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Generalized versus classical normal derivative

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Dedicated to the memory of Professor Csaba Varga

Abstract. Given a bounded domain with Lipschitz boundary, the general Green formula permits to justify that the weak solutions of a Neumann elliptic problem satisfy the Neumann boundary condition in a weak sense. The formula involves a generalized normal derivative. We prove a general result which establishes that the generalized normal derivative of an operator coincides with the classical one, provided that the operator is continuous. This result allows to deduce that, under usual regularity assumptions, the weak solutions of a Neumann problem satisfy the Neumann boundary condition in the classical sense. This information is necessary in particular for applying the strong maximum principle.

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1. Introduction and statement of the result

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain with Lipschitz boundary. Then, it is a consequence of Rademacher Theorem that the outward unit normal $n(x)$ is defined almost everywhere on the boundary $\partial\Omega$ (endowed with the Hausdorff measure H^{N-1}). The normal derivative of a function $u \in C^1(\overline{\Omega})$ is then $\frac{\partial u}{\partial n} = \nabla u \cdot n$ on $\partial\Omega$.

The nonsmooth Green formula ([6], [2]) asserts that

$$\int_\Omega (\operatorname{div} a)\phi \, dx + \int_\Omega a \cdot \nabla \phi \, dx = \int_{\partial\Omega} \gamma_n(a)\gamma(\phi) \, dH^{N-1}$$

for all $\phi \in W^{1,p}(\Omega)$ and all a belonging to

$$V^{p'}(\Omega, \operatorname{div}) = \{a \in L^{p'}(\Omega, \mathbb{R}^N) : \operatorname{div} a \in L^{p'}(\Omega)\}.$$

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Here $p \in (1, +\infty)$ and $p' := \frac{p}{p-1}$ is its Hölder conjugate. The formula involves the classical trace operator $\gamma : W^{1,p}(\Omega) \to W^{\frac{1}{p'},p}(\partial\Omega)$ (see, e.g., [3], [5]) and the generalized normal derivative $\gamma_n : V^{p'}(\Omega, \mathrm{div}) \to W^{-\frac{1}{p'},p'}(\partial\Omega)$ introduced in [6] and [2].

If $\phi \in C^1(\overline{\Omega})$, then due to the classical Green formula we have $\gamma(\phi) = \phi|_{\partial\Omega}$. In fact, it is well known that the equality $\gamma(\phi) = \phi|_{\partial\Omega}$ holds whenever $\phi \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ (see, e.g, [3]).

Similarly, if $a \in C^1(\overline{\Omega}, \mathbb{R}^N)$, then we have $\gamma_n(a) = a \cdot n$. Our main result ensures that this equality holds more generally:

Theorem 1.1. *Let $\gamma_n : V^q(\Omega, \mathrm{div}) \to W^{-\frac{1}{q},q}(\partial\Omega)$ (with $q \in (1, +\infty)$) be the generalized normal derivative. Then, for all $a \in V^q(\Omega, \mathrm{div}) \cap C(\overline{\Omega}, \mathbb{R}^N)$, we have $\gamma_n(a) = a \cdot n$.*

As far as we know, there was no proof of this result in the literature.

This result can be applied to Neumann elliptic boundary value problems driven by the p-Laplacian (or a more general nonlinear operator) for showing that a weak solution $u \in W^{1,p}(\Omega)$ (which belongs in fact to $C^1(\overline{\Omega})$ due to nonlinear regularity theory) satisfies the classical Neumann boundary condition $\frac{\partial u}{\partial n} = 0$. Without the result stated in Theorem 1.1, we can just say that $\gamma_n(|\nabla u|^{p-2}\nabla u) = 0$. The latter equality can be viewed as a Neumann boundary condition in a weak sense. However, it is a key point that for applying the strong maximum principle [9] to u (in order to show for instance that a nonnegative, nontrivial solution is positive on $\overline{\Omega}$), it is necessary to know that the strong Neumann condition $\frac{\partial u}{\partial n} = 0$ holds (the weak one is not sufficient).

The rest of the paper is organized as follows. In Section 2, we present the background on trace operator, generalized normal derivative, and Green formulas. In Section 3, we give the proof of Theorem 1.1. In Section 4, we present the application to Neumann and, more generally, Steklov boundary value problems.

2. Green formulas

In this section, we recall the generalized normal derivative operator defined in [6] and [2]. This operator permits to obtain a nonlinear Green formula, which is crucial for relating weak solutions of quasilinear elliptic problems and their boundary conditions.

Before stating the main definition and the general Green formula (Theorem 2.2), we review other versions of the Green formula involving relatively regular functions and operators.

Recall that $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with Lipschitz boundary $\partial\Omega$. This regularity of the domain implies that we have the $(N-1)$-dimensional Hausdorff measure H^{N-1} on $\partial\Omega$, and the outward unit normal $n(\cdot)$ is defined H^{N-1}-almost everywhere on $\partial\Omega$.

The *classical Green formula* states as follows : if $a \in C^1(\overline{\Omega}, \mathbb{R}^N)$ and $v \in C^1(\overline{\Omega}) (:= C^1(\overline{\Omega}, \mathbb{R}))$, then

$$\int_\Omega (\mathrm{div}\, a) v \, dx + \int_\Omega a \cdot \nabla v \, dx = \int_{\partial\Omega} (a \cdot n) v \, dH^{N-1} \tag{2.1}$$

where $\operatorname{div} a = \sum_{i=1}^{N} \frac{\partial a_i}{\partial x_i}$ and $\nabla v = (\frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_N})$, while "$\cdot$" stands for the scalar product in $\mathbb{R}^N$. For a first generalization of the Green formula, we take v in the Sobolev space $W^{1,p}(\Omega)$ $(p > 1)$ instead of being of class C^1. To this end, the notion of trace is needed:

Theorem 2.1 (see [3, §4.3] **and** [5, §1.5]**).** *There is a unique bounded linear operator* $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega, H^{N-1})$ *which extends the operator* $C^\infty(\overline{\Omega}) \to C(\partial\Omega)$, $v \mapsto v|_{\partial\Omega}$.

Moreover, we have the following properties:

(a) $\gamma(v) = v|_{\partial\Omega}$ *whenever* $v \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$.

(b) *(Green formula) If* $a \in C^1(\overline{\Omega}, \mathbb{R}^N)$ *and* $v \in W^{1,p}(\Omega)$, *then*

$$\int_\Omega (\operatorname{div} a) v \, dx + \int_\Omega a \cdot \nabla v \, dx = \int_{\partial\Omega} (a \cdot n)\gamma(v) \, dH^{N-1}. \tag{2.2}$$

(c) $\ker \gamma = W_0^{1,p}(\Omega)$ *and* $\operatorname{Im} \gamma = W^{\frac{1}{p'},p}(\partial\Omega)$.

In particular, in view of Theorem 2.1 (a)–(b), the Green formula (2.1) remains valid if we assume that $v \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ instead of $v \in C^1(\overline{\Omega})$.

The final stage of the discussion is to replace the assumption that $a \in C^1(\overline{\Omega}, \mathbb{R}^N)$ by a more general one. To this end, for $q > 1$, we define

$$V^q(\Omega, \operatorname{div}) = \{a \in L^q(\Omega, \mathbb{R}^N) : \operatorname{div} a \in L^q(\Omega)\}$$

which is a Banach space for the norm

$$\|a\|_{V^q(\Omega,\operatorname{div})} = \left(\|a\|^q_{L^q(\Omega,\mathbb{R}^N)} + \|\operatorname{div} a\|^q_{L^q(\Omega)}\right)^{\frac{1}{q}}.$$

This requires the definition of a new operator which extends $a \mapsto a \cdot n$ to the space $V^{p'}(\Omega, \operatorname{div})$, where $p' = \frac{p}{p-1}$ is the Hölder conjugate of p.

Theorem 2.2 ([6, 2]**).** *There is a unique bounded linear operator*

$$\gamma_n : V^{p'}(\Omega, \operatorname{div}) \to W^{-\frac{1}{p'},p'}(\partial\Omega) = W^{\frac{1}{p'},p}(\partial\Omega)^*$$

which extends the operator $C^\infty(\overline{\Omega}, \mathbb{R}^N) \to L^\infty(\partial\Omega, H^{N-1})$, $a \mapsto a \cdot n$.

Moreover, we have the following properties:

(a) *(Green formula) If* $a \in V^{p'}(\Omega, \operatorname{div})$ *and* $v \in W^{1,p}(\Omega)$, *then*

$$\int_\Omega (\operatorname{div} a) v \, dx + \int_\Omega a \cdot \nabla v \, dx = \langle \gamma_n(a), \gamma(v) \rangle_{W^{-\frac{1}{p'},p'}(\partial\Omega), W^{\frac{1}{p'},p}(\partial\Omega)}. \tag{2.3}$$

(b) $\operatorname{Im} \gamma_n = W^{-\frac{1}{p'},p'}(\partial\Omega)$.

Remark 2.3. Due to (2.2), (2.3), and the surjectivity of the trace operator $\gamma : W^{1,p}(\Omega) \to W^{\frac{1}{p'},p}(\partial\Omega)$, we have immediately that $\gamma_n(a) = a \cdot n$ whenever $a \in C^1(\overline{\Omega}, \mathbb{R}^N)$.

Example 2.4. (a) If $p = 2$, $u \in W^{1,2}(\Omega)$ is such that $\Delta u := \operatorname{div}(\nabla u) \in L^2(\Omega)$, then the Green formula (2.3) reads as

$$\int_\Omega (\Delta u) v \, dx + \int_\Omega \nabla u \cdot \nabla v \, dx = \langle \gamma_n(\nabla u), \gamma(v) \rangle_{W^{-\frac{1}{2},2}(\partial\Omega), W^{\frac{1}{2},2}(\partial\Omega)}.$$

(b) If $p > 1$ is arbitrary and letting $a = |\nabla u|^{p-2}\nabla u$ for $u \in W^{1,p}(\Omega)$, then the Green formula (2.3) becomes

$$\int_\Omega (\Delta_p u)v\,dx + \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\,dx = \langle \frac{\partial u}{\partial n_p}, \gamma(v)\rangle_{W^{-\frac{1}{p'},p'}(\partial\Omega),W^{\frac{1}{p'},p}(\partial\Omega)}$$

provided that $\Delta_p u \in L^{p'}(\Omega)$, where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator and we denote $\frac{\partial u}{\partial n_p} := \gamma_n(|\nabla u|^{p-2}\nabla u)$. In the case $p \geq 2$, if $u \in C^2(\overline{\Omega})$ then $|\nabla u|^{p-2}\nabla u \in C^1(\overline{\Omega}, \mathbb{R}^N)$ and we get $\frac{\partial u}{\partial n_p} = |\nabla u|^{p-2}\nabla u \cdot n$ (see Remark 2.3). If, moreover, $p = 2$, then $\frac{\partial u}{\partial n_2} = \nabla u \cdot n = \frac{\partial u}{\partial n}$. Thus $\frac{\partial u}{\partial n_p}$ can be seen as a generalized normal derivative.

3. Proof of Theorem 1.1

The proof splits into several steps.

Lemma 3.1. *Let $a \in V^q(\Omega, \operatorname{div}) \cap C(\overline{\Omega}, \mathbb{R}^N)$. Assume that a is the restriction of $a' \in V^q(\Omega', \operatorname{div}) \cap C(\overline{\Omega'}, \mathbb{R}^N)$ for a bounded domain $\Omega' \subset \mathbb{R}^N$ with $\overline{\Omega} \subset \Omega'$. Then, the equality $\gamma_n(a) = a \cdot n$ holds.*

In the following proof, whenever $\rho \in C^\infty_{\mathrm{c}}(\mathbb{R}^N)$ and $h \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, we consider the convolution

$$\rho * h : \mathbb{R}^N \to \mathbb{R},\ x \mapsto \int_{\mathbb{R}^N} \rho(x-y)h(y)\,dy.$$

If $h \in L^q(\Omega')$ then we set $\rho * h = \rho * \bar{h}$ where $\bar{h} \in L^q(\mathbb{R}^N)$ is the extension by zero of h.

Proof of Lemma 3.1. Consider a regularizing sequence $(\rho_k)_{k\geq 1}$, that is,

$$\rho_k \in C^\infty_c(\mathbb{R}^N), \quad \operatorname{supp}\rho_k \subset B(0, \tfrac{1}{k}), \quad \int_{\mathbb{R}^N} \rho_k\,dx = 1, \quad \rho_k \geq 0 \text{ in } \mathbb{R}^N.$$

Choose $k_0 \geq 1$ such that

$$\overline{\Omega + B(0, \tfrac{1}{k_0})} \subset \Omega'. \tag{3.1}$$

Write $a = (a_1, \ldots, a_N)$ and $a' = (a'_1, \ldots, a'_N)$, so that $a_i = a'_i|_{\overline{\Omega}}$ for all $i \in \{1, \ldots, N\}$. Then we set

$$v_k = \rho_k * a' = (\rho_k * a'_1, \ldots, \rho_k * a'_N).$$

Thus, $v_k \in C^\infty(\mathbb{R}^N, \mathbb{R}^N) \cap L^q(\mathbb{R}^N, \mathbb{R}^N)$ (see [1, Théorème IV.15 and Proposition IV.20]) and we have that

$$v_k \to a' \ \text{ in } L^q(\Omega', \mathbb{R}^N) \text{ as } k \to \infty \tag{3.2}$$

(see [1, Théorème IV.22]) and moreover

$$v_k \to a \quad \text{uniformly on } \overline{\Omega} \text{ as } k \to \infty \tag{3.3}$$

(see [1, proof of Proposition IV.21]).

Since $\operatorname{div} a' \in L^q(\Omega')$, we have also that $\rho_k * \operatorname{div} a' \in C^\infty(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ and

$$\rho_k * \operatorname{div} a' \to \operatorname{div} a' \ \text{ in } L^q(\Omega') \text{ as } k \to \infty. \tag{3.4}$$

We claim that

$$\operatorname{div} v_k = \rho_k * \operatorname{div} a' \quad \text{in } \Omega, \;\text{ for all } k \geq k_0. \tag{3.5}$$

The functions on the left- and the right-hand side of (3.5) belong to $C^\infty(\mathbb{R}^N)$, but since we do not know that the partial derivatives $\frac{\partial a_i'}{\partial x_i}$ are defined almost everywhere (though it is the case for $\operatorname{div} a' \in L^q(\Omega')$), we will show (3.5) by reasoning in distributions. So let $\varphi \in C_c^\infty(\Omega)$. We compute

$$\begin{aligned}
\langle \operatorname{div} v_k, \varphi \rangle = \sum_{i=1}^N \Big\langle \frac{\partial(\rho_k * a_i')}{\partial x_i}, \varphi \Big\rangle &= -\sum_{i=1}^N \int_{\mathbb{R}^N} (\rho_k * a_i') \frac{\partial \varphi}{\partial x_i}\, dx \\
&= -\sum_{i=1}^N \int_{\Omega'} a_i' \Big(\check{\rho}_k * \frac{\partial \varphi}{\partial x_i} \Big)\, dx \\
&= -\sum_{i=1}^N \int_{\Omega'} a_i' \frac{\partial(\check{\rho}_k * \varphi)}{\partial x_i}\, dx,
\end{aligned}$$

where we denote $\check{\rho}_k(x) = \rho_k(-x)$ and use [1, Propositions IV.16 and IV.20]. Since $\rho_k \in C_c^\infty(B(0, \frac{1}{k}))$, $\varphi \in C_c^\infty(\Omega)$, and due to (3.1) and the fact that $k \geq k_0$, we have $\check{\rho}_k * \varphi \in C_c^\infty(\Omega')$ (see [1, Proposition IV.18]). Hence

$$\begin{aligned}
\langle \operatorname{div} v_k, \varphi \rangle &= \sum_{i=1}^N \Big\langle \frac{\partial a_i'}{\partial x_i}, \check{\rho}_k * \varphi \Big\rangle = \langle \operatorname{div} a', \check{\rho}_k * \varphi \rangle \\
&= \int_{\Omega'} (\operatorname{div} a')(\check{\rho}_k * \varphi)\, dx \quad (\text{since } \operatorname{div} a' \in L^q(\Omega')) \\
&= \int_{\mathbb{R}^N} (\rho_k * \operatorname{div} a') \varphi\, dx \quad (\text{by [1, Proposition IV.16]}) \\
&= \langle \rho_k * \operatorname{div} a', \varphi \rangle.
\end{aligned}$$

This establishes (3.5).

We have $v_k \in V^q(\Omega, \operatorname{div})$ because $v_k \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$. Formulas (3.2), (3.4), and (3.5) imply that

$$v_k \to a \quad \text{in } V^q(\Omega, \operatorname{div}).$$

Due to the continuity of the operator

$$\gamma_n : V^q(\Omega, \operatorname{div}) \to W^{-1/q,q}(\partial\Omega)$$

we have

$$\gamma_n(v_k) \to \gamma_n(a) \quad \text{in } W^{-1/q,q}(\partial\Omega) \quad \text{as } k \to \infty. \tag{3.6}$$

Since $v_k \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$, we have

$$\gamma_n(v_k) = v_k \cdot n \quad \text{on } \partial\Omega \quad \text{for all } k \tag{3.7}$$

(by definition of γ_n; see Theorem 2.2). By virtue of (3.3), we have

$$v_k \cdot n \to a \cdot n \quad \text{in } L^\infty(\partial\Omega, H^{N-1}) \;\text{ as } k \to \infty.$$

The continuity of the embeddings $L^\infty(\partial\Omega, H^{N-1}) \hookrightarrow L^q(\partial\Omega, H^{N-1}) \hookrightarrow W^{-1/q,q}(\partial\Omega)$ now implies that

$$v_k \cdot n \to a \cdot n \quad \text{in } W^{-1/q,q}(\partial\Omega) \ \text{ as } k \to \infty.$$

Combining this with (3.6) and (3.7), we conclude that

$$\gamma_n(a) = a \cdot n \quad \text{on } \partial\Omega.$$

The proof of the lemma is complete. $\square$

Lemma 3.2. *There is an open covering*

$$\partial\Omega = \bigcup_{i=1}^{m} \Gamma_i,$$

a family of vectors $(\nu_i)_{i=1}^m \subset \mathbb{R}^N$ *and a constant* $\delta > 0$ *such that, for every* $i \in \{1, \ldots, m\}$,

$$U_i(\delta_1, \delta_2) := \{x + t\nu_i : x \in \Gamma_i,\ t \in (-\delta_1, \delta_2)\}$$

is an open subset of $\mathbb{R}^N$ *for all* $\delta_1, \delta_2 \in (0, \delta]$ *and the following inclusions hold:*

$$U_i' := \{x + t\nu_i : x \in \Gamma_i,\ t \in (0, \delta)\} \subset \Omega,$$
$$U_i'' := \{x + t\nu_i : x \in \Gamma_i,\ t \in (-\delta, 0)\} \subset \mathbb{R}^N \setminus \overline{\Omega}.$$

Proof. Fix $x \in \partial\Omega$. Since Ω is assumed to have Lipschitz boundary, there is an open neighborhood $V \subset \mathbb{R}^N$ of x and a Lipschitz map $\chi : \mathbb{R}^{N-1} \to \mathbb{R}$ such that (up to rotating and relabeling the axes)

$$\begin{aligned} V \cap \Omega &= \{(y_1, \ldots, y_N) \in V : \chi(y_1, \ldots, y_{N-1}) < y_N\}, \\ V \cap \partial\Omega &= \{(y_1, \ldots, y_N) \in V : \chi(y_1, \ldots, y_{N-1}) = y_N\} \end{aligned}$$

(see [3, §4.2]). There is $\delta > 0$ and an open neighborhood $W \subset V$ of x such that

$$\bigcup_{y \in W} B(y, \delta) \subset V,$$

where $B(y, \delta)$ stands for the open ball of radius δ with respect to the norm

$$(y_1, \ldots, y_N) \mapsto \max_{1 \le i \le N} |y_i|.$$

Let $\Gamma_x = \Gamma = W \cap \partial\Omega$ and, given $\delta_1, \delta_2 \in (0, \delta]$, let

$$U(\delta_1, \delta_2) = \{y + t\nu : y \in \Gamma,\ t \in (-\delta_1, \delta_2)\}$$

where $\nu = (0, \ldots, 0, 1)$. Note that we have equivalently

$$\begin{aligned} U(\delta_1, \delta_2) = \{&(y_1, \ldots, y_N) \in \mathbb{R}^N : (y_1, \ldots, y_{N-1}, \chi(y_1, \ldots, y_{N-1})) \in W, \\ &y_N - \chi(y_1, \ldots, y_{N-1}) \in (-\delta_1, \delta_2)\}, \end{aligned}$$

which shows that $U(\delta_1, \delta_2)$ is open. Moreover, for all $y = (y_1, \ldots, y_N) \in \Gamma$ and $t \in (-\delta, \delta)$, we have $y + t\nu = (y_1, \ldots, y_{N-1}, y_N + t) \in B(y, \delta) \subset V$ and

$$\chi(y_1, \ldots, y_{N-1}) = y_N \begin{cases} < y_N + t & \text{if } t \in (0, \delta), \\ > y_N + t & \text{if } t \in (-\delta, 0), \end{cases}$$

whence

$$U' := \{y + t\nu : y \in \Gamma,\ t \in (0, \delta)\} \subset \Omega,$$
$$U'' := \{y + t\nu : y \in \Gamma,\ t \in (-\delta, 0)\} \subset \mathbb{R}^N \setminus \overline{\Omega}.$$

By doing the same construction for every $x \in \partial\Omega$ and extracting a finite subcovering from the open covering $\bigcup_{x\in\partial\Omega} \Gamma_x = \partial\Omega$ so obtained, we get a family of open subsets/vectors satisfying the conditions stated in the lemma. □

Lemma 3.2 yields an open neighborhood $U := \bigcup_{i=1}^{m} U_i$ of the boundary $\partial\Omega$, where $U_i := U_i(\delta, \delta)$. Since $\partial\Omega$ is compact, we can find a relatively compact, open neighborhood V of $\partial\Omega$ such that $\overline{V} \subset U$. Let $U_0 := \Omega \setminus \overline{V}$. Then we have an open covering

$$\overline{\Omega} \subset \bigcup_{i=0}^{m} U_i.$$

Let $(\theta_i)_{i=0}^{m}$ be a partition of unity relative to this open covering, i.e.,

- $\theta_i \in C_c^\infty(U_i)$ and $0 \le \theta_i \le 1$ for all $i \in \{0, 1, \ldots, m\}$,
- $\theta_0 + \theta_1 + \ldots + \theta_m = 1$ in $\overline{\Omega}$

(see [1, Lemme IX.3]).

Lemma 3.3. *Let $a \in V^q(\Omega, \mathrm{div}) \cap C(\overline{\Omega}, \mathbb{R}^N)$. For every $i \in \{0, \ldots, m\}$, let $a_i = \theta_i a$ for θ_i as above, so that $a = a_0 + a_1 + \ldots + a_m$. Then:*

(a) *$a_i \in V^q(\Omega, \mathrm{div}) \cap C(\overline{\Omega}, \mathbb{R}^N)$ and $\operatorname{supp} a_i \subset U_i$ for all i.*
(b) *In particular $\operatorname{supp} a_0 \subset \Omega$ and we have $\gamma_n(a_0) = a_0 \cdot n = 0$.*
(c) *If $\gamma_n(a_i) = a_i \cdot n$ for all $i \in \{1, \ldots, m\}$, then $\gamma_n(a) = a \cdot n$.*

Proof. (a) Since $a_i = \theta_i a$ with $\theta_i \in C_c^\infty(U_i)$ and $a \in C(\overline{\Omega}, \mathbb{R}^N)$, we get $a_i \in C(\overline{\Omega}, \mathbb{R}^N)$ and $\operatorname{supp} a_i \subset U_i$. Moreover, we have

$$\operatorname{div} a_i = \theta_i \operatorname{div} a + a \cdot \nabla\theta_i$$

with $\operatorname{div} a \in L^q(\Omega)$, $a \in C(\overline{\Omega}, \mathbb{R}^N)$, and $\theta_i \in C_c^\infty(U_i)$, whence $\operatorname{div} a_i \in L^q(\Omega)$ and, therefore, $a_i \in V^q(\Omega, \mathrm{div})$ for all $i \in \{0, \ldots, m\}$. This shows (a).

(b) In particular, we get $\operatorname{supp} a_0 \subset U_0 \subset \Omega$. This guarantees that, if a_0' denotes the extension by zero of a_0, we have $a_0' \in V^q(\mathbb{R}^N, \mathrm{div}) \cap C(\mathbb{R}^N)$, and by Lemma 3.1 we deduce that $\gamma_n(a_0) = a_0 \cdot n = 0$ on $\partial\Omega$.

(c) Since γ_n is linear and $\gamma_n(a_0) = 0$, we have $\gamma_n(a) = \gamma_n(a_1) + \ldots + \gamma_n(a_m)$. On the other hand, since $a_0 \cdot n = 0$, we have $a \cdot n = a_1 \cdot n + \ldots + a_m \cdot n$. Part (c) of the lemma ensues. □

Lemma 3.4. *Let an open subset $\Gamma \subset \partial\Omega$, a vector $\nu_0 \in \mathbb{R}^N$, and a constant $\delta > 0$ such that*

$$U(\delta_1, \delta_2) := \{x + t\nu_0 : x \in \Gamma,\ t \in (-\delta_1, \delta_2)\}$$

is an open subset of $\mathbb{R}^N$ for all $\delta_1, \delta_2 \in (0, \delta]$, and

$$U' := \{x + t\nu_0 : x \in \Gamma,\ t \in (0, \delta)\} \subset \Omega,$$
$$U'' := \{x + t\nu_0 : x \in \Gamma,\ t \in (-\delta, 0)\} \subset \mathbb{R}^N \setminus \overline{\Omega}.$$

Let $a \in V^q(\Omega, \operatorname{div}) \cap C(\overline{\Omega}, \mathbb{R}^N)$ and suppose that $\operatorname{supp} a \subset U := U(\delta, \delta)$. Then, there is a sequence $(v_k)_{k \geq 1} \subset V^q(\Omega, \operatorname{div}) \cap C(\overline{\Omega}, \mathbb{R}^N)$ satisfying the following properties:

(a) *$v_k \to a$ in $V^q(\Omega, \operatorname{div})$;*
(b) *$v_k \to a$ uniformly on $\overline{\Omega}$;*
(c) *for every $k \geq 1$, v_k is the restriction of $v'_k \in V^q(\Omega_k, \operatorname{div}) \cap C(\overline{\Omega_k}, \mathbb{R}^N)$ for a bounded domain $\Omega_k \subset \mathbb{R}^N$ with $\overline{\Omega} \subset \Omega_k$.*

In particular, by virtue of Lemma 3.1, we have $\gamma_n(v_k) = v_k \cdot n$ for all $k \geq 1$ and finally $\gamma_n(a) = a \cdot n$.

Proof. The final conclusion of the lemma can be justified as follows: on the basis of (c) we can apply Lemma 3.1 which yields $\gamma_n(v_k) = v_k \cdot n$ for all $k \geq 1$. Then, on the one hand, due to (a) and the continuity of γ_n, we have $\gamma_n(v_k) \to \gamma_n(a)$ in $W^{-\frac{1}{q},q}(\partial\Omega)$ as $k \to \infty$. On the other hand, due to (b), we have $v_k \cdot n \to a \cdot n$ in $L^q(\partial\Omega, H^{N-1}) \subset W^{-\frac{1}{q},q}(\partial\Omega)$. Altogether, this yields $\gamma_n(a) = a \cdot n$ as asserted.

Let us now show the rest of the lemma. Let $\epsilon \in (0, \delta)$ small so that

$$\operatorname{supp} a \subset W_\epsilon := \{x + t\nu_0 : x \in \Gamma,\ t \in (-\delta + \epsilon, \delta - \epsilon)\}.$$

Let $U_\epsilon = U(\epsilon, \delta - \epsilon) = \{x + t\nu_0 : x \in \Gamma,\ t \in (-\epsilon, \delta - \epsilon)\}$ and $V_\epsilon = \{x \in \mathbb{R}^N : x + \epsilon\nu_0 \notin \operatorname{supp} a\}$. The union $\Omega_\epsilon := U_\epsilon \cup V_\epsilon$ is then an open subset which contains $\overline{\Omega}$. The latter property can be shown as follows. Let $x \in \overline{\Omega}$ and assume that $x + \epsilon\nu_0 \in \operatorname{supp} a$ (otherwise, we get immediately $x \in V_\epsilon \subset \Omega_\epsilon$). Due to the inclusion $\operatorname{supp} a \subset W_\epsilon$, there are $x' \in \Gamma$ and $t \in (-\delta + \epsilon, \delta - \epsilon)$ such that $x + \epsilon\nu_0 = x' + t\nu_0$, hence $x = x' + (t - \epsilon)\nu_0$. Moreover, since $x \in \overline{\Omega}$, we must have $t - \epsilon \geq 0$. Hence $t - \epsilon \in [0, \delta - 2\epsilon) \subset (-\epsilon, \delta - \epsilon)$ and therefore $x \in U_\epsilon \subset \Omega_\epsilon$.

Now we define $v'_\epsilon \in C(\Omega_\epsilon, \mathbb{R}^N)$ by

$$v'_\epsilon(x) = \begin{cases} a(x + \epsilon\nu_0) & \text{if } x \in U_\epsilon, \\ 0 & \text{if } x \in V_\epsilon. \end{cases}$$

If $x \in U_\epsilon$ then $x + \epsilon\nu_0 \in U' \subset \Omega$ hence $a(x + \epsilon\nu_0)$ is well defined. If $x \in U_\epsilon \cap V_\epsilon$ then $x + \epsilon\nu_0 \notin \operatorname{supp} a$ (due to the definition of V_ϵ), thus $a(x + \epsilon\nu_0) = 0$. This shows that v'_ϵ is well defined and continuous (since $a \in C(\overline{\Omega}, \mathbb{R}^N)$).

Moreover, we have

$$\begin{aligned} \operatorname{div} v'_\epsilon(x) &= \begin{cases} \operatorname{div} a(x + \epsilon\nu_0) & \text{for a.e. } x \in U_\epsilon \\ 0 & \text{for } x \in V_\epsilon \end{cases} \\ &= \overline{\operatorname{div} a}(x + \epsilon\nu_0) \end{aligned} \tag{3.8}$$

where $\overline{\operatorname{div} a} \in L^q(\mathbb{R}^N)$ is the extension by zero of $\operatorname{div} a$. Indeed, if $x \in U_\epsilon$, we have $\operatorname{div} v'_\epsilon(x) = \operatorname{div} a(x + \epsilon\nu_0) = \overline{\operatorname{div} a}(x + \epsilon\nu_0)$. If $x \in V_\epsilon$, then $x + \epsilon\nu_0 \notin \operatorname{supp} a$, hence either we have $x + \epsilon\nu_0 \in \Omega \setminus \operatorname{supp} a$ in which case $\overline{\operatorname{div} a}(x + \epsilon\nu_0) = \operatorname{div} a(x + \epsilon\nu_0) = 0 = \operatorname{div} v'_\epsilon(x)$, or we have $x + \epsilon\nu_0 \notin \Omega$ in which case $\overline{\operatorname{div} a}(x + \epsilon\nu_0) = 0 = \operatorname{div} v'_\epsilon(x)$ (by definition of $\overline{\operatorname{div} a}$). This shows (3.8).

Since $\overline{\operatorname{div} a} \in L^q(\mathbb{R}^N)$ it follows that $\operatorname{div} v'_\epsilon \in L^q(\Omega_\epsilon)$. We define $v_\epsilon := v'_\epsilon|_{\overline{\Omega}}$. Then

$$v_\epsilon \in V^q(\Omega, \operatorname{div}) \cap C(\overline{\Omega}, \mathbb{R}^N).$$

Moreover, v_ϵ satisfies condition (c) of the statement. In addition, in view of (3.8) we can apply [1, Lemme IV.4] which yields

$$\operatorname{div} v_\epsilon \to \operatorname{div} a \quad \text{in } L^q(\Omega) \quad \text{as } \epsilon \to 0.$$

This will show condition (a) of the statement once we will have shown condition (b).

Let $\varepsilon > 0$. Since a is continuous on $\overline{\Omega}$ which is compact, it is uniformly continuous, hence there is $\alpha > 0$ such that

$$(x, y \in \overline{\Omega} \quad \text{and} \quad |x - y| < \alpha) \quad \Longrightarrow \quad |a(x) - a(y)| < \varepsilon,$$

where $|\cdot| : (x_1, \ldots, x_N) \in \mathbb{R}^N \mapsto \max_{1 \le i \le N} |x_i|$ is the infinite norm. Assume ϵ small enough so that $\epsilon \in (0, \alpha)$. For $x \in \overline{\Omega} \cap U_\epsilon$, we deduce that

$$|v_\epsilon(x) - a(x)| = |a(x + \epsilon\nu_0) - a(x)| \le \varepsilon.$$

Now let $x \in \overline{\Omega} \cap V_\epsilon$. If $x \notin \operatorname{supp} a$, then we have

$$|v_\epsilon(x) - a(x)| = 0.$$

If $x \in \operatorname{supp} a$, knowing that $\operatorname{supp} a \subset W_\epsilon$, by definition of W_ϵ we have that $x + \epsilon\nu_0 \in U' \subset \Omega$ (since $x \in U \cap \overline{\Omega}$) and $x + \epsilon\nu_0 \notin \operatorname{supp} a$ (since $x \in V_\epsilon$), hence

$$|v_\epsilon(x) - a(x)| = |0 - a(x)| = |a(x + \epsilon\nu_0) - a(x)| \le \varepsilon.$$

Finally we have shown

$$\|v_\epsilon - a\|_\infty \le \varepsilon.$$

This establishes the convergence

$$v_\epsilon \to a \quad \text{in } C(\overline{\Omega}, \mathbb{R}^N) \quad \text{as } \epsilon \to 0.$$

We obtain condition (b) of the statement. The proof of the lemma is therefore complete. $\square$

Theorem 1.1 follows from the above lemmas. Specifically, Lemma 3.3 shows that it is sufficient to deal with elements a as those considered in Lemma 3.4. Then, the result follows from Lemma 3.4.

Remark 3.5. (a) Our proof of Theorem 1.1 relies on ideas used in [6] for showing that $C^\infty(\overline{\Omega}, \mathbb{R}^N)$ is dense in $V^q(\Omega, \operatorname{div})$.

(b) Theorem 1.1 could be already deduced from Lemma 3.1 if one can show that every element in $V^q(\Omega, \operatorname{div}) \cap C(\overline{\Omega}, \mathbb{R}^N)$ admits an extension to $V^q(\Omega', \operatorname{div}) \cap C(\overline{\Omega'}, \mathbb{R}^N)$ for some larger domain $\Omega' \supset \overline{\Omega}$. We have no indication whether this general extension property holds.

4. Boundary conditions for weak solutions of elliptic Neumann and Steklov problems

In this section, we first consider a Neumann problem involving the Carathéodory functions

$$a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \quad \text{and} \quad f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$$

(i.e., $a(\cdot, s, \xi)$ is measurable for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and $a(x, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega$, and similarly for f). Let p^* be the Sobolev critical exponent given by $p^* = \frac{Np}{N-p}$ if $p < N$ and $p^* = +\infty$ otherwise. In what follows, we assume:

Assumption 4.1. There are constants $r \in (p, p^*)$ and $a_1, a_2, a_3, c_1 \in (0, +\infty)$ such that

$$\begin{aligned} |a(x,s,\xi)| &\leq a_1(|\xi|^{p-1} + |s|^{r/p'} + 1), && (4.1)\\ a(x,s,\xi)\cdot\xi &\geq a_2|\xi|^p - a_3(|s|^r + 1), && (4.2)\\ |f(x,s,\xi)| &\leq c_1(|\xi|^{p-1} + |s|^{r-1} + 1) && (4.3) \end{aligned}$$

for a.e. $x \in \Omega$, all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$.

Parts (4.1) and (4.3) of this assumption guarantee that:

$$\begin{aligned} u \in W^{1,p}(\Omega) &\implies a(x,u,\nabla u) \in L^{p'}(\Omega, \mathbb{R}^N) \text{ and } f(x,u,\nabla u) \in L^{r'}(\Omega)\\ &\implies a(x,u,\nabla u), f(x,u,\nabla u) \in W^{1,p}(\Omega)^* \end{aligned}$$

so that the following definition makes sense.

Definition 4.2. A *weak solution* of the Neumann problem

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) = f(x,u,\nabla u) & \text{in } \Omega,\\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \tag{4.4}$$

is a function $u \in W^{1,p}(\Omega)$ such that the equality

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla v\,dx = \int_\Omega f(x,u,\nabla u)\,v\,dx \tag{4.5}$$

holds for all $v \in W^{1,p}(\Omega)$.

For the moment, the boundary condition "$\frac{\partial u}{\partial n} = 0$" in problem (4.4) is just a notation, in the sense that the normal derivative is a priori not defined for elements in $W^{1,p}(\Omega)$. However, in Proposition 4.6, by using Theorem 1.1, we will show that the boundary condition is satisfied in the classical sense, under suitable regularity conditions on the operator a and the boundary $\partial\Omega$.

In the following lemma, we show that weak solutions to problem (4.4) satisfy a Neumann-type boundary condition in the "weak" sense.

Lemma 4.3. *Assume that $u \in W^{1,p}(\Omega)$ is a weak solution of problem* (4.4). *Then:*

(a) $u \in L^\infty(\Omega)$.

(b) $a(x,u,\nabla u) \in V^{p'}(\Omega, \operatorname{div})$ *and u satisfies the weak Neumann condition*

$$\gamma_n(a(x,u,\nabla u)) = 0 \quad \text{in} \quad W^{-\frac{1}{p'},p'}(\partial\Omega).$$

Proof. Part (a) can be shown by Moser iteration technique; see [4].

(b) First we note that part (a) combined with (4.3) ensures that

$$f(x,u,\nabla u) \in L^{p'}(\Omega).$$

Taking any smooth function $v \in C_{\mathrm{c}}^{\infty}(\Omega)$ as test function, and using the definition of the divergence (as a distribution) and the fact that u is a weak solution of problem (4.4) gives

$$\int_{\Omega} -\operatorname{div} a(x,u,\nabla u)\, v\, dx = \int_{\Omega} a(x,u,\nabla u)\cdot \nabla v\, dx = \int_{\Omega} f(x,u,\nabla u) v\, dx.$$

This implies that $-\operatorname{div} a(x,u,\nabla u) = f(x,u,\nabla u) \in L^{p'}(\Omega)$, and yields in particular

$$a(x,u,\nabla u) \in V^{p'}(\Omega, \operatorname{div})$$

so that $\gamma_n(a(x,u,\nabla u))$ is a well-defined element of $W^{-\frac{1}{p'},p'}(\partial\Omega)$. Now taking an arbitrary $v \in W^{1,p}(\Omega)$ as test function in (4.5) and using the Green formula (2.3), we get

$$\begin{aligned}
&\langle \gamma_n(a(x,u,\nabla u)), \gamma(v)\rangle \\
= \ &\int_{\Omega} \operatorname{div} a(x,u,\nabla u)\, v\, dx + \int_{\Omega} a(x,u,\nabla u)\cdot \nabla v\, dx \\
= \ &-\int_{\Omega} f(x,u,\nabla u)\, v\, dx + \int_{\Omega} f(x,u,\nabla u)\, v\, dx \ = \ 0
\end{aligned}$$

(here, for making the notation easier, we have dropped the reference to the pair $(W^{-\frac{1}{p'},p'}(\partial\Omega), W^{\frac{1}{p'},p}(\partial\Omega))$ in the duality brackets $\langle\cdot,\cdot\rangle$). Since $v \in W^{1,p}(\Omega)$ is arbitrary and the trace map $\gamma : W^{1,p}(\Omega) \to W^{\frac{1}{p'},p}(\partial\Omega)$ is surjective (see Theorem 2.1), it follows that

$$\gamma_n(a(x,u,\nabla u)) = 0 \quad \text{in} \quad W^{-\frac{1}{p'},p'}(\partial\Omega),$$

which concludes the proof. $\square$

In order to apply the regularity theory and relate the generalized normal derivative with the classical one, in Proposition 4.6 and Corollary 4.8 below we assume that the domain Ω has $C^{1,\gamma}$ boundary $\partial\Omega$, for some $\gamma \in (0,1)$, and we also need to strengthen the hypothesis on a.

Assumption 4.4. (a) $a : \overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is continuous and its restriction to $\overline{\Omega}\times\mathbb{R}\times(\mathbb{R}^N\setminus\{0\}) \to \mathbb{R}$ is of class C^1. Moreover, a is of the form

$$a(x,s,\xi) = \alpha(x,s,\xi)\xi$$

with $\alpha : \overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N \to (0,+\infty)$.

(b) There are constants $\mu,\nu \in (0,1)$, $R \in [0,+\infty)$, a nonincreasing map $\kappa_1 : [0,+\infty) \to (0,+\infty)$ and a nondecreasing map $\kappa_2 : [0,+\infty) \to (0,+\infty)$ such that

$$\begin{aligned}
a'_{\xi}(x,s,\xi)\eta\cdot\eta &\geq \kappa_1(|s|)(R+|\xi|)^{p-2}|\eta|^2, \\
\|a'_{\xi}(x,s,\xi)\| &\leq \kappa_2(|s|)(R+|\xi|)^{p-2}, \\
|a(x,s,\eta)-a(y,t,\eta)| &\leq \kappa_2(|s|+|t|)(|x-y|^{\mu}+|s-t|^{\nu})(1+|\eta|)^{p-2}|\eta|
\end{aligned}$$

for all $x,y \in \overline{\Omega}$, $s,t \in \mathbb{R}$, $\xi,\eta \in \mathbb{R}^N$, $\xi \neq 0$. Here $a'_{\xi}(x,s,\cdot)$ denotes the differential of the map $a(x,s,\cdot)$ and $\|\cdot\|$ denotes the norm in the space of linear endomorphisms of $\mathbb{R}^N$.

Example 4.5. For $p > 1$, the mapping $a : (x, s, \xi) \mapsto |\xi|^{p-2}\xi$ satisfies Assumption 4.4 with the map α given by

$$\alpha : (x, s, \xi) \mapsto \begin{cases} |\xi|^{p-2} & \text{if } \xi \neq 0, \\ 1 & \text{if } \xi = 0. \end{cases}$$

(Note that Assumption 4.4 does not require α to be continuous.) This mapping corresponds to the p-Laplacian operator $\Delta_p : u \in W^{1,p}(\Omega) \mapsto \operatorname{div} a(x, u, \nabla u) = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$.

Under Assumptions 4.1 and 4.4, we have:

Proposition 4.6. *Let $u \in W^{1,p}(\Omega)$ be a weak solution of problem* (4.4)*. Then:*

(a) *$u \in C^{1,\lambda}(\overline{\Omega})$ for some $\lambda \in (0, 1)$.*
(b) *$a(x, u, \nabla u) \in V^{p'}(\Omega, \operatorname{div}) \cap C(\overline{\Omega}, \mathbb{R}^N)$ and u satisfies the classical Neumann condition $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$.*

Proof. Part (a) follows from nonlinear regularity theory [7]. The first claim of Part (b) then follows from Lemma 4.3 and the continuity of a in Assumption 4.4. Then, Theorem 1.1 combined with Lemma 4.3 yields

$$a(x, u, \nabla u) \cdot n = \gamma_n(a(x, u, \nabla u)) = 0 \quad \text{on } \partial\Omega.$$

By Assumption 4.4, we have that $a(x, u, \nabla u) = \alpha(x, u, \nabla u)\nabla u$ with $\alpha(x, u, \nabla u) \in (0, +\infty)$, whence finally

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

Note that this equality holds everywhere on $\partial\Omega$. Theorem 1.1 and Lemma 4.3 yield an equality almost everywhere, but in the present proposition due to the regularity assumption on the domain, the outward unit normal n is defined everywhere on $\partial\Omega$ so that the equality makes sense and holds everywhere by continuity. $\square$

We strengthen our assumption in order to apply the strong maximum principle:

Assumption 4.7. (a) The mapping $a(x, s, \xi) = a(x, \xi)$ is independent of the variable s. Moreover, there are constants $d_1, d_2, d_3, \delta \in (0, +\infty)$ such that

$$a'_\xi(x, \xi)\eta \cdot \eta \geq d_1|\xi|^{p-2}|\eta|^2,$$
$$\|a'_\xi(x, \xi)\| \leq d_2|\xi|^{p-2},$$
$$|\xi| < \delta \;\Rightarrow\; \|a'_x(x, \xi)\| \leq d_3|\xi|^{p-1}$$

for all $x \in \overline{\Omega}$, $\xi, \eta \in \mathbb{R}^N$, $\xi \neq 0$.

(b) There is a constant $c > 0$ such that $f(x, s, \xi) \geq -cs^{p-1}$ for a.e. $x \in \Omega$, all $s \in [0, \delta)$, $\xi \in \mathbb{R}^N$.

Under Assumptions 4.1, 4.4, 4.7, we have:

Corollary 4.8. *Let $u \in C^{1,\lambda}(\overline{\Omega})$ be a weak solution of problem* (4.4)*, as in Proposition* 4.6*. Assume that $u \geq 0$ on $\overline{\Omega}$ and $u \not\equiv 0$. Then we have $u > 0$ on $\overline{\Omega}$.*

Proof. By Assumption 4.7 (b), (4.3), and $u \in C^1(\overline{\Omega})$, we find $\tilde{c} > 0$ with

$$\operatorname{div} a(x, \nabla u) \leq \tilde{c} u^{p-1} \quad \text{in } \Omega.$$

This combined with Assumption 4.7 allows us to invoke the strong maximum principle [8, Theorem 8.27], which yields $u > 0$ on Ω and

$$\forall x \in \partial\Omega,\ u(x) = 0 \ \Rightarrow\ \frac{\partial u}{\partial n}(x) < 0.$$

Since we know that $\frac{\partial u}{\partial n}(x) = 0$ for all $x \in \partial\Omega$ (by Proposition 4.6), we get $u(x) > 0$ for all $x \in \partial\Omega$. Whence $u > 0$ on $\overline{\Omega}$ as asserted. $\square$

Finally we consider more general (Steklov-type) boundary conditions. Let $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the growth condition

$$|g(x, s)| \leq c_2(|s|^{\sigma-1} + 1) \quad \text{for a.e. } x \in \partial\Omega, \text{ all } s \in \mathbb{R}, \tag{4.6}$$

for a constant $c_2 > 0$ and some $\sigma \in (1, \frac{(N-1)p}{N-p})$ if $p < N$ and an arbitrary $\sigma \in (1, +\infty)$ if $p \geq N$. Given a, f satisfying respectively (4.1) and (4.3) in Assumption 4.1, we say that $u \in W^{1,p}(\Omega)$ is a weak solution of the problem

$$\begin{cases} -\operatorname{div} a(x, u, \nabla u) = f(x, u, \nabla u) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_a} = g(x, u) & \text{on } \partial\Omega \end{cases} \tag{4.7}$$

if the equality

$$\begin{aligned} \int_\Omega a(x, u, \nabla u) \cdot \nabla v \, dx &= \int_\Omega f(x, u, \nabla u)\, v \, dx \\ &\quad + \int_{\partial\Omega} g(x, \gamma(u))\gamma(v) \, dH^{N-1} \end{aligned} \tag{4.8}$$

holds for all $v \in W^{1,p}(\Omega)$ (there is a continuous embedding $W^{\frac{1}{p'},p}(\partial\Omega) \subset L^\sigma(\partial\Omega, H^{N-1})$, so the definition makes sense).

Proposition 4.9. *Let $u \in C^1(\overline{\Omega})$ be a weak solution of (4.7) such that $a(x, u, \nabla u) \in C(\overline{\Omega}, \mathbb{R}^N)$. Then,*

$$a(x, u, \nabla u) \cdot n = g(x, u) \quad \text{on } \partial\Omega.$$

In particular, if $a(x, u, \nabla u) = |\nabla u|^{p-2}\nabla u$, then $|\nabla u|^{p-2}\frac{\partial u}{\partial n} = g(x, u)$ on $\partial\Omega$.

Proof. Arguing as in the proof of Lemma 4.3, one has $\operatorname{div} a(x, u, \nabla u) = -f(x, u, \nabla u) \in L^{p'}(\Omega)$ hence $a(x, u, \nabla u) \in V^{p'}(\Omega, \operatorname{div})$. For every $v \in W^{1,p}(\Omega)$, by virtue of Theorem 2.2 and formula (4.8), we get

$$\begin{aligned} \langle \gamma_n(a(x, u, \nabla u)), \gamma(v) \rangle &= \int_\Omega a(x, u, \nabla u) \cdot \nabla v \, dx \\ &\quad - \int_\Omega f(x, u, \nabla u)\, v \, dx \\ &= \int_{\partial\Omega} g(x, u)\gamma(v) \, dH^{N-1}, \end{aligned}$$

and Theorem 2.1 (c) yields $\gamma_n(a(x,u,\nabla u)) = g(x,u)$ in $W^{-\frac{1}{p'},p'}(\partial\Omega)$. On the other hand, Theorem 1.1 implies that $\gamma_n(a(x,u,\nabla u)) = a(x,u,\nabla u)\cdot n$ on $\partial\Omega$. The conclusion follows. $\square$

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