Majorization for certain classes of analytic functions defined by convolution structure

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Abstract. In this paper, we investigate majorization properties for certain classes of analytic functions defined by convolution structure. Also we point out some new and known consequences of our main result.

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1. Introduction

Let \( f(z) \) and \( g(z) \) be analytic in the open unit disc \( U = \{ z \in \mathbb{C} : |z| < 1 \} \).

For analytic function \( f(z) \) and \( g(z) \) in \( U \), we say that \( f(z) \) is majorized by \( g(z) \) in \( U \) (see [10]) and write

\[
f(z) \ll g(z) \quad (z \in U),
\]

if there exists a function \( \varphi(z) \), analytic in \( U \) such that

\[
|\varphi(z)| \leq 1 \quad \text{and} \quad f(z) = \varphi(z)g(z) \quad (z \in U).
\]

(1.2)

It may be noted that (1.1) is closely related to the concept of quasi-subordination between analytic functions.

If \( f(z) \) and \( g(z) \) are analytic functions in \( U \), we say that \( f(z) \) is subordinate to \( g(z) \), written symbolically as \( f(z) \prec g(z) \) if there exists a Schwarz function \( w \), which (by definition) is analytic in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) for all \( z \in U \), such that \( f(z) = g(w(z)) \), \( z \in U \). Furthermore, if the function \( g(z) \) is univalent in \( U \), then we have the following equivalence, (see [11, p.4]):

\[
f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).
\]

Let \( A(p) \) denote the class of functions \( f(z) \) of the form:

\[
f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (p \in \mathbb{N} = \{1, 2, \ldots\})
\]

(1.3)
which are analytic and \( p \)-valent in the open unit disc. We note that \( A(1) = A \). Let 
\[ g(z) \in A(p), \]
be given by
\[ g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k, \]
the Hadamard product (or convolution) of \( f(z) \) and \( g(z) \) is given by
\[ (f \ast g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g \ast f)(z). \] (1.4)

For \( \lambda, \ell \geq 0, p \in \mathbb{N}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) and \( f(z), g(z) \in A(p) \), A. O. Mostafa, [12] defined the linear operator \( D_{\lambda,\ell,p}^m(f \ast g) \) as follows:
\[ D_{\lambda,\ell,p}^m(f \ast g) = z^p + \sum_{k=p+1}^{\infty} \left[ \frac{p + \ell + \lambda(k - p)}{p + \ell} \right]^m a_k b_k z^k. \] (1.5)

From (1.5), it is easy to verify that (see [12]),
\[ \lambda z \left( D_{\lambda,\ell,p}^m(f \ast g)(z) \right)' = (\ell + p) D_{\lambda,\ell,p}^{m+1}(f \ast g)(z) - [p(1 - \lambda) + \ell] D_{\lambda,\ell,p}^m(f \ast g)(z). \] (1.6)

We note that:
(i) For \( b_k = 1 \) or \( g(z) = \frac{z^p}{1 - z} \), we have \( D_{\lambda,\ell,p}^m f(z) = I_{p}^m(\lambda,\ell) f(z) \), where the operator \( I_{p}^m(\lambda,\ell) \) was introduced and studied by Cătăș [4], which contains intern the operators \( D_{\lambda,\ell}^p \); (see [2] and [8]) and \( D_{\lambda}^m \) (see [1]).
(ii) For \( b_k = \frac{\alpha_{k-p} \cdots \alpha_{k-p}(\beta_{s})_{k-p}(1)_{k-p}}{\beta_{s}^1 k = p \cdots \beta_{s}^1 k = p}, \) the operator
\[ D_{\lambda,\ell,p}^m(f \ast g)(z) = I_{p,q,r,\lambda}^{m,\ell}(\alpha_1,\beta_1) f(z), \]
where the operator \( I_{p,q,r,\lambda}^{m,\ell}(\alpha_1,\beta_1) f(z) \) was introduced and studied by El-Ashwah and Aouf [6], \( \alpha_1,\alpha_2,\ldots,\alpha_q \) and \( \beta_1,\beta_2,\ldots,\beta_s \) are real or complex number \( \beta_j \in \mathbb{C}\setminus\mathbb{Z}_0 = \{0, -1, -2, \ldots\}; j = 1, \ldots, s\) \((q \leq s + 1; q, s \in \mathbb{N}_0, p \in \mathbb{N}; z \in U\) and
\[ (\theta)_\nu = \frac{\Gamma(\theta + \nu)}{\Gamma(\theta)} = \begin{cases} 1 & (\nu = 0; \theta \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}), \\ \theta(\theta - 1) \ldots (\theta + \nu - 1) & (\nu \in \mathbb{N}; \theta \in \mathbb{C}). \end{cases} \]

Also, for many special operators of the operator \( I_{p,q,r,\lambda}^{m,\ell}(\alpha_1,\beta_1) f(z) \) (see [6]),
(iii) For \( m = 0, b_k = \frac{\alpha_{k-p} \cdots \alpha_{k-p}(\beta_{s})_{k-p}(1)_{k-p}}{\beta_{s}^1 k = p \cdots \beta_{s}^1 k = p}, \) the operator
\[ D_{\lambda,\ell,p}^m(f \ast g)(z) = S_{p,q,s}^j(\gamma; \alpha_1) f(z), \]
where the operator \( S_{p,q,s}^j(\gamma; \alpha_1) f(z) \), was introduced and studied by El-Ashwah [5].
(iv) For \( m = 0 \) and \( b_k = \frac{\Gamma(p+\alpha+k-\beta)\Gamma(k+\beta)}{\Gamma(p+\alpha+k-\beta)\Gamma(k+\alpha+\beta)}, \) the operator \( D_{\lambda,\ell,p}^m(f \ast g)(z) = Q_{p,\beta}^\alpha(f) \)
\((\alpha \geq 0, \beta > -1, p \in \mathbb{N})\), where the operator \( Q_{p,\beta}^\alpha(f) \) was introduced by Liu and Owa [9].

For \( h(z) \) given by
\[ h(z) = z^p + \sum_{k=p+1}^{\infty} c_k z^k \]
A function \( f(z) \in A(p) \) is said to be in the class \( S^{m,j}_{\lambda,\ell,p}(\gamma) \) of \( p \)-valent functions of complex order \( \gamma \neq 0 \) in \( U \), if and only if
\[
\text{Re} \left\{ 1 + \frac{1}{\gamma} \left( \frac{z(D^{m}_{\lambda,\ell,p}(f \ast h)(z))^{(j+1)}}{(D^{m}_{\lambda,\ell,p}(f \ast h)(z))^{(j)}} - p + j \right) \right\} > 0
\]
\((p \in \mathbb{N}; \ j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; \ \ell, \lambda \geq 0; \ \gamma \in \mathbb{C}^*; \ z \in U)\).

Clearly, we have the following relationships:
(i) \( S^{0,0}_{\lambda,\ell,1}(\gamma) = S(\gamma) (\gamma \in \mathbb{C}^*) \),
(ii) \( S^{0,1}_{\lambda,\ell,1}(\gamma) = \kappa(\gamma) (\gamma \in \mathbb{C}^*) \),
(iii) \( S^{0,0}_{\lambda,\ell,1}(1 - \alpha) = S^*(\alpha) (0 \leq \alpha < 1) \).

The classes \( S(\gamma) \) and \( \kappa(\gamma) \) are classes of starlike and convex functions of complex order \( \gamma \neq 0 \) in \( U \) which were studied by Nasr and Aouf [13] and \( S^*(\alpha) \) is the class of starlike functions of order \( \alpha \) in \( U \).

Also, for \( m = 0 \) the operator \( S^{0,0}_{j}(h; \gamma) \) was introduced and studied by El-Ashwah and Aouf [7].

**Definition 1.1.** Let \(-1 \leq B < A \leq 1, p \in \mathbb{N}; j \in \mathbb{N}_0, \gamma \in \mathbb{C}^*\),
\[
|\gamma(A - B) + (p - j)B| < (p - j), \ f \in A(p).
\]
Then \( f \in S^{m,j}_{\lambda,\ell,p}(\gamma; A, B) \), the class of \( p \)-valent functions of complex order \( \gamma \) in \( U \) if and only if
\[
\left\{ 1 + \frac{1}{\gamma} \left( \frac{z(D^{m}_{\lambda,\ell,p}(f \ast h)(z))^{(j+1)}}{(D^{m}_{\lambda,\ell,p}(f \ast h)(z))^{(j)}} - p + j \right) \right\} < \frac{1 + Az}{1 + Bz}.
\]

A majorization problem for the subclasses of analytic function has recently been investigated by Altintas et al. [3] and MacGregor [11]. In this paper we investigate majorization problem for the class \( S^{m,j}_{\lambda,\ell,p}(\gamma; A, B) \) and some related subclasses.

### 2. Main results

Unless otherwise mentioned we shall assume throughout the paper that, \(-1 \leq B < A \leq 1, \ \gamma \in \mathbb{C}^*, \ \ell, \lambda \geq 0, p \in \mathbb{N} \) and \( m, j \in \mathbb{N}_0 \).

**Theorem 2.1.** Let the function \( f \in A(p) \) and suppose that \( g \in S^{m,j}_{\lambda,\ell,p}(\gamma; A, B) \). If \( (D^{m}_{\lambda,\ell,p}(f \ast h)(z))^{(j)} \) is majorized by \( (D^{m+1}_{\lambda,\ell,p}(g \ast h)(z))^{(j)} \) in \( U \), then
\[
\left\{ (D^{m+1}_{\lambda,\ell,p}(f \ast h)(z))^{(j)} \right\} \leq \left\{ (D^{m+1}_{\lambda,\ell,p}(g \ast h)(z))^{(j)} \right\} \quad (|z| < r_1),
\]
where \( r_1 = r_1(p, \gamma, \lambda, \ell, A, B) \) is the smallest positive root of the equation
\[
|\gamma(A - B) + (p + \ell)B| r^3 - [2\lambda |B| + (p + \ell)] r^2 - [\gamma(A - B) + (p + \ell)B + 2\lambda] r + (p + \ell) = 0.
\]

**Proof.** Since \( (g \ast h)(z) \in S^{m,j}_{\lambda,\ell,p}(\gamma; A, B) \), we find from (1.8) that
\[
1 + \frac{1}{\gamma} \left( \frac{z(D^{m}_{\lambda,\ell,p}(g \ast h)(z))^{(j+1)}}{(D^{m}_{\lambda,\ell,p}(g \ast h)(z))^{(j)}} - p + j \right) = \frac{1 + Aw(z)}{1 + Bw(z)},
\]
where \( w \) is analytic in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) (\( z \in U \)). From (2.3), we have
\[
\frac{z(D^m_{\lambda, \ell, p}(g \ast h)(z))^{(j+1)}}{(D^m_{\lambda, \ell, p}(g \ast h)(z))^{(j)}} = \frac{(p - j) + [\gamma(A - B) + (p - j)B]w(z)}{1 + Bw(z)}. \tag{2.4}
\]
In view of
\[
\lambda z \left( D^m_{\lambda, \ell, p}(f \ast g)(z) \right)^{(j+1)} = (p + \ell) \left( D^{m+1}_{\lambda, \ell, p}(f \ast g)(z) \right)^{(j)} - [p(1 - \lambda) + \ell \lambda] \left( D^m_{\lambda, \ell, p}(f \ast g)(z) \right)^{(j)}
\]
\[0 \leq j \leq p; \; p \in \mathbb{N}; \; \lambda > 0; \; z \in U, \tag{2.5}\]
(2.4) immediately yields the following inequality:
\[
\left| \left( D^m_{\lambda, \ell, p}(g \ast h)(z) \right)^{(j)} \right| \leq \frac{(p + \ell)(1 + |B||z|)}{(p + \ell) - |\lambda \gamma(A - B) + (p + \ell)B||z|} \left| \left( D^{m+1}_{\lambda, \ell, p}(g \ast h)(z) \right)^{(j)} \right|. \tag{2.6}
\]
Next, since \( (D^m_{\lambda, \ell, p}(f \ast h)(z))^{(j)} \) is majorized by \( (D^m_{\lambda, \ell, p}(g \ast h)(z))^{(j)} \) in \( U \), from (1.2), we have
\[
(D^m_{\lambda, \ell, p}(f \ast h)(z))^{(j)} = \varphi(z)(D^m_{\lambda, \ell, p}(g \ast h)(z))^{(j)}. \tag{2.7}
\]
Differentiating (2.7) with respect to \( z \), we have
\[
z(D^m_{\lambda, \ell, p}(f \ast h)(z))^{(j+1)} = z\varphi'(z)(D^m_{\lambda, \ell, p}(g \ast h)(z))^{(j)} + z\varphi(z)(D^m_{\lambda, \ell, p}(g \ast h)(z))^{(j+1)}. \tag{2.8}
\]
From (2.5) and (2.8), we have
\[
(D^{m+1}_{\lambda, \ell, p}(f \ast h)(z))^{(j)} = \frac{\lambda z}{p + \ell} \varphi'(z)(D^{m}_{\lambda, \ell, p}(g \ast h)(z))^{(j)} + \varphi(z)(D^{m+1}_{\lambda, \ell, p}(g \ast h)(z))^{(j)}. \tag{2.9}
\]
Thus, by noting that \( \varphi(z) \) satisfies the inequality (see [14]),
\[
\left| \varphi'(z) \right| \leq \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \quad (z \in U),
\]
we see that
\[
\left| (D^{m+1}_{\lambda, \ell, p}(f \ast h)(z))^{(j)} \right| \leq \left| (D^m_{\lambda, \ell, p}(g \ast h)(z))^{(j)} \right| + \frac{\lambda z}{p + \ell} \left| \varphi'(z)(D^m_{\lambda, \ell, p}(g \ast h)(z))^{(j)} \right| + \varphi(z)(D^{m+1}_{\lambda, \ell, p}(g \ast h)(z))^{(j)}. \tag{2.10}
\]
which upon setting
\[
|z| = r \quad \text{and} \quad |\varphi(z)| = \rho \quad (0 \leq \rho \leq 1),
\]
leads us to the inequality
\[
\left| (D^{m+1}_{\lambda, \ell, p}(f \ast h)(z))^{(j)} \right| \leq \frac{\Theta(\rho)}{(1 - r^2)((p + \ell) - |\gamma \lambda(A - B) + (p + \ell)B|r) r} \left| (D^{m+1}_{\lambda, \ell, p}(g \ast h)(z))^{(j)} \right|,
\]
where
\[
\Theta(\rho) = -r\lambda(1 + |B|r)\rho^2 + (1 - r^2)((p + \ell) - |\gamma \lambda(A - B) + (p + \ell)B|r)\rho
\]
\[+r\lambda(1 + |B|r), \tag{2.11}\]

\[\frac{\Theta(\rho)}{(1 - r^2)((p + \ell) - |\gamma \lambda(A - B) + (p + \ell)B|r) r} \left| (D^{m+1}_{\lambda, \ell, p}(g \ast h)(z))^{(j)} \right|
\]
takes its maximum value at $\rho = 1$, with $r_1 = r_1(p, \gamma, \lambda, \ell, A, B)$, where $r_1(p, \gamma, \lambda, \ell, A, B)$ is the smallest positive root of (2.2). Therefore the function $\Phi(\rho)$ defined by

$$
\Phi(\rho) = -\sigma \lambda (1 + |B| \sigma) \rho^2 + (1 - \sigma^2) [(p + \ell) - |\gamma \lambda (A - B) + (p + \ell) B| \sigma] \rho + \sigma \lambda (1 + |B| \sigma)
$$

(2.12)
is an increasing function on the interval $0 \leq \rho \leq 1$, so that

$$
\Phi(\rho) \leq \Phi(1) = (1 - \sigma^2) [(p + \ell) - |\gamma (A - B) + (p + \ell) B| \sigma]
$$

(2.13)

$$(0 \leq \rho \leq 1; 0 \leq \sigma \leq r_0(p, \gamma, \lambda, A, B)).$$

Hence upon setting $\rho = 1$ in (2.12), we conclude that (2.1) holds true for $|z| < r_1 = r_1(p, \gamma, \lambda, \ell, A, B)$, where $r_1(p, \gamma, \lambda, \ell, A, B)$ is the smallest positive root of (2.2). This completes the proof of Theorem 1.

Putting $A = 1$ and $B = -1$ in Theorem 1, we obtain the following result.

**Corollary 2.2.** Let the function $f \in A(p)$ and suppose that $g \in S_{\lambda, \ell, p}^m(\gamma)$. If $(D_{\lambda, \ell, p}^m (f^* h)(z))^{(j)}$ is majorized by $(D_{\lambda, \ell, p}^m (g^* h)(z))^{(j)}$ in $U$, then

$$
| (D_{\lambda, \ell, p}^{m+1} (f^* h)(z))^{(j)} | \leq | (D_{\lambda, \ell, p}^{m+1} (g^* h)(z))^{(j)} | (|z| < r_1),
$$

where $r_1 = r_1(p, \gamma, \lambda, \ell)$ is given by

$$
r_1 = r_1(p, \gamma, \lambda, \ell) = \frac{k - \sqrt{k^2 - 4(p + \ell)} |2\gamma \lambda - (p + \ell)|}{2 |2\gamma \lambda - (p + \ell)|},
$$

(2.14)

where $k = 2\lambda + (p + \ell) + |2\gamma \lambda - (p + \ell)|$.

Putting $A = 1$, $B = -1$ and $p = j = 1$ in Theorem 1, we obtain the following result.

**Corollary 2.3.** Let the function $f \in A$ and suppose that $g \in S_{\lambda, \ell}^{m, 0}(\gamma)$. If $(D_{\lambda, \ell}^m (f^* h)(z))$ is majorized by $(D_{\lambda, \ell}^m (g^* h)(z))$ in $U$, then

$$
| (D_{\lambda, \ell}^{m+1} (f^* h)(z)) | \leq | (D_{\lambda, \ell}^{m+1} (g^* h)(z)) | (|z| < r_2),
$$

where $r_2 = r_2(\gamma, \lambda, \ell)$ is given by

$$
r_2 = r_2(\gamma, \lambda, \ell) = \frac{k - \sqrt{k^2 - 4(1 + \ell)} |2\gamma \lambda - (1 + \ell)|}{2 |2\gamma \lambda - (1 + \ell)|},
$$

(2.15)

where $k = 2\lambda + (1 + \ell) + |2\gamma \lambda - (1 + \ell)|$.

Putting $A = \lambda = 1$, $B = -1$, $m = \ell = 0$, and $h(z) = \frac{z^p}{1-z}$ (or $c_{k+p} = 1$) in Theorem 1, we obtain the following result.

**Corollary 2.4.** Let the function $f \in A(p)$ and suppose that $g \in S_p$. If $f(z)$ is majorized by $g(z)$ in $U$, then

$$
|f'(z)| \leq |g'(z)| (|z| < r_3),
$$

where $r_3 = r_3(p, \gamma)$ is given by

$$
r_3 = r_3(p; \gamma) = \frac{k - \sqrt{k^2 - 4p} |2\gamma - p|}{2 |2\gamma - p|},
$$

where $k = 2 + p + |2\gamma - p|$.
Putting $\gamma = 1$ in Corollary 3, we obtain the following result.

**Corollary 2.5.** Let the function $f \in A(p)$ and suppose that $g \in S_p(\gamma)$. If $f(z)$ is majorized by $g(z)$ in $U$, then

$$|f'(z)| \leq |g'(z)| \quad (|z| < r_4),$$

where $r_4$ is given by

$$r_4 = r_4(p) = \frac{k - \sqrt{k^2 - 4p|2 - p|}}{2|2 - p|},$$

where $k = 2 + p + |2 - p|$

**Remarks 2.6.** (i) Putting $p = 1$ in Corollary 3 we obtain the results obtained by Altintas et al. [3],
(ii) Putting $p = 1$ in Corollary 4 we obtain the results obtained by MacGregor [10].

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