M_{k-1}}{h_k}\right| \cdot |[(t' - z_{k-1}) - (t'' - z_{k-1})] \cdot [(t' - z_{k-1})^2 +$$
$$+ (t' - z_{k-1})(t'' - z_{k-1}) + (t'' - z_{k-1})^2]| +$$
$$+ \frac{1}{2} |M_{k-1}| \cdot |[(t' - z_{k-1}) - (t'' - z_{k-1})] \cdot [(t' - z_{k-1}) +$$
$$+ (t'' - z_{k-1})]| + |m_{k-1}| \cdot |t' - t''|,$$

which, for $f \in H^\mu(\Gamma)$ can be written in the form:

$$\Delta_k F \leqslant A |t' - t''|^\mu + \frac{1}{6}\left|\frac{M_k - M_{k-1}}{h_k}\right| \cdot |t' - t''| \cdot |(t' - z_{k-1})^2 +$$
$$+ (t' - z_{k-1})(t'' - z_{k-1}) + (t'' - z_{k-1})^2| + \tag{14}$$
$$+ \frac{1}{2} |M_{k-1}| \cdot |t' - t''| \cdot |(t' - z_{k-1}) + (t'' - z_{k-1})| + |m_{k-1}| \cdot |t' - t''|.$$

If $t', t'' \in \Gamma_k$ then we have the following three cases:

$$\begin{array}{ll} \text{a)} & z_{k-1} \prec t' \prec t'' \prec z_0 \prec z_k \\ \text{b)} & z_{k-1} \prec t' \prec z_0 \prec t'' \prec z_k \\ \text{c)} & z_{k-1} \prec t' \prec z_0 \prec t'' \prec z_k, \end{array} \tag{15}$$

where $x \prec y$ means that $x$ precede $y$ on $\Gamma$ and $z_0 = (z_{k-1} + z_k)/2$.
Taking into account (10), (11) and (15, a), it follows that

$$|t' - z_{k-1}| \leqslant \frac{|z_k - z_{k-1}|}{2} \leqslant \frac{1}{2}\lambda_k$$
$$|t'' - z_{k-1}| \leqslant \frac{1}{2}\lambda_k \tag{16}$$
$$|t'' - t'| \leqslant \frac{1}{2}\lambda_k.$$

Using these inequalities, from (14) one obtains

$$\Delta_k F \leqslant \frac{1}{2}\lambda_k\left\{A\left(\frac{1}{2}\lambda_k\right)^{\mu-1}+\frac{1}{6}\left|\frac{M_k-M_{k-1}}{h_k}\right|\cdot\left|\frac{\lambda_k^2}{4}+\frac{\lambda_k^2}{4}+\frac{\lambda_k^2}{4}\right|+\right.$$

$$\left.+\frac{1}{2}\,|M_{k-1}|\cdot\left|\frac{1}{2}\lambda_k+\frac{1}{2}\lambda_k\right|+|m_{k-1}|\right\}= \tag{17}$$

$$=\frac{1}{2}\lambda_k\left\{A\left(\frac{1}{2}\lambda_k\right)^{\mu-1}+\frac{\lambda_k^2}{8}\left|\frac{M_k-M_{k-1}}{h_k}\right|+\frac{\lambda_k}{2}\,|M_{k-1}|+|m_{k-1}|\right\}.$$

In the same way, for the case (15,b) one obtains the inequalities

$$|t'-z_{k-1}|\leqslant\frac{\lambda_k}{2}$$

$$|t''-z_{k-1}|\leqslant\frac{\lambda_k}{h} \tag{18}$$

$$|t''-t'|\leqslant\lambda_k$$

that implies

$$\Delta_k F\leqslant A(\lambda_k)^{\mu}+\frac{1}{6}\left|\frac{M_k-M_{k-1}}{h_k}\right|\lambda_k\cdot\left(\frac{\lambda_k^2}{4}+\frac{\lambda_k^2}{4}+\lambda_k^2\right)+$$

$$+\frac{1}{2}\,|M_{k-1}|\cdot\lambda_k\left(\frac{\lambda_k}{2}+\lambda_k\right)+|m_{k-1}|\cdot\lambda_k= \tag{19}$$

$$=\lambda_k\left\{A(\lambda_k)^{\mu-1}+\frac{7}{24}\lambda_k^2\left|\frac{M_k-M_{k-1}}{h_k}\right|+\frac{3}{4}\lambda_k|M_{k-1}|+|m_{k-1}|\right\}$$

respectively for the case (15,c), the inequalities

$$|t'-z_{k-1}|\leqslant\lambda_k$$

$$|t''-z_{k-1}|\leqslant\lambda_k \tag{20}$$

$$|t''-t'|\leqslant\frac{1}{2}\lambda_k$$

and

$$\Delta_k F\leqslant A\left(\frac{1}{2}\lambda_k\right)^{\mu}+\frac{1}{6}\left|\frac{M_k-M_{k-1}}{h_k}\right|\cdot\frac{1}{2}\lambda_k(\lambda_k^2+\lambda_k^2+\lambda_k^2)+$$

$$+\frac{1}{2}\,|M_{k-1}|\cdot\frac{1}{2}\lambda_k(\lambda_k+\lambda_k)+|m_{k-1}|\cdot\frac{1}{2}\lambda_k= \tag{21}$$

$$=\frac{1}{2}\lambda_k\left\{A\left(\frac{1}{2}\lambda_k\right)^{\mu-1}+\frac{\lambda_k^2}{2}\left|\frac{M_k-M_{k-1}}{h_k}\right|+\lambda_k\,|M_{k-1}|+|m_{k-1}|\right\}.$$

So, it is proved the following:

PROPOSITION 1. *Let* $\Gamma$ *be a closed and Jordan rectifiable curve in the open domain* $\Omega$ *from* $\mathbb{R}^2$, $\Delta_\Gamma$ *a partition on* $\Gamma$ *defined by* (1), $f\in H^{\mu}(\Gamma)$ *and* $f_k=f(z_k)$, $k=0,1,\ldots,n+1$.

Then

$$\Delta_k F \leqslant \frac{1}{2}\lambda\left\{A\left(\frac{\lambda}{2}\right)^{\mu_1} + \frac{\lambda^2}{8}\left|\frac{M_k - M_{k-1}}{h_k}\right| + \frac{\lambda}{2}\,|M_{k-1}| + |m_{k-1}|\right\} = L_{k,1} \tag{22}$$

*in the case* (15;a)

$$\Delta_k F \leqslant \lambda\left\{A(\lambda)^{\mu-1} + \frac{7}{24}\lambda^2\left|\frac{M_k - M_{k-1}}{h_k}\right| + \frac{3}{4}\lambda\,|M_{k-1}| + |m_{k-1}|\right\} = L_{k,2} \tag{23}$$

*in the case* (15;b)

$$\Delta_k F \leqslant \frac{1}{2}\lambda\left\{A\left(\frac{\lambda}{2}\right)^{\mu-1} + \frac{1}{2}\lambda^2\left|\frac{M_k - M_{k-1}}{h_k}\right| + \lambda\,|M_{k-1}| + |m_{k-1}|\right\} = L_{k,3} \tag{24}$$

*in the case* (15;c)

*where* $\lambda = \max\ \{\lambda_k \mid k = 1, 2, \ldots, n+1\}$.

Using these rezults we can estimate the norm $||F||_{H^\mu}$. By (8) and (9) we have

$$||F||_{H^\mu} = ||F||_\infty + M_\mu(F) \tag{25}$$

where

$$||F||_\infty = \sup_{t\in\Gamma_k} |F(t)|$$

and

$$M_\mu(F) = \sup_{\substack{t',t''\in\Gamma_k\\ t'\neq t''}} \frac{|F(t') - F(t'')|}{|t' - t''|^\mu}$$

If one denotes by

$$L_k = \max\left\{\frac{L_{k,1}}{|t' - t''|^\mu}, \frac{L_{k,2}}{|t' - t''|^\mu}, \frac{L_{k,3}}{|t' - t''|^\mu}\right\} \tag{26}$$

then it follows:

PROPOSITION 2. *In the conditions of the Proposition* 1, *if* $\sigma \in H^\mu(\Gamma)$ *then*

$$||F||_{H^\mu} \leqslant ||F||_\infty + L_k,\ \ \forall\Gamma_k,\ k = 1, 2, \ldots, n+1.$$

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# RECENZII

J. R. Giles, **Convex Analysis with Application in Differentiation of Convex Functions**, Research Notes in Mathematics, N° 58, Pitman, Boston — London — Melbourne, 1982.

The aim of this book is to study the differentiability properties of convex functions with a special emphasis on their connections with other areas of functional analysis as — geometry of Banach spaces, integration in Banach spaces, Radon-Nikodym property as well as their significance in applications to optimization theory and fixed point theorems. The main themes of the book are: the duality between convex sets and convex functions and the relations between spaces with Radon-Nikodym property and the differentiability properties of convex functions defined on these spaces. The book is selfcontained, the only prerequisities being a knowledge of the basic concepts of functional analysis and topology.

The first two chapters have an introductory character presenting the background material needed in the rest of the book. Chapter 1, Convexity in linear spaces, is concerned with topics as convex sets, convex functions, separation theorems (algebraic theory), while the second chapter, Convexity in linear topological spaces, presents the fundamentals of locally convex space theory.

The core of the book is Chapter 3, The differentiation of convex functions, which is concerned with topics as: lower continuity and lower semi-continuity of convex functions, Gâteaux differentiability in linear and normed spaces (including Kenderov's theorem on weak Asplund spaces) and Fréchet differentiability. The famous Bishop-Phelps theorem on subreflexivity of Banach spaces is presented in detail, in some varied formulations with many and consistent applications. The chapter closes with a section on the exposed structure of convex sets containing Strasewicz-Klee theorem, weak*-Asplund spaces and other topics. The last chapter of the book, Chapter 4, Two convexity problems in the geometry of Banach spaces, is concerned with Mazur's intersection property and convexity of Chebyshev sets, showing that the developed theory applies to prove how topological properties, associated with a norm, characterize convexity of certain sets in the space.

Many exercises, some of them being referred to in the proofs, and oustanding research problems are included in the places where they arise logically in the development of the theory.

The result is a fine book on convex analysis and its applications, which is higly recommended to all interested in this field of research.

S. COBZAŞ

V. Barbu, G. Da Prato, **Hamilton—Jacobi equations in Hilbert spaces**, Research Notes in Mathematics vol. 86, Pitman, Boston — London — Melbourne, 1983, 172 p.

The study of Hamilton—Jacobi equations in infinite dimensional spaces, particularly in Hilbert spaces, is motivated by the theory of deterministic distributed parameter systems and by the control of stochastic distributed systems. The aim of this book is to gather together, in the form of lecture notes, the results obtained in recent years by the authors on the existence and approximation of H—J equations in Hilbert spaces. The investigation is based on two interrelated methods: a constructive approximating method and the dynamic programming method.

In order to make the book self-contained, the authors survey in the first Chapter of the book, entitled Preliminaries, the basic results from convex analysis and from the theory of evolution equations in Banach spaces. Chapter 2, Existence in the class of convex functions, is concerned with initial value problem for the H—J equation $\Phi_t(t, x) + F(\Phi_x(t, x)) - (Ax, \Phi_x(t, x)) = g(t, x), t \in [0, T], x \in D(A), \Phi(0, x) = \Phi_0(x)$. The solution $\Phi$ is a function from $[0, T] \times H$ to $H$, the operator $A : D(A) \subset H \to H$ is the infinitesimal generator of a $C_0$ — semigroup of linear continuous operators on $H$ and $F$, $\Phi_0$, $g$ are real-valued convex functions (with respect to $x$), defined on $H$ and $[0, T] \times H$, respectively. The authors's approach to the problem consists in approximating the operator $\Phi \to F(\Phi_x)$ (which is an accreative operator in an appropriate function space) by $\varepsilon^{-1}(\Phi - (\Phi^* + \varepsilon F)^*$ (here $*$ stands for the Fenchel conjugate) and letting $\varepsilon$ tend to zero in the approximating equation. It is shown that $\Phi \to - F(\Phi_x)$ is the infinitesimal generator of a semigroup of contractions on $C(H)$ (the space of all continuous functions bounded on the balls of $H$). A special attention is paid to the case [illegible] which is studied in §§ 5 and 6 and the Riccati equation (§ 8).

In the third Chapter, Existence theory in nonconvex case, the theory is developped in the nonconvex case. Here is proved an existence theorem for $\Phi$ a locally Lipschitz function and $\Phi_x$ – its generalized Clarke subgradient. Also a direct approach and a variational ones are proposed. A section is devoted to Galerkin method of approximation and applications to synthesis of optimal controls are also included.

In the last chapter of the book, Chapter 4. The dynamic programming equation for stochastic optimal control, the H–J equation is studied in a class of convex functions on a real Hilbert space, verifying suitable growth conditions at infinity. The method proposed here consists in the study of a related optimal control problem of a process governed by a stochastic differential equation.

The book is very well and clearly written. Each chapter is preceded by a section presenting shortly the methods used and the results obtained in the respective chapter.

The book is a valuable contribution to the subject and we recommend it warmly to all interested in this domain of research.

S. COBZAŞ

V. Barbu, **Optimal control of variational inequalities.** Research Notes in Mathematics vol. 100, Pitman, Boston – London – Melbourne, 1984, 298 p.

The book is devoted to the study of first-order necessary conditions of optimality for control problems governed by variational inequalities or by semilinear equations of elliptic and parabolic type. Such problems appears in the study of free boundary value problems in filtration, heat conduction and diffusion theory. More exactly, the book is concerned with a class of nonlinear control systems, $Ay + Fy \ni Bu$ and $y'(t) + Ay(t) + Fy(t) \ni Bu(t)$, $0 \leq t \leq T$, where $A$ is a linear self adjoint positive definite operator acting on the state space $H$ ($H$ is Hilbert space), $F$ is a subgradient operator and $B$ is a linear continuous operator from the space of controls to state space. The book is self-contained, the necessary results from nonlinear analysis and variational inequality theory are collected in the first three chapters of the book.

Chapter 1. Elements of nonlinear analysis, is concerned with maximal monotone operators in Hilbert space, convex analysis, generalized gradients of locally Lipschitz functions and with nonlinear evolution equations in Hilbert space.

Chapters 2 and 3 deal with elliptic variational inequalities (existence and regularity) results and with optimal control of these inequalities.

The main body of the book is formed of the other chapters which are: Chapter 4, Parabolic variational inequalities; Chapter 5, Optimal control of parabolic variational inequalities: distributed controls; Chapter 6, Boundary control of parabolic variational inequalities; Chapter 7, The time-optimal control problem.

The general theory is illustrated by significant examples, which are chosen to be relatively simple, but such they contain the main ideas of the developed theory and which can be used as theoretical models for more sophisticated problems arising in the control of industrial processes.

Written by a leading specialist in the field and based mainly on the results obtained by the author, some of them published for the first time, the book is an excellent guide in this subject as well as a source of inspiration for further investigations and generalizations.

We recommend it warmly to all interested in this field of research and its applications.

S. COBZAŞ

N. H. Pavel, **Differential equations, flow invariance and applications.** Research Notes in Mathematics vol. 113, Pitman, Boston – London – Melbourne, 1984, 246 p.

The book is concerned with existence results for differential equations on closed subsets of Banach spaces and flow invariance of these sets with respect to a differential equation.

In order to make the book self-contained the author collects in the first chapter, entitled Preliminaries of nonlinear analysis, some results on the geometry of Banach spaces strict convexity, uniform convexity, duality mapping and on dissipative and accretive operators.

The second chapter of the book – Differential equations on closed subsets. Flow invariance – begins with the proof of the famous Godunov's theorem that Peano's existence theorem for first order differential equations with continuous right hand side is valid only in finite dimensional Banach spaces. Then, using a sharp version of the classical method of polygonal lines, the author proves some existence theorem for differential equations associated with continuous and dissipative time-dependent domain operators. The flow invariance with respect to a first differential equation is treated in the fourth section of this chapter in the following setting: let $X$ be a Banach space, $G$ a non-void open subset of $X$, $-\infty \leq a < b \leq \infty$ and $A : ]a, b[ \times X \to X$ a continuous function (the values of $A$ at $(t, x)$ is denoted by $A(t)x$) and $t_0 \in ]a, b[$. A closed subset $D$ of $G$ is called flow invariant with respect to the differential equation $u'(t) = A(t) \cdot u(t)$, $u(t_0) = x_0$, if every solution starting in $D$ remains in $D$ as long as it exists. This section contains several necessary and sufficient conditions for the flow invariance of a closed subset $D$ of $G$. In Section 5 the flow invariance is studied in the case of Carathéodory type conditions. Chapter 3, Flow invariance with respect to second-order differential equations and applications to flight space, the flow invariance is studied for secondorder differential equations. By studying the flow invariance of a conic, the author gives interesting applications to the motion of a material point in a Newtonian field (elliptic, hyperbolic and parabolic velocities).

The main tool used in Chapter 4, Flow invariance on Banach manifolds and some optimization problems, is the remark that differentiability preserves transversality. Also, interesting connections with Clarke's tangent cone and Clarke's generalized gradients are established. In the last chapter of the book — Chapter 5, Perturbed differential equations — using methods and results from the theory of one-parameter semigroups of operators acting on a Banach space, the author study semilinear equations of retarded type and semilinear equation with dissipative time-dependent domain perturbations. Some applications to partial differential equations are also included.

Beside the theoretical material the book contains also many interesting applications to chemistry (modelling enzymatic or chemical reactions) to medicine (the spatial spread of bacterial and viral diseases) and to mechanics.

The book is clearly written and is based essentially on the results obtained by the author, some of them in cooperation with D. Motreanu, C. Ursescu, I. I. Vrabie.

We recommend it to all interested in abstract differential equations and their applications.

S. COBZAŞ

I. I. Vrabie, **Compactness Methods for Nonlinear Evolutions**, Pitman Monographs and Surveys in Pure and Applied Mathematics 32, Longman Scientific & Technical, Copublished in the United States with John Wiley & Sons, Inc., New York, 1987, 325 pages.

This volume gives a coherent account of the most significant results concerning the existence problem for strongly nonlinear evolution equation with the main applications to partial differential equations.

It is organised into five chapters, and is characterized by a clear and consistent report which lightens considerably its reading. The presented results are the latest (most of them are due to the author), except the first chapter which introduces the main concepts of nonlinear functional analysis with emphasis on the fundamental results of compactness in $C([a, b]; X)$ and $L([a, b]; X)$ spaces, where $X$ is a real Banach space. Therefore, there are no proofs in the first chapter, but instead ,whenever necessary, there are indicated the appropiate monographs and articles where the relevant details can be found.

In Chapter 2 is introduced the notion of compact semigroup which has several applications. The author presents the proof of the theorem which characterizes the generator of a compact semigroup and also the results concerning the relative compactness of the subset of mild solutions in $C([a, b]; X)$ of nonlinear evolution equations guverned by m-accretive operators. There are given many applications, as for example those concerning the nonlinear diffusion equations and optimal control theory.

Chapter 3 treats the results refering to the existence for nonlinear evolutions gouverned by non-accretive perturbations of m-accretive operators.

The central purpose in Chapter 4 is to study the existence theorems gouverned by perturbations of subdifferentials of proper, convex, lower semicontinous functions. This results are applied for two-equations of Navier-Stokes type.

The last Chapter — using the results of Chapter 2, studies the integro-differential equations of type Volterra and functional-differential equations.

On the whole, the work contains many illustrative examples, which facilitate the understanding of the main notions and results. Well set up and written carefully, the book can be recommended for every mathematician who studies the theory of nonlinear equations.

G. KASSAY

A. E. Djrbashian, F. A. Shamoian, **Topics in the Theory of $A^p_\alpha$ Spaces**, Teubner-Texte zur Mathematik, Band 105, Leipzig 1988, 199 p.

For a function $f(z)$, analytic in the unit disk $D = \{z \in C : |z| \leq 1\}$, denote by $M_p(f, r) = \left[(2\pi)^{-1} \int_{-\pi}^{\pi} |f(re^{it})|^p \cdot dt\right]^{1/p}$, $0 \leq r < 1$, $0 < p < \infty$, the $p$' th integral mean of $f$ and $M_\infty(f, r) = \sup\{|f(z)| : |z| \leq r\}$. The Hardy space $H^p$ is defined as the class of all analytic functions $f$ in $D$, such that $\sup\{M_p(f, r) : 0 \leq r < 1\} < \infty$. A natural way to enlarge the space $H^p$ is to suppose the mean $M_p(f, r)$ belongs only to a weighted $L^p$ – space. The aim of this book is to study the space $A^p_\alpha$, defined for $0 < p < \infty$, and $-1 < \alpha < \infty$ by the condition $\int_0^1 M_p(f, r)^p \cdot (1 - r)^\alpha \cdot r \cdot dr < \infty$ or, equivalently, by the condition $\int_D |f(z)|^p \cdot (1 - |z|)^\alpha \cdot dm_2(z) < \infty$, where $dm_2$ denotes the Lebesgue planar measure. These spaces are called sometimes Bergman spaces but, as point out the authors in the preface of the book, M.M. Djrbashian was the first who in a Note published in 1945 in Doklady Akad. Nauk Armian S.S.R. considered the general spaces $A^p_\alpha$, as defined above, and studied their properties.

The book under review is devoted to a systematical treatment of $A^\alpha_p$ spaces in the unit disk of the complex plain as well in higher dimensions. The space $A^*_\alpha$ of all analytic functions $f(z)$, satisfying $\int_D \log^+ |f(z)| \cdot (1 - |z|)^\alpha \cdot dm_2(z) < \infty$ is also considered. This space is the generalization of the classical Nevanlinna space $N$. The first three chapters — 1. Spaces of area integrable functions in the unit disk, 2. Bounded projections, conjugate harmonic functions and bounded linear functionals on $A^p_\alpha$, 3. Zeros of $A^p_\alpha$ functions— constitute the textbook part, where the authors have collected classical results. The remaining four chapters are devoted to the presentation of the advanced part of the theory, where more recent results, most of them belonging to the authors, are presented. These chapters are: 4. On zeros of analytic functions with restricted growth; 5. Division in $X^\infty_\varphi$ and description of closed ideals; 6. $A^p_\alpha$ spaces of holomorphic functions in higher dimensions; 7. $A^p_\alpha$ spaces of harmonic functions in the unit ball of $R^n$.

The theory of $A^p_\alpha$ spaces have many contrasts with the theory of classical Hardy spaces. For instance, it is well known that the projection from $L^p$ (on the boundary of the unit disk) to $H^p$ is bounded for $1 < p < \infty$. In the case of spaces $A^p_\alpha$ the natural projection from an appropriate weighted $L^p$ – space on $D$ to $A^p_\alpha$ is bounded for all $p$, $1 \leq p < \infty$, and the harmonic conjugacy is a bounded operator for all values of $p$, $0 < p < \infty$ (by M. Riesz theorem, in the classical case this operator is bounded only for $1 < p < \infty$). There are also many unsolved problems concerning the zeros distribution in $A^p_\alpha$ spaces.

The book is the first attempt to collect together results spreaded in various publications and to propose a unified terminology. The authors achieve masterly this task, the result being a clear and throughout introduction to the theory of $A^p_\alpha$ spaces, an important branch of modern complex analysis having many applications in various problems of analysis.

We recommend it warmly to all interested in these problems.

S. COBZAŞ

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