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Hermite-Hadamard type inequalities for product of GA-convex functions via Hadamard fractional integrals

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Abstract. In this paper, some Hermite-Hadamard type inequalities for products of two GA-convex functions via Hadamard fractional integrals are established. Our results about GA-convex functions are analogous generalizations for some other results proved by Pachpette for convex functions.

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1. Introduction

In recent years, very large number of studies of error estimations have been done for Hermite-Hadamard type inequalities. It is known that Hermite-hadamard integral inequality was built on a convex function. In time, Hermite-Hadamrd inequality is developed other kinds of convex functions. For some results which generalize, improve, and extend the Hermite-Hadamard inequality see [1, 7, 10, 18, 20] and references therein.

Hermite-Hadamard type inequalities for products of two convex functions are interesting problem and firstly developed by Pachpatte in [16]. In [17], Pachpette also established Hermite-hadamard type inequalities involving two log-convex functions. In [11], Kırmacı et. al. proved several Hermite-Hadamard type inequalities for products of two convex and s-convex functions. In [19], Sarıkaya et. al. proved some Hermite-Hadamard type inequalities for products of two h-convex functions. In [2], Bakula et. al. established Hermite-Hadamard type inequalities for products of two m-convex and (α, m)-convex functions. In [4, 6], Chen and Wu obtained some Hermite-Hadamard type inequalities for products of two convex and harmonically s-convex functions. In [21], Yin and Qi established some Hermite-Hadamard type inequalities for products of two convex functions. In [5], Chen obtained some new Hermite-Hadamard type inequalities for products of two convex functions via Riemann-Liouville fractional integrals and in [3] he extended this problem to m-convex and (α, m)-convex functions.

In this work, we establish Hermite-Hadamard type inequalities for products of two GA-convex functions via Hadamard fractional integrals. Our results are analogous generalization for some results in [16].

2. Preliminaries

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{2.1}$$

is well known in the literature as Hermite-Hadamard's inequality [8].

In [16], Pachpette established following two Hermite-Hadamard type inequalities for products of convex functions as follows:

Theorem 2.1. *Let f and g be real-valued, non-negative and convex functions on $[a, b]$. Then*

$$\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx \leq \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b) \tag{2.2}$$

and

$$\begin{aligned} 2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \quad \leq \quad & \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx \\ & + \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b) \end{aligned} \tag{2.3}$$

where $M(a,b) = f(a)\,g(a) + f(b)\,g(b)$ and $N(a,b) = f(a)\,g(b) + f(b)\,g(a)$.

Definition 2.2. [14, 15]. A function $f : I \subseteq (0, \infty) \to \mathbb{R}$ is said to be GA-convex (geometric-arithmetically convex) if

$$f(x^t y^{1-t}) \leq tf(x) + (1-t)\,f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

We will now give definitions of the right-hand side and left-hand side Hadamard fractional integrals which are used throughout this paper.

Definition 2.3. [12]. Let $f \in L[a, b]$. The right-hand side and left-hand side Hadamard fractional integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $b > a \geq 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1} f(t)\frac{dt}{t},\ x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \left(\ln\frac{t}{x}\right)^{\alpha-1} f(t)\frac{dt}{t},\ x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt.$$

In [9], İşcan represented Hermite-Hadamard's inequalities for GA-convex functions in fractional integral forms as follows.

Theorem 2.4. *Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a function such that $f \in L[a, b]$ where $a, b \in I$ with $a < b$. If f is a GA-convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\sqrt{ab}\right) \le \frac{\Gamma(\alpha+1)}{2\left(\ln \frac{b}{a}\right)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] \le \frac{f(a)+f(b)}{2} \tag{2.4}$$

with $\alpha > 0$.

In [13], Kunt and İşcan established new Hermite-Hadamard type inequality for GA-convex function in fractional integral forms as follows:

Theorem 2.5. *Let $f : [a, b] \subseteq (0, \infty) \to \mathbb{R}$ be a GA-convex function with $a < b$ and $f \in L[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\sqrt{ab}\right) \le \frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln \frac{b}{a}\right)^\alpha} \left[J_{\sqrt{ab}-}^\alpha f(a) + J_{\sqrt{ab}+}^\alpha f(b)\right] \le \frac{f(a)+f(b)}{2}. \tag{2.5}$$

3. General results

Theorem 3.1. *Let f and $g : [a, b] \subseteq (0, \infty) \to \mathbb{R}$ be non-negative and GA-convex functions with $a < b$ and $f \in L[a, b]$, then the following inequality for fractional integrals hold:*

$$\begin{aligned} &\frac{\Gamma(\alpha+1)}{2\left(\ln \frac{b}{a}\right)^\alpha} \left[J_{a+}^\alpha f(b) g(b) + J_{b-}^\alpha f(a) g(a)\right] \\ \le\ & \left(\frac{\alpha}{\alpha+2} - \frac{\alpha}{\alpha+1} + \frac{1}{2}\right) M(a, b) + \frac{\alpha}{(\alpha+2)(\alpha+1)} N(a, b) \end{aligned} \tag{3.1}$$

where $\alpha > 0$, $M(a, b) = f(a) g(a) + f(b) g(b)$ and $N(a, b) = f(a) g(b) + f(b) g(a)$.

Proof. Since f and g are non-negative and GA-convex functions on $[a, b]$, we have for all $t \in [0, 1]$

$$f(a^t b^{1-t}) \le t f(a) + (1-t) f(b), \tag{3.2}$$

and

$$g(a^t b^{1-t}) \le t g(a) + (1-t) g(b). \tag{3.3}$$

From products of (3.2) and (3.3), we have

$$\begin{aligned} f(a^t b^{1-t}) g(a^t b^{1-t}) \le\ & t^2 f(a) g(a) + (1-t)^2 f(b) g(b) \\ & + t(1-t) \left[f(a) g(b) + f(b) g(a)\right]. \end{aligned} \tag{3.4}$$

Similarly (3.4), we have

$$\begin{aligned} f(a^{1-t}b^{t})g(a^{1-t}b^{t}) &\leq (1-t)^{2} f(a) g(a) + t^{2} f(b) g(b) \\ &\quad + t(1-t)\left[f(a) g(b) + f(b) g(a)\right]. \end{aligned} \tag{3.5}$$

The sum of (3.4) and (3.5), we have

$$\begin{aligned} & f(a^{t}b^{1-t})g(a^{t}b^{1-t}) + f(a^{1-t}b^{t})g(a^{1-t}b^{t}) \\ \leq\ & \left(2t^{2} - 2t + 1\right) M(a,b) + 2t(1-t) N(a,b) \end{aligned} \tag{3.6}$$

Multiplying both sides of (3.6) by $t^{\alpha-1}\frac{\alpha}{2}$, then integrating the obtained inequality with respect to t over $[0,1]$, we have

$$\begin{aligned} & \frac{\alpha}{2}\left[\int_0^1 t^{\alpha-1} f(a^{t}b^{1-t})g(a^{t}b^{1-t})dt + \int_0^1 t^{\alpha-1} f(a^{1-t}b^{t})g(a^{1-t}b^{t})dt\right] \\ =\ & \frac{\alpha}{2}\left[\int_a^b \left(\frac{\ln\frac{b}{u}}{\ln\frac{b}{a}}\right)^{\alpha-1} f(u) g(u) \frac{du}{u\ln\frac{b}{a}} + \int_a^b \left(\frac{\ln\frac{v}{a}}{\ln\frac{b}{a}}\right)^{\alpha-1} f(v) g(v) \frac{du}{v\ln\frac{b}{a}}\right] \\ =\ & \frac{\alpha}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[\int_a^b \left(\ln\frac{b}{u}\right)^{\alpha-1} f(u) g(u) \frac{du}{u} + \int_a^b \left(\ln\frac{v}{a}\right)^{\alpha-1} f(v) g(v) \frac{du}{v}\right] \\ =\ & \frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{a+}^{\alpha} f(b) g(b) + J_{b-}^{\alpha} f(a) g(a)\right] \\ \leq\ & \frac{\alpha}{2}\left[M(a,b)\int_0^1 t^{\alpha-1}\left(2t^{2}-2t+1\right)dt + N(a,b)\int_0^1 t^{\alpha-1} 2t(1-t)\,dt\right] \\ =\ & \left(\frac{\alpha}{\alpha+2} - \frac{\alpha}{\alpha+1} + \frac{1}{2}\right) M(a,b) + \frac{\alpha}{(\alpha+2)(\alpha+1)} N(a,b) \end{aligned}$$

and this completes the proof. □

Remark 3.2. Theorem 3.1 is an analogous generalization of (2.2) for GA-convex functions.

Corollary 3.3. *In Theorem* 3.1*, if we take* $g : [a,b] \to \mathbb{R}$ *as* $g(x) = 1$ *for all* $x \in [a,b]$*, then we have*

$$\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \leq \frac{f(a) + f(b)}{2}$$

which is the right hand side of (2.4)*.*

Corollary 3.4. *In Theorem* 3.1*, if we take* $\alpha = 1$*, then we have*

$$\frac{1}{\ln b - \ln a}\int_a^b f(x) g(x) \frac{dx}{x} \leq \frac{1}{3} M(a,b) + \frac{1}{6} N(a,b)$$

for GA-convex functions.

Theorem 3.5. *Let f and $g : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ be non-negative and GA-convex functions with $a < b$ and $f \in L[a,b]$, then the following inequality for fractional integrals hold:*

$$2f\left(\sqrt{ab}\right)g\left(\sqrt{ab}\right) \leq \frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{a+}^{\alpha}f(b)g(b) + J_{b-}^{\alpha}f(a)g(a)\right]$$
$$+\frac{\alpha}{(\alpha+2)(\alpha+1)}M(a,b) + \left(\frac{\alpha}{\alpha+2} - \frac{\alpha}{\alpha+1} + \frac{1}{2}\right)N(a,b) \tag{3.7}$$

where $\alpha > 0$, $M(a,b) = f(a)g(a) + f(b)g(b)$ and $N(a,b) = f(a)g(b) + f(b)g(a)$.

Proof. It is clear for all $t \in [0,1]$

$$\sqrt{ab} = \sqrt{a^t b^{1-t}.a^{1-t}b^t} = \sqrt{a^t b^{1-t}}\sqrt{a^{1-t}b^t}.$$

Since f and g are non-negative and GA-convex functions on $[a,b]$, we have for all $t \in [0,1]$

$$\begin{aligned}
f\left(\sqrt{ab}\right)g\left(\sqrt{ab}\right) &= f\left(\sqrt{a^t b^{1-t}}\sqrt{a^{1-t}b^t}\right)g\left(\sqrt{a^t b^{1-t}}\sqrt{a^{1-t}b^t}\right) \\
&\leq \frac{1}{4}\left[f\left(a^t b^{1-t}\right) + f\left(a^{1-t}b^t\right)\right]\left[g\left(a^t b^{1-t}\right) + g\left(a^{1-t}b^t\right)\right] \\
&= \frac{1}{4}\left[f\left(a^t b^{1-t}\right)g\left(a^t b^{1-t}\right) + f\left(a^{1-t}b^t\right)g\left(a^{1-t}b^t\right)\right] \\
&\quad +\frac{1}{4}\left[f\left(a^t b^{1-t}\right)g\left(a^{1-t}b^t\right) + f\left(a^{1-t}b^t\right)g\left(a^t b^{1-t}\right)\right] \\
&\leq \frac{1}{4}\left[f\left(a^t b^{1-t}\right)g\left(a^t b^{1-t}\right) + f\left(a^{1-t}b^t\right)g\left(a^{1-t}b^t\right)\right] \\
&\quad +\frac{1}{4}\left[tf(a) + (1-t)f(b)\right]\left[(1-t)g(a) + tg(b)\right] \\
&\quad +\frac{1}{4}\left[(1-t)f(a) + tf(b)\right]\left[tg(a) + (1-t)g(b)\right] \\
&= \frac{1}{4}\left[f\left(a^t b^{1-t}\right)g\left(a^t b^{1-t}\right) + f\left(a^{1-t}b^t\right)g\left(a^{1-t}b^t\right)\right] \\
&\quad +\frac{1}{4}\left\{2t(1-t)\left[f(a)g(a) + f(b)g(b)\right]\right. \\
&\quad \left. +\left(2t^2 - 2t + 1\right)\left[f(a)g(b) + f(b)g(a)\right]\right\} \qquad (3.8)
\end{aligned}$$

Multiplying both sides of (3.8) by $2\alpha t^{\alpha-1}$, then integrating the obtained inequality with respect to t over $[0,1]$, we have

$$\begin{aligned}
2f\left(\sqrt{ab}\right)g\left(\sqrt{ab}\right) &\leq \frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{a+}^{\alpha}f(b)g(b) + J_{b-}^{\alpha}f(a)g(a)\right] \\
&\quad +\frac{\alpha}{(\alpha+2)(\alpha+1)}M(a,b) + \left(\frac{\alpha}{\alpha+2} - \frac{\alpha}{\alpha+1} + \frac{1}{2}\right)N(a,b)
\end{aligned}$$

and this completes the proof. □

Remark 3.6. Theorem 3.5 is an analogous generalization of (2.3) for GA-convex functions.

Corollary 3.7. *In Theorem* 3.5*, if we take* $g:[a,b]\to\mathbb{R}$ *as* $g(x)=1$ *for all* $x\in[a,b]$*, then we have*

$$2f\left(\sqrt{ab}\right)\leq\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]+\frac{f(a)+f(b)}{2}.$$

Corollary 3.8. *In Theorem* 3.5*, if we take* $\alpha=1$*, then we have*

$$2f\left(\sqrt{ab}\right)g\left(\sqrt{ab}\right)\leq\frac{1}{\ln b-\ln a}\int_a^b f(x)\,g(x)\,\frac{dx}{x}+\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b)$$

for GA-convex functions.

Theorem 3.9. *Let* f *and* $g:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ *be non-negative and GA-convex functions with* $a<b$ *and* $f\in L[a,b]$*, then the following inequality for fractional integrals hold:*

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{\sqrt{ab}-}^{\alpha}f(a)\,g(a)+J_{\sqrt{ab}+}^{\alpha}f(b)\,g(b)\right]\\ \leq\;&\left(\frac{\alpha}{4(\alpha+2)}-\frac{\alpha}{2(\alpha+1)}+\frac{1}{2}\right)M(a,b)+\frac{\alpha^2+3\alpha}{4(\alpha+2)(\alpha+1)}N(a,b)\end{aligned}\tag{3.9}$$

where $\alpha>0$*,* $M(a,b)=f(a)\,g(a)+f(b)\,g(b)$ *and* $N(a,b)=f(a)\,g(b)+f(b)\,g(a)$*.*

Proof. Since f and g are non-negative and GA-convex functions on $[a,b]$, multiplying both sides of (3.6) by $t^{\alpha-1}\frac{\alpha}{2^{1-\alpha}}$, then integrating the obtained inequality with respect to t over $\left[0,\frac{1}{2}\right]$, we have

$$\begin{aligned}&\frac{\alpha}{2^{1-\alpha}}\left[\int_0^{\frac{1}{2}}t^{\alpha-1}f(a^tb^{1-t})g(a^tb^{1-t})dt+\int_0^{\frac{1}{2}}t^{\alpha-1}f(a^{1-t}b^t)g(a^{1-t}b^t)dt\right]\\ =\;&\frac{\alpha}{2^{1-\alpha}}\left[\int_{\sqrt{ab}}^b\left(\frac{\ln\frac{b}{u}}{\ln\frac{b}{a}}\right)^{\alpha-1}f(u)\,g(u)\,\frac{du}{u\ln\frac{b}{a}}+\int_a^{\sqrt{ab}}\left(\frac{\ln\frac{v}{a}}{\ln\frac{b}{a}}\right)^{\alpha-1}f(v)\,g(v)\,\frac{du}{v\ln\frac{b}{a}}\right]\\ =\;&\frac{\alpha}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[\int_{\sqrt{ab}}^b\left(\ln\frac{b}{u}\right)^{\alpha-1}f(u)\,g(u)\,\frac{du}{u}+\int_a^{\sqrt{ab}}\left(\ln\frac{v}{a}\right)^{\alpha-1}f(v)\,g(v)\,\frac{du}{v}\right]\\ =\;&\frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{\sqrt{ab}+}^{\alpha}f(b)\,g(b)+J_{\sqrt{ab}-}^{\alpha}f(a)\,g(a)\right]\\ \leq\;&\frac{\alpha}{2^{1-\alpha}}\left[M(a,b)\int_0^{\frac{1}{2}}t^{\alpha-1}\left(2t^2-2t+1\right)dt+N(a,b)\int_0^{\frac{1}{2}}t^{\alpha-1}2t(1-t)\,dt\right]\\ =\;&\left(\frac{\alpha}{4(\alpha+2)}-\frac{\alpha}{2(\alpha+1)}+\frac{1}{2}\right)M(a,b)+\frac{\alpha^2+3\alpha}{4(\alpha+2)(\alpha+1)}N(a,b)\end{aligned}$$

and this completes the proof. □

Remark 3.10. Theorem 3.9 is an other analogous generalization of (2.2) for GA-convex functions.

Corollary 3.11. *In Theorem* 3.9*, if we take* $g : [a, b] \to \mathbb{R}$ *as* $g(x) = 1$ *for all* $x \in [a, b]$*, then we have*

$$\frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J^{\alpha}_{\sqrt{ab}-}f(a) + J^{\alpha}_{\sqrt{ab}+}f(b)\right] \le \frac{f(a)+f(b)}{2}$$

which is the right hand side of (2.5)*.*

Corollary 3.12. *In Theorem* 3.9*, if we take* $\alpha = 1$*, then we have*

$$\frac{1}{\ln b - \ln a}\int_a^b f(x)\,g(x)\,\frac{dx}{x} \le \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b)$$

for GA-convex functions.

Theorem 3.13. *Let* f *and* $g : [a, b] \subseteq (0, \infty) \to \mathbb{R}$ *be non-negative and GA-convex functions with* $a < b$ *and* $f \in L[a, b]$*, then the following inequality for fractional integrals hold:*

$$\begin{aligned}2f\left(\sqrt{ab}\right)g\left(\sqrt{ab}\right) &\le \frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J^{\alpha}_{\sqrt{ab}-}f(a)\,g(a) + J^{\alpha}_{\sqrt{ab}+}f(b)\,g(b)\right] \\ &+\frac{\alpha^2+3\alpha}{4(\alpha+2)(\alpha+1)}M(a,b) + \left(\frac{\alpha}{4(\alpha+2)} - \frac{\alpha}{2(\alpha+1)} + \frac{1}{2}\right)N(a,b) \qquad (3.10)\end{aligned}$$

where $\alpha > 0$, $M(a,b) = f(a)\,g(a) + f(b)\,g(b)$ *and* $N(a,b) = f(a)\,g(b) + f(b)\,g(a)$*.*

Proof. Multiplying both sides of (3.8) by $2^{1+\alpha}\alpha t^{\alpha-1}$, then integrating the obtained inequality with respect to t over $\left[0, \frac{1}{2}\right]$, we have desired result. □

Remark 3.14. Theorem 3.13 is an other analogous generalization of (2.3) for GA-convex functions.

Corollary 3.15. *In Theorem* 3.13*, if we take* $g : [a, b] \to \mathbb{R}$ *as* $g(x) = 1$ *for all* $x \in [a, b]$*, then we have*

$$2f\left(\sqrt{ab}\right) \le \frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J^{\alpha}_{\sqrt{ab}-}f(a) + J^{\alpha}_{\sqrt{ab}+}f(b)\right] + \frac{f(a)+f(b)}{2}.$$

Corollary 3.16. *In Theorem* 3.13*, if we take* $\alpha = 1$*, then we have*

$$2f\left(\sqrt{ab}\right)g\left(\sqrt{ab}\right) \le \frac{1}{\ln b - \ln a}\int_a^b f(x)\,g(x)\,\frac{dx}{x} + \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b)$$

for GA-convex functions.

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