Ulam stability of Volterra integral equation on a generalized metric space

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Abstract. The aim of this paper is to give some Ulam-Hyers stability results for Volterra integral equations on a generalized metric space. In this case we consider the Volterra integral equation in the Krasnoselski-Krein and Naguno-Perron-Van Kampen conditions. Here we present only Ulam-Hyers stability for the Volterra integral equation.

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1. Introduction

The Ulam stability is an important concept in the theory of Volterra integral equations. This problem has been studied by L.P. Castro and A. Ramos [1], N. Cădariu and V. Radu [2], S.M. Jung [3], I.A. Rus [9], [10], I.A. Rus and N. Lungu [11]. But, on a generalized metric spaces this problem has been studied in the papers [1] and [10]. In what follows we shall present Ulam-Hyers stability of a Volterra integral equation on a generalized metric space, N. Lungu [5]. Here, we consider a Volterra integral equation in the Krasnoselski-Krein and Naguno-Perron-Van Kampen conditions. In the present work we consider a generalized metric space $(X, d)$, where $d(x, y) \in \mathbb{R}_+ \cup \{+\infty\}$ is a generalized metric on $X$. For these we need some notions and results from the generalized metric spaces theory.

Let $(X, d)$ be a generalized metric space. On $X$ we have the following equivalence relation:

$$x \sim y \iff d(x, y) < \infty, \ \forall \ x, y \in X.$$ 

Let $X = \bigcup_{\lambda \in \Lambda} X_\lambda$ be the canonical decomposition of $X$ after this equivalence relation. We denote

$$d_\lambda(x, y) = d(x, y)_{|_{X_\lambda \times X_\lambda}}$$
and we have that \((X_\lambda, d_\lambda)\) is a metric space \([7]\).

In this paper we need the following two theorems (see W.A.J. Luxemburg \([6]\), I.A. Rus \([7]\), \([8]\)):

**Theorem 1.1.** Let \((X, d)\) be a generalized complete metric space and \(A : X \to X\) an operator with the property:

\[
\exists \alpha \in [0, 1] \text{ such that } d(x, y) < \infty \Rightarrow d(A(x), A(y)) \leq \alpha d(x, y)
\]

for all \(x, y \in X\).

If there exists \(x_0 \in X\) such that \(d(x_0, A(x_0)) < +\infty\), then the operator \(A\) has at least one fixed point.

**Theorem 1.2.** (Luxemburg-Jung). Let \((X, d)\) be a generalized complete metric space and his canonical decomposition \(X = \bigsqcup X_\lambda\). If \(A : X \to X\) ia a contraction, then the operator \(A\) have in every \(X_\lambda\), for which exists \(u_\lambda\) such that

\[
d(u_\lambda, A(u_\lambda)) < +\infty,
\]

a unique fixed point.

### 2. Ulam-Hyers stability in the generalized Krasnoselskii-Krein conditions

In what follows we shall consider the following integral equation

\[
u(x, y) = h(x, y) + \int_0^x \int_0^y f(s, t, u(s, t)) \, ds \, dt \tag{2.1} 
\]

\[f : [0, a) \times [0, b) \times \mathbb{R} \to \mathbb{R}, \; h : [0, a) \times [0, b) \to \mathbb{R}, \]

\[f \in C([0, a) \times [0, b) \times \mathbb{R}, \mathbb{R}), \]

\[h \in C([0, a), \mathbb{R}) \times C([0, b), \mathbb{R}), \; u \in C([0, a) \times [0, b), \mathbb{R}), \]

\[(x, y) \in [0, a) \times [0, b), \; D = [0, a) \times [0, b).\]

Let \(X\) be the set:

\[X = C(D) \tag{2.2} \]

and the generalized metrics:

\[
d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}
\]

\[
d(u_1, u_2) := \sup_D \frac{|u_1(x, y) - u_2(x, y)|}{(xy)^{p/k}} \tag{2.3} 
\]

for all \(u_1, u_2 \in X\), \(p > 1\), \(k > 0\).

It is known that the space \((X, d)\) is a generalized complete metric space.

Let \(a, b \in (0, \infty]\) and \(\varepsilon > 0\). In what follows we denote by \(A\) the operator

\[A : X \to X
\]

\[A(u)(x, y) := \text{the second part of (2.1)}.
\]

Then the equation (2.1) becomes

\[u(x, y) = A(u)(x, y). \tag{2.4} \]
For the fixed point equation (2.4) we have:

**Definition 2.1.** ([10]) The equation (2.4) is Ulam-Hyers stable if there exists the positive real number $C_f > 0$ such that, for each $\varepsilon \in \mathbb{R}_+^*$ and each solution $v$ of the inequation

$$d(v, Av) \leq \varepsilon$$

there exists a solution $u \in X$ of (2.4) such that

$$d(u, v) \leq C_f \cdot \varepsilon.$$

In this case we have

**Theorem 2.2.** We suppose that:

(i) $f : E \to \mathbb{R}$ is continuous and bounded on $E$, $E = D \times \mathbb{R}$;

(ii) $a < \infty$, $b < \infty$;

(iii) $f$ verifies the generalized Krasnoselski-Krein conditions ([4]):

$$|f(x, y, u_1) - f(x, y, u_2)| \leq \frac{k}{xy}|u_1 - u_2|, \ k > 0$$

(2.6)

$$|f(x, y, u_1) - f(x, y, u_2)| \leq \frac{c}{(xy)^\beta}|u_1 - u_2|^\alpha, \ c > 0$$

(2.7)

$\alpha \in (0, 1)$, $\beta < \alpha$, $k(1 - \alpha)^2 < (1 - \beta)^2$, $\beta < p\sqrt{k}$, $xy \neq 0$, $p^2k(1 - \alpha)^2 < (1 - \beta)^2$, for all $(x, y, u) \in E$.

Then the equation (2.4) is Ulam-Hyers stable.

**Proof.** We consider $X = C(D)$ and $X = \bigcup_{\lambda \in \Lambda} X_\lambda$. Let $v$ be a solution of the inequation (2.5) and there exists $\lambda \in \Lambda$ such that $v \in X_\lambda$. By Luxemburg-Jung theorem (Theorem 1.2), the equation (2.4) has a unique solution $u$ in $X_\lambda$.

From (2.1), (2.5), (2.6) and (2.7), we have:

$$|v(x, y) - u(x, y)| \leq \left|v(x, y) - h(x, y) - \int_0^x \int_0^y f(s, t, v(s, t)) \, ds \, dt\right|$$

$$+ \int_0^x \int_0^y |f(s, t, v(s, t)) - f(s, t, u(s, t))| \, ds \, dt.$$

Hence, from (2.4), (2.6) and (2.7), we have

$$|v(x, y) - u(x, y)| \leq |v(x, y) - A(v)(x, y)| + \int_0^x \int_0^y \frac{k}{st}|v(s, t) - u(s, t)| \, ds \, dt,$$

or

$$|v(x, y) - u(x, y)| \leq |v(x, y) - A(v)(x, y)| + \int_0^x \int_0^y kd(u, v)(st)^{p\sqrt{k}-1} \, ds \, dt,$$

and

$$|v(x, y) - u(x, y)| \leq |v(x, y) - A(v)(x, y)| + kd(u, v) \frac{(xy)^{p\sqrt{k}}}{p^2k},$$

from where we have

$$d(u, v) \leq \varepsilon + \frac{1}{p^2}d(u, v).$$
and
\[ d(u, v) \leq \frac{p^2}{p^2 - 1} \varepsilon \tag{2.9} \]
then
\[ d(u, v) \leq C_f \cdot \varepsilon \]
where
\[ C_f = \frac{p^2}{p^2 - 1}. \]
So, from Definition 2.1, the equation (2.4) is Ulam-Hyers stable.

**Example 2.3.** Let us consider the equation (2.1) in the Krasnoselski-Krein conditions (2.6)+(2.7) and
\[ f(x, y, u) = u(x, y)xye^{-x^2y^2}, \quad h(x, y) = x^2y^2, \]
then \( \alpha = \frac{1}{2}, \beta = \frac{1}{3}, k = 1, p = 2. \)

In this case we have \( c_f = \frac{p^2}{p^2 - 1} \) and for \( p = 2, c_f = \frac{4}{3} \), hence the equation (2.1) is Ulam-Hyers stable.

### 3. Ulam-Hyers stability in the generalized Naguno-Perron-Van Kampen conditions

In this case we consider the integral equation (2.1) in the same conditions. Let \( X = C(D) \) and the generalized metrics
\[ d : X \times X \to \mathbb{R}_+ \cup \{+\infty\} \]
\[ d(u_1, u_2) = \sup_D \frac{|u_1(x, y) - u_2(x, y)|}{(xy)^{p+1}} \tag{3.1} \]
for all \( u_1, u_2 \in X, p > -1. \)

It is known that the space \((X, d)\) is a generalized complete metric space. Here, we consider the stability of the equation (2.4) in the generalized Naguno-Perron-Van Kampen conditions.

**Theorem 3.1.** If we have
(i) \( f : E \to \mathbb{R} \) is continuous and bounded on \( E; \)
(ii) \( a < +\infty, b < +\infty; \)
(iii) \( f \) verifies the generalized Naguno-Perron-Van Kampen conditions ([12]):
\[ |f(x, y, u)| \leq \alpha(xy)^p, \quad p > -1, \quad \alpha > 0. \tag{3.2} \]
\[ |f(x, y, u_1) - f(x, y, u_2)| \leq \frac{c}{(xy)^r} |u_1 - u_2|^q, \quad q \geq 1, \quad c > 0, \tag{3.3} \]
\[ pq + q - r = p, \quad xy \neq 0, \quad \rho = \frac{c(2\alpha)^{q-1}}{(p+1)^2q} < 1, \quad \text{for all } (x, y, u) \in E. \]

Then the equation (2.4) is Ulam-Hyers stable.
Proof. Evidently, in the conditions Naguno-Perron-Van Kampen, by Luxemburg-Jung theorem (Theorem 1.2), the equation (2.4) has a unique solution \( u \) in \( X_\lambda \).

First we observe that \[ |v(x, y) - u(x, y)| \leq \frac{2\alpha}{(p + 1)^2} (xy)^{p+1}. \] (3.4)

From (2.1), (2.5), (3.2), (3.3) we have

\[
|v(x, y) - u(x, y)| \leq |v(x, y) - h(x, y) - \int_0^x \int_0^y f(s, t, v(s, t))dsdt| \\
+ \int_0^x \int_0^y |f(s, t, v(s, t)) - f(s, t, u(s, t))|dsdt.
\]

From (3.3) we have

\[
|v(x, y) - u(x, y)| \leq |v(x, y) - A(v)(x, y)| + \int_0^x \int_0^y c_{st}^r \left| \frac{v(s, t) - u(s, t)}{(st)^{p+1}} \right| \left| \frac{v(s, t) - u(s, t)}{(st)^{q-1}} \right| dsdt \\
\leq |v(x, y) - A(v)(x, y)| + cd(u, v) \int_0^x \int_0^y \frac{(2\alpha)^{q-1}}{(p + 1)^{2(q-1)}} (st)^{pq+q-r} dsdt.
\]

Then we have

\[ d(u, v) \leq d(v, A(v)) + \rho d(u, v) \] (3.5)

and

\[ d(u, v) \leq \frac{\epsilon}{1 - \rho}, \]

then

\[ d(u, v) \leq C_f \cdot \epsilon \]

where

\[ C_f = \frac{1}{1 - \rho}. \]

From Definition 2.1, the equation (2.4) is Ulam-Hyers stable.

Remark 3.2. For every \( \lambda \in \Lambda \) there exists at least a solution \( v \) of (2.5) in \( X_\lambda \) and for each \( v \) exists a unique solution \( u \) of (2.4) which is Ulam-Hyers stable.

Remark 3.3. It is possible that the inequation (2.5) do not have a solution, but in this case the equation (2.4) is Ulam-Hyers stable.

Example 3.4. Let us consider the equation (2.1) in the Naguno-Perron-Van Kampen conditions (3.2)+(3.3), \( p > -1, r = 1, q \geq 1 \).

In this case \( c_f = \frac{1}{1 - \rho} \), where \( \rho = \frac{c(2\alpha)^{q-1}}{(p + 1)^{2q}} \) and the equation (2.1) is Ulam-Hyers stable. If \( \rho = 1 \) then the equation (2.1) is Ulam-Hyers instable.
References


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