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On subclasses of bi-convex functions defined by Tremblay fractional derivative operator

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Abstract. We introduce and investigate new subclasses of analytic and bi-univalent functions defined by modified Tremblay operator in the open unit disk. Also we obtain upper bounds for the coefficients of functions belonging to these classes.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ which are *analytic* in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$ and having the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Also let $\mathcal{S}$ denote the subclass of functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$ (for details, see [7]).

The Koebe One Quarter Theorem (e.g., see [7]) ensures that the image of $\mathbb{U}$ under every univalent function $f(z) \in \mathcal{A}$ contains the disk of radius 1/4. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$\begin{aligned} g(w) = f^{-1}(w) &= w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots \\ &= w + \sum_{n=2}^{\infty} b_n w^n. \end{aligned} \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. We denote by Σ the class of all bi-univalent functions in $\mathbb{U}$ given by the Taylor-Maclaurin series expansion (1.1).

For a brief history and examples of functions in the class Σ, see [19] (see also [5], [6], [11], [25]).

Coefficient bounds for various subclasses of bi-univalent functions were obtained by several authors including Ali *et al.* [2], Caglar et al. [3], Deniz [4], Kumar *et al.* [10], Magesh and Yamini [12], Srivastava *et al.* [17], [18], [22], Sümer Eker [1], [23], [24]. In fact, judging by the remarkable flood of papers on the subject, the pioneering work of Srivastava *et al.* [19] appears to have revived the study of analytic and bi-univalent functions in recent years.

The following definition of fractional derivative will be required in our investigation (see, for details, [13], [14], [20], [21]).

Definition 1.1. The fractional integral of order δ is defined, for a function f, by

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\delta}} d\xi\,; \ (\delta > 0),$$

where f is an analytic function in a simply-connected region of complex z-plane containing the origin, and the multiplicity of $(z-\xi)^{\delta-1}$ is removed by requiring, $log(z-\xi)$ to be real when $z - \xi > 0$.

Definition 1.2. The fractional derivative of order δ is defined, for a function f , by

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\delta}} d\xi \ (0 \le \delta < 1),$$

where f is constrained, and the multiplicity of $(z-\xi)^{-\delta}$ is removed, as in Definition 1.1.

Definition 1.3. Under the hypotheses of Definition 2, the fractional derivative of order $(n+\delta)$ is defined by

$$D_z^{n+\delta} f(z) = \frac{d^n}{dz^n} D_z^{\delta} f(z) \quad (0 \le \delta < 1, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$$

By virtue of Definitions 1.1, 1.2 and 1.3, we have

$$D_z^{-\delta} z^n = \frac{\Gamma(n+1)}{\Gamma(n+\delta+1)} z^{n+\delta} \qquad (n \in \mathbb{N}, \delta > 0)$$

and

$$D_z^{\delta} z^n = \frac{\Gamma(n+1)}{\Gamma(n-\delta+1)} z^{n-\delta} \qquad (n \in \mathbb{N}, 0 \le \delta < 1)$$

Tremblay [26] studied a fractional calculus operator defined in terms of the Riemann-Liouville fractional differential operator. Ibrahim and Jahangiri [9] extended and studied this operator in the complex plane.

Definition 1.4. The Tremblay fractional derivative operator $\mathrm{T}_z^{\mu,\gamma}$ of a function $f \in \mathcal{A}$ is defined, for all $z \in \mathbb{U}$, by

$$\mathrm{T}_z^{\mu,\gamma} f(z) = \frac{\Gamma(\gamma)}{\Gamma(\mu)} z^{1-\gamma} D_z^{\mu-\gamma} z^{\mu-1} f(z)$$

$$(0 < \mu \leq 1;\ 0 < \gamma \leq 1;\ \mu \geq \gamma;\ 0 \leq \mu - \gamma < 1).$$

It is clear that, for $\mu = \gamma = 1$, we have

$$\mathrm{T}_z^{1,1} f(z) = f(z).$$

Example 1.5. Let $f(z) = z^n$. The Tremblay Fractional Derivative of $f(z)$ is:

$$\mathrm{T}_z^{\mu,\gamma} f(z) = \frac{\Gamma(\gamma)}{\Gamma(\mu)} \frac{\Gamma(n+\mu)}{\Gamma(n+\gamma)} z^n,$$

and for $\mu = \gamma = 1$, we have $\mathrm{T}_z^{1,1}(z^n) = z^n$.

Recently in [8], Esa et al. defined modified of Tremblay operator of analytic functions in complex domain as follows:

Definition 1.6. Let $f(z) \in \mathcal{A}$. The modified Tremblay operator denoted by $\mathfrak{T}^{\mu,\gamma} : \mathcal{A} \to \mathcal{A}$ and defined such as:

$$\begin{aligned}
\mathfrak{T}^{\mu,\gamma} f(z) &= \frac{\gamma}{\mu} \mathrm{T}_z^{\mu,\gamma} f(z) \\
&= \frac{\Gamma(\gamma+1)}{\Gamma(\mu+1)} z^{1-\gamma} D_z^{\mu-\gamma} z^{\mu-1} f(z) \\
&= z + \sum_{n=2}^{\infty} \frac{\Gamma(\gamma+1)\Gamma(n+\mu)}{\Gamma(\mu+1)\Gamma(n+\gamma)} a_n z^n.
\end{aligned}$$

The object of the present paper is to introduce a new subclass of the function class Σ by using the modified Tremblay operator and find estimate on the coefficients $|a_2|$ and $|a_3|$ for functions in this class.

We begin by introducing the function class $\mathcal{C}_\Sigma^{\mu,\gamma}(\alpha)$ by means of the following definition.

2. Main results

Definition 2.1. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{C}_\Sigma^{\mu,\gamma}(\alpha)$ $(0 < \mu \leq 1;\ 0 < \gamma \leq 1;\ \mu \geq \gamma;\ 0 \leq \mu - \gamma < 1)$ if the following conditions are satisfied:

$$f \in \Sigma \ \text{ and } \ \left| arg \left(1 + \frac{z(\mathfrak{T}f)''(z)}{\mathfrak{T}f'(z)} \right) \right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1, z \in \mathbb{U}) \tag{2.1}$$

and

$$\left| arg \left(1 + \frac{w(\mathfrak{T}g)''(w)}{\mathfrak{T}g'(w)} \right) \right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1, w \in \mathbb{U}) \tag{2.2}$$

where the function $g(w)$ is given by (1.2).

We first state and prove the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{C}_{\Sigma}^{\mu,\gamma}(\alpha)$.

Theorem 2.2. *If $f(z)$ given by (1.1) be in the class $\mathcal{C}_{\Sigma}^{\mu,\gamma}(\alpha)$, then*

$$\left|a_2\right| \leq \alpha(\gamma+1)\sqrt{\frac{(\gamma+2)}{(\mu+1)[3\alpha(\mu+2)(\gamma+1)+(1-3\alpha)(\mu+1)(\gamma+2)]}} \tag{2.3}$$

and

$$\left|a_3\right| \leq \frac{\alpha(\gamma+2)(\gamma+1)^2}{(\mu+1)(\mu\gamma-\mu+4\gamma+2)}. \tag{2.4}$$

Proof. For f given by (1.1), we can write from (2.1) and (2.2)

$$1+\frac{z(\mathfrak{T}f)''(z)}{\mathfrak{T}f'(z)} = [p(z)]^{\alpha} \tag{2.5}$$

$$1+\frac{w(\mathfrak{T}g)''(w)}{\mathfrak{T}g'(w)} = [q(w)]^{\alpha} \tag{2.6}$$

where $p(z)$ and $q(w)$ are in familiar Caratheódory Class $\mathcal{P}$ (see for details [7]) and have the following series representations:

$$p(z) = 1+p_1z+p_2z^2+p_3z^3+\cdots \tag{2.7}$$

and

$$q(w) = 1+q_1w+q_2w^2+q_3w^3+\cdots. \tag{2.8}$$

Now, equating the coefficients (2.5) and (2.6), we find that

$$2\frac{\mu+1}{\gamma+1}a_2 = \alpha p_1, \tag{2.9}$$

$$\frac{6(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)}a_3 - 4\left(\frac{\mu+1}{\gamma+1}\right)^2 a_2^2 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2}p_1^2, \tag{2.10}$$

$$-2\frac{\mu+1}{\gamma+1}a_2 = \alpha q_1 \tag{2.11}$$

and

$$\frac{6(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)}(2a_2^2-a_3) - 4\left(\frac{\mu+1}{\gamma+1}\right)^2 a_2^2 = \alpha q_2 + \frac{\alpha(\alpha-1)}{2}q_1^2. \tag{2.12}$$

From (2.9) and (2.11), we get

$$p_1 = -q_1 \tag{2.13}$$

and

$$8\left(\frac{\mu+1}{\gamma+1}\right)^2 a_2^2 = \alpha^2(p_1^2+q_1^2). \tag{2.14}$$

Also from (2.10), (2.12) and 2.14, we get

$$a_2^2 = \frac{\alpha^2(p_2+q_2)(\gamma+2)(\gamma+1)^2}{4(\mu+1)\left[3\alpha(\mu+2)(\gamma+1)+(1-3\alpha)(\mu+1)(\gamma+2)\right]}. \tag{2.15}$$

According to the Caratheódory Lemma (see [7]), $|p_n| \leq 2$ and $|q_n| \leq 2$ for $n \in \mathbb{N}$. Now taking the absolute value of (2.15) and applying the Carathéodory Lemma for coefficients p_2 and q_2 we obtain

$$|a_2| \leq \sqrt{\frac{\alpha^2(\gamma+2)(\gamma+1)^2}{(\mu+1)\left[3\alpha(\mu+2)(\gamma+1)+(1-3\alpha)(\mu+1)(\gamma+2)\right]}}.$$

This gives desired bound for $|a_2|$ as asserted in (2.3).
Now, in order to find the bound on $|a_3|$, from (2.12) and (2.10) and (2.13), we can write

$$\left\{\frac{72(\mu+2)^2(\mu+1)^2}{(\gamma+2)^2(\gamma+1)^2} - \frac{48(\mu+2)(\mu+1)^3}{(\gamma+2)(\gamma+1)^3}\right\} a_3 \tag{2.16}$$

$$= \alpha\left\{\left(\frac{12(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)} - \frac{4(\mu+1)^2}{(\gamma+1)^2}\right) p_2 + \frac{4(\mu+1)^2}{(\gamma+1)^2} q_2\right\}$$

$$+\frac{6\alpha(\alpha-1)(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)} p_1^2.$$

If $\alpha = 1$ then

$$|a_3| \leq \frac{(\gamma+2)(\gamma+1)^2}{(\mu+1)(\mu\gamma-\mu+4\gamma+2)}.$$

Now, we consider the case $0 < \alpha < 1$. From (2.16), we can write

$$\left\{\frac{72(\mu+2)^2(\mu+1)^2}{(\gamma+2)^2(\gamma+1)^2} - \frac{48(\mu+2)(\mu+1)^3}{(\gamma+2)(\gamma+1)^3}\right\} Re(a_3) \tag{2.17}$$

$$= \alpha Re\left\{\left(\frac{12(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)} - \frac{4(\mu+1)^2}{(\gamma+1)^2}\right) p_2 + \frac{4(\mu+1)^2}{(\gamma+1)^2} q_2\right\}$$

$$+Re\frac{6\alpha(\alpha-1)(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)} p_1^2.$$

From Herglotz's Representation formula [15] for the functions $p(z)$ and $q(w)$, we have

$$p(z) = \int_0^{2\pi} \frac{1+ze^{-it}}{1-ze^{-it}} d\mu_1(t),$$

and

$$q(w) = \int_0^{2\pi} \frac{1+we^{-it}}{1-we^{-it}} d\mu_2(t),$$

where $\mu_i(t)$ are increasing on $[0, 2\pi]$ and $\mu_i(2\pi) - \mu_i(0) = 1$, $i = 1, 2$.
We also have

$$p_n = 2\int_0^{2\pi} e^{-int} d\mu_1(t), \qquad n = 1, 2, \dots$$

$$q_n = 2\int_0^{2\pi} e^{-int} d\mu_2(t), \qquad n = 1, 2, \dots.$$

Now (2.17) can be written as follows:

$$\left\{\frac{72(\mu+2)^2(\mu+1)^2}{(\gamma+2)^2(\gamma+1)^2} - \frac{48(\mu+2)(\mu+1)^3}{(\gamma+2)(\gamma+1)^3}\right\} Re(a_3)$$

$$= \alpha \left\{ \left(\frac{12(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)} - \frac{4(\mu+1)^2}{(\gamma+1)^2} \right) 2\int_0^{2\pi} \cos 2t d\mu_1(t) + \frac{8(\mu+1)^2}{(\gamma+1)^2} \int_0^{2\pi} \cos 2t d\mu_2(t) \right\}$$
$$- \frac{24\alpha(1-\alpha)(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)} \left[\left(\int_0^{2\pi} \cos t d\mu_1 t \right)^2 - \left(\int_0^{2\pi} \sin t d\mu_1(t) \right)^2 \right]$$
$$\leq 2\alpha \left\{ \left(\frac{12(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)} - \frac{4(\mu+1)^2}{(\gamma+1)^2} \right) \int_0^{2\pi} \cos 2t d\mu_1(t) + \frac{4(\mu+1)^2}{(\gamma+1)^2} \int_0^{2\pi} \cos 2t d\mu_2(t) \right\}$$
$$+ \frac{24\alpha(1-\alpha)(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)} \left(\int_0^{2\pi} \sin t d\mu_1(t) \right)^2$$
$$= 2\alpha \left(\frac{12(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)} - \frac{4(\mu+1)^2}{(\gamma+1)^2} \right) \int_0^{2\pi} (1 - 2\sin^2 t) d\mu_1(t)$$
$$+ \frac{8\alpha(\mu+1)^2}{(\gamma+1)^2} \int_0^{2\pi} (1-2\sin^2 t) d\mu_2(t) + \frac{24\alpha(1-\alpha)(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)} \left(\int_0^{2\pi} \sin t d\mu_1(t) \right)^2.$$

By Jensen's inequality ([16]), we have

$$\left(\int_0^{2\pi} |\sin t| d\mu(t) \right)^2 \leq \int_0^{2\pi} \sin^2 t d\mu(t).$$

Hence

$$\left\{ \frac{72(\mu+2)^2(\mu+1)^2}{(\gamma+2)^2(\gamma+1)^2} - \frac{48(\mu+2)(\mu+1)^3}{(\gamma+2)(\gamma+1)^3} \right\} Re(a_3)$$
$$\leq 2\alpha \left(\frac{12(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)} - \frac{4(\mu+1)^2}{(\gamma+1)^2} \right)$$
$$-4\alpha \left(\frac{12(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)} - \frac{4(\mu+1)^2}{(\gamma+1)^2} \right) \int_0^{2\pi} \sin^2 t d\mu_1(t)$$
$$+ \frac{8\alpha(\mu+1)^2}{(\gamma+1)^2} - \frac{16\alpha(\mu+1)^2}{(\gamma+1)^2} \int_0^{2\pi} \sin^2 t d\mu_2(t)$$
$$+ \frac{24\alpha(1-\alpha)(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)} \int_0^{2\pi} \sin^2 t d\mu_1(t)$$

and thus

$$Re(a_3) \leq \frac{\alpha(\gamma+2)(\gamma+1)^2}{(\mu+1)(\mu\gamma - \mu + 4\gamma + 2)}$$

which implies

$$|a_3| \leq \frac{\alpha(\gamma+2)(\gamma+1)^2}{(\mu+1)(\mu\gamma - \mu + 4\gamma + 2)}.$$

This completes the proof of theorem. □

If we take $\gamma = \mu$, in the Theorem 2.2, we obtain following corollary.

Corollary 2.3. *Let $f(z)$ given by (1.1) be in the class $\mathcal{C}_{\Sigma}^{\mu,\mu}(\alpha)$ $(0 < \alpha \leq 1)$. Then*

$$|a_2| \leq \alpha \qquad \textit{and} \qquad |a_3| \leq \frac{2\alpha}{(\gamma+1)^2}.$$

3. Coefficient estimates for the function class $\mathcal{C}_{\Sigma}^{\mu,\gamma}(\beta)$

Definition 3.1. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{C}_{\Sigma}^{\mu,\gamma}(\beta)$ $(0<\mu\le 1;\ 0<\gamma\le 1;\ \mu\ge\gamma;\ 0\le \mu-\gamma<1)$ if the following conditions are satisfied:

$$f\in\Sigma \qquad \textit{and} \qquad Re\left\{1+\frac{z(\mathfrak{T}f)''(z)}{\mathfrak{T}f'(z)}\right\}>\beta \qquad (0\le\beta<1, z\in\mathbb{U}) \tag{3.1}$$

and

$$Re\left\{1+\frac{w(\mathfrak{T}g)''(w)}{\mathfrak{T}g'(w)}\right\}>\beta \qquad (0\le\beta<1, w\in\mathbb{U}) \tag{3.2}$$

where the function g is inverse of the function f given by (1.2).

For $\gamma=\mu$, the class of $\mathcal{C}_{\Sigma}^{\mu,\gamma}(\beta)$ is reduced to $C_{\Sigma}(\beta)$ of bi-convex of order β $(0\le\beta<1)$, which is introduced by Brannan and Taha [5], [6].

Theorem 3.2. *If $f(z)$ given by (1.1) be in the class $\mathcal{C}_{\Sigma}^{\mu,\gamma}(\beta)$, then*

$$|a_2|\le\sqrt{\frac{(1-\beta)(\gamma+1)^2(\gamma+2)}{(\mu+1)(\mu\gamma-\mu+4\gamma+2)}} \tag{3.3}$$

and

$$|a_3|\le\frac{(1-\beta)(\gamma+1)^2(\gamma+2)}{(\mu+1)(\mu\gamma-\mu+4\gamma+2)}. \tag{3.4}$$

Proof. The inequalities in (3.1) and (3.2) can be written in the following forms :

$$1+\frac{z(\mathfrak{T}f)''(z)}{\mathfrak{T}f'(z)}=\beta+(1-\beta)p(z) \tag{3.5}$$

and

$$1+\frac{w(\mathfrak{T}g)''(w)}{\mathfrak{T}g'(w)}=\beta+(1-\beta)q(w) \tag{3.6}$$

where $p(z)$ and $q(w)$ have the forms (2.7) and (2.8), respectively. As in the proof of Theorem 2.2, by equating coefficients (3.5) and (3.6) yields,

$$2\frac{\mu+1}{\gamma+1}a_2=(1-\beta)p_1, \tag{3.7}$$

$$\frac{6(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)}a_3-4\left(\frac{\mu+1}{\gamma+1}\right)^2a_2^2=(1-\beta)p_2, \tag{3.8}$$

$$-2\frac{\mu+1}{\gamma+1}a_2=(1-\beta)q_1 \tag{3.9}$$

and

$$\frac{6(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)}(2a_2^2-a_3)-4\left(\frac{\mu+1}{\gamma+1}\right)^2a_2^2=(1-\beta)q_2. \tag{3.10}$$

From (3.7) and (3.9) we get

$$p_1=-q_1 \tag{3.11}$$

and

$$8\left(\frac{\mu+1}{\gamma+1}\right)^2a_2^2=(1-\beta)^2(p_1^2+q_1^2). \tag{3.12}$$

Also from (3.8) and (3.10) we obtain

$$\frac{4(\mu+1)(\mu\gamma-\mu+4\gamma+2)}{(\gamma+1)^2(\gamma+2)}a_2^2=(1-\beta)(p_2+q_2). \tag{3.13}$$

Thus, clearly we have

$$|a_2|^2\le\frac{(1-\beta)(\gamma+1)^2(\gamma+2)}{4(\mu+1)(\mu\gamma-\mu+4\gamma+2)}\left(|p_2|+|q_2|\right). \tag{3.14}$$

Applying the Carathéodory Lemma for the coefficients p_2 and q_2 we find the bound on $|a_2|$ as asserted in (3.3).
In order to find the bound on $|a_3|$, we multiply

$$\frac{12(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)}-\frac{4(\mu+1)^2}{(\gamma+1)^2} \text{ and } \frac{4(\mu+1)^2}{(\gamma+1)^2}$$

to the relations (3.8) and (3.10) respectively and on adding them we obtain:

$$\left\{\frac{24(\mu+2)(\mu+1)^2(\mu\gamma-\mu+4\gamma+2)}{(\gamma+1)^3(\gamma+2)^2}\right\}a_3$$

$$=(1-\beta)\left\{\frac{4(\mu+1)(2\mu\gamma+\mu+5\gamma+4)}{(\gamma+2)(\gamma+1)^2}p_2+\frac{4(\mu+1)^2}{(\gamma+1)^2}q_2\right\}. \tag{3.15}$$

Taking the absolute value of (3.15) and applying the Carathéodory Lemma for the coefficients p_2, q_2 we find

$$|a_3|\le\frac{(1-\beta)(\gamma+1)^2(\gamma+2)}{(\mu+1)(\mu\gamma-\mu+4\gamma+2)},$$

which is asserted in (3.4). □

If we take $\gamma=\mu$, in the Theorem 3.2, we obtain following corollary.

Corollary 3.3. [5], [6] *Let $f(z)$ given by (1.1) belong to $C_\Sigma(\beta)$ $(0\le\beta<1)$. Then*

$$|a_2|\le\sqrt{1-\beta} \qquad \textit{and} \qquad |a_3|\le 1-\beta.$$

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