Bounds on third Hankel determinant for certain classes of analytic functions

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Abstract. In this paper, the estimate for the third Hankel determinant $H_{3,1}(f)$ of Taylor coefficients of function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, belonging to certain classes of analytic functions in the open unit disk $\mathbb{D}$, are investigated.

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1. Introduction

Let $H(\mathbb{D})$ denote the class of analytic functions in the open unit disk
\[ \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \]
and $\mathcal{A}$ be the class of functions $f \in H(\mathbb{D})$, having the form
\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}, \quad (1.1) \]
with the standard normalization $f(0) = 0$, $f'(0) = 1$. We denote by $\mathcal{S}$, the subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{D}$, and $\mathcal{P}$ denotes the class of functions $p \in H(\mathbb{D})$ with $\Re(p(z)) > 0$, $z \in \mathbb{D}$.

A function $f \in \mathcal{A}$ is called starlike (with respect to origin 0), if $f$ is univalent in $\mathbb{D}$ and $f(\mathbb{D})$ is a starlike domain. We denote this class of starlike functions by $\mathcal{S}^*$. A function $f \in \mathcal{S}$ maps the unit disk $\mathbb{D}$ onto a convex domain is called convex function, and this class of functions is denoted by $\mathcal{K}$. Let $\mathcal{M}(\lambda)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the inequality
\[ \Re \left( \frac{zf'(z)}{f(z)} \right) < \lambda, \quad z \in \mathbb{D}, \quad (1.2) \]
for some $\lambda (\lambda > 1)$. And let $\mathcal{N}(\lambda)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ if and only if $zf''(z) \in \mathcal{M}(\lambda)$, i.e. $f(z)$ satisfy the inequality

$$\Re \left(1 + \frac{zf''(z)}{f'(z)}\right) < \lambda, \quad z \in \mathbb{D},$$

for some $\lambda (\lambda > 1)$. These classes $\mathcal{M}(\lambda)$ and $\mathcal{N}(\lambda)$ were investigated recently by Nishiwaki and Owa [19] (see also [23]). For $1 < \lambda \leq 4/3$, the classes $\mathcal{M}(\lambda)$ and $\mathcal{N}(\lambda)$ were investigated by Uralegaddi et al. [32].

Throughout the present paper, by $\mathcal{M}$ we always mean the class of functions $\mathcal{M}(3/2)$, and by $\mathcal{N}$ we always mean the class of functions $\mathcal{N}(3/2)$. Ozaki [24] proved that functions in $\mathcal{N}$ are univalent in $\mathbb{D}$. Moreover, if $f \in \mathcal{N}$, then (see e.g. [11, Theorem 1] and [21, p. 196]) one have

$$zf'(z) \prec g(z) = \frac{2(1-z)}{2-z}, \quad z \in \mathbb{D},$$

where $\prec$ denotes the subordination [18]. We see that $g$ above is univalent in $\mathbb{D}$ and maps $\mathbb{D}$ onto the disk $|w - (2/3)| < 2/3$. Thus, functions in $\mathcal{M}$ are starlike in $\mathbb{D}$.

For $f \in \mathcal{A}$ of the form (1.1), a classical problem settled by Fekete and Szegö [9] is to find the maximum value of the coefficient functional $\Phi_\lambda(f) := a_3 - \lambda a_2^2$ for each $\lambda \in [0, 1]$, over the function $f \in \mathcal{S}$. By applying the Löewner method they proved that

$$\max_{f \in \mathcal{S}} |\Phi_\lambda(f)| = \begin{cases} 
1 + 2 \exp \left(\frac{-2\lambda}{1-\lambda}\right), & \lambda \in [0, 1), \\
1, & \lambda = 1.
\end{cases}$$

The problem of calculating the maximum of the coefficient functional $\Phi_\lambda(f)$ for various compact subfamilies of $\mathcal{A}$, as well as $\lambda$ being an arbitrary real or complex number, has been studied by many authors (see e.g. [1, 12, 13, 17, 30, 31]).

We denote by $H_{q,n}(f)$ where $n, q \in \mathbb{N} = \{1, 2, \cdots \}$, the Hankel determinant of functions $f \in \mathcal{A}$ of the form (1.1), which is defined by

$$H_{q,n}(f) = \left| \begin{array}{cccc}
a_n & a_{n+1} & \cdots & a_{n+q-1} \\
a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \\
\end{array} \right|$$

$$\quad(a_1 = 1). \quad (1.4)$$

The Hankel determinant $H_{q,n}(f)$ has been studied by several authors including Cantor [6], Noonan and Thomas [20], Pommerenke [26, 25], Hayman [10], Ehrenborg [8], which are useful, in showing that a function of bounded characteristic in $\mathbb{D}$.

Indeed, $H_{2,1}(f) = \Phi_1(f)$ is the Fekete-Szegö coefficient functional. Many authors have studied the problem of calculating $\max_{f \in \mathcal{F}} |H_{2,2}(f)|$ for various subfamily $\mathcal{F}$ of the class $f \in \mathcal{A}$ (see e.g. [2, 4, 14]). Recently, several authors including Babalola [3], Bansal et al. [5], Prajapat et al. [28], Raza and Malik [29] have obtained the bounds on the third Hankel determinant $H_{3,1}(f)$ for certain families of analytic functions,
which is defined by

\[
H_{3,1}(f) = \begin{vmatrix}
  a_1 & a_2 & a_3 \\
  a_2 & a_3 & a_4 \\
  a_3 & a_4 & a_5 \\
\end{vmatrix}
\]

\[
= a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2). \tag{1.5}
\]

In the present paper, we investigate the bounds on \(H_{3,1}(f)\) for the functions belonging to the classes \(\mathcal{M}\) and \(\mathcal{N}\) defined above. In order to get the main results, we need the following known results.

**Lemma 1.1.** ([16]) If \(p \in \mathcal{P}\) be of the form \(p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n\), then

\[
2c_2 = c_1^2 + x(4 - c_1^2),
\]

and

\[
4c_3 = c_1^3 + 2c_1 x(4 - c_1^2) - c_1 x^2(4 - c_1^2) + 2(4 - c_1^2)(1 - |x|^2)z,
\]

for some \(x, z\) with \(|x| \leq 1\) and \(|z| \leq 1\).

**Lemma 1.2.** ([22, Theorem 1]) If \(f \in \mathcal{N}\) be given by (1.1), then

\[
|a_n| \leq \frac{1}{n(n-1)}, \quad n \geq 2.
\]

The result is sharp for the function \(f_n\) such that \(f_n'(z) = (1 - z^{n-1})^{1/(n-1)}, n \geq 2\).

As it is known that, if \(f(z) \in \mathcal{N}\) then \(zf'(z) \in \mathcal{M}\), therefore from Lemma 1.2, we conclude that

**Lemma 1.3.** If \(f(z) \in \mathcal{M}\) be given by (1.1), then

\[
|a_n| \leq \frac{1}{n-1}, \quad n \geq 2.
\]

The result is sharp for the function \(g_n(z) = z(1 - z^{n-1})^{1/(n-1)}, n \geq 2\).

**Lemma 1.4.** ([22, Corollary 2]) If \(f \in \mathcal{N}\) be given by (1.1), then

\[
|a_3 - a_2^2| \leq 1/4.
\]

Equality is attained for the function \(f\) such that \(f'(z) = (1 - z^2 e^{i\theta})^{1/2}, \theta \in [0, 2\pi]\).

2. Main results

Our first main result is contained in the following theorem:

**Theorem 2.1.** Let the function \(f \in \mathcal{M}\) be given by (1.1), then

\[
|a_3 - a_2^2| \leq 1. \tag{2.1}
\]

The result (2.1) is sharp and equality in (2.1) is attained for the function \(e_1(z) = z - z^2\).
Proof. If the function \( f \in \mathcal{M} \) be given by (1.1), then we may write
\[
\frac{zf'(z)}{f(z)} = \frac{3}{2} - \frac{1}{2}p(z),
\]
where \( p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \) is analytic in \( \mathbb{D} \) and \( \Re(p(z)) > 0 \) in \( \mathbb{D} \). Also, we have \( |c_n| \leq 2 \) for all \( n \geq 1 \) (see [7]). In terms of power series expansion, the last identity is equivalent to
\[
\sum_{n=1}^{\infty} na_n z^n = \left( 1 - \frac{1}{2} \sum_{n=1}^{\infty} c_n z^n \right) \left( \sum_{n=1}^{\infty} a_n z^n \right),
\]
where \( a_1 = 1 \). Equating the coefficients of \( z^n \) on both sides, we deduce that
\[
\begin{align*}
a_2 &= \frac{1}{2} c_1, \\
a_3 &= \frac{1}{8} (c_1^2 - 2c_2), \\
a_4 &= \frac{1}{48} (6c_1 c_2 - 8c_3 - c_1^3).
\end{align*}
\]
Now using Lemma 1.1 for some \( x \) such that \( |x| \leq 1 \), we have
\[
|a_3 - a_2^2| = \left| \frac{1}{8} (c_1^2 - 2c_2) - \frac{1}{4} c_1^2 \right| = \frac{1}{8} |2c_1^2 + x(4 - c_1^2)|.
\]
As \( |c_1| \leq 2 \), taking \( c_1 = c \), assume without restriction that \( c \in [0, 2] \). Hence applying the triangle inequality with \( \mu = |x| \), we obtain
\[
|a_3 - a_2^2| \leq \frac{1}{8} [2c_1^2 + \mu(4 - c^2)] = F_1(c, \mu).
\]
Let \( \Omega = \{(c, \mu) : 0 \leq c \leq 2 \text{ and } 0 \leq \mu \leq 1\} \). Differentiating \( F_1 \) with respect to \( \mu \), we get
\[
\frac{\partial F_1}{\partial \mu} = \frac{1}{8} (4 - c^2) \geq 0 \quad \text{for} \quad 0 \leq \mu \leq 1.
\]
Therefore \( F_1(c, \mu) \) is a non-decreasing function of \( \mu \) on the closed interval \([0, 1]\). Thus, it attains maximum value at \( \mu = 1 \). Let
\[
\max_{0 \leq \mu \leq 1} F_1(c, \mu) = F_1(c, 1) = \frac{c^2 + 4}{8} = G_1(c).
\]
We observe that \( G_1(c) \) is an increasing function in \([0, 2]\), so it will attains maximum value at \( c = 2 \). Next, to find the critical point on the boundary of \( \Omega \), we examine all the four line segments of \( \Omega \). Along the line segment \( c = 2 \) with \( 0 \leq \mu \leq 1 \), we have \( F_1(c, \mu) = F_1(2, \mu) = 1 \), which is a constant, thus every point on the line segment is the critical point. For the line segment \( c = 0 \) with \( 0 \leq \mu \leq 1 \), we have \( F_1(c, \mu) = F_1(0, \mu) = \mu/2 \). For the line segment \( \mu = 0 \) with \( 0 \leq c \leq 2 \), we have \( F_1(c, \mu) = F_1(c, 0) = c^2/4 \), which gives the critical point \((0, 0)\) and \( F_1(0, 0) = 0 \). Also, for the line segment \( \mu = 1 \) with \( 0 \leq c \leq 2 \), we have \( F_1(c, \mu) = F_1(c, 1) = (c^2 + 4)/8 \), which gives another critical point \((0, 1)\) and \( F_1(0, 1) = 1/2 \).

Putting this all together we can conclude that the maximum of \( F_1(c, \mu) \) lie at each point along the line segment \( c = 2 \) with \( 0 \leq \mu \leq 1 \), which can also be verified.
through the mathematica plot of $F_1(c, \mu)$ over the region $\Omega$ given below in the Figure 1. Hence

$$\max_{\Omega} F_1(c, \mu) = F_1(2, \mu) = 1.$$ 

Figure 1. Mapping of $F_1(c, \mu)$ over $\Omega$

To find the extremal function, setting $c_1 = 2$ and $x = 1$ in Lemma 1.1, we find that $c_2 = c_3 = 2$, using these values in (2.3), we get that $a_2 = -1$ and $a_3 = a_4 = 0$, therefore the extremal function would be $e_1(z) = z - z^2$. A simple calculation shows that $e_1(z) \in \mathcal{M}$. This complete the proof of Theorem 2.1.

**Theorem 2.2.** Let the function $f \in \mathcal{M}$ be given by (1.1), then

$$|a_2a_4 - a_3^2| \leq \frac{1}{4}. \quad (2.4)$$

The result (2.4) is sharp and equality is attained for the function

$$e_2(z) = z - \frac{1}{2}z^3 \quad \text{and} \quad e_3(z) = z(1 - z^2)^{1/2}.$$

**Proof.** Using (2.3) and applying Lemma 1.1 for some $x$ and $z$ such that $|x| \leq 1$ and $|z| \leq 1$, we have

$$|a_2a_4 - a_3^2| = \left| -\frac{1}{96}c_1(6c_1c_2 - 8c_3 - c_1^3) - \frac{1}{64}(c_1^2 - 2c_2)^2 \right|$$

$$= \frac{1}{192} \left| -3x^2(4 - c_1^2)^2 + 2c_1^2x(4 - c_1^2) - 4c_1^2x^2(4 - c_1^2) + 8c_1(4 - c_1^2)(1 - |x|^2)z \right|. $$
As \(|c_1| \leq 2\), taking \(c_1 = c\), assume without restriction that \(c \in [0, 2]\). Thus applying the triangle inequality with \(\mu = |x|\), we obtain

\[
|a_2a_4 - a_3^2| \leq \frac{1}{192} \left((4 - c^2)\{3\mu^2(4 - c^2) + 2c^2\mu + 4\mu^2c^2 + 8c(1 - \mu^2)\}\right)
\]

\[
= \frac{1}{192} \left[(4 - c^2)\{(12 - 8c + c^2)\mu^2 + 2c^2\mu + 8c\}\right]
\]

\[
= F_2(c, \mu).
\]

Differentiating \(F_2(c, \mu)\) in the above equation with respect to \(\mu\), we get

\[
\frac{\partial F_2}{\partial \mu} = \frac{(4 - c^2)}{96} \left\{(12 - 8c + c^2)\mu + c^2\right\} \geq 0 \quad \text{for} \quad 0 \leq \mu \leq 1.
\]

Therefore \(F_2(c, \mu)\) is a non-decreasing function of \(\mu\) on closed interval \([0, 1]\). Thus, it attains maximum value at \(\mu = 1\). Let

\[
\max_{0 \leq \mu \leq 1} F_2(c, \mu) = F_2(c, 1) = \frac{16 - c^4}{64} = G_2(c).
\]

We observe that \(G_2(c)\) is a decreasing function in \([0, 2]\), so it will attains maximum value at \(c = 0\). Next, to find the critical point on the boundary of \(\Omega\), we examine all the four line segments of \(\Omega\) by the earlier method used in Theorem 2.1, and we are getting \((0, 0)\), \((2/\sqrt{3}, 0)\) and \((0, 1)\) are the critical points and \(F_2(0, 0) = 0\), \(F_2(2/\sqrt{3}, 0) = 2/9\sqrt{3}\) and \(F_2(0, 1) = 1/4\). Therefore maximum value of \(F_2(c, \mu)\) is obtained by putting \(c = 0\) and \(\mu = 1\), which can also verified through the mathematica plot of \(F_2(c, \mu)\) over \(\Omega\) given below in Figure 2. Hence

\[
\max_{\Omega} F_2(c, \mu) = F_2(0, 1) = \frac{1}{4}.
\]

\[\text{Figure 2. Mapping of } F_2(c, \mu) \text{ over } \Omega\]
Now, to find extremal function, set $c_1 = 0$ and selecting $x = 1$ in Lemma 1.1, we find that $c_2 = 2$ and $c_3 = 0$. Using these values in (2.3), we get $a_2 = a_4 = 0$ and $a_3 = 1/2$, therefore one of the extremal function of (2.4) would be $e_2(z) = z - \frac{1}{2}z^3$. We can also see that equality in (2.4) is attended for the function $e_3(z) = z(1 - z^2)^{1/2} \in \mathcal{M}$. A simple calculation shows that $e_2 \in \mathcal{M}$ and $e_3 \in \mathcal{M}$. This complete the proof of Theorem 2.2.

**Theorem 2.3.** Let the function $f \in \mathcal{M}$ be given by (1.1), then

$$|a_2a_3 - a_4| \leq \frac{2\sqrt{3}}{9}.$$  

(*)

Proof. Using (2.3) and applying Lemma 1.1 for some $x$ and $z$ such that $|x| \leq 1$ and $|z| \leq 1$, we have

$$|a_2a_3 - a_4| = \left| \frac{1}{16}c_1(c_1^2 - 2c_2) + \frac{1}{48}(6c_1c_2 - 8c_3 - c_1^3) \right| = \frac{1}{24} \left| 2c_1x(4 - c_1^2) - c_1x^2(4 - c_1^2) + 2(4 - c_1^2)(1 - |x|^2)z \right|.$$  

As $|c_1| \leq 2$, letting $c_1 = c$, assume without restriction that $c \in [0, 2]$. Hence applying the triangle inequality with $\mu = |x|$, we obtain

$$|a_2a_3 - a_4| \leq \frac{(4 - c^2)}{24} \left[ 2 + 2c\mu + (c - 2)\mu^2 \right] = F_3(c, \mu).$$

To find the maximum of $F_3$ over the region $\Omega$, differentiating $F_3$ with respect to $\mu$ and $c$, we get

$$\frac{\partial F_3}{\partial \mu} = \frac{(4 - c^2)}{12} \left[ c + (c - 2)\mu \right] \quad (2.6)$$

$$\frac{\partial F_3}{\partial c} = \frac{1}{24} \left[ -4c + (8 - 6c^2)\mu + (4 + 4c - 3c^2)\mu^2 \right].$$

(2.7)

A critical point of $F_3(c, \mu)$ must satisfy $\frac{\partial F_3}{\partial \mu} = 0$ and $\frac{\partial F_3}{\partial c} = 0$. The condition $\frac{\partial F_3}{\partial \mu} = 0$ gives $c = \pm 2$ or $\mu = -c/(c-2)$. The interior point $(c, \mu) \in \Omega$ satisfying such condition in only $(0, 0)$, and at that point $(0, 0)$, we have

$$\left( \frac{\partial^2 F_3}{\partial \mu^2} \right) \left( \frac{\partial^2 F_3}{\partial c^2} \right) - \left( \frac{\partial^2 F_3}{\partial c \partial \mu} \right)^2 = 0.$$  

Hence, it is not certain that at $(0, 0)$ function have maximum value in $\Omega$. Since $\Omega$ is closed and bounded and $F_3$ is continuous, the maximum of $F_3$ shall be attained on the boundary of $\Omega$. Along the line segment $c = 2$ with $0 \leq \mu \leq 1$, we have $F_3(c, \mu) = F_3(2, \mu) = 0$, which is a constant. For the line segment $c = 0$ with $0 \leq \mu \leq 1$, we have $F_3(c, \mu) = F_3(0, \mu) = (1 - \mu^2)/3$, which gives the same critical point $(0, 0)$ and $F_3(0, 0) = 1/3$. For the line segment $\mu = 0$ with $0 \leq c \leq 2$, we have $F_3(c, \mu) = F_3(c, 0) = (4 - c^2)/12$, which gives the same critical point $(0, 0)$. Also, for the line segment $\mu = 1$ with $0 \leq c \leq 2$, we have $F_3(c, \mu) = F_3(c, 1) = (4c - c^3)/8,
which gives another critical point \((2/\sqrt{3}, 1)\) on this line and \(F_3(2/\sqrt{3}, 1) = 2\sqrt{3}/9\). Therefore, the point \((0,0)\) and \((2/\sqrt{3}, 1)\) are the only critical points of \(F_3\) over \(\Omega\). Hence, the largest value of \(F_3(c, \mu)\) over the region \(\Omega\) lies at \((2/\sqrt{3}, 1)\) and

\[
\max_{\Omega} F_3(c, \mu) = F_3(2/\sqrt{3}, 1) = \frac{2\sqrt{3}}{9}.
\]

\[\square\]

**Figure 3.** Mapping of \(F_3(c, \mu)\) over \(\Omega\)

**Theorem 2.4.** Let the function \(f \in \mathcal{M}\) be given by (1.1), then

\[
|H_{3,1}(f)| \leq \frac{81 + 16\sqrt{3}}{216}.
\]

**Proof.** Using Lemma 1.3, Theorem 2.1, Theorem 2.2, Theorem 2.3 and the triangle inequality on \(H_{3,1}(f)\), we get

\[
|H_{3,1}(f)| \leq |a_3||a_2a_4 - a_5^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|
\]

\[
\leq \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{2\sqrt{3}}{9} + \frac{1}{4} \cdot 1 = \frac{81 + 16\sqrt{3}}{216}.
\]

This completes the proof of Theorem 2.4. \[\square\]

**Theorem 2.5.** Let the function \(f \in \mathcal{N}\) be given by (1.1), then

\[
|a_2a_3 - a_4| \leq \frac{1}{12}.
\]  

(2.8)

The result (2.8) is sharp and equality in (2.8) is attained for the function \(e_4\) where \(e_4'(z) = (1 - z^3)^{1/3}\).
Proof. Let the function $f \in \mathcal{N}$ be given by (1.1), then by definitions it is clear that $f(z) \in \mathcal{N}$ if and only if $zf'(z) \in \mathcal{M}$, thus replacing $a_n$ by $na_n$ in (2.3), we get

$$a_2 = -\frac{1}{4}c_1, \quad a_3 = \frac{1}{24}(c_1^2 - 2c_2), \quad a_4 = \frac{1}{192}(6c_1c_2 - 8c_3 - c_1^3). \quad (2.9)$$

Now using (2.9) and applying Lemma 1.1 for some $x$ and $z$ such that $|x| \leq 1$ and $|z| \leq 1$, we have

$$|a_2a_3 - a_4| = \left| -\frac{1}{96}c_1(c_1^2 - 2c_2) - \frac{1}{192}(6c_1c_2 - 8c_3 - c_1^3) \right|$$

$$= \frac{1}{192} \left| 3c_1x(4 - c_1^2) - 2c_1x^2(4 - c_1^3) + 4(4 - c_1^2)(1 - |x|^2)z \right|.$$  

As $|c_1| \leq 2$, taking $c_1 = c$, assume without restriction that $c \in [0, 2]$. Hence applying the triangle inequality with $\mu = |x|$, we obtain

$$|a_2a_3 - a_4| \leq \frac{(4 - c^2)}{192} \left[ 4 + 3c\mu + 2(c - 2)\mu^2 \right]$$

$$= F_4(c, \mu).$$

Following the earlier method used in Theorem 2.3, we can show that the global maximum of $F_4(c, \mu)$ over the region $\Omega$ is achieved at $(0, 0)$ and $F_4(0, 0) = 1/12$. This can also be verified through the mathematica plot of $F_4(c, \mu)$ over $\Omega$ given below in Figure 4.

![Figure 4. Mapping of $F_4(c, \mu)$ over $\Omega$](image)

Also observe that equality in (2.8) is attained for the function $e_4$ where

$$e_4'(z) = (1 - z^3)^{1/3}.$$  

A computation shows that $e_4 \in \mathcal{N}$. Hence the result is obtained. \qed
Differentiating the triangle inequality with \( f \) for \( n \) and assuming without restriction that \( c \in [0, 2] \), we get \( |x| \leq 1 \), and we get (2.9) and applying Lemma 1.1 for some \( x \) and \( z \) such that \( |x| \leq 1 \) and \( |z| \leq 1 \), we have

\[
|a_2 a_4 - a_3^2| = \frac{1}{192} \left| -\frac{1}{4} c_1 (6 c_1 c_2 - 8 c_3 - c_4^3) - \frac{1}{3} (c_1^2 - 2 c_2)^2 \right|
\]

As \( |c_1| \leq 2 \), taking \( c_1 = c \), we obtain

\[
|a_2 a_4 - a_3^2| \leq \frac{1}{2304} \left\{ 12 c + 3 c^2 \mu + 2 (8 - 6 c + c^2) \mu^2 \right\}
\]

\[
= F_5(c, \mu).
\]

Differentiating \( F_5(c, \mu) \) with respect to \( \mu \), we get

\[
\frac{\partial F_5}{\partial \mu} = \frac{(4 - c^2)}{2304} \left\{ 4 \mu (c^2 - 6 c + 8) + 3 c^2 \right\} \geq 0 \quad \text{for} \quad 0 \leq \mu \leq 1.
\]

Therefore \( F_5(c, \mu) \) is a non-decreasing function of \( \mu \) on closed interval \([0, 1]\). Thus, it attains maximum value at \( \mu = 1 \). Let

\[
\max_{0 \leq \mu \leq 1} F_5(c, \mu) = F_5(c, 1) = \frac{1}{2304} (64 + 4c^2 - 5c^4) = G_5(c).
\]

We can see that \( G_5(c) \) is an increasing function in \([0, \sqrt{2/5}]\), so \( G_5(c) \) attains maximum value at \( c = \sqrt{2/5} \). Next, to find the critical points on the boundary of \( \Omega \), we examine all the four line segments of \( \Omega \) by the earlier method used in Theorem 2.1 and 2.3, and we get \((0, 0), (2/\sqrt{3}, 0)\) and \((0, 1)\) are the critical points and \( F_5(0, 0) = 0, F_5(2/\sqrt{3}, 0) = 1/36 \sqrt{3} \) and \( F_5(0, 1) = 1/36 \). Therefore \( F_5(c, \mu) \) have maximum value at \( \mu = 1 \) and \( c = \sqrt{2/5} \) in the region \( \Omega \). Thus

\[
\max_{\Omega} F_5(c, \mu) = F_5(\sqrt{2/5}, 1) = \frac{9}{320}.
\]

This completes the proof of Theorem 2.6. \( \square \)

**Remark 2.7.** For \( f \in S \), Thomas [27, p. 166] conjectured that

\[
|H_{2,n}(f)| = |a_n a_{n+2} - a_{n+1}^2| \leq 1, \quad n = 2, 3, 4 \ldots.
\]

Subsequently, Li and Srivastava [15, p. 1040] shown that this conjecture is not valid for \( n \geq 4 \), i.e. conjecture is valid only for \( n = 2, 3 \). From Theorem 2.6, we found that if function \( f \) is member of class \( N \) and having form (1.1), then \( |H_{2,2}(f)| \leq 9/320 \).
Figure 5. Mapping of $F_5(c,\mu)$ over $\Omega$

Since all functions in $\mathcal{N}$ are univalent in $\mathbb{D}$. Therefore, Theorem 2.6 validates the Thomas conjecture when $n = 2$ for the function belonging to the classes $\mathcal{N}$.

**Theorem 2.8.** Let the function $f \in \mathcal{N}$ be given by (1.1), then

$$|H_{3,1}(f)| \leq \frac{139}{5760}.$$  

*Proof.* Using Lemma 1.2, Lemma 1.4, Theorem 2.5, Theorem 2.6 and the triangle inequality on $H_{3,1}(f)$, we get

$$|H_{3,1}(f)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|$$

$$\leq \frac{1}{6}\frac{9}{320} + \frac{1}{12}\frac{1}{12} + \frac{1}{20}\frac{1}{4} = \frac{139}{5760}.$$  

This completes the proof of Theorem 2.8. \qed

**References**


Bounds on third Hankel determinant


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