The Faber polynomial expansion method and its application to the general coefficient problem for some subclasses of bi-univalent functions associated with a certain $q$-integral operator

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Abstract. In our present investigation, we first introduce several new subclasses of analytic and bi-univalent functions by using a certain $q$-integral operator in the open unit disk $U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}$. By applying the Faber polynomial expansion method as well as the $q$-analysis, we then determine bounds for the $n$th coefficient in the Taylor-Maclaurin series expansion for functions in each of these newly-defined analytic and bi-univalent function classes subject to a gap series condition. We also highlight some known consequences of our main results.

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1. Introduction and definitions

Let $\mathcal{A}$ be the class of all functions $f$ which are analytic in the open unit disk

$$U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}$$

and normalized by

$$f(0) = 0 = f'(0) - 1.$$ 

Thus, clearly, the function $f \in \mathcal{A}$ has the following Taylor-Maclaurin series representation:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in U). \quad (1.1)$$

Further, by $\mathcal{S} \subset \mathcal{A}$ we shall denote the class of all functions which are univalent in $U$. 

For two functions \(f, g \in \mathcal{A}\), the function \(f\) is said to be subordinate to the function \(g\) in \(U\), denoted by
\[
f(z) \preceq g(z) \quad (z \in U),
\]
if there exists a function
\[
w \in \mathcal{B}_0 := \{ w : w \in \mathcal{A}, \quad w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in U) \}
\]
such that
\[
f(z) = g(w(z)) \quad (z \in U).
\]
In the case when the function \(g\) is univalent in \(U\), we have the following equivalence:
\[
f(z) \preceq g(z) \quad (z \in U) \iff f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).
\]
Next, for a function \(f \in \mathcal{A}\) given by (1.1) and another function \(g \in \mathcal{A}\) given by
\[
g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (z \in U),
\]
the convolution (or the Hadamard product) of the functions \(f\) and \(g\) is defined by
\[
(f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g \ast f)(z). \tag{1.2}
\]
It is well known that every univalent function \(f\) has an inverse \(f^{-1}\), defined by
\[
f^{-1}(f(z)) = z = f(f^{-1}(z)) \quad (z \in U)
\]
and
\[
f(f^{-1}(w)) = w \quad \left( |w| < r_0(f); \quad r_0(f) \geq \frac{1}{4} \right),
\]
where
\[
f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{1.3}
\]
A function \(f \in \mathcal{A}\) is said to be bi-univalent in \(U\) if both \(f\) and \(f^{-1}\) are univalent in \(U\). We denote the class of all such functions by \(\Sigma\). In recent years, the pioneering work of Srivastava et al. [22] essentially revived the investigation of various subclasses of the analytic and bi-univalent function class \(\Sigma\). In fact, in a remarkably large number of sequels to the pioneering work of Srivastava et al. [22], several different subclasses of the analytic and bi-univalent function class \(\Sigma\) were introduced and studied analogously by the many authors (see, for example, [5], [7], [9], [23], [24], [25], [28] and [29]). However, only non-sharp estimates on the initial coefficients \(|a_2|\) and \(|a_3|\) in the Taylor-Maclaurin series expansion (1.1) were obtained in these recent papers.

The Faber polynomials introduced by Faber [11] play an important rôle in various areas of mathematical sciences, especially in Geometric Function Theory of Complex Analysis (see, for details, [27]). Recently, several authors (see, for example, [13] and [26]; see also [6], [8], [12] and [20]) investigated some interesting and useful properties for analytic functions by applying the Faber polynomial expansion method. Motivated by these and other recent works (see, for example, [1], [14] and [30]), here we make use of the \(q\)-analysis in order to define new subclasses of analytic and bi-univalent
functions in \( \mathbb{U} \) and (by means of the Faber polynomial expansion method) we determine estimates for the general coefficient \( |a_n| \) \((n \geq 3)\) in the Taylor-Maclaurin series expansion \((1.1)\) of functions in each of these subclasses.

We begin by recalling here some basic definitions and other concept details of the \( q \)-calculus \((0 < q < 1)\), which will be used in this paper.

**Definition 1.1.** Let \( q \in (0,1) \) and define the \( q \)-number \([\kappa]_q\) by

\[
[\kappa]_q = \begin{cases} 
1 - q^\kappa & (\kappa \in \mathbb{C}) \\
\sum_{k=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1} & (\kappa = n \in \mathbb{N}),
\end{cases}
\]

where \( \mathbb{N} \) denotes the set of positive integers and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \).

**Definition 1.2.** Let \( q \in (0,1) \) and define the \( q \)-factorial \([n]_q!\) by

\[
[n]_q! = \begin{cases} 
1 & (n = 0) \\
\prod_{k=1}^{n} [k]_q & (n \in \mathbb{N}).
\end{cases}
\]

**Definition 1.3.** (see [15] and [16]) The \( q \)-derivative (or the \( q \)-difference) \( D_q f \) of a function \( f \) is defined, in a given subset of \( \mathbb{C} \), by

\[
(D_q f)(z) = \begin{cases} 
\frac{f(z) - f(qz)}{(1-q)z} & (z \neq 0) \\
f'(0) & (z = 0),
\end{cases}
\]

provided that \( f'(0) \) exists.

We note from Definition 1.3 that

\[
\lim_{q \to 1^-} (D_q f)(z) = \lim_{q \to 1^-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z)
\]

for a function \( f \) which is differentiable in a given subset of \( \mathbb{C} \). It is readily deduced from \((1.1)\) and \((1.4)\) that

\[
(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}.
\]

**Definition 1.4.** The \( q \)-Pochhammer symbol \([\kappa]_{n,q} \) \((\kappa \in \mathbb{C}; \ n \in \mathbb{N}_0)\) is defined as follows:

\[
[\kappa]_{n,q} = \frac{(q^\kappa; q)_n}{(1-q)^n} := \begin{cases} 
1 & (n = 0) \\
[\kappa]_{q}[\kappa+1]_q[\kappa+2]_q \cdots [\kappa+n-1]_q & (n \in \mathbb{N}).
\end{cases}
\]

Moreover, the \( q \)-gamma function \( \Gamma_q(z) \) is defined by the following recurrence relation:

\[
\Gamma_q(z+1) = [z]_q \Gamma_q(z) \quad \text{and} \quad \Gamma_q(1) = 1.
\]
Definition 1.5. [17] For \( f \in A \), let the Ruscheweyh \( q \)-derivative operator be defined as follows:

\[
\mathcal{I}_q^\lambda f(z) = f(z) \ast \mathcal{F}_{q,\lambda+1}(z) \quad (z \in \mathbb{U}; \ \lambda > -1),
\]

where

\[
\mathcal{F}_{q,\lambda+1}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(\lambda + n)}{[n-1]_q! \Gamma_q(\lambda + 1)} z^n = z + \sum_{n=2}^{\infty} \frac{[\lambda + 1]_q, n-1}{[n-1]_q!} z^n
\]

in terms the Hadamard product (or convolution) given by (1.2).

We next define a certain \( q \)-integral operator by using the same technique as that used by Noor [19].

Definition 1.6. For \( f \in A \), let the \( q \)-integral operator \( \mathcal{F}_{q,\lambda} \) be defined by

\[
\mathcal{F}_{q,\lambda+1}(z) \ast \mathcal{F}_{q,\lambda+1}(z) = z D_q f(z).
\]

Then

\[
\mathcal{I}_q^\lambda f(z) = f(z) \ast \mathcal{F}_{q,\lambda+1}^{-1}(z)
\]

\[
= z + \sum_{n=2}^{\infty} \Psi_{n-1} a_n z^n \quad (z \in \mathbb{U}; \ \lambda > -1),
\]

where

\[
\mathcal{F}_{q,\lambda+1}^{-1}(z) = z + \sum_{n=2}^{\infty} \Psi_{n-1} z^n
\]

and

\[
\Psi_{n-1} = \frac{[n]_q! \Gamma_q(\lambda + 1)}{\Gamma_q(\lambda + n)} = \frac{[n]_q!}{[\lambda + 1]_q, n-1}.
\]

Clearly, we have

\[
\mathcal{I}_0^0 f(z) = z D_q f(z) \quad \text{and} \quad \mathcal{I}_q^1 f(z) = f(z).
\]

We note also that, in the limit case when \( q \to 1^- \), the \( q \)-integral operator \( \mathcal{F}_{q,\lambda} \) given by Definition 1.6 would reduce to the integral operator which was studied by Noor [18].

The following identity can be easily verified:

\[
z D_q \left( \mathcal{I}_q^\lambda f(z) \right) = \left( 1 + \frac{[\lambda]_q}{q^\lambda} \right) \mathcal{I}_q^\lambda f(z) - \frac{[\lambda]_q}{q^\lambda} \mathcal{I}_q^{\lambda+1} f(z).
\]

When \( q \to 1^- \), this last identity (1.6) implies that

\[
z \left( \mathcal{I}_q^{\lambda+1} f(z) \right)' = (1 + \lambda) \mathcal{I}_q^\lambda f(z) - \lambda \mathcal{I}_q^{\lambda+1} f(z),
\]

which is the well-known recurrence relation for the above-mentioned integral operator which studied by Noor [18].

The above-defined \( q \)-calculus provides valuable tools that have been extensively used in order to examine several subclasses of \( A \). Even though Ismail et al. [14] were the first to use the \( q \)-derivative operator \( D_q \) in order to study a certain \( q \)-analogue of the class \( S^* \) of starlike functions in \( \mathbb{U} \), yet a rather significant usage of the \( q \)-calculus in the context of Geometric Function Theory of Complex Analysis was
basically furnished and the basic (or $q$-) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava (see, for details, [21, pp. 347 et seq.; see also [23]).

We now introduce the following subclasses of the analytic and bi-univalent function class $\Sigma$.

**Definition 1.7.** For a function $f \in \Sigma$, we say that
\[ f \in R_q (\Sigma, \alpha, \gamma) \quad (0 \leq \alpha < 1; \ \gamma \geq 0) \]
if and only if
\[ \left| D_q f(z) + \gamma z D^2_q f(z) - \frac{1 - \alpha q}{1 - q} \right| < \frac{1 - \alpha}{1 - q} \quad (z \in U) \]
and
\[ \left| D_q g(w) + \gamma w D^2_q g(w) - \frac{1 - \alpha q}{1 - q} \right| < \frac{1 - \alpha}{1 - q} \quad (w \in U). \]

Equivalently, by using the principle of subordination between analytic functions, we can write the above conditions as follows (see, for details, [30]):
\[ D_q f(z) + \gamma z D^2_q f(z) \prec 1 + \left[ 1 + \frac{1 - \alpha (1 + q)}{1 - q} \right] z \quad (z \in U) \]
and
\[ D_q g(w) + \gamma w D^2_q g(w) \prec 1 + \left[ 1 + \frac{1 - \alpha (1 + q)}{1 - q} \right] w \quad (w \in U), \]
respectively, where $g(w) = f^{-1}(w)$ is given by (1.3).

**Definition 1.8.** For a function $f \in \Sigma$, we say that
\[ f \in R_q (\Sigma, \alpha, \gamma, \lambda) \quad (0 \leq \alpha < 1; \ \gamma \geq 0; \ \lambda \geq 0) \]
if and only if
\[ D_q^2 f(z) + \gamma D^2_q f(z) \prec 1 + \left[ 1 + \frac{1 - \alpha (1 + q)}{1 - q} \right] z \quad (z \in U) \]
and
\[ D_q^2 g(w) + \gamma w D^2_q g(w) \prec 1 + \left[ 1 + \frac{1 - \alpha (1 + q)}{1 - q} \right] w \quad (w \in U), \]
where $g(w) = f^{-1}(w)$ is given by (1.3).

### 2. The Faber polynomial expansion method and its applications

In this section, by using the Faber polynomial expansion of a function $f \in A$ of the form (1.1), we observe that the coefficients of its inverse map $g = f^{-1}$ may be expressed as follows (see [4]; see also [13] and [26]):
\[ g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1} (a_2, a_3, \ldots, a_n) w^n, \quad (2.1) \]
\[ K_{n-1}^{-n} = \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!} a_2^{n-3} a_3 \\
+ \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 \\
+ \frac{(-n)!}{(2(-n+2))!(n-5)!} a_2^{n-5} [a_5 + (-n+2)a_3^2] \\
+ \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} [a_6 + (-2n+5)a_3 a_4] \\
+ \sum_{j \geq 7} a_2^{n-j} V_j. \tag{2.2} \]

Here, and in what follows, such expressions as (for example) \((-n)!\) occurring in (2.2) are to be interpreted symbolically by
\[ (-n)! \equiv \Gamma(1-n) := (-n)(-n-1)(-n-2) \cdots (n \in \mathbb{N}_0) \]
and \(V_j \ (7 \leq j \leq n)\) is a homogeneous polynomial in the variables \(a_2, a_3, \cdots, a_n\).

In particular, the first three terms of \(K_{n-1}^{-n}\) are given below:

\[ K_1^{-2} = -2a_2, \quad K_2^{-3} = 3(2a_2^2 - a_3) \]
and
\[ K_3^{-4} = -4(5a_2^3 - 5a_2a_3 + a_4). \]

In general, an expansion of \(K_n^p\) is given by (see, for details, [3])
\[ K_n^p = p a_n + \frac{p(p-1)}{2} E_n^2 + \frac{p!}{(-3)!} E_n^3 + \cdots + \frac{p!}{(p-n)!n!} E_n^p \quad (p \in \mathbb{Z}), \]
where \(\mathbb{Z} := \{0, \pm 1, \pm 2, \cdots\}\) and
\[ E_n^p = E_n^p (a_2, a_3, \cdots). \]

It is clearly seen that
\[ E_n^p (a_1, a_2, \cdots, a_n) = a_1^n, \]
and
\[ E_{n-1}^m (a_2, \cdots, a_n) = \sum_{n=2}^{\infty} \frac{m!(a_2)^{\mu_1} \cdots (a_n)^{\mu_{n-1}}}{\mu_1! \cdots \mu_{n-1}!} (m \leq n). \]

We also have (see [2])
\[ E_{n-1}^{n-1} (a_2, \cdots, a_n) = a_2^{n-1} \]
and
\[ E_n^m (a_1, a_2, \cdots, a_n) = \sum \left( \frac{m!}{\mu_1! \cdots \mu_n!} \right) a_1^{\mu_1} \cdots a_n^{\mu_n} \quad (m \leq n), \]
where \(a_1 = 1\) and the sum is taken over all non-negative integers \(\mu_1, \cdots, \mu_n\) satisfying the following conditions:
\[ \mu_1 + 2\mu_2 + \cdots + n\mu_n = m \]
and
\[ \mu_1 + 2\mu_2 + \cdots + n\mu_n = n. \]
By a similar argument, we note that

\[ E_n(a_1, \cdots, a_n) = E_1^n \]

and that the first and the last polynomials are given by

\[ E_1^n = a_1^n \quad \text{and} \quad E_n^1 = a_n. \]

We now state and prove our main results. Throughout our discussion, the parameters \( L \) and \( M \) are given by

\[ L := [1 - \alpha(1 + q)] \quad \text{and} \quad M := -q. \]

**Theorem 2.1.** For \( 0 \leq \alpha < 1 \) and \( \gamma \geq 0 \), let \( f \in R_q (\Sigma, \alpha, \gamma) \). If

\[ a_m = 0 \quad (2 \leq m \leq n - 1), \]

then

\[ |a_n| \leq \frac{|1 - \alpha + q(1 - \alpha)|}{[n]_q + \gamma [n]_q [n - 1]_q} \quad (n \geq 3). \tag{2.3} \]

**Proof.** For the function \( f \in R_q (\Sigma, \alpha, \gamma) \) of the form (1.1), we have

\[ D_q f(z) + \gamma z D_q^2 f(z) = 1 + \sum_{n=2}^{\infty} \left( [n]_q + \gamma [n]_q [n - 1]_q \right) a_n z^{n-1} \tag{2.4} \]

and, for its inverse map \( g = f^{-1} \), we get

\[ D_q g(w) + \gamma w D_q^2 g(w) = 1 + \sum_{n=2}^{\infty} \left( [n]_q + \gamma [n]_q [n - 1]_q \right) b_n w^{n-1}, \tag{2.5} \]

where

\[ b_n = \frac{1}{[n]_q} K_n^{-1} (a_2, a_3, \cdots, a_n). \]

Since both the function \( f \) and its inverse map \( g = f^{-1} \) are in \( R_q (\Sigma, \alpha, \gamma) \), by the definition of subordination, there exist two Schwarz functions \( p(z) \) and \( q(w) \) given by

\[ p(z) = \sum_{n=1}^{\infty} c_n z^n \quad \text{and} \quad q(w) = \sum_{n=1}^{\infty} d_n w^n \quad (z, w \in U), \]

so that we have

\[ D_q f(z) + \gamma z D_q^2 f(z) = \frac{1 + L \rho(z)}{1 + M \rho(z)} \]

\[ = 1 - \sum_{n=1}^{\infty} (L - M) K_n^{-1} (c_1, c_2, \cdots, c_n, M) z^n \tag{2.6} \]

and

\[ D_q g(w) + \gamma w D_q^2 g(w) = \frac{1 + L \rho(w)}{1 + M \rho(w)} \]

\[ = 1 - \sum_{n=1}^{\infty} (L - M) K_n^{-1} (d_1, d_2, \cdots, d_n, M) w^n. \tag{2.7} \]
In general, for any $p \in \mathbb{N}$ and $n \geq 2$, we have the following expansion of $K^p_n(k_1, k_2, \ldots, k_n, \mathcal{M})$ (see [3] and [4]):

$$K^p_n(k_1, k_2, \ldots, k_n, \mathcal{M}) = \frac{p!}{(p-n)!n!} k^n \mathcal{M}^{n-1} + \frac{p!}{(p-n+1)!(n-2)!} k^{n-2}k_2 \mathcal{M}^{n-2}$$

$$+ \frac{p!}{(p-n+2)!(n-3)!} k^{n-3}k_3 \mathcal{M}^{n-3}$$

$$+ \frac{p!}{(p-n+3)!(n-4)!} k^n [k_4 \mathcal{M}^{n-4} + \frac{p-n+3}{2} k^2 \mathcal{M}]$$

$$+ \frac{p!}{(p-n+4)!(n-5)!} k^n [k_5 \mathcal{M}^{n-5} + (p-n+4)k_3k_4 \mathcal{M}]$$

$$+ \sum_{j \geq 6} k^{n-1}X_j, \quad (2.8)$$

where $X_j$ is a homogeneous polynomial of degree $j$ in the variables $k_1, k_2, \ldots, k_n$.

For the coefficients of the Schwarz functions $p(z)$ and $q(w)$, we have (see [10])

$$|c_n| \leq 1 \quad \text{and} \quad |d_n| \leq 1.$$  

Thus, upon comparing with the corresponding coefficients in (2.4) and (2.6), we find that

$$(\lfloor n \rfloor_q + \gamma \lfloor n \rfloor_q \lfloor n-1 \rfloor_q) a_n = -(\mathcal{L} - \mathcal{M})K^{-1}_{n-1}(c_1, c_2, \ldots, c_{n-1}, \mathcal{M}). \quad (2.9)$$

Similarly, in view of the corresponding coefficients in (2.5) and (2.7), we have

$$(\lfloor n \rfloor_q + \gamma \lfloor n \rfloor_q \lfloor n-1 \rfloor_q) b_n = -(\mathcal{L} - \mathcal{M})K^{-1}_n(d_1, d_2, \ldots, d_n, \mathcal{M}). \quad (2.10)$$

We note for

$$a_m = 0 \quad (2 \leq m \leq n-1) \quad \text{and} \quad b_n = -a_n,$$

that

$$(\lfloor n \rfloor_q + \gamma \lfloor n \rfloor_q \lfloor n-1 \rfloor_q) a_n = -(\mathcal{L} - \mathcal{M})c_{n-1} \quad (2.11)$$

and

$$-(\lfloor n \rfloor_q + \gamma \lfloor n \rfloor_q \lfloor n-1 \rfloor_q) a_n = -(\mathcal{L} - \mathcal{M})d_{n-1} \quad (2.12)$$

Taking the moduli in (2.11) and (2.12), we thus obtain

$$|a_n| \leq \frac{|\mathcal{L} - \mathcal{M}|}{\lfloor n \rfloor_q + \gamma \lfloor n \rfloor_q \lfloor n-1 \rfloor_q} |c_{n-1}|$$

$$= \frac{|\mathcal{L} - \mathcal{M}|}{\lfloor n \rfloor_q + \gamma \lfloor n \rfloor_q \lfloor n-1 \rfloor_q} |d_{n-1}|.$$  

Therefore, we have

$$|a_n| \leq \frac{|1 - \alpha + q(1 - \alpha)|}{\lfloor n \rfloor_q + \gamma \lfloor n \rfloor_q \lfloor n-1 \rfloor_q} \quad (n \geq 3),$$

which completes the proof of the assertion (2.3) of Theorem 2.1.  \[ \square \]
If we let $q \to 1−$ in Theorem 2.1 above, we obtain the following known result given by Srivastava et al. [26].

**Corollary 2.2.** (see [26]) Let $f$ given by (1.1) be in the class $\mathcal{R}^{\alpha,\gamma}_\Sigma$ $(0 \leq \alpha < 1; \gamma \geq 0)$.

If

$$a_m = 0 \quad (2 \leq m \leq n - 1),$$

then

$$|a_n| \leq \frac{2(1 - \alpha)}{n[1 + \gamma(n - 1)]} \quad (n \in \mathbb{N} \setminus \{1, 2\}).$$

**Theorem 2.3.** For $0 \leq \alpha < 1$ and $0 \leq \gamma$, let $f \in \mathcal{R}_q(\Sigma, \alpha, \gamma)$. Then

$$|a_2| \leq \min \left\{ \frac{|1 - \alpha + q(1 - \alpha)|}{[2]_q + \gamma [2]_q [1]_q}, \sqrt{\frac{2(1 + q)|1 - \alpha + q(1 - \alpha)|}{[2]_q [3]_q + \gamma [3]_q [2]_q}} \right\},$$

$$|a_3| \leq \min \left\{ \frac{|1 - \alpha + q(1 - \alpha)|}{[1]_q + [1]_q} \left( \frac{[2]_q |1 - \alpha + q(1 - \alpha)|}{([2]_q + \gamma [2]_q [1]_q)^2} + \frac{2}{[3]_q + \gamma [3]_q [2]_q} \right), \frac{2 (q + 2)|1 - \alpha + q(1 - \alpha)|}{([1]_q + [1]_q)([3]_q + \gamma [3]_q [2]_q)} \right\},$$

$$|a_3 - [2]_q a_2^2| \leq \frac{(1 + q)|1 - \alpha + q(1 - \alpha)|}{[3]_q + \gamma [3]_q [2]_q},$$

and

$$\left| a_3 - \frac{[2]_q}{[1]_q + [1]_q} a_2^2 \right| \leq \frac{2|1 - \alpha + q(1 - \alpha)|}{([1]_q + [1]_q)([3]_q + \gamma [3]_q [2]_q)}.$$  

**Proof.** Upon setting $n = 2$ and $n = 3$ in (2.9) and (2.10), respectively, we get

$$\left( [2]_q + \gamma [2]_q [1]_q \right) a_2 = -(\mathcal{L} - \mathcal{M})c_1, \quad (2.13)$$

$$\left( [3]_q + \gamma [3]_q [2]_q \right) a_3 = -(\mathcal{L} - \mathcal{M})(\mathcal{M}c_1^2 - c_2), \quad (2.14)$$

$$- \left( [2]_q + \gamma [2]_q [1]_q \right) a_2 = -(\mathcal{L} - \mathcal{M})d_1 \quad (2.15)$$

and

$$\left( [3]_q + \gamma [3]_q [2]_q \right) \left( [2]_q a_2^2 - a_3 \right) = -(\mathcal{L} - \mathcal{M})(\mathcal{M}d_1^2 - d_2). \quad (2.16)$$
From (2.13) and (2.15), we have

$$|a_2| \leq \frac{|L - M|}{[2]_q + \gamma [2]_q [1]_q} |c_1|$$

$$= \frac{|L - M|}{[2]_q + \gamma [2]_q [1]_q} |d_1|$$

$$\leq \frac{|1 - \alpha + q(1 - \alpha)|}{[2]_q + \gamma [2]_q [1]_q}. \quad (2.17)$$

Adding (2.14) and (2.16), we find that

$$[2]_q [3]_q + \gamma [3]_q [2]_q a_2^2 = -(L - M) [M (c_1^2 + d_1^2) - (c_2 + d_2)], \quad (2.18)$$

which, upon taking the moduli on both sides, yields

$$|a_2|^2 = \frac{2 |L - M| (|M| + 1)}{[2]_q [3]_q + \gamma [3]_q [2]_q}. \quad (2.19)$$

This last equation can be written as follows:

$$|a_2| \leq \sqrt{\frac{2(1 + q)|1 - \alpha + q(1 - \alpha)|}{[2]_q [3]_q + \gamma [3]_q [2]_q}}. \quad (2.20)$$

Now, in order to find $|a_3|$, by subtracting (2.16) from (2.14), we obtain

$$a_3 = \frac{(L - M) [M (d_1^2 - c_1^2) - (c_2 - d_2)]}{([1]_q + [1]_q) ([3]_q + \gamma [3]_q [2]_q)} + \frac{[2]_q}{([1]_q + [1]_q)} a_2^2. \quad (2.21)$$

Taking the moduli in (2.20) and using the fact that $d_1^2 = c_1^2$, we have

$$|a_3| \leq \frac{2 |L - M| ([1]_q + [1]_q) ([3]_q + \gamma [3]_q [2]_q)}{([1]_q + [1]_q) ([3]_q + \gamma [3]_q [2]_q)} + \frac{[2]_q}{[1]_q + [1]_q} |a_2|^2. \quad (2.22)$$

Using (2.17) in (2.21), we obtain

$$|a_3| \leq \frac{|1 - \alpha + q(1 - \alpha)|}{[1]_q + [1]_q}$$

$$\cdot \left( \frac{[2]_q |1 - \alpha + q(1 - \alpha)|}{[2]_q + \gamma [2]_q [1]_q} \right)^2 + \frac{2}{[3]_q + \gamma [3]_q [2]_q}. \quad (2.23)$$

Again, by using the equation (2.19) in (2.21), we have

$$|a_3| \leq \frac{2 (q + 2) |1 - \alpha + q(1 - \alpha)|}{([1]_q + [1]_q) ([3]_q + \gamma [3]_q [2]_q)}. \quad (2.24)$$

We also find from (2.16) that

$$|a_3 - [2]_q a_2^2| \leq \frac{1 + q}{[3]_q + \gamma [3]_q [2]_q}. \quad (2.25)$$
From (2.20) and using the fact that $d_1^2 = c_1^2$, we have
\[ a_3 - \frac{[2]_q}{[1]_q + [1]_q} a_2^2 = \frac{(\mathcal{L} - \mathcal{M}) (c_2 - d_2)}{([1]_q + [1]_q) \left( [3]_q + \gamma [3]_q \frac{[2]_q}{[1]_q} \right)}. \tag{2.24} \]

Finally, by taking the moduli in (2.24), we finally obtain
\[ \left| a_3 - \frac{[2]_q}{[1]_q + [1]_q} a_2^2 \right| \leq \frac{2 |1 - \alpha + q(1 - \alpha)|}{\left( [1]_q + [1]_q \right) \left( [3]_q + \gamma [3]_q \frac{[2]_q}{[1]_q} \right)}. \]

The proof of Theorem 2.3 is thus completed. \qed

In the limit case when $q \to 1^-$, Theorem 2.3 yields the following bounds on $|a_2|$ and $|a_3|$ given by Srivastava et al. [26].

**Corollary 2.4.** (see [26]) Let $f$ given by (1.1) be in the class $\mathcal{R}_\Sigma^\alpha,\gamma$ $(0 \leq \alpha < 1; \gamma \geq 0)$.

Then
\[ a_2 \leq \begin{cases} \sqrt{\frac{2(1 - \alpha)}{3(1 + 2\gamma)}} & \left( 0 \leq \alpha \leq \frac{1 + 2\gamma - 2\gamma^2}{3(1 + 2\gamma)} \right) \\ \frac{1 - \alpha}{1 + \gamma} & \left( \frac{1 + 2\gamma - 2\gamma^2}{3(1 + 2\gamma)} \leq \alpha < 1 \right) \end{cases} \]

and
\[ a_3 \leq \frac{2(1 - \alpha)}{3(1 + 2\gamma)}. \]

**Theorem 2.5.** For $0 \leq \alpha < 1$ and $0 \leq \gamma$, let $f \in \mathcal{R}_q (\Sigma, \alpha, \gamma, \lambda)$. If
\[ a_m = 0 \quad (2 \leq m \leq n - 1), \]
then
\[ |a_n| \leq \frac{|1 - \alpha + q(1 - \alpha)| [\lambda + 1]_{q,n-1}}{\left( [n]_q + \gamma [n]_q [n - 1]_q \right) [n]_q!} \quad (n \geq 3). \tag{2.25} \]

**Proof.** For the function $f \in \mathcal{R}_q (\Sigma, \alpha, \gamma, \lambda)$ of the form (1.1), we have
\[ D_q \mathcal{I}_q^\lambda f (z) + \gamma z D_q^2 \mathcal{I}_q^\lambda f (z) = 1 + \sum_{n=2}^{\infty} \left( [n]_q + \gamma [n]_q [n - 1]_q \right) \Psi_{n-1} a_n z^{n-1}. \tag{2.26} \]

Also, for its inverse mapping $g = f^{-1}$, we have
\[ D_q \mathcal{I}_q^\lambda g(w) + \gamma w D_q^2 \mathcal{I}_q^\lambda g(w) = 1 + \sum_{n=2}^{\infty} \left( [n]_q + \gamma [n]_q [n - 1]_q \right) \Psi_{n-1} b_n w^{n-1}, \tag{2.27} \]
where
\[ b_n = \frac{1}{[n]_q} K_{n-1}^{-1} (a_2, a_3, \cdots, a_n). \]

Since, both \( f \) and its inverse \( g = f^{-1} \) are in the function class \( \mathcal{R}_q (\Sigma, \alpha, \gamma, \lambda) \), by the definition of subordination, there exist two Schwarz functions \( p(z) \) and \( q(w) \) given by
\[
p(z) = \sum_{n=1}^{\infty} c_n z^n \quad \text{and} \quad q(w) = \sum_{n=1}^{\infty} d_n w^n \quad (z, w \in U),
\]
so that we have
\[
D_q I_\lambda f(z) + \gamma z D_q^2 I_\lambda f(z) = 1 + \frac{\mathcal{L} p(z)}{1 + \mathcal{M} p(z)} = 1 - \sum_{n=1}^{\infty} (\mathcal{L} - \mathcal{M}) K_n^{-1} (c_1, c_2, \cdots, c_n, \mathcal{M}) z^n \quad (2.28)
\]
and
\[
D_q I_\lambda g(w) + \gamma w D_q^2 I_\lambda g(w) = 1 + \frac{\mathcal{L} q(w)}{1 + \mathcal{M} q(w)} = 1 - \sum_{n=1}^{\infty} (\mathcal{L} - \mathcal{M}) K_n^{-1} (d_1, d_2, \cdots, d_n, \mathcal{M}) w^n. \quad (2.29)
\]

In general, for any \( p \in \mathbb{N} \) and \( n \geq 2 \), an expansion of
\[ K_p^n (k_1, k_2, \cdots, k_n, \mathcal{M}) \]
is given by (2.8) (see [3] and [4]). Moreover, the coefficients of the Schwarz functions \( p(z) \) and \( q(w) \) are constrained by (see [10])
\[
|c_n| \leq 1 \quad \text{and} \quad |d_n| \leq 1.
\]
Thus, upon comparing the corresponding coefficients in (2.26) and (2.28), we find that
\[
\left( [n]_q + \gamma [n]_q [n-1]_q \right) \Psi_{n-1} a_n = - (\mathcal{L} - \mathcal{M}) K_{n-1}^{-1} (c_1, c_2, \cdots, c_{n-1}, \mathcal{M}). \quad (2.30)
\]
Similarly, by comparing the corresponding coefficients in (2.27) and (2.29), we have
\[
\left( [n]_q + \gamma [n]_q [n-1]_q \right) \Psi_{n-1} b_n = - (\mathcal{L} - \mathcal{M}) K_{n-1}^{-1} (d_1, d_2, \cdots, d_n, \mathcal{M}). \quad (2.31)
\]
We note also that, for
\[ a_m = 0 \quad (2 \leq m \leq n - 1) \quad \text{and} \quad b_n = -a_n,
\]
we have
\[
\left( [n]_q + \gamma [n]_q [n-1]_q \right) \Psi_{n-1} a_n = - (\mathcal{L} - \mathcal{M}) c_{n-1} \quad (2.32)
\]
and
\[- \left( [n]_q + \gamma [n]_q [n-1]_q \right) \Psi_{n-1} a_n = -(\mathcal{L} - \mathcal{M}) d_{n-1}.\] (2.33)

Finally, by taking the moduli in (2.32) and (2.33), we obtain
\[|a_n| \leq \frac{|\mathcal{L} - \mathcal{M}|}{(n)_q + \gamma [n]_q [n-1]_q} \Psi_{n-1} |e_{n-1}|\]
\[= \frac{|\mathcal{L} - \mathcal{M}|}{(n)_q + \gamma [n]_q [n-1]_q} |d_{n-1}|.\]

Consequently, we have
\[|a_n| \leq \frac{|1 - \alpha + q(1 - \alpha)| [\lambda + 1]_q, n-1}{(n)_q + \gamma [n]_q [n-1]_q} |n|_q! (n \geq 3),\]
which completes the proof of the assertion (2.25) of Theorem 2.5. □

**Theorem 2.6.** For $0 \leq \alpha < 1$ and $\gamma \geq 0$, let $f \in \mathcal{R}_q (\Sigma, \alpha, \gamma, \lambda)$. Then
\[|a_2| \leq \min \left\{ \frac{|1 - \alpha + q(1 - \alpha)| [\lambda + 1]_q, 1}{(2)_q + \gamma [2]_q [1]_q} [2]_q!, \right.\]
\[\left. \sqrt{\frac{2(1 + q) |1 - \alpha + q(1 - \alpha)| [\lambda + 1]_q, 2}{(2)_q \left( [3]_q + \gamma [3]_q [2]_q \right)} [3]_q!} \right\}, \] (2.34)

\[|a_3| \leq \min \left\{ \frac{|1 - \alpha + q(1 - \alpha)| (([\lambda + 1]_q, 1)^2 [2]_q |1 - \alpha + q(1 - \alpha)|}{(2)_q!^2 \left( [2]_q + \gamma [2]_q [1]_q \right)^2} + \frac{2[\lambda + 1]_q, 2}{([3]_q + \gamma [3]_q [2]_q) [3]_q!}, \right.\]
\[\left. \frac{2 (q + 2) |1 - \alpha + q(1 - \alpha)| [\lambda + 1]_q, 2}{([1]_q + [1]_q) \left( [3]_q + \gamma [3]_q [2]_q \right) [3]_q!} \right\}, \] (2.35)

\[|a_3 - [2]_q a_2|^2 \leq \frac{(1 + q) |1 - \alpha + q(1 - \alpha)| [\lambda + 1]_q, 2}{([3]_q + \gamma [3]_q [2]_q) [3]_q!}, \] (2.36)

and
\[|a_3 - \left( \frac{[2]_q}{[1]_q + [1]_q} \right) a_2|^2 \leq \frac{2 |1 - \alpha + q(1 - \alpha)| [\lambda + 1]_q, 2}{([1]_q + [1]_q) \left( [3]_q + \gamma [3]_q [2]_q \right) [3]_q!}. \] (2.37)
Proof. Upon setting \( n = 2 \) and \( n = 3 \) in (2.30) and (2.31), respectively, we have
\[
\left( [2]_q + \gamma [2]_q [1]_q \right) \Psi_1 a_2 = -(\mathcal{L} - \mathcal{M}) c_1, \tag{2.38}
\]
\[
\left( [3]_q + \gamma [3]_q [2]_q \right) \Psi_2 a_3 = -(\mathcal{L} - \mathcal{M})(\mathcal{M}c_1^2 - c_2), \tag{2.39}
\]
\[- \left( [2]_q + \gamma [2]_q [1]_q \right) \Psi_1 a_2 = -(\mathcal{L} - \mathcal{M}) d_1, \tag{2.40}
\]
and
\[
\left( [3]_q + \gamma [3]_q [2]_q \right) \Psi_2 \left( [2]_q a_2^2 - a_3 \right) = -(\mathcal{L} - \mathcal{M})(\mathcal{M}d_1^2 - d_2). \tag{2.41}
\]
Making use of (2.38) and (2.40), we find that
\[
|a_2| \leq \frac{|\mathcal{L} - \mathcal{M}|}{\left( [2]_q + \gamma [2]_q [1]_q \right) \Psi_1} |c_1|
\]
\[
= \frac{|\mathcal{L} - \mathcal{M}|}{\left( [2]_q + \gamma [2]_q [1]_q \right) \Psi_1} |d_1|
\]
\[
\leq |1 - \alpha + q(1 - \alpha)||\lambda + 1|_{q,1} \left( [2]_q + \gamma [2]_q [1]_q \right) [2]_q! . \tag{2.42}
\]
Also, by adding (2.39) and (2.41), we have
\[
\left( [3]_q + \gamma [3]_q [2]_q \right) \Psi_2 \left( [2]_q a_2^2 - a_3 \right) = -(\mathcal{L} - \mathcal{M}) \left[ \mathcal{M} \left( c_1^2 + d_1^2 \right) - (c_2 + d_2) \right] . \tag{2.43}
\]
Now, if we take the moduli in both sides of (2.43), we obtain
\[
|a_2|^2 = \frac{2 |\mathcal{L} - \mathcal{M}| (|\mathcal{M}| + 1)}{[2]_q \left( [3]_q + \gamma [3]_q [2]_q \right) \Psi_2},
\]
so that
\[
|a_2| \leq \sqrt{\frac{2(1 + q)|1 - \alpha + q(1 - \alpha)||\lambda + 1|_{q,2}}{[2]_q \left( [3]_q + \gamma [3]_q [2]_q \right) [3]_q!}} . \tag{2.44}
\]

In order to find \( |a_3| \), we subtract (2.41) from (2.39). We thus obtain
\[
a_3 = \frac{(\mathcal{L} - \mathcal{M}) \left[ \mathcal{M} \left( d_1^2 - c_1^2 \right) - (c_2 - d_2) \right]}{\left( [1]_q + [1]_q \right) \left( [3]_q + \gamma [3]_q [2]_q \right) \Psi_2} + \left( \frac{[2]_q}{\left( [1]_q + [1]_q \right) \Psi_2} \right) a_2^2, \tag{2.45}
\]
which, after taking the moduli and using the fact that \( d_1^2 = c_1^2 \),
yields
\[
|a_3| \leq \frac{2 |\mathcal{L} - \mathcal{M}|}{ \left( [1]_q + [1]_q \right) \left( [3]_q + \gamma [3]_q [2]_q \right) \Psi_2} + \left( \frac{[2]_q}{\left( [1]_q + [1]_q \right) \Psi_2} \right) |a_2|^2 . \tag{2.46}
\]
Using (2.42) in (2.46), we have
\[ |a_3| \leq \frac{|1 - \alpha + q(1 - \alpha)|}{[1]_q + [1]_q} \left( \frac{([\lambda + 1]_q, 1)^2}{([2]_q + \gamma [2]_q [1]_q)^2} \left( [2]_q + \gamma [2]_q [1]_q \right) \right. \\
\left. + \frac{2[\lambda + 1]_q, 2}{([3]_q + \gamma [3]_q [2]_q) [3]_q !} \right). \tag{2.47} \]

Again, by using (2.44) in (2.46), we get
\[ |a_3| \leq \frac{2(q + 2)|1 - \alpha + q(1 - \alpha)|}{([1]_q + [1]_q)} \left( [3]_q + \gamma [3]_q [2]_q \right) [3]_q !. \]

It follows from (2.41) that
\[ |a_3 - [2]_q a_2^2| \leq \frac{(1 + q)|1 - \alpha + q(1 - \alpha)|}{([3]_q + \gamma [3]_q [2]_q) [3]_q !}. \]

Using the fact that
\[ d_1^2 = c_1^2 \]
in (2.45), we have
\[ a_3 - \left( \frac{[2]_q}{[1]_q + [1]_q} \right) a_2^2 = \frac{\mathcal{L} - \mathcal{M} (c_2 - d_2)}{([1]_q + [1]_q) ([3]_q + \gamma [3]_q [2]_q) \Psi_2}. \tag{2.48} \]

By taking the moduli on both sides of (2.48), we finally obtain
\[ |a_3 - \left( \frac{[2]_q}{([1]_q + [1]_q)} \right) a_2^2| \leq \frac{2|1 - \alpha + q(1 - \alpha)|}{([1]_q + [1]_q) ([3]_q + \gamma [3]_q [2]_q) [3]_q !}, \]
which completes the proof of Theorem 2.6. \hfill \Box

3. Concluding remarks and observations

Here, in our present investigation, we have successfully applied the Faber polynomial expansion method as well as the q-analysis in our study of several new subclasses of analytic and bi-univalent functions by using a certain q-integral operator in the open unit disk \( U \). We have derived bounds for the \( n \)th coefficient in the Taylor-Maclaurin series expansion for functions in each of these newly-defined analytic and bi-univalent function classes subject to a gap series condition. By means of corollaries of our main theorems, we have also highlighted some known consequences of our main results, which were given recently by Srivastava et al. [26].
References


The Faber polynomial expansion method and its application


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