Stud. Univ. Babeş-Bolyai Math. 66(2021), No. 2, 307–319
DOI: 10.24193/subbmath.2021.2.07

Approximations of the solution of a stochastic Ginzburg-Landau equation

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Dedicated to Professor Gheorghe Coman on the occasion of his 85th anniversary.

Abstract. This paper presents a method to approximate the solution of a stochastic Ginzburg-Landau equation with multiplicative noise term. Error estimates for the approximation of the solution are given.

Mathematics Subject Classification (2010): 60H15, 34K28, 35Q56.

Keywords: Stochastic Ginzburg-Landau equation, power-type nonlinearity, multiplicative noise.

1. Introduction

The complex Ginzburg-Landau equation on a bounded domain G in $\mathbb{R}$ or $\mathbb{R}^2$ with sufficiently regular boundary ∂G is

$$\begin{aligned} &dX(t) = (a_1 + \mathrm{i}a_2)\Delta X(t)dt + (\lambda_1 + \mathrm{i}\lambda_2)|X(t)|^2 X(t)dt + \gamma X(t)dt, \\ &X(0) = X_0, \end{aligned} \tag{1.1}$$

where $X : G \times [0,\infty) \to \mathbb{C}$ and $a_1, a_2, \lambda_1, \lambda_2, \gamma$ are certain real parameters.

Throughout this paper i is the imaginary unit, $\mathrm{Re}z$, $\mathrm{Im}z$ are, respectively, the real part and imaginary part of a complex number z, $\bar{z}$ denotes its complex conjugate and $|z| = \sqrt{(\mathrm{Re}z)^2 + (\mathrm{Im}z)^2}$ its modulus. Further we use the notations: $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and $\mathbb{N}^* := \{1, 2, \ldots\}$.

Equation (1.1) is a nonlinear Schrödinger type equation with complex coefficients and power-type nonlinearity. Different forms of this evolution equation have applications in physics, see for example [5, 6, 8, 10, 11, 12, 13].

In this paper we consider a method to approximate the solution of the following stochastic complex Ginzburg-Landau evolution equation in dimension one perturbed by a multiplicative noise term

$$dX(t) = \mathrm{i}a\Delta X(t)\,dt - \lambda|X(t)|^2X(t)\,dt + \gamma X(t)dt + \mathrm{i}\sum_{k=1}^{\infty} b_k(t)X(t)\,dW_k(t) \qquad (1.2)$$

with initial condition $X(0) = X_0$ and homogeneous Neumann boundary condition. Here, X is a complex-valued stochastic process depending on $t \in [0,T]$ and $x \in G \subset \mathbb{R}$, $a \in \mathbb{R}^*$, $\lambda, \gamma, T > 0$ are fixed, $(W_k)_{k\in\mathbb{N}}$ is a sequence of independent real-valued Wiener processes and $(b_k)_{k\in\mathbb{N}}$ is a sequence of real-valued functions, whose properties will be detailed later. The stochastic equation (1.2) corresponds to the case when in the deterministic equation (1.1) the parameters are $a_1 = 0, a_2 = a \in \mathbb{R}^*$ and $\lambda_1 = -\lambda < 0, \lambda_2 = 0, \gamma > 0$.

The existence of the solution of the stochastic Ginzburg-Landau equations is studied for example in [9, 1, 2, 5] with different noise terms than in our paper. In [9] the Galerkin method for the stochastic equation is used, while in [1] there are studied mild solutions and Strichartz' estimates are applied. In [5] the equation is studied on a three dimensional torus and has an additive noise term. A similar noise term is considered in [2], where the martingale solution is investigated.

In this paper we prove the existence of the solution by using a deterministic Ginzburg-Landau type equation. Moreover, we present a method to approximate the solution of (1.2) and give error estimates for this approximation. In the context of computer simulations error estimates are very important.

The paper has the following structure: Section 2 contains some notations, preliminary results, and the variational formulation of the stochastic, as well as the deterministic Ginzburg-Landau equation. In Section 3 we prove the existence and uniqueness of the considered evolution equation. In the last section we give some approximation results and error estimates for both the solution of the deterministic and stochastic Ginzburg-Landau equation.

2. Preliminaries

For simplicity we take the domain $G = (0,1)$. Consider the complex Hilbert spaces $H := L^2(0,1)$ and $V := H^1(0,1)$, the inner product in H is given by

$$(u,v) := \int_0^1 u(x)\,\bar{v}(x)\,dx, \qquad \text{for all } u, v \in H,$$

while the inner product in V is

$$(u,v)_V := \int_0^1 \left[u(x)\,\bar{v}(x) + \frac{du}{dx}(x)\,\frac{d\bar{v}}{dx}(x)\right] dx, \qquad \text{for all } u, v \in V.$$

The corresponding norms in H and V are $\|\cdot\|$ and $\|\cdot\|_V$, respectively. Furthermore, let V^* be the dual space of V and $\langle\cdot,\cdot\rangle$ the duality pairing of V^* and V.

Let $A : V \to V^*$ be the operator defined by

$$\langle Au, v\rangle := \int_0^1 \frac{du}{dx}(x)\,\frac{d\bar{v}}{dx}(x)\,dx, \qquad \text{for all } u, v \in V. \qquad (2.1)$$

Let $(\mu_k)_{k\in\mathbb{N}}$ be the increasing sequence of real eigenvalues and let $(h_k)_{k\in\mathbb{N}}$ be the corresponding eigenfunctions of A with respect to homogeneous Neumann boundary conditions. The eigenfunctions $(h_k)_{k\in\mathbb{N}}$ form an orthonormal system in H and they are orthogonal in V. Obviously, for all $u \in H$ and all $v \in V$, it holds that

$$u = \sum_{k=1}^{\infty}(u, h_k)h_k, \qquad Av = \sum_{k=1}^{\infty}\mu_k(v, h_k)h_k\,,$$

and

$$\operatorname{Im}\langle Av, v\rangle = 0, \tag{2.2}$$

$$\operatorname{Re}\langle Av, v\rangle = \langle Av, v\rangle = \sum_{k=1}^{\infty}\mu_k\,|(v, h_k)|^2 \geq 0.$$

Moreover,

$$\|Av\|_{V^*} \leq \|v\|_V, \quad \text{for all } v \in V. \tag{2.3}$$

Recall that $V \hookrightarrow H$ is a compact embedding, (V, H, V^*) is a triplet of rigged Hilbert spaces (Gelfand triple), and $\langle Au, v\rangle = (Au, v)$, for each $u, v \in V$, such that $Au \in H$.

In [4, Lemma 1.1] it is stated that

$$\sup_{x\in[0,1]}|v(x)|^2 \leq \|v\|\left(\|v\| + 2\left\|\frac{dv}{dx}\right\|\right) \leq 2\|v\|_V^2, \text{ for all } v \in V. \tag{2.4}$$

Recall that $H^1(0,1) \hookrightarrow C[0,1]$.

For each $n \in \mathbb{N}^*$ set $H_n := \operatorname{sp}\{h_1, h_2, \ldots, h_n\}$ with the norm being induced from H. The norms $\|\cdot\|$ and $\|\cdot\|_V$ are equivalent on H_n.

The map $\Pi_n : H \to H_n$ defined by $\Pi_n h := \sum_{k=1}^{n}(h, h_k)h_k$ is the orthogonal projection of H onto H_n. Then, $\Pi_n h = h$, for each $h \in H_n$, and $\|\Pi_n h\| \leq \|h\|$, for each $h \in H$. Moreover, for each $h \in H_n$ it holds

$$Ah = \sum_{k=1}^{n}\mu_k(h, h_k)h_k \in H_n,$$

and then

$$\langle Ah, h\rangle = (Ah, h) = \sum_{k=1}^{n}\mu_k\,|(h, h_k)|^2 = \|h\|_V^2 - \|h\|^2, \tag{2.5}$$

$$(Ah, Ah) = \sum_{k=1}^{n}\mu_k^2\,|(h, h_k)|^2\,, \text{ for each } h \in H_n. \tag{2.6}$$

Further, we mention some results used throughout the paper.

Lemma 2.1. *Let $u, v \in V$ such that $Av \in H$ and $|u|^2 v \in V$, then*

$$\operatorname{Re}(|u|^2 v, Av) \geq 0. \tag{2.7}$$

Especially, if $|v|^2 v \in V$, it holds

$$\operatorname{Re}(|v|^2 v, Av) \geq 0. \tag{2.8}$$

Proof. We have

$$\begin{aligned}\mathrm{Re}(Av,|u|^2v) &= \mathrm{Re}\int_0^1 \frac{dv}{dx}(x)\frac{d|u|^2\bar{v}}{dx}(x)\,dx\\ &= \mathrm{Re}\int_0^1 \left(\bar{v}(x)\frac{dv}{dx}(x)\frac{d|u|^2}{dx}(x)+|u(x)|^2\left|\frac{dv}{dx}(x)\right|^2\right)dx\\ &= \int_0^1 \left(\left(\frac{1}{2}\frac{d|v|^2}{dx}(x)\right)\frac{d|u|^2}{dx}(x)+|u(x)|^2\left|\frac{dv}{dx}(x)\right|^2\right)dx \geq 0.\end{aligned}$$

Therefore,

$$\mathrm{Re}(|u|^2v,Av)=\mathrm{Re}\overline{(|u|^2v,Av)}=\mathrm{Re}(Av,|u|^2v)\geq 0.$$

□

Lemma 2.2. *Let $z_1,z_2\in\mathbb{C}$. Then the following inequalities hold*

$$\left||z_1|^2z_1-|z_2|^2z_2\right|\leq 3\left(|z_1|^2+|z_2|^2\right)|z_1-z_2|; \tag{2.9}$$

$$Re\left(\left(|z_1|^2z_1-|z_2|^2z_2\right)(\bar{z}_1-\bar{z}_2)\right)\geq 0. \tag{2.10}$$

Proof. See, e.g., [9, Lemma 7.2, Lemma 7.3]. □

Lemma 2.3. *Let S be a bounded set in $L^2([0,T];V)$, which is equicontinuous in $C([0,T];V^*)$. Then, S is relatively compact in $L^2([0,T];H)$.*

Proof. We use [14, Theorem 4.1] applied for $V\hookrightarrow H\hookrightarrow V^*$, where $V\hookrightarrow H$ is compact. □

In what follows we assume that $(\Omega,\mathcal{F},P)$ is a complete probability space and $(W_k)_{k\in\mathbb{N}}$ a sequence of independent real-valued standard Brownian motions on $[0,T]$ generating an increasing family of σ-algebras $\left(\mathcal{F}_t\right)_{t\in[0,T]}$. For each $k\geq 1$, let $b_k:[0,T]\to\mathbb{R}$ be square integrable functions such that

$$\sum_{k=1}^{\infty}\int_0^T b_k^2(s)\,ds<\infty. \tag{2.11}$$

Throughout the paper let $\lambda,\gamma,T>0$, $a\in\mathbb{R}^*$, $X_0\in V$ be fixed.

Definition 2.4. An $\left(\mathcal{F}_t\right)_{t\in[0,T]}$ adapted process

$$X\in L^2(\Omega;C([0,T];H))\cap L^4(\Omega\times[0,T];V)$$

is called a **variational solution of the stochastic Ginzburg-Landau equation** (1.2) with initial condition $X_0\in V$ if

$$\begin{aligned}(X(t),v)=&(X_0,v)-\mathrm{i}a\int_0^t\langle AX(s),v\rangle ds-\lambda\int_0^t(|X(s)|^2X(s),v)ds\\ &+\gamma\int_0^t(X(s),v)ds+\mathrm{i}\sum_{k=1}^{\infty}\int_0^t b_k(s)(X(s),v)dW_k(s)\end{aligned} \tag{2.12}$$

holds for all $t\in[0,T]$, $v\in V$, and a.e. $\omega\in\Omega$.

Remark 2.5. Let (2.11) be satisfied. Recall that the real-valued stochastic integral with respect to countably many Brownian motions

$$R(t) := \sum_{k=1}^{\infty} \int_0^t b_k(s) dW_k(s), \qquad t \in [0,T],$$

is a continuous square integrable martingale with respect to the filtration $\big(\mathcal{F}_t\big)_{t\in[0,T]}$, see [3, Lemma 2.1], having the quadratic variation equal to

$$[R]_t = \sum_{k=1}^{\infty} \int_0^t b_k^2(s) ds, \ t \in [0,T].$$

Moreover, for each $U \in L^2(\Omega; C([0,T];H))$ the H-valued stochastic integral

$$I(t) := \sum_{k=1}^{\infty} \int_0^t b_k(s) U(s)\, dW_k(s), \qquad t \in [0,T],$$

is a continuous square integrable H-valued martingale with respect to the filtration $\big(\mathcal{F}_t\big)_{t\in[0,T]}$ and

$$E\Big(\|I(t)\|^2\Big) = E \sum_{k=1}^{\infty} \int_0^t b_k^2(s) \|U(s)\|^2 ds, \ t \in [0,T].$$

Similar to the method from the paper [7], we associate to (2.12) a deterministic equation, which has the same initial condition $X_0 \in V$. For this we denote

$$Y(t) := \exp\left(-\gamma t - \frac{1}{2}\sum_{k=1}^{\infty} \int_0^t b_k^2(s)\, ds - \mathrm{i} \sum_{k=1}^{\infty} \int_0^t b_k(s)\, dW_k(s)\right)$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$. The $\big(\mathcal{F}_t\big)_{t\in[0,T]}$ adapted real-valued process $(Y(t))_{t\in[0,T]}$ is the solution of the following stochastic linear differential equation

$$Y(t) = 1 - \gamma \int_0^t Y(s)\, ds - \sum_{k=1}^{\infty} \int_0^t b_k^2(s) Y(s)\, ds - \mathrm{i} \sum_{k=1}^{\infty} \int_0^t b_k(s) Y(s)\, dW_k(s)$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$, where the stochastic integral in this equation is a real-valued continuous martingale (see Remark 2.5). For all $t \in [0,T]$ let

$$B(t) := \frac{1}{|Y(t)|^2} = \exp\left(2\gamma t + \sum_{k=1}^{\infty} \int_0^t b_k^2(s)\, ds\right), \tag{2.13}$$

and then $0 < B(t) \le B(T) < \infty$ for all $t \in [0,T]$.

Definition 2.6. If $Z \in C([0,T];H)) \cap L^4([0,T];V)$ satisfies the evolution equation

$$(Z(t), v) = (X_0, v) - \mathrm{i}a \int_0^t \langle AZ(s), v\rangle\, ds - \lambda \int_0^t B(s)(|Z(s)|^2 Z(s), v)\, ds \tag{2.14}$$

for all $t \in [0,T]$ and all $v \in V$, then Z is called a **variational solution of the deterministic Ginzburg-Landau type equation**.

3. Existence results

Theorem 3.1. *There exists a unique solution*

$$Z \in C([0,T];H) \cap L^4([0,T];V)$$

of (2.14). *Moreover, the following inqualities hold*

$$\sup_{t\in[0,T]} \|Z(t)\|^2 \leq \|X_0\|^2,$$

$$\int_0^T \|Z(t)\|_V^2 dt \leq T\|X_0\|_V^2, \quad \int_0^T \|Z(t)\|_V^4 dt \leq T\|X_0\|_V^4.$$

Proof. Fix $n \in \mathbb{N}^*$. We consider the finite dimensional deterministic equation corresponding to (2.14)

$$(Z_n(t), h_j) = (X_0, h_j) - \mathrm{i}a\int_0^t \langle AZ_n(s), h_j\rangle\, ds - \lambda\int_0^t B(s)(|Z_n(s)|^2 Z_n(s), h_j)\, ds, \quad (3.1)$$

for all $t \in [0,T]$, $j \in \{1,...,n\}$. In what follows we study the existence and uniqueness of the solution $Z_n \in C([0,T];H_n)$ of (3.1).

From (2.2) and (2.10) it follows by standard arguments that the solution of (3.1) is unique in $C([0,T];H_n)$, and that the solution of (2.14) is unique in $C([0,T];H) \cap L^4([0,T];V)$.

The existence of $Z_n \in C([0,T];H_n)$ follows from the finite dimensional theory for differential equations with locally Lipschitz nonlinearities. Note that (2.9) assures that the nonlinearity in (3.1) is locally Lipschitz. The solution is global on $[0,T]$ by the estimate (3.4) below: By taking the complex conjugate in (3.1) we have

$$\overline{(Z_n(t), h_j)} = \overline{(X_0, h_j)} + \mathrm{i}a\int_0^t \overline{\langle AZ_n(s), h_j\rangle} ds - \lambda\int_0^t B(s)\overline{(|Z_n(s)|^2 Z_n(s), h_j)} ds \quad (3.2)$$

for all $t \in [0,T]$, $j \in \{1,...,n\}$.

For $z \in \mathbb{C}$ we recall the following identities

$$z - \bar{z} = 2\mathrm{i}\ \mathrm{Im} z \text{ and } z + \bar{z} = 2\ \mathrm{Re} z. \quad (3.3)$$

By using (3.1), (3.2), the chain rule for the product

$$(Z_n(\cdot), h_j) \cdot \overline{(Z_n(\cdot), h_j)},\ j \in \{1,...,n\},$$

as well as (3.3) and the property

$$\|Z_n(t)\|^2 = \sum_{j=1}^n |(Z_n(t), h_j)|^2 \in \mathbb{R},\ t \in [0,T],$$

we obtain for all $t \in [0,T]$

$$\|Z_n(t)\|^2 = \|\Pi_n X_0\|^2 + 2a\mathrm{Im}\int_0^t \langle AZ_n(s), Z_n(s)\rangle\, ds$$
$$- 2\lambda\mathrm{Re}\int_0^t B(s)(|Z_n(s)|^2 Z_n(s), Z_n(s))\, ds.$$

By (2.2) and (2.8) we conclude

$$\sup_{t\in[0,T]} \|Z_n(t)\|^2 \le \|X_0\|^2. \tag{3.4}$$

Further we obtain estimates for $\sup_{t\in[0,T]} \|Z_n(t)\|_V^2$. Using (3.1) and (3.2) as above, we have for all $t \in [0,T]$ and $j \in \{1, ..., n\}$

$$\begin{aligned}\mu_j|(Z_n(t),h_j)|^2 =& \mu_j|(X_0,h_j)|^2 + 2a\mathrm{Im}\int_0^t \mu_j^2|(Z_n(s),h_j)|^2 ds \\ &- 2\lambda\mathrm{Re}\int_0^t B(s)(|Z_n(s)|^2 Z_n(s), \mu_j(Z_n(s),h_j)h_j)\, ds.\end{aligned}$$

Summing up from $j=1$ to n, then, by (2.5) and (2.6), we obtain for all $t \in [0,T]$

$$(AZ_n(t), Z_n(t)) = (A\Pi_n X_0, \Pi_n X_0) - 2\lambda\mathrm{Re}\int_0^t B(s)(|Z_n(s)|^2 Z_n(s), AZ_n(s))ds.$$

By (2.8) and (2.5) it follows for all $t \in [0,T]$

$$\|Z_n(t)\|_V^2 - \|Z_n(t)\|^2 \le (A\Pi_n X_0, \Pi_n X_0) \le \langle AX_0, X_0\rangle = \|X_0\|_V^2 - \|X_0\|^2.$$

Therefore by (3.4) it follows

$$\sup_{t\in[0,T]} \|Z_n(t)\|_V^2 \le \|X_0\|_V^2. \tag{3.5}$$

Then, we conclude $Z_n \in L^4([0,T];V)$.
By (3.1) and (2.3) we have for all $r, t \in [0,T]$ with $r < t$

$$\begin{aligned}\|Z_n(t) - Z_n(r)\|_{V^*}^2 &\le 2\int_r^t \|AZ_n(s)\|_{V^*}^2 ds + 2\lambda^2 \int_r^t B^2(s)\||Z_n(s)|^2 Z_n(s)\|_{V^*}^2 ds \\ &\le 2\int_r^t \|Z_n(s)\|_V^2 ds + 2\lambda^2 C B^2(T)\int_r^t \||Z_n(s)|^2 Z_n(s)\|^2 ds,\end{aligned}$$

where C is the embedding constant of $H \hookrightarrow V^*$. Moreover, by (2.4) we write for all $r, t \in [0,T]$ with $r < t$

$$\int_r^t \||Z_n(s)|^2 Z_n(s)\|^2 ds = \int_r^t \left(\int_0^1 |Z_n(s)|^6 dx\right) ds \le 4\int_r^t \|Z_n(s)\|_V^4 \|Z_n(s)\|^2 ds.$$

Using these estimates, as well as (3.4) and (3.5), it follows for all $r, t \in [0,T]$ with $r < t$

$$\|Z_n(t) - Z_n(r)\|_{V^*}^2 \le 2(t-r)\|X_0\|_V^2 \left(1 + 4\lambda^2 C B^2(T)\|X_0\|_V^2 \|X_0\|^2\right).$$

We observe that $S := (Z_n)_{n\ge1}$ is equicontinuous in $C([0,T];V^*)$ and it is bounded in $L^2([0,T];V)$ and also in $L^4([0,T];V)$ (see (3.5)).

It follows that there exist $U \in L^2([0,T];H) \cap L^4([0,T];V)$ and a subsequence $(Z_{n_k})_{k\ge1}$ which is:
• strongly convergent in $L^2([0,T];H)$ to U (by Lemma 2.3)
and
• weakly convergent in $L^4([0,T];V)$ and, also in $L^2([0,T];V)$, to U.

Recall that $L^4([0,T];V) \hookrightarrow L^2([0,T];V) \hookrightarrow L^2([0,T];H)$; these are reflexive Banach spaces and we can use [15, Proposition 21.23(i), Proposition 21.35(c)].

Since $(Z_{n_k})_{k\geq 1}$ is strongly convergent to U in $L^2([0,T];H)$, one can prove by using (2.9) that $\big(|Z_{n_k}|^2 Z_{n_k}\big)_{k\geq 1}$ is weakly convergent to $|U|^2U$ in $L^2([0,T];H)$. In (3.1) we take n_k instead of n, then let $k \to \infty$, and using the above convergence results, we get for all $j \in \mathbb{N}^*$ that

$$(U(t), h_j) = (X_0, h_j) - \mathrm{i}a\int_0^t \langle AU(s), h_j\rangle\, ds - \lambda\int_0^t B(s)(|U(s)|^2U(s), h_j)\, ds \qquad (3.6)$$

holds for a.e. $t \in [0,T]$.

There exists an H-valued function that is equal to U for a.e. $t \in [0,T]$ and is equal to the right side of (3.6) for all $t \in [0,T]$. This function we denote by Z. By the properties of U we have

$$Z \in C([0,T];H) \cap L^4([0,T];V)$$

and Z is the solution of (2.14).

The estimate

$$\sup_{t\in[0,T]} \|Z(t)\|^2 \leq \|X_0\|^2$$

is obtained similarly to (3.4) by using (2.12). By the weak convergence of $(Z_{n_k})_{k\geq 1}$ to Z in $L^2([0,T];V)$ and also in $L^4([0,T];V)$, we get from (3.5)

$$\int_0^T \|Z(t)\|_V^2 dt \leq \liminf_{k\to\infty} \int_0^T \|Z_{n_k}(t)\|_V^2 dt \leq T\|X_0\|_V^2,$$

$$\int_0^T \|Z(t)\|_V^4 dt \leq \liminf_{k\to\infty} \int_0^T \|Z_{n_k}(t)\|_V^4 dt \leq T\|X_0\|_V^4.$$

□

Remark 3.2. In [13, Chapter IV, Theorem 5.1] and [8, Chap. 10, Théorème 10.1] the reader may find alternative ideas for the proof of the existence of the solution of (2.14). The purpose of our detailed proof, using classical methods from partial differential equations, was to obtain the estimates stated in Theorem 3.1, which will be used in the computation of error bounds in Section 4.

Theorem 3.3. *There exists a unique variational solution of* (2.12)

$$X \in L^2(\Omega; C([0,T];H)) \cap L^4(\Omega \times [0,T];V).$$

Moreover, $X \in C([0,T];H) \cap L^4([0,T];V)$ *for a.e.* $\omega \in \Omega$ *and the following inqualities hold for a.e.* $\omega \in \Omega$

$$\sup_{t\in[0,T]} \|X(t)\|^2 \leq B(T)\|X_0\|^2,$$

$$\int_0^T \|X(t)\|_V^2 dt \leq TB(T)\|X_0\|_V^2, \quad \int_0^T \|X(t)\|_V^4 dt \leq TB^2(T)\|X_0\|_V^4.$$

Proof. By the Itô formula and the uniqueness of the solution of (2.14) one has that

$$X(t) := Z(t)Y^{-1}(t), \qquad \text{for all } t \in [0,T] \text{ and a.e. } \omega \in \Omega,$$

is the unique solution of (2.12). The estimates for X follow from Theorem 3.1 and we also have

$$E \sup_{t\in[0,T]} \|X(t)\|^2 \leq B(T)\|X_0\|^2,$$

$$E\int_0^T \|X(t)\|_V^2 dt \leq TB(T)\|X_0\|_V^2, \quad E\int_0^T \|X(t)\|_V^4 dt \leq TB^2(T)\|X_0\|_V^4. \qquad \square$$

4. Approximation of the solution

We will approximate the solution of (2.12) by a sequence of stochastic processes $(X_N)_{N\geq 1}$, where, for each $N \in \mathbb{N}^*$, we consider $X_N := Z_N Y^{-1}$, Z_N being the solution of the following *linearized deterministic problem* in variational formulation

$$(Z_N(t), v) = (X_0, v) - \mathrm{i}a\int_0^t \langle AZ_N(s), v\rangle\, ds - \lambda\int_0^t B(s)(|Z_{N-1}(s)|^2 Z_N(s), v)\, ds, \quad (4.1)$$

for all $t \in [0,T]$ and all $v \in V$. We take $Z_0 := X_0$.

Theorem 4.1. *For each $N \in \mathbb{N}^*$ there exists a unique solution*

$$Z_N \in C([0,T]; H) \cap L^4([0,T]; V)$$

of (4.1).

Proof. The result is obtained successively: Let $N \geq 1$. If

$$Z_{N-1} \in C([0,T]; H) \cap L^4([0,T]; V), \text{ then } Z_N \in C([0,T]; H) \cap L^4([0,T]; V)$$

is a solution of (4.1).

The existence and uniqueness of the solution of (4.1) is proved analogously to Theorem 3.1. We use (2.7), Lemma 2.3, and the Galerkin method associated to (4.1) in order to obtain the following estimates for each $N \in \mathbb{N}^*$

$$\sup_{t\in[0,T]} \|Z_N(t)\|^2 \leq \|X_0\|^2,$$

$$\int_0^T \|Z_N(t)\|_V^2 dt \leq T\|X_0\|_V^2, \quad \int_0^T \|Z_N(t)\|_V^4 dt \leq T\|X_0\|_V^4. \qquad \square$$

Theorem 4.2. *For each $N \in \mathbb{N}^*$ it holds*

$$\sup_{t\in[0,T]} \|Z_N(t) - Z(t)\|^2 \leq \frac{3\lambda TB(T)}{2^{N-4}}\|X_0\|^2\|X_0\|_V^2 \exp\left(20\lambda TB(T)\|X_0\|_V^2\right).$$

Proof. By using (4.1) and (2.14), we have

$$\begin{aligned}\|Z_N(t) - Z(t)\|^2 &= 2a\mathrm{Im}\int_0^t \langle AZ_N(s) - AZ(s), Z_N(s) - Z(s)\rangle\, ds \qquad (4.2)\\ &- 2\lambda\mathrm{Re}\int_0^t B(s)(|Z_{N-1}(s)|^2 Z_N(s) - |Z(s)|^2 Z(s), Z_N(s) - Z(s))\, ds,\end{aligned}$$

for all $t \in [0,T]$. We define

$$e(t) = \exp\left(-20\lambda \int_0^t B(s)\|Z(s)\|_V^2 ds\right), \qquad \text{for all } t \in [0,T].$$

Then, by (4.2) and (2.2), we have for all $t \in [0,T]$

$$\begin{aligned} & e(t)\|Z_N(t) - Z(t)\|^2 = \qquad (4.3)\\ & - 2\lambda \mathrm{Re} \int_0^t e(s)B(s)(|Z_{N-1}(s)|^2 Z_N(s) - |Z(s)|^2 Z(s), Z_N(s) - Z(s))\, ds \\ & - 20\lambda \int_0^t e(s)B(s)\|Z(s)\|_V^2 \|Z_N(s) - Z(s)\|^2 ds. \end{aligned}$$

We compute

$$\begin{aligned} & - \mathrm{Re}(|Z_{N-1}|^2 Z_N - |Z|^2 Z, Z_N - Z) \\ & = -\mathrm{Re}(|Z_{N-1}|^2 (Z_N - Z), Z_N - Z) + \mathrm{Re}((|Z|^2 - |Z_{N-1}|^2) Z, Z_N - Z). \end{aligned}$$

Denote $Q = Z_{N-1}, R = Z_N$ (we omit writing the dependence on s and x). Due to the definition of the scalar product $(\cdot,\cdot)$ in H, we write

$$\begin{aligned} & - \mathrm{Re}(|Z_{N-1}|^2 (Z_N - Z), Z_N - Z) = - \int_0^1 |Q|^2 |R - Z|^2 dx \\ & = - \int_0^1 \Big(|Q - Z|^2 + |Z|^2 + 2\mathrm{Re}[(Q - Z)\bar{Z}]\Big)|R - Z|^2 dx \\ & \le \int_0^1 \Big(- |Q - Z|^2 |R - Z|^2 - |Z|^2 |R - Z|^2 + 2|Q - Z||Z||R - Z|^2 \Big) dx \\ & \le \int_0^1 \Big(- \frac{1}{2}|Q - Z|^2 |R - Z|^2 + |Z|^2 |R - Z|^2 \Big) dx, \end{aligned}$$

as well as

$$\begin{aligned} & \mathrm{Re}((|Z|^2 - |Z_{N-1}|^2) Z, Z_N - Z) = \mathrm{Re} \int_0^1 (|Z|^2 - |Q|^2) Z(\bar{R} - \bar{Z}) dx \\ & = \mathrm{Re} \int_0^1 \Big(- |Q - Z|^2 - 2\mathrm{Re}[(Q - Z)\bar{Z}]\Big) Z(\bar{R} - \bar{Z}) dx \\ & \le \int_0^1 \Big(\frac{1}{2}|Q - Z|^2 |R - Z|^2 + |Z|^2 |R - Z|^2 + \frac{3}{2}|Z|^2 |Q - Z|^2\Big) dx\, . \end{aligned}$$

Then,

$$\begin{aligned} & - 2\lambda \mathrm{Re}(|Z_{N-1}|^2 Z_N - |Z|^2 Z, Z_N - Z) \\ & \le \lambda \int_0^1 \Big(4|Z|^2 |Z_N - Z|^2 + 3|Z|^2 |Z_{N-1} - Z|^2\Big) dx \\ & \le 8\lambda \|Z\|_V^2 \|Z_N - Z\|^2 + 6\lambda \|Z\|_V^2 \|Z_{N-1} - Z\|^2, \end{aligned}$$

where in the last inequality we apply (2.4). Using the result obtained above, we get, by (4.3), for all $t \in [0,T]$

$$e(t)\|Z_N(t) - Z(t)\|^2 + 12\lambda \int_0^t e(s)B(s)\|Z(s)\|_V^2 \|Z_N(s) - Z(s)\|^2 ds \qquad (4.4)$$
$$\leq 6\lambda \int_0^t e(s)B(s)\|Z(s)\|_V^2 \|Z_{N-1}(s) - Z(s)\|^2 ds.$$

This implies, for each $N \in \mathbb{N}^*$

$$\int_0^T e(s)B(s)\|Z(s)\|_V^2 \|Z_N(s) - Z(s)\|^2 ds$$
$$\leq \frac{1}{2} \int_0^T e(s)B(s)\|Z(s)\|_V^2 \|Z_{N-1}(s) - Z(s)\|^2 ds$$
$$\ldots \leq \frac{1}{2^N} \int_0^T e(s)B(s)\|Z(s)\|_V^2 \|X_0 - Z(s)\|^2 ds$$
$$\leq \frac{1}{2^{N-1}} B(T)\big(\|X_0\|^2 + \sup_{s\in[0,T]} \|Z(s)\|^2\big) \int_0^T \|Z(s)\|_V^2 ds$$
$$\leq \frac{TB(T)}{2^{N-2}} \|X_0\|^2 \|X_0\|_V^2.$$

Note that in the last inequality we take into consideration the estimates from Theorem 3.1. Using the above result in (4.4), we obtain for each $N \in \mathbb{N}^*$

$$\sup_{t\in[0,T]} \|Z_N(t) - Z(t)\|^2 \leq 6\lambda \frac{TB(T)}{2^{N-3} e(T)} \|X_0\|^2 \|X_0\|_V^2$$
$$\leq \frac{3\lambda TB(T)}{2^{N-4}} \|X_0\|^2 \|X_0\|_V^2 \exp\left(20\lambda TB(T)\|X_0\|_V^2\right).$$

□

Applying Theorem 3.3 and Theorem 4.2, we obtain the main result of our paper.

Theorem 4.3. *For a.e. $\omega \in \Omega$ and for each $N \in \mathbb{N}^*$ let*

$$X_N(t) := Z_N(t)Y^{-1}(t), \quad \textit{for all } t \in [0,T].$$

The following approximation result holds for a.e. $\omega \in \Omega$ and all $N \in \mathbb{N}^*$

$$\sup_{t\in[0,T]} \|X_N(t) - X(t)\|^2 \leq \frac{3T\lambda B^2(T)}{2^{N-4}} \|X_0\|^2 \|X_0\|_V^2 \exp\left(20\lambda TB(T)\|X_0\|_V^2\right),$$

where

$$B(T) = \exp\left(2\gamma T + \sum_{k=1}^{\infty} \int_0^T b_k^2(s)\, ds\right).$$

In particular, this implies

$$P\Big(\lim_{N\to\infty} \sup_{t\in[0,T]} \|X_N(t) - X(t)\|^2 = 0\Big) = 1$$

and

$$\lim_{N\to\infty} E \sup_{t\in[0,T]} \|X_N(t) - X(t)\|^2 = 0.$$

Remark 4.4. We can obtain similar results as in this paper, if:
1) we consider homogeneous Dirichlet or periodic boundary conditions;
2) instead of the nonlinear term $|X|^2X$ we take $|X|^{2\sigma}X$, where $\sigma \geq 1$, or combined power-type nonlinearities such as $|X|^{2\sigma_1}X + |X|^{2\sigma_2}X$, where $\sigma_1, \sigma_2 \geq 1$;
3) $\gamma \leq 0$;
4) in (2.12) the operator $-\mathrm{i}aA$ is replaced by $-(a_1 + \mathrm{i}a_2)A$, where $a_2 \in \mathbb{R}^*$ and $a_1 > 0$;
5) for each $k \geq 1$, we assume $b_k : \Omega \times [0,T] \to \mathbb{R}$ to be $\left(\mathcal{F}_t\right)_{t\in[0,T]}$ adapted processes satisfying

$$E\left(\exp\left(3\sum_{k=1}^{\infty}\int_0^T b_k^2(s)\,ds\right)\right) < \infty.$$

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