King-type operators related to squared Szász-Mirakyan basis

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Abstract. In this paper we study some approximation properties of a sequence of positive linear operators defined by means of the squared Szász-Mirakyan basis and prove that these operators behave better than the classical Szász-Mirakyan operators.

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1. Introduction

The operators defined by

\[ S_n(f, x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0, \infty), \quad n = 1, 2, \ldots \]

where \( s_{n,k} \) are

\[ s_{n,k}(x) = e^{-nx} \frac{(nx)^k}{k!}, \]

were introduced and studied independently by Mirakyan [14], Favard [3] and Szász [17]. They usually are referred to as Szász-Mirakyan operators and the functions \( s_{n,k} \) form the Szász-Mirakyan basis or the Poisson distribution.

Motivated by the article of Gavrea and Ivan [4] we study the following operators

\[ A_n(f, x) = \frac{\sum_{k=0}^{\infty} s_{n,k}^2(x) f\left(\frac{k}{n}\right)}{\sum_{k=0}^{\infty} s_{n,k}^2(x)}, \quad x \geq 0, \quad n = 1, 2, \ldots \] (1.1)
Herzog [5] introduced and studied the following sequence of positive linear operators

\[ A_\nu^n(f, x) = \begin{cases} 
\frac{1}{I_\nu(nx)} \sum_{k=0}^{\infty} \frac{(nx)^{2k+\nu}}{\Gamma(k+1)\Gamma(k+1+\nu)} \cdot f \left( \frac{2k}{n} \right), & x > 0 \\
 f(0), & x = 0 
\end{cases} \]

where \( I_\nu \) is the modified Bessel function defined by

\[ I_\nu(t) = \sum_{k=0}^{\infty} \frac{(nt)^{2k+\nu}}{\Gamma(k+1)\Gamma(k+1+\nu)}. \]

For \( \nu = 0 \) the operators \( A_\nu^n \) can be written in terms of the operators (1.1) by

\[ A_0^n(f, x) = A_n(f \circ g^{-1}, g(x)), \]

where \( g \) is the function defined by \( g(x) = x/2, x \geq 0. \)

The author of [5] studied the operators \( A_\nu^n \) in polynomial and exponential weight spaces (see also [6]), but did not point out how well behave these operators compared to the Szász-Mirakyan operators.

In this paper, we show that \( A_n \) are King-type operators [12] preserving the functions \( e_0 \) and \( e_2 \) and so extending the class of Szász-Mirakyan type operators which preserve some polynomial functions [2, 18]. We also prove that the error of approximation of a function \( f \) by \( A_n f \) is smaller than the error of approximation by the classical Szász-Mirakyan operators. In the final part of the paper, we present some approximation properties of \( (A_n) \), showing what functions can be uniformly approximated by these operators and what is the order of the convergence by giving a quantitative Voronovskaya theorem. A similar study for Bernstein operators was done recently in [4, 9] and for Baskakov operators in [10].

2. Some properties of the operators

Let us notice first that the operators \( A_n \) preserve the functions \( e_0 \) and \( e_2 \) (we denote as usual \( e_k(x) = x^k \)). From the relation (1.1) we can easily see that

\[ A_n(e_0, x) = e_0(x) = 1. \]

From the following relation

\[ \sum_{k=0}^{\infty} s_{n,k}^2(x) \cdot \frac{k^2}{n^2} e^{-2nx} \sum_{k=0}^{\infty} \frac{(nx)^{2k}}{(k!)^2} \cdot \frac{k^2}{n^2} = x^2 e^{-2nx} \sum_{k=1}^{\infty} \frac{(nx)^{2k-2}}{[(k-1)!]^2} \]

\[ = x^2 e^{-2nx} \sum_{i=0}^{\infty} \frac{(nx)^{2i}}{(i!)^2} = x^2 \sum_{i=0}^{\infty} s_{n,i}^2(x), \]

we deduce that \( A_n(e_2, x) = e_2(x) = x^2 \), for every \( x \geq 0 \). In fact, only for \( \nu = 0 \), the general operators \( A_\nu^n \) do preserve the function \( e_2 \). This can be seen from the following
relation obtained in [5]

\[ A_\nu^n(e_2, x) = x^2 \cdot \frac{I_{\nu+2}(nx)}{I_\nu(nx)} + \frac{2x}{n} \cdot \frac{I_{\nu+1}(nx)}{I_\nu(nx)} \]

and the recurrence relation (9.6.26) of [1]

\[ I_{\nu-1}(t) - I_{\nu+1}(t) = \frac{2\nu}{t} I_\nu(t). \]

We have

\[ A_\nu^n(e_2, x) = x^2 - \frac{2x\nu}{n} \cdot \frac{I_{\nu+1}(nx)}{I_\nu(nx)}. \]

So, \( A_\nu^n(e_2) = e_2 \) if and only if \( \nu = 0 \).

Let us denote

\[ \mu_{n,k}(x) = A_n((e_1 - x)^k, x), \quad k = 0, 1, 2, \ldots \]

the central moments of the operators \( A_n \), which will be very important in our study.

Next let us observe that

\[ \mu_{n,2}(x) = -2x\mu_{n,1}(x). \quad (2.1) \]

Indeed,

\[ \mu_{n,2}(x) = A_n(e_2, x) - 2xA_n(e_1, x) + x^2A_n(e_0, x) = 2x^2 - 2xA_n(e_1, x) = -2x\mu_{n,1}(x). \]

**Lemma 2.1.** For every \( x \in (0, \infty) \) we have

\[ \lim_{n \to \infty} 4n \cdot \mu_{n,1}(x) = -1 \quad (2.2) \]

\[ \lim_{n \to \infty} 2n \cdot \mu_{n,2}(x) = x. \quad (2.3) \]

**Proof.** Because of the relation (2.1) it suffices to prove (2.2).

Let us denote

\[ K_n(x) = \sum_{k=0}^\infty s_{n,k}^2(x). \quad (2.4) \]

The function \( K_n \) was expressed [15] in terms of the modified Bessel function \( I_0 \) by

\[ K_n(x) = e^{-2nx}I_0(2nx). \quad (2.5) \]

Using the well-known relation

\[ s_{n,k}'(x) = s_{n,k}(x) \cdot \frac{k - nx}{x} \]
we have
\[
2n \cdot \mu_{n,1}(x) = 2n \left( \sum_{k=0}^{\infty} s_{n,k}(x) \cdot \frac{k}{n} \right) = 2 \sum_{k=0}^{\infty} \frac{s_{n,k}(x)(k - nx)}{K_n(x)}
\]
\[
= \frac{2x \sum_{k=0}^{\infty} s_{n,k}(x)s'_{n,k}(x)}{K_n(x)} = \frac{xK'(x) - x}{K_n(x)} = \frac{2nx[I'_0(2nx) - I_0(2nx)]}{I_0(2nx)}.
\]

We have obtained a formula expressing the central moment of order 1 in terms of the modified Bessel function \( I_0 \):
\[
\mu_{n,1}(x) = x \left( \frac{I'_0(2nx)}{I_0(2nx)} - 1 \right). \tag{2.6}
\]

For every \( x \in (0, \infty) \) the quantity \( t = 2nx \) grows to infinity when \( n \) tends to infinity. Using the asymptotic relations (9.7.1) and (9.7.3) from Abramowitz and Stegun [1]
\[
I_0(t) \sim \frac{e^t}{\sqrt{2\pi t}} \left( 1 + \frac{1}{8t} + \frac{9}{2(8t)^2} + \ldots \right) \quad (t \to \infty) \tag{2.7}
\]
\[
I'_0(t) \sim \frac{e^t}{\sqrt{2\pi t}} \left( 1 - \frac{3}{8t} - \frac{15}{2(8t)^2} - \ldots \right) \quad (t \to \infty)
\]
we obtain
\[
\mu_{n,1}(x) \sim -\frac{1}{4n} - \frac{1}{32n^2x} - \frac{15}{1024n^3x^2} - \ldots \quad (n \to \infty)
\]
which proves (2.2). \( \square \)

Lemma 2.2. The sequence \((n \cdot \mu'_{n,1}(x))\) converges to zero for every \( x > 0 \).

Proof. Computing the derivative of \( \mu_{n,1} \) we obtain
\[
\mu'_{n,1}(x) = \frac{I'_0(2nx)}{I_0(2nx)} - 1 + 2nx \cdot \frac{I''_0(2nx)I_0(2nx) - [I'_0(2nx)]^2}{[I_0(2nx)]^2}.
\]
Using the relation \( tI''_0(t) + I'_0(t) - tI_0(t) = 0 \) (see (9.6.1) from [1]), we have
\[
\mu'_{n,1}(x) = 2nx - 1 - 2nx \frac{[I'_0(2nx)]^2}{[I_0(2nx)]^2}.
\]
The asymptotic relations (2.7) show that
\[
\mu'_{n,1}(x) \sim -\frac{29}{128(2nx)^2} + \frac{31}{1024(2nx)^3} + \ldots \quad (n \to \infty)
\]
and this proves the assertion stated in the lemma. \( \square \)

Lemma 2.3. For every \( x \geq 0 \) we have
\[
\mu_{n,2}(x) \leq S \cdot \frac{x}{n}, \tag{2.8}
\]
where $S$ is defined by

$$S = \sup_{x > 0} \left( x - \frac{x^2}{\frac{1}{2} + \sqrt{x^2 + \frac{9}{4}}} \right) = 0.67038 \ldots$$

**Proof.** Using (2.6) and (2.1) the central moment of order 2 can be expressed by

$$\mu_{n,2}(x) = 2x^2 \left( 1 - \frac{I_0'(2nx)}{I_0(2nx)} \right).$$

To prove (2.8) it is enough to prove that

$$t \left( 1 - \frac{I_0'(t)}{I_0(t)} \right) < S, \quad t > 0.$$ 

Using inequality (73) of [16] we have

$$\frac{tI_0'(t)}{I_0(t)} > \frac{t^2}{\frac{1}{2} + \sqrt{\frac{9}{4} + t^2}}.$$ 

But this proves that

$$t \left( 1 - \frac{I_0'(t)}{I_0(t)} \right) < t - \frac{t^2}{\frac{1}{2} + \sqrt{\frac{9}{4} + t^2}} \leq S. \quad \square$$

**Remark 2.4.** Because the second central moment of the usual Szász-Mirakyan operators is $\frac{x^2}{n}$, inequality (2.8) proves that the central moment of order 2 of the operators (1.1) is smaller than the classical Szász-Mirakyan operators. In addition, we use the estimation

$$|L_n(f, x) - f(x)| \leq (1 + n\mu_{n,2}(x)) \cdot \omega \left( f, \frac{1}{\sqrt{n}} \right),$$

which is valid for every sequence of positive linear operators ($L_n$) preserving constants functions and for every uniformly continuous function $f$. This estimation proves that the error by approximating $f$ with $A_n f$ is smaller than the error of approximation by the classical Szász-Mirakyan operators.

We prove in the next Lemma that $A_n$ satisfy a differential equation. This equation is similar to the relation satisfied by the so called exponential type operators (see [13, 11]).

**Lemma 2.5.** For every $f \in C[0,1]$ and $x \in (0,1)$ we have

$$\left( A_n(f, x) \right)' = \frac{2n}{x} \left[ A_n(f \cdot (e_1 - xe_0), x) - A_n(e_1 - xe_0, x) \cdot A_n(f, x) \right].$$

(2.9)

**Proof.** Using again

$$s_{n,k}'(x) = s_{n,k}(x) \cdot \frac{k - nx}{x}$$
we get
\[
\frac{\left(\sum_{i=0}^{n} s_{n,i}^2(x) \right)'}{\sum_{i=0}^{n} s_{n,i}^2(x)} = \frac{2s_{n,k}(x)s'_{n,k}(x)}{\sum_{i=0}^{n} s_{n,i}^2(x)} - \frac{2s_{n,k}^2(x)\sum_{i=0}^{n} s_{n,i}(x)s'_{n,i}(x)}{\left(\sum_{i=0}^{n} s_{n,i}^2(x)\right)^2}
\]
\[
= \frac{2s_{n,k}^2(x)}{\sum_{i=0}^{n} s_{n,i}^2(x)} \cdot \left( \frac{k - nx}{x} - \frac{\sum_{i=0}^{n} s_{n,i}(x) \frac{i}{x}}{\sum_{i=0}^{n} s_{n,i}^2(x)} \right)
\]
\[
= \frac{2n}{x} \cdot \frac{s_{n,k}^2(x)}{\sum_{i=0}^{n} s_{n,i}^2(x)} \cdot \left( \frac{k}{n} - \frac{\sum_{i=0}^{n} s_{n,i}(x) \frac{i}{n}}{\sum_{i=0}^{n} s_{n,i}^2(x)} \right).
\]

We obtain
\[
(A_n(f, x))' = \frac{2n}{x} \cdot A_n(f \cdot (e_1 - A_n(e_1, x)), x)
\]
which is equivalent with (2.9). \qed

Lemma 2.6. We have for every \(x > 0\)
\[
\lim_{n \to \infty} (2n)^2 \cdot \mu_{n,4}(x) = 3x^2.
\]

Proof. Using Lemma 2.2 and (2.1) the following limit holds true for every \(x > 0\)
\[
\lim_{n \to \infty} 2n \cdot \mu'_{n,2}(x) = \lim_{n \to \infty} -4n\mu_{n,1}(x) - 4nx\mu'_{n,1}(x) = 1.
\]
In relation (2.9) we take \(f = (e_1 - xe_0)^k\) and we obtain the recurrence relation
\[
(\mu_{n,k}(x))' + k \cdot \mu_{n,k-1}(x) = \frac{2n}{x} \cdot [\mu_{n,k+1}(x) - \mu_{n,1}(x) \cdot \mu_{n,k}(x)],
\]
which is similar to the relation (2.7) of Ismail and May [11]. Using (2.10) we get
\[
2n\mu_{k+1}(x) = x\mu'_{n,k}(x) + kx\mu_{n,k-1}(x) + 2n\mu_{n,1}(x)\mu_{n,k}(x), \quad k = 1, 2, \ldots
\]
For \(k = 2\) we have
\[
2n\mu_3(x) = x\mu'_{n,2}(x) + 2x\mu_{n,1}(x) + 2n\mu_{n,1}(x)\mu_{n,2}(x).
\]
Multiplying this equality with $2n$ and using the relations (2.2) and (2.3), we have for every $x$

$$\lim_{n \to \infty} 4n^2 \cdot \mu_{n,3}(x) = \frac{x}{2}.$$  

For $k = 3$, the recurrence (2.10) becomes

$$\mu'_{n,3}(x) + 3\mu_{n,2}(x) = \frac{2n}{x} \cdot [\mu_{n,4}(x) - \mu_{n,1}(x)\mu_{n,3}(x)].$$

Multiplying with $2n$ and letting $n$ tend to infinity we get

$$\lim_{n \to \infty} 4n^2 \cdot \mu_{n,4}(x) = 3x^2,$$

for every $x > 0$, if $2n\mu'_{n,3}(x) \to 0$. We prove this convergence.

Applying the derivative to the relation (2.10) for $k = 2$ we get

$$2n\mu'_{n,3}(x) = 2n\mu_{n,1}(x)\mu'_{n,2}(x) + 2n\mu_{n,2}(x)\mu'_{n,1}(x)$$

$$+ \mu'_{n,2}(x) + x\mu''_{n,2}(x) + 2x\mu'_{n,1}(x) + 2\mu_{n,1}(x).$$

It remains to prove that $\mu''_{n,2}$ converges to zero.

Applying the derivative twice to the relation (2.1), the sequence $(\mu''_{n,2})$ converges to zero if and only if the sequence $\mu''_{n,1}$ converges to zero. But applying the derivative to the relation (2.10) for $k = 1$ we obtain

$$2n\mu'_{n,2}(x) = 4n\mu_{n,1}(x)\mu'_{n,1}(x) + \mu'_{n,1}(x) + x\mu''_{n,1}(x) + 1.$$ 

Using that $2n\mu'_{n,2}(x) \to 1$ we obtain that $\mu''_{n,1} \to 0$ and the lemma is proved. □

3. Some approximation results

In order to give some approximation results for the operators $A_n$, let us introduce some notation.

For $\alpha \geq 0$, we denote by $C_{\theta,\alpha}$ the space of all continuous functions defined on the positive half-line $f : (0, \infty) \to \mathbb{R}$ with the property that exists a constant $M > 0$ such that $|f(x)| \leq Me^{\alpha\theta(x)}$, for every $x > 0$. We denote with $C_\theta$ the union of all spaces $C_{\theta,\alpha}$.

Let us observe that for $\theta(x) = x$, the functions $A_n f$ exist for every $f \in C_{\theta,\alpha}$. To prove this, it is enough to prove that $A_n(e^{\alpha x})$ exist. We will prove more in the next lemma.

Lemma 3.1. The sequence $A_n(e^{\alpha x}, x)$ converges pointwise to the function $e^{\alpha x}$.

Proof. We have

$$A_n(e^{\alpha x}, x) = \frac{I_0(2nxe^{\alpha x})}{I_0(2nx)}.$$

For a fixed $x \in (0, \infty)$ we use the asymptotic relation (2.7) and we obtain

$$A_n(e^{\alpha x}, x) \sim \frac{e^{2nxe^{\alpha x}}}{\sqrt{2\pi} \cdot 2nxe^{\alpha x}} \cdot \sqrt{\frac{2\pi}{2n}} \cdot 2nx \sim e^{2nxe^{\alpha x}} \sim e^{\alpha x} \quad (n \to \infty). \quad \Box$$
Remark 3.2. The Lemma 3.1 implies that for a fixed \( x > 0 \) we have
\[
A_n(\max(e^{\alpha t}, e^{\alpha x}), x) \leq M_\alpha(x),
\]
for every \( n \in \mathbb{N} \). Indeed, for \( x > 0 \), there is \( n_0 \) such that
\[
|A_n(e^{\alpha t}, x) - e^{\alpha x}| \leq 1,
\]
for every \( n \geq n_0 \).

We obtain for every \( n \geq n_0 \)
\[
A_n(\max(e^{\alpha t}, e^{\alpha x}), x) \leq A_n(e^{\alpha t} + e^{\alpha x}, x) \leq 1 + 2e^{\alpha x}.
\]
The inequality (3.1) is true for
\[
M_\alpha(x) = 1 + 2e^{\alpha x} + \max_{n \leq n_0} A_n(\max(e^{\alpha t}, e^{\alpha x}), x).
\]

Remark 3.3. As was pointed out in Remark 7.2.1 of [6], the function \( A_nf \) does not necessarily belong to the space \( C_{\theta,\alpha} \) when \( f \) belong to the space \( C_{\theta,\alpha} \), for \( \theta(x) = x \).

We prove that for \( \theta(x) = \sqrt{x} \), this condition is satisfied as in the case of the classical Szász-Mirakyan operators (see [7]).

Lemma 3.4. There is a constant \( M_\alpha > 0 \) not depending on \( n \) or \( x \) such that
\[
A_n(e^{\alpha \sqrt{t}}, x) \leq M_\alpha e^{\alpha \sqrt{x}},
\]
for every \( x > 0 \), \( \alpha \geq 0 \) and \( n \in \mathbb{N} \).

Proof. We need to prove that \( A_n(e^{\alpha (\sqrt{t} - \sqrt{x})}, x) \) is bounded.
Starting from the inequality
\[
\sqrt{t} - \sqrt{x} = \frac{t - x}{\sqrt{t} + \sqrt{x}} \leq \frac{t - x}{\sqrt{x}}, \quad x > 0
\]
we obtain that
\[
A_n(e^{\alpha (\sqrt{t} - \sqrt{x})}, x) \leq A_n(e^{\frac{\alpha (t - x)}{\sqrt{x}}}, x) = \frac{A_n(e^{\frac{\alpha t}{\sqrt{x}}}, x)}{e^{\alpha \sqrt{x}}} = \frac{I_0(2nx e^{\frac{\alpha}{2n\sqrt{x}}})}{I_0(2nx) \cdot e^{\alpha \sqrt{x}}}.
\]

Using again (2.7) we deduce the existence of a constant \( t_0 > 0 \) such that
\[
\frac{e^t}{2\sqrt{2\pi t}} < I_0(t) < \frac{3e^t}{2\sqrt{2\pi t}}, \quad \text{for every} \quad t > t_0.
\]

So, for \( x > \frac{t_0}{2n} \) and \( n \in \mathbb{N} \)
\[
A_n(e^{\alpha (\sqrt{t} - \sqrt{x})}, x) \leq 3 \frac{e^{2nx e^{\frac{\alpha}{2n\sqrt{x}}}} \cdot \sqrt{2\pi} \cdot 2nx}{e^{2nx} \cdot e^{\alpha \sqrt{x}}} \cdot \sqrt{2\pi} \cdot 2nx e^{\frac{\alpha}{2n\sqrt{x}}}
\]
\[
\leq 3 \exp \left( 2nx(e^{\frac{\alpha}{2n\sqrt{x}}} - 1) - \alpha \sqrt{x} \right).
\]

Using the inequality \( e^u - 1 \leq u + u^2 e^u, \ u \geq 0 \), we obtain
\[
A_n(e^{\alpha (\sqrt{t} - \sqrt{x})}, x) \leq 3 \exp \left( 2nx \cdot \frac{\alpha}{2n \sqrt{x}} + 2nx \cdot \frac{\alpha^2}{4n^2 x} e^{\frac{\alpha}{2n \sqrt{x}}} - \alpha \sqrt{x} \right)
\]
\[
= 3 \exp \left( \frac{\alpha^2}{2n} e^{\frac{\alpha}{2n \sqrt{x}}} \right) \leq 3 \exp \left( \frac{\alpha^2}{2} e^{\frac{\alpha}{\sqrt{t_0}}} \right).
\]
Consider now the case when \( x \) is smaller than \( \frac{t_0}{2n} \). In this case, we need only prove that \( A_n(e^{\alpha \sqrt{\frac{t}{n}}}, x) \) is bounded. Because \( \sqrt{k} \leq k \), for every \( k = 0, 1, 2, \ldots \) and \( I_0(2nx) \geq 1 \) we obtain
\[
A_n(e^{\alpha \sqrt{\frac{t}{n}}}, x) \leq A_n(e^{t\alpha \sqrt{\frac{t}{n}}}, x) = \frac{I_0(2nx e^{\frac{\alpha \sqrt{t}}{n}})}{I_0(2nx)} \leq I_0 \left( 2nx e^{\frac{\alpha \sqrt{t}}{n}} \right) \leq I_0 \left( t_0 e^{\frac{\alpha \sqrt{t}}{n}} \right).
\]

We need the following general result.

**Theorem 3.5** ([8]). Let \( m \) be a nonnegative integer and let \( f \in C_{\theta, \alpha} \) such that \( f \) is \( m \) times continuously differentiable with \( f^{(m)} \in C_{\theta, \alpha} \). Then
\[
\left| L_n(f, x) - \sum_{k=0}^{m} \frac{f^{(k)}(x)}{k!} \cdot \mu_n(k) \right| \leq \frac{1}{m!} \left( A_{n,m}(x) + \frac{B_{n,m}(x)}{\delta_n} \right) \omega_{\varphi, \theta, \alpha} \left( f^{(m)}, \delta_n \right)
\]
where
\[
A_{n,m}(x) = L_n \left( \max \left( e^{\alpha \theta(t)}, e^{\alpha \theta(x)} \right) |t - x|^m, x \right)
\]
\[
B_{n,m}(x) = L_n \left( \max \left( e^{\alpha \theta(t)}, e^{\alpha \theta(x)} \right) |t - x|^m \cdot |\varphi(t) - \varphi(x)|, x \right)
\]
\[
\omega_{\varphi, \theta, \alpha}(f, \delta) = \sup_{x, t \in I, |\varphi(t) - \varphi(x)| \leq \delta} \frac{|f(t) - f(x)|}{\max \left( e^{\alpha \theta(t)}, e^{\alpha \theta(x)} \right)}
\]
and \( \varphi \) is a continuous and strictly increasing function on \( I \) such that \( \theta \circ \varphi^{-1} \) is uniformly continuous on \( \varphi(I) \).

**Theorem 3.6.** Let \( \theta(x) = \varphi(x) = \sqrt{x} \). For every \( f \in C_{\theta, \alpha} \) there is a constant \( M > 0 \) independent of \( n \) and \( x \) such that
\[
|A_n(f, x) - f(x)| \leq Me^{\alpha \sqrt{x}} \cdot \omega_{\varphi, \theta, \alpha} \left( f, \frac{1}{\sqrt{n}} \right)
\]
for every \( x > 0 \) and \( n \in \mathbb{N} \).

**Proof.** We apply Theorem 3.5 for \( m = 0 \) and \( \delta_n = \frac{1}{\sqrt{n}} \). Using inequality (3.2) we easily obtain that \( A_{n,0}(x) \leq C_1 e^{2\alpha \sqrt{x}} \), for every \( x > 0 \), for some constant \( C_1 > 0 \). Using the Cauchy-Schwarz inequality for positive linear operators the quantity \( B_{n,0}(x) \) is bounded by
\[
B_{n,0}(x) \leq \sqrt{A_n(\max(e^{2\alpha \sqrt{\frac{t}{n}}}, e^{2\alpha \sqrt{x}}), x) \cdot \sqrt{A_n(|\varphi(t) - \varphi(x)|^2, x)}}.
\]
Using inequalities (3.3) and (2.8) we have for \( x > 0 \)
\[
A_n(|\varphi(t) - \varphi(x)|^2, x) \leq \frac{1}{x} \cdot \mu_n(2,x) \leq \frac{S}{n}.
\]
Using again (3.2), the inequality
\[
\sqrt{n} \cdot B_{n,0}(x) \leq C_2
\]
is true for every \( x > 0 \) and \( n \geq 1 \), where \( C_2 \) is some constant independent of \( n \) and \( x \). \( \Box \)
Corollary 3.7. For every function $f$ such that $g(x) = e^{-x}f(x^2)$ is uniformly continuous on $(0, \infty)$ we have
\[
\lim_{n \to \infty} \sup_{x > 0} e^{-\alpha \sqrt{x}} |A_n(f, x) - f(x)| = 0.
\]

Proof. Because $g$ is uniformly continuous, $\omega_{\varphi, \theta, \alpha}(f, 1/\sqrt{n}) \to 0$ when $n \to \infty$ (see [8]).

Theorem 3.8. For $\alpha \geq 0$, $\theta(x) = x$ and $\varphi(x) = x$ let $f \in C_{\theta, \alpha}$ be a twice continuously differentiable function such that $f'' \in C_{\theta, \alpha}$. Then
\[
\left| A_n(f, x) - f(x) - \mu_{n, 1}(x)f'(x) - \frac{\mu_{n, 2}(x)}{2} f''(x) \right| \\
\leq \frac{1}{2} \left( \sqrt{\mu_{n, 4}(x) M_{2 \alpha}(x)} + \sqrt{n} \cdot 4 \sqrt{M_{4 \alpha}(x)} \cdot 4 \sqrt{[\mu_{n, 4}(x)]^3} \right) \cdot \omega_{\varphi, \theta, \alpha} \left( f'', \frac{1}{\sqrt{n}} \right),
\]
for every $x > 0$ and $n \in \mathbb{N}$.

Proof. We use Theorem 3.5 for $m = 2$ and $\delta_n = \frac{1}{\sqrt{n}}$. We have
\[
A_{n, 2}(x) \leq \sqrt{A_{n} \left( \max(e^{2 \alpha t}, e^{2 \alpha x}) \right)} \cdot \sqrt{A_{n}(\|t - x\|^4, x)} \leq \sqrt{\mu_{n, 4}(x) M_{2 \alpha}(x)}.
\]
Using Hölder inequality for $p = 4$ and $q = 4/3$ we obtain
\[
B_{n, 2}(x) = A_{n} \left( \max(e^{\alpha t}, e^{\alpha x}) \|t - x\|^3, x \right) \\
\leq (A_{n} \left( \max(e^{4 \alpha t}, e^{4 \alpha x}) \right))^{\frac{1}{4}} \cdot (A_{n} \left( \|t - x\|^4, x \right))^{\frac{3}{4}} \\
\leq \sqrt[4]{M_{4 \alpha}(x)} \cdot 4 \sqrt{\mu_{n, 4}(x)}.
\]

Corollary 3.9. For every $f \in C_{\theta, \alpha}$, with $\theta(x) = x$ such that $f''$ exists and
\[
g(x) = e^{-x}f''(x)
\]
is uniformly continuous on $(0, \infty)$ and for every $x > 0$
\[
\lim_{n \to \infty} n[A_n(f, x) - f(x)] = -\frac{1}{4} \cdot f'(x) + \frac{x}{4} \cdot f''(x).
\]

Proof. Because $g$ is uniformly continuous on $(0, \infty)$, the quantity $\omega_{\varphi, \theta, \alpha} \left( f'', \frac{1}{\sqrt{n}} \right)$ tends to zero as $n$ goes to infinity. We multiply with $n$ the inequality proved in Theorem 3.8 and we take the limit as $n$ tends to infinity, using Lemma 2.6 and the relations (2.2) and (2.3). The right-hand side of this inequality is 0.

Problem 3.10. We propose the reader to study the general operators
\[
L_n(f, x) = \frac{\sum_{k=0}^{\infty} g(s_{n, k}(x)) f \left( \frac{k}{n} \right)}{\sum_{k=0}^{\infty} g(s_{n, k}(x))}, \quad x \geq 0, \ n = 1, 2, \ldots
\]
For \( g(x) = x \) we obtain the classical Szász-Mirakyan operators. For \( g(x) = x^2 \) we have the operators studied in this paper. It would be interesting to study the operators for \( g(x) = x^m \), related to the Rényi entropy and for \( g(x) = x \ln x \), related to the Shannon entropy.

References

[14] Mirakjan, G., Approximation des fonctions continues au moyen de polynomes de la forme \( e^{-nx} \sum_{k=0}^{m} C_{k,n} x^k \), Dokl. Akad. Nauk, 31(1941), 201-205.
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