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On Hadamard-type inequalities for m-convex functions via Riemann-Liouville fractional integrals

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Abstract. In this paper we prove the Hadamard-type inequalities for m-convex functions via Riemann-Liouville fractional integrals and the Hadamard-type inequalities for convex functions via Riemann-Liouville fractional integral are deduced. Also we find connections with some well known results related to the Hadamard inequality.

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1. Introduction

Following L'Hospital's and Leibniz's first inquiries, fractional calculus was primarily a study reserved for the best minds in mathematics. Fourier, Euler, Laplace were among those who were interested in fractional calculus and its mathematical consequences [15]. Euler and Liouville developed their thoughts about the computation of non-integer order integrals and derivatives. Many initiate, using their own notation and methodology, definitions that fit the concept of a non-integer order integral or derivative. The most well-known of these definitions that have been popularized in the subject of fractional calculus are the Riemann-Liouville and the Grunwald-Letnikov definition [4, 12]. In [18] Riemann-Liouville fractional integrals are defined as follows:

Definition 1.1. Let $f \in L_1[a, b]$. Then Riemann-Liouville fractional integrals of order $\alpha > 0$ with $a \geq 0$ are defined as:

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a \tag{1.1}$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b. \tag{1.2}$$

For further details one may see [15, 16, 17, 9, 8, 13, 19].
Convex functions play a vital role in the mathematical analysis. They have been considered for defining and finding new dimensions of analysis. In [20] Toader define the concept of m-convexity, an intermediate between usual convexity and star shape function.

Definition 1.2. A function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be m-convex, where $m \in [0, 1]$, if we have

$$f\left(tx + m(1-t)y\right) \leq tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

If we take $m = 1$, then we recapture the concept of convex functions defined on $[0, b]$ and if we take $m = 0$, then we get the concept of starshaped functions on $[0, b]$. We recall that $f : [0, b] \to \mathbb{R}$ is called *starshaped* if

$$f(tx) \leq tf(x) \text{ for all } t \in [0, 1] \text{ and } x \in [0, b].$$

Denote by $K_m(b)$ the set of the m-convex functions on $[0, b]$ for which $f(0) < 0$, then one has

$$K_1(b) \subset K_m(b) \subset K_0(b),$$

whenever $m \in (0, 1)$. Note that in the class $K_1(b)$ are only convex functions $f : [0, b] \to \mathbb{R}$ for which $f(0) \leq 0$ (see [5]).

Example 1.3. [14] The function $f : [0, \infty) \to \mathbb{R}$, given by

$$f(x) = \frac{1}{12}\left(4x^3 - 15x^2 + 18x - 5\right)$$

is $\frac{16}{17}-$convex function but it is not convex function.

For more results and inequalities related to m-convex functions one can consult for example [7, 5, 11, 2, 16] along with references.
Let $f : I \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then the following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.3}$$

is well known in literature as the Hadamard inequality.

For more refinements, generalizations and inequalities related to (1.3), see [1, 2, 3, 16, 6].

In [19], Sarikaya et al. proved the following Hadamard-type inequalities for Riemann-Liouville fractional integrals.

Theorem 1.4. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is a convex function on [a,b], then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{(\frac{a+b}{2})+}f(b) + J^\alpha_{(\frac{a+b}{2})-}f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.4}$$

with $\alpha > 0$.

Theorem 1.5. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$. If $|f'|^q$ is convex on $[a, b]$ for $q \geq 1$, then the following inequality for fractional integrals holds:*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}[J^\alpha_{(\frac{a+b}{2})+}f(b) + J^\alpha_{(\frac{a+b}{2})-}f(a)] - f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{b-a}{4(\alpha+1)}\left(\frac{1}{2(\alpha+2)}\right)^{\frac{1}{q}}\Big[((\alpha+1)|f'(a)|^q + (\alpha+3)|f'(b)|^q)^{\frac{1}{q}} \tag{1.5}$$
$$+ ((\alpha+3)|f'(a)|^q + (\alpha+1)|f'(b)|^q)^{\frac{1}{q}}\Big].$$

Theorem 1.6. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$. If $|f'|^q$ is convex on $[a, b]$ for $q > 1$, then the following inequality for fractional integral holds:*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}[J^\alpha_{(\frac{a+b}{2})+}f(b) + J^\alpha_{(\frac{a+b}{2})-}f(a)] - f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{b-a}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f'(a)|^q + 3|f'(b)|q}{4}\right)^{\frac{1}{q}} + \left(\frac{3|f'(a)|^q + |f'(b)|^q}{4}\right)^{\frac{1}{q}}\right] \tag{1.6}$$
$$\leq \frac{b-a}{4}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}[|f'(a)| + |f'(b)|],$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

In this paper we generalize the fractional Hadamard-type inequalities (1.4), (1.5) and (1.6) for m-convex function via Riemann-Liouville fractional integrals and show that these inequalities are the special cases of our results. Also we find some well known results.

2. Hadamard-type inequalities for m-convex functions via fractional integrals

Start with the following result.

Theorem 2.1. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is a m-convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+mb}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^\alpha}\left[J^\alpha_{(\frac{a+mb}{2})+}f(mb) + m^{\alpha+1}J^\alpha_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right] \tag{2.1}$$
$$\leq \frac{\alpha}{4(\alpha+1)}\left[f(a) - m^2 f\left(\frac{a}{m^2}\right)\right] + \frac{m}{2}\left[f(b) + mf\left(\frac{a}{m^2}\right)\right]$$

with $\alpha > 0$.

Proof. From m-convexity of f we have,

$$f\left(\frac{x+my}{2}\right) \leq \frac{f(x) + mf(y)}{2}. \tag{2.2}$$

Put $x = \frac{t}{2}a + m\frac{(2-t)}{2}b, y = \frac{(2-t)}{2m}a + \frac{t}{2}b$ for $t \in [0,1]$. Then $x, y \in [a,b]$ and above inequality gives,

$$2f\left(\frac{a+mb}{2}\right) \leq f\left(\frac{t}{2}a + m\frac{2-t}{2}b\right) + mf\left(\frac{2-t}{2m}a + \frac{t}{2}b\right), \tag{2.3}$$

multiplying both sides of above inequality with $t^{\alpha-1}$, and integrating over $[0,1]$ we have,

$$\begin{aligned}
&\frac{2}{\alpha}f\left(\frac{a+mb}{2}\right)\\
&\leq \int_0^1 t^{\alpha-1} f\left(\frac{t}{2}a + m\frac{2-t}{2}b\right)dt + m\int_0^1 t^{\alpha-1} f\left(\frac{2-t}{2m}a + \frac{t}{2}b\right)dt\\
&= \int_{mb}^{\frac{a+mb}{2}} \left(\frac{2}{mb-a}(mb-u)\right)^{\alpha-1} f(u)\frac{2du}{a-mb}\\
&+ m^2\int_{\frac{a}{m}}^{\frac{a+mb}{2m}} \left(\frac{2}{b-\frac{a}{m}}(v-\frac{a}{m})\right)^{\alpha-1} f(v)\frac{2dv}{mb-a}\\
&= \frac{2^\alpha\Gamma(\alpha)}{(mb-a)^\alpha}\left[J^\alpha_{(\frac{a+mb}{2})+}f(mb) + m^{\alpha+1}J^\alpha_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right],
\end{aligned}$$

from which one has

$$f\left(\frac{a+mb}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^\alpha}\left[J^\alpha_{(\frac{a+mb}{2})+}f(mb) + m^{\alpha+1}J^\alpha_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right]. \tag{2.4}$$

On the other hand m-convexity of f gives

$$\begin{aligned}
&f\left(\frac{t}{2}a + m\frac{2-t}{2}b\right) + mf\left(\frac{2-t}{2m}a + \frac{t}{2}b\right)\\
&\leq \frac{t}{2}\left[f(a) - m^2 f\left(\frac{a}{m^2}\right)\right] + m\left[f(b) + mf\left(\frac{a}{m^2}\right)\right],
\end{aligned}$$

multiplying both sides of above inequality with $t^{\alpha-1}$, and integrating over $[0,1]$ we have,

$$\begin{aligned}
&\int_0^1 t^{\alpha-1} f\left(\frac{t}{2}a + m\frac{2-t}{2}b\right)dt + m\int_0^1 t^{\alpha-1} f\left(\frac{2-t}{2m}a + \frac{t}{2}b\right)dt\\
&\leq \frac{1}{2}\left[f(a) - m^2 f\left(\frac{a}{m^2}\right)\right]\int_0^1 t^\alpha dt + m\left[f(b) + mf\left(\frac{a}{m^2}\right)\right]\int_0^1 t^{\alpha-1}dt\\
&\int_{mb}^{\frac{a+mb}{2}} \left(\frac{2}{mb-a}(mb-u)\right)^{\alpha-1} f(u)\frac{2du}{a-mb}\\
&+ m^2\int_{\frac{a}{m}}^{\frac{a+mb}{2m}} \left(\frac{2}{b-\frac{a}{m}}(v-\frac{a}{m})\right)^{\alpha-1} f(v)\frac{2dv}{mb-a}\\
&\leq \frac{1}{2(\alpha+1)}\left[f(a) - m^2 f\left(\frac{a}{m^2}\right)\right] + \frac{m}{\alpha}\left[f(b) + mf\left(\frac{a}{m^2}\right)\right]
\end{aligned}$$

from which one has

$$\begin{aligned}&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+mb}{2})+}f(mb)+m^{\alpha+1}J^{\alpha}_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right]\\&\leq\frac{\alpha}{4(\alpha+1)}\left[f(a)-m^2f\left(\frac{a}{m^2}\right)\right]+\frac{m}{2}\left[f(b)+mf\left(\frac{a}{m^2}\right)\right].\end{aligned}\tag{2.5}$$

Combining inequality (2.4) and inequality (2.5) we get inequality (2.1) . □

Remark 2.2. If we take $m = 1$, Theorem 2.1 gives inequality (1.4) of Theorem 1.4 and putting $\alpha = 1$ along with $m = 1$ in Theorem 2.1 we get the classical Hadamard inequality.

For next results we need the following lemma.

Lemma 2.3. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $f' \in L[a, b]$, then the following equality for fractional integrals holds:*

$$\begin{aligned}&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+mb}{2})+}f(mb)+m^{\alpha+1}J^{\alpha}_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right]\\&-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\\&=\frac{mb-a}{4}\left[\int_0^1 t^{\alpha}f'\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)dt-\int_0^1 t^{\alpha}f'\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)dt\right].\end{aligned}\tag{2.6}$$

Proof. One can note that

$$\begin{aligned}&\frac{mb-a}{4}\left[\int_0^1 t^{\alpha}f'\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)dt\right]\\&=\frac{mb-a}{4}\left[-\frac{2}{mb-a}f\left(\frac{a+mb}{2}\right)\right.\\&\left.-\frac{2\alpha}{(a-mb)}\int_{mb}^{\frac{a+mb}{2}}\left(\frac{2}{mb-a}(mb-x)\right)^{\alpha-1}\frac{2}{a-mb}f(x)dx\right]\end{aligned}\tag{2.7}$$

$$=\frac{mb-a}{4}\left[-\frac{2}{mb-a}f\left(\frac{a+mb}{2}\right)+\frac{2^{\alpha+1}\Gamma(\alpha+1)}{(mb-a)^{\alpha+1}}J^{\alpha}_{(\frac{a+mb}{2})-}f(mb)\right].\tag{2.8}$$

Similarly

$$\begin{aligned}&-\frac{mb-a}{4}\left[\int_0^1 t^{\alpha}f'\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)dt\right]\\&=-\frac{mb-a}{4}\left[\frac{2m}{mb-a}f\left(\frac{a+mb}{2m}\right)-\frac{2^{\alpha+1}m^{\alpha+1}\Gamma(\alpha+1)}{(mb-a)^{\alpha+1}}J^{\alpha}_{(\frac{a+mb}{2m})+}f\left(\frac{a}{m}\right)\right].\end{aligned}\tag{2.9}$$

Adding (2.7) and (2.9) one has (2.6). □

Using the above lemma we give the following Hadamard-type inequality.

Theorem 2.4. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$. If $|f'|^q$ is m-convex on $[a,b]$ for $q \geq 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\Bigg|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+mb}{2})+}f(mb)+m^{\alpha+1}J^{\alpha}_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right]\\
&-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\Bigg|\\
&\leq \frac{mb-a}{4(\alpha+1)}\left(\frac{1}{2(\alpha+2)}\right)^{\frac{1}{q}}\Big[\left((\alpha+1)\,|f'(a)|^q+m\,(\alpha+3)\,|f'(b)|^q\right)^{\frac{1}{q}}\\
&+\left(m\,(\alpha+3)\,|f'\left(\frac{a}{m^2}\right)|^q+(\alpha+1)\,|f'(b)|^q\right)^{\frac{1}{q}}\Big].
\end{aligned} \tag{2.10}$$

with $\alpha > 0$.

Proof. From Lemma 2.3 and m-convexity of $|f'|^q$ and for $q = 1$ we have

$$\begin{aligned}
&\Bigg|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+mb}{2})+}f(mb)+m^{\alpha+1}J^{\alpha}_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right]\\
&-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\Bigg|\\
&\leq \frac{mb-a}{4}\int_0^1 t^{\alpha}\left(\left|f'\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)\right|dt+\left|f'\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)\right|\right)dt.\\
&=\frac{mb-a}{4}\left(\frac{m}{\alpha+1}\left[|f'(b)|+|f'\left(\frac{a}{m^2}\right)|\right]\right.\\
&\left.+\left[|f'(a)|-m|f'\left(\frac{a}{m^2}\right)|+|f'(b)|-m|f'(b)|\right]\right).
\end{aligned}$$

For $q > 1$ we proceed as follows. Using Lemma 2.3 we have

$$\begin{aligned}
&\Bigg|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+mb}{2})+}f(mb)+m^{\alpha+1}J^{\alpha}_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right]\\
&-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\Bigg|\\
&\leq \frac{mb-a}{4}\int_0^1 t^{\alpha}\left|f'\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)\right|dt+\int_0^1 t^{\alpha}\left|f'\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)\right|dt.
\end{aligned}$$

Using power mean inequality we get

$$\begin{aligned}
&\Bigg|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+mb}{2})+}f(mb)+m^{\alpha}J^{\alpha}_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right]\\
&-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\Bigg|\\
&\leq \frac{mb-a}{4}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{p}}\left[\left[\int_0^1 t^{\alpha}\left|f'\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right.
\end{aligned}$$

$$+\left[\int_0^1 t^{\alpha}\left|f'\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\Bigg].$$

m-convexity of $|f'|^q$ gives

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+mb}{2})+}f(mb)+m^{\alpha+1}J^{\alpha}_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|$$
$$\leq\frac{mb-a}{4}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{p}}\left[\left[\int_0^1 t^{\alpha}\left(\frac{t}{2}|f'(a)|^q+m\frac{2-t}{2}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\int_0^1 t^{\alpha}\left(m\frac{2-t}{2}|f'\left(\frac{a}{m^2}\right)|^q+\frac{t}{2}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}\right]$$
$$=\frac{mb-a}{4(\alpha+1)}\left(\frac{1}{2(\alpha+2)}\right)^{\frac{1}{q}}\left[\left((\alpha+1)\,|f'(a)|^q+m\,(\alpha+3)\,|f'(b)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(m\,(\alpha+3)\,|f'\left(\frac{a}{m^2}\right)|^q+(\alpha+1)\,|f'(b)|^q\right)^{\frac{1}{q}}\right].$$

Hence the proof is complete. □

Remark 2.5. If we take $m=1$ in Theorem 2.4, we get inequality (1.5) of Theorem 1.5 and if we take $\alpha=q=1$ along with $m=1$ in Theorem 2.4, then inequality (2.10) gives the following result.

Corollary 2.6. *With the assumptions of Theorem* 2.4 *we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{(b-a)}{8}\left(|f'(a)|+|f'(b)|\right). \tag{2.11}$$

Theorem 2.7. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) with $a<b$. If $|f'|^q$ is m-convex on $[a,b]$ for $q>1$, then the following inequality for fractional integral holds:*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+mb}{2})+}f(mb)+m^{\alpha+1}J^{\alpha}_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|$$
$$\leq\frac{mb-a}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f'(a)|^q+3m|f'(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3m|f'\left(\frac{a}{m^2}\right)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{mb-a}{4}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}\left[|f'(a)|+|f'(b)|+3m\left(|f'\left(\frac{a}{m^2}\right)|+|f'(b)|\right)\right], \tag{2.12}$$

with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 2.3 we have

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+mb}{2})+}f(mb)+m^{\alpha+1}J^{\alpha}_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|$$
$$\leq\frac{mb-a}{4}\left[\int_0^1 t^{\alpha}\left|f'\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)\right|dt+\int_0^1 t^{\alpha}\left|f'\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)\right|dt\right].$$

From the *Hölder's* inequality we get

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+mb}{2})+}f(mb)+m^{\alpha+1}J^{\alpha}_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|$$
$$\leq\frac{mb-a}{4}\left[\left[\int_0^1 t^{\alpha p}dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f'\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\int_0^1 t^{\alpha p}dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f'\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right].$$

m-convexity of $|f'|^q$ gives

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+mb}{2})+}f(mb)+m^{\alpha+1}J^{\alpha}_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|$$
$$\leq\frac{mb-a}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left[\int_0^1\left(\frac{t}{2}|f'(a)|^q+m\frac{2-t}{2}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\int_0^1\left(m\frac{2-t}{2}|f'(\frac{a}{m^2})|^q+\frac{t}{2}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}\right]$$
$$=\frac{mb-a}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left[\frac{|f'(a)|^q+3m|f'(b)|^q}{4}\right]^{\frac{1}{q}}+\left[\frac{3m|f'\left(\frac{a}{m^2}\right)|^q+|f'(b)|^q}{4}\right]^{\frac{1}{q}}\right].$$

For the second inequality of (2.12) we use Minkowski's inequality as

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+mb}{2})+}f(mb)+m^{\alpha+1}J^{\alpha}_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|$$
$$\leq\frac{mb-a}{16}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}\left[[|f'(a)|^q+3m|f'(b)|^q]^{\frac{1}{q}}+\left[3m|f'\left(\frac{a}{m^2}\right)|^q+|f'(b)|^q\right]^{\frac{1}{q}}\right]$$

$$\leq \frac{mb-a}{4}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}\left[|f'(a)|+|f'(b)|+3m\left(|f'\left(\frac{a}{m^2}\right)|+|f'(b)|\right)\right]. \qquad \square$$

Remark 2.8. If we take $m = 1$ in Theorem 2.7, we get inequality (1.6) of Theorem 1.6 and if we take $\alpha = 1$ along with $m = 1$ in Theorem 2.7, then inequality (2.12) gives the following result.

Corollary 2.9. *With the assumptions of Theorem* 2.7 *we have*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\ &\leq \frac{b-a}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[(|f'(a)|^q+3|f'(b)|^q)^{\frac{1}{q}}+(3|f'(a)|^q+|f'(b)|^q)^{\frac{1}{q}}\right].\end{aligned} \tag{2.13}$$

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