Differential sandwich theorem for certain class of analytic functions associated with an integral operator

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Abstract. In this paper we obtain some applications of first order differential subordination and superordination result involving an integral operator for certain normalized analytic function.

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1. Introduction and preliminaries

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0,$$

which are analytic and univalent in the open unit disk $U = \{z : |z| < 1\}$.

If $f$ and $g$ are analytic functions in $U$, we say that $f$ is subordinate to $g$ in $U$, written symbolically as $f \prec g$ or $f(z) \prec g(z)$ if there exists a Schwarz function $w(z)$ analytic in $U$, with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$, $z \in U$. In particular, if the function $g$ is univalent in $U$, the subordination $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$ (see [2], [3]).

For the function $f$ given by (1.1) and $g \in A$ given by $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, the Hadamard product (or convolution) of $f$ and $g$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z).$$
The set of all functions $f$ that are analytic and injective on $U - E(f)$, denote by $Q$ where

$$E(f) = \{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$, (see [4]).

If $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and $h$ is univalent in $U$ with $q \in Q$. In [3] Miller and Mocanu consider the problem of determining conditions on admissible functions $\psi$ such that

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \quad (1.2)$$

implies that $p(z) \prec q(z)$ for all functions $p \in \mathcal{H}[a,n]$ that satisfy the differential subordination (1.2).

Let $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ and $h \in \mathcal{H}$ with $q \in \mathcal{H}[a,n]$. In [4] and [5] is studied the dual problem and determined conditions on $\phi$ such that

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \quad (1.3)$$

implies $q(z) \prec p(z)$ for all functions $p \in Q$ that satisfy the above subordination. They also found conditions so that the functions $q$ is the largest function with this property, called the best subordinant of the subordination (1.3).

Let $\mathcal{H}(U)$ be the class of analytic functions in the open unit disc.

For $n$ a positive integer and $a \in \mathbb{C}$ let

$$\mathcal{H}[a,n] = \{ f \in \mathcal{H} : f(z) = a + a_nz^n + \ldots \}.$$  

The integral operator $I^m$ of a function $f$ is defined in [6] by

$$I^0f(z) = f(z),$$

$$I^1f(z) = If(z) = \int_0^z f(t)t^{-1}dt,$$

$$\ldots$$

$$I^mf(z) = I(I^{m-1}f(z)), \quad z \in U.$$  

**Lemma 1.1.** [3] Let $q$ be univalent in $U$, $\zeta \in \mathbb{C}^*$ and suppose that

$$\Re \left\{ 1 + \frac{q''(z)}{q'(z)} \right\} > \max \left\{ 0, -\Re \left( \frac{1}{\zeta} \right) \right\}. \quad (1.4)$$

If $p$ is analytic in $U$ with $p(0) = q(0)$ and

$$p(z) + \zetazp'(z) \prec q(z) + \zeta q'(z) \quad (1.5)$$

then $p(z) \prec q(z)$ and $q$ is the best dominant.

**Lemma 1.2.** [3] Let the function $q$ be univalent in the unit disk and let $\theta, \varphi$ be analytic in domain $D$ containing $q(U)$ with $\varphi(w) \neq 0$, where $w \in q(U)$. Set

$$Q(z) = zq'(z)\varphi(q(z)) \quad \text{and} \quad h(z) = \theta(q(z)) + Q(z).$$

Suppose that

- $Q$ is starlike univalent in $U$;
- $\Re \left\{ \frac{zh'(z)}{Q(z)} \right\} > 0$, for $z \in U.$
If \( p \) is analytic with \( p(0) = q(0) \), \( p(U) \subseteq D \) and
\[
\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z))
\] (1.6)
then \( p(z) \prec q(z) \) and \( q \) is the best dominant.

**Lemma 1.3.** [1] Let \( q \) be convex in the unit disc \( U \), \( q(0) = a \) and \( \zeta \in \mathbb{C} \), \( \Re(\zeta) > 0 \).
If \( p \in \mathcal{H}[a, 1] \cap Q \) and \( p(z) + \zeta z p'(z) \) is univalent in \( U \) then
\[
q(z) + \zeta z q'(z) \prec p(z) + \zeta z p'(z)
\] (1.7)
implies \( q(z) \prec p(z) \) and \( q \) is the best subordinant.

**Lemma 1.4.** [2] Let the function \( q \) be convex and univalent in the unit disc \( U \) and \( \theta \) and \( \varphi \) be analytic in a domain \( D \) containing \( q(U) \). Suppose that
1. \( \Re \left\{ \frac{\theta'(q(z))}{\varphi(q(z))} \right\} > 0 \) for \( z \in U \) and
2. \( Q(z) = z q'(z) \varphi(q(z)) \) is starlike univalent in \( U \).
If \( p \in \mathcal{H}[q(0), 1] \cap Q \) with \( p(U) \subseteq D \) and \( \theta(p(z)) + zp'(z)\varphi(p(z)) \) is univalent in \( U \) and
\[
\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(p(z)) + zp'(z)\varphi(p(z))
\] (1.8)
then \( q(z) \prec p(z) \) and \( q \) is the best subordinant.

## 2. Main results

**Theorem 2.1.** Let \( q \) be univalent in \( U \), with \( q(0) = 1 \) and \( q(z) \neq 0 \) for all \( z \in U \), and let \( \sigma \in \mathbb{C}^* \), \( f \in A \) and suppose that \( f \) and \( g \) satisfy the next conditions:
\[
\frac{I^{m+1}(f(z))}{z} \neq 0, z \in U
\] (2.1)
and
\[
\Re \left\{ 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right\} > 0 \text{ for } z \in U.
\] (2.2)
If
\[
\frac{I^m(f(z))}{I^{m+1}(f(z))} < 1 + \frac{zq'(z)}{\sigma q(z)},
\] (2.3)
then
\[
\left( \frac{I^{m+1}(f(z))}{z} \right)^\sigma \prec q(z)
\]
and \( q \) is the best dominant of (2.3).

**Proof.** Let
\[
p(z) = \left( \frac{I^{m+1}(f(z))}{z} \right)^\sigma, z \in U.
\] (2.4)
Because the integral operator \( I^m \) satisfies the identity \( z \left[ I^{m+1}(f(z)) \right]' = I^m(f(z)) \) and the function \( p(z) \) is analytic in \( U \), by differentiating (2.4) logarithmically with respect to \( z \), we obtain
\[
\frac{zp'(z)}{p(z)} = \sigma \left( \frac{I^m(f(z))}{I^{m+1}(f(z))} - 1 \right).
\] (2.5)
In order to prove our result we will use Lemma 1.2. In this lemma we consider
\[ \theta(w) = 1 \text{ and } \varphi(w) = \frac{1}{\sigma w}, \]
then \( \theta \) is analytic in \( \mathbb{C} \) and \( \varphi(w) \neq 0 \) is analytic in \( \mathbb{C}^* \). Also, if we let
\[ Q(z) = zq'(z)\varphi(q(z)) = \frac{zq'(z)}{\sigma q(z)}, \]
and
\[ h(z) = \theta(q(z)) + Q(z) = 1 + \frac{zq'(z)}{\gamma \sigma q(z)} \]
from (2.2) we see that \( Q(z) \) is a starlike function in \( U \). We also have
\[ \Re \left\{ \frac{zh'(z)}{Q(z)} \right\} = \Re \left\{ 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right\} > 0 \text{ for } z \in U \]
and then, by using Lemma 1.2 we deduce that subordination (2.3) implies \( p(z) \prec q(z) \)
and the function \( q \) is the best dominant of (2.3).

Taking \( q(z) = \frac{1 + Az}{1 + Bz} \) \((-1 \leq B < A \leq 1)\) in Theorem 2.1, it easy to check that the assumption
\[ p(z) + \frac{1}{\sigma}zp'(z) \prec q(z) + \frac{\alpha}{\sigma}zq'(z) \]
holds, hence we obtain the next result.

**Corollary 2.2.** Let \( \sigma \in \mathbb{C}^* \) and \( f \in \mathbb{A} \). Suppose
\[ \frac{I^{m+1}(f(z))}{z} \neq 0, z \in U. \]
If
\[ \frac{I^{m}(f(z))}{I^{m+1}(f(z))} \prec 1 + \frac{z(A - B)}{\sigma (1 + Az) (1 + Bz)}, \]
then
\[ \left( \frac{I^{m+1}(f(z))}{z} \right)^{\sigma} \prec \frac{1 + Az}{1 + Bz} \]
and \( q(z) = \frac{1 + Az}{1 + Bz} \) is the best dominant.

Taking \( q(z) = \frac{1 + z}{1 - z} \) in Theorem 2.1, it easy to check that the assumption
\[ p(z) + \frac{1}{\sigma}zp'(z) \prec q(z) + \frac{\alpha}{\sigma}zq'(z) \]
holds, hence we obtain the next result.

**Corollary 2.3.** Let \( \sigma \in \mathbb{C}^* \) and \( f \in \mathbb{A} \). Suppose
\[ \frac{I^{m+1}(f(z))}{z} \neq 0, z \in U. \]
If
\[ \frac{I^{m}(f(z))}{I^{m+1}(f(z))} \prec 1 + \frac{2z}{\sigma (1 - z) (1 + z)}, \]
then

\[
\left( \frac{I^{m+1}(f(z))}{z} \right)^{\sigma} < \frac{1+z}{1-z}
\]

and \( q(z) = \frac{1+z}{1-z} \) is the best dominant.

**Theorem 2.4.** Let \( q \) be univalent in \( U \), with \( q(0) = 1 \). Let \( \sigma \in \mathbb{C}^* \) and \( t,\nu,\eta \in \mathbb{C} \) with \( \nu + \eta \neq 0 \). Let \( f \in A \) and suppose that \( f \) and \( g \) satisfy the next conditions

\[
\frac{\nu I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\nu + \eta)z} \neq 0, z \in U
\]  

(2.6)

and

\[
\text{Re} \left\{ 1 + \frac{2q''(z)}{q'(z)} \right\} > \max \{0, -\text{Re} t\}, z \in U.
\]  

(2.7)

If

\[
\psi(z) = t \left[ \frac{\nu I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\nu + \eta)z} \right]^{\sigma} + \sigma \left[ \frac{vz(I^{m+1}(f(z)))' + z\eta(I^{m+2}(f(z)))'}{\nu I^{m+1}(f(z)) + \eta I^{m+2}(f(z))} - 1 \right]
\]  

(2.8)

and

\[
\psi(z) < tq(z) + \frac{zq'(z)}{q(z)}
\]  

(2.9)

then

\[
\left[ \frac{\nu I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\nu + \eta)z} \right]^{\sigma} < q(z)
\]

and \( q \) is the best dominant.

**Proof.** Let

\[
p(z) = \left[ \frac{\nu I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\nu + \eta)z} \right]^{\sigma}, z \in U.
\]  

(2.10)

According to (2.3) the function \( p(z) \) is analytic in \( U \) and differentiating (2.10) logarithmically with respect to \( z \), we obtain

\[
\frac{zp'(z)}{p(z)} = \sigma \left[ \frac{vz(I^{m+1}(f(z)))' + z\eta(I^{m+2}(f(z)))'}{\nu I^{m+1}(f(z)) + \eta I^{m+2}(f(z))} - 1 \right]
\]  

(2.11)

and hence

\[
zp'(z) = \sigma \left[ \frac{v I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\nu + \eta)z} \right]^{\sigma} \left[ \frac{vz(I^{m+1}(f(z)))' + z\eta(I^{m+2}(f(z)))'}{\nu I^{m+1}(f(z)) + \eta I^{m+2}(f(z))} - 1 \right].
\]

In order to prove our result we will use Lemma 1.2. In this lemma we consider

\[
\theta(w) = tw \quad \text{and} \quad \varphi(w) = \frac{1}{w}
\]
then $\theta$ is analytic in $\mathbb{C}$ and $\varphi(w) \neq 0$ is analytic in $\mathbb{C}^*$. Also if we let

$$Q(z) = zq'(z)\varphi(q(z)) = \sigma \left[ \frac{vz (\text{I}^{m+1} (f(z)))' + z\eta (\text{I}^{m+2} (f(z)))'}{v\text{I}^{m+1} (f(z)) + \eta\text{I}^{m+2} (f(z))} - 1 \right]$$

and

$$h(z) = \theta(q(z)) + Q(z) = t \left[ \frac{v\text{I}^{m+1} (f(z)) + \eta\text{I}^{m+2} (f(z))}{(v + \eta)z} \right]^{\sigma} + \sigma \left[ \frac{vz (\text{I}^{m+1} (f(z)))' + z\eta (\text{I}^{m+2} (f(z)))'}{v\text{I}^{m+1} (f(z)) + \eta\text{I}^{m+2} (f(z))} - 1 \right]$$

from (2.6) we see that $Q(z)$ is a starlike function in $U$. We also have

$$\text{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} = \text{Re} \left\{ t + 1 + \frac{zq''(z)}{q'(z)} \right\} > 0 \text{ for } z \in U$$

and then, by using Lemma 1.2 we deduce that the subordination (2.9) implies $p(z) < q(z)$.

Taking $q(z) = \frac{1+A}{1+Bz}$ ($-1 \leq B < A \leq 1$) in Theorem 2.4 and according to

$$\frac{zp'(z)}{p(z)} = \sigma \left( \frac{\text{I}^{m+1} (f(z))}{\text{I}^{m+2} (f(z))} - 1 \right)$$

the condition (2.7) becomes $\max\{0, -\text{Re}(t)\} \leq \frac{1-|B|}{1+|B|}$. Hence, for the special case $v = 1$ and $\eta = 0$ we obtain the following result.

**Corollary 2.5.** Let $t \in \mathbb{C}$ with $\max\{0, -\text{Re}(t)\} \leq \frac{1-|B|}{1+|B|}$. Let $f \in \mathcal{A}$ and suppose that

$$\frac{\text{I}^{m+1} (f(z))}{z} \neq 0, z \in U.$$

If

$$t \left[ \frac{\text{I}^{m+1} (f(z))}{z} \right]^{\sigma} + \sigma \left[ \frac{z (\text{I}^{m+1} (f(z)))'}{\text{I}^{m+1} (f(z))} - 1 \right] < t \frac{1+A}{1+Bz} + \frac{(1-B)z}{(1+A)(1+Bz)}$$

then

$$\left( \frac{\text{I}^{m+1} (f(z))}{z} \right)^{\sigma} < \frac{1+A}{1+Bz}$$

and $q(z) = \frac{1+A}{1+Bz}$ is the best dominant.

Taking $v = m = 1$, $\eta = 0$ and $q(z) = \frac{1+z}{1-z}$ in Theorem 2.1, we obtain the next result.

**Corollary 2.6.** Let $f \in \mathcal{A}$ and suppose that $\frac{I^2(f(z))}{z} \neq 0, z \in U$, $\sigma \in \mathbb{C}^*$. If

$$t \left[ \frac{I^2 (f(z))}{z} \right]^{\sigma} + \sigma \left[ \frac{z (I^2 (f(z)))'}{I^2 (f(z))} - 1 \right] < t \frac{1+z}{1-z} + \frac{2z}{(1+z)(1-z)}$$

then

$$\left( \frac{I^2 (f(z))}{z} \right)^{\sigma} < \frac{1+z}{1-z}$$

and $q(z) = \frac{1+z}{1-z}$ is the best dominant.
Theorem 2.7. Let $q$ be convex in $U$, with $q(0) = 1$. Let $\sigma \in \mathbb{C}^*$ and $t, \nu, \eta \in \mathbb{C}$ with $\nu + \eta \neq 0$ and $\text{Re} t > 0$. Let $f \in A$ and suppose that $f$ satisfies the next conditions:

$$\frac{\nu I^{m+1} (f(z)) + \eta I^{m+2} (f(z))}{(\nu + \eta) z} 
eq 0, z \in U$$

and

$$\left[ \frac{\nu I^{m+1} (f(z)) + \eta I^{m+2} (f(z))}{(\nu + \eta) z} \right]^\sigma \in H [q(0), 1] \cap Q.$$  

If the function $\psi$ given by (2.8) is univalent in $U$ and

$$tq(z) + \frac{zq'(z)}{q(z)} < \psi(z),$$

then

$$q(z) < \left[ \frac{\nu I^{m+1} (f(z)) + \eta I^{m+2} (f(z))}{(\nu + \eta) z} \right]^\sigma$$

and $q(z)$ is the best subordinant of (2.14).

Proof. Let

$$p(z) = \left[ \frac{\nu I^{m+1} (f(z)) + \eta I^{m+2} (f(z))}{(\nu + \eta) z} \right]^\sigma, z \in U.$$  

According to (2.12) the function $p(z)$ is analytic in $U$ and differentiating (2.15) logarithmically with respect to $z$, we obtain

$$\frac{zp'(z)}{p(z)} = \sigma \left[ \frac{\nu z (I^{m+1} (f(z)))' + z\eta (I^{m+2} (f(z)))'}{\nu I^{m+1} (f(z)) + \eta I^{m+2} (f(z))} - 1 \right].$$

In order to prove our result we will use Lemma 1.4. In this lemma we consider

$$Q(z) = zq'(z)\varphi(q(z)) = \sigma \left[ \frac{\nu z (I^{m+1} (f(z)))' + z\eta (I^{m+2} (f(z)))'}{\nu I^{m+1} (f(z)) + \eta I^{m+2} (f(z))} - 1 \right]$$

and

$$h(z) = \theta(q(z)) + Q(z)$$

$$= t \left[ \frac{\nu I^{m+1} (f(z)) + \eta I^{m+2} (f(z))}{(\nu + \eta) z} \right]^\sigma + \sigma \left[ \frac{\nu z (I^{m+1} (f(z)))' + z\eta (I^{m+2} (f(z)))'}{\nu I^{m+1} (f(z)) + \eta I^{m+2} (f(z))} - 1 \right]$$

from (2.12) we see that $Q(z)$ is a starlike function in $U$. We also have

$$\text{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} = \text{Re} \left\{ t + 1 + \frac{zq''(z)}{q'(z)} \right\} > 0 \text{ for } z \in U$$

and then, by using Lemma 1.4 we deduce that the subordination (2.14) implies $q(z) < p(z)$ and the proof is completed. □
Corollary 2.8. Let $q_1, q_2$ are two convex functions in $U$, with $q_1(0) = q_2(0) = 1$, $\sigma \in \mathbb{C}^*$, $t, \nu, \eta \in \mathbb{C}$ with $\nu + \eta \neq 0$ and $\text{Re} \ t > 0$. Let $f \in \mathcal{A}$ and suppose that $f$ satisfies the next conditions:

$$\frac{\nu I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\nu + \eta) z} \neq 0, z \in U$$

and

$$\left(\frac{\nu I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\nu + \eta) z}\right)^\sigma \in \mathcal{H}[q_1(0), 1] \cap Q.$$

If the function $\psi(z)$ given by (2.8) is univalent in $U$ and

$$tq_1(z) + \frac{zq_1'(z)}{q_1(z)} \prec \psi(z) \prec tq_2(z) + \frac{zq_2'(z)}{q_2(z)}$$

then

$$q_1(z) \prec \left(\frac{\nu I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\nu + \eta) z}\right)^\sigma \prec q_2(z) \quad (2.17)$$

and $q_1, q_2$ are respectively, the best subordinant and the best dominant of (2.17).

References


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