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A generalized Ekeland's variational principle for vector equilibria

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Abstract. In this paper, we establish an Ekeland-type variational principle for vector valued bifunctions defined on complete metric spaces with values in locally convex spaces ordered by closed convex cones. The main improvement consists in widening the class of bifunctions for which the variational principle holds. In order to prove this principle, a weak notion of continuity for vector valued functions is considered, and some of its properties are presented. We also furnish an existence result for vector equilibria in absence of convexity assumptions, passing through the existence of approximate solutions of an optimization problem.

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1. Introduction

Ekeland's variational principle (see [11]) has many applications in nonlinear analysis and optimization, see [1, 4, 2, 3, 5, 6], [7], [14], [19], [10] and the reference therein. Blum, Oettli [8] and Théra [18] showed that their existence result for a solution of an equilibrium problem is equivalent to Ekeland-type variational principle for bifunctions. Several authors have extended the Ekeland's variational principle to the case with a vector valued bifunction taking values in an ordered vector space, see [7], [2], [6], [15]. Araya et. al. [6] established a version of Ekeland's variational principle for vector valued bifunctions, which is expressed by the existence of a strict approximate minimizer for a weak vector equilibrium problem.

By a weak vector equilibrium problem we understand the problem of finding $\overline{x} \in X$ such that

$$f(\overline{x}, y) \notin -intK, \quad \text{for all } y \in X,$$

where $f : X \times X \to Y$ is a given bifunction, (X, d) is a complete metric space and (Y, K) is a Hausdorff topological vector space, ordered by the closed convex cone K.

Recall that $K \subseteq Y$ is said to be closed and convex cone if K is closed, $\alpha K \subseteq K$ for all $\alpha > 0$ and $K + K \subseteq K$.
The approach given in Araya et. al. [6] is based on the assumption that the equilibrium bifunction f satisfies the following triangle property:

$$f(x, y) + f(y, z) \in f(x, z) + K, \quad \text{for all } x, y, z \in X. \tag{1.1}$$

We stress the fact that (1.1) is a rather strong condition and it is rarely verified when the equilibrium problem is a variational inequality, see [10].
Motivated and inspired by [10], in this paper we shall give an improvement of Theorem 2.1 in Araya et. al. [6]. We widen the class of the vector bifunctions for which the Ekeland's variational principle is applicable. Further, some sufficient conditions for existence of equilibria which do not involve any convexity concept, neither for the domain nor for the bifunction are given, under a relaxed continuity concept for the vector functions.
The rest of the paper is organized as follows. In Section 2 we collect some definitions and results needed for further investigations. A weak notion of continuity for the vector valued functions is also studied and some of its properties are presented. Sections 3 and 4 are devoted to Ekeland's principles for the vector valued functions and bifunctions. Section 5 is devoted to an existence result for the weak vector equilibria where the vector bifunctions satisfy a property which generalizes the triangle inequality.

2. Preliminaries

Throughout this paper, unless otherwise specified, we assume that (X, d) is a complete metric space, (Y, K) is a locally convex Hausdorff topological vector space ordered by the nontrivial closed convex cone $K \subseteq Y$ with $intK \neq \emptyset$, where $intK$ denotes the topological interior of K, as follows:

$$x \leq_K y \Leftrightarrow y - x \in K.$$

We agree that any cone contains the origin, according to the following definition.

Definition 2.1. The set $K \subseteq Y$ is called a cone iff $\lambda x \in K$ for all $x \in K$ and $\lambda \geq 0$. The cone K is pointed iff $K \cap (-K) = \{0\}$; proper iff $K \neq Y$ and $K \neq \{0\}$.

Let $k_0 \in K \setminus (-K)$. The nonlinear scalarization function [20] (see also [16]) $z_{K,k_0} : Y \to [-\infty, \infty]$ is defined as

$$z_{K,k_0}(y) = \inf\{r \in \mathbb{R} \mid y \in rk_0 - K\}.$$

We present some properties of the scalarization function which will be used in the sequel.

Lemma 2.2. [16] *For each $r \in \mathbb{R}$ and $y \in Y$, the following statements are true:*

(i) z_{K,k_0} *is proper;*
(ii) z_{K,k_0} *is lower semicontinuous;*
(iii) z_{K,k_0} *is sublinear;*
(iv) z_{K,k_0} *is K monotone;*
(v) $z_{K,k_0}(y) \leq r \Leftrightarrow y \in rk_0 - K$*;*

(vi) $z_{K,k_0}(y) > r \Leftrightarrow y \notin rk_0 - K$;
(vii) $z_{K,k_0}(y) \geq r \Leftrightarrow y \notin rk_0 - intK$;
(viii) $z_{K,k_0}(y) < r \Leftrightarrow y \in rk_0 - intK$;
(ix) $z_{K,k_0}(y + \lambda k_0) = z_{K,k_0}(y) + \lambda$, *for every* $y \in Y$ *and* $\lambda \in \mathbb{R}$.

As a corollary of the lemma above, Göpfert et al. [13] presented the following nonconvex separation theorem, see also [16].

Lemma 2.3. [13] *Assume that* Y *is a topological vector space,* K *a closed solid convex and* $A \subset Y$ *a nonempty set such that* $A \cap (-intK) = \emptyset$. *Then* z_{K,k_0} *is a finite valued continuous function such that*

$$z_{K,k_0}(-y) < 0 \leq z_{K,k_0}(x) \quad \text{for all } x \in A \quad \text{and } y \in intK,$$

moreover $z_{K,k_0}(x) > 0$ *for all* $x \in intA$.

In the vector valued case there are several possible extensions of the scalar notion of *lower semicontinuity*, see [9]. We recall here the concept of (k_0, K)-lower semicontinuity introduced by Chr. Tammer [19] which will be used in the sequel. This concept is weaker than the K-*lower semicontinuity* which was introduced by Borwein et. al. [9] (see also [12], [17] and [21].)

Definition 2.4. [19] A function $\varphi : X \longrightarrow Y$ is said to be:
(i) (k_0, K)-*lower semicontinuous* if for all $r \in \mathbb{R}$, the set $\{x \in X : \varphi(x) \in rk_0 - K\}$ is closed;
(ii) (k_0, K)-*upper semicontinuous* if for all $r \in \mathbb{R}$, the set $\{x \in X : \varphi(x) \in rk_0 + K\}$ is closed;
(iii) (k_0, K)-*continuous* if it is both (k_0, K)-lower semicontinuous as well as (k_0, K)-upper semicontinuous.

The function $\varphi : X \longrightarrow Y$ is said to be K-*bounded* below if there exists $\overline{y} \in Y$ such that $\varphi(X) \subseteq \overline{y} + K$.
In [19], the following assertion was proved.

Lemma 2.5. [19]
(i) *If* φ *is* (k_0, K)-*lower semicontinuous, then* $z_{K,k_0} \circ \varphi$ *is lower semicontinuous;*
(ii) *If* φ *is* (k_0, K)-*upper semicontinuous, then* $z_{K,k_0} \circ \varphi$ *is upper semicontinuous.*

Remark 2.6. It is well known that the sum of two K-lower semicontinuous mappings is not a K-lower semicontinuous mapping in general, see [7]. Due to the following example, we can obtain a similar conclusion for the (k_0, K)-lower semicontinuity, i.e., if $\varphi : X \longrightarrow Y$ is (k_0, K)-lower semicontinuous, the function $\varphi(\cdot) - \varphi(x)$, where $x \in X$ is fixed, is not necessary (k_0, K)-lower semicontinuous.

Example 2.7. Let us consider $X = \mathbb{R}^2$, $Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$. Define $\varphi : X \to Y$ as:

$$\varphi(x) = \begin{cases} (1, -2), & x_1 > 0,\ x_2 \in \mathbb{R}, \\ (x_1, x_1), & x_1 \leq 0,\ x_2 \in \mathbb{R}, \end{cases}$$

where $x = (x_1, x_2)$.
This function is (k_0, K)-lower semicontinuous with $k_0 = (1, 1)$. Now take $x = (1, 0)$.

We will prove that the function $\varphi(\cdot)-\varphi(x)$ is not (k_0,K)-lower semicontinuous. Take also $r=1$ and consider the set

$$L=\{y\in X:\varphi(y)-\varphi(x)\in(1,1)-K\}.$$

It is easy to observe that $y_n=(\frac{1}{n},\frac{2}{n})\in L$, $n\in\mathbb{N}$, and $y_n\to y_0$, where $y_0=(0,0)$. On the other hand,

$$\varphi(y_0)-\varphi(x)=(0,0)-(1,-2)=(-1,2)\notin(1,1)-K.$$

Hence $y_0\notin L$, which shows that the set L is not closed, i.e., the conclusion.

In what follows, we will furnish some properties for this kind of continuity for the vector functions.

Proposition 2.8. *If $\varphi:X\longrightarrow Y$ is (k_0,K)-lower semicontinuous, then the function $-\varphi$ is (k_0,K)-upper semicontinuous.*

Theorem 2.9. *If $\varphi:X\longrightarrow Y$ is (k_0,K)-lower semicontinuous and*

$$\varphi(X)\subset\bigcup_{t\in\mathbb{R}}\{tk_0\},$$

then the function $\varphi(\cdot)-\varphi(x)$, where $x\in X$ is fixed, is (k_0,K)-lower semicontinuous.

Proof. Let us fix $x_0\in X$ and consider the function $\delta:X\to Y$ defined by

$$\delta(y)=\varphi(y)-\varphi(x_0),\ y\in X.$$

Fix also $r\in\mathbb{R}$ and consider the set $S=\{y\in X:\varphi(y)-\varphi(x_0)\in rk_0-K\}$.
We will prove that this set is closed.
Since $\varphi(X)\subset\bigcup_{t\in\mathbb{R}}\{tk_0\}$, it follows that, for $x_0\in X$, there exists $t_0\in\mathbb{R}$ such that $\varphi(x_0)=t_0k_0$. We obtain

$$S=\{y\in X:\varphi(y)\in(r+t_0)k_0-K\}.$$

Since $r,t_0\in\mathbb{R}$ are fixed and φ is (k_0,K)-lower semicontinuous, it follows the set S is closed, i.e., the conclusion. ☐

Corollary 2.10. *If $\varphi:X\longrightarrow Y$ is (k_0,K)-lower semicontinuous and*

$$\varphi(X)\subset\bigcup_{t\in\mathbb{R}}\{tk_0\},$$

then the function $\varphi(x)-\varphi(\cdot)$, where $x\in X$ is fixed, is (k_0,K)-upper semicontinuous.

3. Ekeland's variational principle for the vector functions

This section deals with an Ekeland's variational principle for the vector valued functions. Inspired by the results obtained in Theorem 3.1 Araya [5], we are able to present our result when the vector function is (k_0,K)-lower semicontinuous.

Theorem 3.1. *If $\varphi:X\to Y$ is (k_0,K)-lower semicontinuous is such that*

(i) *for each $x\in X$, there exists $\overline{y}\in Y$ such that $(\varphi(X)-\varphi(x))\cap(\overline{y}-intK)=\emptyset$;*

(ii) $\varphi(X) \subset \bigcup_{t\in\mathbb{R}} \{tk_0\}$,

then, for every given $\varepsilon > 0$ *and for every* $\widehat{x} \in X$ *there exists* $\overline{x} \in X$ *such that:*

(a) $\varphi(\overline{x}) - \varphi(\widehat{x}) + \varepsilon d(\overline{x}, \widehat{x})k_0 \in -K$*;*

(b) $\varphi(x) - \varphi(\overline{x}) + \varepsilon d(\overline{x}, x)k_0 \notin -K$*, for every* $x \in X, x \neq \overline{x}$*.*

Proof. Let us consider the functional

$$z_{K,k_0} : Y \to [-\infty, \infty],$$

defined by

$$z_{K,k_0}(y) = \inf\{r \in \mathbb{R} \mid y \in rk_0 - K\}.$$

For each $x \in X, \varepsilon > 0$ consider the set

$$S(x) = \{y \in X \mid \ y = x \text{ or } z_{K,k_0}(\varphi(y) - \varphi(x)) + \varepsilon d(x, y) \leq 0\}$$

It is obviously that $x \in S(x)$, therefore $S(x) \neq \emptyset$ for all $x \in X$. By Theorem 2.9, since φ is a (k_0, K)-lower semicontinuous function, then the function $\delta(\cdot) = \varphi(\cdot) - \varphi(x)$, where $x \in X$ is fixed, is also (k_0, K)-lower semicontinuous. From Lemma 2.5 it follows that $z_{K,k_0} \circ \delta$ is lower semicontinuous and $d(x, y)$ is continuous, therefore $S(x)$ is closed for every $x \in X$.

Now we show that $z_{K,k_0}(\varphi(X) - \varphi(x)) := \cup_{y\in X}\{z_{K,k_0}(\varphi(y) - \varphi(x)\})$ is bounded from below for all $x \in X$. By assumption (i) and Lemma 2.3 we have that

$$0 \leq z_{K,k_0}(\varphi(y) - \varphi(x) - \overline{y}), \text{ for all } y \in X.$$

Using (iii) of Lemma 2.2, we get

$$-\infty < -z_{K,k_0}(-\overline{y}) < z_{K,k_0}(\varphi(y) - \varphi(x)) \text{ for any } y \in X,$$

which implies that $z_{K,k_0}(\varphi(X) - \varphi(x))$ is bounded from below.

Let define the real valued function

$$v(x) = \inf_{y\in S(x)} z_{K,k_0}(\varphi(y) - \varphi(x)). \tag{3.1}$$

and set $x = \widehat{x} \in X$. Since $z \circ \delta$ is bounded below, we have

$$v(\widehat{x}) = \inf_{y\in S(\widehat{x})} z_{K,k_0}(\varphi(y) - \varphi(\widehat{x})) > -\infty.$$

Starting from $\widehat{x} \in X$, a sequence x_n of points of X can be defined such that $x_{n+1} \in S(x_n)$ such that

$$z_{K,k_0}(\varphi(x_{n+1}) - \varphi(x_n)) \leq v(x_n) + \frac{1}{n+1}.$$

Let us take $y \in S(x_{n+1}) \setminus \{x_{n+1}\}$. It follows that

$$z_{K,k_0}(\varphi(y) - \varphi(x_{n+1})) + \varepsilon d(x_{n+1}, y) \leq 0. \tag{3.2}$$

Since $x_{n+1} \in S(x_n)$, we also have

$$z_{K,k_0}(\varphi(x_{n+1}) - \varphi(x_n)) + \varepsilon d(x_{n+1}, x_n) \leq 0. \tag{3.3}$$

Adding (3.2) and (3.3) we obtain

$$z_{K,k_0}(\varphi(x_{n+1}) - \varphi(x_n)) + z_{K,k_0}(\varphi(y) - \varphi(x_{n+1})) + \varepsilon d(x_{n+1}, x_n) + \varepsilon d(x_{n+1}, y) \leq 0.$$

Using the triangle inequality for the distance and taking into account that z_{K,k_0} is sublinear, it follows that

$$z_{K,k_0}(\varphi(y)-\varphi(x_n))+\varepsilon d(x_n,y)\le 0 \Longleftrightarrow y\in S(x_n).$$

Therefore, $y\in S(x_n)$ implies that $S(x_{n+1})\subseteq S(x_n)$. In particular,

$$\begin{aligned} v(x_{n+1}) &= \inf_{y\in S(x_{n+1})} z_{K,k_0}(\varphi(y)-\varphi(x_{n+1})) \ge \inf_{y\in S(x_n)} z_{K,k_0}(\varphi(y)-\varphi(x_n)) \\ &\ge \inf_{y\in S(x_n)} z_{K,k_0}(\varphi(y)-\varphi(x_n)) - z_{K,k_0}(\varphi(x_{n+1})-\varphi(x_n)) \\ &= v(x_n) - z_{K,k_0}(\varphi(x_{n+1})-\varphi(x_n)) \ge -\frac{1}{n+1} \end{aligned} \tag{3.4}$$

Thus, for $y\in S(x_{n+1})\setminus\{x_{n+1}\}$, from (3.1), (3.2) and (3.4) we obtain

$$\varepsilon d(x_{n+1},y)\le -z_{K,k_0}(\varphi(y)-\varphi(x_{n+1}))\le -v(x_n+1)\le \frac{1}{n+1}\to 0 \text{ as } n\to\infty,$$

which entails

$$diam(S(x_n))\to 0 \text{ as } n\to\infty.$$

Since the sets $S(x_n)$ are closed and $S(x_{n+1})\subseteq S(x_n)$ we obtain from this that the intersection of the sets $S(x_n)$ is a singleton $\{\overline{x}\}$ and $S(\overline{x})=\{\overline{x}\}$. This implies that $\overline{x}\in S(\widehat{x})$, or equivalently

$$z_{K,k_0}(\varphi(\overline{x})-\varphi(\widehat{x}))\le -\varepsilon d(\widehat{x},\overline{x}).$$

From Lemma 2.2 (v), it follows that

$$\varphi(\overline{x})-\varphi(\widehat{x})+\varepsilon d(\widehat{x},\overline{x})k_0\in -K.$$

Therefore, (a) holds. Moreover, if $x\ne\overline{x}$, then $x\notin S(\overline{x})$, and we get

$$z_{K,k_0}(\varphi(x)-\varphi(\overline{x}))> -\varepsilon d(x,\overline{x}).$$

Using again Lemma 2.2 (vi) we have

$$\varphi(x)-\varphi(\overline{x})\notin -\varepsilon d(x,\overline{x})k_0-K, \text{ for all } x\ne\overline{x}, \tag{3.5}$$

which is the conclusion (b) of our theorem. □

Remark 3.2. In Araya [5], an important assumption is

$$(H)\quad \{y\in X \mid \varphi(y)-\varphi(x)+d(x,y)k_0\in -K\} \text{ is closed for every } x\in X.$$

On the other hand, we use the (k_0,K)-lower semicontinuity for the function φ.
Before going further, we spend some time discussing on the comparison between the condition (H) and the (k_0,K)-lower semicontinuity. Taking into account Example 2.7 we can observe that if the function φ is (k_0,K)-lower semicontinuous, not necessary satisfies condition (H).
However, if the function φ satisfies the condition (H) then is not necessary (k_0,K)-lower semicontinuous, as the following example shows.

Example 3.3. Let $X = [0,1]$, $Y = l_\infty$ and $\varphi : X \to Y$ defined as

$$\varphi(x) = \begin{cases} \left(\frac{1}{x+1}, \frac{1}{x+2}, \ldots, \frac{1}{x+n}, \ldots\right), & x \neq 0; \\ (2, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots), & x = 0. \end{cases}$$

The ordering cone is $K_{l_\infty} = \{y \in l_\infty \mid y_i \geq 0 \text{ for all } i \in \mathbb{N}\}$ and has nonempty interior. Considering $k_0 = \left(1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\right)$ and $r = 1$, by Definition 2.4, taking $x_n \to 0, x_n \in S$, it is easy to observe that the set

$$S = \{x \in X : \varphi(x) \in rk_0 - K\}$$

is not closed, $0 \notin S$. On the other hand, φ satisfies the condition (H). Concluding, no one implies the other.

4. Ekeland's variational principle for the vector bifunctions

Araya et al. [6] obtained a vectorial version of Ekeland's variational principle for the bifunctions related to an equilibrium problem. They used the triangle inequality in order to obtain the desired result. Further, instead the triangle inequality property a suitable approximation from below of the bifunction f is required.
Let $f : X \times X \to Y$ be a bifunction. Consider the following *property*: there exists $\varphi : X \to Y$ such that

$$(P) \quad f(x,y) \in \varphi(y) - \varphi(x) + K \text{ for all } x, y \in X.$$

Property (P) is more general than the triangle inequality:

$$(T) \quad f(x,z) + f(z,y) \in f(x,y) + K, \text{ for all } x, y, z \in X.$$

Indeed, take in triangle inequality, for example, $\varphi_{\widehat{x}} = f(\widehat{x}, \cdot)$, where $\widehat{x} \in X$ is fixed, and property (P) follows.
We illustrate that the property (P) is more general than the triangle inequality considering the following example.

Example 4.1. Let $X = [0,1]$ and $Y = l_\infty$ and $f : X \times X \to Y$ defined as:

$$f(x,y) = \begin{cases} y\left(\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots\right), & x \neq \frac{1}{2},\ y \neq \frac{1}{2}; \\ (0, 0, \ldots, 0, \ldots), & x = \frac{1}{2},\ y \neq \frac{1}{2}; \\ (1-x)\left(\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots\right), & x \neq \frac{1}{2},\ y = \frac{1}{2}; \\ \left(\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots\right), & x = \frac{1}{2},\ y = \frac{1}{2}. \end{cases}$$

The ordering cone is $K_{l_\infty} = \{y \in l_\infty \mid y_i \geq 0 \text{ for all } i \in \mathbb{N}\}$. The function f does not satisfy the triangle inequality; take $x = 1, y = \frac{1}{2}$ and $z = \frac{1}{4}$. We obtain

$$f\left(1, \frac{1}{2}\right) + f\left(\frac{1}{2}, \frac{1}{4}\right) \notin f\left(1, \frac{1}{4}\right) + K.$$

On the other hand, there exists $\varphi : X \to Y$, namely

$$\varphi(x) = \begin{cases} \left(\frac{x}{2}, \frac{x}{4}, \ldots, \frac{x}{2^n}, \ldots\right), & x \neq \frac{1}{2}; \\ \left(\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots\right), & x = \frac{1}{2}, \end{cases}$$

such that the property (P) is satisfied.

The following result extends Theorem 2.1 in [6].

Theorem 4.2. *Let $f : X \times X \to Y$ and assume that*

(i) *there exists $\varphi : X \to Y$ (k_0, K)-lower semicontinuous such that*

$$f(x,y) \in \varphi(y) - \varphi(x) + K, \text{ for all } x, y \in X;$$

(ii) *for each $x \in X$, there exists $\overline{y} \in Y$ such that $(\varphi(X) - \varphi(x)) \cap (\overline{y} - intK) = \emptyset$;*

(iii) *for each $x \in X$, $\{y \in X \mid (\varphi(y) - \varphi(x)) + d(x,y)k_0 \in -K\}$ is closed.*

Then, for every $\varepsilon > 0$ and for every $\widehat{x} \in X$, there exists $\overline{x} \in X$ such that

(a) $\varphi(\overline{x}) - \varphi(\widehat{x}) + \varepsilon d(\overline{x}, \widehat{x})k_0 \in -K$;

(b) $f(\overline{x}, x) + \varepsilon d(\overline{x}, y)k_0 \notin -K$, *for all $x \in X$, $x \neq \overline{x}$.*

Proof. The function φ satisfies all the assumptions of Theorem 3.1 in [5]. Then there exists $\overline{x} \in X$ such that item (a) is verified. From the property (P) we have

$$f(\overline{x}, x) - \varphi(x) + \varphi(\overline{x}) \in K, \text{ for all } x \in X,$$

and by item (iii) of Theorem 3.1 we get

$$\varphi(x) - \varphi(\overline{x}) + \varepsilon d(\overline{x}, x)k_0 \notin -K, \text{ for every } x \in X, x \neq \overline{x}.$$

Adding these two relations we obtain item (b) of the theorem. □

Remark 4.3. We have to remark the fact that we do not need the assumption

$$f(x,x) = 0,$$

see Theorem 2.1 in [6].

We present now the following vectorial form of equilibrium version of Ekeland-type variational principle, result which extends similar results from the literature, see [6], [7] and [2].

Theorem 4.4. *Let $f : X \times X \to Y$ such that*

(i) *there exists $\varphi : X \to Y$ (k_0, K)-lower semicontinuos such that*

$$f(x,y) \in \varphi(y) - \varphi(x) + K, \text{ for all } x, y \in X;$$

(ii) *for each $x \in X$, there exists $\overline{y} \in Y$ such that $(\varphi(X) - \varphi(x)) \cap (\overline{y} - intK) = \emptyset$;*

(iii) $\varphi(X) \subset \bigcup_{t \in \mathbb{R}} \{tk_0\}$.

Then, for every $\varepsilon > 0$ and for every $\widehat{x} \in X$, there exists $\overline{x} \in X$ such that

(a) $\varphi(\overline{x}) - \varphi(\widehat{x}) + \varepsilon d(\overline{x}, \widehat{x})k_0 \in -K$;

(b) $f(\overline{x}, x) + \varepsilon d(\overline{x}, x)k_0 \notin -K$, *for all $x \in X$, $x \neq \overline{x}$.*

Proof. The idea of the proof is like in Theorem 4.2 and is based on Theorem 3.1. □

There are many cases where Theorem 2.1 [6] cannot be applied but all the assumptions of Theorem 4.4 are satisfied.

Example 4.5. Let $X = [0, 2]$, $Y = \mathbb{R}^2$ and $f : X \times X \to Y$ defined as:

$$f(x,y) = \begin{cases} (y, 2y), & x > 0,\ y > 0; \\ (2 - x, 0), & x > 0,\ y = 0; \\ (y + 2, y), & x = 0,\ y > 0; \\ (0, 0), & x = 0,\ y = 0. \end{cases}$$

The ordering cone of Y is $K = \mathbb{R}^2_+$. The function f does not satisfy the triangle inequality; take $x = 2, y = 0$ and $z = 1$. We obtain

$$f(2, 0) + f(0, 1) \notin f(2, 1) + K.$$

On the other hand, there exists $\varphi : X \to Y$, namely

$$\varphi(x) = (x, 0),$$

such that φ is (k_0, K)-lower semicontinuous with $k_0 = (1, 0)$.
Moreover, $\varphi(X) \subset \bigcup_{t\in\mathbb{R}} \{tk_0\}$ and the property (P) is satisfied.
We notice that $x = 1$ is a solution for the weak equilibria.

5. Existence solutions for the weak equilibria

The settings for this section are the same like in the section before.
Using Theorem 3.1, we are able to show the nonemptiness of the solution set of the weak equilibria without any convexity requirements on the set X and the function f, going through the existence of approximate solutions of an optimization problem.
The next statement provides the existence of solution of an optimization problem when the domain is compact.

Theorem 5.1. *If C is a nonempty compact subset of X, $\varphi : C \to Y$ is (k_0, K)-lower semicontinuous such that*

(i) *for each $x \in C$, there exists $\overline{y} \in Y$ such that $(\varphi(C) - \varphi(x)) \cap (\overline{y} - intK) = \emptyset$;*
(ii) *$\varphi(C) \subset \bigcup_{t\in\mathbb{R}} \{tk_0\}$;*

then there exists $\overline{x} \in C$ such that $\varphi(y) - \varphi(\overline{x}) \notin -intK$, for every $y \in C$.

Proof. From Theorem 3.1, for each $n \in \mathbb{N}$, there exists $x_n \in C$ such that

$$\varphi(y) - \varphi(x_n) + \frac{1}{n} d(x_n, y)k_0 \notin -K, \quad \text{for all } y \in C,\ y \neq x_n.$$

By Lemma 2.2 (vi), we have

$$z_{K,k_0}(\varphi(y) - \varphi(x_n)) + \frac{1}{n} d(x_n, y) > 0, \quad \text{for all } y \in C,\ y \neq x_n \text{ and } n \in \mathbb{N}.$$

Since C is compact, we can choose a subsequence $\{x_{n_k}\}$ of x_n such that $x_{n_k} \to \overline{x} \in C$ as $k \to \infty$. Then, since $\varphi(y) - \varphi(\cdot)$, where $y \in C$ is fixed, is (k_0, K)-upper semicontinuous, we obtain that $z_{K,k_0}(\varphi(y) - \varphi(\cdot))$ is upper semicontinuous, see Lemma 2.5. Hence,

$$z_{K,k_0}(\varphi(y)-\varphi(\overline{x})) \geq \limsup_{k\to\infty}(z_{K,k_0}(\varphi(y)-\varphi(x_{n_k}))+\frac{1}{n_k}d(x_{n_k},x)) \geq 0, \quad \text{for all } y \in C.$$

Therefore, again by Lemma 2.2 (vii), it follows

$$\varphi(y) - \varphi(\overline{x}) \notin -intK, \quad \text{for all } y \in C,$$

and thus, $\overline{x}$ is a solution for an optimization problem. □

The next result gives sufficient conditions for the existence of solutions when we move to the wider class of bifunctions which satisfies the property (P).

Theorem 5.2. *Let C be a nonempty compact subset of X, $f : C \times C \to Y$ a bifunction which satisfies property (P) with respect to $\varphi : C \to Y$ which is (k_0, K)-lower semicontinuous. Assume that:*

(i) *for each $x \in C$, there exists $\overline{y} \in Y$ such that $(\varphi(C) - \varphi(x)) \cap (\overline{y} - intK) = \emptyset$;*

(ii) $\varphi(C) \subset \bigcup_{t\in\mathbb{R}}\{tk_0\}$,

Then there exists $\overline{x} \in C$ such that $f(\overline{x}, y) \notin -intK$, for every $y \in C$.

Proof. The proof is based on Theorem 5.1 taking into account the property (P). □

6. Concluding remarks

In this paper, we widen the class of vector bifunctions for which Ekeland's variational principle holds and obtain a result which improves the main result in Araya et. al [6]. In the literature, when dealing with vector equilibrium problems and the existence of their solutions, the most used assumptions are the convexity of the domain and the generalized convexity and monotonicity, together with some weak continuity assumptions of the vector function. In this paper, we focus on conditions that do not involve any convexity concept, neither for the domain nor for the bifunction involved. Sufficient conditions for the weak vector equilibria with bifunctions which satisfy property (P), in the absence of the convexity, are given for compact domains.

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