A refinement of an inequality due to Ankeny and Rivlin

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Abstract. Let \( p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu} \) be a polynomial of degree \( n \),
\[
M(p, R) := \max_{|z|=R \geq 0} |p(z)|, \quad \text{and} \quad M(p, 1) := M(p).
\]
Then by well-known result due to Ankeny and Rivlin [1], we have
\[
M(p, R) \leq \left( \frac{R^n + 1}{2} \right) M(p), \quad R \geq 1.
\]
In this paper, we sharpen and generalizes the above inequality by using a result due to Govil [5].


Keywords: Inequalities, polynomials, maximum modulus.

1. Introduction

Let \( \mathcal{P}_n := \left\{ p(z); p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu} \right\} \) be a class of polynomial of degree \( n \). Let
\[
\max_{|z|=R} |p(z)| = M(p, R) \quad \text{and} \quad M(p, 1) = M(p).
\]
Then from maximum modulus principle, \( M(p, R) \) is a strictly increasing function and for \( 0 \leq R < \infty \). Also, it is a simple deduction from the maximum modulus principle (see [10, p. 158, Problem 269]) that for \( R \geq 1 \),
\[
M(p, R) \leq R^n M(p). \quad (1.1)
\]
The result is best possible and equality holds if and only if \( p(z) = \lambda z^n \), where \( \lambda \) being a complex number.

For \( p \in \mathcal{P}_n \) not vanishing in the interior of unit circle, Ankeny and Rivlin [1] sharpened inequality (1.1), by proving following result.
Theorem 1.1. If \( p \in \mathcal{P}_n \) and \( p(z) \neq 0 \) for \(|z| < 1\), then for \( R \geq 1\),

\[
M(p, R) \leq \left( \frac{R^n + 1}{2} \right) M(p), \quad R \geq 1.
\]

The above inequality is sharp and equality holds for polynomial

\[ p(z) = \alpha + \beta z^n, \quad |\alpha| = |\beta|. \]

Since the equality in (1.2) holds only for \( p(z) = \alpha + \beta z^n \), which satisfy

\[ |\beta| = \frac{1}{2} M(p), \]

therefore it should possible to improve the bound (1.2) for the polynomial not satisfying (1.3). Govil [5] solve this problem by proving the following result.

Theorem 1.2. If \( p \in \mathcal{P}_n \) and \( p(z) \neq 0 \) for \(|z| < 1\), then for \( R \geq 1\),

\[
M(p, R) \leq \left( \frac{R^n + 1}{2} \right) M(p) - \frac{n}{2} \left( \frac{M(p)^2 - 4|a_n|^2}{M(p)} \right) \left\{ \frac{(R - 1)M(p)}{M(p) + 2|a_n|} \right\} - \ln \left( 1 + \frac{(R - 1)M(p)}{M(p) + 2|a_n|} \right)
\]

(1.4)

The result is best possible and the equality holds for \( p(z) = (\lambda + \mu z^n) \), \( \lambda \) and \( \mu \) being complex numbers with \(|\lambda| = |\mu|\).

The other extension and generalization of Theorem 1.1 has been mentioned in the various article, e.g. Aziz [2], Aziz and Mohammad [3], Milovanović, Mitrinović and Rassias [8], Govil [6], Govil, Qazi and Rahman [7] and Rahman and Schmeisser [12], Tripathi [13] etc.

2. Main results

In this paper, we prove the following improved generalization of Theorem 1.2 for the class of Lacunary type of polynomial

\[ p(z) = a_0 + \sum_{\nu=\mu}^{n} a_{\nu} z^\nu. \]

Theorem 2.1. If \( p(z) = a_0 + \sum_{\nu=\mu}^{n} a_{\nu} z^\nu \) is a polynomial of degree \( n \) and \( p(z) \neq 0 \) for \(|a| < k, k \geq 1\), then for \( R > r \geq 1\),

\[
|\{p(Re^{i\theta})\}^s| \leq \frac{(R^n s - r^n s)}{1 + k^\mu} \{M(p)\}^s - \frac{n}{1 + k^\mu} \{M(p)\}^s \left( 1 - \frac{(1 + k^\mu)|a_n|}{M(p)} \right) h(n)
\]

\[ + |\{p(re^{i\theta})\}^s|, \]

(2.1)
where
\[ h(n) = \left( \frac{R^n - r^n}{n} \right) + \sum_{k=1}^{n-1} \left( \frac{R^{n-k} - r^{n-k}}{n-k} \right) (-1)^k \left( \frac{(1+k\mu)|a_n|}{M(p)} + 1 \right)^k \left( \frac{(1+k\mu)|a_n|}{M(p)} \right)^{k-1} \]
\[ + (-1)^n \left( \frac{(1+k\mu)|a_n|}{M(p)} + 1 \right)^n \left( \frac{(1+k\mu)|a_n|}{M(p)} \right)^{n-1} \ln \left( \frac{R(M(p)) + (1+k\mu)|a_n|}{r(M(p)) + (1+k\mu)|a_n|} \right) \]
for \( n \geq 1 \) and \( h(0) = 0 \).

On taking \( s = 0, \mu = 1, r = 1 \) and \( k = 1 \), we have the following application of above Theorem 2.1.

**Corollary 2.2.** If \( p \in \mathcal{P}_n \) and \( p(z) \neq 0 \) for \( |z| < 1 \), then for \( R \geq 1 \),
\[ |p(Re^{i\theta})| \leq \frac{(R^n + 1)}{2} M(p) - \frac{n}{2} M(p) \left( 1 - \frac{2|a_n|}{M(p)} \right) h(n), \]
(2.2)
where
\[ h(n) = \left( \frac{R^n - 1}{n} \right) + \sum_{k=1}^{n-1} \left( \frac{R^{n-k} - 1}{n-k} \right) (-1)^k \left( \frac{2|a_n|}{M(p)} + 1 \right) \left( \frac{2|a_n|}{M(p)} \right)^{k-1} \]
\[ + (-1)^n \left( \frac{2|a_n|}{M(p)} + 1 \right)^n \left( \frac{2|a_n|}{M(p)} \right)^{n-1} \ln \left( 1 + \frac{(R-1)M(p)}{M(p) + 2|a_n|} \right) \]
for \( n \geq 1 \) and \( h(0) = 0 \).

**Remark 2.3.** From Lemma 3.7, we get \( 0 \leq h(n) \). Using this in Corollary 2.2, we get
\[ |p(Re^{i\theta})| \leq \frac{(R^n + 1)}{2} M(p) - \frac{n}{2} M(p) \left( 1 - \frac{2|a_n|}{M(p)} \right) h(n) \leq \frac{(R^n + 1)}{2} M(p), \]
which shows that Corollary 2.2, clearly refines Theorem 1.1 due to Ankeny and Rivlin [1].

**Remark 2.4.** From Lemma 3.7, we have \( h(1) \leq h(n) \). Using this inequality in Corollary 2.2, we get
\[ |p(Re^{i\theta})| \leq \frac{(R^n + 1)}{2} M(p) - \frac{n}{2} M(p) \left( 1 - \frac{2|a_n|}{M(p)} \right) h(n) \leq \frac{(R^n + 1)}{2} M(p), \]
(2.3)
and,
\[ h(1) = (R - 1) - \left( 1 + \frac{2|a_n|}{M(p)} \right) \ln \left( 1 + \frac{(R-1)M(p)}{M(p) + 2|a_n|} \right). \]
(2.4)
Substitute the value of \( h(1) \) in (2.3), we get
\[ |p(Re^{i\theta})| \leq \left( \frac{R^n + 1}{2} \right) M(p) - \frac{n}{2} \left( \frac{M(p)^2}{M(p)} - 4|a_n|^2 \right) \left\{ \frac{(R-1)M(p)}{M(p) + 2|a_n|} \right\}, \]
which is Theorem 1.2 due to Govil [5].

By taking $\mu = 1$ in inequality (2.1), we obtain the following results.

**Corollary 2.5.** If $p \in P_n$ and $p(z) \neq 0$ for $|z| < k, k \geq 1$, then for $R > r \geq 1$,

$$
\left| \left\{ p(Re^{i\theta}) \right\}^s \right| \leq \frac{(R^{ns} - r^{ns})}{1 + k} \{M(p)\}^s - \frac{n}{1 + k} \{M(p)\}^s \left( 1 - \frac{(1 + k)|a_n|}{M(p)} \right) h(n) + \left| \left\{ p(re^{i\theta}) \right\}^s \right|,
$$

(2.5)

where

$$
h(n) = \left( \frac{R^n - r^n}{n} \right) + \sum_{k=1}^{n-1} \left( \frac{R^{n-k} - r^{n-k}}{n-k} \right) (-1)^k \left( \frac{(1 + k)|a_n|}{M(p)} + 1 \right) \left( \frac{(1 + k)|a_n|}{M(p)} \right)^{k-1}
$$

$$
+ (-1)^n \left( \frac{(1 + k)|a_n|}{M(p)} + 1 \right) \left( \frac{(1 + k)|a_n|}{M(p)} \right)^{n-1} \ln \left( \frac{R(M(p)) + (1 + k)|a_n|}{r(M(p)) + (1 + k)|a_n|} \right)
$$

for $n \geq 1$ and $h(0) = 0$.

**Remark 2.6.** We also have some other application Theorem 2.1, by taking $s = 0$, $k = 1$ and $r = 1$ respectively.

### 3. Lemmas

For the proof of theorem, we need the following lemmas. Our first lemma is a well-known generalization of Schwarz’s lemma (see for example [9, p. 167]).

**Lemma 3.1.** If $f(z)$ is analytic inside and on the circle $|z| = 1$, $f(0) = a$, where $|a| < f$, then

$$
|f(z)| \leq M(f) \left( \frac{M(f)|z| + |a|}{|a||z| + M(f)} \right).
$$

(3.1)

**Lemma 3.2.** If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree $n$, then for $|z| = R \geq 1$,

$$
|p(z)| \leq \left( \frac{|a_n| R + M(p)}{M(p) R + |a_n|} \right) M(p) R^n.
$$

(3.2)

The proof follows easily on applying Lemma 3.1 to the function $T(z) = z^n p(1/z)$ and noting that $M(T) = M(p)$ (for details see [12, Lemma 2]).

From Lemma 3.2, one immediately gets:

**Lemma 3.3.** If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree $n$, then for $|z| = R \geq 1$,

$$
|p(z)| \leq \left( 1 - \frac{M(p) - |a_n|(R - 1)}{M(p) R + |a_n|} \right) M(p) R^n.
$$

(3.3)

The following result is due to Chan and Malik [4].
Lemma 3.4. If \( p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v \) is a polynomial of degree \( n \), and \( p(z) \neq 0 \) for \( |z| < k, k \geq 1 \), then
\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + k^n} M(p). \tag{3.4}
\]

Lemma 3.5. If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \), and let \( r \geq 1 \), then
\[
\left(1 - \frac{(x-|a_n|)(r-1)}{rx + n|a_n|}\right)x \tag{3.5}
\]
is an increasing function of \( x \), for \( x > 0 \).

The proof of above lemma is straightforward using derivative test, so we omit the detail proof.

Lemma 3.6. Let
\[
h(n) = \int_{r}^{R} \frac{(t-1)(t^{n-1})}{t+a} dt \text{ for } n \geq 1.
\]
Then
\[
h(n) = \left(\frac{R^n - r^n}{n}\right) + \sum_{k=1}^{n-1} \left(\frac{R^{n-k} - r^{n-k}}{n-k}\right) (-1)^k (a+1)a^{k-1}
+ (-1)^n (a+1)a^{n-1} \ln \left(\frac{R+a}{r+a}\right).
\]

Proof. We define the function \( f(n) = \int_{r}^{R} \frac{t^n}{t+a} dt \) for \( n \geq 0 \). It is easy to see that
\[
h(n) = f(n) - f(n-1) \text{ for } n \geq 1.
\]
We can obtain
\[
f(n) + af(n-1) = \int_{r}^{R} \frac{t^n + at^{n-1}}{t+a} dt
= \int_{r}^{R} \frac{t^{n-1}(t+a)}{t+a} dt = \frac{R^n - r^n}{n} = g(n), \quad \text{(say)}.
\]
Then
\[
f(n) = g(n) - af(n-1). \tag{3.6}
\]
Solving the recurrence relation (3.6), we get
\[
f(n) = \sum_{k=0}^{n-1} g(n-k)(-1)^k a^k + (-1)^n a^n f(0), \tag{3.7}
\]
where
\[
f(0) = \int_{1}^{R} \frac{1}{r+a} dr = \ln \left(\frac{R+a}{r+a}\right).
\]
Now, Substituting the value of \( f(0) \) in (3.7), we get
\[
f(n) = \sum_{k=0}^{n-1} g(n-k)(-1)^k a^k + (-1)^n a^n \ln \left( \frac{R+a}{r+a} \right), n \geq 0. \tag{3.8}
\]
Using \( h(n) = f(n) - f(n-1) \) and value of \( g(n) \), we have Lemma 3.6 for \( n \geq 1 \). \( \square \)

**Lemma 3.7.** Let
\[
h(n) = \int_r^R \frac{(t-1)(t^{n-1})}{t+a} dt \text{ for } n \geq 1.
\]
Then \( h(n) \) is a non-negative increasing function of \( n \) for \( n \geq 1 \).

**Proof.** Let
\[
f(n) = \int_r^R \frac{r^n}{r+a} dr \text{ for } n \geq 0.
\]
It is easy to see that \( h(n) = f(n) - f(n-1) \) for \( n \geq 1 \). For \( n \geq 1 \),
\[
f(n) - f(n-1) = \int_1^R \frac{(r-1)(r^{n-1})}{r+a} dr \geq \int_1^R \frac{(r-1)(r^{n-2})}{r+a} dr = f(n-1) - f(n-2)
\]
as \( r^{n-1} \geq r^{n-2} \) for \( r \geq 1 \). Therefore,
\[
h(n) = f(n) - f(n-1) \geq f(n-1) - f(n-2) = h(n-1).
\]
Therefore, \( h(n) \) is an increasing function of \( n \) for \( n \geq 1 \).
Also, \( h(n) = f(n) - f(n-1) \geq 0 \) for \( n \geq 0 \) as
\[
\int_r^R \frac{(t-1)(t^{n-1})}{t+a} dt \geq 0
\]
for \( n \geq 1 \) and \( h(0) = 0 \). Therefore, \( h(n) \geq 0 \) and is an increasing function of \( n \) for \( n \geq 0 \). \( \square \)

**4. Proof of the Theorem**

**Proof of Theorem 2.1.** For each \( \theta, 0 \leq \theta < 2\pi \), we have
\[
|\{p(Re^{i\theta})\}^s - \{p(re^{i\theta})\}^s| = \left| \int_r^R \frac{d}{dt}\{p(te^{i\theta})\}^s dt \right| \leq \int_r^R s|\{p(te^{i\theta})\}^{s-1}||p'(te^{i\theta})|dt,
\]
\[
\leq \{M(p)\}^{s-1} \int_r^R t^{ns-n} s|p'(te^{i\theta})|dt
\]
\[
\leq \{M(p)\}^{s-1} \int_r^R \left\{ 1 - \frac{(M(p') - n|a_n|)(t - 1)}{n|a_n| + tM(p')} \right\} M(p')dt, \tag{4.1}
\]
by using Lemma 3.3 for the polynomial \( p'(z) \), which is of degree \( n - 1 \). We can see, from Lemma 3.5, the integrand in (4.1) is an increasing function of \( M(p') \).
Now, applying Lemma 3.4 to inequality (4.1), we get for $0 \leq \theta < 2\pi$,
\[
|\{p(Re^{i\theta})\}^s - \{p(re^{i\theta})\}^s| \\
\leq \{M(p)\}^{s-1} \int_r^R \frac{st^{sn-1}}{1+ (n+1)\mu} \left( 1 - \frac{n}{1+k\mu} \frac{M(p)}{n|a_n|} (t-1) \right) \frac{n}{1+k\mu} M(p) dt \\
= \frac{(R^n - r^n)}{1+k\mu} \{M(p)\}^s - \frac{n}{1+k\mu} \{M(p)\}^s (1 - \frac{n}{M(p)} \left( 1 + \frac{k\mu}{M(p)} |a_n| \right) ) h(n) \\
+ |\{p(re^{i\theta})\}^s|,
\]
where
\[
h(n) = \left( \frac{R^n - r^n}{n} \right) + \sum_{k=1}^{n-1} \left( \frac{R^n - r^n - R^{n-k} - r^{n-k}}{n-k} \right) (-1)^k \left( \frac{(1+k\mu)}{M(p)} |a_n| + 1 \right) \left( \frac{(1+k\mu)}{M(p)} |a_n| \right)^{k-1} \\
+ (-1)^n \left( \frac{(1+k\mu)}{M(p)} |a_n| + 1 \right) \left( \frac{(1+k\mu)}{M(p)} |a_n| \right)^{n-1} \ln \left( \frac{R M(p)}{r M(p)} + (1+k\mu) |a_n| \right)
\]
for $n \geq 1$ and $h(0) = 0$.

5. Computation

For the polynomial $p(z) = (z-2)^2$, $p(z) \neq 0$ for $|z| < 1$ and $M(p) = 9$. Then, for $R = 3$, exact value of $M(p, R)$ is 25. Using Theorem 1.2,
\[
M(p, R) \leq 45 - 7 \ast (2 - 11/9 \log(29/11)) = 39.29
\]
Using Corollary 2.2 of Theorem 2.1,
\[
M(p, R) \leq 45 - 7 \ast (4 - 22/9 + 22/81 \log(29/11)) = 32.26
\]

References


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