THE CONTINUITY OF THE METRIC PROJECTION OF A FIXED POINT ONTO MOVING CLOSED-CONVEX SETS IN UNIFORMLY-CONVEX BANACH SPACES

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We will show that a result similar to Hölder continuity in Hilbert spaces of the metric projections of a fixed point onto a pseudo-Lipschitz continuous family of closed convex sets [6] holds for uniformly-convex Banach spaces. The continuity of the metric projections with respect to perturbations play an important role in the sensitivity analysis of variational inequalities in Hilbert spaces [1, 3, 4, 6, 7] and hence in a wide range of nonlinear optimization, evolution and boundary value problems. The results from this paper offer us the possibility of extending the studies involving the metric projections in a larger class of spaces.

We denote by \((\Lambda, d)\) a metric space and by \(X\) a uniformly-convex Banach space. We suppose \(X^*\) locally-uniformly-convex. Let \(\omega_0, x_0 \in X\), \(\lambda_0 \in \Lambda\), and their neighborhoods \(\Omega_0 = B(\omega_0, r)\) (the closed ball centered at \(\omega_0\) and radius \(r\)) of \(\omega_0\), \(\Lambda_0\) of \(\lambda_0\). Let \(C : \Lambda_0 \rightrightarrows X\) be a set-valued mapping with nonempty, closed, convex values. Let us consider the following problem:

- for \(\lambda \in \Lambda_0\) and \(\omega \in \Omega_0\) find \(x(\omega, \lambda) = P_{C(\lambda)}(\omega) \in C(\lambda)\) such that

\[
\|\omega - x(\omega, \lambda)\| = \min_{x \in C(\lambda)} \|\omega - x\|. \tag{1}
\]

In our context such an element exists for all \(\omega \in \Omega_0\) and \(\lambda \in \Lambda_0\) and satisfies

\[
\langle J(\omega - x(\omega, \lambda)), x - x(\omega, \lambda) \rangle \leq 0, \quad \forall \ x \in C(\lambda), \tag{2}
\]

where \(J\) is the normalized duality mapping.

(2) is equivalent with

\[
0 \in -J(\omega - x(\omega, \lambda)) + N_{C(\lambda)}(x(\omega, \lambda)), \tag{3}
\]
where

\[ N_{C(\lambda)}(x) = \{ x^* \in X^* : \langle x^*, y - x \rangle \leq 0, \ \forall \ y \in C(\lambda) \} \]

is the normal cone to the set \( C(\lambda) \) at the point \( x \).

Hence we need to study the sensitivity with respect to \( \lambda \) of the following generalized equation:

\[ 0 \in -J(\omega - x) + N_{C(\lambda)}(x). \quad (4) \]

For Theorem 1 it is enough to consider \((\Omega, d)\) be a metric space, \( \omega_0 \in \Omega \) and \( \Omega_0 \) be a neighborhood of \( \omega_0 \). Let \( f : X_0 \times \Omega_0 \to X^* \) be a single-valued mapping.

**Definition 1.** The mappings \( f(\cdot, \omega) \) are \( \varphi \)-monotone on \( X_0 \) for all \( \omega \in \Omega_0 \), if there exists an increasing function \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \), with \( \varphi(r) > 0 \) when \( r > 0 \), such that

\[ \langle f(x_1, \omega) - f(x_2, \omega), x_1 - x_2 \rangle \geq \varphi(\|x_1 - x_2\|)|x_1 - x_2|, \]

for all \( x_1, x_2 \in X_0 \) and \( \omega \in \Omega_0 \).

The following proposition shows that the \( \varphi \)-monotonicity assumption is a natural one in uniformly-convex Banach spaces.

**Proposition 1.** [5] A Banach space \( X \) is uniformly-convex if and only if for each \( R > 0 \) there exists an increasing function \( \varphi_R : \mathbb{R}_+ \to \mathbb{R}_+ \), with \( \varphi_R(r) > 0 \) when \( r > 0 \), such that the normalized duality mapping \( J : X \rightharpoonup X^* \), defined by

\[ J(x) = \{ x^* \in X^* : \langle x^*, x \rangle = \|x\|^2, \|x\| = \|x^*\| \}, \]

is \( \varphi_R \)-monotone in \( B(0, R) \).

**Definition 2.** \( C \) is pseudo-continuous around \((\lambda_0, x_0) \in \text{Graph} C\) if there exist neighborhoods \( V \subset \Lambda_0 \) of \( \lambda_0 \), \( W \subset X_0 \) of \( x_0 \) and there exists a function \( \beta : \mathbb{R}_+ \to \mathbb{R}_+ \) continuous at 0, with \( \beta(0) = 0 \), such that

\[ C(\lambda_1) \cap W \subset C(\lambda_2) + \beta(d(\lambda_1, \lambda_2)) \ B(0, 1) \quad (5) \]
for all $\lambda_1, \lambda_2 \in V$.

If the function $\beta$ is defined as $\beta(r) = Lr$, with $L \geq 0$, ([2]) then we say that $C$ is pseudo-Lipschitz continuous around $(\lambda_0, x_0)$.

**Theorem 1.** Let us suppose that:

a) $0 \in f(x_0, \omega_0) + N_{C(\lambda_0)}(x_0)$;
b) $f$ is continuous on $X_0 \times \Omega_0$;
c) the mappings $f(\cdot, \omega)$ are $\varphi$-monotone in $X_0$ for all $\omega \in \Omega_0$;
d) $C$ is pseudo-continuous around $(\lambda_0, x_0)$.

Then there exist neighborhoods $\Lambda_1$ of $\lambda_0$, $\Omega_1$ of $\omega_0$ and a unique continuous mapping $x : \Omega_1 \times \Lambda_1 \to X_0$, such that $x(\omega_0, \lambda_0) = x_0$ and $x(\omega, \lambda)$ is a solution of the variational inequality

$$0 \in f(x, \omega) + N_{C(\lambda)}(x),$$

for all $(\omega, \lambda) \in \Omega_1 \times \Lambda_1$.

**Proof.** Let us note that assumptions b), c) imply that $\varphi(r) \to 0$, $\varphi(r)r \to 0$ iff $r \to 0$. We choose positive constants $s, r, \varepsilon$ such that $B(x_0, s) \subset X_0$, $B(\lambda_0, \varepsilon) \subset \Lambda_0$, $B(\omega_0, r) \subset \Omega_0$, $\beta(d(\lambda, \lambda_0)) \leq s$ for all $\lambda \in B(\lambda_0, \varepsilon)$ and the pseudo-continuity of $C$ to be written as:

$$C(\lambda_1) \cap B(x_0, s) \subset C(\lambda_2) + \beta(d(\lambda_1, \lambda_2)) B(0, 1)$$

for all $\lambda_1, \lambda_2 \in B(\lambda_0, \varepsilon)$.

Let $\lambda \in B(\lambda_0, \varepsilon)$ and $\omega \in B(\omega_0, r)$ be arbitrarily chosen. Then the inclusion

$$x_0 \in C(\lambda_0) \cap B(x_0, s) \subset C(\lambda) + Ld(\lambda, \lambda_0) B(0, 1)$$

implies the existence of an $u_\lambda \in C(\lambda)$ such that

$$\|x_0 - u_\lambda\| \leq \beta(d(\lambda, \lambda_0)) \leq s.$$
This means that \( C(\lambda) \cap B(x_0, s) \) is nonempty for all \( \lambda \in B(\lambda_0, \varepsilon) \). Corollary 32.35 from [8] shows that the variational inequality

\[
0 \in f(x, \omega) + N_{C(\lambda) \cap B(x_0, s)}(x)
\]

has a unique solution \( x(\omega, \lambda) \in C(\lambda) \cap B(x_0, s) \). So

\[
\langle f(x(\omega, \lambda), \omega), u - x(\omega, \lambda) \rangle \geq 0
\]

for all \( u \in C(\lambda) \cap B(x_0, s) \).

The pseudo-Lipschitz continuity of the set-valued mapping \( C \) implies that for \( x(\omega, \lambda) \) there exists an element \( u_0 \in C(\lambda_0) \) such that \( ||x(\omega, \lambda) - u_0|| \leq \beta(d(\lambda, \lambda_0)) \).

Using the \( \varphi \)-monotonicity of \( f(\cdot, \omega) \) we obtain

\[
\varphi(||x(\omega, \lambda) - x_0||) ||x(\omega, \lambda) - x_0|| \leq \\
\leq \langle f(x(\omega, \lambda), \omega) - f(x_0, \omega), x(\omega, \lambda) - x_0 \rangle \leq \\
\leq \langle f(x(\omega, \lambda), \omega) - f(x_0, \omega), x(\omega, \lambda) - x_0 \rangle + \langle f(x_0, \omega_0), u_0 - x_0 \rangle + \\
+ \langle f(x(\omega, \lambda), \omega), u_0 - x(\omega, \lambda) \rangle = \\
= \langle f(x(\omega, \lambda), \omega), u_0 - x_0 \rangle + \langle f(x_0, \omega), u_0 - x(\omega, \lambda) \rangle + \\
+ \langle f(x_0, \omega_0) - f(x_0, \omega), u_0 - x_0 \rangle \leq \\
\leq ||f(x(\omega, \lambda), \omega)|| ||u_0 - x_0|| + ||f(x_0, \omega)|| ||u_0 - x(\omega, \lambda)|| + \\
+ ||f(x_0, \omega_0) - f(x_0, \omega)|| ||u_0 - x_0||.
\]

Assumption \textit{a) implies that} \( ||f(x_0, \omega_0)|| < \infty \), and hence using the continuity of \( f \), we can suppose that \( ||f(x, \omega)|| \leq M < \infty \), for all \( x \in B(x_0, s) \) and \( \omega \in B(\omega_0, r) \).

We know also that

\[
||u_0 - x_0|| \leq ||u_0 - x(\omega, \lambda)|| + ||x(\omega, \lambda) - x_0|| \leq \\
\leq \beta(d(\lambda, \lambda_0)) + s.
\]

So,

\[
\varphi(||x(\omega, \lambda) - x_0||) ||x(\omega, \lambda) - x_0|| \leq \\
\leq 2M \beta(d(\lambda, \lambda_0)) + ||f(x_0, \omega_0) - f(x_0, \omega)|| \beta(d(\lambda, \lambda_0)) + s.
\]
This means that \( x(\omega, \lambda) \to x_0 \), when \((\omega, \lambda) \to (\omega_0, \lambda_0)\). Thus we can choose neighborhoods \( \Omega_1 \subset B(\omega_0, r) \) of \( \omega_0 \) and \( \Lambda_1 \subset B(\lambda_0, \varepsilon) \) of \( \lambda_0 \) such that \( x(\omega, \lambda) \in \text{int} B(x_0, s) \), for all \((\omega, \lambda) \in \Omega_1 \times \Lambda_1\). Hence

\[
0 \in f(x(\omega, \lambda), \omega) + N_{C(\lambda)}(x(\omega, \lambda)),
\]

because

\[
N_{C(\lambda)}(x(\omega, \lambda)) = N_{C(\lambda) \cap B(x_0, s)}(x(\omega, \lambda)),
\]

for all \((\omega, \lambda) \in \Omega_1 \times \Lambda_1\).

Let us choose \( \lambda_1, \lambda_2 \in \Lambda_1 \) and \( \omega_1, \omega_2 \in \Omega_1 \).

For \( x(\omega_1, \lambda_1) \in C(\lambda_1) \cap B(x_0, s) \) there exists \( u_2 \in C(\lambda_2) \), such that

\[
||x(\omega_1, \lambda_1) - u_2|| \leq \beta(d(\lambda_1, \lambda_2)).
\]

For \( x(\omega_1, \lambda_2) \in C(\lambda_2) \cap B(x_0, s) \) there exists \( u_1 \in C(\lambda_1) \) such that

\[
||x(\omega_1, \lambda_2) - u_1|| \leq \beta(d(\lambda_1, \lambda_2)).
\]

Then

\[
\varphi(||x(\omega_1, \lambda_1) - x(\omega_1, \lambda_2)||) ||x(\omega_1, \lambda_1) - x(\omega_1, \lambda_2)|| \leq \leq \langle f(x(\omega_1, \lambda_1), \omega_1) - f(x(\omega_1, \lambda_2), \omega_1), x(\omega_1, \lambda_1) - x(\omega_1, \lambda_2) \rangle +
\]

\[
+ \langle f(x(\omega_1, \lambda_1), \omega_1), u_1 - x(\omega_1, \lambda_1) \rangle +
\]

\[
+ \langle f(x(\omega_1, \lambda_2), \omega_1), u_2 - x(\omega_1, \lambda_2) \rangle =
\]

\[
= \langle f(x(\omega_1, \lambda_1), \omega_1), u_1 - x(\omega_1, \lambda_2) \rangle +
\]

\[
+ \langle f(x(\omega_1, \lambda_2), \omega_1), u_2 - x(\omega_1, \lambda_1) \rangle \leq
\]

\[
\leq 2M \beta(d(\lambda_1, \lambda_2)).
\]

Hence we obtain that \( x(\omega_1, \lambda_1) \to x(\omega_1, \lambda_2) \), when \( \lambda_1 \to \lambda_2 \), uniformly for all \( \omega_1 \in \Omega_1 \).

We have also that

\[
\varphi(||x(\omega_1, \lambda_2) - x(\omega_2, \lambda_2)||) ||x(\omega_1, \lambda_2) - x(\omega_2, \lambda_2)|| \leq \leq \langle f(x(\omega_1, \lambda_2), \omega_1) - f(x(\omega_2, \lambda_2), \omega_1), x(\omega_1, \lambda_2) - x(\omega_2, \lambda_2) \rangle +
\]

\[
+ \langle f(x(\omega_1, \lambda_2), \omega_1), x(\omega_2, \lambda_2) - x(\omega_1, \lambda_2) \rangle +
\]

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\[ + \langle f(x(\omega_2, \lambda_2), \omega_2), x(\omega_1, \lambda_2) - x(\omega_2, \lambda_2) \rangle = \]
\[ = \langle f(x(\omega_2, \lambda_2), \omega_2) - f(x(\omega_2, \lambda_2), \omega_1), x(\omega_1, \lambda_2) - x(\omega_2, \lambda_2) \rangle \leq \]
\[ \leq ||f(x(\omega_2, \lambda_2), \omega_2) - f(x(\omega_2, \lambda_2), \omega_1)|| ||x(\omega_1, \lambda_2) - x(\omega_2, \lambda_2)||. \]

Thus \[x(\omega_1, \lambda_2) \to x(\omega_2, \lambda_2), \text{ when } \omega_1 \to \omega_2.\]

The two convergence imply the continuity of \(x(\cdot, \cdot)\) at \((\omega_2, \lambda_2)\). This point being chosen arbitrarily the continuity hold in \(\Omega_1 \times \Lambda_1.\)

As a corollary of the previous theorem we can prove the continuity of the metric projection with respect to perturbations.

Let \(\Omega = X\) and \(\omega_0 \in X\).

**Corollary 1.** Let us suppose that:

i) \(x_0 = P_{C(\lambda)}(\omega_0);\)

ii) \(C\) is pseudo-continuous around \((\lambda_0, x_0)\).

Then there exists neighborhoods \(\Omega'_0\) of \(\omega_0\), \(\Lambda'_0\) of \(\lambda_0\), such that \(x(\cdot, \cdot) = P_{C(\cdot)}(\cdot)\) is continuous on \(\Omega'_0 \times \Lambda'_0\) and hence \(x(\omega, \cdot) = P_{C(\cdot)}(\omega)\) is continuous on \(\Lambda'_0\) for all \(\omega \in \Omega'_0\).

**Proof.** In the case of a uniformly-convex Banach space with locally-uniformly convex dual the normalized duality mapping is single-valued, \(\varphi\)-monotone on each closed-ball and continuous from the strong topology of \(X\) to the strong topology of \(X^*\).

So, we can define the mapping \(f(x, \omega) = -J(\omega - x)\) and we can use Theorem 1 to prove the continuity of \(x(\cdot, \cdot)\) on \(\Omega'_0 \times \Lambda'_0\).

Hence for all \(\omega \in \Omega'_0\) the metric projections \(P_{C(\lambda)}(\omega)\) vary continuously with respect to \(\lambda\) on \(\Lambda'_0\).

As we have seen, even when \(C\) is pseudo-Lipschitz continuous, this continuity is not the same \(\frac{1}{2}\)-Hölder type as in [6], because the normalized duality mapping is not strongly-monotone in a general uniformly-convex Banach spaces.

In the case of a Hilbert space, the \(\frac{1}{2}\)-Hölder-continuity with respect to \(\lambda\) is a consequence of Theorem 1 and Corollary 1.
References


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