Application of Ruscheweyh $q$-differential operator to analytic functions of reciprocal order

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Abstract. The core object of this paper is to define and study new class of analytic function using Ruscheweyh $q$-differential operator. We also investigate a number of useful properties such as inclusion relation, coefficient estimates, subordination result, for this newly subclass of analytic functions.

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1. Introduction

Quantum calculus ($q$-calculus) is simply the study of classical calculus without the notion of limits. The study of $q$-calculus attracted the researcher due to its applications in various branches of mathematics and physics, see detail [8]. Jackson [10, 12] was the first to give some application of $q$-calculus and introduced the $q$-analogue of derivative and integral. Later on Aral and Gupta [5, 6, 7] defined the $q$-Baskakov Durrmeyer operator by using $q$-beta function while the author’s in [2, 3, 4] discussed the $q$-generalization of complex operators known as $q$-Picard and $q$-Gauss-Weierstrass singular integral operators. Recently, Kanas and Răducanu [13] defined $q$-analogue of Ruscheweyh differential operator using the concepts of convolution and then studied some of its properties. The application of this differential operator was further studied by Mohammed and Darus [1] and Mahmood and Sokół [14]. The aim of the current paper is to define a new class of analytic functions of reciprocal order involving $q$-differential operator.

Let $A$ be the class of functions having the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$  \hspace{1cm} (1.1)
which are analytic in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). Let \( \mathcal{M}(\alpha) \) denote a subclass of \( \mathcal{A} \) consisting of functions which satisfy the inequality

\[
\Re \frac{zf'(z)}{f(z)} < \alpha \quad (z \in U),
\]

for some \( \alpha (\alpha > 1) \). And let \( \mathcal{N}(\alpha) \) be the subclass of \( \mathcal{A} \) consisting of functions \( f \) which satisfy the inequality:

\[
\Re \frac{(zf'(z))'}{f'(z)} < \alpha \quad (z \in U),
\]

for some \( \alpha (\alpha > 1) \). These classes were studied by Owa et al. [16, 18]. Shams et al. [20] have introduced the \( k \)-uniformly starlike \( SD(k, \alpha) \) and \( k \)-uniformly convex \( CD(k, \alpha) \) of order \( \alpha \), for some \( k (k \geq 0) \) and \( \alpha (0 \leq \alpha < 1) \). Using these ideas in above defined classes, Junichi et al. [17] introduced the following classes.

**Definition 1.1.** Let \( f \in \mathcal{A} \). Then \( f \) is said to be in class \( MD(k, \alpha) \) if it satisfies

\[
\Re \frac{zf'(z)}{f(z)} < k \left| \frac{zf'(z)}{f(z)} - 1 \right| + \alpha \quad (z \in U),
\]

for some \( \alpha (\alpha > 1) \) and \( k (k \leq 0) \).

**Definition 1.2.** An analytic function \( f \) of the form (1.1) belongs to the class \( ND(k, \alpha) \), if and only if

\[
\Re \frac{(zf'(z))'}{f'(z)} < k \left| \frac{(zf'(z))'}{f'(z)} - 1 \right| + \alpha \quad (z \in U),
\]

for some \( \alpha (\alpha > 1) \) and \( k (k \leq 0) \).

If \( f \) and \( g \) are analytic in \( U \), we say that \( f \) is subordinate to \( g \), written as \( f \prec g \) or \( f(z) \prec g(z) \), if there exists a Schwarz function \( w \), which is analytic in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) such that \( f(z) = g(w(z)) \). Furthermore, if the function \( g(z) \) is univalent in \( U \), then we have the following equivalence holds, see [11, 15].

\[
f(z) \prec g(z) \quad (z \in U) \iff f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).
\]

For two analytic functions

\[
f(z) = \sum_{n=1}^{\infty} a_n z^n \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \quad (z \in U),
\]

For \( t \in \mathbb{R} \) and \( q > 0, q \neq 1 \), the number \( [t, q] \) is defined in [14] as

\[
[t, q] = \frac{1 - q^t}{1 - q}, \quad [0, q] = 0.
\]

For any non-negative integer \( n \) the \( q \)-number shift factorial is defined by

\[
[n, q]! = [1, q][2, q][3, q] \cdots [n, q], \quad ([0, q]! = 1).
\]

We have \( \lim_{q \to 1} [n, q] = n \). Throughout in this paper we will assume \( q \) to be fixed number between 0 and 1.

The \( q \)-derivative operator or \( q \)-difference operator for \( f \in \mathcal{A} \) is defined as

\[
\partial_q f(z) = \frac{f(qz) - f(z)}{z(q - 1)}, \quad z \in U.
\]
It can easily be seen that for \( n \in \mathbb{N} := \{1, 2, 3, \ldots \} \) and \( z \in \mathbb{U} \)
\[
\partial_q z^n = [n, q] z^{n-1}, \quad \partial_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n, q] a_n z^{n-1}.
\]
The \( q \)-generalized Pochhammer symbol for \( t \in \mathbb{R} \) and \( n \in \mathbb{N} \) is defined as
\[
[n, q] = \left[ t, q \right] \left[ t + 1, q \right] \left[ t + 2, q \right] \cdots \left[ t + n - 1, q \right],
\]
and for \( t > 0 \), the \( q \)-gamma function is defined as
\[
\Gamma_q(t) = [1, q] \Gamma_q(t) \quad \text{and} \quad \Gamma_q(1) = 1.
\]

**Definition 1.3.** [14] For a function \( f(z) \in \mathcal{A} \), the Ruscheweyh \( q \)-differential operator is defined as
\[
\mathcal{D}_q^\mu f(z) = \phi(q, \mu + 1; z) \ast f(z) = z + \sum_{n=2}^{\infty} \Phi_{n-1} a_n z^n, \quad (z \in \mathbb{U} \text{ and } \mu > -1),
\]
where
\[
\phi(q, \mu + 1; z) = z + \sum_{n=2}^{\infty} \Phi_{n-1} z^n,
\]
and
\[
\Phi_{n-1} = \frac{\Gamma_q(\mu + n)}{[n-1, q]! \Gamma_q(\mu + 1)} = \frac{[\mu + 1, q]_{n-1}}{[n-1, q]!}.
\]

From (1.2), it can be seen that \( L_0^q f(z) = f(z) \) and \( L_1^q f(z) = z \partial_q f(z) \),
and
\[
L_m^q f(z) = \frac{z \partial^m_q \left(z^{m-1} f(z)\right)}{[m, q]!}, \quad (m \in \mathbb{N}).
\]

\[
\lim_{q \to 1^-} \phi(q, \mu + 1; z) = \frac{z}{(1 - z)^{\mu + 1}},
\]
and
\[
\lim_{q \to 1^-} \mathcal{D}_q^\mu f(z) = f(z) \ast \frac{z}{(1 - z)^{\mu + 1}}.
\]
This shows that in case of \( q \to 1^- \), the Ruscheweyh \( q \)-differential operator reduces to the Ruscheweyh differential operator \( D^\mu \left(f(z)\right) \) (see [19]). From (1.2) the following identity can easily be derived.
\[
z \partial \mathcal{D}_q^\mu f(z) = \left(1 + \frac{[\mu, q]}{q^\mu}\right) \mathcal{D}_q^\mu f(z) - \frac{[\mu, q]}{q^\mu} \mathcal{D}_q^\mu f(z).
\]

If \( q \to 1^- \), then
\[
z \left(\mathcal{D}_q^\mu f(z)\right)' = (1 + \mu) \mathcal{D}_q^\mu f(z) - \mu \mathcal{D}_q^\mu f(z).
\]
Now using the Ruscheweyh \( q \)-differential operator, we define the following class.
Definition 1.4. Let $f \in \mathcal{A}$. Then $f$ is in the class $K^{\mathcal{D}}_{q}(k, \alpha, \gamma)$ if

$$
\Re \left\{ 1 + \frac{1}{\gamma} \left( \frac{z \partial_q^{\mu} f(z)}{D^{\mu}_q f(z)} - 1 \right) \right\} < k \left| \frac{1}{\gamma} \left( \frac{z \partial_q^{\mu} f(z)}{D^{\mu}_q f(z)} - 1 \right) \right| + \alpha,
$$

for some $k (k \leq 0)$, $\alpha (\alpha > 1)$ and for some $\gamma \in \mathbb{C} \setminus \{0\}$.

We note that $\mathcal{L}^{\mathcal{D}}_{2}(1, 1, \alpha) = \mathcal{M}(\alpha)$ and $\mathcal{L}^{\mathcal{D}}_{1}(1, 1, \alpha) = \mathcal{N}(\alpha)$, the classes introduced by Owa et al. [16, 18]. When we take $\gamma = 1, 2, c = 1$, and $a = 1$ the class $K^{\mathcal{D}}_{q}(k, \alpha, \gamma)$ reduces to the classes $\mathcal{M}^{\mathcal{D}}(k, \alpha)$ and $\mathcal{N}^{\mathcal{D}}(k, \alpha)$ (see [17]). For $1 < \alpha < 4/3$ the classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ were investigated by Uralegaddi et al. [21].

2. Preliminary results

Lemma 2.1. [9] For a positive integer $t$, we have

$$
\sigma \sum_{j=1}^{t} \frac{(\sigma)_{j-1}}{(j-1)!} = \frac{(\sigma)_{t}}{(t-1)!}.
$$

Proof. Consider

$$
\begin{align*}
\sigma \sum_{j=1}^{t} \frac{(\sigma)_{j-1}}{(j-1)!} &= \sigma \left( 1 + \frac{\sigma}{1} + \frac{(\sigma)_{2}}{2!} + \frac{(\sigma)_{3}}{3!} + \frac{(\sigma)_{4}}{4!} + \cdots + \frac{(\sigma)_{t-1}}{(t-1)!} \right) \\
&= \sigma (1 + \sigma) \left( 1 + \frac{\sigma}{2} + \frac{\sigma(\sigma + 2)}{2 \times 3} + \cdots + \frac{\sigma(\sigma + 2) \cdots (\sigma + t - 2)}{2 \times \cdots \times (t - 1)} \right) \\
&= \sigma (1 + \sigma) \frac{(\sigma + 2)}{2} \left( 1 + \frac{\sigma}{3} + \cdots + \frac{\sigma(\sigma + 3) \cdots (\sigma + t - 2)}{3 \times 4 \times \cdots \times (t - 1)} \right) \\
&= \sigma (1 + \sigma) \frac{(\sigma + 2)}{2} \frac{(\sigma + 3)}{3} \left( 1 + \frac{\sigma}{4} + \cdots + \frac{\sigma(\sigma + 4) \cdots (\sigma + t - 2)}{4 \times \cdots \times (t - 1)} \right) \\
&= \sigma (1 + \sigma) \frac{(\sigma + 2)}{2} \frac{(\sigma + 3)}{3} \frac{(\sigma + 4)}{4} \left( 1 + \frac{\sigma}{5} + \cdots + \frac{\sigma \cdots (\sigma + t - 2)}{5 \times 6 \times \cdots \times (t - 1)} \right) \\
&= \sigma (1 + \sigma) \frac{(\sigma + 2)}{2} \frac{(\sigma + 3)}{3} \frac{(\sigma + 4)}{4} \cdots \left( 1 + \frac{\sigma}{t-1} \right) \\
&= \sigma (1 + \sigma) \frac{(\sigma + 2)}{2} \frac{(\sigma + 3)}{3} \frac{(\sigma + 4)}{4} \cdots \left( \frac{\sigma + (t - 1)}{t - 1} \right) \\
&= \frac{(\sigma)_{t}}{(t-1)!}.
\end{align*}
$$

□
3. Main results

With the help of the definition of $\mathcal{KD}_q(k, \alpha, \gamma)$, we prove the following results.

**Theorem 3.1.** If $f(z) \in \mathcal{KD}_q(k, \alpha, \gamma)$, then

$$f(z) \in \mathcal{KD}_q(0, \frac{\alpha - k}{1 - k}, \gamma).$$

**Proof.** Because $k \leq 0$, we have

$$\Re \left\{ 1 + \frac{1}{\gamma} \left( \frac{z \partial_q \mathcal{D}^\mu_q f(z)}{\mathcal{D}^\mu_q f(z)} - 1 \right) \right\} < k \left| \frac{1}{\gamma} \left( \frac{z \partial_q \mathcal{D}^\mu_q f(z)}{\mathcal{D}^\mu_q f(z)} - 1 \right) \right| + \alpha,$$

$$\leq k \Re \left( \frac{1}{\gamma} \left( \frac{z \partial_q \mathcal{D}^\mu_q f(z)}{\mathcal{D}^\mu_q f(z)} - 1 \right) \right) + \alpha - k,$$

which implies that

$$(1 - k) \Re \frac{1}{\gamma} \left( \frac{z \partial_q \mathcal{D}^\mu_q f(z)}{\mathcal{D}^\mu_q f(z)} - 1 \right) < \alpha - k.$$

After simplification, we obtain

$$\Re \left[ 1 + \frac{1}{\gamma} \left( \frac{z \partial_q \mathcal{D}^\mu_q f(z)}{\mathcal{D}^\mu_q f(z)} - 1 \right) \right] < \frac{\alpha - k}{1 - k}, \quad (k \leq 0, \ \alpha > 1 \text{ and }).$$

(3.1)

This completes the proof. □

**Theorem 3.2.** If $f(z) \in \mathcal{KD}_q(k, \alpha, \gamma)$ and if $f(z)$ has the form (1.1), then

$$|a_n| \leq \frac{(\sigma)_{n-1}}{(n-1)! \Phi_{n-1}},$$

where

$$\sigma = \frac{2|\gamma| (\alpha - 1)}{q(1 - k)}.$$  

(3.2)

(3.3)

**Proof.** Let us define a function

$$p(z) = \frac{(\alpha - k) - (1 - k) \left[ 1 + \frac{1}{\gamma} \left( \frac{z \partial_q \mathcal{D}^\mu_q f(z)}{\mathcal{D}^\mu_q f(z)} - 1 \right) \right]}{\alpha - 1}. $$

(3.4)

Then $p(z)$ is analytic in $U$, $p(0) = 1$ and $\Re \{p(z)\} > 0$ for $z \in U$. We can write

$$\left[ 1 + \frac{1}{\gamma} \left( \frac{z \partial_q \mathcal{D}^\mu_q f(z)}{\mathcal{D}^\mu_q f(z)} - 1 \right) \right] = \frac{(\alpha - k) - (\alpha - 1)p(z)}{1 - k}.$$  

(3.5)

If we take $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, then (3.5) can be written as

$$z \partial_q \mathcal{D}^\mu_q f(z) - \mathcal{D}^\mu_q f(z) = -\frac{\gamma (\alpha - 1)}{1 - k} \left( \mathcal{D}^\mu_q f(z) \sum_{n=1}^{\infty} p_n z^n \right).$$

this implies that

$$\sum_{n=2}^{\infty} q [n - 1] \Phi_{n-1} a_n z^n = -\frac{\gamma (\alpha - 1)}{1 - k} \left( \sum_{n=1}^{\infty} \Phi_{n-1} a_n z^n \right) \left( \sum_{n=1}^{\infty} p_n z^n \right).$$
Using Cauchy product \( \left( \sum_{n=1}^{\infty} x_n \right) \cdot \left( \sum_{n=1}^{\infty} y_n \right) = \sum_{j=1}^{\infty} \sum_{k=1}^{j} x_k y_{k-j} \), we obtain

\[
q [n-1] \Phi_{n-1} a_n z^n = -\frac{\gamma (\alpha - 1)}{1 - k} \sum_{n=2}^{\infty} \left( \sum_{j=1}^{n-1} \Phi_{j-1} a_j p_{n-j} \right) z^n.
\]

Comparing the coefficients of \( nth \) term on both sides, we obtain

\[
a_n = -\frac{\gamma (\alpha - 1)}{q [n-1] \Phi_{n-1} (1 - k)} \sum_{j=1}^{n-1} \Phi_{j-1} a_j p_{n-j}.
\]

By taking absolute value and applying triangle inequality, we get

\[
|a_n| \leq \frac{|\gamma| (\alpha - 1)}{q [n-1] \Phi_{n-1} (1 - k)} \sum_{j=1}^{n-1} \Phi_{j-1} |a_j| |p_{n-j}|.
\]

Applying the coefficient estimates \( |p_n| \leq 2 \) \( (n \geq 1) \) for Caratheodory functions [11], we obtain

\[
|a_n| \leq \frac{2 |\gamma| (\alpha - 1)}{q [n-1] \Phi_{n-1} (1 - k)} \sum_{j=1}^{n-1} \Phi_{j-1} |a_j|,
\]

where \( \sigma = 2|\gamma|(\alpha - 1)/q(1 - k) \). To prove (3.2) we apply mathematical induction. So for \( n = 2 \), we have from (3.6)

\[
|a_2| \leq \frac{\sigma}{\Phi_1} = \frac{(\sigma)_{2-1}}{[2-1]!\Phi_{2-1}},
\]

which shows that (3.2) holds for \( n = 2 \). For \( n = 3 \), we have from (3.6)

\[
|a_3| \leq \frac{\sigma}{[3-1] \Phi_{3-1}} \{1 + \Phi_1 |a_2|\},
\]

using (3.7), we have

\[
|a_3| \leq \frac{\sigma}{[2 \Phi_2 (1 + \sigma)]} \frac{(\sigma)_{3-1}}{[3-1] \Phi_{3-1}},
\]

which shows that (3.2) holds for \( n = 3 \). Let us assume that (3.2) is true for \( n \leq t \), that is,

\[
|a_t| \leq \frac{(\sigma)_{t-1}}{[t-1]!\Phi_{t-1}} \quad j = 1, 2, \ldots, t.
\]

(3.8)
Using (3.6) and (3.8), we have

\[ |a_{t+1}| \leq \frac{\sigma}{t\Phi_t} \sum_{j=1}^{t} \Phi_{j-1} |a_j| \]
\[ \leq \frac{\sigma}{t\Phi_t} \sum_{j=1}^{t} \psi_{j-1} \frac{(\sigma)_{j-1}}{|j-1|!} \Phi_{j-1} \]
\[ = \frac{\sigma}{t\Phi_t} \sum_{j=1}^{t} \frac{(\sigma)_{j-1}}{|j-1|!} . \]

Applying (2.1), we have

\[ |a_{t+1}| \leq \frac{1}{t\Phi_t} \frac{(\sigma)_t}{|t-1|!} \]
\[ = \frac{1}{\Phi_t} \frac{(\sigma)_t}{|t|!} . \]

Consequently, using mathematical induction, we have proved that (3.2) holds true for all \( n, n \geq 2 \). This completes the proof. \( \square \)

**Theorem 3.3.** If a function \( f \in KD_q(k, \alpha, \gamma) \), then

\[ \frac{z \partial_q \mathcal{D}_q^\mu f(z)}{\mathcal{D}_q^\mu f(z)} < 1 + 2 (\alpha_1 - 1) - \frac{2 (\alpha_1 - 1)}{1 - z} \quad (z \in \mathbb{U}), \]  
\[ \alpha_1 = \frac{\alpha - k}{1 - k} . \]

**Proof.** If \( f(z) \in KD_q(k, \alpha, \gamma) \), then by (3.1),

\[ \Re \left\{ 1 + \frac{1}{\gamma} \left( \frac{z \partial_q \mathcal{D}_q^\mu f(z)}{\mathcal{D}_q^\mu f(z)} - 1 \right) \right\} < \alpha_1 . \]

Then there exists a Schwarz function \( w(z) \) such that

\[ \frac{\alpha_1 - \left\{ 1 + \frac{1}{\gamma} \left( \frac{z \partial_q \mathcal{D}_q^\mu f(z)}{\mathcal{D}_q^\mu f(z)} - 1 \right) \right\}}{\alpha_1 - 1} = \frac{1 + w(z)}{1 - w(z)} , \]  
(3.12)

and

\[ \Re \left\{ \frac{1 + w(z)}{1 - w(z)} \right\} > 0, \quad (z \in \mathbb{U}) . \]

Therefore, from (3.12), we obtain

\[ \frac{z \partial_q \mathcal{D}_q^\mu f(z)}{\mathcal{D}_q^\mu f(z)} = 1 + \gamma (\alpha_1 - 1) \left( 1 - \frac{1 + w(z)}{1 - w(z)} \right) . \]

This gives

\[ \frac{z \partial_q \mathcal{D}_q^\mu f(z)}{\mathcal{D}_q^\mu f(z)} = 1 + 2\gamma (\alpha_1 - 1) - \frac{2\gamma (\alpha_1 - 1)}{1 - w(z)} . \]
and hence
\[ \frac{z \partial_q \mathcal{D}_q^\mu f(z)}{\mathcal{D}_q^\mu f(z)} < 1 + 2\gamma (\alpha_1 - 1) - \frac{2\gamma (\alpha_1 - 1)}{1 - z} \quad (z \in \mathbb{U}). \]
which was required in (3.9). \hfill \Box

**Theorem 3.4.** If function \( f \in \mathcal{KD}_q(k, \alpha, \gamma) \), then we have
\[
\frac{1 - [1 + 2\gamma (\alpha_1 - 1)] r}{1 - r} \leq \Re \left\{ \frac{z \partial_q \mathcal{D}_q^\mu f(z)}{\mathcal{D}_q^\mu f(z)} \right\} \leq \frac{1 + [1 + 2\gamma (\alpha_1 - 1)] r}{1 + r}, \tag{3.13}
\]
for \(|z| = 1 < 1\) and \(\alpha_1\) is defined by (3.10).

**Proof.** By the virtue of Theorem (3.3), let us take the function \( \phi(z) \) defined by
\[
\phi(z) = 1 + 2\gamma (\alpha_1 - 1) - \frac{2\gamma (\alpha_1 - 1)}{1 - z} \quad (z \in \mathbb{U}).
\]
Letting \( z = re^{i\theta} (0 \leq r < 1) \), we see that
\[
\Re \phi(z) = 1 + 2\gamma (\alpha_1 - 1) + \frac{2\gamma (1 - \alpha_1) (1 - r \cos \theta)}{1 + r^2 - 2r \cos \theta}.
\]
Let us define
\[
\psi(t) = \frac{1 - rt}{1 + r^2 - 2rt} \quad (t = \cos \theta).
\]
Since \( \psi'(t) = \frac{r (1 - r^2)}{(1 + r^2 - 2rt)^2} \geq 0 \), because \( r < 1 \). Therefore we get
\[
1 + 2\gamma (\alpha_1 - 1) - \frac{2\gamma (\alpha_1 - 1)}{1 - r} \leq \Re \phi(z) \leq 1 + 2\gamma (\alpha_1 - 1) - \frac{2\gamma (\alpha_1 - 1)}{1 + r}.
\]
After simplification, we have
\[
\frac{1 - [1 + 2\gamma (\alpha_1 - 1)] r}{1 - r} \leq \Re \phi(z) \leq \frac{1 + [1 + 2\gamma (\alpha_1 - 1)] r}{1 + r}.
\]
Since we note that \( \frac{z \partial_q \mathcal{D}_q^\mu f(z)}{\mathcal{D}_q^\mu f(z)} \sim \phi(z), (z \in \mathbb{U}) \) by Theorem 3.3 and \( \phi(z) \) is analytic in \( \mathbb{U} \), we proved the inequality (3.13). \hfill \Box

**Theorem 3.5.** If \( f \in \mathcal{A} \) satisfies
\[
\left| \frac{z \partial_q \mathcal{D}_q^\mu f(z)}{\mathcal{D}_q^\mu f(z)} - 1 \right| < \frac{(\alpha - 1) |\gamma|}{(1 - k)} \quad z \in \mathbb{U}, \tag{3.14}
\]
for some \( k (k \leq 0), \alpha (\alpha > 1) \) and \( \gamma \in \mathbb{C} \setminus \{0\} \). Then \( f \in \mathcal{KD}_q(k, \alpha, \gamma) \).
Proof.

\[
\left| \frac{z\partial_q D^\mu_q f(z)}{D^\mu_q f(z)} - 1 \right| < \frac{(\alpha - 1)|\gamma|}{1 - k}
\]

\[
\Rightarrow \left| \frac{1}{\gamma} \left( \frac{z\partial_q D^\mu_q f(z)}{D^\mu_q f(z)} - 1 \right) \right| < \frac{\alpha - 1}{1 - k}
\]

\[
\Rightarrow (1 - k) \left| \frac{1}{\gamma} \left( \frac{z\partial_q D^\mu_q f(z)}{D^\mu_q f(z)} - 1 \right) \right| + 1 < \alpha
\]

\[
\Rightarrow \Re \left\{ 1 + \frac{1}{\gamma} \left( \frac{z\partial_q D^\mu_q f(z)}{D^\mu_q f(z)} - 1 \right) \right\} + 1 < k \left| \frac{1}{\gamma} \left( \frac{z\partial_q D^\mu_q f(z)}{D^\mu_q f(z)} - 1 \right) \right| + \alpha
\]

\[
\Rightarrow f \in LD^k_b(a,c,\beta)
\]

\[\square\]

Corollary 3.6. Let \( f \in A \) be of the form (1.1) and satisfies

\[
\left| \sum_{n=2}^{\infty} [n-1] a_n z^{n-1} \right| < \frac{(\alpha - 1)|\gamma|}{q(1 - k)} \quad z \in \mathbb{U},
\]

(3.15)

for some \( k (k \leq 0) \), \( \beta (\beta > 1) \) and for some \( b \in \mathbb{C} \setminus \{0\} \). Then \( f \in KD_q(k,\alpha,\gamma) \).

Proof. We have

\[
D^\mu_q f(z) = z + \sum_{n=2}^{\infty} \Phi_{n-1} a_n z^n
\]

and by (1.5)

\[
z\partial_q D^\mu_q f(z) = z + \sum_{n=2}^{\infty} [n] \Phi_{n-1} a_n z^n.
\]

Therefore, (3.14) follows immediately (3.15). \[\square\]

Theorem 3.7. Let \( f \in A \) be of the form (1.1) and satisfies

\[
\sum_{n=2}^{\infty} ([n-1]+y) |\Phi_{n-1}| |a_n| < y \quad z \in \mathbb{U},
\]

(3.16)

for some \( k (k \leq 0) \), \( \beta (\beta > 1) \) and for some \( b \in \mathbb{C} \setminus \{0\} \) where

\[
y = \frac{(\alpha - 1)|\gamma|}{q(1 - k)} > 0.
\]

Then \( f \in KD_q(k,\alpha,\gamma) \).
Proof. We have
\[
\sum_{n=2}^{\infty} \left( |[n-1] + y| \Phi_{n-1} ||a_n|| < y \right)
\]
\[
\Rightarrow \sum_{n=2}^{\infty} \left( |[n-1] + y| \Phi_{n-1} ||a_n|| < y - y \sum_{n=2}^{\infty} \Phi_{n-1} ||a_n|| \right)
\]
\[
\Rightarrow 0 < y - y \sum_{n=2}^{\infty} |\Phi_{n-1}||a_n||
\]
\[
\Rightarrow 0 < y - y \sum_{n=2}^{\infty} |\Phi_{n-1}||a_n||z^{n-1}|
\]
\[
\Rightarrow 0 < y \left| 1 + \sum_{n=2}^{\infty} \Phi_{n-1}a_n z^{n-1} \right|(3.17)
\]

We have
\[
\sum_{n=2}^{\infty} \left( |[n-1] + y| \Phi_{n-1} ||a_n|| < y \right)
\]
\[
\Rightarrow \sum_{n=2}^{\infty} \left( |[n-1] + y| \Phi_{n-1} ||a_n||z^{n-1}\right) < y
\]
\[
\Rightarrow \sum_{n=2}^{\infty} |[n-1] \Phi_{n-1} ||a_n||z^{n-1} | < y - y \sum_{n=2}^{\infty} |\Phi_{n-1}||a_n||z^{n-1}| \]
\[
\Rightarrow \sum_{n=2}^{\infty} |[n-1] \Phi_{n-1} a_n z^{n-1} | < y \left| 1 + \sum_{n=2}^{\infty} \Phi_{n-1}a_n z^{n-1} \right|
\]
\[
\Rightarrow \left| \sum_{n=2}^{\infty} |[n-1] \Phi_{n-1} a_n z^{n-1} | \right| < y,
\]
because of (3.17). By (3.15) it follows \( f \in \mathcal{LD}_{b}(a,c,\beta) \).

References


Application of Ruscheweyh $q$-differential operator


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