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CLASSES OF n-α-CLOSE-TO-CONVEX FUNCTIONS

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REZUMAT. — Clase de funcţii n-α-aproape convexe. Se introduc clase noi de funcţii univalente, care generalizează unele clase definite de H. S. Al-Amiri [1] şi S. Ruscheweyh [6] şi se stabilesc unele proprietăţi de incluziune între aceste clase.

1. **Introduction** Let A be the class of functions $f(z)$, analytic in the unit disc U with $f(0) = f'(0) - 1 = 0$. As in [2] we denote by $K_{n,\alpha}(\delta)$ the class of functions $f \in A$ which satisfy

$$\mathrm{Re}\left[(1-\alpha)\frac{D^{n+1}f(z)}{D^n f(z)} + \alpha\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right] > \delta, \quad z \in U,$$

where $\alpha \geqslant 0$, $\delta < 1$ and $D^n f(z) = \frac{z}{(1-z)^{n-1}} * f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!}$, where $(*)$ stands for the Hadamard product. Some results concerning the classes $K_{n,\alpha}(\delta)$ and $Z_n(\delta) \equiv K_{n,0}(\delta)$ are presented in [2]; note that the classes $K_{n,\alpha}\left(\frac{1}{2}\right)$ and $Z_n\left(\frac{1}{2}\right)$ were introduced by *H. S. Al-Amiri* [1] and *S. Ruscheweyh* [6] respectively.

Let $AC_n(\delta)$ be the class of functions $f \in A$ which satisfy

$$\mathrm{Re}\,\frac{D^{n+1}f(z)}{D^{n+1}g(z)} > \delta, \quad z \in U, \quad \text{where } g \in Z_{n+1}(\delta),$$

and we call this class the class of *n-close-to-convex functions of order* δ.

Let $C_{n,\alpha}(\delta)$ be the class of functions $f \in A$ which satisfy

$$\mathrm{Re}\left[(1-\alpha)\frac{D^{n+1}f(z)}{D^{n+1}g(z)} + \alpha\frac{D^{n+2}f(z)}{D^{n+2}g(z)}\right] > \delta, \ z \in U, \text{ where}$$

$g \in Z_{n+2}(\delta)$, and we call this class the class of *n-α-close-to-convex functions of order* δ.

Note that $C_{n,1}(\delta) = AC_{n+1}(\delta)$, $C_{n,0}(\delta) = AC_n(\delta)$, $Z_{n+1}(\delta) \subset AC_n(\delta)$ and $Z_{n+2}(\delta) \cup C_{n,\alpha}(\delta)$; the classes $C_{n,\alpha}\left(\frac{1}{2}\right) = C_n(\alpha)$, $AC_n\left(\frac{1}{2}\right) = C_n$ were introduced by H. S. Al-Amri [1], who proved that $C_n(\alpha) \subset C_n$, $\alpha \geqslant 0$ and $C_n(\alpha) \subset C_n(\beta)$, $\alpha > \beta \geqslant 0$. In this paper we shall study some properties of these classes and several particular results will also be given.

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2. **Preliminaries.** Let f and g be regular in U. We say that f is subordinate to g, written $f(z) \prec g(z)$, if g is univalent, $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

We will need the following theorems to prove our main results.

THEOREM A. Let $\frac{1}{2} \leqslant \delta < 1$ *and* $n \in N$; *then* $Z_{n+1}(\delta) \subset Z_n(\delta_n^*)$ *where*

$\delta_n^* = 1/F\left(1,\ 2(n+2)(1-\delta),\ n+2;\ \frac{1}{2}\right)$, *and this result is sharp.*

This theorem is a particular case of Theorem 3[2], when $\alpha = 1$.

THEOREM B. [5]. *Let* $\beta > 0$, $\beta + \gamma > 0$ *and* $-\frac{\gamma}{\beta} \leqslant \delta < 1$. *Then the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = \frac{1-(1-2\delta)z}{1+z} \equiv h_\delta(z),\quad q(0) = 1$$

has a univalent solution is U, *given by*

$$q(z) = \frac{1}{\beta Q(z)} - \frac{\gamma}{\beta}$$

where $Q(z) = \int\limits_0^1 \left(\frac{1-z}{1-rz}\right)^{2\beta(1-\delta)} t^{\beta+\gamma-1}\ dt,\ z \in U,$ *and* $q(z) \prec \frac{1-(1-2\delta)z}{1+z}$.

If $p(z) = 1 + p_1 z + \ldots$ *is regular in* U *and satisfies the differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1-(1-2\delta)z}{1+z}$$

then $p(z) \prec q(z)$ *and this subordination is sharp.*

THEOREM C. [3]. *Let* $d\mu(t)$ *be a positive measure on* $[0, 1]$ *and let* $Q(z, t)$ *be a complex-valued function defined on* $U \times [0, 1]$, *such that* $Q(z, \cdot)$ *is integrable on* $[0, 1]$ *for each* $z \in U$. *Suppose that* $\operatorname{Re} Q(z, t) > 0$ *in* U, $Q(-r, t)$ *is real and*

$$\operatorname{Re} \frac{1}{Q(z,r)} \geqslant \frac{1}{Q(-r,t)} \quad for\ |z| \leqslant r < 1$$

and $t \in [0, 1]$.

$$If\ Q(z) = \int\limits_0^1 Q(z, t)\, d\mu(t),\ then\ \operatorname{Re} \frac{1}{Q(z)} \geqslant \frac{1}{Q(-r)}\ for\ |z| \leqslant r.$$

THEOREM D. [4]. *Let* $h, q \in H(U)$ *be univalent in* U *and suppose* $q \in H(\bar{U})$. *If* $\psi: \mathbf{C}^3 \to \mathbf{C}$ *satisfies:*

a) ψ *is analytic in a domain* $D \subset \mathbf{C}^3$,

b) $(q(0), 0, 0) \in D$ *and* $\psi(q(0), 0, 0) \in h(U)$,

c) $\psi(r, s, t) \notin D$ *when* $(r, s, t) \in D$, $r = q(\zeta)$, $s = m\zeta q'(\zeta)$,

$\mathrm{Re}(1 + t/s) \geqslant m\,\mathrm{Re}\,(1 + \zeta q''(\zeta)/q'(\zeta))$, *where* $|\zeta| = 1$, $m \geqslant 1$, *then for all* $p \in H(U)$ *so that* $(p(z), zp'(z), z^2p''(z)) \in D$, $z \in U$ *we have:*

$$\psi(p(z), zp'(z), z^2p''(z)) \prec h(z) \Rightarrow p(z) \prec q(z).$$

3. **Main results.** *THEOREM* 1. *Let* $\alpha \geqslant 0$ *and* $\frac{1}{2} \leqslant \delta < 1$; *if* $f \in C_{n,\alpha}(\delta)$ *related to* $g \in Z_{n+2}(\delta)$ *then* $f \in AC_n(\delta^*_{n+1})$ *related to* $g \in Z_{n+1}(\delta^*_{n+1})$, *where* $\delta^*_{n+1} = 1/F\left(1,\ 2(n+3)(1-\delta),\ n+3;\ \frac{1}{2}\right)$.

Proof. Using Theorem A we obtain $g \in Z_{n+1}(\delta^*_{n+1})$ i.e.

$$\frac{D^{n+2}g(z)}{D^{n+1}g(z)} \prec h_{\delta_{n+1}}(z).$$

Let $p(z) = D^{n+1}f(z)/D^{n+1}g(z)$; then $p(0) = 1$ and using
$z(D^nf(z))' = (n+1)D^{n+1}f(z) - nD^nf(z)$, $n \in \mathbf{N}$ we obtain that $f \in C_{n,\alpha}(\delta)$ is equivalent to $p(z) + \alpha(z)\,zp'(z) \prec h_\delta(z)$, where $\alpha(z) = \frac{\alpha}{n+2}\frac{D^{n+1}g(z)}{D^{n+2}g(z)}$.
Without loss of generality we can assume that p and h_δ satisfy the conditions of the theorem on the closed disc $\bar{U}$; if not we can replace $p(z)$ by $p_r(z) = p(rz)$ and $h_r(z)$ by $h_{\delta,r}(z) = h_\delta(rz)$, $0 < r < 1$, and these new functions satisfy the conditions of the theorem on $\bar{U}$. We would then prove $p_r(z) \prec h_{\delta,r}(z)$ for all $0 < r < 1$ and by letting $r \to 1^-$, we have $p(z) \prec h_\delta(z)$.

Because $g \in Z_{n+1}(\delta^*_{n+1})$, for $\alpha > 0$ we have $\mathrm{Re}\,\alpha(z) > 0$, $z \in U$. Let $\psi(r, s) = r + \alpha(z)s$ which is analytic in C^2 and $\psi(h_\delta(0), 0) = h_\delta(0) \in h_\delta(U)$. A simple calculus shows that

$$\mathrm{Re}\,\frac{\psi_0 - h_\delta(\zeta_0)}{\zeta_0 h'_\delta(\zeta_0)} = m_0\,\mathrm{Re}\,\alpha(z) > 0,\ \text{where}$$

$$\psi_0 = h_\delta(\zeta_0) + \alpha(z)m_0\zeta_0h'_\delta(\zeta_0),\ m_0 \geqslant 1,\ |\zeta_0| = 1.$$

Using this fact together with the fact that $\zeta_0h'_\delta(\zeta_0)$ is an outward normal to the boundary of the convex domain $h_\delta(U)$ we conclude that $\psi_0 \notin h_\delta(U)$ and using Theorem D we have $p(z) \prec h_\delta(z)$. A simple calculus shows that $\delta^*_{n+1} \leqslant \delta$ hence $p(z) \prec h_{\delta^*_{n+1}}(z)$ and the proof of the theorem is complete.

Remarks. 1°. For $\delta = \frac{1}{2}$ we obtain $\delta^*_{n+1} = \frac{1}{2}$ and the above result becomes Theorem 1 [1].

2° Theorem 1 shows that if $\frac{1}{2} \leqslant \delta < 1$ and $\alpha \geqslant 0$, the $C_{n,\alpha}(\delta) \subset AC_n(\delta^*_{n+1})$. Taking $\alpha = 1$ we obtain $AC_{n+1}(\delta) \subset AC_n(\delta^*_{n+1})$.

COROLLARY 1. *Let* $\frac{1}{2} \leqslant \delta < 1$ *and* $\alpha > \beta \geqslant 0$. *Then* $C_{n,\alpha}(\delta) \subset C_{n,\beta}(\delta^*_{n+1})$ *where* $\delta^*_{n+1} = 1/F\left(1,\ 2(n+3)(1-\delta),\ n+3;\frac{1}{2}\right)$.

Proof. If $\beta = 0$; using Theorem 1 we have

$$C_{n,\alpha}(\delta) \subset AC_n(\delta^*_{n+1}) = C_{n,0}(\delta^*_{n+1}).$$

If $\beta \neq 0$, using Theorem 1 and $\delta^*_{n+1} \leqslant \delta$ we obtain that if

$$f \in C_{n,\alpha}(\delta) \text{ then } \operatorname{Re}\left[(1-\beta)\frac{D^{n+1}f(z)}{D^{n+1}g(z)} + \beta\frac{D^{n+2}f(z)}{D^{n+2}g(z)}\right] =$$

$$= \frac{\beta}{\alpha}\left\{\operatorname{Re}\left[(1-\alpha)\frac{D^{n+1}f(z)}{D^{n+1}g(z)} + \alpha\frac{D^{n+2}f(z)}{D^{n+2}g(z)}\right] + \left(\frac{\alpha}{\beta}-1\right)\operatorname{Re}\frac{D^{n+1}f(z)}{D^{n+1}g(z)}\right\} >$$

$$> \frac{\beta}{\alpha}\left(\delta + \left(\frac{\alpha}{\beta}-1\right)\delta^*_{n+1}\right) \geqslant \delta^*_{n+1}.$$

Let $\gamma \in \mathbf{C}$ with $\operatorname{Re}\gamma > -1$ and $b_\gamma(z) = \sum_{j=1}^{\infty}\frac{\gamma+1}{\gamma+j}z^j$.

In [6], S. Ruscheweyh showed that if $\operatorname{Re}\gamma \geqslant \frac{n-1}{2}$ and $f \in Z_n\left(\frac{1}{2}\right)$ then $f * b_\gamma \in Z_n\left(\frac{1}{2}\right)$. Our next theorem presents a result concerning this function.

THEOREM 2. *Let* $\gamma > -1$ *and* $\delta_0 = \max\left\{\frac{n-\gamma}{n+1},\ \frac{2n-\gamma}{2(n+1)}\right\} \leqslant \delta < 1$. *If* $f \in Z_n(\delta)$ *then* $f*b_\gamma \in Z_n(\tilde{\delta}(n,\gamma,\delta))$, *where*

$$\tilde{\delta}(n,\gamma,\delta) = \frac{1}{n+1}\left(\frac{\gamma+1}{F\left(1, 2(n+1)(1-\delta),\ \gamma+2;\frac{1}{2}\right)} - \gamma + n\right)$$

and this result is sharp.

Proof. Let $F(z) = f(z) * b_\gamma(z)$; using the well-known formulas [6]:

$$z(D^k f(z))' = (k+1)D^{k+1}f(z) - kD^k(z),\ k \in \mathbf{N}$$

$$z(D^kF(z))' = (\gamma+1)D^kf(z) - \gamma D^kf(z),\ \operatorname{Re}\gamma > -1,\ k \in \mathbf{N}$$

we obtain that $f \in Z_n(\delta)$ is equivalent to

$$p(z) + \frac{zp'(z)}{(n+1)p(z)+\gamma-n} \prec h_\delta(z), \text{ where } p(z) = \frac{D^{n+1}F(z)}{D^nF(z)}.$$

Considering the differential equation

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma'}=h_\delta(z) \text{ where } \beta=n+1,\ \gamma'=\gamma-n$$

and using Theorem B we deduce that this equation has the univalent solution

$$q(z)=\frac{1}{\beta Q(z)}-\frac{\gamma'}{\beta},\ \text{where}\ Q(z)=\int_0^1\left(\frac{1-z}{1-}\right)^{2\beta(1-\delta)}t^{\beta+\gamma'-1}dt.$$

We also have $p(z)\prec q(z)$ and $q(z)$ is the best dominant.
Using a method similar to that of P. T. Mocanu, D. Ripeanu and I. Şerb [5] we show tha $\inf\{\operatorname{Re} q(z): z\in U\}=q(-1)$.

We use the following well-known formulas:

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}F(a,b,c;z),\ \text{with}\ c>b>0$$

$$F(a,b,c;z)=F(b,a,c;z)$$

$$F(a,b,c;z)=(1-z)^{-a}F\left(a,\ c-b,\ c;\frac{z}{z-1}\right)$$ which hold for all

$z\in \mathbb{C}\setminus(1,+\infty)$.

If $\delta_0<\delta<1$, we denote $a=2\beta(1-\delta)$, $b=\beta+\gamma'$, $c-b=1$ and using the above relations we deduce

$$Q(z)=\frac{1}{\beta+\gamma'}F\left(1,a,c;\frac{z}{z-1}\right).$$

Since $c>a>0$ we obtain that $Q(z)=\int_0^1\frac{1-z}{1-(1-t)z}d\mu(t),\quad z\in U$ where

$$d\mu(t)=\frac{\Gamma(b)}{\Gamma(a)\Gamma(c-a)}t^{a-1}(1-t)^{c-a-1}\,dt$$ is a positive measure.

If we let $Q(z,t)=\frac{1-z}{1-(1-t)z}$, then $\operatorname{Re} Q(z,t)>0$,

$Q(-r,t)\in\mathbf{R}$ for $0\leqslant r<1$, $t\in[0,1]$ and $\operatorname{Re}\frac{1}{Q(z,t)}\geqslant\frac{1}{Q(-r,t)}$

for $|z| \leqslant r < 1$, $t \in [0, 1]$. By using Theorem C we deduce

$$\operatorname{Re} \frac{1}{Q(z)} \geqslant \frac{1}{Q(-r)}, \quad |z| \leqslant r < 1$$

and by letting $r \to 1^-$ we have $\operatorname{Re} \frac{1}{Q(z)} > \frac{1}{Q(-1)}$, $z \in U$.

Then, by letting $\delta \to \delta_0^+$ we obtain our result.

Now we will prove that our result is sharp. If we let $\frac{D^{n+1}F(z)}{D^n F(z)} = q(z)$ we obtain $\frac{z(D^n F(z))'}{D^n F(z)} = (n+1)q(z) - n \equiv \tilde{q}(z)$, and letting $D^n F(z) = \varphi(z)$, $\varphi(0) = \varphi'(0) - 1 = 0$ we deduce $\frac{z\varphi'(z)}{\varphi(z)} = \tilde{q}(z)$, $\tilde{q}(0) = 1$. This last differential equation has the regular solution $\varphi(z) = z \exp \int_0^z \frac{\tilde{q}(t) - 1}{t}\, dt$, hence $D^n F(z) = z \exp (n+1) \int_0^z \frac{q(t) - 1}{t}\, dt \equiv G_n(z)$. Because $z(D^{n-1}F(z))' + (n-1)D^{n-1}F(z) = nD^n F(z)$ we deduce $D^{n-1}F(z) + \frac{1}{n-1} z(D^{n-1}F(z))' = \frac{n}{n-1} G_n(z)$, $G_n(0) = 0$, $n > 1$ hence $D^{n-1}F(z) = \frac{n}{z^{n-1}} \int_0^z G_n(t) t^{n-2}\, dt \equiv G_{n-1}(z)$. A simple calculus shows that

$$D^{n-2}F(z) = \frac{n-1}{z^{n-2}} \int_0^z G_{n-1}(t)\, t^{n-3} dt \equiv G_{n-2}(z)$$

$$D^1 F(z) = \frac{2}{z} \int_0^z G_2(t) dt \equiv G_1(z),$$

and $F(z) = \int_0^z \frac{G_1(t)}{t}\, dt$; since $zF'(z) = (1+\gamma)f(z) - \gamma F(z)$ we conclude that $f(z) = \frac{1}{1+\gamma} (\gamma F(z) + zF'(z))$ is the extremal function and this completes the proof of our theorem.

COROLLARY 2. *If* $\gamma \geqslant \max\left\{\frac{n-1}{2},\ n-1\right\}$, *then* $f \in Z_n\left(\frac{1}{2}\right)$ *implies that* $f * b_\gamma \in Z_n\left(\tilde{\delta}\left(n, \gamma; \frac{1}{2}\right)\right)$, *where* $\tilde{\delta}\left(n, \gamma, \frac{1}{2}\right) = \frac{1}{n+1}\left[\frac{\gamma+1}{F\left(1, n+1, \gamma+2; \frac{1}{2}\right)} - \gamma + n\right] > \frac{1}{2}$, *and this result is sharp.*

Proof. Taking $\delta = \frac{1}{2}$ in Theorem 2 we obtain the first part of this corollary; the relation $\tilde{\delta}\left(n, \gamma, \frac{1}{2} > \frac{1}{2}\right)$ is equivalent to $F\left(1, n+1, \gamma+2; \frac{1}{2}\right) < \frac{2(\gamma+1)}{2\gamma-n+1}$ and a simple calculus shows that the last inequality holds.

Taking $n = 0$ and $n = 1$ in Theorem 2 we obtain respectively:

COROLLARY 3. *If* $\gamma > -1$, $\max\left\{-\gamma, -\frac{\gamma}{2}\right\} \leqslant \delta < 1$ *and* $f \in A$, *then* $\operatorname{Re} \frac{zf'(z)}{f(z)} > \delta$, $z \in U$ *implies* $\operatorname{Re} \frac{zF'(z)}{F(z)} > \tilde{\delta}(0, \gamma, \delta)$, $z \in U$, *where* $F(z) = f(z) * b_\gamma(z)$ *and* $\tilde{\delta}(0, \gamma, \delta) = \dfrac{\gamma+1}{F\left(1, 2(1-\delta), \gamma+2; \frac{1}{2}\right)} - \gamma$.

COROLLARY 4. *If* $\gamma > -1$, $\max\left\{\frac{1-\gamma}{2}, \frac{2-\gamma}{4}\right\} \leqslant \delta < 1$ *and* $f \in A$, *then* $\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 2\delta - 1$, $z \in U$ *implies* $\operatorname{Re}\left(1 + \frac{zF''(z)}{F'(z)}\right) > 2\tilde{\delta}(1, \gamma, \delta) - 1$, $z \in U$ *where* $F(z) = f(z) * b_\gamma(z)$ *and*

$$\tilde{\delta}(1, \gamma, \delta) = \frac{1}{2}\left[\frac{\gamma+1}{F\left(1, 4(1-\delta), \gamma+2; \frac{1}{2}\right)} - \gamma + 1\right].$$

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STUDIUL RESTULUI ÎNTR-O FORMULĂ DE APROXIMARE A FUNCŢIILOR DE DOUĂ VARIABILE CU AJUTORUL UNUI OPERATOR DE TIP FAVARD-SZÁSZ

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ABSTRACT. — Study on the Rest in an Approximation Formula of the Two-Variable Functions by means of a Favard-Szász-Type Operator. One class of positive, linear operators is constructed on the multitude of the functions $f: D \to R \to$, where $D = \{(x, y) : x \geq 0, y \geq 0\}$, by relation (2). An analysis is further made on the rest in an approximation formula of functions by means of these operators.

1. In [1] D. D. Stancu dă o metodă de construcţie a unei clase de operatori liniari pozitivi depinzînd de un parametru real α, definind pentru orice funcţie $f: I \to \mathbf{R}$, I fiind un interval al axei reale, aplicaţia

$$(L_m^{\langle\alpha\rangle} f)(x) = \frac{1}{\varphi_m^{\langle\alpha\rangle}(0)} \sum_{k=0}^{\infty} (-1)^k \frac{x^{[k, -\alpha]}}{k!} D_\alpha^k \varphi_m^{\langle\alpha\rangle}(x) f(x_{m,k}), \tag{1}$$

unde $x \in I = [0, a]$, $a > 0$, $x^{[k, -\alpha]} = x(x + \alpha) \ldots (x + (k - 1)\alpha)$, $x_{m,k} \in \in JCI$, $(\varphi_m^{\langle\alpha\rangle})$ fiind un şir de funcţii depinzînd de α, analitice într-un domeniu D care conţine discul $|z - a| \leq a$ şi care pot fi dezvoltate în serie Newton convergente pe D, D_α^k fiind operatori diferenţe Nörbund

$$D_\alpha^k g(x) = D_\alpha(D^{k-1} g(x)),\ D_\alpha g(x) = \frac{g(x + \alpha) - g(x)}{\alpha},\ D_\alpha^0 g(x) = g(x).$$

Luînd în particular

$$\varphi_m^{\langle\alpha\rangle}(x) = (1 + \alpha m)^{-\frac{x}{\alpha}},\ x_{m,k} = \frac{}{m}$$

pentru care

$$D_\alpha^k \varphi_m^{\langle\alpha\rangle}(x) = (-1)^k \left(\frac{m}{1 + \alpha m}\right)^k (1 + \alpha m)^{-\frac{x}{\alpha}}$$

se obţine operatorul de tip Favard-Szász

$$(L_m^{\langle\alpha\rangle} f)(x) = \sum_{k=0}^{\infty} w_{m,k}^{\langle\alpha\rangle}(x) f\left(\frac{k}{m}\right),$$

unde $w_{m,k}^{\langle\alpha\rangle}(x) = (1 + \alpha m)^{-\frac{x}{\alpha}} \left(\frac{m}{1 + \alpha m}\right)^k \frac{x^{[x, -\alpha]}}{k!}$.

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2. În această lucrare construim o clasă de operatori liniari pozitivi pe mulțimea funcțiilor

$$f : D \to \mathbf{R}, \text{ unde } D = \{(x, y) : x \geq 0,\ y \geq 0\}$$

și facem o analiză a restului într-o formulă de aproximare a funcțiilor prin acești operatori.

Fie α, β doi parametri reali, m, n numere naturale.

Definim operatorul $L_{m,n}^{\langle\alpha,\beta\rangle}$ prin

$$(L_{m,n}^{\langle\alpha,\beta\rangle} f)(x, y) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} v_{m,k}^{\langle\alpha\rangle}(x) w_{n,j}^{\langle\beta\rangle}(y) f\left(\frac{k}{m}, \frac{j}{n}\right), \tag{2}$$

unde

$$v_{m,k}^{\langle\alpha\rangle}(x) = (1 + \alpha m)^{-\frac{x}{\alpha}} \left(\frac{m}{1 + \alpha m}\right)^k \cdot \frac{x^{[k,-\alpha]}}{k!}, \tag{3}$$

$$w_{n,j}^{\langle\beta\rangle}(y) = (1 + \beta_n)^{-\frac{y}{\beta}} \left(\frac{n}{1 + \beta n}\right)^j \cdot \frac{y^{[j,-\beta]}}{j!} \tag{3'}$$

Este evident că operatorul $L_{m,n}^{\langle\alpha,\beta\rangle}$ este un operator liniar pozitiv pentru $\alpha > 0$, $\beta > 0$.

Notînd $$(R_{m,n}^{\langle\alpha,\beta\rangle} f)(x, y) = f(x, y) - (L_{m,n}^{\langle\alpha,\beta\rangle} f)(x, y),$$

avem $$f(x, y) = (L_{m,n}^{\langle\alpha,\beta\rangle} f)(x, y) + (R_{m,n}^{\langle\alpha,\beta\rangle} f)(x, y) \tag{4}$$

și scopul este de a obține forme lucrative ale restului $(R_{m,n}^{\langle\alpha,\beta\rangle} f)$ (x, y) în această formulă de aproximare.

3. Înainte de toate, fie funcțiile $l_{ij}(n, y) = x^i y^j$, $i = 0, 1, 2$; $j = 0, 1, 2$. Avem

$$(L_{m,n}^{\langle\alpha,\beta\rangle} l_{00})(x, y) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} v_{m,k}^{\langle\alpha\rangle}(x) w_{n,j}^{\langle\beta\rangle}(y) =$$

$$= \left[\sum_{k=0}^{\infty} (1 + \alpha m)^{-\frac{x}{\alpha}} \left(\frac{m}{1 + \alpha n}\right)^k \cdot \frac{x(x + \alpha) \ldots (x + (k - 1)\alpha)}{k!}\right] \times$$

$$\times \left[\sum_{j=0}^{\infty} (1 + \beta_n)^{-\frac{y}{\beta}} \left(\frac{n}{1 + \beta n}\right)^j \cdot \frac{y(y + \beta) \ldots y + (y - 1)\beta)}{j!}\right] = 1.$$

Un calcul similar, arată

$$(L_{m,n}^{\langle\alpha,\beta\rangle} l_{10})(x, y) = x, \quad (L_{m,n}^{\langle\alpha,\beta\rangle} l_{01}(x, y) = y,$$

$$(L_{m,n}^{\langle\alpha,\beta\rangle} l_{11})(x, y) = xy,$$

$$(L_{m,n}^{\langle\alpha,\beta\rangle} l_{02})(x, y) = y^2 + \frac{1+\beta n}{n} y,$$

$$(L_{m,n}^{\langle\alpha,\beta\rangle} l_{20})(x, y) = x^2 + \frac{1+\alpha n}{m} x,$$

$$(L_{m,n}^{\langle\alpha,\beta\rangle} l_{12})(x, y) = xy^2 + \frac{1+\beta n}{n} x\cdot y, \quad (L_{m,n}^{\langle\alpha,\beta\rangle} l_{21})(x, y) = x^2y + \frac{1+\alpha m}{m} xy,$$

$$(L_{m,n}^{\langle\alpha,\beta\rangle} l_{22})(x, y) = x^2y^2 + \left(\frac{1+\alpha m}{m} + \frac{1+\beta n}{n}\right) xy(x+y) + \frac{(1+\alpha m)(1+\beta n)}{mn} x$$

Prin urmare

$$(R_{m,n}^{\langle\alpha,\beta\rangle} l_{ij})(x, y) = 0, \text{ pentru } (i, j) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\},$$

$$(R_{m,n}^{\langle\alpha,\beta\rangle} l_{12})(x, y) = -\frac{1+\beta n}{n} xy,$$

$$(R_{m,n}^{\langle\alpha,\beta\rangle} l_{21})(x, y) = -\frac{1+\alpha m}{m} xy,$$

$$(R_{m,n}^{\langle\alpha,\beta\rangle} l_{22})(x, y) = -\left(\frac{1+\alpha m}{m} + \frac{1+\beta n}{n}\right) xy(x+y) - \frac{(1+\alpha m)(1+\beta n)}{mn} xy.$$

Din aceasta, rezultă că

$$\lim_{m,n\to\infty} (L_{m,n}^{\langle\alpha,\beta\rangle} l_{ij})(x, y) = l_{ij}(x, y), \text{ pentru}$$
$$i, j \in \{0, 1, 2\}.$$

4. Fie Y fixat. Folosind notaţiile şi formulele (19) din [1], obţinem

$$f(x, y) = (L_m^{\langle\alpha\rangle} f(\cdot, y))(x) + (R_m^{\langle\alpha\rangle} f(\cdot, y))(x),$$

cu

$$(L_m^{\langle\alpha\rangle} f(\cdot, y))(x) = \sum_{k=0}^{\infty} v_{m,k}^{\langle\alpha\rangle}(x) f\left(\frac{k}{m}, y\right) \text{ şi}$$

$$(R_m^{\langle\alpha\rangle} f(\cdot, y))(x) = -\sum_{k=0}^{\infty} \frac{x+k\alpha}{m} v_{m,k}^{\langle\alpha\rangle}(x)\left[x, \frac{k}{m}, \frac{k+1}{m}; f(\cdot, y)\right] =$$

$$= -\frac{1+\alpha m}{m} x \sum_{k=0}^{\infty} v_{m,k}^{\langle\alpha\rangle}(x+\alpha)\left[x, \frac{k}{m}, \frac{k+1}{m}; f(\cdot, y)\right].$$

Dezvoltînd pe $f\left(\frac{k}{m}, y\right)$ după aceeași formulă, avem

$$f\left(\frac{k}{m}, y\right) = \sum_{j=0}^{\infty} w_{n,j}^{\langle\beta\rangle}(y) f\left(\frac{k}{m}, \frac{j}{n}\right) + \left(R_n^{\langle\beta\rangle} f\left(\frac{k}{m}, \cdot\right)\right)(y),$$

unde

$$\left(R_n^{\langle\beta\rangle} f\left(\frac{k}{m}, \cdot\right)\right)(y) = -\sum_{j=0}^{\infty} \frac{y + j\beta}{n} w_{n,j}^{\langle\beta\rangle}(y) \left[y, \frac{j}{n}, \frac{j+1}{n}; f\left(\frac{k}{m}, \cdot\right)\right].$$

Rezultă

$$f(x, y) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} v_{m,k}^{\langle\alpha\rangle}(x) \cdot w_{n,j}^{\langle\beta\rangle}\left(\frac{k}{m}, \frac{j}{n}\right) -$$

$$- \sum_{k=0}^{\infty} \frac{x + k\alpha}{m} v_{n,k}^{\langle\beta\rangle}(x) \left[x, \frac{k}{m}, \frac{k+1}{m}; f(\cdot, y)\right] -$$

$$- \sum_{j=0}^{\infty} \frac{y + j\beta}{n} w_{n,j}^{\langle\beta\rangle}(y) \cdot \left[y, \frac{j}{n}, \frac{j+1}{n}; f(x, \cdot)\right] -$$

$$- \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(x + k\alpha)(y + j\beta)}{mn} v_{m,k}^{\langle\alpha\rangle}(x) w_{n,j}^{\langle\beta\rangle}(y) \left[\begin{matrix} x, & \frac{k}{m}, & \frac{k+1}{n} \\ y, & \frac{j}{n}, & \frac{j+1}{n} \end{matrix}; f\right]$$

și deci

$$(R_{m,n}^{\langle\alpha,\beta\rangle} f)(x, y) = -\sum_{k=0}^{\infty} \frac{x + k\alpha}{m} v_{m,k}^{\langle\alpha\rangle}(x) \left[x, \frac{k}{m}, \frac{k+1}{m}; f(\cdot, y)\right] -$$

$$- \sum_{j=0}^{\infty} \frac{y + j\beta}{n} w_{n,j}^{\langle\beta\rangle}(y) \left[y, \frac{j}{n}, \frac{j+1}{n}; f(x, \cdot)\right] -$$

$$- \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(x + k\alpha)(y + j\beta)}{mn} v_{m,k}^{\langle\alpha\rangle}(x) w_{n,j}^{\langle\beta\rangle} \cdot \left[\begin{matrix} x, & \frac{k}{m}, & \frac{k+1}{m} \\ y, & \frac{j}{n}, & \frac{j+1}{n} \end{matrix}; f\right] \qquad (5)$$

unde $\left[\begin{matrix} x, & \frac{k}{m}, & \frac{k+1}{m} \\ y, & \frac{j}{n}, & \frac{j+1}{n} \end{matrix}; f\right]$ reprezintă diferența divizată bidimensională a lui f pe sistemul de puncte indicate.

Presupunînd că diferențele divizate de ordinul al doilea ale lui f în raport cu x, pentru fiecare y fixat, şi în raport cu y, pentru fiecare x fixat, şi diferențele bidimensionale sînt egal mărginite de o aceeaşi constantă M, rezultă

$$|(R_{m,n}^{\langle\alpha,\beta\rangle} f)(x, y)| \leqslant M\left(\frac{1+\alpha m}{m} x + \frac{1+\beta n}{n} y + \frac{1+\alpha n+\beta n+\alpha\beta mn}{mn}\right).$$

5. Să presupunem că funcția f are derivate parțiale continue pînă la ordinul al patrulea pe $(0, \infty) \times (0, \infty)$.

Atunci, conform [3 formula (11)], avem

$$(R^{\langle\alpha\rangle} f(\cdot, y))(x) = -\frac{1+\alpha m}{2m} x \frac{\partial^2 f}{\partial x^2}(\xi, y).$$

Reluînd calculul prin care am dedus formula (5), obţinem:

$$(R_{m,n}^{\langle\alpha,\beta\rangle} f)(x, y) = -\frac{1+\alpha m}{2m} x \frac{\partial^2 f}{\partial x^2}(\xi, y) - \frac{1+\beta n}{2n} y \frac{\partial^2 f}{\partial y^2}(x, \eta) -$$

$$-\frac{(1+\alpha n)(1+\beta n)}{4mn} xy \frac{\partial^4 f}{\partial x^2 \partial y^2}(\xi, \eta). \tag{6}$$

6. În [1], D. D. Stancu arată că dacă funcția $f: [0, \infty) \to \mathbf{R}$ are derivate pînă la ordinul al doilea pe $[0, \infty)$, atunci restul $(R_m^{\langle\alpha\rangle} f)(x)$ din formula

$$f(x) = (L_m^{\langle\alpha\rangle} f)(x) + (R_m^{\langle\alpha\rangle} f)(x) \tag{7}$$

poate fi pus sub forma

$$(R_m^{\langle\alpha\rangle} f)(x) = \int_0^\infty G_m^{\langle\alpha\rangle}(t, x) f''(t) dt, \tag{8}$$

unde

$$G_m^{\langle\alpha\rangle}(t, x) = (R_m^{\langle\alpha\rangle} \psi_x)(t), \quad \psi_x(t) = (x - t)_+.$$

$R_m^{\langle\alpha\rangle} \psi_x$ fiind restul relativ la variabila x, t fiind fixat.

Vom nota însă, prin analogie.

$$H_n^{\langle\beta\rangle}(\tau, \eta) = (R_n^{\langle\beta\rangle} \psi)(\tau).$$

Vom presupune că funcția f are derivate parțiale pînă la ordinul IV inclusiv. Aplicînd formulele (7), (8), pentru Y fixat, avem

$$f(x, y) = \sum_{k=0}^\infty v_{m,k}^{\langle\alpha\rangle}(x) f\left(\frac{k}{m}, y\right) + \int_0^\infty G_m^{\langle\alpha\rangle}(t, x) \frac{\partial^2 f}{\partial t^2}(t, y) dt. \tag{9}$$

Exprimînd cu această formulă $f\left(\frac{k}{m}, y\right)$, schimbînd rolul variabilelor, obţinem

$$f\left(\frac{k}{m}, y\right)=\sum_{j=0}^{\infty} w_{n,j}^{\langle\beta\rangle}(y) f\left(\frac{k}{m}, \frac{j}{n}\right)+\int_0^{\infty} H_n^{\langle\beta\rangle}(\tau, y) \frac{\partial^2 f}{\partial \tau^2}\left(\frac{k}{m}, \tau\right) d\tau \tag{10}$$

Formula (9), aplicată funcţiei $\frac{\partial^2 f}{\partial y^2}$, ne conduce la

$$\sum_{k=0}^{\infty} v_{m,k}^{\langle\alpha\rangle}(x) \frac{\partial^2 f}{\partial \tau^2}\left(\frac{k}{m}, \tau\right)=\frac{\partial^2 f}{\partial \tau^2}(x, \tau)-\int_0^{\infty} G_m^{\langle\alpha\rangle}(t, x) \frac{\partial^{(IV)} f}{\partial t^2 \partial \tau^2}(t, \tau) d\tau \tag{11}$$

Inlocuind (10) şi (11) în (9), obţinem

$$f(x, y)=\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} v_{m,k}^{\langle\alpha\rangle}(x) \cdot w_{n,j}^{\langle\beta\rangle}(y) \cdot f\left(\frac{k}{m}, \frac{j}{n}\right)+\int_0^{\infty} G_m^{\langle\alpha\rangle}(t, x) \frac{\partial^2 f}{\partial t^2}(t, y) dt +$$

$$+\int_0^{\infty} H_n^{\langle\beta\rangle}(\tau, y) \frac{\partial^2 f}{\partial \tau^2}(x, \tau) d\tau-\int_0^{\infty}\int_0^{\infty} G_m^{\langle\alpha\rangle}(t, x) H_n^{\langle\beta\rangle}(\tau, y) \cdot \frac{\partial^{(IV)} f}{\partial t^2 \partial \tau^2}(t, \tau) dt d\tau,$$

ceea ce conduce la expresia integrală a restului din formula (4) :

$$(R_{m,n}^{\langle\alpha,\beta\rangle} f)(x, y)=\int_0^{\infty} G_m^{\langle\alpha\rangle}(t, x) \cdot \frac{\partial^2 f}{\partial t^2}(t, y) dt+\int_0^{\infty} H_n^{\langle\beta\rangle}(\tau, y) \cdot \frac{\partial^2 f}{\partial \tau^2}(x, \tau) d\tau -$$

$$-\int_0^{\infty}\int_0^{\infty} G_m^{\langle\alpha\rangle}(t, x) H_n^{\langle\beta\rangle}(\tau, y) \cdot \frac{\partial^{(IV)} f}{\partial t^2 \partial \tau^2}(t, \tau) dt d\tau \tag{12}$$

Sîntem recunoscători prof. D.D. Stancu, care ne-a îndemnat să întreprindem acest studiu şi ale cărui indicaţii ne-am înlesnit realizarea lui.

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ON STRONGLY-STARLIKE AND STRONGLY-CONVEX FUNCTIONS

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REZUMAT. — Asupra funcţiilor tare stelate şi tare convexe. Rezultatul principal al lucrării este conţinut în următoarea teoremă.

TEOREMA 1. Dacă $0 < \alpha \leqslant 2$ şi f este o funcţie olomorfă în discul unitate U, $f(0) = f'(0) - 1 = 0$, care satisface condiţia (2), atunci

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \alpha \frac{\pi}{2} \quad z \in U.$$

1. Introduction. Let A be the class of analytic functions f in the unit disc $U = \{z;\ |z| < 1\}$, which are normalized by $f(0) = f'(0) - 1 = 0$. A function $f \in A$ is called stronglystarlike (strongly-convex) of order α, $0 < \alpha \leqslant 1$, if $|\arg [zf'(z)/f(z)]| < \alpha\pi/2$ $(|arg[1 + zf''(z)/f'(z)]| < \alpha\pi/2)$, for $z \in U$. If $\alpha = 1$ these concepts reduce to the well-known concepts of starlikeness and convexity, respectively.

For $0 < \alpha \leqslant 1$ it is easy to show that each stronglyconvex function of order α is strongly — starlike of order α, i.e. the following implication

$$f \in A \text{ and} \left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \alpha \frac{\pi}{2} \Rightarrow \left|\arg \frac{zf'(z)}{f(z)}\right| < \alpha \frac{\pi}{2}$$

holds. In terms of subordination this implication can be written as follows

$$f \in A \text{ and } 1 + \frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \Rightarrow \frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \tag{1}$$

where $0 < \alpha \leqslant 1$. This result fails if $\alpha > 1$.

In the case $\alpha = 1$, we improved (1) by the following „open door' theorem [3].

THEOREM A. *If* $f \in A$ *satisfies*

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z} + \frac{2z}{1-z^2},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}.$$

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Geometrically, Theorem A shows that if $1 + zf''(z)/f'(z)$ lies in the complex plane slit along the half-lines $u = 0,\ v \geqslant \sqrt{3}$ and $u = 0,\ v \leqslant -\sqrt{3}$, then $zf'(z)/f(z)$ lies in the right half-plane, i.e. the function f is starlike.

In this paper we improve (1) by the following result, which holds for all $\alpha \in (0, 2]$.

THEOREM 1. *If* $0 < \alpha \leqslant 2$ *and* $f \in A$ *satisfies*

$$1 + \frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha} + \frac{2\alpha z}{1-z^2} \tag{2}$$

then

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}.$$

2. **Preliminaries.** If F and G are analytic functions in U, then F is subordinate to G, written $F \prec G$, or $F(z) \prec G(z)$, if G is univalent, $F(0) = G(0)$ and $F(U) \subset G(U)$.

We will need the following two lemmas to prove Theorem 1.

LEMMA 1 [1, p. 128]. *Let q be analytic and injective on $U \setminus E(q)$, where $E(q) \subset \partial U$ consists of a finite number of isolated singularities. Let p be analytic in U, with $p(0) = q(0)$. If there exist points $z_0 \in U$ and $\zeta_0 \in \partial U \setminus E(q)$, such that $q'(\zeta_0) \neq 0$, $p(z_0) = q(\zeta_0)$* and *$p(|z| < |z_0|) \subset q(U)$, then*

$$z_0 p'(z_0) = m\zeta_0 q'(\zeta_0),$$

where $m \geqslant 1$.

LEMMA 2 *Let P be an analytic function in U such that*

$$P(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha} + \frac{2\alpha z}{1-z^2} \equiv h(z), \quad 0 < \alpha \leqslant 2. \tag{3}$$

If p is analytic in U, $p(0) = 1$ and satisfies the differential equation

$$zp'(z) + P(z)p(z) = 1, \tag{4}$$

then

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha}.$$

Proof. If we let $q(z) = [(1-z)/(1+z)]^{\alpha}$, then

$$h(z) = \frac{1}{q(z)} - \frac{zq'(z)}{q(z)}.$$

The domain $h(U)$ is symmetric with respect to the real axis. Therefore, if $z = e^{i\theta}$, then in order to obtain the boundary of $h(U)$ it is sufficient to suppose $0 \leqslant \theta \leqslant \pi$.

Letting ctg $(\theta/2) = t$ and $h(e^{i\theta}) = u + iv$, we find

$$\begin{cases} u = u(t) = At^{\alpha} \\ v = v(t) = Bt^{\alpha} + \dfrac{\alpha(1+t^2)}{2t}, \quad t \geqslant 0. \end{cases} \tag{5}$$

where $A = \cos(\alpha\pi/2)$ and $B = \sin(\alpha\pi/2)$.

If $\alpha = 1$, then $u = 0$ and $v \geqslant \sqrt{3}$ and we find that $h(U)$ is the complex plane slit along the half-lines $u = 0$, $v \geqslant \sqrt{3}$ and $u = 0$, $v \leqslant -\sqrt{3}$.

We note that $A > 0$ (i.e. $u > 0$), for $0 < \alpha < 1$ and $A < 0$ (i.e. $u < 0$), for $1 < \alpha \leqslant 2$. In the last two cases it is possible to eliminate the parameter t in (5) and to express v as a function of u. Actually we find

$$v = \frac{B}{A}u + \frac{\alpha}{2}\left[\left(\frac{u}{A}\right)^{1/\alpha} + \left(\frac{u}{A}\right)^{-1/\alpha}\right], \quad \begin{cases} u > 0, \text{ for } 0 < \alpha < 1 \\ u < 0, \text{ for } 1 < \alpha \leqslant 2 \end{cases} \tag{6}$$

We also have $v(0) = v(\infty) = \infty$.

In all cases we deduce that h is univalent in U.

Now we suppose that $0 < \alpha < 2$ and the solution p of (3) is not subordinate to q. Then there exist points $z_0 \in U$ and $\zeta_0 \in \partial U$ such that $p(z_0) = q(\zeta_0)$ and $p(|z| < |z_0|) \subset q(U)$. Since $q(-1) = \infty$, it is clear that $\zeta_0 \neq -1$. Suppose $\zeta_0 = 1$. Then $p(z_0) = 0$ is the corner of the sector $q(U)$. If $p'(z_0) = 0$, by letting $z = z_0$ in (4), we obtain a contradiction. If $p'(z_0) \neq 0$ and $0 < \alpha < 1$, then the image by p of the circle $|z| = |z_0|$ cannot pass through the corner $w = 0$ without itself having a corner, which contradicts $p'(z_0) \neq 0$. If $p'(z_0) \neq 0$ and $1 \leqslant \alpha < 2$, then is it easy to show that

$$(3-\alpha)\frac{\pi}{2} \leqslant \arg[z_0 p'(z_0)] \leqslant (1+\alpha)\frac{\pi}{2},$$

which shows that Re $[z_0 p'(z_0)] \leqslant 0$. Hence, if we let $z = z_0$ in (4), we again obtain a contradiction. All the above contradictions show that $\zeta_0 \neq 1$. Therefore we can apply Lemma 1 to obtain $z_0 p'(z_0) = m\,\zeta_0 q'(\zeta_0)$ and from (4) we deduce

$$P(z_0) = \frac{1}{q(\zeta_0)} - \frac{m\,\zeta_0 q'(\zeta_0)}{q(\zeta_0)} \equiv Q(\zeta_0, m). \tag{7}$$

If we let $\zeta_0 = e^{i\theta}$, we can suppose $0 \leqslant \theta \leqslant \pi$. Letting ctg $(\theta/2) = t$, we obtain

$$Q(\zeta_0, m) = u(t) + iV(t),$$

where

$$V(t) = v(t) + \frac{(m-1)\,\alpha\,(1+t^2)}{2t}, \quad t \geqslant 0,$$

and $u(t)$, $v(t)$ are given by (5).

Since $m \geqslant 1$, we deduce $V(t) \geqslant v(t)$, which shows that $P(z_0) = Q(\zeta_0, m) \notin h(U)$. This contradicts the condition (3).

Therefore we must have $p \prec q$.

In the case $\alpha = 2$, the result can be obtained by a limiting procedure.

3. **Proof of Theorem 1.** Let $f \in A$ satisfy (2) and let $g(z) = zf'(z)$. From (2) we deduce $g(z)/z \neq 0$, for $z \in U$. Therefore the functions $p(z) = f(z)/g(z)$ and $P(z) = zg'(z)/g(z)$ are analytic in U. Moreover p satisfies the differential equation (4). Since the condition (2) is equivalent to (3), by Lemma 2 we deduce $p(z) \prec [(1-z)/(1+z)]^\alpha$, which is equivalent to $[zf'(z)/f(z)] \prec [(1+z)/(1-z)]^\alpha$. This completes the proof of Theorem 1.

The above proof shows that Theorem 1 can be stated in the following equivalent form.

THEOREM 2. *Let* $0 < \alpha \leqslant 2$ *and let* $g \in A$ *satisfy*

$$\frac{zg'(z)}{g(z)} \prec \left(\frac{1+z}{1-z}\right)^\alpha + \frac{2\alpha z}{1-z^2}.$$

If

$$f(z) = \int_0^z \frac{g(t)}{t}\, dt$$

then $f \in A$, $f(z)/z \neq 0$ *and*

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^\alpha.$$

If we integrate the differential equation

$$1 + \frac{zf''(z)}{f'(z)} = P(z),$$

we easily obtain another equivalent form of Theorem 1.

THEOREM 3. *If the analytic function* P *satisfies in* U *the condition* (3), *then*

$$\left|\arg \int_z^1 \left(\exp \int_z^{tz} \frac{P(w)-1}{w}\, dw\right) dt\right| < \alpha\frac{\pi}{2}, \text{ for } |z| < 1.$$

4. **Particular cases.**

a) For $\alpha = 1/2$ the equation (6) becomes

$$v = u + \frac{u^2}{2} + \frac{1}{8u^2}, \quad u > 0.$$

It is easy to show that $v > 1$. Hence from Theorem 1 we deduce the following result.

COROLLARY 1.1. *If $f \in A$ and*

$$\left| \operatorname{Im} \frac{zf''(z)}{f'(z)} \right| \leqslant 1, \quad z \in U,$$

then

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi}{4}, \quad z \in U.$$

For example, if we take $f(z) = e^z - 1$, we obtain

$$\left| \arg \frac{z}{e^z - 1} \right| < \frac{\pi}{4}, \text{ for } |z| < 1.$$

Using Theorem 3, we obtain the following equivalent form of Corollary 1.1.

COROLLARY 3.1. *If Q is analytic in* U, *$Q(0) = 0$ and $|\operatorname{Im} Q(z)| \leqslant 1$ in U, then*

$$\left| \arg \int_0^1 \left(\exp \frac{2}{\pi} \int_z^{tz} \frac{Q(w)}{w} dw \right) dt \right| < \frac{\pi}{4}, \text{ for } |z| < 1.$$

For example, if we take

$$Q(z) = \frac{2}{\pi} \log \frac{1+z}{1-z},$$

we obtain

$$\left| \arg \int_0^1 \left(\exp \frac{2}{\pi} \int_z^{tz} \frac{1}{w} \log \frac{1+w}{1-w} dw \right) dt \right| < \frac{\pi}{4}, \text{ for } |z| < 1.$$

b) For $\alpha = 1$ Theorem 1 reduces to Theorem A, In this case the equations (5) become

$$u = 0 \text{ and } v = t + \frac{1}{2}\left(t + \frac{1}{t}\right)' \quad t > 0.$$

c) For $\alpha = 2$ the equation (6) becomes

$$v = \sqrt{-u} + \frac{1}{\sqrt{-u}}, \quad u < 0.$$

Since $v \geqslant 2$, from Theorem 1 we deduce the following result, COROLLARY 1.2. *If $f \in A$ and*

$$\left|\operatorname{Im} \frac{zf''(z)}{f'(z)}\right| < 2, \qquad z \in U,$$

then

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \pi, \qquad z \in U.$$

Using Theorem 3, we obtain the following equivalent form of Corollary 1.2.

COROLLARY 3.2. *If Q is analytic in U, $Q(0) = 0$ and $|\operatorname{Im} Q(z)| < 2$ in U, then*

$$\left|\arg \int_0^1 \left[\exp \int_z^{tz} \frac{Q(w)}{w}\, dw\right] dt\right| < \pi, \qquad \text{for } |z| < 1.$$

5. **Remark.** If $0 < \alpha \leqslant 1$, Theorem 1 is a particular case of a more general result recently obtained in [2], by using a „subordination chain" technique. The present proof is elementary; moreover for $1 < \alpha \leqslant 2$ Theorem 1 cannot be deduced from the result in [2], since the subordination chain condition is not satisfied in this case.

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SIMPLEX-LIKE METHOD FOR THE PARETO MINIMUM SOLUTIONS OF AN INCONSISTENT SYSTEM

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REZUMAT. Metoda simplex pentru soluţii minime Pareto ale unui sistem inconsistent. În 1977 s-a definit o clasă de funcţii convexe în medie de ordinul α (αm-convexe) [2] şi s-a arătat că orice funcţie αm-convexă este pseudo convexă. În această notă se arată că în anumite condiţii orice funcţie pseudo convexă este αm-convexă cu un număr α determinat.

1. **Introduction.** Recently [6] we have defined Pareto minimum solutions of an inconsistent system and we have shown that there is a strong connection between these extremal approximate solutions of a system and a multicriterion optimization problem. In [6] we gave some properties of the Pareto minimum solutions of an inconsistent semi-infinite linear system.

In this paper we are going to present a simplex-like technique to generate extreme Pareto minimum solutions of an inconsistent linear system.

2. **Pareto minimum solutions of a system.** Let I be an index set and $f_i : \mathbf{R}^n \to \mathbf{R}$, $i \in I$. Consider the system

$$f_i(x) = 0, \ i \in I. \tag{1}$$

If $I = \{1, 2, \ldots, m\}$ and

$$f_i(x) = \sum_{j=1} a_{ij} x_j - b_i, \quad i \in I \tag{2}$$

then system (1) becomes a finite linear system:

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i \in I \tag{3}$$

or

$$Ax = b, \tag{3}$$

where $A = (a_{ij})$ is a $m \times n$ real matrix and $b = (b_i)$ — a column matrix of the type $m \times 1$

If $I = \mathbf{N}$ and f_i are given in (2), the system (3) is a semiinfinit linear system.

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DEFINITION 1. *Vector* $x \in \mathbf{R}^n$ *is called Pareto minimum solution (or Pareto minimum point) of the system* (1) *if there is no* $y \in \mathbf{R}^n$ *such that*

(i) $$\forall i \in I \Rightarrow |f_i(y)| \leqslant |f_i(x)|$$

(ii) $$\exists i_0 \in I, \qquad |f_{i_0}(y)| < |f_{i_0}(x)|$$

DEFINITION 2. *Vector* $x \in \mathbf{R}^n$ *is called weak Pareto minimum solution of the system* (1) *if there is no* $y \in \mathbf{R}^n$ *such that*

$$\forall i \in I \Rightarrow |f_i(y)| < |f_i(x)|$$

If we denote by $P(f, I)$, $P_w(f, I)$ the set of all Pareto minimum and weak Pareto minimum solutions of the system (1) respectively, then obviously

$$P(f, I) \subset P_w(f, I)$$

The converse inclusion generally does not hold (see [6]).

3. **Equivalent multicriterion optimizatuin problem.** Consider $X \subseteq \mathbf{R}^n$, $f: X \to \mathbf{R}^m$, $g: X \to \mathbf{R}^p$. Assume that

$$S = \{x \in X : g(x) \leqslant 0\} \neq \emptyset.$$

We remind that $x^0 \in S$ is called Pareto minimum solution (efficient solution) or Pareto minim point on S of the vectorminimization problem:

$$f(x) \to \min$$

subject to

$$g(x) \leqslant 0, \ x \in X$$

if there is no $x \in S$ such that

$$f(x) \leqslant f(x^0), \ f(x) \neq f(x^0).$$

Assume that $I = \{1, 2, \ldots, m\}$ and consider the following vector-minimization problem (V.P):

$$(u_1, u_2, \ldots, u_m) \to \min \tag{4}$$

subject to

$$|f_i(x)| \leqslant u_i, \ i \in I, \ x \in \mathbf{R}^n, \ u \in \mathbf{R}^+_m$$

THEOREM 1. ([6, *Theorem* 1]). $x_0 \in P(f, I)$ *if and only if* (x^0, u^0) *is a Pareto minimum solution to the problem* $(V.P)$, *where*

$$u_0 = (|f_1(x_0)|, \ldots, |f_m(x^0)|)$$

In what follows we shall deal with the case when

$$f_i(x) = \sum_{j=0}^{n} a_{ij} x_j - b_i, \ i \in I = \{1, 2, \ldots, m\}$$

i.e. we shall consider the Pareto minimum solutions of the system

$$Ax = b, \tag{6}$$

where $A \in \mathfrak{M}_{m\times n}(\mathbf{R})$ and $b \in \mathfrak{M}_{m\times 1}(\mathbf{R})$.

We denote by $P(A, b)$ the set of all Pareto minimum solutions of the system (6).

In the present case the problem (V.P) becomes a Pareto linear program (PLP) : minimize $u \in \mathbf{R}^m$ subject to

$$|Ax - b| \leqslant u, \ x \in \mathbf{R}^n, \ u \in \mathbf{R}^m_+$$

From Theorem 1 it follows

Corollary 1 shows that to find $x^0 \in P(A, b)$ is equivalent to find a Pareto minimum solution to the vector minimization problem (PLP).

DEFINITION 3. *An approximate solution* $x^* \in \mathbf{R}^n$ *of the system* (*1*) *is called the least squares solution of* (*1*) *if*

$$\sum_{i=1}^{m} f_i^2(x^*) = \inf_{x\in\mathbf{R}^n} \sum_{i=1}^{m} f_i^2(x)$$

THEOREM 2. *Problem* (PLP) *is always consistent and* $P(A, b) \neq \emptyset$ *for each matricsis* A *and* b.

Proof. Consider $x^* \in \mathbf{R}^n$ the least squares solution to (6), i.e. a solution of the system

$$\sum_{i=1}^{m} f_i(x)a_{ij} = 0, \ j = 1, 2, \ldots, n$$

which always exists. Then

$$(x^*, u_1^*, u_2^*, \ldots, u_m^*) \in \mathbf{R}^{n+m},$$

where

$$u_i^* = |f_i(x^*)|, \ i = 1, 2, \ldots, m$$

is a feasible solution to the problem (PLP), since $u_i^* \geqslant 0$, $i = 1, 2, \ldots, m$.

But if x^* is the least squares solution to (6), then $x^* \in P(A, b)$. Indeed, if $x^* \notin P(A, b)$, then there is $x \in \mathbf{R}^n$ such that (i)—(ii) are satisfied. Then

$$\sum_{i=1}^{m} f_i^2(x) < \sum_{i=1}^{m} f_i^2(x^*),$$

which contradicts the fact that x^* is the least squares solution to the system (6).

4. **Optimality criteria.** To solve the problem (PLP) it is convenient to express our multiobjective programming problem in th following simplex-like tableau

$$\begin{array}{r|cc|c|} & -x & -u & 1 \\ \hline y = & A & -E_m & b \\ \bar{y} = & -A & -E_m & -b \\ \hline f = & 0 & -E_m & 0 \\ \hline \end{array} \tag{7}$$

where $E_m \in \mathcal{M}_{m\times m}$ $(\mathbf{R})$ is the unit matrix and $\bar{y} = (y_{m+1} \ldots y_{2m})^T$.

Without loss of generality, we assume that

$$\operatorname{rank} A = n,$$

otherwise system (6) is generally consistent and every its solution is always a Pareto minimum solution (see [6, Theorem 4])

Then, after n Jordan elimination steps (J.e.s.) we can eliminate variables x. Assume that $x_1, x_2, \ldots, x_n$ were eliminated from the first n lines. Then we get the tableau

$$\begin{array}{r|ccc|c|} & -y_1 \ldots -y_n & -u_1 \ldots -u_m & & 1 \\ \hline y_{n+1} = & & & & \\ \vdots & & & & \\ y_m = & & A_1 & & b_1 \\ y_{m+1} = & & & & \\ \vdots & & & & \\ y_{2m} & & & & \\ \hline f = & 0 & & -E_m & 0 \\ \hline \end{array}$$

in which we have omitted the first n lines corresponding to the variables x, writing separately

$$x = B^{-1}b' - B^{-1}y', \tag{8}$$

where

$$B = (a_{ij})_{i,j=1}^n, \quad b' = (b_1 \ldots b_n)^T, \quad y' = (y_1 \ldots y_n)^T$$

Continuing with the first stage of the simple algorithm, to determine a basic feasible solution (b.f.s.) to the problem (PLP) (that in view of Theorem 2 exists) we get the tableau

$$\begin{array}{r|c|c|} & -z & 1 \\ \hline v = & D & d \\ \hline f = & C & c \\ \end{array} \tag{9}$$

corresponding to the canonical vector-minimization problem (CPLP):

$$f(z) = -Cz + c \to \min$$

$$Dz \leqslant d, \ z \geqslant 0$$

where $D \in \mathcal{M}_{(2m-n)\times(n+m)}$, $C \in \mathcal{M}_{m\times(n+m)}$ and $d \in \mathbf{R}^{2m-n}$, $c \in \mathbf{R}^m$.

To simplify the notation, we have denoted by $z \in \mathbf{R}_+^{n+m}$ and $v \in \mathbf{R}_+^{2m-n}$ the nonbasic and basic variables of the problem (CPLP) respectively.

THEOREM 3. [4, *Theorem 1*]. *Let* $(0, d)$ $(d \geqslant 0)$ *be a b.f.s. given in* (9) *and let* $Q = \{i : d_i = 0\}$. *Then* $(0, d)$ *is Pareto minimum solution to* (CPLP) *if and only if*

$$\begin{aligned} &Cu \geqslant 0, \ Cu \neq 0 \\ &D_Q u \leqslant 0, \ u \geqslant 0 \end{aligned} \tag{10}$$

is inconsistent, where

$$D_Q = (d_{ij}), \ i \in Q, \ j = 1, 2, \ldots, n+m$$

Remark 1. If $(0, d)$ is a non-degenerate b.f.s., then $Q = \emptyset$, and (10) becomes

$$Cu \geqslant 0, \ Cu \neq 0, \ u \geqslant 0 \tag{10'}$$

We denote by

$$a^{i\cdot} = (a_{i1}, a_{i2}, \ldots, a_{in}), \ a^{\cdot j} = (a_{1j}, a_{2j}, \ldots, a_{mj})^T$$

the row-vector and column-vector of the matrix $A = (a_{ij}) \in \mathcal{M}_{m\times m}$ respectively.

Now, let $i_1, i_2, \ldots, i_k$ $(1 \leqslant k \leqslant m)$ be distinct numbers of I such that

$$c_{i_1j} \leqslant 0, \ j \in J_0 = \{1, 2, \ldots, n+m\}, \ J_1 = \{j \in J_0 : c_{i_1j} = 0\}$$

$$c_{i_2j} \leqslant 0, \ j \in J_1, \ J_2 = \{j \in J_1 : c_{i_2j} = 0\}$$

. . . .

$$c_{j_kj} \leqslant 0, \ j \in J_{k-1}, J_k = \{j \in J_{k-1} : c_{i_kj} = 0\}.$$

Obviously

$$J_0 \supseteq J_1 \supseteq J_2 \supseteq \ldots \supseteq J_k$$

THEOREM 4. *If* $J_0, J_1, \ldots, J_{k-1}$ *are nonempty and* $J_k = \emptyset$, *then* $(0, d)$ *is a Pareto minimum solution to* (*CPLP*).

Proof. If $(0, d)$ is not a Pareto minimum solution to (CPLP) then there is $(\bar{z}, \bar{v})$ such that

$$C\bar{z} \geqslant 0, \ C\bar{z} \neq 0. \tag{11}$$

Let $H = \{j \in J_0 : \bar{z}_j > 0\}$. From (11) it is clear that $\emptyset \neq H \subseteq J_1$. Indeed, if there is $h \in H \setminus J_1$, then from $c_{i_1 h} < 0$ it follows

$$\sum_{j \in H} c_{i_1 j} \bar{z}_j < 0$$

contradicting (11). Therefore $H \subseteq J_1$.

Denote by

$$s = \max \{h \leqslant \{1, 2, \ldots, k\} : H \subseteq J_h\}$$

Then we have $1 \leqslant s \leqslant k$, $J_{s+1} \neq J_s$.

Since

$$c_{i_{s+1} j} < 0,\ j \in J_s \setminus J_{s+1}$$

it follows

$$c^{i_{s+1}} \cdot \bar{z} = \sum_{j \in J_s} c_{i_{s+1}} \bar{z}_j < 0$$

which again contradicts (11).

From the proof of Theorem 4 it follows

COROLLARY 1. *If* $J_0, J_1, \ldots, J_{m-1}$ *are non empty, then* $(0, d)$ *is a Pareto minimum solution to* $(CPLP)$.

COROLLARY 2. *Let* $c^i \leqslant 0$ *and* $J_1 = \{j \leqslant J_0 : c_{ij} = 0\}$.
If there is $s \in I \setminus \{i\}$ *such that*

$$\forall j \in J_1 \Rightarrow c_{sj} < 0$$

then $(0, d)$ *is a Pareto minimum solution to* $(CPLP)$.

5. **Description of the algorithm.** The general outline of the algorithm following from Theorems 3 and 4 is as follows.

Step 0. Starting from (7) eliminate variables x and construct Tableau (9).

Step 1. Starting from Tableau (9) proceed to a b.f.s.

Step 2. Set $i := 1$, $i_i := i_1$, $J_i := J_0 = \{1, 2, \ldots, n + m\}$

Step 3. Minimize

$$f_{i_i}(z) = - \sum_{j \in J_i} c_{ij} z_j \text{ on } S = \{z : Dz \leqslant d,\ z \geqslant 0\}$$

Step 4. Set $i := i + 1$; $i_i := i_{i+1}$, $J_i := J_{i+1} = \{j \in J_i : c_{ij} = 0\}$.

Step 5. If $J_i = \emptyset$ or $i + 1 \geqslant m + 1$, then go to Step 6, else go to Step 3.

Step 6. Calculate x^* from (8) for $(v, z) = (0, d)$ and terminate.

Remark 2. To minimize $f_i(z)$ at the Step 3 we can use the simplex algorithm corresponding to the first $2m - n$ rows, objective function f_i in the Tableau (9) and column $j \in J_i$.

Remark 3. To compute another Pareto minimum solution to the system (6) we have to itrate the algorithm by changing the initial index i_1 in the Step 2.

Remark 4. To generate all extreme Pareto minimum solutions to (6) we can apply the method given in [3], by taking, for instance, supercriterion

$$F(x, u) = \sum_{i=1}^{m} u_i.$$

6. **A numerical example.** Consider the following inconeistent linear system

$$\begin{aligned} x_1 \quad\quad &= 0 \\ x_2 &= 0 \\ x_1 + x_2 &= 1 \end{aligned}$$

The initial simplex tableau is the following

	$-x_1$	$-x_2$	$-u_1$	$-u_2$	$-u_3$	1
$y_1 =$	1	0	-1	0	0	0
$y_2 =$	0	1	0	-1	0	0
$y_3 =$	1	1	0	0	-1	1
$y_4 =$	-1	0	-1	0	0	0
$y_5 =$	0	-1	0	-1	0	0
$y_6 =$	-1	-1	0	0	-1	-1
$f_1 =$	0	0	-1	0	0	0
$f_2 =$	0	0	0	-1	0	0
$f_3 =$	0	0	0	0	-1	0

Step 0. Eliminating x_1 and x_2, after two Jordan steps (J.S) we get the tableau

	$-y_1$	$-y_2$	$-u_1$	$-u_2$	$-u_3$	1
$y_3 =$	-1	-1	1	1	-1	1
$y_4 =$	1	0	-2	0	0	0
$y_5 =$	0	1	0	-2	0	0
$y_6 =$	1	1	-1	-1	-1	-1
$f_1 =$	0	0	-1	0	0	0
$f_2 =$	0	0	0	-1	0	0
$f_3 =$	0	0	0	0	-1	0

$$x_1 = -y_1 + u_1$$
$$x_2 = -y_2 + u_2$$

Step 1. After one J.s. we get the tableau

	$-y_1$	$-y_2$	$-u_1$	$-y_6$	$-u_3$	1
$y_3 =$	0	0	0	1	-2	0
$y_4 =$	1	0	-2	0	0	0
$y_5 =$	-2	-1	2	-2	2	2
$u_2 =$	-1	-1	1	-1	1	1
$f_1 =$	0	0	-1	0	0	0
$f_2 =$	-1	-1	1	-1	1	1
$f_3 =$	0	0	0	0	-1	0

which gives a b.f.s. $y_1 = y_2 = u_1 = y_6 = u_3 = 0$; $y_3 = y_4 = 0$, $y_5 = 2$, $u_2 = 1$.

Step 2. Set $i_1 := 1$, $J_1 = \{1, 2, 4, 5\}$.

Step 3. Minimize

$$y_1 + y_2 + y_6 - u_3 \qquad (f_2(z) \rightarrow \min)$$

After one J.s. we get the tableau

	$-y_1$	$-y_2$	$-u_1$	$-y_6$	$-u_2$	1
$y_3 =$	-2	-2	2	-1	2	2
$y_4 =$	1	0	-2	0	0	0
$y_5 =$	0	1	0	0	-2	0
$u_3 =$	-1	-1	1	-1	1	1
$f_1 =$	0	0	-1	0	0	0
$f_2 =$	0	0	0	0	-1	0
$f_3 =$	-1	-1	1	-1	1	1

Step 4. $i_2 := 2$, $J_2 = \{1, 2, 4\}$

Step 5. $J_3 = \emptyset$.

Step 6. A Pareto minimum solution is given by the b.f.s.

$$y_1^0 = y_0^2 = u_1^0 = y_6^0 = u_2^0 = 0$$

$$y_3^0 = 2,\ y_4^0 = 0,\ y_5^0 = 0,\ u_3^0 = 1$$

i.e.

$$x_1^0 = -y_1^0 + u_1^0$$

$$x_2^0 = -y_2^0 + u_2^0$$

Therefore $x^0 = (0, 0)$ is the first extreme Pareto minimum solution of the given system.

Iterating the algorithm by taking $i_1 := 2$, after one J.s. we get another extreme Pareto minimum solution $x^1 = (1, 0)$, which is written from the tableau

	$-y_1$	$-y_2$	$-u_3$	$-y_6$	$-u_2$	1
$y_3 =$	0	0	−2	1	0	0
$y_4 =$	−1	−2	2	−2	2	2
$y_5 =$	0	1	0	0	−2	0
$u_1 =$	−1	−1	1	−1	1	1
$f_1 =$	−1	−1	1	−1	1	1
$f_2 =$	0	0	0	0	−1	0
$f_3 =$	0	0	−1	0	0	0

in wich $i_2 := 3$ and $J_1 = 1,2,3,4$, $J_2 = \emptyset$.

The last iteration of the algorithm, starting with $i_1 := 3$ and taking $i_2 := 1$, $i_3 := 2$, gives the last extreme Pareto minimum solution $x^2 = (0,1)$.

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NUMERICAL METHODS IN FUZZY HIERARCHICAL PATTERN RECOGNITION

I. Cluster Substructure of a Fuzzy Class

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ABSTRACT. — In Part I, a multilevel fuzzy classification is introduced. The cluster substructure of a fuzzy class is described by a fuzzy partition of this class. A refinement relation between fuzzy partitions is defined. Some convexity properties for fuzzy partitions are given. A generalization of the Fuzzy ISODATA clustering algorithm is developed. A stratified classification may be obtained using this algorithm.

In Part II, a fuzzy hierarchy is defined. A divisive clustering algorithm to obtain a binary fuzzy hierarchy is given. The algorithm represents an effective technique for identifying the cluster structure of a data set.

1. **Introduction.** This paper presents a fuzzy hierarchical approach of the pattern recognition problem. The main task in pattern recognition is the class identification. The most real-world classes are fuzzy in nature. The classical sets are therefore not appropriate to describe such classes. A class of patterns may be conceived as a fuzzy set. The cluster structure of a collection X of patterns will be given by a set of disjoint fuzzy sets which form a fuzzy partition of X. The cluster substructure of a fuzzy class C may be described by a fuzzy partition of C. The hierarchical structure of X is given by a chain of fuzzy partitions ordered by the refinement relation. This chain generates a binary fuzzy hierarchy. Hierarchies may be obtained agglomeratively or divisively. In this paper a fuzzy divisive procedure to build a fuzzy hierarchy is developed. The classes are subdivided as long as is necessary to produce the final objective classification.

A decomposition criterion which permits to retain in the hierarchy only „real” clusters is used. In this way no *a priori* knowledge concerning the optimal number of clusters is required. The method gives therefore a solution of the cluster validity problem.

The method is more powerful than the one-level classification methods because it permits a more intimate exploration of the cluster substructure. No evaluation of a validity functional [1, 4, 14, 15] is needed. Our approach is essentially different from the hierarchial clustering methods based on fuzzy relations. In the method of Bezdek and Harris [2], for example, fuzzy relations are used to obtain classical hierarchies. As the author knows, the present procedure is the unique which produces a hierarchy of fuzzy classes.

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2. **Prerequisites.** Let $X = \{x_1, \ldots, x_p\}$ be a set of patterns. Every pattern x_i is specified by the values of d features. $x_{ij} \in \mathbf{R}$ represents the value of the j-th feature with respect to $x_i \cdot x_i$ may be thus considered as a vector (or point) in $\mathbf{R}^d$.

A fuzzy set on X is a function $A : X \to [0, 1]$. We denote by $L(X)$ the class of all fuzzy sets on X. The set operations of fuzzy sets are defined using the triangular norms (t-norms) and t-conorms (see for instance) [11]). In this paper we consider the t-norm $T(x, y) = \max(x + y - 1, 0)$ and the t-conorm $S(x, y) = \min(x + y, 1)$.

Let A and B be two fuzzy sets from $L(X)$. The reunion $A \cup B$ is defined by

$$(A \cup B)(x) = \min(A(x) + B(x), 1), \qquad \forall x \in X.$$

The intersection $A \cap B$ is defined by

$$A \cap B)(x) = \max(A(x) + B(x) - 1, 0), \qquad \forall x \in X$$

The inclusion on $L(X)$ is defined as usually

$$A \subseteq B \text{ if } A(x) \leqslant B(x), \qquad \forall x \in X.$$

The family $A_1, \ldots, A_n$, $n \geqslant 2$, of fuzzy sets is called disjoint [5] iff

$$\left(\bigcup_{i=1}^{j} A_i\right) \cap A_{j+1} = \emptyset, \qquad j = 1, \ldots, n-1,$$

where $\emptyset(x) = 0, \qquad \forall x \in X.$

The family $A_1, \ldots, A_n$ of fuzzy sets is said to be a fuzzy partition of the fuzzy set C iff it is disjoint and its reunion is just C. A_i is an atom or member of the partition. It is not difficult to prove [5] that the family $A_1 \ldots, A_n$ is a fuzzy partition of C iff $\sum_i A_i(x) = C(x)$, $\forall x \in X$. For $C = X$ this equality is just Ruspini's definition of a fuzzy partition. We denote by $F(C)$ $(F_n(C))$ the class of all fuzzy partitions of C (having n atoms).

Let P, Q be from $F(C)$. Q is said to be a *refinement* of P, $P \prec Q$, iff every atom of P is a reunion of some atoms of Q.

It is easy to see that if $P = \{A_1, \ldots, A_n\}$, $P \in F_n(C)$ and $Q_i \in F(A_i)$, then $\{Q_1, \ldots, Q_n\} = Q \in F(C)$ and $P \prec Q$. The refinement relation is an order relation on $F(C)$ [11].

3. **Convexity propeties.** Let M_{np} be the linear space of real $(m \times p)$ matrices. Any fuzzy partition $P = \{A_1, \ldots, A_n\}$, $P \in F_n(C)$ may be characterized by matrices in M_{np}. Let m_{ij} be the ij-th element of the matrix M and define

$$U_n(C) = \left\{M \in M_{np} \,|\, m_{ij} \in [0, 1], \sum_{j=1}^{n} m_{ij} = C(x_j), \forall j\right\}.$$

There is an isomorphism $f: F_n(C) \to U_n(C)$, defined by $f(P) = M$ where $m_{ij} = A_i(x_j)$. Throughout this section we identify a fuzzy partitions with the matrix associated to it by this isomorphism.

A fuzzy partition is called non-degenerate iff none of its atoms is empty, i.e. $\sum_j A_i(x_j) > 0$, for every i. P is degenerate iff $\sum_j A_i(x_j) \geqslant 0$, for each i. Let $F_{n0}(C)$ be the set of all degenerate fuzzy partitions of C having n atoms. We denote by by $F_{nh}(X)$ $(F_{nh0}(X))$ the set of all non-degenerate (degenerate) classical or hard partitions of X. Using the established isomorphism we may speak about the convex combination of fuzzy partitions. We are now able to state the next convexity property.

PROPOSITION 1. *The sets $F_n(C)$ and $F_{n0}(C)$, where $C \in L(X)$, $C \neq \emptyset$, are convex.*

Proof. Let us consider $P_j = \{A_1^j, \ldots, A_n^j\}$, $P_j \in F_n(C)$, and $a_1, \ldots, a_k \geqslant 0$, $\sum_{j=1}^{k} a_j = 1$. We define the convex combination $B_i(x) = \sum_{j=1}^{k} a_j A_i^j(x)$ and denote $Q = \{B_1, \ldots, B_n\}$. We have thus $\sum_{i=1}^{n} B_i(x) = \sum_{i=1}^{n} \sum_{j=1}^{k} a_j A_i^j(x) = \sum_{j=1}^{k} a_j \sum_{i=1}^{n} A_i^j(x) = \sum_{j=1}^{k} a_i \cdot C(x) = C(x)$, for every x from X. It follows thus that Q is in $F_n(C)$.

It is not difficult to see that the convex hull of $F_{nh}(X)$ is a subset of $F_n(X)$. The inclusion conv $F_{nh}(X) \subset F_n(X)$ is strict [3]. The next proposition proves this affirmation. It gives a necessary and sufficient condition that a fuzzy partitions from $F_n(X)$ admits a convex decomposition with non-degenerate hard partitions.

PROPOSITION 2. *Let $P = \{A_1, \ldots, A_n\}$ be from $F_n(X)$. P is in* conv $F_{nh}(X$ *if and only if* $\sum_{x \in X} A_i(x) \geqslant 1$, $i = 1, \ldots, n$.

Proof. Necessity. If $P \in \operatorname{conv} F_{nh}(X)$ then there exist $a_1, \ldots, a_k \geqslant 0$, $\sum_{j=1}^{k} a_j = 1$ and $Q_j = \{B_1^j, \ldots, B_n^j\}$, Q_j from $F_{nh}(X)$, $j = 1, \ldots, k$, such that $A_i(x) = \sum_{j=1}^{k} a_j B_i^j(x)$, for every x from X. Q_i is non-degenerate and thus $\sum_x B_i^j(x) \geqslant 1$, for every i, j. We may write

$$\sum_x A_i(x) = \sum_{j=1}^{k} a_j \sum_x B_i^j(x) \geqslant \sum_{j=1}^{k} a_j = 1.$$

For sufficiency, an algorithm for the convex decomposition of every fuzzy partitions P with $\sum_{i=1}^{n} A_i(x) \geqslant 1$ has been elaborated. Every partition in the obtained convex decomposition is non-degenerate. Because of its technical character this algorithm is omitted here. It will be presented in a further paper.

Remark. The theorem gives „the additional property P in $F_n(X)$ needs to distinguish it as a member of conv $F_{nh}(X)$" required in [3]. It may play a central role in clustering by convex decomposition.

Example. Let us consider $X = \{x_1, x_2, x_3\}$ and $P = \{A_1, A_2\}$ the fuzzy partition of X given by $A_1(x_j) = \lambda$, $\forall j$, $A_2(x_j) = 1 - \lambda$, $\forall j$. The associated matrix is

$$f(P) = \begin{pmatrix} \lambda & \lambda & \lambda \\ 1-\lambda & 1-\lambda & 1-\lambda \end{pmatrix}.$$

According with Proposizion 2, P admits a convex decomposition with non-degenerate hard partitions iff $3\lambda \geqslant 1$ and $3(1 - \lambda) \geqslant 1$, i.e. $\lambda \in [1/3, 2/3]$.

For $\lambda = 1/2$ the decomposition is

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} +$$

$$+ \frac{1}{6}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

This result contradicts the affirmation made in [3] that P (or $f(P)$) is not in conv $F_{2h}(X)$.

4. Multilevel classification. The cluster structure of the set of patterns $X = \{x_1, \ldots, x_p\}$, $x_i \in \mathbf{R}^d$, may be described by a fuzzy partition of X. A class of patterns (or a cluster) corresponds to an atom A_i of a fuzzy partition P of X. In the following sections, we'll refer to the atom A_i as fuzzy class or the cluster A_i.

In our two-level classification model the cluster substructure of the fuzzy class A_i is given by a fuzzy partition of A_i. We may consider a multilevel fuzzy classification in which the cluster substructure of a fuzzy class C from a level l is described by a fuzzy partition P of C. The atoms of P belong to the level $l + 1$.

Let $P = \{A_1, \ldots, A_n\}$ be a fuzzy partition of the fuzzy class C. A fuzzy class A_i is represented by a prototype $L_i \in \mathbf{R}^d$. $D(x_j, L_i)$ denotes a dissimilarity index measuring the degree in which x_j differs from the prototype L_i. D is a function $D : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$ such that

(i) $$D(x, y) \geqslant 0, \quad D(x, x) = 0, \qquad \forall x, y.$$

(ii) $$D(x, y) = D(y, x), \qquad \forall x, y.$$

D may be a squared distance on $\mathbf{R}^d$.

In order to define a local dissimilarity D_i which depends on A we consider a distance d on $\mathbf{R}$ given by

$$d_i(x, y) = \begin{cases} \min (A_i(x), \ A_i(y))\, d(x, y), & \text{if } x, y \in X \\ A_i(x)\, d(x, y) & \text{if } x \in X,\ y \neq X \\ d(x, y) & \text{if } x, y \notin X \end{cases}$$

The local dissimilarity is thus

$$D_i(x, y) = d_i^2(x, y), \qquad \forall x, y.$$

If $L_i \in X$ then $A_i(L_i) = \max\limits_{x \in X} A_i(x)$ and therefore we may write

$$D_i(x_j, L_i) = d_i^2(x_j, L_i) = (A_i(x_j))^2 d^2(x_j, L_i).$$

The inadequacy $I(A_i, L_i)$ between the fuzzy class A_i and its prototype L_i may be defined as

$$I(A_i, L_i) = \sum_{j=1}^{p} D_i(x_j, L_i) = \sum_{j=1}^{p} (A_i(x_j))^2 d^2(x_j, L_i).$$

The inadequacy $J(P, L)$ between the partition P and its representation $L = (L_1, \ldots, L_n)$ is given by

$$J(P, L) = \sum_{i=1}^{n} I(A_i, L_i) = \sum_{i=1}^{n} \sum_{j=1}^{p} (A_i(x_j))^2\, d^2(x_j, L_i),$$

where J is a function $J : F_n(C) \times \mathbf{R}^{dn} \to \mathbf{R}$.

The detection of the cluster substructure of the fuzzy class C reduces to search for the partition $P \in F_n(C)$ and its representation $L \in \mathbf{R}^{dn}$ which minimize $J(P, L)$. The optimal fuzzy partition is thus obtained as the solution of the minimization problem:

$$\begin{cases} \text{minimize } J(P, L) \\ P \in F_n(C) \\ L \in \mathbf{R}^{dn} \end{cases} \tag{1}$$

The next proposition gives a local solution of this problem:

PROPOSITION 3. i) $P \in F_n(C)$ *is a minimum of the function* $J(\cdot, L)$ *if and only if*

$$A_i(x_j) = \frac{C(x_j)}{\sum\limits_{k=1}^{n} \frac{d^2(x_j, L_i)}{d^2(x_j, L_k)}}, \qquad \forall i, j. \tag{2}$$

ii) $L \in \mathbf{R}^{dn}$ *is a minimum of the function* $J(P, \cdot)$ *if and only if*

$$L_i = \frac{\sum_{j=1}^{p} (A_i(x_j))^2 \cdot x_j}{\sum_{j=1}^{p} (A_i(x_j))^2}, \qquad \forall i. \tag{3}$$

Proof. For necessity the Lagrange multipliers method is used. For sufficiency one shows that the Hessians associated with $J(\cdot, L)$ and $J(P, \cdot)$ are positive definite.

Remark. The Picard iteration with (2) and (3) is used to obtain a local solution of the problem (1). The starating point in the Picard iteration may be an arbitrary choice for P or an arbitrary choice for L. For $C = X$ this procedure reduces to the well-known Fuzzy ISODATA algoritm [1, 12]. For this reason we will call it Generalized Fuzzy ISODATA (GFI) algorithm. Using the GFI algoritm a stratified classification of the pattern set may be obtained. In the second part of this paper a hierarchical divisive procedure to detect the optimal cluster structure in the data set X will be given.

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ON A GENERAL TYPE OF CONVEXITY

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REZUMAT. — Asupra unui tip general de convexitate. În lucrare se dă o caracterizare operatorilor integrali (2) ce conservă funcţiile S-convexe definite în [5].

In their book [5], A. W. Roberts and D. E. Varberg have proposed, for an independent study project, the following general notion of convexity. Let S be a subset of $I \times I$ (where $I = [0, 1]$) and $D = [0, b]$. The function $f: D \to \mathbf{R}$ is said to be S-convex it it verifies the relation:

$$f(sx + ty) \leqslant s \cdot f(x) + t \cdot f(y) \tag{1}$$

for any $(s, t) \in S$ and any $x, y \in D$.

The set of all S-convex functions defined on D is denoted by $K(S)$. Theoretically S can be a subset of $\mathbf{R}^2$ and a S-convex function can be defined on some subsets of a linear space. But even in the case given before can appear some complications. For example, from (1) we can see that $s + t \leqslant 1$ for any $(s, t) \in S$. Otherwise b must be infinite because $(s \cdot t) \cdot x \in D$ for $x \in D$.

Apart from the well known examples of S-convexity given in [5], let us to mention here another one, given by us in [7]. For a given $m \in I$, we say that the function $f: D \to \mathbf{R}$ is m-convex if:

$$f(sx + m(1 - s)y) \leqslant s \cdot f(x) + m(1 - s) \cdot f(y)$$

for any $x, y \in D$ and any $s \in I$. A function is m-convex if and only if it is S_m- convex, where:

$$S_m = \{(s, t) : s \in I,\ t = m(1 - s)\}.$$

As follows from Lemma 2, m-convexity is a notion intermediate to convexity ($m = 1$) and starshapendness ($m = 0$). So it may be considered similar to a notion given for complex functions by P. T. Mocanu in [4].

For $s = t = 0$, from (1) we have $f(0) \leqslant 0$, that we suppose to be valid for any function which appears in what follows.

To answer to some questions from [5], we consider the following relation between sets: $S < S'$ if for any $(s, t) \in S$ there is an $(s, t') \in S'$ such that $t \leqslant t'$. We put $0 < S$ for $I \times \{0\} < S$.

LEMMA 1. *If $0 < S$, any S-convex function f is starshaped.*

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Proof. For any $s \in I$, ther is a $t \geqslant 0$ such that $(s, t) \in S$. So, for any $x \in D$, we have:

$$f(sx) = f(sx + t \cdot 0) \leqslant s \cdot f(x) + t \cdot f(0) \leqslant s \cdot f(x).$$

LEMMA 2. *If* $0 < S < S'$, then $K(S) \supset K(S')$.

Proof. Let f be in $K(S')$ and x, y in D. For any $(s, t) \in S$ there is a $(s, t') \in S$ such that $t' \geqslant t$. Hence:

$$f(sx + ty) = f(sx + t'(t/t')y) \leqslant sf(x) + t'f((t/t')y) \leqslant sf(x) + tf(y).$$

Remark 1. As $s + t \leqslant 1$ for $(s, t) \in S$, we deduce that the usual convexity is the most restrictive.

COROLLARY 1. *If* $0 < S$ *and* $G \subset S$, *where*:

$$G = \{(s, t_s) : s \in I,\ t_s = \inf\{t : (s, t) \in S\}\},$$

then $K(S) = K(G)$.

Remark 2. This property gives an answer, at least partial, to the question on the minimality of the set S which determines a class $K(S)$.

But our central objective in this note is related to another problem. In [2] A. M. Bruckner and E. Ostrow have proved that the integral mean:

$$F(f)(x) = \frac{1}{x}\int_0^x f(v)\,\mathrm{d}v$$

preserves the convexity, the starshapendness and the superadditivity of the function f. In [3] it is considered a more general mean:

$$F_g(f)(x) = \frac{1}{g(x)}\int_0^x g'(v)\,f(v)\,dv. \tag{2}$$

In [6] we have obtained a characterization of the weight-functions g which give integral means F_g that preserve the above properties. We want to extend now this characterization to the case of S-convexity.

THEOREM. *The function* $F_g(f)$ *is* S-*convex for any* S-*convex function* f *if and only if the function* g *is of the form*:

$$g(x) = k \cdot x^a,\ k \in \mathbf{R},\ a > 0. \tag{3}$$

Proof. The function $f_0(x) = cx$ is S-convex for any real c. Hence so must be also the function:

$$F_0(x) = F_g(f_0)(x) = \frac{c}{g(x)}\int_0^x g'(v)\cdot v\,dv.$$

But, c being of arbitrary sign, this happens if and only if, for $c = 1$:

$$F_0(sx + t \cdot y) = s \cdot F_0(x) + t \cdot F_0(y)$$

for $(s, t) \in S$; $x, y \in D$. Thus (see [11]) $F_0(x) = bx$ and so g must be of the form (3). If $a > 0$, (2) is not defined for $f(x) = c$.

Conversely, if g is given by (3), then (2) becomes:

$$F_a(f)(x) = \frac{a}{x^a} \int_0^x v^{a-1} \cdot f(v)\, dv. \tag{4}$$

aking the substitution (given in [3]): $v = x \cdot w^{1/a}$, from (4) we get:

$$F_a(f)(x) = \int_0^1 f(x \cdot w^{1/a})\, dw.$$

If f is in $K(S)$, for any $(s, t) \in S$ and any $x, y \in D$, we have:

$$F_a(f)(sx + ty) = \int_0^1 f((sx + ty)w^{1/a})\, dw \leqslant s \int_0^1 f(xw^{1/a})\, dw +$$

$$+ t \int_0^1 f(yw^{1/a})\, dw = s \cdot F_a(f)(x) + t \cdot F_a(f)(y)$$

that is $F_a(f)$ is also in $K(S)$.

If we denote:

$$M^a K(S) = \{f : F_a(f) \in K(S)\}$$

we have thus the following:

COROLLARY 2. *If* $0 < S < S'$ *and* $a > 0$, *then*:

$$\begin{array}{ccc} K(S') & \subset & K(S) \\ \cap & & \cap \\ M^a K(S') & \subset & M^a K(S). \end{array}$$

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FURTHER REMARKS ON THE FIXED POINT STRUCTURES

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REZUMAT. — Alte observații asupra structurilor de punct fix. În [5] am realizat o teorie a structurilor de punct fix în spații metrice. În prezenta lucrare se extind aceste rezultate în cazul unei mulțimi oarecare.

1. **Introduction.** The purpose of this paper is to improve the results given in [5].

We follow terminologies and notations in [4] and [5].

2. **Fixed point structures.** Let X be a nonempty set and $Y \in P(X)$. We denote by $\mathbf{M}(Y)$ the set of all mapping $f: Y \to Y$.

DEFINITION 2.1. A triple (X, S, M) is a fixed point structure if

(i) $S \subset P(X)$ is a nonempty subset of $P(X)$,

(ii) $M: P(X) \to \bigcup_{Y \in P(X)} \mathbf{M}(Y)$, $Y \mapsto M(Y) \subset \mathbf{M}(Y)$, is a mapping such that, if $Z \subset Y$, then

$$M(Z) \supset \{f|_Z : f \in M(Y) \text{ and } f(Z) \subset Z\},$$

(iii) Every $Y \in S$ has the fixed point property with respect to $M(Y)$.

Now, let us consider some simple examples.

Example 2.1. X is a nonempty set, $S = \{\{x\} | x \in X\}$, and $M(Y) = \mathbf{M}(Y)$

Example 2.2. (Bourbaki-Birkhoff). $(X, \leqslant)$ is an ordered set, $S = \{Y \in P(X) | (Y, \leqslant)$ has a maximal element$\}$ and $M(Y) = \{f: Y \to Y | x \leqslant f(x)$, for all $x \in Y\}$.

Example 2.3. (Knaster, Tarski, Birkhoff). $(X, \leqslant)$ is a complete lattice, $S = \{Y \in P(X) | (Y, \leqslant)$ is a complete sublattice of $X\}$ and $M(Y) = \{f: Y \to Y | f$ is order-preserving mapping$\}$.

Example 2.4. (Banach, Caccioppoli). (X, d) is a complete metric space, $S = P_{cl}$ and $M(Y) = \{f: Y \to Y | f$ is a contraction$\}$.

Example 2.5. (Niemytzki, Edelstein). (X, d) is a complete metric space, $S = P_{cp}(X)$ and $M(Y) = \{f: Y \to Y | f$ is a contractive mapping$\}$.

Example 2.6. (Schauder) X is a Banach space, $S = P_{cp,cv}(X)$ and $M(Y) = C(Y, Y)$.

Example 2.7. (Dotson). X is a Banach space, $S = P_{cp,st}(X)$ and $M(Y) = \{f: Y \to Y | f$ is a nonexpansive mapping$\}$.

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Example 2.8. (Browder). X is a Hilbert space, $S = P_{b,cl,cv}(X)$ and $M(Y) = \{f: Y \to Y \mid f$ is a nonexpansive mapping$\}$.

Example 2.9. (Schauder). X is a Banach space, $S = P_{b,cl,cv}(X)$ and $M(Y) = \{f: Y \to Y \mid f$ is completely continuous$\}$.

Example 2.10. (Tychonov). X is a locally convex space, $S = P_{cp,cv}(X)$ and $M(Y) = C(Y, Y)$.

3. **Mappings with the intersection property.** Now, let us introduce

DEFINITION 3.1. Let X be a nonempty set, $Z \subset P(X)$ and $Z \neq \emptyset$. A mapping $\theta: Z \to \mathbf{R}_+$ has the intersection property if $Y_n \in Z$, $Y_{n+1} \subset Y_n$, $n \in \mathbf{N}$ and $\lim\limits_{n\to\infty} \theta(Y_n) = 0$ implies $Y_\infty := \bigcap\limits_{n \in \mathbf{N}} Y_n \neq \emptyset$ and $\theta(Y_\infty) = 0$.

For some examples of mappings with the intersection property see [5]. Consider, however

Example 3.1. Let (X, d) be a complete metric space. If $x_1, x_2, x_3 \in X$, then we denote by $\delta_2(x_1, x_2, x_3)$ the area of the triangle $\triangle(x_1, x_2, x_3)$. For $Y \in P_b(X)$ let

$$\delta_2(Y) := \sup \{\delta_2(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in Y\}.$$

If Z is the set of all connected and bounded subset of X, then $\delta_2: Z \to \mathbf{R}_+$, $Y \mapsto \delta_2(Y)$, has the intersection property.

For some properties of the mappings with the intersection property see [5].

4. **Compatibility with the fixed point structures.** Let us give

DEFINITION 4.1. Let (X, S, M) be a fixed point structure, $\theta: Z \to \mathbf{R}_+$ $(S \subset Z \subset P(X))$, $\eta: Z \to Z$. The pair (θ, η) is compatible with (X, S, M) if

(i) there exists Z_1, $S \subset Z_1 \subset Z$, such that $\theta|_{Z_1}$ has the intersection property,

(ii) η is a closure operator,

(iii) $\theta(\eta(Y)) = \theta(Y)$, for all $Y \in Z$,

(iv) $F_\eta \cap Z_0 \subset S$.

Now we illustrate this definition by some examples.

Example 4.1. Let X be a Banach space, $S = P_{cp,cv}(X)$, $M(Y) = C(Y, Y)$, $Z = P_b(X)$, $Z_1 = P_{b,cl}(X)$, $\theta = \alpha_k$, and $\eta(A) = \overline{co}A$, $A \in Z$.

Example 4.2. Let (X, S, M) be as in Example 4.1. $\theta = \gamma$ (measure of non compact-convexity), $\eta(A) = \bar{A}$, $A \in P_b(X)$.

5. (θ, φ)**-contractions**

DEFINITION 5.1. Let X be a set, $Z \subset P(X)$, $Z \neq \emptyset$, $\theta: Z \to \mathbf{R}_+$ and $\varphi: \mathbf{R}_+ \to \mathbf{R}_+$ a comparison function. A mapping $f: Y \to X$ is a (θ, φ) — contraction $(Y \subset X)$ if

(i) $A \in P(Y) \cap Z$ implies $f(A) \in Z$,

(ii) $\theta(f(A)) \leqslant \varphi(\theta(A))$ for all $A \in Z \cap I(f)$.

Now we have.

THEOREM 5.1. *Let* (θ, η) *be a compatible pair with the fixed point structure* (X, S, M). *Let* $Y \in F_{\eta/Z_1}$ *and* $f \in M(Y)$. *If* f *is a* (θ, φ) — *contraction, then* $F_f \neq \emptyset$ *and* $\theta(F_f) = 0$.

Proof. Let $Y_1 = \eta(f(Y))$. Since $Y \in F_{\eta/Z_1}$, we have $Y_1 \subset Y$. Let $Y_2 = \eta(f(Y_1)), \ldots, Y_n = \eta(f(Y_n)), \ldots$ We denote $A_\infty := \bigcap_{n \in \mathbf{N}} Y_n$. From the Definition 4.1. we have $Y_\infty \neq \emptyset$, $\theta(Y_\infty) = 0$ and $Y_\infty \in F_\eta$. On the other hand $Y_n \in I(f)$ and $Y_\infty \in I(f)$. These imply $Y_\infty \in S$. Since $f \in M(Y)$ and (X, S, M) is a fixed point structure we have $F_f \neq \infty$.

From $f(F_f) = F_f$ and the condition (ii) in Definition 5.1. we have $\theta(F_f) = 0$.

For some consequences of this general result see [5].

6. θ-condensing mappings

DEFINITION 6.1. Let X be a set, $Z \subset P(X)$, $Z \neq \emptyset$, and $\theta : Z \to \mathbf{R}_+$. A mapping $f : Y \to X$ is a θ-condensing mapping if

(i) $A \in P(Y) \cap Z$ implies $f(A) \in Z$,

(ii) $A \in I(f)$, $\theta(A) \neq 0$ implies $\theta(f(A)) < \theta(A)$.

We have

THEOREM 6.1. *Let* (θ, η) *be a compatible pair with the fixed point structure* (X, S, M). *Let* $Y \in F_{\eta/Z_1}$ *and* $f \in M(Y)$. *If*

(i) $A \in Z$, $x \in Y$ *imply* $A \cup \{x\} \in Z$,

(ii) $\theta(A \cup \{x\}) = \theta(A)$ *for all* $A \in Z$, $x \in Y$,

(iii) f *is* θ-*condensing.*

Then $F_f \neq \emptyset$ *and* $\theta(F_f) = 0$.

Proof. The proof is the same as the proof of the Theorem 5.1. in [5]

For some consequences of this theoreme see [5].

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NOTE ON A CONJECTURE IN PRIME NUMBER THEORY

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ABSTRACT. — Let p_n be the n-th natural prime number and let $\alpha_n = \sqrt{p_{n+1}} - \sqrt{p_n}$, $n \geq 1$. One presents some asymptotic estimates for $(\alpha_n)_{n \geq 1}$. Relations among the diverses conjectures in the prime number theory are also considered.

In what follows we shall denote by p_n the n^{th} prime number. Consider the sequences $(d_n)_{n \geq 1}$, $(\alpha_n)_{n \geq 1}$ defined by $d_n = p_{n+1} - p_n$ and $\alpha_n = \sqrt{p_{n+1}} - \sqrt{p_n}$. It is well-known that $\limsup\limits_{n\to\infty} d_n = +\infty$ (see [1], [7]). From this point of view the sequence $(\alpha_n)_{n \geq 1}$ has a different behaviour than $(d_n)_{n \geq 1}$. In this sense we begin with the following conjecture:

CONJECTURE 1. *The following inequality holds*

$$\alpha_n < 1 \tag{1}$$

for every natural numbers n.

The inequality (1) hs verified on a computer Felix C-256 with a programm in FΦRTRAN IV for all prime numbers $\leq 10^6 + 3$, so for the first 78.500 primes. The numerical tests and the programm was accomplished by Dan Grecu from Politehnic Institute of Cluj-Napoca.

Our next theorem contains some remarks abouth the lim inf and lim sup of some sequences which contain the difference $\alpha_n = \sqrt{p_{n+1}} - \sqrt{p_n}$.

THEOREM 1. *If* $\beta \in [0, 1/2)$ *then*

$$\liminf_{n\to\infty} (n \ln n)^{\beta} \cdot \alpha_n = 0 \tag{2}$$

$$\limsup_{n\to\infty} (n/\ln n)^{1/2} \cdot \alpha_n = \infty \tag{3}$$

Proof. To prove the relation (2), following the method of [8], we consider the function $f\colon [1, \infty) \to [1, \infty)$, $f(x) = x^{\alpha}$, $\alpha \in (0, 1)$. Denote $\lambda_n = \sqrt{p_n}$. It is clear that $\lambda_1 < \lambda_2 < \ldots$ and using the inequality $tf(t) \leq \lambda_{n+1} f(\lambda_{n+1})$ for $t \in [\lambda_n, \lambda_{n+1}]$ we obtain

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1} f(\lambda_{n+1})} \leq \sum_{n=1}^{\infty} \int_{\lambda_n}^{\lambda_{n+1}} \frac{dt}{tf(t)} = \int_{\lambda_1}^{\infty} \frac{dt}{t^{1+\alpha}} < +\infty \tag{4}$$

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But, ob the other hand, we have:

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1} f(\lambda_n)} - \sum_{n=1}^{\infty} \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1} f(\lambda_{n+1})} = \sum_{n=1}^{\infty} \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right)\left(\frac{1}{f(\lambda_n)} - \frac{1}{f(\lambda_{n+1})}\right) <$$

$$< \sum_{n=1}^{\infty} \left(\frac{1}{f(\lambda_n)} - \frac{1}{f(\lambda_{n+1})}\right) = \frac{1}{f(\lambda_1)} \tag{5}$$

and one obtains

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1} f(\lambda_n)} < +\infty \tag{6}$$

$$\sum_{n=1}^{\infty} \frac{\sqrt{p_{n+1}} - \sqrt{p_n}}{\sqrt{p_{n+1}}(\sqrt{p_n})^{\alpha}} < +\infty \tag{7}$$

In the following, using the divergence of the series $\sum_{n=1}^{\infty} 1/p_n$ ([1], [3] p. 135, [8]) we get easy

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{p_n p_{n+1}}} = +\infty. \tag{8}$$

From (7) and (8) it follows

$$\liminf_{n\to\infty} \left(\frac{\sqrt{p_{n+1}} - \sqrt{p_n}}{\sqrt{p_{n+1}}(\sqrt{p_n})^{\alpha}}\right) \Big/ \frac{1}{\sqrt{p_n p_{n+1}}} = 0$$

This is equivalent with $\liminf_{n\to\infty} p_n^{\frac{1-\alpha}{2}} \cdot \alpha_n = 0$. Taking into account that $p_n \sim n \ln n$ ([10] p. 153) we get the relation (2) where $\beta = \frac{1-\alpha}{2} \in [0, {}^1/_2)$.

To prove the relation (3) we use the inequality of Rankin ([2] p. 355, [6] p. 99):

$$p_{n+1} - p_n > C \ln p_n \frac{\ln \ln p_n \cdot \ln \ln \ln \ln p_n}{(\ln \ln \ln p_n)^2} \tag{9}$$

which is true for an infinity of natural numbers n.

We have successively

$$\alpha_n = \frac{p_{n+1} - p_n}{\sqrt{p_{n+1}} + \sqrt{p_n}} > \frac{p_{n+1} - p_n}{2\sqrt{p_{n+1}}} > C \frac{\ln p_n}{\sqrt{p_{n+1}}} \cdot \frac{\ln \ln p_n \cdot \ln \ln \ln \ln p_n}{(\ln \ln \ln p_n)^2}$$

for an infinity of natural indices. It follows

$$\frac{\sqrt{p_{n+1}}}{\ln p_n} \cdot \alpha_n > C \cdot \frac{\ln \ln p_n \cdot \ln \ln \ln \ln p_n}{(\ln \ln \ln p_n)^2} \tag{10}$$

so that

$$\limsup_{n\to\infty} \frac{\sqrt{p_{n+1}}}{\ln p_n} \cdot \alpha_n = +\infty \tag{11}$$

Using again the asymptotic relation $p_n \sim n \ln n$ we get immediately

$$\limsup_{n\to\infty} \frac{n \ln n}{\ln (n \ln n)} \cdot \alpha_n = +\infty \tag{12}$$

and after an elementary calculation we obtain the relation (3).

Remarks. 1. Fro the relation (2) we give a proof similar to that given for the relation (3) using instead of (9) the inequality of B o m b i e r i [4]

$$p_{n+1} - p_n < (0{,}46 \ldots) \ln p_n \tag{13}$$

which is true for an infinity of natural numbers n. We have

$$\alpha_n = \frac{p_{n+1} - p_n}{\sqrt{p_{n+1}} + \sqrt{p_n}} < \frac{p_{n+1} - p_n}{2\sqrt{p_n}} < K \frac{\ln p_n}{\sqrt{p_n}}, \quad K = 0{,}46 \ldots$$

for an infinity of n.

It follows

$$p_n^{\beta} \cdot \alpha_n < \mathrm{K} \frac{\ln p_n}{p_n^{1/2-\beta}} \tag{14}$$

from where one obtains

$$\liminf_{n\to\infty} p_n^{\beta} \cdot \alpha_n = 0 \tag{15}$$

and, consequently, using the relation $p_n \sim n \ln n$ we have (2). This method has the disadvantage that it uses the strong inequality (13) and this inequality has a very difficult proof.

2. One sets, in a natural way, the question if the relation (2) remains true in the limit case $\beta = 1/2$. It is very surprising the fact that in this case we have:

$$\liminf_{n\to\infty} \sqrt{p_n} \cdot \alpha_n \geqslant 1 \tag{16}$$

If $\liminf\limits_{n\to\infty} \sqrt{p_n} \cdot \alpha_n < 1$, then there is a positive number such that for an infinity of natural numbers n one has

$$\sqrt{p_n} \cdot \alpha_n \leqslant 1 - \varepsilon \tag{17}$$

The inequality (17) is equivalent with

$$\frac{p_{n+1} - p_n}{1 - \varepsilon} \leqslant 1 + \sqrt{p_{n+1}/p_n}$$

But $p_{n+1} - p_n \geqslant 2$ and it follows

$$(1+\varepsilon)/(1-\varepsilon) \leqslant \sqrt{p_{n+1}/p_n} \tag{18}$$

But this relation is in contradiction with the known fact that $\lim_{n\to\infty} p_{n+1}/p_n = 1$.

3. From the relation (2) one obtains $\liminf_{n\to\infty} \alpha_n = 0$, a result proved by L. Panaitopol [7].

Let us recall other three conjectures in prime number theory.

CONJECTURE 2 ([11]). *One has the equality*

$$\lim_{n\to\infty} \alpha_n = 0 \tag{19}$$

CONJECTURE 3 (A. Schinzel [9]). *For* $x \geqslant 8$ *between* x *and* $x + (\ln x)^2$ *there is a prime number.*

CONJECTURE 4 ([3] p. 73). *For* $n \geqslant 1$ *the interval* $[n^2, (n+1)^2]$ *contains a prime number.*

In connection with Conjecture 4 the best known result is due to M. N. Huxley (see [5]) and states that there is a prime number between n^2 and $n^2 + n\theta$ for every $\theta > 7/6$ and $n \geqslant n_0(\theta)$, where $n_0(\theta)$ is a sufficiently great natural number.

The following theorem establishes the connections of our Conjecture 1 with the above mentioned conjectures.

THEOREM 2. *The following implications hold:*

$$\begin{array}{ccc} & C\,3 & \Rightarrow C\,2 \\ & \Downarrow & \\ C\,1 \Rightarrow & C\,4 & \end{array}$$

C 2 $\Rightarrow$ C 1 *when n is sufficiently great.*

C 2 $\Rightarrow$ C 4 *when n is sufficiently great.*

Proof. „C 3 $\Rightarrow$ C 2". Supposing that C 3 is true, it follows $p_n < p_{n+1} < p_n + (\ln p_n)^2$, so that

$$\alpha_n < \sqrt{p_n + (\ln p_n)^2} - \sqrt{p_n} = \frac{(\ln p_n)^2}{\sqrt{p_n + (\ln p_n)^2} + \sqrt{p_n}} < \frac{(\ln p_n)^2}{\sqrt{p_n}}$$

which implies C 2.

„C 3 $\Rightarrow$ C 4". By the Conjecture 3, in the interval $[n^2, (n+1)^2]$ there is a prime number. But we have the inequality $n^2 + 4(\ln n)^2 < n^2 + 2n + 1$, for every natural number n, which implies C 4.

„C 1 $\Rightarrow$ C 4". If C 4 is not true then there is a natural number n such that $p_k < n^2 < (n+1)^2 < p_{k+1}$. Then an elementary calculation shows that $\sqrt{p_{k+1}} - \sqrt{p_k} = \alpha_k > 1$, which is a contradiction.

„C 2 $\Rightarrow$ C 1 for sufficiently great n" is clear.

„C 2 ⇒ C 4 for sufficiently great n". We obtain easily this implication taking into account that C 1 ⇒ C 4.

In conection with Conjecture 1 it presents also interest the proof of the weaker statement that the sequence $(\alpha_n)_{n\geq 1}$ is bounded.

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ON GENERALIZED MEASURES OF THE AMOUNT OF INFORMATION BASED ON THE STRATIFIED RANDOM SAMPLE

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ABSTRACT. - In this paper we define a new measure of the amount of information associated with a generalized random variable as well as the measures based on the stratified random sample.

1. **Generalized measures of the amount of information.** Let $\{\Omega, \mathfrak{B}, P\}$ be a probability space, that is, Ω an arbitrary nonempty set, called the set of elementary events; $\mathfrak{B}$ a σ — algebra of subsets of Ω, containing Ω itself, the elements of $\mathfrak{B}$ being called events; and P a probability measure, that is, a nonnegative and additive set function, defined on $\mathfrak{B}$, for which $P(\Omega) = 1$.

Let

$$\Delta_N^* = \left\{ \mathfrak{P} = (p_1, p_2, \ldots, p_N);\ p_i > 0,\ i = \overline{1,N}, \sum_{i=1}^{N} p_i = 1 \right\}, \tag{1.1}$$

be the set of all probability distributions associated with a discrete finite random variable X.

S h a n n o n [8] introduced a measure of information by the quantity

$$H(\mathfrak{P}) = H(X) = -\sum_{i=1}^{N} p_i \log_2 p_i, \tag{1.2}$$

called entropy of the distribution $\mathfrak{P}$ (or, entropy of the random variable X).

Measure (1.2) satisfies the additivity

$$H(\mathfrak{P} * \mathfrak{Q}) = H(\mathfrak{P}) + H(\mathfrak{Q}), \tag{1.3}$$

where

$$\mathfrak{P} * \mathfrak{Q} = (p_1q_1, \ldots, p_1q_N, \ldots, p_Nq_1, \ldots, p_Nq_N) \in \Delta_{NN}^* \tag{1.4}$$

is direct product of the distributions $\mathfrak{P}$ and $\mathfrak{Q}$, $\mathfrak{P}, \mathfrak{Q} \in \Delta_N^*$.

R é n y i [7] introduced a generalization of the notion of a random variable.

DEFINITION 1. An incomplete random variable X, is a function $\xi = \xi(\omega)$ measurable with respect to the measure on $\mathfrak{B}$ and defined on a subset Ω_1 of Ω, where $\Omega_1 \in \mathfrak{B}$ and $P(\Omega_1) > 0$.

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The only difference between an ordinary random variable (ξ is an ordinary or complete random variable if $P(\Omega_1) = 1$) and an incomplete random variable is thus that the latter is not necessarily defined for every $\omega \in \Omega$. Therefore, for a incomplete random variable we have $0 < P(\Omega_1) < 1$.

DEFINITION 2. If $0 < P(\Omega_1) \leqslant 1$, then random variable ξ, defined on the Ω_1, is a generalized random variable. The distribution of a generalized random variable X will called a generalized probability distribution.

In this sense, the ordinary distributions can be considered as a particular case of a latter.

We denote by

$$w(\mathfrak{P}) = \sum_{i=1}^{N} p_i, \tag{1.5}$$

the weight of the distribution $\mathfrak{P}$.

Using the above definitions it follows that:

— if $w(\mathfrak{P}) = 1$, then $\mathfrak{P}$ is an ordinary distribution;

— if $0 < w(\mathfrak{P}) < 1$, then $\mathfrak{P}$ is an incomplete distribution;

— if $0 < w(\mathfrak{P}) \leqslant 1$, then $\mathfrak{P}$ is a generalized probability distribution

Also, we denote by

$$\Delta_N = \{\mathfrak{P} = (p_1, p_2, \ldots, p_N);\ p_i > 0,\ i = \overline{1,N},\ 0 < w(\mathfrak{P}) \leqslant 1\}, \tag{1.6}$$

the set of all finite discrete generalized probability distributions.

DEFINITION 3. [5] The measure of the amount of information, associated to a generalized random variable X, have the form

$$H_{\alpha^*}(\mathfrak{P}) = H_{\alpha^*}(X) = -\frac{1}{\alpha^*}\log_2\left(\sum_{i=1}^{N} q_i \cdot p_i^{\alpha^*}\right), \tag{1.7}$$

where

$$q_i = \frac{p_i^{\beta+a_i}}{\sum\limits_{j=1}^{N} p_j^{\beta+a_j}},\ i = \overline{1,N},\ \sum_{i=1}^{N} q_i = 1, \tag{1.8}$$

$$\alpha^* = \frac{\alpha - n}{n},\ \alpha^* \in (-1, 0) \cup (0, \infty);\ \alpha > 0,\ \alpha \neq n,\ n \geqslant 1, \tag{1.9}$$

$$\beta + a_i \geqslant 1,\ i = \overline{1,N},\ \mathfrak{P} \in \Delta_N. \tag{1.10}$$

This measure can be called the measure of the information of order α/n and of type $\{\beta + a_i\}$, associated to the generalized propability distribution $\mathfrak{P}$, [4].

Remark 1. The measure (1.7) is a generalized measure of the amount of information in the Daroczy's sense [1] that is

$$H_{\alpha^*}(\mathfrak{P}) = -\log_2 M_{\alpha^*}(\mathfrak{P}) = H_{\varphi}(\mathfrak{P})_f, \tag{1.11}$$

where

$$M_{\varphi}(\mathfrak{P})_f = M_{\alpha^*}(\mathfrak{P}) = \left(\sum_{i=1}^{N} q_i \cdot p_i^{\alpha^*}\right)^{1/\alpha^*} = \left(\frac{\sum_{i=1}^{N} p_i^{\beta+a_i} \cdot p_i^{\alpha^*}}{\sum_{i=1}^{N} p_i^{\beta+a_i}}\right)^{1/\alpha^*}, \tag{1.12}$$

represents the weighted mean associated to the generalized probability distribution $\mathfrak{P}$, and f, φ represent the weight function, respectively, the representation function, namely

$$f(t) = t^{\beta+a_i},\ \varphi(t) = t^{\alpha^*},\ t \in (0,\ 1]. \tag{1.13}$$

In the paper [6] to prove a theorem whence follows the form (1.11) of the measure (1.7), the additivity property (1.3), as well as the properties of the functions f and φ which were considered by Daróczy.

2. **Measures of the amount of informazion based on the stratified random sample.** Let C be a population (collectivity) and X a common property of hers elements. We want to study this collectivity relativ to this common property (characteristic).

Because, in general, the population C is heterogeneous in comparison with the characteristic X, we consider a stratification of the population C so that to obtain an homogeneity in each strata (subpopulation).

We assume that the population C is divided into N mutually exclusive subpopulations $C_1, C_2, \ldots, C_N$,

$$C = C_1 \cup C_2 \cup \ldots \cup C_N. \tag{2.1}$$

We denote by $m = M(X) = M(X \mid C)$, $\sigma^2 = D^2(X) = D^2(X \mid C)$ the expectation and the variance of the random variable (of the characteristic) X relating to whole population C. Also, we denote by

$$m_i = M(X \mid C_i),\ \sigma_i^2 = D^2(X \mid C_i),\ i = \overline{1,N}, \tag{2.2}$$

the expectations and variances of the same characteristic X but relating to the subpopulation (strata) C_i.

Let

$$p_i = P(X = x \mid x \in C_i),\ i = \overline{1,N},$$

be the probability (the proportion) that a certain element x of the population C to belong to the strata C_i. We assume that all these probabilities are known.

A sample S from C obtained by taking random samples of size n_1 from C_1, of size n_2 from C_2, ..., of size n_N from C_N is called a stratified sample of total size

$$n = n_1 + n_2 + \ldots + n_N. \tag{2.3}$$

In this paper, we discuss the stratified random sampling for estimation of the population expectation m if we assume that σ^2, m_i, σ_i^2 and p_i, $i = \overline{1,N}$, are known. Alsom, using these specifications as well as the Definition 3, we shall define measures of the amount of information based on the stratified random sample.

DEFINITION 4. If the size of sample, n, satisfies the relations

$$n = \frac{n_1}{p_1} = \frac{n_2}{p_2} = \dots = \frac{n_N}{p_N}, \tag{2.4}$$

then the sample S is called a represenative sample. Also, if n from Definition 3 has just this semnification, then about the measure (1.7) we shall say that is *representative*.

DEFINITION 5. If the sizes of sample n_i, $i = \overline{1,N}$, to determine from the condition that the sample mean

$$\overline{X} = \sum_{i=1}^{N} p_i \overline{X}_i, \tag{2.5}$$

where

$$\overline{X}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} x_{ij}, \quad i = \overline{1,N}, \tag{2.5'}$$

(x_{ij} are elements from the strata C_i), to be an efficient estimator for unknown expectation m, that is,

$$n_i = \frac{n p_i \cdot \sigma_i}{\sum\limits_{j=1}^{N} p_j \cdot \sigma_j}, \quad i = \overline{1,N}, \tag{2.6}$$

then about the measure (1.7) we shall say that is *optimum*.

Therefore, the representative information of sample, $H_S^R(X)$, may be written in the form

$$H_S^R(X) = -\log_2 M_{\alpha_R^*}(X), \tag{2.7}$$

where

$$M_{\alpha_R^*}(X) = \left(\sum_{i=1}^{N} q_i \cdot p_i^{\alpha_R^*}\right)^{1/\alpha_R^*}, \tag{1.8}$$

and

$$\alpha_R^* = \frac{\alpha \cdot p_i}{n_i} - 1, \quad \forall i,\ i = \overline{1,N}. \tag{2.9}$$

In the same way, the optimum information of sample, $H_S^0(X)$, has the form

$$H_R^0(X) = -\log_2 M_{\alpha_0^*}(X), \tag{2.10}$$

where

$$M_{\alpha_0^*}(X) = \left(\sum_{i=1}^{N} q_i \cdot p_i^{\alpha_0^*}\right)^{1/\alpha_0^*}, \tag{2.11}$$

and

$$\alpha_0^* = \frac{p_i \cdot \sigma_i}{n_i \sum_{j=1}^{N} p_j \cdot \sigma_j} - 1, \quad \forall\, i,\ i = \overline{1, N}. \tag{2.12}$$

Now, we shall compare, between them, the measures (2.7) and (2.10). But, to compare these measures means, in fact, to compare the generalized means corresponding to (2.8) and (2.11).

Remark 2. Because the parameters (2.9) and (2.12) are independent by the index i, then when $i = \overline{1, N}$, it follows that among the standard deviations σ_1, $\sigma_2, \ldots, \sigma_N$, exist at least an index i so that

$$\sigma_i > \bar{\sigma} = \sum_{j=1}^{N} p_j\ \sigma_j, \tag{2.13}$$

where $\bar{\sigma}$ represents just the mean value of the standard deviations $\sigma_1, \sigma_2, \ldots, \sigma_N$.

If the inequality (2.13) is realized, then

$$\alpha_0^* - \alpha_R^* = \frac{\alpha \cdot p_i}{n_i}\left(\frac{\sigma_i}{\sigma} - 1\right) > 0, \tag{2.14}$$

and hence

$$\alpha_R^* < \alpha_0^*. \tag{2.15}$$

THEOREM. *If the parameters* α_R^* *and* α_0^* *are in the relation* (2.15), *then*

$$M_{\alpha_R^*}(X) < M_{\alpha_0^*}(X) \tag{2.16}$$

and

$$H_S^0(X) < H_S^R(X). \tag{2.17}$$

Proof. Because the parameters α_R^* and α_0^* belong to the set $A = (-1, 0) \cup \cup (0, \infty)$ we shall distinguish two cases.

Case 1. $0 < \alpha_R^* < \alpha_0^*$ and $\alpha_R^* = \gamma \cdot \alpha_0^*$, $0 < \gamma < 1$.

If we denote

$$u_i = q_i \cdot p_i^{\alpha_0^*}, \quad v_i = q_i. \quad i = \overline{1, N}, \tag{2.18}$$

then

$$q_i \cdot p_i^{\alpha_R^*} = u_i^{\gamma} \cdot v_i^{1-\gamma} \tag{2.19}$$

and therefore

$$\sum_{i=1}^{N} q_i \cdot p_i^{\alpha_R^*} = \sum_{i=1}^{N} u_i^{\gamma} \cdot v_i^{1-\gamma}. \qquad (2.20)$$

Using the inequality [2]

$$b_1^{p_1} \cdot b_2^{p_2} \cdot \ldots \cdot b_N^{p_N} < p_1 b_1 + b_2 p_2 + \ldots + p_N n_N, \qquad (2.21)$$

which is a generalization of the inequality between the arithmetic and geometric means of N nonnegative numbers, we obtain

$$\frac{u_i^{\gamma}}{\left(\sum\limits_{j=1}^{N} u_j\right)^{\gamma}} \cdot \frac{v_i^{1-\gamma}}{\left(\sum\limits_{j=1}^{N} v_i\right)^{1-\gamma}} < \gamma \cdot \frac{u_i}{\sum\limits_{j=1}^{N} u_j} + (1-\gamma) \cdot \frac{v_j}{\sum\limits_{j=1}^{N} v_j}. \qquad (2.22)$$

Summing this inequality over i, we obtain

$$\frac{\sum\limits_{i=1}^{N} u_i^{\gamma} \cdot v_i^{1-\gamma}}{\left(\sum\limits_{j=1}^{N} u_j\right)^{\gamma} \cdot \left(\sum\limits_{j=1}^{N} v_j\right)^{1-\gamma}} < \sum_{i=1}^{N} \left[\gamma \cdot \frac{u_i}{\sum\limits_{j=1}^{N} u_j} + (1-\gamma) \cdot \frac{v_i}{\sum\limits_{j=1}^{N} v_j}\right] = \gamma + (1-\gamma) = 1, \qquad (2.23)$$

and hence the inequality

$$\sum_{i=1}^{N} u_i^{\gamma} \cdot v_i^{1-\gamma} < \left(\sum_{i=1}^{N} u_i\right)^{\gamma} \cdot \left(\sum_{i=1}^{N} v_i\right)^{1-\gamma}. \qquad (2.24)$$

In view of (2.18) and if we effectuate the calculations we obtain just the inequality (2.16) and hence the inequality (2.17).

Case 2: $-1 < \alpha_R^* < \alpha_0^* < 0$.

Using the relations [3]

$$M_{-\alpha_R^*}(X) = \frac{1}{\left(\sum\limits_{i=1}^{N} q_i \left(\frac{1}{p_i}\right)^{\alpha_R^*}\right)^{1/\alpha_R^*}} \qquad (2.25)$$

$$M_{-\alpha_0^*}(X) = \frac{1}{\left(\sum\limits_{i=1}^{N} q_i \left(\frac{1}{p_i}\right)^{\alpha_0^*}\right)^{1/\alpha_0^*}}, \qquad (2.26)$$

the proof of this case is similar to the preceding case.

Remark 3. Using the Remark 2, that is, the facts that parameters α_R^* and α_0^* are independent by the index i, is possible, also, to find an index s so that

$$\sigma_s < \sigma = \sum_{j=1}^{N} p_j \cdot \sigma_j, \qquad (2.27)$$

or, an index r se that

$$\sigma_r = \bar{\sigma} = \sum_{j=1}^{N} p_j \cdot \sigma_j. \tag{2.28}$$

And in these cases the Theorem is also true, only that the inequalities (2.16) and (2.17) will be

$$M_{\alpha_R^*}(X) > M_{\alpha_0^*}(X),\ H_S^0(X) > H_S^R(X), \tag{2.29}$$

respectively,

$$M_{\alpha_R^*}(X) = M_{\alpha_0^*}(X),\ H_S^0(X) = H_S^R(X). \tag{2.30}$$

Remark 4. If the characteristic X follows a uniform distribution relating to each strata C_i, $i = \overline{1, N}$, namely $p_i = \frac{1}{N}$, $i = \overline{1, N}$, and $\sigma_1 = \sigma_2 = \ldots = \sigma_N$, then

$$H_S^0(X) = H_S^R(X) = \log_2 N, \tag{2.31}$$

where $\log_2 N$ is hist the Shannon's information associated with a uniform distribution.

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ON A CLASS OF MULTIVARIATE LINEAR POSITIVE APPROXIMATING OPERATORS

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REZUMAT. — Asupra unei clase de operatori de aproximare liniari pozitivi multidimensionali. În această lucrare se prezintă o extindere multidimensională a unui operator liniar pozitiv, de tip Bernstein, introdus şi studiat, în anul 1983, în lucrarea [5], în cazul unidimensional. Se dau evaluări ale restului formulei de aproximare corespunzătoare şi se estimează ordinul de aproximare, folosind modulul de continuitate multidimensional.

1. In our earlier paper [5] we have introduced and investigated the approximation properties of a linear positive operator L_m^r, of Bernstein type, depending on a non-negative integer parameter r, m being a natural number such that $m > 2r$. This operator, which maps into itself the Banach space $C[0, 1]$ of real-valued continuous functions on $[0, 1]$, is defined explicitly by

$$(L_m^r f)(x) := \sum_{k=0}^{m} w_{m,k}^r(x) f\left(\frac{k}{m}\right), \tag{1}$$

where

$$w_{m,k}^r(x) := \begin{cases} \binom{m-r}{k} x^k(1-x)^{m-r-k+1} & \text{if } 0 \leq k < r \\ \binom{m-r}{k} x^k(1-x)^{m-r-k+1} & \text{if } r \leqslant k \leqslant m-r \\ \binom{m-r}{k-r} x^{k-r+1}(1-x)^{m-k} & \\ \binom{m-r}{k-r} x^{k-r+1}(1-x)^{m-k} & \text{if } m-r < k \leqslant m. \end{cases} \tag{2}$$

It is easy to see that if $r = 0$ or $r = 1$ then this operator reduces to the Bernstein operator B_m, defined by

$$(B_m f)(x) := \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right), \tag{3}$$

where

$$p_{m,k}(x) := \binom{m}{k} x^k (1-x)^{m-k}. \tag{4}$$

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As we have shown, one can express $(L_m^r f)(x)$ in terms of the Bernstein fundamental polynomials (4) in the following form

$$(L_m^r f)(x) = \sum_{k=0}^{m-r} p_{m-r,k}(x)\left[(1-x)f\left(\frac{k}{m}\right) + xf\left(\frac{k+r}{m}\right)\right]. \tag{5}$$

In this paper we present a multivariate extension of the operator L_m^r and investigate how it can be used in the multivariate constructive approximation theory.

In the Euclidian space $E_s \equiv \mathbf{R}^s$ of all s-tuples of real numbers $(x_1, x_2 \dots, x_s)$ we consider the s-dimensional unit cube

$$\Omega_s = \{(x_1, x_2, \dots, x_s) \in \mathbf{R}^s \mid 0 \leqslant x_i \leqslant 1, \quad i = 1(1)s\}.$$

To any real-valued function f defined on Ω_s and arbitrary vector of non-negative integer components $(r_1, r_2, \dots, r_s)$ $(m_s > 2r_i.\ i = 1(1)s)$, we define the s-dimensional linear positive operator

$$L_{m_1,\dots,m_s}^{r_1,\dots,r_s}$$

by the following formula

$$\left(L_{m_1,\dots,m_s}^{r_1,\dots,r_s} f\right)(x_1, \dots, x_s) := \tag{6}$$

$$= \sum_{k_1=0}^{m_1-r_1} \dots \sum_{k_s=0}^{m_s-r_s} p_{m_1-r_1,k_1}(x_1) \dots p_{m_s-r_s,k_s}(x_s) \cdot \left(Q_{m_1,\dots,m_s,k_1,\dots,k_s}^{r_1,\dots,r_s} f\right)(x_1, \dots, x_s),$$

where

$$\left(Q_{m_1,\dots,m_s,k_1,\dots,k_s}^{r_1,\dots,r_s} f\right)(x_1, \dots, x_s) :$$

$$= (1-x_1)\dots(1-x_s)f\left(\frac{k_1}{m_1}, \dots, \frac{k_s}{m_s}\right) +$$

$$+ x_1(1-x_2)\dots(1-x_s)f\left(\frac{k_1+r_1}{m_1}, \frac{k_2}{m_2}, \dots, \frac{k}{m_s}\right) +$$

$$+ \dots + (1-x_1)\dots(1-x_{s-1})x_s f\left(\frac{k_1}{m_1}, \dots, \frac{k_{s-2}}{m_{s-2}}, \frac{k_{s-1}+r_{s-1}}{m_{s-1}}, \frac{k_s}{m_s}\right) +$$

$$+ x_1x_2(1-x_3)\dots(1-x_s)f\left(\frac{k_1+r_1}{m_1}, \frac{k_2+r_2}{m_2}, \frac{k_3}{m_3}, \dots, \frac{k_s}{m_s}\right) + \dots +$$

$$+ (1-x_1)\dots(1-x_{s-2})x_{s-1}x_s f\left(\frac{k_1}{m_1}, \dots \frac{k_{s-2}}{m_{s-2}}, \frac{k_{s-1}+r_{s-1}}{m_{s-1}}, \frac{k_s+r_s}{m_s}\right) +$$

$$+ \ldots + (1 - x_1)x_2 \ldots x_s f\left(\frac{k_1}{m_1}, \frac{k_2 + r_2}{m_2}, \ldots, \frac{k_s + r_s}{m_s}\right) +$$

$$+ x_1(1 - x_2)x_3 \ldots x_s f\left(\frac{k_1 + r_1}{m_1}, \frac{k_2}{m_2}, \frac{k_3 + r_3}{m_3}, \ldots, \frac{k_s + m_s}{m_s}\right) +$$

$$+ \ldots + x_1 \ldots x_{s-1}(1 - x_s) f\left(\frac{k_1 + r_1}{m_1}, \ldots, \frac{k_{s-1} + r_{s-1}}{m_{s-1}}, \frac{k_s}{m_s}\right).$$

One observes that for $r_1 = \ldots = r_s = 0$ this operator reduces to the s-dimensio nalBernstein operator $B_{m_1, \ldots, m_s}$, defined by

$$(B_{m_1, \ldots, m_s} f)(x_1, \ldots, x_s) : =$$

$$\sum_{k_1=0}^{m_1} \ldots \sum_{k_s=0}^{m_s} p_{m_1, k_1}(x_1) \ldots p_{m_s, k_s}(x_s) f\left(\frac{k_1}{m_1}, \ldots, \frac{k_s}{m_s}\right).$$

It should be noticed that the higher-dimensional analogous of the operator L_m^r from (1) can also be expressed under the following form

$$\left(L_{m_1, \ldots, m_s}^{r_1, \ldots, r_s} f\right)(x_1, \ldots, x_s) =$$

$$= \sum_{k_1=0}^{m_n} \ldots \sum_{k_s=0}^{m_s} w_{m_1, \ldots, m_s, k_1, \ldots, k_s}^{r_1, \ldots, r_s}(x_1, \ldots, x_s) f\left(\frac{k_1}{m_1}, \ldots, \frac{k_s}{m_s}\right),$$

where

$$w_{m_1, \ldots, m_s, k_1, \ldots, k_s}^{r_1, \ldots, r_s}(x_1, \ldots, x_s) = \prod_{i=1}^{s} w_{m_i, k_i}^{r_i}(x_i).$$

3. By using formula (6) we can find easily the values of the operator $L_{m_1, \ldots, m_s}^{r_1, \ldots, r_s}$ applied to the test functions $e_{i_1, \ldots, i_s}$, defined — for any point $(x_1, \ldots, x_s) \in \Omega_s$ — by

$$e_{i_1, \ldots, i_s}(x_1, \ldots, x_r) = x_1^{i_1} \ldots x_s^{i_s} \; (0 \leqslant i_1 + \ldots + i_s \leqslant 2).$$

We have

$$\left(L_{m_1, \ldots, m_s}^{r_1, \ldots, r_s} e_{i_1 \ldots, i_s}\right)(x_1, \ldots, x_s) = x_1^{i_1} \ldots . x_s^{i_s} (i_p = 0,1 \; ; \; p = 1(1)s), \tag{7}$$

and

$$\left(L_{m_1, \ldots, m_s}^{r_1, \ldots, r_s} e_{i_1}, \ldots, i_s\right)(x_1, \ldots, x_s) = x_p^2 + \left[1 + \frac{r_p(r_p - 1)}{m_p}\right] \frac{x_p(1 - x_p)}{m}, \tag{8}$$

for

$$i_p = 2,\ i_1 = \ldots = i_{p-1} = i_{p+1} = \ldots = i_s = 0,\ p = 1(1)\,s.$$

Appealing to the known Bohman-Korovkin-Volkov uniform convergence criterion, we may formulate the following convergence theorem.

THEOREM 1. *If $f\ C\ (\Omega_s)$, then we have*

$$\lim_{m_1, \ldots, m_s \to \infty} L^{r_1, \ldots, r_s}_{m_1, \ldots, m_s} f = f,$$

uniformly on the unit s-cube Ω_s.

4. We now proceed to determine the expression of the remainder of the approximation formula of a function $f \in C(\Omega_s)$ by means of the s-dimensional Bernstein type operator introduced in this paper:

$$f(x_1, \ldots, x_s) = \left(L^{r_1, \ldots, r_s}_{m_1, \ldots, m_s} f\right)(x_1, \ldots, x_s) + \left(R^{r_1, \ldots, r_s}_{m_1, \ldots, m_s} f\right)(x_1, \ldots, x_s). \qquad (9)$$

One observes first that the polynomial $L^{r_1, \ldots, r_s}_{m_1, \ldots, m_s} f$, which — according to (7) — reproduces the linear functions, interpolates the function f at the vertices of the s-cube Ω. This is the reason that formula (9) has the degree of exactness (**1**, ..., **1**), as can be easily seen from the equalities (7) and (8).

Now refering to an expression of the remainder given in the one-dimensional case in our paper [5] and to a generalization of the formula (7.2) for the remainders, presented in our earlier paper [3], in the bivariate linear approximation formulas, we can formulate the following.

THEOREM 2. *If $f \in C(\Omega_s)$, then for any point $(x_1, \ldots, x_s) \in \Omega_s$, the remainder of the approximation formula* (9) *can be expressed, by means of one and two-dimensional divided differences, in the following form*

$$\left(R^{r_1, \ldots, r_s}_{m_1, \ldots, m_s} f\right)(x_1, \ldots, x_s) = \qquad [10)$$

$$= \sum_{i=1}^{s} \left[1 + \frac{r_i(r_i - 1)}{m_i}\right] \cdot \frac{x_i(1 - x_i)}{m_i} \left[\xi^{(i)}_{m_i},\ \eta^{(i)}_{m_i},\ \zeta^{(i)}_{m_i};\ f(x_n, \ldots, x_{i-1}, t_i, x_{i+1}, \ldots, x_s)\right]_{t_i} -$$

$$- \sum_{\substack{i, j=1 \\ (i<j)}}^{s} \left[1 + \frac{r_i(r_i - 1)}{m_i}\right]\left[1 + \frac{r_j(r_j - 1)}{m_j}\right] \frac{x_i(1 - x_i)}{m_i} \cdot \frac{x_j(1 - x_j)}{m_j}\Bigg) \cdot$$

$$\cdot \left[\begin{matrix} \xi^{(i)}_{m_i}, & \eta^{(i)}_{m_i}, & \zeta^{(i)}_{m_i} \\ \xi^{(i)}_{m_j}, & \eta^{(j)}_{m_j}, & \zeta^{(j)}_{m_j} \end{matrix};\ f(x_1, \ldots, x_{i-1}, t_i, x_{i+1}, \ldots, x_{j-1}, t_j, x_{j+1}, \ldots, x_s)\right]_{t_i, t_i}$$

$$-\left[1+\frac{r_1(r_1-1)}{m_1}\right]\cdots\left[1+\frac{r_s(r_s-1)}{m_s}\right]\frac{x_1(1-x_1)}{m_1}\cdots\frac{x_s(1-x_s)}{m_s}\cdot$$

$$\left[\begin{matrix}\xi^{(1)}_{m_1}, & \eta^{(1)}_{m_1}, & \zeta^{(1)}_{m_1} \\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot \\ \xi^{(s)}_{m_s}, & \eta^{(s)}_{m_s}, & \zeta^{(s)}_{m_s}\end{matrix}\ ;\ f(t_1, \ldots, t_s)\right]$$

where $\xi^{(j)}_{m_i}, \eta^{(j)}_{m_i}, \zeta^{(j)}_{m_i}$ $(i, j = 1(1)s)$ *are certain points in* $(0,1)$.

If we take into account formula (6.8) from our paper [3], which corresponds to the extension to several variables of a Peano-Milne integral representation formula of a linear functional having a certain degree of exactness, we can state the following theorem.

THEOREM 3. *If the function f has continuous partial derivatives of second order on Ω_s, then the remainder of the approximation formula (9) can be represented, for any point of Ω_s, under the following form*

$$\left(R^{r_1,\ldots,r_s}_{m_1,\ldots,m_s} f\right)(x_1, \ldots, x_s) =$$

$$= \sum_{i=1}^{s} \int_0^1 G^{r_i}_{m_i}(t_i ; x_i) f_{x_i^2}\, dt_i - \sum_{\substack{i,j=1 \\ (i<j)}}^{s} \int_0^1\int_0^1 G^{r_i}_{m_i}(t_j ; x_j) G^{r_j}_{m_j}(t_j ; x_j) f_{x_i^2 x_j^2} \cdot dt_j\, dt_j +$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots +$$

$$+ (-1)^{s-1} \int_0^1 \cdots \int_0^1 G^{r_1}_{m_1}(t_1 ; x_1) \ldots G^{r_s}_{m_s}(t_s ; x_s) f_{x_1^2 \ldots x_s^2} \cdot dt_1 \ldots dt_s,$$

where

$$G^{r_i}_{m_i}(t_i ;\ x_i) = (R^{r_i}_{m_i} \varphi_{x_i})(t_i),$$

$$\varphi_{x_i}(t_i ; x_i) = (x_i - t_i)_+ = \frac{1}{2}\left[x_i - t_i + |x_i - t_i|\right]$$

and $R^{r_i}_{m_i}$ represents the remainder in the approximation formula of $f(x_1, \ldots, x_s)$ by means of $L^{r_i}_{m_i} f$, $L^{r_i}_{m_i}$ being the one-diensional linear positive operator corresponding to the argument x_i.

If we use the explicit expressions given in [5] for these Peano kernels one can see that their values on Ω are nonpositive on $[0,1]$, so that appealing to the mean value theorem of the integral calculus, or using the Cauchy mean value theorem for the divided differences occuring in formula (10), we can state.

THEOREM 4. *If the function f is twice continuous differentiable in Ω_s, with respect to each of its arguments, then for any point $(x_1, \ldots, x_s) \in \Omega$ there exists a point $(\xi_1, \xi_2, \ldots, \xi_s)$ in this domain such that*

$$\left(R^{r_1, \ldots, r_s}_{m_1, \ldots, m_s} f\right)(x_1, \ldots, x_s) = -\sum_{i=1}^{s}\left[1 + \frac{r_i(r_i - 1)}{m_i}\right]\frac{x_i(1 - x_i)}{2m_i} f_{\xi_i^2} -$$

$$-\sum_{\substack{i,j=1\\(i<j)}}^{s}\left[1 + \frac{r_i(r_i - 1)}{m_i}\right]\left[1 + \frac{r_j(r_j - 1)}{m_j}\right]\frac{x_i(1 - x_i)}{2m_i}\cdot\frac{x_j(1 - x_j)}{2m_j} f_{\xi_i^2\xi_j^2} -$$

$$-\left[1 + \frac{r_1(r_1 - 1)}{m_1}\right]\cdots\left[1 + \frac{r_s(r_s - 1)}{m_s}\right]\frac{x_1(1 - x_1)}{2m_1}\cdots\frac{x_s(1 - x_s)}{2m_s} f_{\xi_1^2\cdots\xi_s^2}.$$

We note that in the special case $s = 2$ this result has been given in our recent paper [6], while in the case $s = 2$, $r_1 = 0$, $r_2 = 0$ or $r_1 = 1$, $r = 1$ all these results were found in our earlier paper [4].

5. We can give also an asymptotic estimate of the remainder in the approximation formula (9), which corresponds to a result of Voronovskaja in the case of the Bernstein operator.

THEOREM 5. *If for the function $f : \Omega \to \mathbf{R}$, at an interior point $(x_1, \ldots, x_s)$ of Ω_s the second total differential $d^2f(x_1, \ldots, x_s)$ exists, then we have the asymptotic formula*

$$\left(R^{r_1, \ldots, r_s}_{m_1, \ldots, m_s} f\right)(x_1, \ldots, x_s) =$$

$$= \sum_{i=1}^{s}\left[1 + \frac{r_i(r_i - 1)}{m}\right]\frac{x_i(1 - x_i)}{2m} f_{x_i^2}(x_1, \ldots, x_s) + \sum_{p=1}^{s}\frac{1}{m_p}\cdot\varepsilon^{r_1, \ldots, r_s}_{m_1, \ldots, m_s},$$

where $\varepsilon^{r_1, \ldots, r_s}_{m_1, \ldots, m_s}$ thend to 0 as $m_1, \ldots, m_s$ thend to ∞.

Proof. Let $(t_1, \ldots, t_s) \in \Omega_s$. It is known that because of our hypotheses on f, there exists a function $g : \Omega_s \to \mathbf{R}$ such that we have $g(t_1, \ldots, t_s) \to 0$ as $t_1 \to x_1, \ldots, t_s \to x_s$, while $f(t_1, \ldots, t_s)$ may be expanded, according to Taylor's formula, in the following form

$$f(t_1, \ldots, t_s) = f(x_1, \ldots, x_s) + \sum_{i=1}^{s}(t_i - x_i)f_{x_i}(x_1, \ldots, x_s) +$$

$$+\frac{1}{2}\sum_{i=1}^{s}(t_i - x_i)^2 f_{x_i^2}(x_1, \ldots, x_s) + \sum_{i,j=1}^{s}(t_i - x_i)(t_j - x_j)f_{x_ix_j}(x_1, \ldots, x_s) +$$

$$+\left[\sum_{i=1}^{s}(t_i - x_i)^2\right](t_1, \ldots, t_s).$$

If we insert here $t_i = k_i/m_i$, multiply by the fundamental polynomial

$$w^{r_1, \ldots, r_s}_{m_1, \ldots, m_s, k_1, \ldots, k_s}(x_1, \ldots, x_s),$$

and take into consideration the equalities (7), (8), we obtain

$$\left(L^{r_1, \ldots, r_s}_{m_1, \ldots, m_s} f\right)(x_1, \ldots, x_s)$$

$$f(x_1, \ldots, x_s) + \sum_{i=1}^{s} \left[1 + \frac{r_i(r_i - 1)}{m_i}\right] \frac{x_i(1 - x_i)}{2m_i} f_{x_i^2}(x_1, \ldots, x_s) +$$

$$+ \rho^{r_1, \ldots, r_s}_{m_1, \ldots, m_s}(x_1, \ldots, x_s),$$

where

$$\rho^{r_1, \ldots, r_s}_{m_1, \ldots, m_s}(x_1, \ldots, x_s) =$$

$$= \sum_{k_1=0}^{m_1} \cdots \sum_{k_s=0}^{m_s} w^{r_1, \ldots, r_s}_{m_1, \ldots, m_s k_1, \ldots, k_s}(x_1, \ldots, x_s) \left[\sum_{p=1}^{s} \left(\frac{i_p}{m_p} - x_p\right)^2\right] \cdot g\left(\frac{k_1}{m_1}, \ldots, \frac{k_s}{m_s}\right).$$

Since $g(t_1, \ldots, t_s) \to 0$ as $t_1 \to x_1, \ldots, t \to x_s$, it follows that for any ε positive we can choose the positive numbers $\delta_1, \ldots, \delta_s$ such that $|g(t_1, \ldots, t_s)| < \varepsilon$ whenever $|t_i - x_i| \leq \delta_i$, $i = 1(1)s$. By replacing $t_i = k_i/m_i$, $i = 1(1)s$, we can write

$$\left|g\left(\frac{k_1}{m_1}, \ldots, \frac{k_s}{m_s}\right)\right| < \varepsilon \text{ when } \left|\frac{k_i}{m_i} - x_i\right| \leq \delta_i, \quad i = 1(1)s.$$

Since

$$\left|\rho^{r_1, \ldots, r_s}_{m_1, \ldots, m_s}\right|(x_1, \ldots, x_s) \leq \sum_{k_1=0}^{m_1} \cdots \sum_{k_s=0}^{m_s} w^{r_1, \ldots, r_s}_{m_1, \ldots, m_s k_1, \ldots, k_s}(x_1, \ldots, x_s) \cdot$$

$$\cdot \left[\sum_{p=1}^{s} \left(\frac{i_p}{m_p} - x_p\right)^2\right] \cdot \left|g\left(\frac{k_1}{m_1}, \ldots, \frac{k_s}{m_s}\right)\right|,$$

we may proceed further in like manner as in the case of one variable [1]. One concludes that there exist

$$\varepsilon^{r_1, \ldots, r_s}_{m_1, \ldots, m_s, p} = \varepsilon^{r_1, \ldots, r_s}_{m_1, \ldots, m_s, p}(x_1, \ldots, x_s)$$

tending to 0 as $m_1, \ldots, m_s$ tend to ∞, so that we can write

$$\rho^{r_1, \ldots, r_s}_{m_1, \ldots, m_s}(x_1, \ldots, x_s) \sum_{p=1}^{s} \frac{1}{m_p} \cdot \varepsilon^{r_1, \ldots, r_s}_{m_1, \ldots, m_s, p}.$$

6. We now discuss the estimation of the order of approximation of a function $f \in C(\Omega_s)$ by means of the operator considered in this paper, in order to see the speed of convergence of these operators.

We shall make use of the modulus of continuity ω on Ω_s, defined by

$$\omega(f;\ \delta_1, \ldots, \delta_s) = \max |f(x_1'', \ldots, x_s'') - f(x_1', \ldots, x_s')|,$$

where $\delta_1 > 0, \ldots, \delta_p > 0$, while $(x_1', \ldots, x_s')$ and $(x_1'', \ldots, x_s'')$ are points from Ω_s, so that

$$|x_i'' - x_i'| \leqq \delta_i, \quad i = 1(1)s.$$

We shall now establish

THEOREM 6. *If $f \in C(\Omega_s)$, then we have*

$$\left|f(x_1, \ldots, x_s) - \left(L_{m_1, \ldots, m_s}^{r_1, \ldots, r_s} f\right)(x_1, \ldots, x_s)\right| \leqq$$
$$\leqq \left[1 + \sum_{i=1}^{s} \frac{1}{\alpha_i} \sqrt{1 + \frac{r_i(r_i - 1)}{m_i}}\right] \omega\left(f;\ \alpha_1 \sqrt{\frac{x_1(1 - x_1)}{m_1}}, \ldots, \alpha_s \sqrt{\frac{x_s(1 - x_s)}{m_s}}\right) \tag{11}$$

where $\alpha_1, \ldots, \alpha_s$ are any positive constants.

Proof. Because we have on Ω_s:

$$w_{m_1, \ldots, m_s, k_1, \ldots, k_s}^{r_1, \ldots, r_s} \geqq 0$$

and

$$L_{m_1, \ldots, m_s}^{r_1, \ldots, r_s} e_{0, \ldots 0} = e_0,$$

we can write

$$\left|f(x_1, \ldots, x_s) - \left(L_{m_1, \ldots, m_s}^{r_1, \ldots, r_s}\right)(x_1, \ldots, x_s)\right| \leqq$$
$$\leqq \sum_{k_1=0}^{m_1} \cdots \sum_{k_s=0}^{m_s} w_{m_1, \ldots, m_s, k_1, \ldots, k_s}^{r_1, \ldots, r_s}(x_1, \ldots, x_s)\left|f(x_1, \ldots, x_s) - f\left(\frac{k_1}{m_1}, \ldots, \frac{k_s}{m_s}\right)\right|.$$

We shall use now the following two properties of the modulus of continuity

$$|f(x_1'', \ldots, x_s'') - f(x_1', \ldots, x_s')| \leqq \omega(f;\ |x_1'' - x_1'|, \ldots, |x_s'' - x_s'|),$$
$$\omega(f;\ \lambda_1\delta_1, \ldots, \lambda_s\delta_s) \leqq (1 + \lambda_1 + \ldots + \lambda_s)\omega(f;\ \delta_1, \ldots, \delta_s).$$

Since

$$\left|f(x_1, \ldots, x_s) - f\left(\frac{k_1}{m_1}, \ldots, \frac{k_s}{m_s}\right)\right| \leqq \omega\left(f;\ \frac{1}{\delta_1}\left|x_1 - \frac{k_1}{m_1}\right|\delta_1, \ldots, \frac{1}{\delta_s}\left|x_s - \frac{k_s}{m_s}\right|\delta_s\right) \leqslant$$
$$\leqslant \left(1 + \sum_{i=1}^{s} \frac{1}{\delta_i}\left|x_i - \frac{k_i}{m_i}\right|\right)\omega(f;\ \delta_1, \ldots, \delta_s),$$

it follows that we can write

$$\left|f(x_1, \ldots, x_s) - \left(L^{r_1, \ldots, r_s}_{m_1, \ldots, m_s} f\right)(x_1, \ldots, x_s)\right| \leqslant$$

$$\leqq \left(1 + \sum_{i=1}^{s} \sum_{k_i=0}^{m_i} \frac{1}{\delta_i} \left| x_i - \frac{k_i}{m_i} \right| w^{r_i}_{m_i, k_i}(x_i)\right) \omega(f; \delta_1, \ldots, \delta_s).$$

In accordance with the Cauchy-Schwarz inequality and with the identities (7) and (8), we have

$$\sum_{k_i=0}^{m_i} w^{r_i}_{m_i, k_i}(x_i) \left| x_i - \frac{k_i}{m_i} \right| \leqq \left[\sum_{k_i=0}^{m_i} w^{r_i}_{m_i, k_i}(x_i) \left(x_i - \frac{k_i}{m_i}\right)^2\right]^{1/2} \leqq$$

$$\leqslant \sqrt{\left[1 + \frac{r_i(r_i - 1)}{m_i}\right] \frac{x_i(1 - x_i)}{m_i}}\,.$$

By using these inequalities and selecting

$$\delta_p = \alpha_p \sqrt{\frac{x_p(1 - x_p)}{m_p}}, \qquad p = 1(1)s,$$

$\alpha_1, \ldots, \alpha_s$ being any positive constants, we finally get the inequality (11).

Now, since for any $(x_1, \ldots, x_s) \in \Omega_s$ we have $x_p(1 - x_p) \leqq 1/4$, we can select $\alpha_1 = \ldots = \alpha_s = 2$ and we obtain the following result.

THEOREM 7. *If $f \in C(\Omega_s)$ in the maximum norm over Ω_s we have*

$$\left\| f - L^{r_1, \ldots, r_s}_{m_1, \ldots, m_s} f \right\|$$

$$\leqq \left(1 + \frac{1}{2} \sum_{p=1}^{s} \sqrt{1 + \frac{r_p(r_p - 1)}{m_p}}\right) \omega\left(f; \frac{1}{\sqrt{m_1}}, \ldots, \frac{1}{\sqrt{m_s}}\right).$$

In the particular case $s = 1$, $r_1 = 0$ or $r_1 = 1$, this inequality reduces to the inequality given in 1935 by T. Popoviciu [2] for the Bernstein polynomials.

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ON A PROBLEM OF AREOLAR MECHANICS

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ABSTRACT. — In this paper a method is given for determining, in polar coordinates, the linear accelerations on plane curves, considering the functions r, θ as zero order accelerations, and the derivatives $\dot{r}, \dot{\theta}$ as first order accelerations. At the same time the areolar accelerations of the mobile body are also determined. The differential equation solution is obtained by introducing some unknown functions, of the t-time variable, called „direct connexion functions".

1. Introduction. The real development of complex phenomena cannot be comprised in simple differential equations. The simplicity disappears when the progression of the phenomenon, in all its extent, is slow or fast. In this case, the easy determination of these out of common equations' solutions disappears, and new and pretentious methods must often be resorted to.

In this paper a method is given for determining linear accelerations on plane curves, considering functions r, θ as zero order accelerations, and the derivatives $\dot{r}, \dot{\theta}$ as first order accelerations. At the same time, the areolar accelerations of the mobile body in curvilinear motion are also determined.

2. Description of the method. Let be the areolar differential of the motion of a mobile body on a plane curvilinear trajectory

$$a_2(t)\ddot{A} + a_1(t)\dot{A} + a_0(t)A = f(t), \tag{1}$$

with the given initial conditions $\overset{(i)}{A}(0) = \overset{(i)}{A}_0$, $(i = 0, 1)$, where $A(t)$ is an area.

By denoting with $R = R(\theta)$ the polar equation of the plane trajectory, the elementary area dA has the expression

$$dA = \frac{1}{2} R^2(\theta)d\theta,$$

or

$$dA = \frac{1}{2} r^2(t)\dot{\theta}(t)dt, \tag{2}$$

where $r(t) = R[\theta(t)]$ and $\theta(t)$ are the polar coordinates of the mobile body instantaneous position.

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By integrating (2), it follows

$$A(t) = A_0 + \frac{1}{2}\int_0^t r^2(s)\dot{\theta}(s)ds, \tag{3}$$

and the derivatives of A are

$$\dot{A}(t) = \frac{1}{2} r^2(t)\dot{\theta}(t), \quad \ddot{A}(t) = \frac{1}{2}(2\dot{r}r\dot{\theta} + r^2\ddot{\theta}). \tag{4}$$

By introducing the „direct connexion functions" [3]

$$\omega_{i+1,\,i}(t), \quad \bar{\omega}_{i+1,\,i}(t), \quad \omega_{2,\,0}(t); \quad (i = 0,\ 1),$$

we have the relations

$$\dot{r} = \bar{\omega}_{0,\,1}r, \quad \ddot{r} = \bar{\omega}_{2,\,1}\dot{r}, \tag{5}$$

$$\dot{\theta} = \omega_{1,\,0}\theta, \quad \ddot{\theta} = \omega_{2,\,1}\dot{\theta}, \quad \ddot{\theta} = \omega_{2,\,1}\omega_{1,\,0}\theta. \tag{6}$$

By integrating the first relation (5) and the second one (6), one obtains

$$r(t) = r_0 \exp\left[\int_0^t \bar{\omega}_{1,\,0}(s)ds\right], \tag{7}$$

$$\dot{\theta}(t) = \dot{\theta}_0 \exp\left[\int_0^t \omega_{2,\,1}(s)ds\right]. \tag{8}$$

By observing (5) and (6), relations (4) become

$$\dot{A} = \frac{1}{2} r^2\dot{\theta}, \tag{9}$$

$$\ddot{A} = \frac{1}{2} r^2\dot{\theta}(2\bar{\omega}_{1,\,0} + \omega_{2,\,1}). \tag{10}$$

By substituting (8) in (3), (9), and (10), it follows

$$A(t) = A_0 + \frac{1}{2}\dot{\theta}_0 \int_0^t r^2(s) \exp\left[\int_0^s \omega_{2,\,1}(\sigma)d\sigma\right] ds, \tag{11}$$

$$\dot{A}(t) = \frac{1}{2}\dot{\theta}_0 r^2(t) \exp\left[\int_0^t \omega_{2,\,1}(s)ds\right], \tag{12}$$

$$\ddot{A}(t) = \frac{1}{2}\dot{\theta}_0(2\bar{\omega}_{1,\,0} + \omega_{2,\,1})r^2(t) \exp\left[\int_0^t \omega_{2,\,1}(s)ds\right]. \tag{13}$$

By observing (11), (12), and (13), equation (1) becomes

$$\frac{1}{2}\,\dot{\theta}_0 r^2(t)\,\exp\left[\int_0^t \omega_{2,1}(s)ds\right]\{a_2(t)\,[2\bar{\omega}_{1,0}(t)+\omega_{2,1}(t)]+a_1(t)\}+$$

$$+\,a_0(t)\left\{A_0+\frac{1}{2}\,\dot{\theta}_0\int_0^s r^2(s)\exp\left[\int_0^t \omega_{2,1}(\sigma)d\sigma\right]ds\right\}=f(t). \tag{14}$$

By integrating expression (8), one obtains

$$\theta(t)=\theta_0+\dot{\theta}_0\int_0^t \exp\left[\int_0^s \omega_{2,1}(\sigma)d\sigma\right]ds. \tag{15}$$

By substituting (7) in the first relation (5), we have

$$\dot{r}(t)=r_0\bar{\omega}_{1,0}(t)\exp\left[\int_0^t \bar{\omega}_{1,0}(s)ds\right]. \tag{16}$$

By observing (8), the second relation (6) becomes

$$\ddot{\theta}(t)=\dot{\theta}_0\omega_{2,1}(t)\exp\left[\int_0^t \omega_{2,1}(s)ds\right]. \tag{17}$$

From the third relation (6) and from

$$\ddot{\theta}=\omega_{2,0}\theta, \tag{18}$$

it follows

$$\omega_{2,0}(t)=\omega_{2,1}(t)\omega_{1,0}(t). \tag{19}$$

By substituting (15) in the first relation (6), and from (18), we have

$$\dot{\theta}(t)=\omega_{1,0}(t)\left\{\theta_0+\dot{\theta}_0\int_0^t \exp\left[\int_0^s \omega_{2,1}(\sigma)d\sigma\right]ds\right\}, \tag{20}$$

$$\ddot{\theta}(t)=\omega_{2,0}(t)\left\{\theta_0+\dot{\theta}_0\int_0^t \exp\left[\int_0^s \omega_{2,1}(\sigma)d\sigma\right]ds\right\}. \tag{21}$$

By equalizing (17) with (21), one obtains

$$\dot{\theta}_0\omega_{2,1}(t)\exp\left[\int_0^t \omega_{2,1}(s)ds\right]-\omega_{2,0}(t)\left\{\theta_0+\dot{\theta}_0\int_0^t \exp\left[\int_0^s \omega_{2,1}(\sigma)d\sigma\right]ds\right\}=0. \tag{22}$$

From (8) and (20) it results

$$\dot{\theta}_0 \exp\left[\int_0^t \omega_{2,1}(s)ds\right] - \omega_{1,0}(t)\left\{\theta_0 + \dot{\theta}_0 \int_0^t \exp\left[\int_0^s \omega_{2,1}(\sigma)d\sigma\right]ds\right\} = 0. \tag{23}$$

By substituting (16) in the second relation (5), it is obtained

$$\ddot{r}(t) = r_0 \bar{\omega}_{2,1}(t)\bar{\omega}_{1,0}(t) \exp\left[\int_0^t \bar{\omega}_{1,0}(s)ds\right]. \tag{24}$$

By integrating the second relation (5), it follows

$$\dot{r}(t) = \dot{r}_0 \exp\left[\int_0^t \bar{\omega}_{2,1}(s)ds\right]. \tag{25}$$

From (16) and (25), one obtains

$$r_0 \bar{\omega}_{1,0}(t) \exp\left[\int_0^t \bar{\omega}_{1,0}(s)ds\right] - \dot{r}_0 \exp\left[\int_0^t \bar{\omega}_{2,1}(s)ds\right] = 0. \tag{26}$$

The expressions (7), (11), (12), (13), (14), (15), (16), (19), (20), (21), (22), (23), (24) and (26) make up a system (S) of 14 equations with 14 unknown quantities

$$\overset{(i)}{r},\ \overset{(i)}{\theta},\ \overset{(i)}{A},\ (i = 0, 1, 2),\ \omega_{i+1,i},\ \bar{\omega}_{i+1,i},\ (i = 0, 1),\ \omega_{2,0}.$$

For determining the solution, the initial conditions $\overset{(i)}{r}(0) = \overset{(i)}{r}_0$, $(i = 0, 1, 2)$, $\theta(0) = \theta_0$ are also given.

The constant $\ddot{A}_0$ results from (1), for $t = 0$.

The constant $\dot{\theta}_0$ has the value

$$\dot{\theta}_0 = 2\dot{A}_0 r_0^{-2}.$$

The value $\ddot{\theta}_0$ is obtained from the second relation (4), for $t = 0$. From (5) and (6), for $t = 0$, we have

$$\bar{\omega}_{1,0}(0) = \dot{r}_0(r_0)^{-1}, \quad \bar{\omega}_{2,1}(0) = \ddot{r}_0(\dot{r}_0)^{-1}$$

$$\omega_{1,0}(0) = \dot{\theta}_0(\theta_0)^{-1}, \quad \omega_{2,1}(0) = \ddot{\theta}_0(\dot{\theta}_0)^{-1}.$$

For $t = 0$, from (19) it follows

$$\omega_{2,0}(0) = \omega_{2,1}(0)\omega_{1,0}(0).$$

Knowing the functions $\overset{(i)}{r}(t)$ and $\overset{(i)}{\theta}(t)$, $(i = 0, 1, 2)$, the module of the speed and acceleration of the mobile body is determined

$$|\bar{v}| = [\dot{r}^2 + (r\dot{\theta})^2]^{\frac{1}{2}},$$

$$|\bar{a}| = [(\ddot{r} - r\dot{\theta}^2)^2 + (2\dot{r}\dot{\theta} + r\ddot{\theta})^2]^{\frac{1}{2}}.$$

The functions $r = r(t)$ and $\theta = \theta(t)$ are parametrical equations, in polar coordinates, ale the mobile body trajectory, which lead to the polar equation of the plane curve, by eliminating parameter t.

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ON SATURATED Π-FORMATIONS

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ABSTRACT. — A theorem giving necessary and sufficient conditions for a π-formation to be saturated is proved in the paper.

1. **Preliminaries.** It is the aim of this note to prove a theorem which gives necessary and sufficient conditions for a π-formation (i.e. a π-closed formation) to be saturated.

All groups considered are finite. Let π be a set of primes, π' the complement to π in the set of all primes and $O_{\pi'}(G)$ the largest normal π'-subgroup of a group G.

We first give some useful definitions.

DEFINITION 1.1. ([4], [5], [7]) a) *A class $\mathfrak{X}$ of groups is a homomorph if $\mathfrak{X}$ is epimorphically closed, i.e. if $G \in \mathfrak{X}$ and N is a normal subgroup of G, then $G/N \in \mathfrak{X}$.*

b) *A homomorph $\mathfrak{F}$ is a formation if $G/N_1 \in \mathfrak{F}$, $G/N_2 \in \mathfrak{F}$ implies $G/N_1 \cap N_2 \in \mathfrak{F}$.*

c) *A formation $\mathfrak{F}$ is saturated if $\mathfrak{F}$ is Frattini closed, i.e. $G/\Phi(G) \in \mathfrak{F}$ implies $G \in \mathfrak{F}$, where $\Phi(G)$ denotes the Frattini subgroup of G.*

d) *A group G is primitive if G has a maximal subgroup H with* $\mathrm{core}_G H = 1$, *where* $\mathrm{core}_G H = \bigcap \{H^g / g \in G\}$.

e) *A homomorph $\mathfrak{F}$ is a Schunck class if $\mathfrak{F}$ is primitively closed, i.e. if any group G, all of whose primitive factor groups are in $\mathfrak{F}$, is itself in $\mathfrak{F}$.*

DEFINITION 1.2. ([4]) *Let $\mathfrak{F}$ be a class of groups, G a group and H a subgroup of G. H is an $\mathfrak{F}$-covering subgroup of G if:* (i) $H \in \mathfrak{F}$; (ii) $H \leqslant K \leqslant G$, $K_0 \trianglelefteq K$, $K/K_0 \in \mathfrak{F}$ *imply* $K = HK_0$.

DEFINITION 1.3. a) ([3]) *A group is π — solvable if every chief factor is either a solvable π — group or a π' — group. For π the set of all primes, we obtain the notion of solvable group.*

b) *A class $\mathfrak{F}$ of groups is π — closed if*

$$G/O_{\pi'}(G) \in \mathfrak{F} \Rightarrow G \in \mathfrak{F}.$$

A π-closed homomorph, formation, respectively Schunck class is called π — *homomorph*, π — *formation*, respectively π — *Schunck class.*

In the proof of the main theorem we need the following results:

LEMMA 1.4. ([4]) *If $\mathfrak{X}$ is a homomorph, G a group, N a normal subgroup of G, K/N an $\mathfrak{X}$-covering subgroup of G/N and H an $\mathfrak{X}$ — covering subgroup of K, then H is an $\mathfrak{X}$ — covering subgroup of G.*

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LEMMA 1.5. ([1]) *A solvable minimal normal subgroup of a group is abelian.*

LEMMA 1.6. ([1]) *If S is a maximal subgroup of G with* $\mathrm{core}_G\, S = 1$ *and N is a minimal normal subgroup of G, then $G = SN$ and $S \cap N = 1$.*

LEMMA 1.7. ([5]) *Let $\mathfrak{F}$ be a class of groups. $\mathfrak{F}$ is a saturated formation if and only if $\mathfrak{F}$ is a Schunck class and a formation.*

LEMMA 1.8. ([2]) *Let $\mathfrak{F}$ be a π —* homomorph. Then $\mathfrak{F}$ is a Schunck class if and only if any π — solvable group has $\mathfrak{F}$ — covering subgroups.

2. The main result

THEOREM 2.1. *Let $\mathfrak{F}$ be a π formation. The following conditions are equivalent:*

(1) *$\mathfrak{F}$ is saturated*;

(2) *if G is a π — solvable group and $G \notin \mathfrak{F}$, but for the minimal normal subgroup N of G we have $G/N \in \mathfrak{F}$, then N has a complement in G*;

(3) *any π — solvable group G has $\mathfrak{F}$ — covering subgroups.*

Proof.

(1) $\Rightarrow$ (2). $\mathfrak{F}$ being a saturated π — formation, $\mathfrak{F}$ is, by 1.7., a π — Schunck class. Hence, applying 1.8., G has an $\mathfrak{F}$ — covering subgroup H. We shall prove that H is a complement of N in G. Indeed, $HN = G$, because of 1.2. (ii) used for $H \leqslant G = G$, $N \trianglelefteq G$, $G/N \in \mathfrak{F}$. Further we have $H \cap N = 1$, as the following shows. Since G is π — solvable, N is either a solvable π — group or a π' — group. Let us suppose that N is a π' — group. It follows that $N \leqslant O_{\pi'}(G)$ and we have

$$G/O_{\pi'}(G) \simeq (G/N)/(O_{\pi'}(G)/N).$$

But $G/N \in \mathfrak{F}$, hence $G/O_{\pi'}(G) \in \mathfrak{F}$, which implies, by the π — closure of $\mathfrak{F}$, the contradiction $G \in \mathfrak{F}$. So N is a solvable π — group. By 1.5., N is abelian. It follows that $H \cap N \trianglelefteq G$. Since $H \cap N \neq N$ ($H \cap N = N$ leads to $N \subseteq H$, hence $G = HN = H$, in contradiction with $G \notin$ F and $H \in \mathfrak{F}$), we have $H \cap N = 1$.

(2) $\Rightarrow$ (3). By induction on $|G|$. Two cases are possible:

1) $G \in \mathfrak{F}$. Then G is its own $\mathfrak{F}$ — covering subgroup.

2) $G \notin \mathfrak{F}$. Let N be a minimal normal subgroup of G. By the induction, G/N has an $\mathfrak{F}$ — covering subgroup E/N. We can have two possibilities:

2a) $G/N \in \mathfrak{F}$. Then $E/N = G/N$. By (2), N has a complement V in G. Again two cases are possible:

2a$_1$) $\mathrm{core}_G V \neq 1$. The induction shows that $G/\mathrm{core}_G V$ has an $\mathfrak{F}$ — covering subgroup $H/\mathrm{core}_G V$. Let us suppose that $H = G$. Then $G/\mathrm{core}_G V \in \mathfrak{F}$. Hence $G/N \cap \mathrm{core}_G V \in \mathfrak{F}$, because $\mathfrak{F}$ is a formation. But V being a complement of N in G, we have $N \cap \mathrm{core}_G V = 1$. It follows the contradiction $G \in \mathfrak{F}$. So $H < G$. By the induction, H has an $\mathfrak{F}$ — covering subgroup $\bar{H}$. By 1.4., $\bar{H}$ is an $\mathfrak{F}$ — covering subgroup of G.

2a$_2$) $\mathrm{core}_G V = 1$. We shall prove that V is an $\mathfrak{F}$ — covering subgroup of G. Since

$$V \simeq V/V \cap N \simeq VN/N = G/N,$$

we have $V \in \mathfrak{F}$. Let now

$$V \leqslant K \leqslant G,\ K_0 \trianglelefteq K,\ K/K_0 \in \mathfrak{F}.$$

We shall prove that $K = VK_0$. It is easy to see that V is a maximal subgroup in G. Indeed, $V < G$, for $V \in \mathfrak{F}$ but $G \notin \mathfrak{F}$. Further, if $V \leqslant V^* < G$, supposing $V < V^*$, it follows that there is an element $v^* \in V^* \setminus V$; but $G = VN$ implies $v^* = vn$, with $v \in V$ and $n \in N$. We obtain that $n = v^{-1}v^* \in V \cap N$. Since $V \cap N = 1$, $n = 1$. So $v^* = v \in V$, a contradiction. Hence $V = V^*$ and V is maximal in G. It follows that we have either $K = V$ or $K = G$. In the first alternative, $K = KK_0 = VK_0$. If $K = G$, we notice that $K_0 \neq 1$, for if $K_0 = 1$ it follows the contradiction $G \in \mathfrak{F}$. Let M be a minimal normal subgroup of G with $M \subseteq K_0$. We are in the hypotheses of 1.6.: V is a maximal subgroup of G with $\mathrm{core}_G V = 1$ and M is a minimal normal subgroup of G. Hence $G = VM$. It follows that $K = G = VM = VK_0$.

2b) $G/N \notin \mathfrak{F}$. Then $E/N < G/N$, hence $E < G$. By the induction, E has an $\mathfrak{F}$ — covering subgroup $\overline{E}$. Applying 1.4, $\overline{E}$ is an $\mathfrak{F}$ — covering subgroup of G.

(3) $\Rightarrow$ (1). By 1.8., $\mathfrak{F}$ is a Schunck class. Since $\mathfrak{F}$ is a formation, 1.7. implies that $\mathfrak{F}$ is saturated.

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NUMERICAL RESULTS FOR THE FREE CONVECTION FLOW FROM A VERTICAL PLATE WITH GENERALISED WALL TEMPERATURE DISTRIBUTION

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ABSTRACT. — The problem of natural convection over a semi-infinite vertical flat plate with non-uniform wall temperature is studied by using a numerical method. The wall derivates of the universal functions for the Prandtl numbers 0.733 and 7 are tabulated. Such tabulations serve to calculate the heat transfer and skin friction from the plate.

Introduction. As is well known, the problem of natural convection boundary layer flow over a semi-infinite vertical flat plate is one of the most basic problems in the study of heat transfer over external surfaces and numerous papers dealing with various physical or mathematical aspects of this problem have been published. An excellent review article concerning this problem is given by Jaluria [1].

Recently Kundu [4] has considered a special form of the problem of free convection flow over a vertical semi-infinite flat plate, viz., that of a wall with a temperature distribution of the form

$$T_w - T_\infty = \bar{x}^n \sum_{i=1}^{r} A_i \bar{x}^i \qquad (1)$$

where A_i are constants, T_w is the wall temperature, T_∞ is the ambient temperature and $\bar{x}$ measures the distance along the plate from the leading edge. However, the derived differential equations have not been analytically or numerically solved in Kundu's paper. It is, therefore, there aim of his Research Note to complete Kundu's problem by giving a numerical solution shooting techniques employing the fourth order Runge-Kutta routines as outlined by Soundalgekar, Takhar and Singh [3]. At the same time, we shall correct some misprints in his derived equations. The first and second-order wall derivatives of the universal functions are given in a table. It is worth mentioning that having a numerical solution is very helpful in evaluation of both data and approximate methods in design, and in other further calculations, such as those related to instability.

Basic ecuations. The present problem is formulated on the basis of a semi-infinite vertical surface with the origin at the leading edge. The x-axis is vertically upward and y is perpendicular to the plate. Employing the Boussinesq approximation and neglecting the viscous term in the energy equation, the governing differential equations for the solution of natural convection

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flow past a semiinfinite vertical flat plate with variable wall temperature can be written, in terms of dimensionless quantities, as

$$\frac{\partial u}{\partial x} + \frac{\partial y}{\partial y} = 0 \tag{2a}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \theta + \frac{\partial^2 u}{\partial y^2} \tag{2b}$$

$$u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} = \frac{1}{\mathrm{Pr}} \frac{\partial^2 \theta}{\partial y^2} . \tag{2c}$$

The boundary conditions of the problem are

$$\begin{aligned} &u = v = 0, \qquad \theta = \theta_w(x) \qquad \text{at} \quad y = 0 \\ &u, \quad \theta \to 0 \qquad\qquad\qquad\quad \text{as} \quad y \to \infty \end{aligned} \tag{2d}$$

Here u, v are the velocity components along x, y-axes; θ is the temperature and Pr is the Prandtl number. The dimensionless quantities in equations (2) are related to their corresponding dimensional variables through the following definitions:

$$\begin{gathered} x = \bar{x}/L, \; y = \bar{y}/L, \; u = \bar{u}L/\nu, \; v = \bar{v}L/\nu \\ \theta = (g\beta L^3/\nu^2)(T - T_\infty) \end{gathered} \tag{3}$$

where L is there reference length and other physical quantities have their usual meaning.

Next, to reduce equations (2) to ordinary differential ones, we introduce the following variables, after Kundu [2]:

$$\psi = 4(a_0/4)^{1/4} x^{(n+3)/4} \sum_{i=0}^{r} x^i f_i(\eta) \tag{4a}$$

$$\theta = x^n \sum_{i=0}^{r} a_i \, x^i \, \theta_i \, (\eta) \tag{4b}$$

where the stream function ψ is defined by

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x} \tag{5}$$

and

$$\eta = (a_0/4)^{1/4} \; x^{(n-1)/4} \; y \tag{6}$$

is an independent variable.

Substitution of equations (4) into (2) then yields the following system of ordinary differential equations:

$$a_0 \sum_{i=0}^{r} (2n + 2 + 4i) f'_i \, f'_{r-i} - a_0 \sum_{i=0}^{r} (n + 3 + 4i) f_i f''_{r-i} = a_0 f''' + a_r \, \theta_r \quad (7a)$$

$$4 \sum_{i=0}^{r} (n + i) \, a_i \, \theta_i \, f'_{r-i} - \sum_{i=0}^{r} (n + 3 + 4\, i) \, a_{r-i} f_i \theta'_{r-i} = a_s \theta''_r / \mathrm{Pr}$$

subject to the boundary conditions

$$f_r = f'_r = 0, \quad \theta = 1 \qquad \text{at} \qquad \eta = 0$$

$$f'_r, \; \theta \to 0 \qquad \text{as} \qquad \eta \to \infty$$

In the above equations primes denote differentiation with respect to η.

Analysis and results. To shorten the paper, we give in the Table 1 the wall temperature distributions, the stream function transformations and the first 24 differential equations of the problem. The universal functions and their surface derivatives needed for the evaluations of flow and heat transfer parameters are computed for Prandtl numbers of 0.733 and 7 respectively when $n = 0$.

Table 1. Functions and differential equations for different r

r	No. of equations to be solved	Functions and differential equations
0	2	$\theta_w = a_0 x^n, \; f_0, \theta_0$ $(2n + 2) f_0'^2 - (n + 3) f_0 f'' = f_0''' + \theta_0$ $\theta'' + \mathrm{Pr}\,[(n + 3) f_0 \theta' - 4n f'_0 \theta_0] = 0$
1	$2 + 2 = 4$	$\theta_w = x^n (a_0 + a_1 x), \; f_1 = (a_1/a_0) F_{11}, \; \theta_1 = \Phi_{11}$ $(4n + 8) f'_0 F'_{11} - (n + 3) f_0 F''_{11} - (n + 7) f''_0 F_{11} =$ $= F'''_{11} + \Phi_{11}$ $4[n \theta_0 F'_{11} + (n + 1) \, \Phi_{11} f'_0] - [(n + 3) f_0 \, \Phi'_{11} + (n +$ $+ 7) F_{11} \theta'_0] = \Phi''_{11} / \; \mathrm{Pr}$
2	$4 + 4 = 8$	$\theta_w = x^n (a_0 + a_1 x + a_2 x^2), \; f_2 = (a_1/a_0)^2 F_{21} + (a_2/a_0) F_{22}$ $\theta_2 = (a_1/a_2 a_0) \, \Phi_{21} + \; \Phi_{22}$ $(4n + 12)(f'_2 F'_{91} + F_{11}'^2/2) - (n + 3) f_0 F''_{21} - (n + 7) F_{11} F''_{11} -$ $- (n + 11) f'' F_{21} = F'''_{21} + \Phi_{21}$ $(4n + 12) f'_0 F'_{22} - (n + 3) f_0 F''_{22} - (n + 11) f''_0 F_{22} =$ $= F'''_{22} \, \Phi_{22}$ $4[n \theta_0 F'_{21} + (n + 1) \Phi_{11} F'_{11} + (n + 2) \, \Phi_{21} f'_0] - [(n + 3)$ $f_0 \, \Phi'_{21} + (n + 7) F_{11} \, \Phi'_{11} + (n + 11) F_{21} \theta'_0] = \Phi''_{21} / \mathrm{Pr}$ $4[n \theta_0 F'_{22} + (n + 2) \, \Phi''_{21} f'_0] - [(n + 3) f_0 \, \Phi'_{22} + (n + 11)$ $F_{22} \Phi'_0] = \; \Phi''_{22} / \mathrm{Pr}$

3 $8 + 6 = 14$

$$\theta_w = x^n(a_0 + a_1x + a_2x^2 + a_3x^3), \quad f_1 = (a_1/a_0)^3F_{31} + (a_2a_1/a_0^2)F_{32} + (a_3/a_0)F_{33},$$

$$\theta_3 = (a_1^3/a_3a_0^2)\,\Phi_{31} + a_2a_1/a_3a_0)\Phi_{32} + \Phi_{33}$$

$$(4n + 16)(f_0'F_{31}' + F_{11}'F_{21}') - (n + 3)f_0F_{32}'' - (n+7)F_{11}F_{22}'' - (n + 11)F_{21}F_{11}'' - (n + 15)F_{31}f_0'' = F_{31}''' + \Phi_{31}$$

$$(4n + 16)(f_0'F_{32}' + F_{11}'F_{22}') - (n + 3)f_0F_{32}'' - (n+7)F_{11}F_{22}'' - (n + 11)F_{22}F_{11}'' - (n + 15)F_{32}f_0'' = F_{32}''' + \Phi_{32}$$

$$(4n + 16)f_0'F_{33}' - (n + 3)f_0F_{33}'' - (n + 15)F_{33}f_0'' = F_{33}''' + \Phi_{33}$$

$$4[n\theta_0F_{31}' + (n+1)\,\Phi_{11}F_{21}' + (n + 2)\Phi_{21}F_{11}' + (n + 3)\Phi_{31}f_0'] - ([n + 3)\,\Phi_{31}'f_0 + (n + 7)\,\Phi_{21}'F_{11} + (n + 11)\,\Phi_{11}'F_{21} + (n + 15)\,\theta_0'F_{31}] = \Phi_{31}''/\boxed{\Pr}$$

$$4[n\theta_0F_{32}' + (n + 11)\,\Phi_{11}F_{22}' + (n+2)\,\Phi_{22}F_{11}' + (n+3)\Phi_{32}f_0'] - [(n + 3)\,\Phi_{32}'f_0 + (n + 7)\,\Phi_{22}'F_{11} + (n + 11)\,\Phi_{11}'F_{22} + (n + 15)\,\theta_0'F_{32}] = \Phi_{32}''/\boxed{\Pr}$$

$$4[n\theta_0F_{33}' + (n+3)\,\Phi_{33}f_0'] - [(n+3)\,\Phi_{33}'f_0 + (n + 15)\,\theta_0'F_{33}] = \Phi_{33}''/\boxed{\Pr}$$

4 $14 + 10 = 24$

$$\theta_w = x^n(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4), \quad f_4 = (a_1/a_0)F_{41} + (a_1^2a_2/a_0^3)F_{40} + (a_1a_3/a_0^2)F_{43} + (a_4/a_0)F_{44} + (a_2^2/a_0^2)F_{45},$$

$$\theta_4 = (a_1^4/a_4a_3^3)\,\Phi_{41} + (a_1a_3/a_4a_0)\,\Phi_{43} + \Phi_{44} + (a_2^2/a_4a_0)\,\Phi_{45}$$

$$(4n + 20)(f_0'F_{41}' + F_{11}'F_{31}' + F'^2/2) - (n + 3)f_0F_{41}'' - (n + 7)F_{11}F_{31}'' - (n+11)F_{21}''F_{21} - (n + 15)F_{31}F_{11}'' - (n + 19)F_{41}f_0'' = F_{41}''' + \Phi_{41}$$

$$(4n + 20)(f_0'F_{42}' + F_{11}'F_{32}' + F_{21}'F_{22}') - (n + 3)f_0F_{42}'' - (n + 7)\,F_{11}F_{32}'' - (n + 11)\,(F_{22}''F_{21} + F_{22}F_{21}'') - (n + 15)F_{11}''F_{32} - (n + 19)f_0''F_{42} = F_{42}''' + \Phi_{42}$$

$$(4n + 20)(f_0'F_{43}' + F_{11}'F_{33}') - (n+3)(f_0F_{43}'' - (n + 7)F_{11}F_{33}'' - (n + 15)F_{11}''F_{32} - (n + 19)f_0'F_{42} = F_{42}''' + \Phi_{42}$$

$$(4n + 20)f_0'F_{44}' - (n + 3)f_0F_{44}'' - (n + 19)f_0''F_{44} = F_{44}''' + \Phi_{44}$$

$$(4n + 20)(f_0'F_{45}' + F_{22}'^2/2) - (n + 3)f_0F_{45}'' - (n + 11)F_{22}F_{22}'' - (n + 19)f_0''F_{45} = F_{45}''' + \Phi_{45}$$

$$4[n\theta_0F_{41}' + (n + 1)\Phi_{11}F_{31}' + (n + 2)\Phi_{21}F_{21}' + (n + 3)\,\Phi_{31}F_{11}' + (n + 4)\,\Phi_{41}f_0'] - [(n + 3)f_0\Phi_{41}' + (n + 7)\,F_{11}\Phi_{31}' + (n + 11)F_{21}\Phi_{21}' + (n + 15)F_{31}\Phi_{11}' + (n + 19)\,F_{41}\theta_0'] = \Phi_{41}''/\boxed{\Pr}$$

$$4[n\theta_0F_{42}' + (n + 1)\,\Phi_{11}F_{32}' + (n + 2)(\Phi_{22}F_{21}' + \Phi_{21}F_{22}') + (n + 3)\,\Phi_{32}F_{11}' + (n + 4)f_0'\,\Phi_{42}] - [(n + 3)f_0\Phi_{42}' + (n + 7)\,F_{11}\Phi_{32}' + (n + 11)\,(F_{21}\Phi_{22}' + F_{22}\Phi_{21}') + (n + 15)F_{32}\Phi_{11}' + (n + 19)F_{42}\theta_0'] = \Phi_{42}''/\boxed{\Pr}$$

$$4[n\theta_0F'_{43}+(n+1)\Phi_{11}F'_{33}+(n+3)\Phi_{33}F'_{11}+(n+4)f'_0\Phi'_{43}]-$$
$$-[(n+3)f_0\Phi'_{43}+(n+7)F_{11}\Phi'_{33}+(n+15)F_{33}\Phi'_{11}+$$
$$+(n+19)F_{43}\theta'_0]=\Phi''_{43}/\mathrm{Pr}$$

$$4[n\theta_0F'_{44}+(n+4)\Phi_{44}f'_0]-[(n+3)f_0\Phi'_{44}+(n+19)F_{44}\theta'_0]=$$
$$=\Phi''_{44}/\mathrm{Pr}$$

$$4[n\theta_0F'_{45}+(n+2)\Phi_{22}F'_{22}+(n+4)\Phi_{45}f'_0]-[(n+3)f_0\Phi'_{45}+$$
$$+(n+11)F_{22}\Phi'_{22}+(n+19)F_{45}\theta'_0]=\Phi''_{45}/\mathrm{Pr}$$

The boundary conditions on the problem are

$$\left.\begin{array}{lll} F_{rj}=F'_{rj}=0 & \text{for all } j \\ \Phi_{rr}=1, \quad \Phi_{rj}=0 & \text{for } j\neq r \end{array}\right\} \text{ at } \eta=0 \tag{9a}$$

$$F'_{rj}=0, \quad \Phi_{rj}=0 \quad \text{for all } j \quad \text{at } \eta\to\infty. \tag{9b}$$

In Table 2 we present the wall derivatives of the universal functions from which the rate of heat transfer and the skin friction can be calculated. Such tabulations serve as a reference against which other approximate solutions can be compared. We note that our results for $f''_0(0)$ and $\theta'_0(0)$ which correspond to the case of an isothermal flat plate agree very closely with those of Ostrach [4]. We also mention that for those who wish to follow similar configuration problems, the present tabulated data provide enough test cases for checking the computer program.

The velocity and temperature profiles associated with some universal functions are illustrated in Figures 1 and 2 for Pr = 0.733.

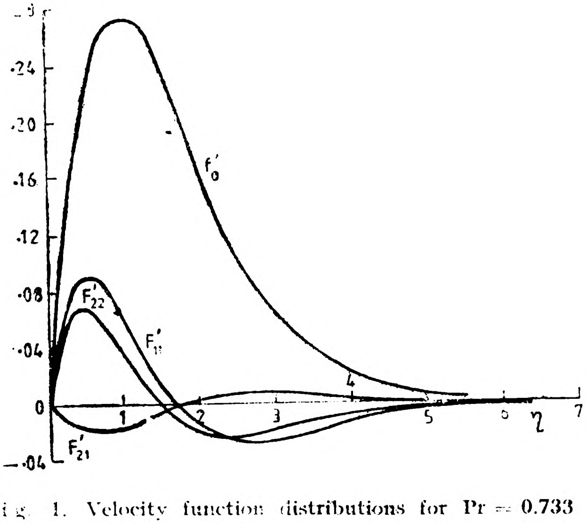

Fig. 1. Velocity function distributions for Pr = 0.733

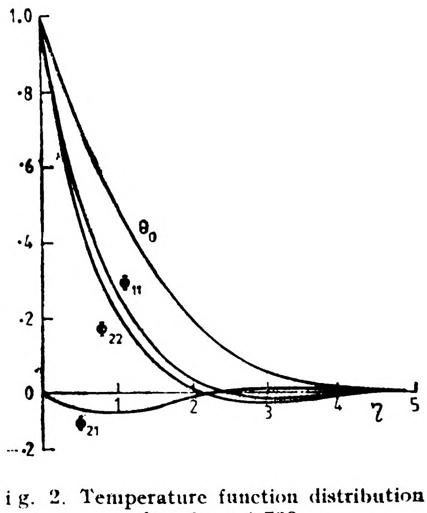

Fig. 2. Temperature function distribution for Pr = 0.733

Table 2

Values of the derivatives at the plate for $n = 0$

	Pr 0.733	Pr = 7
$f_0''(0)$	0.6741	0.4494
$\theta_0'(0)$	−0.5079	−1.0508
$F_{11}''(0)$	0.3873	0.2563
$\Phi_{11}'(0)$	−0.9286	−1.8539
$F_{21}''(0)$	−0.0382	−0.0251
$\Phi_{21}'(0)$	−0.1194	−0.2432
$F_{22}''(0)$	0.3361	0.2221
$\Phi_{22}'(0)$	−1.1175	−2.1941
$F_{31}''(0)$	−0.0021	−0.0024
$\Phi_{31}'(0)$	0.0492	0.0740
$F_{32}''(0)$	−0.0670	−0.0440
$\Phi_{32}'(0)$	−0.2563	−0.5193
$F_{33}''(0)$	0.3051	0.2016
$\Phi_{33}'(0)$	−1.2625	2.4557
$F_{41}''(0)$	0.0033	0.0018
$\Phi_{41}'(0)$	0.0016	−0.0130
$F_{42}''(0)$	0.0335	0.0218
$\Phi_{42}'(0)$	0.0864	0.1765
$F_{43}''(0)$	−0.0613	−0.0406
$\Phi_{43}'(0)$	−0.2724	−0.5498
$F_{44}''(0)$	0.2834	0.1874
$\Phi_{44}'(0)$	−1.3824	−2.6729
$F_{45}''(0)$	−0.0295	−0.0194
$\Phi_{45}'(0)$	−0.1297	−0.2636

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CRONICĂ

I. Publicaţii ale seminariilor de cercetare ale catedrelor de Matematică (serie de preprinturi)

Preprint 1—1985, Seminar of Functional Analysis and Numerical Methods.
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II. Participări la manifestări ştiinţifice

1. Probleme actuale ale cosmologiei, Timişoara, 25 mai 1985: A Pál, Problema neutrinilor masivi şi a „materiei exotice cenuşii" în cosmologie şi astronomie galactică.
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3. Seminarul de Ecuaţii diferenţiale şi control optimal, Bucureşti, septembrie, 1985: I. A. Rus, Realizări clujene în domeniul ecuaţiilor diferenţiale.
4. Conferinţa naţională de geometrie şi topologie, Călimăneşti, 26—29 septembrie 1985: P. Enghiş, Probleme asupra unor conexiuni semi-simetrice şi sfert-simetrice: P. Enghiş, M. Tarină, Asupra unor conexiuni aproape produs: A. Vasiu, Asupra unei clase de prostructuri şi structuri de incidenţă cu elemente învecinate.
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10. Şedinţele de comunicări lunare ale Facultăţii de Matematică din Bucureşti: P. Mocanu, Demonstraţia lui L. de Branges a conjecturii lui Bieberbach.
11. Şedinţele de comunicări ale Secţiei de Matematică a Academiei R.S.R.: L. Lupşa, Mulţimi convexe faţă de o mulţime dată; P. Mocanu, Operatorii integrali şi funcţii stelate.

III. Manifestări ştiinţifice organizate de catedrele de matematică

1. Şedinţele de comunicări lunare ale catedrelor de matematică.
2. Seminarul itinerant de ecuaţii funcţionale, aproximare şi convexitate, Cluj-Napoca, 23—25 mai 1985.
3. Conferinţa Naţională de Algebră, 6—8 iunie, 1985.
4. Conferinţa de ecuaţii diferenţiale, Cluj-Napoca, 21—23 noiembrie 1985.
5. Simpozionul de informatică şi aplicaţii, Cluj-Napoca, 6 decembrie, 1985.

RECENZII

R. Mneimmé, F. Testard, **Introduction à la théorie des groupes de Liee classique,** Hermann Paris, Collection Méthodès, 1986, p. 345

C'est un ouvrage distingué par l'intention des auteurs à réaliser un exposé de la théorie de groupes de Lie plus accessible que celui de la plupart des livres consacrés. La table de la matière dont nous rappellons les principaux chapitres rend compte sur cet projet. À savoir on y présente.

1. Les premières propriétés des groupes GL(n, K), (K = R ou C) 2. Groupes topologiques opérant sur un ensemble. Application à l'étude de la topologie de GL(n, K). 3. La function exponentielle. Applications. 4. Étude des groupes orthogonaux. 5. Étude des groupes unitaires; géométries réelle et symplectique associées. 6. Étude des groupes symplectiques. 7. Intégration sur les variétés. Polynomes harmoniques.

Le livre continent aussi une liste de problèmes, un index terminologique, un index des notations et une bibliographie éssentielle.

La théorie générale vient d'etre ilustrée par des exemples concretes du domaine des groupes classiques, dont certaines propriétés sont traités d'une manière originale, inédite.

Le texte est adréssé aux étudiants qui préparent la licence en topologie et géométrie diffèrentielle, mais il offre une lecture instructive et attrayante à tous ceux qui s'interessent sur le sujet.

M. ŢARINĂ

L. Lovasz, M. D. Plummer, **Matching Theory,** Akadémiai Kiado, Budapest, 1986, 544 + XXXIII pp.

This book deals with the matchings (sets of edges without common points) in graphs. In the theory of matchings a lot of applied problems can be modelled, from which the entire theory was really born.

A complete treatment of this and related subjects is divided into twelve chapters. These chapters are the followings: 1. Matchings in bipartite graphs, 2. Flow theory, 3. Size and structure of maximum matchings, 4. Bipartite graphs with perfect matchings, 5. General graphs with perfect matchings, 6. Some graph-theoretical problems related to matchings, 7. Matchings and linear programmings, 8. Determinants and matchings, 9. Matching algorithms, 10. The f-factor problem, 11. Matroid matching, 12. Vertex packing and covering, and References with an impressive number of titles. Algorithmical aspects are also considered.

This well-written book is recommanded to all, who are interested in matching problems.

Z. KÁSA

F. Gécseg — M. Steinby, Tree Automata Akadémiai Kiado, Budapest 1984, 235 pages.

The book presents a rigorous mathematical discussion of the theory of tree automata, recognizable forests and tree transformations using, primarily, the language of universal algebra. It consists of four chapters. The first one contains topics of universal algebra, lattice, theory, finite automata and formal languages. Chapters II—IV present the basic results of tree automata theory: tree recognizers, tree grammars, recognizable forests and context-free languages, tree transducers and tree transformations.

The book is a good and systematic presentation of the results of the subject presented above and it is recomended to all who are interested in this field.

M. FRENŢIU

INTREPRINDEREA POLIGRAFICĂ CLUJ,
Municipiul Cluj-Napoca, Cd. nr. 577/1986

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