Fekete-Szegö problems for generalized Sakaguchi type functions associated with quasi-subordination

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Abstract. In the present paper, the authors introduce a generalized Sakaguchi type non-Bazilevic function class $M_{\lambda,\beta}^{\phi,s,t}(\phi,s,t)$ of analytic functions involving quasi-subordination and obtain bounds for the Fekete-Szegö functional $|a_3-\mu a_2^2|$ for the functions belonging to the above and associated classes. Some important and useful special cases of the main results are also pointed out.

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1. Introduction and preliminaries

Let $\mathcal{A}$ be the class of analytic functions in the open unit disk:

$$U := \{z \in \mathbb{C} : |z| < 1\}$$

having the normalized power series expansion given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in U). \quad (1.1)$$

A function $f(z) \in \mathcal{A}$ is said to be univalent in $U$ if $f(z)$ is one-to-one in $U$. As usual, we denote by $S$ the subclass of $\mathcal{A}$ consisting of univalent functions in $U$ (see [3]).

For two functions $f$ and $g$ in $\mathcal{A}$, we say that $f$ is subordinate to $g$ in $U$, and write as

$$f \prec g \text{ in } U \quad \text{or} \quad f(z) \prec g(z) \quad (z \in U),$$

if there exists a Schwarz function $w(z)$, analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$ such that

$$f(z) = g(w(z)) \quad (z \in U). \quad (1.2)$$
If the function $g$ is univalent in $U$, then
\[
f(z) \prec g(z) \quad (z \in U) \iff f(0) = g(0) \text{ and } f(U) \subset g(U).
\]

For a brief survey on the concept of subordination, we refer to the works in [3, 10, 13, 27].

Further, a function $f(z)$ is said to be quasi-subordinate to $g(z)$ in the unit disk $U$ if there exists the functions $\varphi(z)$ and $w(z)$ (with constant coefficient zero) which are analytic and bounded by one in the unit disk $U$ such that
\[
\frac{f(z)}{\varphi(z)} \prec g(z) \quad (z \in U).
\]

We denote the quasi-subordination by
\[
f(z) \prec_q g(z) \quad (z \in U).
\]

Also, we note that quasi-subordination (1.4) is equivalent to
\[
f(z) = \varphi(z)g(w(z)) \quad (z \in U).
\]

One may observe that when $\varphi(z) \equiv 1$ ($z \in U$), the quasi-subordination $\prec_q$ becomes the usual subordination $\prec$. If we put $w(z) = z$ in (1.5), then the quasi-subordination (1.5) becomes the majorization. In this case, we have
\[
f(z) \prec_q g(z) \implies f(z) = \varphi(z)g(z) \implies f(z) \ll g(z) \quad (z \in U).
\]

The concept of majorization is due to MacGregor [12] and quasi-subordination is thus a generalization of the usual subordination as well as the majorization. The work on quasi-subordination is quite extensive which includes some recent expository investigations in [1, 7, 9, 14, 21, 22].

Recently, Frasin [5] introduced and studied a generalized Sakaguchi type classes $S_1(\alpha, s, t)$ and $T(\alpha, s, t)$. A function $f(z) \in A$ is said to be in the class $S_1(\alpha, s, t)$ if it satisfies
\[
\Re \left[ \frac{(s-t)zf'(z)}{\varphi(z)} \right] > \alpha
\]
for some $\alpha$ ($0 \leq \alpha < 1$), $s, t \in \mathbb{C}$, $|s-t| \leq 1$, $s \neq t$ and $z \in U$.

We also denote by $T(\alpha, s, t)$, the subclass of $A$ consisting of all functions $f(z)$ such that $zf'(z) \in S(\alpha, s, t)$. For $s = 1$, the class $S(\alpha, 1, t)$ becomes the subclass $S^*(\alpha, t)$ studied by Owa et al. [17, 18]. If $t = -1$ in $S(\alpha, 1, t)$, then the class $S(\alpha, 1, -1) = S_s(\alpha)$ was introduced by Sakaguchi [23] and is called Sakuguchi function of order $\alpha$ (see [2, 17]), whereas $S_s(0) \equiv S_s$ is the class of starlike functions with respect to symmetrical points in $U$. Further, $S(\alpha, 1, 0) \equiv S^*(\alpha)$ and $T(\alpha, 1, 0) \equiv C(\alpha)$ are the familiar classes of starlike functions of order $\alpha$ ($0 \leq \alpha < 1$) and convex function of order $\alpha$ ($0 \leq \alpha < 1$), respectively.

Obradovic [16] introduced a class of functions $f \in A$ which satisfies the inequality:
\[
\Re \left[ f'(z) \left( \frac{z}{f(z)} \right)^{1+\lambda} \right] > 0 \quad (0 < \lambda < 1; z \in \mathbb{U}),
\]
and he calls such functions as functions of non-Bazilevič type.
By \( \mathcal{P} \), we denote the class of functions \( \phi \) analytic in \( \mathbb{U} \) such that \( \phi(0) = 1 \) and \( \Re(\phi(z)) > 0 \).

Ma and Minda [11] unified various subclasses of starlike and convex functions for which either of the quantity \( \frac{zf'(z)}{f(z)} \) or \( 1 + \frac{zf''(z)}{f'(z)} \) is subordinate to a more general subordination function. They introduced a class \( S^*(\phi) \) defined by

\[
S^*(\phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z) \ (z \in \mathbb{U}) \right\},
\]

where \( \phi \in \mathcal{P} \) and \( \phi(\mathbb{U}) \) is symmetrical about the real axis and \( \phi'(0) > 0 \). A function \( f \in S^*(\phi) \) is called a Ma and Minda starlike function with respect to \( \phi \).

Recently, Sharma and Raina [25] introduced and studied a generalized Sakaguchi type non-Bazilevic function class \( \mathcal{G}_q^\lambda(\phi, b) \). A function \( f(z) \in \mathcal{A} \) is said to be in the class \( \mathcal{G}_q^\lambda(\phi, b) \) if it satisfies the condition that

\[
\left\lfloor f'(z) \left( \frac{(1 - b)z}{f(z) - f(bz)} \right)^\lambda - 1 \right\rfloor \prec_q (\phi(z) - 1) \quad (1 \neq b \in \mathbb{C}, |b| \leq 1, \lambda \geq 0; z \in \mathbb{U}).
\]

Motivated by aforementioned works, we introduce here a new subclass of \( \mathcal{A} \) which is defined as follows:

**Definition 1.1.** Let \( \phi \in \mathcal{P} \) be univalent and \( \phi(\mathbb{U}) \) symmetrical about the real axis and \( \phi'(0) > 0 \). For \( s, t \in \mathbb{C}, s \neq t, |s - t| \leq 1, \lambda, \beta \geq 0 \), a function \( f \in \mathcal{A} \) is said to be in the class \( \mathcal{M}_q^{\lambda, \beta}(\phi, s, t) \) if it satisfies the condition that

\[
\left\lfloor (1 - \beta)f'(z) + \beta \frac{f(z)}{z} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\rfloor \left\lfloor \frac{(s - t)z}{f(sz) - f(tz)} \right\rfloor^\lambda \prec (\phi(z) - 1) \quad (z \in \mathbb{U}),
\]

where the powers are considered to be having only principal values.

By specializing the parameters \( \lambda, \beta, s, \) and \( t \) in Definition 1.1 above, we obtain various subclasses which have been studied recently. To illustrate these subclasses, we observe the following:

(i) When \( \beta = 0, s = 1 \), then the class \( \mathcal{M}_q^{0, \lambda}(\phi, 1, t) = \mathcal{G}_q^\lambda(\phi, t) \) which was studied recently by Sharma and Raina [25].

(ii) Next, when \( \beta = t = 0, \lambda = s = 1; \beta = \lambda = t = 0, s = 1 \) and \( \lambda = \beta = s = 1, t = 0 \); then the classes \( \mathcal{M}_q^{1,0}(\phi, 1, 0), \mathcal{M}_q^{0,0}(\phi, 1, 0) \) and \( \mathcal{M}_q^{1,-1}(\phi, 1, 0) \) which, respectively, reduce to the classes \( S_q^*(\phi), \mathcal{R}_q(\phi) \) and \( \mathcal{C}_q(\phi) \) were studied earlier by Mohd and Darus [14].

From the Definition 1.1, it follows that \( f \in \mathcal{M}_q^{\lambda, \beta}(\phi, s, t) \) if and only if there exists an analytic function \( \varphi(z) \) with \( |\varphi(z)| \leq 1 \ (z \in \mathbb{U}) \) such that

\[
\left\lfloor (1 - \beta)f'(z) + \beta \frac{f(z)}{z} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\rfloor \left\lfloor \frac{(s - t)z}{f(sz) - f(tz)} \right\rfloor^\lambda \prec (\phi(z) - 1) \quad (z \in \mathbb{U}).
\]

If we set \( \varphi(z) \equiv 1 \ (z \in \mathbb{U}) \) in (1.11), then the class \( \mathcal{M}_q^{\lambda, \beta}(\phi, s, t) \) is denoted by \( \mathcal{M}^{\lambda, \beta}(\phi, s, t) \) satisfying the condition that

\[
\left\lfloor (1 - \beta)f'(z) + \beta \frac{f(z)}{z} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\rfloor \left\lfloor \frac{(s - t)z}{f(sz) - f(tz)} \right\rfloor^\lambda \prec (\phi(z) \quad (z \in \mathbb{U}).
\]
It may be noted that for $\beta = 0$, $s = \lambda = 1$ and for real $t$, the class $M^{1,0}(\phi, 1, t) = S^\gamma(\phi, t)$ which was studied by Goyal and Goswami [6].

It is well-known (see [3]) that for $f \in S$ given by (1.1), there holds a sharp inequality for the functional $|a_3 - a_2^2|$. Fekete-Szegö [4] obtained sharp upper bounds for $|a_3 - \mu a_2^2|$ for $f \in S$ when $\mu$ is real and thus the determination of the sharp upper bounds for such a nonlinear functional for any compact family $F$ of functions in $S$ is popularly known as the Fekete-Szegö problem for $F$. Fekete-Szegö problems for several subclasses of $S$ have been investigated by many authors including [19, 20, 24]; see also [26].

The aim of this paper is to obtain the coefficient estimates including a Fekete-Szegö inequality of functions belonging to the classes $M^{\lambda,\beta}_{q,s,t}(\phi,s,t)$ and $M^{\lambda,\beta}_{q,s,t}(\phi,s,t)$ and the class involving the majorization. Some consequences of the main results are also pointed out.

We need the following lemma in our investigations.

**Lemma 1.2.** ([8, p.10]) Let the Schwarz function $w(z)$ be given by

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots \quad (z \in \mathbb{U}),$$

(1.13)

then

$$|w_1| \leq 1, \quad |w_2 - \mu w_1^2| \leq 1 + (|\mu| - 1)|w_1|^2 \leq \max\{1, |\mu|\},$$

where $\mu \in \mathbb{C}$. The result is sharp for the function $w(z) = z$ or $w(z) = z^2$.

## 2. Main results

Let $f \in A$ of the form (1.1), then for $s, t \in \mathbb{C}, \ |s - t| \leq 1, s \neq t$, we may write that

$$\frac{f(sz) - f(tz)}{s - t} = z + \sum_{n=2}^{\infty} \gamma_n a_n z^n,$$

(2.1)

where

$$\gamma_n = \frac{s^n - t^n}{s - t} = s^{n-1} + s^{n-2}t + \cdots + t^{n-1} \quad (n \in \mathbb{N}).$$

(2.2)

Therefore for $\lambda \geq 0$, we have

$$\left[ \frac{(s-t)z}{f(sz) - f(tz)} \right]^\lambda = 1 - \lambda \gamma_2 a_2 z + \lambda \left[ \frac{\lambda + 1}{2} \gamma_2 a_2^2 - \gamma_3 a_3 \right] z^2 + \cdots.$$ 

(2.3)

Unless otherwise stated, throughout the sequel, we assume that

$$\lambda \gamma_n \neq (n - 1)^2 \beta + n;$$

and that for real $s, t$:

$$\lambda \gamma_n < (n - 1)^2 \beta + n, \quad n = 2, 3, 4, \ldots.$$

Let the function $\phi \in P$ be of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots \quad (B_1 \in \mathbb{R}, B_1 > 0),$$

(2.4)

and $\varphi(z)$ analytic in $\mathbb{U}$ be of the form

$$\varphi(z) = c_0 + c_1 z + c_2 z^2 + \cdots (c_0 \neq 0).$$

(2.5)
We now state and prove our first main result.

**Theorem 2.1.** Let the function \( f \in \mathcal{A} \) of the form (1.1) be in the class \( \mathcal{M}_q^{\lambda, \beta}(\phi, s, t) \), then

\[
|a_2| \leq \frac{B_1}{2 + \beta - \lambda \gamma_2},
\]

(2.6)

and for any \( \mu \in \mathbb{C} \):

\[
|a_3 - \mu a_2^2| \leq \frac{B_1}{|3 + 4\beta - \lambda \gamma_3|} \max \left\{ 1, \left| \frac{B_2}{B_1} - B_1 R \right| \right\},
\]

(2.7)

where

\[
R = \frac{(3 + 4\beta - \lambda \gamma_3)\mu}{(2 + \beta - \lambda \gamma_2)^2} - \frac{\lambda (4 + 2\beta - (1 + \lambda)\gamma_2) \gamma_2 + 4\beta}{2(2 + \beta - \lambda \gamma_2)^2}
\]

(2.8)

and \( \gamma_n \ (n \in \mathbb{N}) \) is given by (2.2). The result is sharp.

**Proof.** Let \( f \in \mathcal{M}_q^{\lambda, \beta}(\phi, s, t) \). In view of Definition 1.1, there exists then a Schwarz function \( w(z) \) given by (1.13) and an analytic function \( \varphi(z) \) given by (2.5) such that

\[
\left[ (1 - \beta)f'(z) + \beta \frac{f(z)}{z} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] \left[ \frac{(s - t)z}{f(sz) - f(tz)} \right]^\lambda - 1 = \varphi(z)(\phi(w(z)) - 1),
\]

(2.9)

which can be expressed as

\[
\varphi(z)(\phi(w(z)) - 1) = (c_0 + c_1 z + c_2 z^2 + \cdots) (B_1 w_1 z + (B_1 w_2 + B_2 w_1^2) z^2 + \cdots) = c_0 B_1 w_1 z + \{ c_0 (B_1 w_2 + B_2 w_1^2) + c_1 B_1 w_1 \} z^2 + \cdots.
\]

(2.10)

Using now the series expansions for \( f'(z) \) and \( f''(z) \) from (1.1), we obtain that

\[
(1 - \beta)f'(z) + \beta \frac{f(z)}{z} \left[ 1 + \frac{zf''(z)}{f'(z)} \right] = 1 + (2 + \beta) a_2 z + ((3 + 4\beta) a_3 - 2\beta a_2^2) z^2 + \cdots.
\]

(2.11)

Thus, it follows from (2.3) and (2.11) that

\[
\left[ (1 - \beta)f'(z) + \beta \frac{f(z)}{z} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] \left[ \frac{(s - t)z}{f(sz) - f(tz)} \right]^\lambda - 1 = (2 + \beta - \lambda \gamma_2) a_2 z + (3 + 4\beta - \lambda \gamma_3) a_3 - \lambda \left( 2 + \beta - \frac{1 + \lambda}{2} \gamma_2 \right) \gamma_2 a_2^2 - 2\beta a_2^2 z^2 + \cdots.
\]

(2.12)

Making use of (2.10) and (2.12) in (2.9) and equating the coefficients of \( z \) and \( z^2 \) in the resulting expression, we get

\[
(2 + \beta - \lambda \gamma_2) a_2 = c_0 B_1 w_1
\]

(2.13)

and

\[
(3 + 4\beta - \lambda \gamma_3) a_3 - \lambda \left[ 2 + \beta - \frac{1 + \lambda}{2} \gamma_2 \right] \gamma_2 a_2^2 - 2\beta a_2^2 = c_0 (B_1 w_2 + B_2 w_1^2) + c_1 B_1 w_1.
\]

(2.14)

Now (2.13) yields that

\[
a_2 = \frac{c_0 B_1 w_1}{2 + \beta - \lambda \gamma_2}.
\]

(2.15)
From (2.14), we have
\[
a_3 = \frac{B_1}{3 + 4\beta - \lambda\gamma_3} \left[ c_1 w_1 \right.
\]
\[+ c_0 \left\{ w_2 + \left( \frac{c_0\lambda \left( 1 + \frac{2+\beta-\gamma_2}{2(\beta-\lambda\gamma_2)} \right)}{2(2+\beta-\lambda\gamma_2)} \frac{2\beta c_0 B_1}{(2+\beta-\lambda\gamma_2)^2} + \frac{B_2}{B_1} \right) w_1^2 \right\}. \tag{2.16}
\]

Hence, for any complex number \( \mu \), we have
\[
a_3 - \mu a_2^2 = \frac{B_1}{3 + 4\beta - \lambda\gamma_3} \left[ c_1 w_1 \right.
\]
\[+ c_0 \left\{ w_2 + \left( \frac{\lambda(4+2\beta-(1+\lambda)\gamma_2)\gamma_2 + 4\beta}{2(2+\beta-\lambda\gamma_2)^2} c_0 B_1 + \frac{B_2}{B_1} \right) w_1^2 \right\]. \tag{2.17}
\]
where \( R \) is given by (2.8).

Since \( \varphi(z) \) given by (2.5) is analytic and bounded in the open unit disk \( U \), hence upon using [15, p. 172], we have for some \( y \) \( |y| \leq 1 \):
\[
|c_0| \leq 1 \text{ and } c_1 = (1 - c_0^2) y. \tag{2.18}
\]

Putting the value of \( c_1 \) from (2.18) into (2.17), we finally get
\[
a_3 - \mu a_2^2 = \frac{B_1}{3 + 4\beta - \lambda\gamma_3} \left[ w_2 + \left( 1 + \frac{B_1 B_2}{B_1^2 w_1^2} \right) c_0 - (B_1 w_1^2 R + w_1 y) c_0 \right]. \tag{2.19}
\]

If \( c_0 = 0 \), then (2.19) gives
\[
|a_3 - \mu a_2^2| \leq \frac{B_1}{3 + 4\beta - \lambda\gamma_3}. \tag{2.20}
\]

On the other hand, if \( c_0 \neq 0 \), then we consider
\[
T(c_0) = w_2 + \left( \frac{B_2}{B_1} w_1^2 \right) c_0 - (B_1 w_1^2 R + w_1 y) c_0^2. \tag{2.21}
\]

The expression (2.21) is a quadratic polynomial in \( c_0 \) and hence analytic in \( |c_0| \leq 1 \). The maximum value of \( |T(c_0)| \) is attained at \( c_0 = e^{i\theta} \) \( (0 \leq \theta < 2\pi) \), and hence, we have
\[
\max |T(c_0)| = \max_{0 \leq \theta < 2\pi} |T(e^{i\theta})| = |T(1)| = \left| w_2 - \left( B_1 R - \frac{B_2}{B_1} \right) w_1^2 \right|. \tag{2.22}
\]

Thus from (2.19), we get
\[
|a_3 - \mu a_2^2| \leq \frac{B_1}{3 + 4\beta - \lambda\gamma_3} \left| w_2 - \left( B_1 R - \frac{B_2}{B_1} \right) w_1^2 \right|, \tag{2.22}
\]
and in view of Lemma 1.2, we obtain that
\[
|a_3 - \mu a_2^2| \leq \frac{B_1}{3 + 4\beta - \lambda\gamma_3} \max \left\{ 1, \left| B_1 R - \frac{B_2}{B_1} \right| \right\}. \tag{2.23}
\]
The desired assertion (2.7) follows now from (2.20) and (2.23). The result is sharp for the function $f(z)$ given by

$$
[(1 - \beta)f'(z) + \beta \frac{f(z)}{z} \left(1 + \frac{zf''(z)}{f'(z)}\right)] \left[\frac{(s-t)z}{f(sz) - f(tz)}\right]^\lambda = \phi(z)
$$
or

$$
[(1 - \beta)f'(z) + \beta \frac{f(z)}{z} \left(1 + \frac{zf''(z)}{f'(z)}\right)] \left[\frac{(s-t)z}{f(sz) - f(tz)}\right]^\lambda = \phi(z^2).
$$

This completes the proof of Theorem 2.1. □

By setting $\beta = t = 0$, $\lambda = s = 1$ in the Theorem 2.1, we obtain the following sharp results for the subclass $S_\lambda^*(\phi)$.

**Corollary 2.2.** Let $f \in A$ of the form (1.1) be in the class $S_\lambda^*(\phi)$, then

$$|a_2| \leq B_1,$$

and for any $\mu \in \mathbb{C}$:

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{2} \max \left\{1, \left|\frac{B_2}{B_1} + (1 - 2\mu)B_1\right|\right\}.
$$

The result is sharp.

Next, putting $\beta = \lambda = s = 1$ and $t = 0$ in Theorem 2.1, we obtain the following sharp results for the class $C_\lambda^*(\phi)$.

**Corollary 2.3.** Let $f \in A$ of the form (1.1) belong to the class $C_\lambda^*(\phi)$, then

$$|a_2| \leq \frac{B_1}{2},$$

and for any $\mu \in \mathbb{C}$:

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{6} \max \left\{1, \left|\frac{B_2}{B_1} + \left(1 - \frac{3\mu}{2}\right)B_1\right|\right\}.
$$

The result is sharp.

Further, by putting $\beta = \lambda = t = 0$ and $s = 1$ in Theorem 2.1, we get the following sharp results for the class $R_\lambda^*(\phi)$.

**Corollary 2.4.** Let $f \in A$ of the form (1.1) belong to the class $R_\lambda^*(\phi)$, then

$$|a_2| \leq \frac{B_1}{2},$$

and for any $\mu \in \mathbb{C}$:

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{3} \max \left\{1, \left|\frac{B_2}{B_1} - \frac{3\mu}{4}B_1\right|\right\}.
$$

The result is sharp.

**Remark 2.5.** The Fekete-Szegö type inequalities mentioned above for the classes $S_\lambda^*(\phi)$, $C_\lambda^*(\phi)$ and $R_\lambda^*(\phi)$ improve similar results obtained earlier in [14].

The next theorem gives the result for the class $M^{\lambda,\beta}(\phi, s, t)$. 
Theorem 2.6. Let \( f \in A \) of the form (1.1) belong to the class \( M^{\lambda,\beta}(\phi, s, t) \), then

\[
|a_2| \leq \frac{B_1}{|2 + \beta - \lambda \gamma_2|},
\]

and for any \( \mu \in \mathbb{C} \):

\[
|a_3 - \mu a_2^2| \leq \frac{B_1}{|3 + 4\beta - \lambda \gamma_3|} \max \left\{ 1, \left| \frac{B_2}{B_1} - B_1 R \right| \right\},
\]

where \( R \) is given by (2.8) and \( \gamma_n (n \in \mathbb{N}) \) is given by (2.2). The result is sharp.

Proof. The proof is similar to Theorem 2.1. Let \( f \in M^{\lambda,\beta}(\phi, s, t) \). If \( \varphi(z) \equiv 1 \), then (2.5) gives \( c_0 = 1 \) and \( c_n = 0 \) \((n \in \mathbb{N})\). Therefore, in view of (2.15), (2.17) and by an application of Lemma 1.2, we obtain the desired assertion. The result is sharp for the function \( f(z) \) given by

\[
(1 - \beta)f'(z) + \beta f(z) \left( 1 + zf''(z) \right) \left[ \frac{(s - t)z}{f(sz) - f(tz)} \right] = \phi(z)
\]
or

\[
(1 - \beta)f'(z) + \beta f(z) \left( 1 + zf''(z) \right) \left[ \frac{(s - t)z}{f(sz) - f(tz)} \right]^\lambda = \phi(z^2).
\]

The next theorem gives the result based on majorization.

Theorem 2.7. Let \( s, t \in \mathbb{C}, s \neq t, |s - t| \leq 1 \).

If a function \( f \in A \) of the form (1.1) satisfies

\[
(1 - \beta)f'(z) + \beta \frac{f(z)}{z} \left( 1 + z f''(z) \right) \left[ \frac{(s - t)z}{f(sz) - f(tz)} \right]^\lambda - 1 \ll \varphi(z) - 1 \quad (z \in \mathbb{U}),
\]

then

\[
|a_2| \leq \frac{B_1}{|2 + \beta - \lambda \gamma_2|},
\]

and for any \( \mu \in \mathbb{C} \):

\[
|a_3 - \mu a_2^2| \leq \frac{B_1}{|3 + 4\beta - \lambda \gamma_3|} \max \left\{ 1, \left| \frac{B_2}{B_1} - B_1 R \right| \right\},
\]

where \( R \) is given by (2.8) and \( \gamma_n (n \in \mathbb{N}) \) is defined as (2.2). The result is sharp.

Proof. Assume that (2.24) holds true. Hence, by the definition of majorization there exists an analytic function \( \varphi(z) \) given by (2.5) such that for \( z \in \mathbb{U} \) we have

\[
(1 - \beta)f'(z) + \beta \frac{f(z)}{z} \left( 1 + z f''(z) \right) \left[ \frac{(s - t)z}{f(sz) - f(tz)} \right]^\lambda - 1 = \varphi(z)(\varphi(z) - 1). \tag{2.25}
\]

Following similar steps as in the proof of Theorem 2.1 and by setting \( w(z) \equiv 1 \), so that \( w_1 = 1 \) and \( w_n = 0 \), \( n \geq 2 \), we obtain

\[
a_2 = \frac{c_0 B_1}{2 + \beta - \lambda \gamma_2},
\]
so that

\[ |a_2| \leq \frac{B_1}{2 + \beta - \lambda \gamma_2} \]

and

\[ a_3 - \mu a_2^2 = \frac{B_1}{3 + 4\beta - \lambda \gamma_3} \left[ c_1 + \frac{B_2}{B_1} c_0 - B_1 c_0^2 R \right]. \quad (2.26) \]

On putting the value of \( c_1 \) from (2.18) in (2.26), we get

\[ a_3 - \mu a_2^2 = \frac{B_1}{3 + 4\beta - \lambda \gamma_3} \left[ y + \frac{B_2}{B_1} c_0 - (B_1 R + y) c_0^2 \right]. \quad (2.27) \]

If \( c_0 = 0 \), then (2.27) yields

\[ |a_3 - \mu a_2^2| \leq \frac{B_1}{|3 + 4\beta - \lambda \gamma_3|}. \quad (2.28) \]

But if \( c_0 \neq 0 \), then we define the function

\[ H(c_0) := y + \frac{B_2}{B_1} c_0 - (B_1 R + y) c_0^2. \quad (2.29) \]

The expression (2.29) is a polynomial in \( c_0 \) and hence analytic in \(|c_0| \leq 1\). The maximum value of \(|H(c_0)|\) occurs at \( c_0 = e^{i\theta} \) \((0 \leq \theta < 2\pi)\), and we have

\[ \max_{0 \leq \theta < 2\pi} H(e^{i\theta}) = |H(1)|. \]

From (2.27), we get

\[ |a_3 - \mu a_2^2| \leq \frac{B_1}{|3 + 4\beta - \lambda \gamma_3|} \left| B_1 R - \frac{B_2}{B_1} \right|. \quad (2.30) \]

Thus, the assertion of Theorem 2.7 follows from (2.28) and (2.30). The result is sharp for the function given by

\[ (1 - \beta) f'(z) + \beta f(z) \left( 1 + \frac{zf''(z)}{f'(z)} \right) \left[ \frac{(s - t)z}{f(sz) - f(tz)} \right]^\lambda = \phi(z) \quad (z \in U). \]

This completes the proof of Theorem 2.7. \( \square \)

Next, we determine the bounds for the functional \(|a_3 - \mu a_2^2|\) for real \( \mu, s, t \) for the class \( \mathcal{M}^{\lambda,\beta}_q(\phi, s, t) \).

**Corollary 2.8.** Let the function \( f \in \mathcal{A} \) given by (1.1) be in the class \( \mathcal{M}^{\lambda,\beta}_q(\phi, s, t) \), then (for real values of \( \mu, s, t \)):

\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_1}{3 + 4\beta - \lambda \gamma_3} \left[ B_1 Q + \frac{B_2}{B_1} \right] & \mu \leq \alpha_1, \\
\frac{B_1}{3 + 4\beta - \lambda \gamma_3} & \alpha_1 \leq \mu \leq \alpha_1 + 2\rho, \\
- \frac{B_1}{3 + 4\beta - \lambda \gamma_3} \left[ B_1 Q + \frac{B_2}{B_1} \right] & \mu \geq \alpha_1 + 2\rho, \end{cases} \quad (2.31) \]

where

\[ \alpha_1 = \frac{\lambda(4 + 2\beta - (1 + \lambda)\gamma_2)\gamma_2 + 4\beta}{2(3 + 4\beta - \lambda \gamma_3)} - \frac{(2 + \beta - \lambda \gamma_2)^2}{(3 + 4\beta - \lambda \gamma_3)} \left( \frac{1}{B_1} - \frac{B_2}{B_1^2} \right). \quad (2.32) \]
When one of its rotations.

\( \mu \) rotations, while for \( \mu \) (iii) Equality holds for

\( \alpha \) and \( \gamma \) (n \in \mathbb{N}) is given by (2.2). Each of the estimates in (2.31) is sharp.

Proof. For \( s, t, \mu \in \mathbb{R} \), the above bounds can be obtained from (2.7), respectively, under the following cases:

\[ B_1R - \frac{B_2}{B_1} \leq -1, \quad -1 \leq B_1R - \frac{B_2}{B_1} \leq 1 \text{ and } B_1R - \frac{B_2}{B_1} \geq 1, \]

where \( R \) is given by (2.8). We also note the following:

(i) When \( \mu < \alpha_1 \) or \( \mu > \alpha_1 + 2\rho \), then the equality holds if and only if \( w(z) = z \) or one of its rotations.

(ii) When \( \alpha_1 < \mu < \alpha_1 + 2\rho \), then the inequality holds if and only if \( w(z) = z^2 \) or one of its rotations.

(iii) Equality holds for \( \mu = \alpha_1 \) if and only if \( w(z) = \frac{z(z+\epsilon)}{1+\epsilon z} (0 \leq \epsilon \leq 1) \) or one of its rotations, while for \( \mu = \alpha_1 + 2\rho \), the equality holds if and only if \( w(z) = -\frac{z(z+\epsilon)}{1+\epsilon z} (0 \leq \epsilon \leq 1) \), or one of its rotations.

The second part of assertion in (2.31) can be improved further.

Theorem 2.9. Let \( f \in A \) of the form (1.1) belong to the class \( \mathcal{M}^{\lambda, \beta}_q(\phi, s, t) \), then (for \( s, t, \mu \in \mathbb{R} \) \( \alpha_1 \leq \mu \leq \alpha_1 + 2\rho \))

\[ |a_3 - \mu a_2^2| + (\mu - \alpha_1)|a_2|^2 \leq \frac{B_1}{3 + 4\beta - \lambda \gamma_3} \quad (\alpha_1 \leq \mu \leq \alpha_1 + \rho) \]  

(2.34)

and

\[ |a_3 - \mu a_2^2| + (\alpha_1 + 2\rho - \mu)|a_2|^2 \leq \frac{B_1}{3 + 4\beta - \lambda \gamma_3} \quad (\alpha_1 + \rho, \mu < \alpha_1 + 2\rho) \]  

(2.35)

where \( \alpha_1 \) and \( \rho \) are given by (2.32) and (2.33), respectively, and \( \gamma_3 \) is given by (2.2).

Proof. Let \( f \in \mathcal{M}^{\lambda, \beta}_q(\phi, s, t) \). For \( s, t, \mu \in \mathbb{R} \) and \( \alpha_1 \leq \mu \leq \alpha_1 + \rho \), and in view of (2.15) and (2.22), we get

\[ |a_3 - \mu a_2^2| + (\mu - \alpha_1)|a_2|^2 \leq \frac{B_1}{3 + 4\beta - \lambda \gamma_3} \cdot \left[ |w_2| - \frac{B_1(3 + 4\beta - \lambda \gamma_3)}{(2 + \beta - \lambda \gamma_2)^2} (\mu - \alpha_1 - \rho)|w_1|^2 + \frac{B_1(3 + 4\beta - \lambda \gamma_3)}{(2 + \beta - \lambda \gamma_2)^2} (\mu - \alpha_1)|w_1|^2 \right]. \]

Hence, by virtue of Lemma 1.2, we have

\[ |a_3 - \mu a_2^2| + (\mu - \alpha_1)|a_2|^2 \leq \frac{B_1}{3 + 4\beta - \lambda \gamma_3} \left[ 1 - |w_1|^2 + |w_1|^2 \right], \]

which yields the assertion (2.34).

If \( \alpha_1 + \rho < \mu < \alpha_1 + 2\rho \), then again from (2.15) and (2.22) and Lemma 1.2, we obtain

\[ |a_3 - \mu a_2^2| + (\alpha_1 + 2\rho - \mu)|a_2|^2 \leq \frac{B_1}{3 + 4\beta - \lambda \gamma_3}. \]
\[
\left| w_2 \right| + \frac{B_1(3 + 4\beta - \lambda\gamma_3)}{(2 + \beta - \lambda\gamma_2)^2} (\mu - \alpha_1 - \rho) \left| w_1 \right|^2 + \frac{B_1(3 + 4\beta - \lambda\gamma_3)}{(2 + \beta - \lambda\gamma_2)^2} (\alpha_1 + 2\rho - \mu) \left| w_1 \right|^2 \leq \frac{B_1}{3 + 4\beta - \lambda\gamma_3} [1 - \left| w_1 \right|^2 + \left| w_1 \right|^2],
\]
which gives the estimate (2.35).

We conclude this paper by remarking that the above theorems include several previously established results for particular values of the parameters \(\lambda, s, t\) and \(\beta\). Thus, if we set \(\beta = 0, s = 1\) in Theorems 2.1 and 2.6 above, we arrive at the Fekete-Szegö type inequalities for the classes \(G^\lambda_\phi(\phi, t)\) and \(G^\lambda(\phi, t)\), respectively, studied by Sharma and Raina [25]. Further, the majorization result and improvement of bounds given by Theorems 2.7 and 2.9 provide extensions of similar results due to Sharma and Raina [25].

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