Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 1, 161–170
DOI: 10.24193/subbmath.2023.1.12

Radius of starlikeness through subordination

Asha Sebastian and Vaithiyanathan Ravichandran

Abstract. A normalized function f on the open unit disc is starlike (or convex) univalent if the associated function $zf'(z)/f(z)$ (or $1+zf''(z)/f'(z)$) is a function with positive real part. The radius of starlikeness or convexity is usually obtained by using the estimates for functions with positive real part. Using subordination, we examine the radius of various starlikeness, in particular, radii of Janowski starlikeness and starlikeness of order β, for the function f when the function f is either convex or $(zf'(z) + \alpha z^2 f''(z))/f(z)$ lies in the right-half plane. Radii of starlikeness associated with lemniscate of Bernoulli and exponential functions are also considered.

Mathematics Subject Classification (2010): 30C80, 30C45.

Keywords: Univalent functions, convex functions, starlike functions, subordination, radius of starlikeness.

1. Introduction

Let $\mathcal{A}$ be the class of all functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

analytic on the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{S}$ be its subclass consisting of univalent functions. The Bieberbach conjecture (and now de Branges theorem [4]) states that the coefficients of $f \in \mathcal{S}$ satisfy the inequality $|a_n| \leq n$ for $n \geq 2$ and it led to the study of several geometrically defined classes such as the class of starlike functions, denoted by $\mathcal{S}^*$ and the class of convex functions, denoted by $\mathcal{K}$. These classes and other subclasses can be unified by subordination and convolution. The concept of subordination was introduced by Lindelöf [9]. A function f analytic in $\mathbb{D}$ is subordinate to an analytic function g in $\mathbb{D}$, written $f \prec g$, if there exists a Schwarz

The first author is supported by a Research Fellowship from NIT Tiruchirappalli. The authors are thankful to the referee for his useful comments.
Received 21 February 2020; Revised 09 March 2020.

function $w : \mathbb{D} \to \mathbb{D}$ such that $f(z) = g(w(z))$ for all $z \in \mathbb{D}$. When g is univalent in $\mathbb{D}$, the subordination $f \prec g$ holds if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. The convolution or Hadamard product of two functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n$$

in $\mathcal{A}$ is defined by

$$(f * g)(z) := \sum_{n=0}^{\infty} a_n b_n z^n.$$

Motivated by earlier works on unifying various subclasses of starlike and convex functions, Shanmugam [26] introduced and studied convolutions properties (using results of [25]) of the class

$$\mathcal{S}^*_g(\varphi) := \{f \in \mathcal{A} : z(f * g)'(z)/(f * g)(z) \prec \varphi(z)\}$$

where φ is a convex function and g is a fixed function in the class $\mathcal{A}$. When $g(z)$ is $z/(1-z)$ and $z/(1-z)^2$, the subclass $\mathcal{S}^*_g(\varphi)$ becomes the classes

$$\mathcal{S}^*(\varphi) := \{f \in \mathcal{A} : zf'(z)/f(z) \prec \varphi(z)\}$$

and

$$\mathcal{K}(\varphi) := \{f \in \mathcal{A} : 1 + zf''(z)/f'(z) \prec \varphi(z)\}$$

respectively. Ma and Minda [11] studied the distortion, growth theorems for these classes where φ is a starlike function. We are interested in few special choices of φ. When $\varphi(z) = (1 + (1-2\alpha)z)(1-z)^{-1}$, $0 \leq \alpha < 1$, the classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ are the classes of starlike and convex functions of order α introduced by Robertson [24]. The classes $\mathcal{S}^*(0) = \mathcal{S}$ and $\mathcal{K}(0) = \mathcal{K}$ are respectively the well-known classes of starlike and convex functions. For example, when $-1 \leq B < A \leq 1$, the class

$$\mathcal{S}^*[A, B] := \mathcal{S}^*\big((1 + Az)/(1 + Bz)\big)$$

is the class of Janowski starlike functions and the class

$$\mathcal{K}[A, B] := \mathcal{K}\big((1 + Az)/(1 + Bz)\big)$$

is the class of Janowski convex functions considered by several authors [5, 21, 22]. We are also interested in the class $\mathcal{S}^*_L = \mathcal{S}^*(\sqrt{1+z})$ studied by Sokół and Stankiewicz [28] and $\mathcal{S}^*_e = \mathcal{S}^*(e^z)$ studied by Mendiratta *et al.* [12]. These classes were studied in [2, 1, 3, 18, 6, 14].

Let $\alpha > 1$, $0 \leq \beta < 1$ and $\beta \geq 1/2 - 1/(2\alpha)$. Let $\varphi_p : \mathbb{D} \to \mathbb{C}$ be defined by

$$\varphi_p(z) := (1-\alpha)\frac{1 + (1-2\beta)z}{(1-z)} + \alpha\left(\frac{1 + (1-2\beta)z}{(1-z)}\right)^2 + \alpha\frac{2(1-\beta)z}{(1-z)^2}. \tag{1.1}$$

The image of the unit disk $\mathbb{D}$ under the function $\varphi_p(z) = u + \mathrm{i}\, v$ is the exterior of parabola given by

$$v^2 = -\frac{(1 - \alpha(1-2\beta))^2(2-2\beta)}{\alpha(3-2\beta)}\left(u - (\alpha\beta\,(\beta - 1/2) + \beta - \alpha/2)\right)$$

with its vertex at $(\alpha\beta(\beta-1/2)+\beta-\alpha/2,0)$. Note that it includes the right half plane. If $\beta = 1/2-1/(2\alpha)$, the region $\varphi_p(\mathbb{D})$ becomes the entire complex plane with a slit along the negative real axis from $-\big((2\alpha^2-\alpha+1)/4\alpha\big)$ to $-\infty$. Also the condition $\beta \geq 1/2-1/(2\alpha)$ restricts the range of β to $(0,1/2)$. We are mainly concerned with the class $\mathcal{S}^*_{\alpha,\beta}$ of all functions $f\in\mathcal{A}$, with $f(z)/z\neq 0$, satisfying

$$\frac{zf'(z)+\alpha z^2 f''(z)}{f(z)} \prec \varphi_p(z) \tag{1.2}$$

where the function φ_p is defined in (1.1). Singh and Gupta [27, Corollary 4.1] have shown that $\mathcal{S}^*_{\alpha,\beta}\subseteq\mathcal{S}^*_{\beta}$. This extends the results of Li and Owa [8], Padmanabhan [17] and Ravichandran *et al.* [23]. These functions were also studied in [7, 10, 15, 16, 19, 20].

For two families $\mathcal{G}$ and $\mathcal{F}$ of $\mathcal{A}$, the $\mathcal{G}$-radius of $\mathcal{F}$, denoted by $R_{\mathcal{G}}(\mathcal{F})$ is the largest number R such that $r^{-1}f(rz)\in\mathcal{G}$ for $0<r\leq R$, and for all $f\in\mathcal{F}$. Whenever $\mathcal{G}$ is characterised by a geometric property P, the number R is also referred to as the radius of property P for the class $\mathcal{F}$. If the class $\mathcal{F}$ is clear from the context, then we just write $R_{\mathcal{G}}(\mathcal{F})$ as $R_{\mathcal{G}}$. Using the theory of differential subordination developed by Miller and Mocanu [13], we determine radius constants for functions in the classes $\mathcal{S}^*_{\alpha,\beta}$ and $\mathcal{K}$ to belong to various subclass of starlike functions, in particular, to the class of Janowski starlike functions and the starlike functions of order β as well as to the classes of starlike functions associated with lemniscate of Bernoulli and the exponential functions. The results are shown to be sharp by explicitly showing the extremal function. The class $\mathcal{S}^*_{\alpha,\beta}$ for suitable α,β is a subclass of starlike functions of order β and the class of convex functions $\mathcal{K}$ is a subclass of functions starlike of order 1/2. These observations lead us to discuss radius constants of functions in the class $\mathcal{S}^*(\beta)$ in Lemma 1.2. It is then applied to find radius constants for functions in the classes $\mathcal{S}^*_{\alpha,\beta}$ and $\mathcal{K}$.

Various radii constants for the class $\mathcal{S}^*_{\alpha,\beta}$ are given in the following:

Theorem 1.1. *The following sharp radius results hold for the class $\mathcal{S}^*_{\alpha,\beta}$:*

(i) For $-1\leq B<A\leq 1$, the $\mathcal{S}^*[A,B]$ radius

$$R_{\mathcal{S}^*[A,B]}=\min\{1,\ (A-B)/\left(|A+B-2\beta B|+2(1-\beta)\right)\}.$$

(ii) For $0\leq\gamma<1,\ \gamma>\beta$, the $\mathcal{S}^*(\gamma)$ radius $R_{\mathcal{S}^*(\gamma)}=(1-\gamma)/(1+\gamma-2\beta)$.

(iii) The $\mathcal{S}_L$ radius $R_{\mathcal{S}_L}=(\sqrt{2}-1)/\left(\sqrt{2}+1-2\beta\right)$.

(iv) The $\mathcal{S}^*_e$ radius $R_{\mathcal{S}^*_e}=(e-1)/\left(e+1-2\beta\right)$.

The idea of the proof is to use inclusion results for the class $\mathcal{S}^*_{\alpha,\beta}$ with the class of starlike functions of order β. Singh and Gupta [27, Corollary 4.1] have shown that $\mathcal{S}^*_{\alpha,\beta}\subseteq\mathcal{S}^*(\beta)$. In order to use this inclusion, we first find the various radii for the class of starlike functions of order β in the following:

Lemma 1.2. *The following sharp radius results hold for the class $\mathcal{S}^*(\beta)$:*

(i) For $-1\leq B<A\leq 1$, the $\mathcal{S}^*[A,B]$ radius

$$R_{\mathcal{S}^*[A,B]}=\min\{1,\ (A-B)/\left(|A+B-2\beta B|+2(1-\beta)\right)\}.$$

(ii) For $0\leq\gamma<1,\ \gamma>\beta$, the $\mathcal{S}^*(\gamma)$ radius $R_{\mathcal{S}^*(\gamma)}=(1-\gamma)/(1+\gamma-2\beta)$.

(iii) *The* $\mathcal{S}_L$ *radius* $R_{\mathcal{S}_L} = (\sqrt{2}-1)/\left(\sqrt{2}+1-2\beta\right)$.
(iv) *The* $\mathcal{S}_e^*$ *radius* $R_{\mathcal{S}_e^*} = (e-1)/(e+1-2\beta)$.

Theorem 1.1 follows from this lemma except for the sharpness. To find the extremal function $\tilde{f}$ for the class $\mathcal{S}^*_{\alpha,\beta}$, write $\tilde{f}$ as

$$\tilde{f}(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

and determine the coefficients a_n from

$$\frac{z\tilde{f}'(z)}{\tilde{f}(z)}\left(1+\alpha\frac{z\tilde{f}''(z)}{f_1'(z)}\right) = \varphi_p(z) \tag{1.3}$$

where φ_p is given by (1.1). Writing

$$C = 2(2\alpha-\beta) - 4\alpha\beta,\ D = 2(\alpha+\beta) + 2\alpha\beta(2\beta-3) - 1,$$

the equation (1.3) readily gives

$$\begin{aligned} a_2 &= \frac{C+2}{1+2\alpha} = 2(1-\beta) \\ a_n &= \frac{\big(C+2(n-1)+2\alpha(n-1)(n-2)\big)}{(1+n\alpha)(n-1)}a_{n-1} \\ &\quad + \frac{\big(D-(n-2)-\alpha(n-2)(n-3)\big)}{(1+n\alpha)(n-1)}a_{n-2}. \end{aligned}$$

Calculating the coefficients a_n from the above recurrence relation, we see that the extremal function $\tilde{f}$ is the generalised Koebe's function given by

$$\tilde{f}(z) = \frac{z}{(1-z)^{2-2\beta}}. \tag{1.4}$$

Interestingly, it is the extremal of the class $\mathcal{S}^*(\beta)$ and hence the sharpness of our theorem follows trivially.

It is also well-known that a convex function is starlike of order $1/2$ and so the class $\mathcal{K}$ of convex function is contained in the class $\mathcal{S}^*(1/2)$ of starlike functions of order $1/2$. This inclusion and Lemma 1.2 together readily yields the following radii results for the class of convex functions:

Corollary 1.3. *The following sharp radius results hold for the class* $\mathcal{K}$*:*

(i) *For* $-1 \le B < A \le 1$*, the* $\mathcal{S}^*[A,B]$ *radius*
$$R_{\mathcal{S}^*[A,B]} = \min\{1,\ (A-B)/(1+|A|)\}.$$
(ii) *For* $0 \le \gamma < 1,\ \gamma > 1/2$*, the* $\mathcal{S}^*(\gamma)$ *radius* $R_{\mathcal{S}^*(\gamma)} = (1-\gamma)/\gamma$.
(iii) *The* $\mathcal{S}_L$ *radius* $R_{\mathcal{S}_L} = 1-1/\sqrt{2} \approx 0.2929$.
(iv) *The* $\mathcal{S}_e^*$ *radius* $R_{\mathcal{S}_e^*} = 1-1/e \approx 0.6321$.

The method of convolution can also be applied to find radius problems of various classes. Corollary 1.3 (ii) requires the largest number ρ such that the function $l_\rho : \mathbb{D} \to \mathbb{C}$ is a starlike of order $\gamma \ge 1/2$, where $f_\rho(z) = f(z) * l_\rho(z)$. Here $l(z) = z/(1-z)$ is the convolution identity and the functions $f_\rho,\ l_\rho : \mathbb{D} \to \mathbb{C}$ are defined respectively by $f_\rho(z) = f(\rho z)/\rho$ and $l_\rho(z) = z/(1-\rho z)$. This is equivalent to find the number ρ such that $\mathrm{Re}(\rho z/(1-\rho z)) > \gamma - 1$. It follows by simple computation that $\rho = (1-\gamma)/\gamma$, since the real part of the function $(\rho z/(1-\rho z))$ attains minimum at $z = -1$.

2. Proof of Lemma 1.2

Let the function $f \in \mathcal{S}^*(\beta)$. Then, it follows that

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{(1-z)}.$$

Define the function $f_\rho : \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ by $f_\rho(z) := f(\rho z)/\rho$. For this function, we immediately get

$$\frac{zf_\rho'(z)}{f_\rho(z)} \prec \frac{1+(1-2\beta)\rho z}{(1-\rho z)}. \tag{2.1}$$

(i) Let $-1 \le A < B \le 1$ and the functions $p,\ q : \mathbb{D} \longrightarrow \mathbb{C}$ be defined by,

$$p(z) = \frac{1+(1-2\beta)z}{(1-z)} \quad \text{and} \quad q(z) = \frac{1+Az}{1+Bz}. \tag{2.2}$$

From (2.2), it follows that $p^{-1}(w) = (w-1)/(w+1-2\beta)$ and hence

$$p^{-1} \circ q(z) = \frac{q(z)-1}{q(z)+1-2\beta} = \frac{(A-B)z}{(A+B-2\beta B)z+2(1-\beta)}. \tag{2.3}$$

The values taken by $p^{-1} \circ q(z)$ in (2.3) leads us in finding ρ through two different cases.

Case 1. If $(A-B)/(|A+B-2\beta B|+2(1-\beta)) \ge 1$, then, by (2.3), we have

$$|p^{-1} \circ q(z)| \ge \frac{A-B}{|A+B-2\beta B|+2(1-\beta)} \ge 1 \quad (z \in \partial\mathbb{D}).$$

This shows that $z \prec p^{-1}(q(z))$ and hence $p(z) \prec q(z)$. This shows that $\rho = 1$.

Case 2. If $(A-B)/(|A+B-2\beta B|+2(1-\beta)) \le 1$, then it follows from (2.2) that

$$\begin{aligned} R_{\mathcal{S}^*[A,B]} &= \min_{|z|=1} \left|p^{-1} \circ q(z)\right| \\ &= \min_{|z|=1} \left|\frac{(A-B)z}{(A+B-2\beta B)z+2-2\beta}\right| \\ &= \frac{A-B}{|A+B-2\beta B|+2(1-\beta)}. \end{aligned} \tag{2.4}$$

Thus, for $0 < \rho \le R_{\mathcal{S}^*[A,B]}$, we have $p(\rho z) \prec q(z)$. By (2.1), it follows that $zf_\rho'(z)/f_\rho(z) \prec p(\rho z) \prec q(z)$ or $f_\rho \in \mathcal{S}^*[A,B]$. Thus, the $\mathcal{S}^*[A,B]$ radius of the class $S^*(\beta)$ is at least $R_{\mathcal{S}^*[A,B]}$.

To show the sharpness, consider the function $\tilde{f} : \mathbb{D} \to \mathbb{C}$ defined by

$$\tilde{f}(z) = \frac{z}{(1-z)^{2-2\beta}}. \tag{2.5}$$

At the point $z = R_{\mathcal{S}^*[A,B]}$, the function $\tilde{f}$ satisfies

$$\left|\frac{(z\tilde{f}'(z)/\tilde{f}(z) - 1}{A - B(z\tilde{f}'(z)/\tilde{f}(z)}\right| = 1$$

and hence the result is sharp.

(ii) Let $0 \leq \gamma < 1$. We consider two cases depending on the values γ, namely, $\gamma \leq \beta$ and $\gamma \geq \beta$. Since $\mathcal{S}^*(\gamma) = \mathcal{S}^*[1-2\gamma, -1]$, substituting $A = 1-2\gamma$ and $B = -1$ in (2.4), we obtain $\rho = 1$ when $\gamma \leq \beta$ and the required $\rho = R_{\mathcal{S}^*(\gamma)}$ when $\gamma \geq \beta$.

(iii) For $0 < a < \sqrt{2}$, by [2, Lemma 2.2], we have

$$\{w \in \mathbb{C} : |w - a| < \sqrt{2} - a\} \subseteq \{w \in \mathbb{C} : \left|w^2 - 1\right| < 1\}. \tag{2.6}$$

Let the functions $p, q : \mathbb{D} \to \mathbb{C}$ be defined by

$$p(z) := \frac{1 + 2(1-\beta)z}{1-z} \quad \text{and} \quad q(z) := \sqrt{1+z}. \tag{2.7}$$

It is evident from (2.7) that

$$p^{-1}(q(z)) = \frac{\sqrt{1+z} - 1}{\sqrt{1+z} + 1 - 2\beta} = \left(1 + \frac{2(1-\beta)}{\sqrt{1+z} - 1}\right)^{-1}. \tag{2.8}$$

By (2.6), we have $\left|\sqrt{1+z} - 1\right| \geq \sqrt{2} - 1$ and so

$$1 + \frac{2(1-\beta)}{\left|\sqrt{1+z} - 1\right|} \leq 1 + \frac{2(1-\beta)}{\sqrt{2} - 1}. \tag{2.9}$$

Substituting (2.9) in (2.8), it follows that

$$\rho = \min_{|z|=1} \left|p^{-1} \circ q(z)\right| = \min_{|z|=1} \left|\left(1 + \frac{2(1-\beta)}{\sqrt{1+z} - 1}\right)^{-1}\right|. \tag{2.10}$$

This is equivalent to

$$\rho = \left(\max_{|z|=1} \left|1 + \frac{2(1-\beta)}{\sqrt{1+z} - 1}\right|\right)^{-1} = \left(1 + \frac{2(1-\beta)}{\sqrt{2} - 1}\right)^{-1}. \tag{2.11}$$

Therefore, we have

$$\frac{1 + 2(1-\beta)\rho z}{1 - \rho z} \prec \sqrt{1+z}.$$

By (2.1), this proves that the function $f_\rho \in \mathcal{S}_L$.
At the point $z = R_{\mathcal{S}_L}$, the function $\tilde{f}$ defined in (2.5) satisfies

$$\left|\left(\frac{z\tilde{f}'(z)}{\tilde{f}(z)}\right)^2 - 1\right| = \left|\left(1 + \frac{2(1-\beta)z}{1-z}\right)^2 - 1\right| = 1.$$

(iv) Let the functions p and q be defined as

$$p(z) := \frac{1 + (1-2\beta)z}{1-z} \quad \text{and} \quad q(z) := e^z, \quad z \in \mathbb{D}. \tag{2.12}$$

It is apparent from (2.12) that

$$p^{-1}(q(z)) = \frac{e^z - 1}{e^z + 1 - 2\beta} = \left(1 + \frac{2(1-\beta)}{e^z - 1}\right)^{-1}. \tag{2.13}$$

Let $\lambda = 2(1-\beta),\ 0 \le \beta \le 1$. On the boundary of the unit disc $\mathbb{D}$, we have

$$\begin{aligned} \left|1 + \frac{\lambda}{e^z - 1}\right|^2 &= \left|1 + \frac{\lambda}{e^{\cos\theta}\cos(\sin\theta) - 1 + \mathrm{i}\, e^{\cos\theta}\sin(\sin\theta)}\right|^2 \\ &= \frac{e^{2\cos\theta} + 2(\lambda-1)e^{\cos\theta}\cos(\sin\theta) + (\lambda-1)^2}{e^{2\cos\theta} - 2e^{\cos\theta}\cos(\sin\theta) + 1}. \end{aligned} \tag{2.14}$$

Substituting $\cos\theta = x$ in (2.14), we get

$$\begin{aligned} \left|1 + \frac{\lambda}{e^z - 1}\right|^2 &= \frac{e^{2x} + 2(\lambda-1)e^x\cos(\sqrt{1-x^2}) + (\lambda-1)^2}{e^{2x} - 2e^x\cos(\sqrt{1-x^2}) + 1} \\ &= \frac{g(x,\lambda)}{g(x,0)} \end{aligned} \tag{2.15}$$

where

$$g(x,\lambda) := e^{2x} - 2e^x\cos(\sqrt{1-x^2}) + 1. \tag{2.16}$$

Let $-1 \le x \le 1,\ 0 \le \lambda \le 2$ and the function S be defined by

$$S(x) := g(x,\lambda)g(1,0) - g(x,0)g(1,\lambda).$$

Using (2.16) in $S(x)$, it can be seen that

$$\begin{aligned} S(x) = {} & 2x(e^2 + \lambda - 1)e^x\cos(\sqrt{1-x^2}) - (2e + \lambda - 2)e^{2x} \\ & - e(2(\lambda-1) - e(\lambda-2)). \end{aligned} \tag{2.17}$$

Define the function s by

$$s(x) := 2x(e^2 + \lambda - 1)e^x\cos(\sqrt{1-x^2}) - (2e + \lambda - 2)e^{2x}.$$

The function $s'(x)$ is an increasing function. Therefore it has at most one zero, say η. Also $s''(x) > 0$, this shows that η is a local minima. Thus, the maximum of s occurs at $x = \pm 1$. At $x = -1$,

$$s(-1) = -2(e - e^{-2}) - \lambda e^{-1}(e^{-1} + 2) \le 0.$$

These observations together with (2.17) lead us to the fact that $S(x) \le 0$, or equivalently, the function h defined by $h(x) := g(x,\lambda)/g(x,0)$ satisfies $h(x) \le h(1)$. Therefore, the maximum of $h(x)$ occurs at $x = 1$, and, by (2.15),

$$\left|1 + \frac{2(1-\beta)}{e^z - 1}\right| \le \left|1 + \frac{2(1-\beta)}{e - 1}\right|. \tag{2.18}$$

From the definition of ρ, it follows from (2.13) that

$$\rho = \min_{|z|=1} \left|p^{-1} \circ q(z)\right| = \min_{|z|=1} \left|\left(1 + \frac{2(1-\beta)}{e^z - 1}\right)^{-1}\right|.$$

From (2.18), it is clear that

$$\rho = \left(\max_{|z|=1} \left| 1 + \frac{2(1-\beta)}{e^z - 1} \right| \right)^{-1} = \left(1 + \frac{2(1-\beta)}{e-1} \right)^{-1}. \tag{2.19}$$

This proves that

$$\frac{1 + (1-2\beta)\rho z}{1 - \rho z} \prec e^z$$

and so the function $f_\rho \in \mathcal{S}^*_e$.

At the point $z = R_{\mathcal{S}^*_e}$, the function $\tilde{f}$ defined in (2.5) satisfies

$$\left| \log \frac{z\tilde{f}'(z)}{\tilde{f}(z)} \right| = \left| \log \left(1 + \frac{2(1-\beta)z}{1-z} \right) \right| = 1.$$

This completes the proof of the lemma.

Let $-1 \leq B < A \leq 1$ and $-1 \leq D < C \leq 1$. In 1997, Gangadharan and Ravichandran [5] discussed the $\mathcal{S}^*[A,B]$ radius of the class $\mathcal{S}^*[C,D]$ and shown that

$$R_{\mathcal{S}^*[A,B]}(\mathcal{S}^*[C,D]) = \min\left\{1, (A-B)/(C-D+|AD-BC|)\right\}.$$

Lemma 1.2*(i)* is indeed a particular case when $C = 1-2\delta$ and $D = -1$. The radius determined in Corollary [5, pp.305] is exactly the same as Lemma 1.2*(ii)*. Theorem [2, pp.6562] determined the $\mathcal{S}_L$ radius of $\mathcal{S}^*[A,B]$ when $B \leq 0$. When $A = 1-2\delta, B = -1$, their result gives

$$R_{\mathcal{S}_L} = \min\left\{1, (\sqrt{2}-1)/(1-\delta+\sqrt{(1-\delta)^2+(\sqrt{2}-1)(\sqrt{2}+1-2\delta)}\ )\right\}$$

and it is same as the radius in Lemma 1.2*(iii)*. Mendiratta *et al.* [12] discussed subordination theorems and radii constants for the functions in the class $\mathcal{S}^*(e^z)$. They determined the $\mathcal{S}^*_e$ radius of $f \in \mathcal{S}^*[A,B]$. By substituting $A = 1-2\delta, B = -1$ in Theorem [12, pp.381], the radius obtained is our Lemma 1.2*(iv)*.

References

[1] Ali, R.M., Cho, N.E., Jain, N.K., Ravichandran, V., *Radii of starlikeness and convexity for functions with fixed second coefficient defined by subordination*, Filomat, **26**(2012), no. 3, 553–561.

[2] Ali, R.M., Jain, N.K., Ravichandran, V., *Radii of starlikeness associated with the lemniscate of Bernoulli and the left-half plane*, Appl. Math. Comput., **218**(2012), no. 11, 6557–6565.

[3] Aouf, M.K., Dziok J., Sokół, J., *On a subclass of strongly starlike functions*, Appl. Math. Lett., **24**(2011), no. 1, 27–32.

[4] de Branges, L., *A proof of the Bieberbach conjecture*, Acta Math., **154**(1985), 137–152.

[5] Gangadharan, A., Ravichandran, V., Shanmugam, T.N., *Radii of convexity and strong starlikeness for some classes of analytic functions*, J. Math. Anal. Appl., **211**(1997), no. 1, 301–313.

[6] Khatter, K., Ravichandran, V., Sivaprasad Kumar, S., *Starlike functions associated with exponential function and the lemniscate of Bernoulli*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **113**(2019), no. 1, 233–253.

[7] Lewandowski, Z.A., Miller, S., Złotkiewicz, E., *Generating functions for some classes of univalent functions*, Proc. Amer. Math. Soc., **56**(1976), no. 1, 111–117.

[8] Li, J.-L., Owa, S., *Sufficient conditions for starlikeness*, Indian J. Pure Appl. Math., **33**(2002), no. 3, 313–318.

[9] Lindelöf, E., *Mémoire sur certaines inégalités dans la théorie des fonctions monogènes et sur quelques propriétés nouvelles de ces fonctions dans le voisinage d'un point singulier essentiel*, Acta Soc. Sci. Fenn., **35**(1908), no. 7.

[10] Liu, M.-S., Zhu, Y.-C., Srivastava, H.M., *Properties and characteristics of certain subclasses of starlike functions of order β*, Math. Comput. Modelling, **48**(2008), no. 3-4, 402–419.

[11] Ma, W.C., Minda, D., *A unified treatment of some special classes of univalent functions*, Proceedings of the Conference on Complex Analysis (Tianjin), Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA., (1992), 157–169.

[12] Mendiratta, R., Nagpal, S., Ravichandran, V., *On a subclass of strongly starlike functions associated with exponential function*, Bull. Malays. Math. Sci. Soc. , **38**(2015), no. 1, 365–386.

[13] Miller, S.S., Mocanu, P.T., *Differential subordinations and univalent functions*, Michigan Math. J., **28**(1981), no. 2, 157–172.

[14] Naz, A., Nagpal, S., Ravichandran, V., *Starlikeness associated with the exponential function*, Turkish J. Math., **43**(2019), no. 3, 1353–1371.

[15] Nunokawa, M., *On a sufficient condition for multivalently starlikeness*, Tsukuba J. Math., **18**(1994), no. 1, 131–134.

[16] Obradović, M., Joshi, S.B., *On certain classes of strongly starlike functions*, Taiwanese J. Math., **2**(1998), no. 3, 297–302.

[17] Padmanabhan, K.S., *On sufficient conditions for starlikeness*, Indian J. Pure Appl. Math., **32**(2001), no. 4, 543–550.

[18] Paprocki, E., Sokół, J., *The extremal problems in some subclass of strongly starlike functions*, Zeszyty Nauk. Politech. Rzeszowskiej Mat., (1996), no. 20, 89–94.

[19] Ramesha, C., Kumar, S., Padmanabhan, K.S., *A sufficient condition for starlikeness*, Chinese J. Math., **23**(1995), no. 2, 167–171.

[20] Ravichandran, V., *Certain applications of first order differential subordination*, Far East J. Math. Sci. (FJMS), **12**(2004), no. 1, 41–51.

[21] Ravichandran, V., Hussain Khan, M., Silverman, H., *Radius problems for a class of analytic functions*, Demonstratio Math., **39**(2006), no. 1, 67–74.

[22] Ravichandran, V., Rønning, F., Shanmugam, T.N., *Radius of convexity and radius of starlikeness for some classes of analytic functions*, Complex Variables Theory Appl., **33**(1997), no. 1-4, 265–280.

[23] Ravichandran, V., Selvaraj, C., Rajalaksmi, R., *Sufficient conditions for starlike functions of order α*, JIPAM. J. Inequal. Pure Appl. Math., **3**(2002), no. 5, Article 81, 6.

[24] Robertson, M.I.S., *On the theory of univalent functions*, Ann. of Math., **37**(1936), no. 2, 374–408.

[25] Ruscheweyh, S., Sheil-Small, T., *Hadamard products of Schlicht functions and the Pólya-Schoenberg conjecture*, Comment. Math. Helv., **48**(1973), 119–135.

[26] Shanmugam, T.N., *Convolution and differential subordination*, Internat. J. Math. Math. Sci., **12**(1989), no. 2, 333–340.

[27] Singh, S., Gupta, S., *A differential subordination and starlikeness of analytic functions*, Appl. Math. Lett., **19**(2006), no. 7, 618–627.

[28] Sokół, J., Stankiewicz, J., *Radius of convexity of some subclasses of strongly starlike functions*, Zeszyty Nauk. Politech. Rzeszowskiej Mat., (1996), no. 19, 101–105.

Asha Sebastian
Department of Mathematics,
National Institute of Technology,
Tiruchirappalli-620015, India
e-mail: `ashasebastian13@gmail.com`

Vaithiyanathan Ravichandran
Department of Mathematics,
National Institute of Technology,
Tiruchirappalli-620015, India
e-mail: `vravi68@gmail.com; ravic@nitt.edu`