New subclasses of bi-univalent functions defined by multiplier transformation

Saurabh Porwal and Shivam Kumar

Abstract. In the present paper, we introduce new subclasses of the function class $\sum$ of bi-univalent functions by using multiplier transformation. Furthermore, we obtain estimates on the coefficients $|a_2|, |a_3|$ and $|a_4|$ for functions of this class. Relevant connections of the results presented here with various well-known results are briefly indicated.

Mathematics Subject Classification (2010): 30C45.

Keywords: Analytic, univalent, bi-univalent functions, multiplier transformation.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) which are also univalent in $U$.

It is well known that every $f \in \mathcal{S}$ has an inverse $f^{-1}$, defined by

$$f^{-1}(f(z)) = z, \ (z \in U)$$

and

$$f \left( f^{-1}(\omega) \right) = \omega, \ \left( |\omega| < r_0(f), \ r_0(f) \geq \frac{1}{4} \right),$$

where

$$f^{-1}(\omega) = \omega - a_2 \omega^2 + (2a_2^2 - a_3) \omega^3 - (5a_2^3 - 5a_2a_3 + a_4) \omega^4 + \ldots$$

A function $f(z) \in \mathcal{A}$ is said to be bi-univalent in $U$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $U$. Let $\Sigma$ denote the class of bi-univalent functions in $U$ given by
the Taylor-Maclaurin series expansion (1.1). Examples of functions in the class Σ are
\[ \frac{1}{1-z}, \log(1-z), \frac{1}{2} \log \left( \frac{1+z}{1-z} \right) \] and so on.

However, the familiar Koebe function is not a member of Σ. Other common examples of functions in \( \mathcal{S} \) such as \( z - z^2 \) and \( \frac{1}{1-z^2} \) are also not members of Σ.

Lewin [7] first investigated the bi-univalent function class Σ showed that \( |a_2| < 1.5 \).

Subsequently, Brannan and Clunie [2] conjectured that \( |a_2| \leq \sqrt{2} \).

Netanyahu [9], on the other hand, showed that \( \max_{f \in \Sigma} |a_2| = \frac{4}{3} \).

The coefficient estimate problems for bi-univalent function class Σ is an interesting problem of Geometric Function Theory. Several researchers e.g. (see [1], [6], [11], [12], [14], [16], [17]), introduced the various new subclasses of the bi-univalent function class Σ and obtained non-sharp bounds on the first two coefficients \( |a_2| \) and \( |a_3| \). Recently, Mishra and Soren [8] obtain coefficient bounds for bi-starlike analytic functions and improve results of [3].

In order to prove our main results, we shall require the following lemma due to [10].

**Lemma 1.1.** If \( h \in P \) then \( |c_k| \leq 2 \) for each \( k \), where \( P \) is the family of all functions \( h \) analytic in \( U \) for which \( \Re \{ h(z) \} > 0 \),

\[ h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots \text{ for } z \in U. \]

2. **Coefficient bounds for the function class** \( B_\Sigma(n, \beta, \lambda, \mu) \)

Cho and Srivastava [5], (see also [4]), introduced the operator \( I_\lambda^n : \mathcal{A} \to \mathcal{A} \) defined for the function \( f(z) \) of the form (1.1) as

\[ I_\lambda^n f(z) = z + \sum_{k=2}^{\infty} \left( \frac{k + \lambda}{1 + \lambda} \right)^n a_k z^k, \]

where \( n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) and \( -1 < \lambda \leq 1 \). For \( \lambda = 1 \), the operator \( I^n_\lambda \equiv I^n \) was studied by Uralegaddi and Somanatha [15] and for \( \lambda = 0 \) the operator \( I^n_\lambda \) reduces to well-known Sălăgean operator introduced by Sălăgean [13].

**Definition 2.1.** A function \( f(z) \) given by (1.1) is said to be in the class \( B_\Sigma(n, \beta, \lambda, \mu) \), \( (n \in \mathbb{N}_0, 0 < \beta \leq 1, \mu \geq 1, -1 < \lambda < 1) \), if the following conditions are satisfied:

\[ f \in \Sigma \text{ and } \left| \arg \left\{ \frac{(1-\mu)I_\lambda^n f(z) + \mu I_\lambda^{n+1} f(z)}{z} \right\} \right| < \frac{\beta \pi}{2}, \text{ (} z \in U \text{)} \quad (2.1) \]

and

\[ \left| \arg \left\{ \frac{(1-\mu)I_\lambda^n g(\omega) + \mu I_\lambda^{n+1} g(\omega)}{\omega} \right\} \right| < \frac{\beta \pi}{2}, \text{ (} \omega \in U \text{)}, \quad (2.2) \]

where the function \( g(\omega) \) is given by

\[ g(\omega) = \omega - a_2 \omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \ldots \quad (2.3) \]
Theorem 2.2. Let the function $f(z)$ given by (1.1) be in the class $B_{\Sigma}(n, \beta, \lambda, \mu)$, $n \in \mathbb{N}_0$, $0 < \beta \leq 1$ and $\mu \geq 1$, $-1 < \lambda \leq 1$. Then

$$|a_2| \leq \frac{2\beta}{\sqrt{2M_3\beta + M_2^2(1-\beta)}},$$  \hspace{1cm} (2.4)

and

$$|a_3| \leq \frac{2\beta}{M_3},$$  \hspace{1cm} (2.5)

and

$$|a_4| \leq \left\{ \begin{array}{ll}
\frac{\beta}{M_4} & \left[2 + \frac{4(1-\beta)}{3} \left\{ \frac{2M_3\beta + (1-2\beta)M_2^2}{2M_3\beta + M_2^2(1-\beta)} \right\} \right], \quad (0 < \beta \leq \frac{1}{2}) \\
\frac{\beta}{M_4} & \left[2 + \frac{4(1-\beta)}{3} \left\{ \frac{2M_3\beta + (2\beta-1)M_2^2}{2M_3\beta + M_2^2(1-\beta)} \right\} \right], \quad (\frac{1}{2} \leq \beta \leq 1) \end{array} \right. ,$$  \hspace{1cm} (2.6)

Proof. It follows from (2.1) and (2.2) that

$$\frac{(1-\mu)I_\lambda^nf(z) + \mu I_\lambda^{n+1}f(z)}{z} = [p(z)]^\beta,$$  \hspace{1cm} (2.7)

and

$$\frac{(1-\mu)I_\lambda^ng(\omega) + \mu I_\lambda^{n+1}g(\omega)}{\omega} = [q(\omega)]^\beta,$$  \hspace{1cm} (2.8)

where $p(z)$ and $q(\omega)$ in $P$ and have the forms

$$p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \ldots$$  \hspace{1cm} (2.9)

and

$$q(\omega) = 1 + q_1\omega + q_2\omega^2 + q_3\omega^3 + \ldots$$  \hspace{1cm} (2.10)

Now, equating the coefficients in (2.7) and (2.8), we obtain

$$M_2a_2 = \beta p_1,$$  \hspace{1cm} (2.11)

$$M_3a_3 = \beta p_2 + \frac{\beta(\beta-1)}{2} p_1^2,$$  \hspace{1cm} (2.12)

$$M_4a_4 = \beta p_3 + \beta(\beta-1)p_1p_2 + \frac{\beta(\beta-1)(\beta-2)}{6} p_1^3,$$  \hspace{1cm} (2.13)

$$-M_2a_2 = \beta q_1,$$  \hspace{1cm} (2.14)

$$M_3(2a_2^2 - a_3) = \beta q_2 + \frac{\beta(\beta-1)}{2} q_1^2,$$  \hspace{1cm} (2.15)

$$-M_4(5a_2^3 - 5a_2a_3 + a_4) = \beta q_3 + \beta(\beta-1)q_1q_2 + \frac{\beta(\beta-1)(\beta-2)}{6} q_1^3,$$  \hspace{1cm} (2.16)
where
\[ M_k = \left[ (1 - \mu) \left( \frac{k + \lambda}{1 + \lambda} \right)^n + \mu \left( \frac{k + \lambda}{1 + \lambda} \right)^{n+1} \right]. \]

From (2.11) and (2.14), we obtain
\[ p_1 = -q_1. \quad (2.17) \]

We shall obtain estimates on \(|p_1|\) for use in the estimates of \(|a_2|\), \(|a_3|\) and \(|a_4|\). For this purpose we first add (2.12) and (2.15), we get
\[ 2M_3a_2^2 = \beta(p_2 + q_2) + \frac{\beta(\beta - 1)}{2}(p_1^2 + q_1^2). \]

Using (2.17) in last equation, we have
\[ 2M_3a_2^2 = \beta(p_2 + q_2) + \beta(\beta - 1)p_1^2. \]

Putting \(a_2 = \frac{\beta p_1}{M_2}\) from (2.11), we have after simplification
\[ p_1^2 = \frac{(p_2 + q_2) M_2^2}{2M_3\beta + M_2^2(1 - \beta)}. \quad (2.18) \]

By applying \(|p_2| \leq 2\) and \(|q_2| \leq 2\), we get
\[ |p_1| \leq \frac{2M_2}{\sqrt{2M_3\beta + M_2^2(1 - \beta)}}. \]

Therefore
\[ |a_2| \leq \frac{2\beta}{\sqrt{2M_3\beta + M_2^2(1 - \beta)}}. \]

To find a bound on \(|a_3|\) we may express \(a_3\) in terms of \(p_1, p_2, q_1\) and \(q_2\). For this purpose we first subtract (2.15) from (2.12), we get
\[ 2M_3(a_3 - a_2^2) = \beta(p_2 - q_2) + \frac{\beta(\beta - 1)}{2}(p_1^2 - q_1^2). \]

Using (2.17) in last equations, we have
\[ 2M_3a_3 = 2M_3a_2^2 + \beta(p_2 - q_2). \quad (2.19) \]

Putting \(a_2 = \frac{\beta p_1}{M_2}\) from (2.11) and using (2.18), we get
\[
2M_3a_3 = 2M_3 \left( \frac{\beta p_1}{M_2} \right)^2 + \beta(p_2 - q_2) \\
= \frac{2M_3\beta^2 (p_2 + q_2)}{2M_3\beta + M_2^2(1 - \beta)} + \beta(p_2 - q_2) \\
= \beta \left[ \frac{(4M_3\beta + M_2^2(1 - \beta)) p_2 - M_2^2(1 - \beta)q_2}{2M_3\beta + M_2^2(1 - \beta)} \right].
\]

Using the inequalities \(|p_2| \leq 2\) and \(|q_2| \leq 2\) and after simple calculation, we have
\[ |a_3| \leq \frac{2\beta}{M_3}. \]
We shall next find an estimates on $|a_4|$. At first we shall derive a relation connecting $p_1, p_2, p_3, q_1, q_2$ and $q_3$. To this end, first we add the equations (2.13) and (2.16), we get

$$-M_4 \left(5a_2^3 - 5a_2a_3\right) = \beta(p_3 + q_3) + \beta(\beta - 1)(p_1p_2 + q_1q_2) + \frac{\beta(\beta - 1)(\beta - 2)}{6}(p_1^3 + q_1^3).$$

Using (2.17) and (2.19), we have

$$\frac{5M_4\beta(p_2 - q_2)a_2}{2M_3} = \beta(p_3 + q_3) + \beta(\beta - 1)p_1(p_2 - q_2).$$

Using (2.11) and after simple calculation we have

$$p_1(p_2 - q_2) = \frac{2M_2M_3(p_3 + q_3)}{5M_4\beta + 2M_2M_3(1 - \beta)}. \quad (2.20)$$

We wish to express $|a_4|$ in terms of $p_1, p_2, p_3, q_1, q_2$ and $q_3$. For this purpose subtracting equation (2.16) from (2.13), we get

$$M_4 \left(2a_4 + 5a_2^3 - 5a_2a_3\right) = \beta(p_3 - q_3) + \beta(\beta - 1)p_1(p_2 + q_2) + \frac{\beta(\beta - 1)(\beta - 2)}{3}p_1^3.$$

Again using equations (2.11) and (2.18), we get

$$2M_4a_4 = \frac{5M_4\beta}{2M_3} \frac{p_1}{M_2} (p_2 - q_2) + \beta(p_3 - q_3) + \beta(\beta - 1)p_1(p_2 + q_2)
+ \frac{\beta(\beta - 1)(\beta - 2)}{3}p_1^2 \frac{(p_2 + q_2)M_2^2}{2M_3\beta + M_2^2(1 - \beta)}.$$

Using (2.20) we have

$$2M_4a_4 = \beta \left[ \frac{5M_4\beta(p_3 + q_3)}{5M_4\beta + 2M_2M_3(1 - \beta)} + p_3 - q_3 + (\beta - 1)p_1(p_2 + q_2)
+ \frac{(\beta - 1)(\beta - 2)}{3}p_1^2 \frac{(p_2 + q_2)M_2^2}{2M_3\beta + M_2^2(1 - \beta)} \right]
+ (1 - \beta)p_1(p_2 + q_2) \left\{ \frac{(2 - \beta)M_2^2}{3(2M_3\beta + M_2^2(1 - \beta))} - 1 \right\}.$$

Using Lemma 1.1 and after simple calculation, we have

$$|a_4| \leq \left\{ \begin{array}{ll}
\frac{\beta}{M_4} \left[ 2 + \frac{4(1 - \beta)}{3} \left\{ \frac{2M_3\beta + (1 - 2\beta)M_2^2}{2M_3\beta + M_2^2(1 - \beta)} \right\} \right], & (0 < \beta \leq \frac{1}{2}) \\
\frac{\beta}{M_4} \left[ 2 + \frac{4(1 - \beta)}{3} \left\{ \frac{2M_3\beta + (2\beta - 1)M_2^2}{2M_3\beta + M_2^2(1 - \beta)} \right\} \right], & (\frac{1}{2} \leq \beta \leq 1). 
\end{array} \right.$$
3. Coefficient bounds for the function class $H_{Σ}(n, γ, λ, µ)$

**Definition 3.1.** A function $f(z)$ given by (1.1) is said to be in the class $H_{Σ}(n, γ, λ, µ)$, $(n ∈ N₀, 0 ≤ γ < 1, µ ≥ 1, −1 < λ ≤ 1)$, if the following conditions are satisfied:

$$f ∈ Σ \text{ and } \Re \left\{ \frac{(1 − µ) I_λ^n f(z) + µ I_λ^{n+1} f(z)}{z} \right\} > γ, \ (z ∈ U) \quad (3.1)$$

and

$$\Re \left\{ \frac{(1 − µ) I_λ^n g(ω) + λ I_λ^{n+1} g(ω)}{ω} \right\} > γ, \ (ω ∈ U), \quad (3.2)$$

where the function $g$ is defined by (2.3).

By specializing the parameters in the class $H_{Σ}(n, γ, λ, µ)$, we obtain the following known subclasses studied earlier by various researchers

1. $H_{Σ}(n, γ, 0, µ) ≡ H_{Σ}(n, γ, µ)$ studied by Porwal and Darus [11].
2. $H_{Σ}(0, γ, 0, µ) ≡ H_{Σ}(γ, µ)$ studied by Frasin and Aouf [6].
3. $H_{Σ}(0, γ, 0, 1) ≡ H_{Σ}(γ)$ studied by Srivastava et al. [14].

**Theorem 3.2.** Let the function $f(z)$ given by (1.1) be in the class $H_{Σ}(n, γ, λ, µ)$, $n ∈ N₀, 0 ≤ γ < 1, µ ≥ 1, −1 < λ ≤ 1$. Then

$$|a_2| ≤ \sqrt{\frac{2(1 − γ)}{M_3}}, \quad (3.3)$$

$$|a_3| ≤ \frac{2(1 − γ)}{M_3}, \quad (3.4)$$

and

$$|a_4| ≤ \frac{2(1 − γ)}{M_4}. \quad (3.5)$$

**Proof.** It follows from (3.1) and (3.2) that there exists $p(z) ∈ P$ and $q(ω) ∈ P$ such that

$$\frac{(1 − µ) I_λ^n f(z) + µ I_λ^{n+1} f(z)}{z} = γ + (1 − γ)p(z) \quad (3.6)$$

and

$$\frac{(1 − µ) I_λ^n g(ω) + λ I_λ^{n+1} g(ω)}{ω} = γ + (1 − γ)q(ω), \quad (3.7)$$

where $p(z)$ and $q(ω)$ have the forms (2.9) and (2.10), respectively. Equating coefficients in (3.6) and (3.7) yields

$$M_2 a_2 = (1 − γ)p_1, \quad (3.8)$$

$$M_3 a_3 = (1 − γ)p_2, \quad (3.9)$$

$$M_4 a_4 = (1 − γ)p_3, \quad (3.10)$$

$$−M_2 a_2 = (1 − γ)q_1, \quad (3.11)$$

$$M_3 (2a_2^2 − a_3) = (1 − γ)q_2 \quad (3.12)$$

and

$$−M_4 (5a_2^3 − 5a_2a_3 + a_4) = (1 − γ)q_3. \quad (3.13)$$
From (3.8) and (3.11), we have
\[ p_1 = -q_1. \]  
(3.14)

Adding equation (3.9) and (3.14), we get
\[ 2M_3a_2^2 = (1 - \gamma)(p_2 + q_2). \]

Putting \( a_2 = \frac{(1-\gamma)p_1}{M_2} \) from (3.8), we have
\[ p_1^2 = \frac{(p_2 + q_2)M_2^2}{2M_3(1 - \gamma)}. \]  
(3.15)

By applying the inequalities \(|p_2| \leq 2\) and \(|q_2| \leq 2\), we get
\[ |p_1| \leq \sqrt{\frac{2}{M_3(1 - \gamma)}M_2}. \]

Therefore
\[ |a_2| \leq \sqrt{\frac{2(1 - \gamma)}{M_3}}. \]

To obtain a bound on \(|a_3|\) we wish express in terms of \(p_1, p_2, q_1\) and \(q_2\). For this purpose subtracting (3.12) from (3.9), we get
\[ 2M_3(a_3 - a_2^2) = (1 - \gamma)(p_2 - q_2). \]  
(3.16)

Using (3.8), (3.15) and after simple calculation, we have
\[ 2M_3a_3 = 2(1 - \gamma)p_2. \]

Using \(|p_2| \leq 2\) we have
\[ |a_3| \leq \frac{2(1 - \gamma)}{M_3}. \]

We shall next find an estimate on \(|a_4|\). At first we shall derive a relation connecting \(p_1, p_2, p_3, q_1, q_2\) and \(q_3\). To this end, we first add the equations (3.10) and (3.13), we get
\[ -M_4(5a_2^3 - 5a_2a_3) = (1 - \gamma)(p_3 + q_3). \]

Using (3.8), (3.16) and after simple calculation, we get
\[ p_1(p_2 - q_2) = \frac{2M_2M_3(p_3 + q_3)}{5M_4(1 - \gamma)}. \]  
(3.17)

We wish to express \(a_4\) in terms of \(p_1, p_2, p_3, q_1, q_2\) and \(q_3\). Now subtracting (3.13) from (3.10), we get
\[ M_4(2a_4 + 5a_2^3 - 5a_2a_3) = (1 - \gamma)(p_3 - q_3). \]

Using (3.8), (3.16), (3.17) and after simple calculation
\[ 2M_4a_4 = 2(1 - \gamma)p_3. \]

Using inequality \(|p_3| \leq 2\), we have
\[ |a_4| \leq \frac{2(1 - \gamma)}{M_4}. \]  
□
Remark 3.3. If we put $\lambda = 0$ in Theorems 2.2 and 3.2, then our estimate on $|a_3|$ improves the corresponding results of Porwal and Darus [11].

Remark 3.4. If we put $n = 0$, $\lambda = 0$ in Theorems 2.2 and 3.2, then our estimate on $|a_3|$ improves the corresponding results due to Frasin and Aouf [6].

Remark 3.5. If we put $n = 0$, $\lambda = 0$, $\mu = 1$ in Theorems 2.2 and 3.2, then our estimate on $|a_3|$ improves the corresponding results due to Srivastava et al. [14].

Remark 3.6. Sharp estimates for the coefficients $|a_2|$, $|a_3|$ and other coefficients of functions belonging to the classes investigated in this paper are yet open problems. Indeed it would be of interest even to find estimates (not necessarily sharp) for $|a_n|$, $n \geq 5$.

Acknowledgement. The authors are thankful to the referee for his valuable comments and observations which helped in improving the paper.

References


Saurabh Porwal
Lecturer Mathematics
Sri Radhey Lal Arya Inter College,
Aihan, Hathras-204101,
(U.P.), India

e-mail: saurabhjcb@rediffmail.com

Shivam Kumar
Department of Mathematics
UIET, CSJM University, Kanpur-208024
(U.P.), India