Existence and multiplicity of solutions to the Navier boundary value problem for a class of \((p(x), q(x))\)-biharmonic systems

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Abstract. In this article, we study the following problem with Navier boundary conditions.

\[
\begin{align*}
\Delta(a(x, \Delta u)) &= F_u(x, u, v), \quad \text{in } \Omega \\
\Delta(a(x, \Delta v)) &= F_v(x, u, v), \quad \text{in } \Omega, \\
u = v = \Delta u = \Delta v &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

By using the Mountain Pass Theorem and the Fountain Theorem, we establish the existence of weak solutions of this problem.

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1. Introduction

In recent years, the study of differential equations and variational problems with \(p(x)\)-growth conditions was an interesting topic, which arises from nonlinear electrorheological fluids and elastic mechanics. In that context we refer the reader to Ruzicka [15], Zhikov [20] and the reference therein; see also [4, 7, 8, 5].

Fourth-order equations appears in many context. Some of these problems come from different areas of applied mathematics and physics such as Micro Electro-Mechanical systems, surface diffusion on solids, flow in Hele-Shaw cells (see [10]). In addition, this type of equations can describe the static from change of beam or the sport of rigid body.

In [1] the authors studied a class of \(p(x)\)-biharmonic of the form

\[
\begin{align*}
\Delta(|\Delta u|^{p(x)-2}\Delta u) &= \lambda |u|^{q(x)-2}u \quad \text{in } \Omega, \\
u &= \Delta u = 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
where $\Omega$ is a bounded domain in $\mathbb{R}^N$, with smooth boundary $\partial \Omega$, $N \geq 1$, $\lambda \geq 0$.

In [3], A. El Amrouss and A. Ourraoui considered the below problem and using variational methods, by the assumptions on the Carathéodory function $f$, they establish the existence of Three solutions the problem of the form

$$\Delta(|\Delta u|^{p(x)-2}\Delta u) + a(x)u|^{p(x)-2}u = f(x, u) + \lambda g(x, u) \quad \text{in } \Omega,$$

$$Bu = Tu = 0 \quad \text{on } \partial \Omega.$$ 

Inspired by the above references, the work of L. Li [11] and [14], the aim of this article is to study the existence and multiplicity of weak solutions for $(p(x), q(x))$—biharmonic type system

$$\begin{cases}
\Delta(a(x, \Delta u)) = F_u(x, u, v), \quad \text{in } \Omega \\
\Delta(a(x, \Delta v)) = F_v(x, u, v), \quad \text{in } \Omega, \\
u = \Delta u = 0, \quad v = \Delta v = 0 \quad \text{on } \partial \Omega,
\end{cases} \quad (1.1)$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$, $N \geq 1$,

$$\Delta_{p(x)}^2 u := \Delta(|\Delta u|^{p(x)-2}\Delta u),$$

is the $p(x)$—biharmonic operator, $p, q$ are continuous functions on $\overline{\Omega}$ with

$$\inf_{x \in \Omega} p(x) > \max \left\{1, \frac{N}{2}\right\}, \quad \inf_{x \in \Omega} q(x) > \max \left\{1, \frac{N}{2}\right\}$$

and $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a function such that $F(., s, t)$ is continuous in $\overline{\Omega}$, for all $(s, t) \in \mathbb{R}^2$, $F(x, \ldots)$ is $C^1$ in $\mathbb{R}^2$ for every $x \in \Omega$, and $F_u, F_v$ denote the partial derivative of $F$, with respect to $u, v$ respectively such that

$$(F_1) \quad \text{For all } (x, s, t) \in \Omega \times \mathbb{R}^2, \quad \text{we assume}$$

$$\lim_{|s| \to 0} \frac{F_u(x, s, t)}{|s|^{p(x)-1}} = 0, \quad \lim_{|t| \to 0} \frac{F_v(x, s, t)}{|s|^{q(x)-1}} = 0.$$

$$(F_2) \quad \text{For all } (x, s, t) \in \Omega \times \mathbb{R}^2, \quad \text{we assume}$$

$$F(x, s, t) = o(|s|^{p(x)-1} + |t|^{q(x)-1}) \quad \text{as } |(s, t)| \to \infty.$$ 

$$(F_3) \quad \text{There exists } u > 0, \quad \text{such that } F(x, u, v) > 0 \quad \text{for a.e } x \in \Omega$$

$$(F_4) \quad \text{There exist } \lambda > 0 \quad \text{such that } F(x, s, t) \geq \lambda(|s|^{\alpha(x)} - |t|^{\beta(x)}) \quad \text{for all } (s, t) \in \mathbb{R}^2,$$

with

$$\alpha^- > r^+, \quad 1 < \beta^- \leq \beta^+ < r^-.$$

$$(F_5) \quad \text{For all } (x, s, t) \in \Omega \times \mathbb{R}^2 \quad F(x, -s, -t) = -F(x, s, t).$$

Let $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ to be a continuous potential derivative with respect to $\xi$ of the mapping $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ where $a = DA = A'$, with the assumption as below

$$(A_1) \quad A(x, 0) = 0 \quad \text{for all } x \in \Omega,$$

$$(A_2) \quad a(x, \xi) \leq C_1(1 + |\xi|^{r(x)-1}), \quad C_1 > 0 \quad \text{and } r^- > p^+, \quad r^- > q^+.$$
(A₃) A is $r(x)$-uniformly convex: there exists a constant $k > 0$ such that

$$A \left( x, \frac{\xi + \eta}{2} \right) \leq \frac{1}{2} A(x, \xi) + \frac{1}{2} A(x, \eta) - k|\xi - \eta|^{r(x)},$$

for all $x \in \Omega$, $\xi, \eta \in \mathbb{R}^N$.

(A₄) A is $r(x)$-subhomogeneous, for all $(x, \xi) \in \Omega \times \mathbb{R}^N$,

$$|\xi|^{r(x)} \leq a(x, \xi) \leq r(x)A(x, \xi).$$

(A₅) For all $(x, s) \in \Omega \times \mathbb{R}^N$ $a(x, -s) = -a(x, s)$.

The main results of this paper are the following theorems.

**Theorem 1.1.** Assume that $(A_1) - (A_4)$ and $(F_1) - (F_3)$ hold. Then the problem (1.1) has two weak solutions.

**Theorem 1.2.** Assume that $(A_1) - (A_5)$ and $(F_1) - (F_5)$ hold. Then the problem (1.1) has a sequence of weak solutions such that $\phi(\pm (u_k, v_k)) \to +\infty$, as $k \to +\infty$ with $\phi$ is an energy associated of the problem (1.1) defined in (2.2).

This paper is organized as three sections. In Section 2, we recall some basic properties of the variable exponent Lebesgue-Sobolev spaces. In Section 3 we give the proof of main results.

## 2. Preliminaries

To study $p(x)$-Laplacian problems, we need some results on the spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$, and properties of $p(x)$-Laplacian, which we use later. Let $\Omega$ be a bounded domain of $\mathbb{R}^N$, denote

$$C_+^{(\Omega)} = \{ h(x); h(x) \in C(\overline{\Omega}), h(x) > 1, \forall x \in \Omega \}.$$

For any $h \in C_+^{(\Omega)}$, we define

$$h^+ = \max\{h(x); x \in \Omega\}, \quad h^- = \min\{h(x); x \in \Omega\};$$

For any $p \in C_+^{(\Omega)}$, we define the *variable exponent Lebesgue space*

$$L^{p(x)}(\Omega) = \left\{ u; u \text{ is a measurable real-valued function such that} \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\},$$

endowed with the so-called *Luxemburg norm*

$$|u|_{p(x)} = \inf \left\{ \mu > 0; \int_{\Omega} \frac{|u(x)|^{p(x)}}{\mu} dx \leq 1 \right\}.$$

Then $(L^{p(x)}(\Omega), | \cdot |_{p(x)})$ becomes a Banach space.
Proposition 2.1 ([9]). The space \((L^p(x))(\Omega), \| \cdot \|_{p(x)}\) is separable, uniformly convex, reflexive and its conjugate space is \(L^{q(x)}(\Omega)\) where \(q(x)\) is the conjugate function of \(p(x)\), i.e.,

\[
\frac{1}{p(x)} + \frac{1}{q(x)} = 1,
\]

for all \(x \in \Omega\). For \(u \in L^p(x)(\Omega)\) and \(v \in L^{q(x)}(\Omega)\), we have

\[
\left| \int_{\Omega} uv \, dx \right| \leq \left( \frac{1}{p(x)} + \frac{1}{q(x)} \right) |u|_{p(x)} |v|_{q(x)} \leq 2 |u|_{p(x)} |v|_{q(x)}.
\]

The Sobolev space with variable exponent \(W^{k,p(x)}(\Omega)\) is defined as

\[ W^{k,p(x)}(\Omega) = \{ u \in L^p(x)(\Omega) : D^\alpha u \in L^p(x)(\Omega), |\alpha| \leq k \}, \]

where

\[
D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_N^{\alpha_N}} u,
\]

with \(\alpha = (\alpha_1, \ldots, \alpha_N)\) is a multi-index and \(|\alpha| = \sum_{i=1}^{N} \alpha_i\). The space \(W^{k,p(x)}(\Omega)\) equipped with the norm

\[
\|u\|_{k,p(x)} = \sum_{|\alpha| \leq k} |D^\alpha u|_{p(x)},
\]

also becomes a separable and reflexive Banach space. For more details, we refer the reader to [6, 9, 13]. Denote

\[
p_k^* (x) = \begin{cases} \frac{Np(x)}{N-kp(x)} & \text{if } kp(x) < N, \\ +\infty & \text{if } kp(x) \geq N \end{cases}
\]

for any \(x \in \bar{\Omega}\), \(k \geq 1\).

Proposition 2.2 ([9]). For \(p, r \in C_+ (\bar{\Omega})\) such that \(r(x) \leq p_k^*(x)\) for all \(x \in \bar{\Omega}\), there is a continuous embedding

\[ W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega). \]

If we replace \(\leq\) with \(<\), the embedding is compact.

We denote by \(W^{k,p(x)}_0(\Omega)\) the closure of \(C_0^\infty (\Omega)\) in \(W^{k,p(x)}(\Omega)\). Then the function space \(\left( \left( W^{2,p(x)}(\Omega) \cap W^{1,p(x)}_0(\Omega) \right), \|u\|_{p(x)} \right)\) is a separable and reflexive Banach space, where

\[
\|u\|_{p(x)} = \inf \{ \mu > 0 : \int_{\Omega} \left( \frac{\Delta u(x)}{\mu} \right)^{p(x)} \leq 1 \}.
\]

Remark 2.3. According to [[18] Theorem 4.4.], the norm \(\| \cdot \|_{2,p(x)}\) is equivalent to the norm \(\| \cdot \|_{p(x)}\) in the space \(X\). Consequently, the norms \(\| \cdot \|_{2,p(x)}, \| \cdot \|\) and \(\| \cdot \|_{p(x)}\) are equivalent.

Proposition 2.4 ([2]). If we denote \(\rho(u) = \int_{\Omega} |\Delta u|^{p(x)} \, dx\), then for \(u, u_n \in X\), we have

\[
(1) \|u\|_p < 1 \text{ (respectively } 1; > 1) \iff \rho(u) < 1 \text{ (respectively } 1; > 1); \]
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(2) \( \|u\|_p \leq 1 \Rightarrow \|u\|_{p^+}^p \leq \rho(u) \leq \|u\|_{p^-}^p \); 
(3) \( \|u\|_p \geq 1 \Rightarrow \|u\|_{p^-}^p \leq \rho(u) \leq \|u\|_{p^+}^p \); 
(4) \( \|u_n\|_p \to 0 \) (respectively \( \to \infty \)) \( \iff \rho(u_n) \to 0 \) (respectively \( \to \infty \)).

Note that the weak solutions of problem (1.1) are considered in the generalized
Sobolev space
\[ X = \left( W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \right) \times \left( W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \right) \]
equipped with the norm
\[ \|(u,v)\| = \max\{\|u\|_{p(x)}, \|u\|_{q(x)}\}. \]

\textbf{Remark 2.5 (see [19]).} As the Sobolev space \( X \) is a reflexive and separable Banach
space, there exist \((e_n)_{n \in \mathbb{N}^*} \subseteq X\) and \((f_n)_{n \in \mathbb{N}^*} \subseteq X^*\) such that \( f_n(e_l) = \delta_{nl} \) for any \( n,l \in \mathbb{N}^* \) and
\[ X = \text{span}\{e_n : n \in \mathbb{N}^*\}, \quad X^* = \text{span}\{f_n : n \in \mathbb{N}^*\}^w. \]

For \( k \in \mathbb{N}^* \), denote by
\[ X_k = \text{span}\{e_k\}, \quad Y_k = \oplus_{j=1}^k X_j, \quad Z_k = \oplus_{j=1}^\infty X_j. \]
For every \( m > 1 \), \( u,v \in L^m(\Omega) \), we define
\[ |(u,v)|_m := \max\{|u|_m, |v|_m\}. \]

\textbf{Lemma 2.6 (See [8]).} Define
\[ \beta_k := \sup\{|(u,v)|_m; \|(u,v)\| = 1, (u,v) \in Z_k\}, \]
where \( m := \max(p(x), q(x)) \). Then, we have
\[ \lim_{k \to \infty} \beta_k = 0. \]

\section*{2.1. Existence and multiplicity of weak solutions}

\textbf{Definition 2.7.} We say that \((u,v) \in X\) is weak solution of (1.1) if
\[ \int_\Omega a(x, \Delta u) \Delta \varphi dx + \int_\Omega a(x, \Delta v) \Delta \varphi dx = \int_\Omega F_u(x,u,v) \varphi dx + \int_\Omega F_v(x,u,v) \varphi dx, \quad (2.1) \]
for all \( \varphi \in X \).

The functional associated to (1.1) is given by
\[ \phi(u,v) = \int_\Omega A(x, \Delta u) dx + \int_\Omega A(x, \Delta v) dx - \int_\Omega F(x,u,v) dx, \quad (2.2) \]
It should be noticed that under the condition \((F_1) - (F_2)\) the functional \( \phi \) is of class
\( C^1(X, \mathbb{R}) \) and
\[ \phi'(u,v)(\psi, \varphi) = \int_\Omega a(x, \Delta u) \Delta \psi dx + \int_\Omega a(x, \Delta v) \Delta \varphi dx \]
\[ - \int_\Omega F_u(x,u,v) \psi dx - \int_\Omega F_v(x,u,v) \varphi dx, \quad \forall (\psi, \varphi) \in X. \]

(2.3)
Then, we know that the weak solution of (1.1) corresponds to critical point of the functional $\phi$.

**Definition 2.8.** We say that

1. The $C^1$-functional $\phi$ satisfies the Palais-Smale condition (in short $\text{(PS)}$ condition) if any sequence $(u_n)_{n \in \mathbb{N}} \subseteq X$ for which, $(\phi(u_n))_{n \in \mathbb{N}} \subseteq \mathbb{R}$ is bounded and $\phi'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence.

2. The $C^1$-functional $\phi$ satisfies the Palais-Smale condition at the level $c$ (in short $\text{(PS)}_c$ condition) for $c \in \mathbb{R}$ if any sequence $(u_n)_{n \in \mathbb{N}} \subseteq X$ for which, $\phi(u_n) \to c$ and $\phi'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence.

3. The $C^1$-functional $\phi$ satisfies the $\text{(PS)}^*_c$ condition for $c \in \mathbb{R}$ if any sequence $(u_n)_{n \in \mathbb{N}} \subseteq X$ for which, $u_n \in Y_n$ for each $n \in \mathbb{N}$, $\phi(u_n) \to c$ and $\phi'|_{Y_n}(u_n) \to 0$ as $n \to \infty$ with $Y_n$, $n \in \mathbb{N}$ as defined in Remark 2.5, has a subsequence convergent to a critical point of $\phi$.

**Remark 2.9.** It is easy to see that if $\phi$ satisfies the $\text{(PS)}$ condition, then $\phi$ satisfies the $\text{(PS)}_c$ condition for every $c \in \mathbb{R}$.

**Proof of Theorem 1.1.** To prove Theorem 1.1, we shall use the Mountain Pass theorem [16]. We first start with the following lemmas.

**Lemma 2.10.** Under the assumptions $(F_1)$-$(F_3)$ and $(A_1)$-$(A_3)$ $\phi$ is sequentially weakly lower semi continuous and coercive.

**Proof.** By $(F_1)$-$(F_2)$, we see that

$$|F(x, s, t)| \leq C_3(1 + |s|^{p(x)} + |t|^{q(x)}), \forall (s, t) \in \mathbb{R}^2. \quad (2.4)$$

By the compact embeddings

$$X \hookrightarrow L^{p(x)}(\Omega), \ X \hookrightarrow L^{q(x)}(\Omega),$$

we deduce that $w \mapsto \int_{\Omega} F(x, w)dx$ is sequentially lower semi continuous $\forall w \in \mathbb{R}^2$.

Since

$$w \mapsto \int_{\Omega} A(x, \Delta u)dx + \int_{\Omega} A(x, \Delta v)dx$$

is convex uniformly, so it is sequentially lower semi continuous.

Now we prove that $\phi$ is coercive. From $(F_2)$ for $\varepsilon$ small enough, there exist $\delta > 0$ such that

$$|F(x, s, t)| \leq \varepsilon(|s|^{p(x)} + |t|^{q(x)}), \text{ for } |(s, t)| > \delta,$$

and thus we have

$$|F(x, s, t)| \leq \varepsilon(|s|^{p(x)} + |t|^{q(x)}) + \max_{||(s, t)|| \leq \delta} |F(x, s, t)| |(s, t)|, \forall (s, t) \in \mathbb{R}^2,$$
for a.e \( x \in \Omega \). Consequently, for \( \|(u, v)\| > 1 \) we obtain

\[
\phi(u, v) \geq \int_{\Omega} A(x, \Delta u) dx + \int_{\Omega} A(x, \Delta v) dx \\
- \varepsilon \int_{\Omega} |u|^{p(x)} dx - \varepsilon \int_{\Omega} |v|^{q(x)} dx - \max_{|u, v| \leq \delta} F(x, u, v) \int_{\Omega} |(u, v)| dx \\
\geq \int_{\Omega} \frac{1}{r(x)} |\Delta u|^{r(x)} dx + \int_{\Omega} \frac{1}{r(x)} |\Delta v|^{r(x)} dx \\
- C\varepsilon \int_{\Omega} |u|^{p(x)} dx - C\varepsilon \int_{\Omega} |v|^{q(x)} dx - \max_{|u, v| \leq \delta} F(x, u, v) \int_{\Omega} |(u, v)| dx
\]

\[
\geq \frac{1}{p^+} \max \left( \|u\|_{r(x)}^{-}, \|v\|_{r(x)}^{-} \right) - 2C\varepsilon \max \left( \|u\|_{p(x)}^{p^+}, \|v\|_{q(x)}^{q^+} \right) \\
- C\varepsilon \max_{|u, v| \leq \delta} F(x, u, v) \max \left( \|u\|_{p(x)}^{p^+}, \|v\|_{q(x)}^{q^+} \right).
\]

Therefore, \( \phi \) is coercive and has a global minimizer \((\overline{u}, \overline{v})\) which is a nontrivial because by \((F_3)\)

\[
\phi(\overline{u}, \overline{v}) \leq \phi(u, v) < 0.
\]

**Lemma 2.11.** Under the assumptions \((F_1)-(F_3)\) and \((A_1)-(A_4)\). Then \( \phi \) satisfies the Palais-smale condition.

**Proof.** Let \( w_n = (u_n, v_n) \subset X \) be a Palais-smale sequence, then

\[
\phi'(w_n) \to 0 \text{ in } X^*, \ \phi(w_n) \to l \in \mathbb{R}.
\]

We show that \((w_n)\) is bounded. By \((A_5)\) we have

\[
\phi(w_n) = \int_{\Omega} A(x, \Delta u_n) dx + \int_{\Omega} A(x, \Delta v_n) dx - \int_{\Omega} F(x, u_n, v_n) dx
\]

\[
\geq \int_{\Omega} \frac{1}{r(x)} |\Delta u_n|^{r(x)} dx + \int_{\Omega} \frac{1}{r(x)} |\Delta v_n|^{r(x)} dx - \int_{\Omega} F(x, u_n, v_n) dx,
\]

and we get

\[
\phi'(u_n, v_n)(u_n, v_n) = \int_{\Omega} a(x, \Delta u_n) \Delta u_n dx + \int_{\Omega} a(x, \Delta v_n) \Delta v_n dx \\
- \int_{\Omega} F_{u_n}(x, u_n, v_n) u_n dx - \int_{\Omega} F_{v_n}(x, u_n, v_n) v_n dx \\
\leq \int_{\Omega} r(x) A(x, \Delta u_n) dx + \int_{\Omega} r(x) A(x, \Delta v_n) dx \\
- \int_{\Omega} F_{u_n}(x, u_n, v_n) u_n dx - \int_{\Omega} F_{v_n}(x, u_n, v_n) v_n dx.
\]

Using the fact that \( F_s, F_t \in C(\Omega \times \mathbb{R}^2, \mathbb{R}) \) and with \((F_1)-(F_2)\), for \( \varepsilon > 0 \) there exists \( \delta > 0 \) and \( \eta > 0 \) such that

\[
|F_s(x, s, t)| \leq \varepsilon |s|^{p(x)-1}, \ |F_t(x, s, t)| \leq \varepsilon |t|^{q(x)-1},
\]
and

$$|F(x, s, t)| \leq \varepsilon(|s|^p + |t|^q),$$

for all $|s, t| \leq \delta$, and for all $|s, t| \geq \eta$.

Then we have

$$|F_s(x, s, t)s| \leq \varepsilon|s|^p, \quad |F_t(x, s, t)t| \leq \varepsilon|t|^q, \quad (2.5)$$

and

$$|F(x, s, t)| \leq \varepsilon(|s|^p + |t|^q),$$

for all $|s, t| \leq \delta$, and for all $|s, t| \geq \eta$.

It yields,

$$\frac{1}{2r^+} \sigma (u_n, v_n)(u_n, v_n)$$

$$\geq \frac{1}{2r^+} \int_{\Omega} r(x)A(x, \Delta u_n)dx - \frac{1}{2r^+} \int_{\Omega} r(x)A(x, \Delta v_n)dx$$

$$+ \frac{1}{2r^+} \left[ \int_{\Omega} F_{u_n}(x, u_n, v_n)u_n dx + \int_{\Omega} F_{v_n}(x, u_n, v_n)v_n dx \right]$$

$$\geq \frac{1}{2r^+} \int_{\Omega} r(x)A(x, \Delta u_n)dx - \frac{1}{2r^+} \int_{\Omega} r(x)A(x, \Delta v_n)dx$$

$$+ \frac{1}{2r^+} \left[ \int_{\Omega} F_{u_n}(x, u_n, v_n)u_n dx + \int_{\Omega} F_{v_n}(x, u_n, v_n)v_n dx \right].$$

Thus,

$$\phi(u_n, v_n) - \frac{1}{2r^+} \sigma (u_n, v_n)(u_n, v_n)$$

$$\geq \int_{\Omega} A(x, \Delta u_n)dx + \int_{\Omega} A(x, \Delta v_n)dx - \int_{\Omega} F(x, u_n, v_n)dx$$

$$- \frac{1}{2r^+} \int_{\Omega} r(x)A(x, \Delta u_n)dx - \frac{1}{2r^+} \int_{\Omega} r(x)A(x, \Delta v_n)dx$$

$$- \int_{\Omega} F(x, u_n, v_n)dx + \frac{1}{2r^+} \left[ \int_{\Omega} F_{u_n}(x, u_n, v_n)u_n dx + \int_{\Omega} F_{v_n}(x, u_n, v_n)v_n dx \right]$$

$$\geq \frac{1}{2} \left[ \int_{\Omega} |\Delta u_n|^p dx + \int_{\Omega} |\Delta v_n|^q dx \right] - \int_{\Omega} F(x, u_n, v_n)dx$$

$$+ \frac{1}{2r^+} \left[ \int_{\Omega} F_{u_n}(x, u_n, v_n)u_n dx + \int_{\Omega} F_{v_n}(x, u_n, v_n)v_n dx \right]$$

$$\geq \frac{1}{2} \max \left( \|u_n\|_{p(x)}^p, \|v_n\|_{q(x)}^q \right) - (C\varepsilon + \varepsilon) \int_{\Omega} |u_n|^p dx - (C\varepsilon + \varepsilon) \int_{\Omega} |v_n|^q dx.$$
we deduce
\[
\phi(u_n, v_n) - \frac{1}{2r^+} \phi'(u_n, v_n)(u_n, v_n) \geq \frac{1}{2} \max(\|u_n\|_{r(x)}, \|v_n\|_{r(x)}) - 2(C'\varepsilon + \varepsilon)\|(u_n, v_n)\|
\]
\[
\geq \left[\frac{1}{2} - 2(C'\varepsilon + \varepsilon)\right] \|(u_n, v_n)\|,
\]
where \(C'\) is positive constant.

For \(\varepsilon\) small enough with \(R = \frac{1}{2} - 2(C'\varepsilon + \varepsilon) > 0\), we get
\[
\|(u_n, v_n)\| \leq \frac{1}{R} \left(\phi(u_n, v_n) - \frac{1}{2r^+} \phi'(u_n, v_n)(u_n, v_n)\right).
\]
Since \(\phi(u_n, v_n)\) is bounded and \(\phi'(u_n, v_n)(u_n, v_n) \to 0\) as \(n \to \infty\), then \((u_n, v_n)\) is bounded in \(X\), passing to a subsequence, so \((u_n, v_n) \to (u, v)\) in \(X\) and \((u_n, v_n) \to L^{p(x)}(\Omega) \times L^{q(x)}(\Omega)\). We show that \((u_n, v_n) \to (u, v)\) in \(X\).

\[
\phi'(u_n, v_n)((u_n, v_n) - (u, v)) = \int_\Omega a(x, \Delta u_n)\Delta(u_n - u)\,dx + \int_\Omega a(x, \Delta v_n)\Delta(v_n - v)\,dx
\]
\[
- \int_\Omega F_{u_n}(x, u_n, v_n)(u_n - u)\,dx - \int_\Omega F_{v_n}(x, u_n, v_n)(v_n - v)\,dx.
\]

Since
\[
\left|\int_\Omega a(x, \Delta u_n)\Delta(u_n - u)\,dx + \int_\Omega a(x, \Delta v_n)\Delta(v_n - v)\,dx\right|
\]
\[
= |\phi'(u_n, v_n)((u_n, v_n) - (u, v)) + \int_\Omega F_{u_n}(x, u_n, v_n)(u_n - u)\,dx
\]
\[
+ \int_\Omega F_{v_n}(x, u_n, v_n)(v_n - v)\,dx|
\]
\[
\leq \|\phi'(u_n, v_n)\|_{X'} \|(u_n, v_n) - (u, v)\|
\]
\[
+ \int_\Omega |F_{u_n}(x, u_n, v_n)||u_n - u|\,dx + \int_\Omega |F_{v_n}(x, u_n, v_n)||v_n - v|\,dx.
\]

By (2.5), we have
\[
\int_\Omega |F_{u_n}(x, u_n, v_n)||u_n - u|\,dx + \int_\Omega |F_{v_n}(x, u_n, v_n)||v_n - v|\,dx
\]
\[
\leq \varepsilon \int_\Omega (|u_n - u|^{p(x)} + |v_n - v|^{q(x)})\,dx,
\]
we get
\[
\limsup_{n \to +\infty} \left(\int_\Omega a(x, \Delta u_n)\Delta(u_n - u)\,dx + \int_\Omega a(x, \Delta v_n)\Delta(v_n - v)\,dx\right) \leq 0.
\]
Since \(a(x, \xi)\) is of \((S_+)\) type, we see that \((u_n, v_n) \to (u, v)\) in \(X\).
Now, we verified the conditions of Mountain Pass Theorem. By H"older's inequality, from (F1) there exists $\delta > 0$ such that

$$|F(x, u, v)| \leq \left| \int_0^u F_s(x, s, v)\,dx + \int_0^v F_t(x, 0, t)\,dx + F(x, 0, 0) \right|$$

$$\leq \varepsilon \left| \int_0^u |s|^{p(x)-1}\,dx + \int_0^v |t|^{q(x)-1}\,dx \right| + |F(x, 0, 0)|$$

$$\leq \varepsilon (|u|^{p(x)} + |v|^{q(x)}) + M,$$

for all $|u, v| \leq \delta$, with $M := \max_{x \in \Omega} F(x, 0, 0)$ and by (F2), there exists $M(\delta) > 0$ such that

$$|F(x, u, v)| \leq M(\delta)(|u|^{p(x)} + |v|^{q(x)})$$

for $|(u, v)| > \delta$.

Therefore, for $\|(u, v)\| = \rho$ small enough, we have

$$\phi(u, v) \geq \int_{\Omega} A(x, \Delta u)\,dx + \int_{\Omega} A(x, \Delta v)\,dx - \varepsilon \int_{|(u, v)| < \delta} \left( |u|^{p(x)} + |v|^{q(x)} \right)\,dx$$

$$- M(\delta) \int_{|(u, v)| > \delta} \left( |u|^{p(x)} + |v|^{q(x)} \right) - M\text{meas}\{(u, v) < \delta\}$$

$$\geq \frac{1}{\rho^+} \max \left( \| u \|^+ \| v \|^+ \right)$$

$$- \min(\varepsilon C, M(\delta)C') \max \left( \| u \|^-, \| v \|^- \right) - M\text{meas}\{(u, v) < \delta\}$$

$$= g(\rho).$$

There exists $\theta > 0$ such that $g(\rho) > \theta > 0$. Since $\phi(0, 0) = 0$, we conclude that $\phi$ satisfies the conditions of Mountain Pass Theorem. Then there exists $(\overline{w}_2, \overline{v}_2)$ such that $\phi'(\overline{w}_2, \overline{v}_2) = 0$.

**Proof of Theorem 1.2.** To prove Theorem 1.2, above, will be based on a variational approach, using the critical points theory, we shall prove that the $C^1$-functional $\phi$ has a sequence of critical values. The main tools for this end are “Fountain theorem” (see Willem [16, Theorem 6.5]) which we give below.

**Theorem 2.12 (“Fountain theorem”, [16]).** Let $X$ be a reflexive and separable Banach space, $\phi \in C^1(X, \mathbb{R})$ be an even functional and the subspaces $X_k, Y_k, Z_k$ as defined in remark 2.5. If for each $k \in \mathbb{N}^*$ there exist $\rho_k > r_k > 0$ such that

1. $\inf_{x \in Z_k, \|x\|=r_k} \phi(x) \to +\infty$ as $k \to \infty$,
2. $\max_{x \in Y_k, \|x\|=\rho_k} \phi(x) \leq 0$,
3. $I$ satisfies the $(PS)_c$ condition for every $c > 0$.

Then $I$ has a sequence of critical values tending to $+\infty$.

According to Lemma 2.6, (F5) and (A5), $\Phi \in C^1(X, \mathbb{R})$ is an even functional. We will prove that if $k$ is large enough, then there exist $\rho_k > \nu_k > 0$ such that

$$b_k := \inf \{ \Phi(u)/u \in Z_k, \|u\| = \nu_k \} \to +\infty \text{ as } k \to +\infty;$$

$$a_k := \max \{ \Phi(u)/u \in Y_k, \|u\| = \rho_k \} \to 0 \text{ as } k \to +\infty.$$
For any \((u, v) \in Z_k, \|v\|_{q(x)} > 1, \|u\|_{p(x)} > 1\) and \(\|(u, v)\| = \eta_k\), \(\eta_k\) will be specified later, by \((2.4)\) we have

\[
\phi(u, v) = \int_{\Omega} A(x, \Delta u)dx + \int_{\Omega} A(x, \Delta v)dx - \int_{\Omega} F(x, u, v)dx
\]

\[
\geq \frac{1}{r^+} \max \left(\|u\|_{r(x)}^{-},\|v\|_{r(x)}^{-}\right) - \int_{\Omega} C_3(1 + |u|^{p(x)} + |v|^{q(x)})dx
\]

\[
\geq \frac{1}{r^+} \max \left(\|u\|_{r(x)}^{-},\|v\|_{r(x)}^{-}\right) - C_3 \int_{\Omega} dx - C_3 \int_{\Omega} |u|^{p(x)}dx - C_3 \int_{\Omega} |v|^{q(x)}dx
\]

\[
\geq \frac{1}{r^+} \|(u, v)\|^{r^{-}} - C_3(\beta_k\|(u, v)\|)^{p^{+}} - C_3(\beta_k\|(u, v)\|)^{q^{+}} - C_3|\Omega|
\]

\[
\geq \frac{1}{r^+} \|(u, v)\|^{r^{-}} - C_4\beta_k\|(u, v)\|^m - C_3|\Omega|
\]

where \(m\) is defined in Lemma 2.6. We fix

\[
\eta_k = \left(\frac{1}{r^+C_4\beta_k}\right)^{\frac{1}{m-r^-}} \to +\infty \text{ as } k \to +\infty.
\]

Consequently

\[
\phi(u, v) \geq \eta_k \left[\frac{1}{r^+\eta_k^{r^{-}}-1} - C_4\beta_k\eta_k^{m-1}\right] - C_3|\Omega|.
\]

Then,

\[
\phi(u, v) \to +\infty \text{ as } k \to +\infty.
\]

**Proof of (2.7).** From \((F_{4})\), there exists \(\lambda > 0\) such that

\[
F(x, s, t) \geq \lambda(|s|^{\alpha(x)} - |t|^{\beta(x)}),
\]

with \(\alpha^- > r^+, \beta^+ < r^-\).

Therefore, by Lemma 2.1 \([12]\) and Lemma 3.1 \([17]\), for any \(\omega := (u, v) \in Y_k\) with \(\|\omega\| = 1\) and \(1 < t = \rho_k\), we have

\[
\phi(t\omega) = \int_{\Omega} A(x, t\Delta u)dx + \int_{\Omega} A(x, t\Delta v)dx - \int_{\Omega} F(x, t\omega)dx
\]

\[
\leq \int_{\Omega} tr(x)A(x, \Delta u)dx + \int_{\Omega} tr(x)A(x, \Delta v)dx
\]

\[
- \lambda \int_{\Omega} |tu|^{\alpha(x)}dx + \lambda \int_{\Omega} |tv|^{\beta(x)}dx
\]

\[
\leq t^{r^+} \left[\int_{\Omega} A(x, \Delta u)dx + \int_{\Omega} A(x, \Delta v)dx\right]
\]

\[
- \lambda t^{\alpha^-} \int_{\Omega} |u|^{\alpha(x)}dx + \lambda t^{\beta^-} \int_{\Omega} |v|^{\beta(x)}dx.
\]

By \(\alpha^- > r^+ > \beta^-\) and \(\text{dim} Y_k < \infty\), we conclude that \(\phi(tu, tv) \to -\infty\) as \(\|t\omega\| \to +\infty\) for \(\omega \in Y_k\). By applying the fountain Theorem, we achieved the proof of Theorem 1.2.
References


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