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Existence and topological structure of solution sets for φ-Laplacian impulsive stochastic differential systems

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Abstract. In this article, we present results on the existence and the topological structure of the solution set for initial-value problems relating to the first-order impulsive differential equation with infinite Brownian motions are proved. The approach is based on nonlinear alternative Leray-Schauder type theorem in generalized Banach spaces.

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1. Introduction

Differential equations with impulses were considered for the first time by Milman and Myshkis [18] and then followed by a period of active research which culminated with the monograph by Halanay and Wexler [13]. Many phenomena and evolution processes in physics, chemical technology, population dynamics, and natural sciences may change state abruptly or be subject to short-term perturbations. These perturbations may be seen as impulses. Impulsive problems arise also in various applications in communications, mechanics (jump discontinuities in velocity), electrical engineering, medicine and biology fields. A comprehensive introduction to the basic theory is well developed in the monographs by Benchohra et al [3], Graef *et al* [11], Laskshmikantham *et al.* [1], Samoilenko and Perestyuk [26].For instance, in the periodic treatment of some diseases, impulses correspond to the administration of a drug treatment or a missing product.In environmental sciences, impulses correspond to seasonal changes of the water level of artificial reservoirs.
Random differential and integral equations play an important role in characterizing

many social, physical, biological and engineering problems; see for instance the monograph of Da Prato and Zabczyk [7], Gard [9], Gikhman and Skorokhod [10], Sobzyk [27] and Tsokos and Padgett [28]. For example, a stochastic model for drug distribution in a biological system was described by Tsokos and Padgett [28] to a closed system with a simplified heat, one organ or capillary bed, and re-circulation of a blood with a constant rate of flow, where the heart is considered as a mixing chamber of constant volume. For the basic theory concerning stochastic differential equations see the monographs of Wu *et al* [30], Bharucha-Reid [4], Mao[16], Øksendal, [20], Tsokos and Padgett [28], Da Prato and Zabczyk [7].
In this paper, we study the existence theory for initial-value problems with impulse effects.

$$\begin{cases} (\phi(x'(t)))' &= f^1(t,x(t),y(t))dt \\ & \quad + \sum_{l=1}^{\infty} \sigma_l^1(t,x(t),y(t))dW^l(t), \ t\in[0,T], t\neq t_k, \\ (\phi(y'(t)))' &= f^2(t,x(t),y(t))dt \\ & \quad + \sum_{l=1}^{\infty} \sigma_l^2(t,x(t),y(t))dW^l(t), \ t\in[0,T], t\neq t_k, \\ \Delta x(t) &= I_k(x(t_k)), \ \Delta x'(t) = I_k^1(x'(t_k)), \ t=t_k, \ k=1,2,\dots,m, \\ \Delta y(t) &= \overline{I}_k(y(t_k)), \ \Delta y'(t) = \overline{I}_k^2(y'(t_k)), \ t=t_k, \ k=1,2,\dots,m, \\ x(0) &= A_0, \quad y(0)=B_0, \\ x'(0) &= A_1, \quad y'(0)=B_1, \end{cases} \tag{1.1}$$

where $0=t_0<t_1<\ldots<t_m<t_{m+1}=T, \ \ J:=[0,T]$. $f_l^1, f_l^2 : J\times\mathbb{R}^2\to\mathbb{R}$ is a given function, $\sigma_l^1,\sigma_l^2 : J\times\mathbb{R}^2\to\mathbb{R}$ is a given function and W^l is an infinite sequence of independent standard Brownian motions, $l=1,2,\dots$ and $\phi:\mathbb{R}\to\mathbb{R}$ is a suitable monotone homeomorphism, $I_k^1, \overline{I}_k^1, \overline{I}_k^2, I_k^2 \in C(\mathbb{R},\mathbb{R})$, $(k=1,2,\dots,m)$ and $A_j, B_j\in\mathbb{R}$ for each $j=0,1$, $\Delta x|_{t=t_k} = x(t_k^+)-x(t_k^-), \Delta y|_{t=t_k} = y(t_k^+)-y(t_k^-)$ and $\Delta x'|_{t=t_k} = x'(t_k^+)-x'(t_k^-), \Delta y'|_{t=t_k} = y'(t_k^+)-y'(t_k^-)$. The notations $y(t_k^+)=\lim_{h\to 0^+} y(t_k+h)$ and $y(t_k^-)=\lim_{h\to 0^+} y(t_k-h)$ stand for the right and the left limits of the function y at $t=t_k$, respectively. Set

$$\begin{cases} f_i(.,x,y) = (f_1^i(.,x,y), f_2^i(.,x,y),\ldots), \\ \|f_i(.,x,y)\| = \Big(\sum_{l=1}^{\infty}(f_l^i)^2(.,x,y)\Big)^{\frac{1}{2}}, \end{cases} \tag{1.2}$$

where $i=1,2, \ f_i(.,x,y)\in l^2$ for all $x\in\mathbb{R}$.
This paper is organized as follows: In Section 2, we introduce all the back- ground material used in this paper such as stochastic calculus. In Section 3, to provide some existence results and to establish the compactness of solution sets to the above problems are quoted.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $(\mathcal{F} = \mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions (i.e. right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets). Assume $W(t)$ is an infinite sequence of independent standard Brownian motions, defined on $(\Omega, \mathcal{F}, \mathbb{P})$ that is, $W(t) = (W^1(t), W^2(t), \ldots)^T$. An $\mathbb{R}$-valued random variable is an $\mathcal{F}$-measurable function $x(t) : \Omega \to \mathbb{R}$ and the collection of random variables

$$S = \{x(t, \omega) : \Omega \to \mathbb{R} |\ t \in J\}$$

is called a stochastic process. Generally, we write $x(t)$ instead of $x(t, \omega)$.

Definition 2.1. An $\mathcal{F}$-adapted process X on $[0, T] \times \Omega$ is elementary processes if for a partition $\phi = \{t = 0 < t_1 < \ldots < t_n = T\}$ and $(\mathcal{F}_{t_i})$-measurable random variables $(X_{t_i})_{i<n}$, X_t satisfies

$$X_t(\omega) = \sum_{i=0}^{n-1} X_i(\omega)\chi_{[t_i,t_{i+1})}(t), \quad \text{for} \quad 0 \leq t \leq T, \quad \omega \in \Omega.$$

The Itô integral of the simple process X is defined as

$$\int_0^T X(s)dW^l(s) = \sum_{i=0}^{n-1} X_l(t_i)(W^l(t_{i+1}) - W^l(t_i)), \tag{2.1}$$

whenever $X_{t_i} \in L^2(\mathcal{F}_{t_i})$ for all $i \leq n$.

The following result is one of the elementary properties of square-integrable stochastic processes [20, 16].

Lemma 2.2. *(***Itô Isometry for Elementary Processes***) Let $(X_l)_{l\in\mathbb{N}}$ be a sequences of elementary processes. Assume that*

$$\int_0^T E|X(s)|^2 ds < \infty,$$

where $|X|^2 = \left(\sum_{l=1}^{\infty} X_l^2\right)$. Then

$$E\left(\sum_{l=1}^{\infty} \int_0^T X_l(s)dW^l(s)\right)^2 = E\left(\sum_{l=1}^{\infty} \int_0^T X_l^2(s)ds\right). \tag{2.2}$$

Remark 2.3. For a square integrable stochastic process X on $[0, T]$, its Itô integral is defined by

$$\int_0^T X(s)dW(s) = \lim_{n\to\infty} \int_0^T X_n(s)dW(s),$$

taking the limit in L^2, with X_n is defined in definition 2.1. Then the Itô isometry holds for all Itô-integrable X.

The next result is known as the Burholder-Davis-Gundy inequalities. It was first proved for discrete martingales and $p > 0$ by Burkholder [5] in 1966. In 1968, Millar [17] extended the result to continuous martingales. In 1970, Davis [8] extended the result for discrete martingales to $p = 1$. The extension to $p > 0$ was obtained independently by Burkholder and Gundy [6] in 1970 and Novikov [19] in 1971.

Theorem 2.4. [23] *For each $p > 0$ there exist constants $c_p, C_p \in (0, \infty)$, such that for any progressive process x with the property that for some $t \in [0, \infty)$, $\int_0^t X_s^2 ds < \infty$ a.s, we have*

$$c_p E\left(\int_0^t X_s^2 ds\right)^{\frac{p}{2}} \leq E\left(\sup_{s\in[0,t]} \int_0^t X_s dW(s)\right)^p \leq C_p E\left(\int_0^t X_s^2 ds\right)^{\frac{p}{2}}. \tag{2.3}$$

2.1. Some results on fixed point theorems and set-valued analysis

The classical Banach contraction principle was extended for contractive maps on spaces endowed with vector-valued metric space by Perov [21] in 1964 and Precup [22].

For $x, y \in \mathbb{R}^n$, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for all $i = 1, \dots, n$.

Also $|x| = (|x_1|, \dots, |x_n|)$ and $\max(x, y) = \max(\max(x_1, y_1), \dots, \max(x_n, y_n))$.

If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c$ for each $i = 1, \dots, n$.

Definition 2.5. Let X be a nonempty set. A vector-valued metric on X is a map $d : X \times X \to \mathbb{R}^n$ with the following properties:

(i) $d(u, v) \geq 0$ for all $u, v \in X$; if $d(u, v) = 0$ then $u = v$;

(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$;

(iii) $d(u, v) \leq d(u, w) + d(w, v)$ for all $u, v, w \in X$.

The pair (X, d) is said to be a generalized metric space.

For $r = (r_1, \dots, r_n) \in \mathbb{R}_+^n$, we will denote by

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\},$$

the open ball centered in x_0 with radius r and

$$\overline{B(x_0, r)} = \{x \in X : d(x_0, x) \leq r\}$$

the closed ball centered in x_0 with radius r. We mention that for generalized metric space, the notation of open subset, closed set, convergence, Cauchy sequence and completeness are similar to those in usual metric spaces.

Definition 2.6. A generalized metric space (X, d), where

$$d(x, y) := \begin{pmatrix} d_1(x, y) \\ \cdots \\ d_n(x, y) \end{pmatrix},$$

is complete if (X, d_i) is a complete metric space for every $i = 1, \dots, n$.

Definition 2.7. The map $f : J \times X \to X$ is said to be L^2-Caratheodory if

i) $t \mapsto f(t, u)$ is measurable for each $u \in X$;

ii) $u \mapsto f(t,u)$ is continuous for almost all $t \in J$;
iii) For each $q > 0$, there exists $\alpha_q \in L^1(J, \mathbb{R}^+)$ such that

$$E|f(t,u)|_X^2 \leq \alpha_q \text{ , for all } u \in X \text{ such that } E|u|_X^2 \leq q \text{ and for } a.e. \quad t \in J.$$

Lemma 2.8 (Grönwall-Bihari [2]). *Let $I = [0,b]$ and let $u, g : I \to \mathbb{R}$ be positive continuous functions. Assume there exist $c > 0$ and a continuous nondecreasing function $h : [0,\infty) \to (0,+\infty)$ such that*

$$u(t) \leq c + g(s)h(u(s))ds, \quad \forall t \in I.$$

Then

$$u(t) \leq H^{-1}\Big(\int_p^t g(s)ds\Big), \quad \forall t \in I,$$

provided

$$\int_c^{+\infty} \frac{dy}{h(y)} > \int_p^q g(s)ds,$$

where H^{-1} refers to inverse of the function $H(u) = \int_c^u \frac{dy}{h(y)}$ for $u \geq c$.

In the paper [14], the case of a single system of differential equations was analyzed based on the technique of applying the nonlinear alternative of Leray-Schauder type. In the present paper we extend these results to the more general case of coupled stochastic differential systems with infinite Brownian motions, and we will apply a different technique to obtain our results.
Next, we quote the version of nonlinear alternative Leary-Schauder type theorem in generalized Banach space[29].

Theorem 2.9. *Let $C \subset E$ be a closed convex subset and $U \subset C$ a bounded open neighborhood of zero (with respect to topology of C). If $N : \overline{U} \to E$ is compact continuous then*

i) *Either N has a fixed point in $\overline{U}$, or*
ii) *There exists $x \in \partial U$ such that $x = \lambda N(x)$ for some $\lambda \in (0,1)$.*

3. Main results

Let $J_k = (t_k, t_{k+1}],\ k = 1, 2, \ldots, m$. In order to define a solution for Problem (1.1), consider the following space of piece-wise continuous functions.

Let us introduce the spaces

$$H_2([0,T]; L^2(\Omega,\mathbb{R})) = \{x : J \to L^2(\Omega,\mathbb{R}) \text{ , } x\,|_{(t_k,t_{k+1}]} \in C((t_k,t_{k+1}], L^2(\Omega,\mathbb{R})),$$
$$k = 1, 2, .., m \text{ and there exist } x(t_k^+) \text{ for } k = 1, 2, .., m\},$$

and

$$H_2'([0,T]; L^2(\Omega,\mathbb{R})) = \{x : J \to L^2(\Omega,\mathbb{R}) \text{ , } x\,|_{(t_k,t_{k+1}]} \in C^1((t_k,t_{k+1}], L^2(\Omega,\mathbb{R})),$$
$$k = 1, 2, .., m \text{ and there exist } x(t_k^+) \text{ for } k = 1, 2, .., m\}.$$

It is clear that $H_2([0,T];L^2(\Omega,\mathbb{R}))$ endowed with the norm

$$\|x\|_{H_2}=\sup_{s\in[0,T]}(E|x(s,.)|^2)^{\frac{1}{2}}.$$

It is easy to see that H_2' is a Banach space with the norm $\|x\|_{H_2'}=\|x\|_{H_2}+\|x'\|_{H_2}$. Finally, let the space

$$PC=\{x:[0,T]\to L^2(\Omega,\mathbb{R})\quad \text{and}\, x\mid_J\in H_2' \text{ such that}$$
$$\sup_{t\in[0,T]}E|x(t,.)|^2<\infty \text{ almost surely}\},$$

endowed with the norm

$$\|x\|_{PC}=\sup_{s\in[0,T]}(E|x(s,.)|^2)^{\frac{1}{2}}.$$

It is not difficult to check that PC is a Banach space with norm $\|\cdot\|_{PC}$.

Let us now prove the existence and uniqueness of solutions to our problem which will be obtained by applying the Leary-Schauder fixed point theorem. To this end we first need to introduce the following hypotheses:

$(H1)$ $f^i,\sigma^i:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is an Carathéodory function and $E|\phi^{-1}(X)|^2\le\phi^{-1}(E|X|^2)$ with $X\in\mathbb{R}$, $I_k,\bar{I}_k\in C(\mathbb{R},\mathbb{R})$.

$(H2)$ There exist constants $\overline{a}_i,\overline{b}_i,c_i\in\mathbb{R}^+$ such that each

$$|f^i(t,x,y)|^2\le\overline{a}_i|x|^2+\overline{b}_i|y|^2+c_i,\ i=1,2.$$

for all $x,y\in\mathbb{R}$, and a.e. $t\in J$.

$(H3)$ There exist constants $\overline{\alpha}_i\in\mathbb{R}^+$ and $\overline{\beta}_i,\overline{c}_i\in\mathbb{R}^+$ such that

$$\|\sigma^i(t,x,y)\|^2\le\overline{\alpha}_i|x|^2+\overline{\beta}_i|y|^2+\overline{c}_i,\quad i=1,2$$

for all $x,y\in\mathbb{R}$, and a.e. $t\in J$.

Theorem 3.1. *Assume that* $(H1)$-$(H3)$ *hold. Then, problem* (1.1) *has at least one solution and the solution set*

$$S_c=\{(x,y)\in PC\times PC:(x,y)\ \textit{is a solution of}\ (1.1)\}$$

is compact.

Proof. The proof involves several steps.

Step 1. Consider the problem

$$\begin{cases}(\phi(x'(t)))' &= f^1(t,x(t),y(t))dt+\displaystyle\sum_{l=1}^{\infty}\sigma_l^1(t,x(t)),y(t))dW^l(t),\ t\in[0,t_1],\\ (\phi(y'(t)))' &= f^2(t,x(t),y(t))dt+\displaystyle\sum_{l=1}^{\infty}\sigma_l^2(t,x(t)),y(t))dW^l(t),\ t\in[0,t_1],\\ x(0)=A_0 &,\quad y(0)=B_0,\\ x'(0)=A_1 &,\quad y'(0)=B_1.\end{cases}\tag{3.1}$$

Let

$$\widehat{C}_{t_0}=\{x:[0,t_1]\to L^2(\Omega,\mathbb{R})\quad ,x\mid_{[0,t_1]}\in C^1([0,t_1],L^2(\Omega,\mathbb{R})),\quad k=1,2,..,m,$$

and there exists

$$x(t_1^+) \quad \text{for} \quad k = 1, 2, .., m\},$$

with

$$C_{t_0} = \{x : [0, t_1] \to L^2(\Omega, \mathbb{R}) \qquad \text{and}\, x\mid_{[0,t_1]} \in \widehat{C}_{t_0} \text{ such that}$$

$$\sup_{t\in[0,t_1]} E|x(t, .)|^2 < \infty \text{ almost surely}\},$$

Consider the operator

$$P^0 : C_{t_0} \times C_{t_0} \to C_{t_0} \times C_{t_0}$$

defined by

$$P^0(x, y) = (P_1^0(x, y), P_2^0(x, y)),\ (x, y) \in C_{t_0} \times C_{t_0}$$

where

$$\begin{cases} P_1^0(x,y) &= A_0 + \displaystyle\int_0^t \phi^{-1}\Big(\phi(A_1) + \int_0^s f^1(r, x(r), y(r))dr + \\ & \quad + \displaystyle\sum_{l=1}^{\infty} \int_0^s \sigma_l^1(r, x(r), y(r))dW^l(r)\Big)ds, \quad t \in [0, t_1],\ a.e.\ \omega \in \Omega. \\ P_2^0(x,y) &= B_0 + \displaystyle\int_0^t \phi^{-1}\Big(\phi(B_1) + \int_0^s f^2(r, x(r), y(r))dr + \\ & \quad + \displaystyle\sum_{l=1}^{\infty} \int_0^s \sigma_l^2(r, x(r), y(r))dW^l(r)\Big)ds, \quad t \in [0, t_1],\ a.e.\ \omega \in \Omega. \end{cases} \tag{3.2}$$

Clearly, the fixed points of $P^0 = (P_1^0, P_2^0)$ are solutions of the problem (3.1).
To apply the nonlinear alternative of Leray-Schauder type, we first show that P^0 is completely continuous. The proof will be given in several steps.
Claim 1. P^0 sends bounded sets into bounded sets in $C_{t_0} \times C_{t_0}$. Indeed, it is enough to show that for any $q > 0$, there exists a positive constant κ such that for each

$$(x, y) \in B_q = \{(x, y) \in C_{t_0} \times C_{t_0} : \sup_{t\in[0,t_1]} E|x(t, \cdot)|^2 \le q,\ \sup_{t\in[0,t_1]} E|x(t, \cdot)|^2 \le q\},$$

we have

$$\|P^0(x, y)\| \le \kappa = (\kappa_1, \kappa_2).$$

Then for each $t \in [0, t_1]$, we have

$$E|P_1^0(x, y)|^2 \le 2E|A_0|^2 + 2\int_0^t E\Big|\phi^{-1}\Big(\phi(A_1) + \int_0^s f^1(r, x(r), y(r))dr + \sum_{l=1}^{\infty} \int_0^s \sigma_l^1(r, x(r), y(r))dW^l(r)\Big)ds\Big|^2.$$

From Lemma 2.4, we obtain

$$E\Big|\phi(A_1)+\int_0^s f^1(r,x(r),y(r))dr+\sum_{l=1}^{\infty}\int_0^s \sigma_l^1(r,x(r),y(r))dW^l(r)\Big|^2$$
$$\leq 3|\phi(A_1)|_X^2+3t_1\int_0^s(\overline{a}_1|x(r)|_X^2+\overline{b}_1|y(r)|^2+c_1)dr$$
$$+3C_2\int_0^s(\overline{\alpha}_1|x(r)|^2+\overline{\beta}_1|y(r)|^2+\overline{c}_1)dr,$$

it follows that

$$E\Big|\phi(A_1)+\int_0^s f^1(r,x(r),y(r))dr+\sum_{l=1}^{\infty}\int_0^s \sigma_l^1(r,x(r),y(r))dW(r)\Big|^2\in\overline{B}(0,l_1),$$

where

$$l_1=3E|\phi(A_1)|^2+3t_1\int_0^s(\overline{a}_1E|x(r)|^2+\overline{b}_1E|y(r)|^2+c_1)dr$$
$$+3C_2\int_0^s(\overline{\alpha}_1E|x(r)|^2+\overline{\beta}_1E|y(r)|^2+\overline{c}_1)dr.$$

Since ϕ^{-1} is continuous,

$$\sup_{\eta_1\in\overline{B}(0,l_1)}|\phi^{-1}(\eta_1)|<\infty.$$

Thus

$$E|P_1^0(x,y)|^2\leq 2E|A_0|^2+2t_1\sup_{\eta_1\in\overline{B}(0,l_1)}|\phi^{-1}(\eta_1)|:=\kappa_1.$$

Similarly,

$$E|P_2^0(x,y)|^2\leq 2E|B_0|^2+2t_1\sup_{\eta_2\in\overline{B}(0,l_2)}|\phi^{-1}(\eta_2)|:=\kappa_2,$$

where

$$l_2=3E|\phi(B_1)|^2+3t_1\int_0^s(\overline{a}_2E|x(r)|^2+\overline{b}_2E|y(r)|^2+c_2)dr$$
$$+3C_2\int_0^s(\overline{\alpha}_2E|x(r)|^2+\overline{\beta}_2E|y(r)|^2+\overline{c}_2)dr.$$

Since ϕ^{-1} is continuous,

$$\sup_{\eta_1\in\overline{B}(0,l_1)}|\phi^{-1}(\eta_1)|<\infty.$$

Claim 2. P^0 maps bounded sets into equicontinuous sets. Let $l_1, l_2 \in [0, t_1]$, $l_1 < l_2$ and B_q be a bounded set of $C_{t_0} \times C_{t_0}$ as in Claim 1. Let $(x, y) \in B_q$. Then

$$\begin{aligned}
E|(P_1^0(x,y))'(t)|^2 &= E\Big|\phi^{-1}\Big(\phi(A_1) + \int_0^t f^1(s, x(s), y(s))dr \\
&\quad + \sum_{l=1}^{\infty} \int_0^t \sigma_l^1(s, x(s), y(s))dW^l(s)\Big) - \phi^{-1}(\phi(A_1))\Big|^2 \\
&\leq E\Big|\phi^{-1}\Big(\phi(A_1) + \int_0^t f^1(s, x(s), y(s))dr \\
&\quad + \sum_{l=1}^{\infty} \int_0^t \sigma_l^1(s, x(s), y(s))dW^l(s)\Big)\Big|^2 + E\Big|A_1\Big|^2 \\
&\leq \sup_{\eta_1 \in \overline{B}(0,l_1)} |\phi^{-1}(\eta_1)| + E|A_1| := r'.
\end{aligned}$$

Using the mean value theorem, we obtain

$$E|(P_1^0(x,y))(l_2) - (P_1^0(x,y))(l_1)| = E|(P_1^0(x,y))'(\xi, \overline{\xi})(l_2 - l_1)| \leq r'|l_2 - l_1|.$$

As $l_2 \to l_1$ the right-hand side of the above inequality tends to zero.
Similarly,

$$\begin{aligned}
E|(P_2^0(x,y))'(t)|_X^2 &= E\Big|\phi^{-1}\Big(\phi(B_1) + \int_0^t f^2(s, x(s), y(s))dr \\
&\quad + \sum_{l=1}^{\infty} \int_0^t \sigma_l^2(s, x(s), y(s))dW^l(s)\Big) - \phi^{-1}(\phi(B_1))\Big|^2 \\
&\leq \Big|\phi^{-1}\Big(\phi(B_1) + \int_0^t f^2(s, x(s), y(s))dr \\
&\quad + \sum_{l=1}^{\infty} \int_0^t \sigma_l^2(s, x(s), y(s))dW^l(s)\Big)\Big|^2 + \Big|B_1\Big|^2 \\
&\leq \sup_{\eta_2 \in \overline{B}(0,l_2)} |\phi^{-1}(\eta_2)| + |B_1| := r'.
\end{aligned}$$

Using the mean value theorem, we obtain

$$E|(P_2^0(x,y))(l_2) - (P_2^0(x,y))(l_1)| = E|(P_2^0(x,y))'(\xi, \overline{\xi})(l_2 - l_1)| \leq r'|l_2 - l_1|.$$

As $l_2 \to l_1$ the right-hand side of the above inequality tends to zero.
Claim 3. P^0 is continuous. Let $(x_n, y_n)_{n \in \mathbb{N}}$ be a sequence such that $(x_n, y_n) \to (x, y)$ in $C_{t_0} \times C_{t_0}$. Then there is an integer q such that

$$\sup_{t \in [0,t_1]} E|x_n(t, \cdot)|^2 \leq q, \quad \sup_{t \in [0,t_1]} E|y_n(t, \cdot)|^2 \leq q \leq q \text{ for all } n \in \mathbb{N}$$

and

$$\sup_{t \in [0,t_1]} E|x(t, \cdot)|^2 \leq q, \quad \sup_{t \in [0,t_1]} E|y(t, \cdot)|^2 \leq q, \ (x_n, y_n) \in B_q \text{ and } (x, y) \in B_q.$$

Then for each $t \in [0, t_1]$, we have

$$\begin{aligned} E|P_1^0(x_n, y_n) - P_1^0(x, y)|_X^2 \quad &\leq \int_0^t E\Big|\phi^{-1}\Big(\phi(A_1) + \int_0^s f^1(r, x(r), y(r))dr \\ &\quad + \sum_{l=1}^{\infty} \int_0^s \sigma_l^1(r, x(r), y(r))dW^l(r)\Big) \\ &\quad -\phi^{-1}\Big(\phi(A_1) + \int_0^s f^1(r, x(r), y(r))dr) \\ &\quad + \sum_{l=1}^{\infty} \int_0^s \sigma^1(r, x(r), y(r))dW^l(r)\Big)\Big|^2 ds. \end{aligned}$$

Using the dominated convergence theorem, we have

$$E\Big|\phi(A_1) + \int_0^s f^1(r, x_n(r), y_n(r))dr + \sum_{l=1}^{\infty} \int_0^s \sigma^1(r, x_n(r), y_n(r))dW^l(r),$$

$$-\phi(A_1) - \int_0^s f(r, x(r), y(r))dr - \sum_{l=1}^{\infty} \int_0^s \sigma^1(r, x(r), y(r))dW^l(r)\Big|_X^2 \to 0 \quad \text{as } n \to \infty,$$

since ϕ^{-1} is continuous. Then using the dominated convergence theorem, we have

$$\begin{aligned} &\sup_{t\in[0,t_1]} E|P_1^0(x_n, y_n) - P_1^0(x, y)|^2 \\ &\leq \int_0^{t_1} E|\phi^{-1}[\phi(B) + \int_0^s f^1(r, x_n, y_n)dr + \sum_{l=1}^{\infty} \int_0^s \sigma^1(r, x_n(r), y_n(r))dW^l(r)] \\ &- \phi^{-1}[\phi(B) + \int_0^s f^1(r, x, y)dr + \sum_{l=1}^{\infty} \int_0^s \sigma_l^1(r, x(r), y(r))dW^l(r)]^2 ds \to 0, \end{aligned}$$

as $n \to \infty$. Thus P_1^0 is continuous.
Similarly,

$$\begin{aligned} &\sup_{t\in[0,t_1]} E|P_2^0(x_n, y_n) - P_2^0(x, y)|^2 \\ &\leq \int_0^{t_1} E|\phi^{-1}[\phi(B) + \int_0^s f^2(r, x_n, y_n)dr + \sum_{l=1}^{\infty} \int_0^s \sigma_l^2(r, x_n(r), y_n(r))dW^l(r)] \\ &- \phi^{-1}[\phi(B) + \int_0^s f^2(r, x, y)dr + \sum_{l=1}^{\infty} \int_0^s \sigma_l^2(r, x(r), y(r))dW(r)]|^2 ds \to 0, \end{aligned}$$

as $n \to \infty$. Thus P_2^0 is continuous.

Claim 4. *Apriori* estimate. Now we show that there exists a constant M_0 such that $\sup_{t\in[0,t_1]} E|x(t,\cdot)|_X^2 \leq M_0$ where (x, y) is a solution of the problem (3.1). Let (x, y) a solution of (3.1):

$$\begin{cases} x(t) &= A_0 + \displaystyle\int_0^t \phi^{-1}\Big(\phi(A_1) + \int_0^s f^1(r,x(r),y(r))dr + \\ & \quad + \displaystyle\sum_{l=1}^{\infty} \int_0^s \sigma_l^1(r,x(r),y(r))dW^l(r)\Big)ds, \quad t \in [0,t_1],\ a.e.\ \omega \in \Omega. \\ y(t) &= B_0 + \displaystyle\int_0^t \phi^{-1}\Big(\phi(B_1) + \int_0^s f^2(r,x(r),y(r))dr + \\ & \quad + \displaystyle\sum_{l=1}^{\infty} \int_0^s \sigma_l^2(r,x(r),y(r))dW^l(r)\Big)ds, \quad t \in [0,t_1],\ a.e.\ \omega \in \Omega. \end{cases} \tag{3.3}$$

From Lemma 2.4, we obtain

$$\begin{aligned} E|x(t)|^2 &\leq E|A_0|^2 + \int_0^t E\Big|\phi^{-1}\Big(\phi(A_1) + \int_0^s f^1(r,x(r),y(r))dr \\ &\quad + \sum_{l=1}^{\infty}\int_0^s \sigma^1(r,x(r),y(r))dW^l(r)\Big)\Big|^2 ds \\ &\leq 2E|A_0|^2 + 2t_1 \sup_{\eta_1 \in \overline{B}(0,l_1)} |\phi^{-1}(\eta_1)| =: M_0 \end{aligned}$$

where

$$\begin{aligned} l_1 = 3E|\phi(A_1)|^2 &+ 3t_1 \int_0^s (\overline{a}_1 E|x(r)|_X^2 + \overline{b}_1 E|y(r)|^2 + c_1)dr \\ &+ 3C_2 \int_0^s (\overline{\alpha}_1 E|x(r)|^2 + \overline{\beta}_1 E|y(r)|^2 + \overline{c}_1)dr. \end{aligned}$$

Thus,

$$\sup_{t\in[0,t_1]} E|x(t)|^2 \leq M_0,$$

and

$$\begin{aligned} E|y(t)|^2 &\leq E|B_0|^2 + \int_0^t E\Big|\phi^{-1}\Big(\phi(B_1) + \int_0^s f^2(r,x(r),y(r))dr \\ &\quad + \sum_{l=1}^{\infty}\int_0^s \sigma^2(r,x(r),y(r))dW^l(r)\Big)\Big|^2 ds \\ &\leq 2E|B_0|^2 + 2t_1 \sup_{\eta_2 \in \overline{B}(0,l_2)} |\phi^{-1}(\eta_2)| =: M_0 \end{aligned}$$

where

$$\begin{aligned} l_2 = 3E|\phi(A_1)|^2 &+ 3t_1 \int_0^s (\overline{a}_1 E|x(r)|^2 + \overline{b}_1 E|y(r)|^2 + c_1)dr \\ &+ 3C_2 \int_0^s (\overline{\alpha}_1 E|x(r)|^2 + \overline{\beta}_1 E|y(r)|^2 + \overline{c}_1)dr. \end{aligned}$$

Thus,

$$\sup_{t\in[0,t_1]} E|y(t)|^2 \leq M_0.$$

Set

$$U = \{y \in C([0,t_1],\mathbb{R}) : \sup_{t\in[0,t_1]} E|x(t)|^2 < M_0 + 1, \ \sup_{t\in[0,t_1]} E|y(t)|^2 < M_0 + 1\}.$$

As a consequence of Claims 1-4 and the Ascoli-Arzela theorem, we can conclude that the map $P^0 : \overline{U} \to C_{t_0} \times C_{t_0}$ is compact. From the choice of U there is no $(x,y) \in \partial U$ such that $(x,y) = \lambda P^0(x,y)$ for any $\lambda \in (0,1)$. And from the consequence of the nonlinear alternative of Leray-Schauder we deduce that P^0 has a fixed point denoted by $(x_0, y_0) \in \overline{U}$ which is solution of the problem (3.1).

Step 2. Now consider the problem

$$\begin{cases} (\phi(x'(t)))' &= f^1(t,x(t),y(t))dt + \sum_{l=1}^{\infty} \sigma_l^1(t,x(t)),y(t))dW(t), \ t \in (t_1,t_2], \\ (\phi(y'(t)))' &= f^2(t,x(t),y(t))dt + \sum_{l=1}^{\infty} \sigma_l^2(t,x(t)),y(t))dW(t), \ t \in (t_1,t_2], \\ x(t_1^+) &= x_0(t_1^-) + I_1(x_0(t_1^-)) \quad , x'(t_1^+) = x_0'(t_1^-) + I_1^1(x_0(t_1^-)), \\ y(t_1^+) &= y_0(t_1^-) + \overline{I}_1(x_0(t_1^-)) \quad , y'(t_1^+) = y_0'(t_1^-) + \overline{I}_1^2(y_0(t_1^-)). \end{cases} \tag{3.4}$$

Let

$$\widehat{C}_{t_1} = \{x : (t_1,t_2] \to L^2(\Omega,\mathbb{R}), \ x\mid_{(t_1,t_2]} \in C^1((t_1,t_2], L^2(\Omega,\mathbb{R})), \ k = 1,2,..,m$$

and there exists

$$x(t_2^+) \quad \text{for} \quad k = 1,2,..,m\},$$

with

$$D_{t_1} = \{x : (t_1,t_2] \to L^2(\Omega,\mathbb{R}) \qquad \text{and } \ x(t)\mid_{(t_1,t_2]} \in \widehat{C}_{t_1} \qquad \text{such that}$$

$$\sup_{t\in(t_1,t_2]} E|x(t,.)|^2 < \infty \ \text{ almost surely}\}.$$

Set

$$C_1 = C_{t_0} \cap D_{t_1}.$$

Consider the operator $P^1 : C_1 \times C_1 \to C_1 \times C_1$ defined by

$$P^1(x,y) = (P_1^1(x,y), P_2^1(x,y)), \ (x,y) \in C_1 \times C_1.$$

It is clear that all solutions of (3.4) are fixed points of the multi-valued operator $P_i^1 : C_1 \times C_1 \to C_1$, for each $i = 1, 2$ defined by

$$\begin{cases} P_1^1(x,y) &= A_3 + \int_{t_1}^t \phi^{-1}\Big(\phi(A_4) + \int_{t_1}^s f^1(r,x(r),y(r))dr + \\ & \quad + \sum_{l=1}^{\infty}\int_0^s \sigma_l^1(r,x(r),y(r))dW^l(r)\Big)ds, \quad t \in (t_1,t_2], \ a.e.\ \omega \in \Omega. \\ \\ P_2^1(x,y) &= B_3 + \int_{t_1}^t \phi^{-1}\Big(\phi(B_4) + \int_{t_1}^s f^2(r,x(r),y(r))dr + \\ & \quad + \sum_{l=1}^{\infty}\int_0^s \sigma^2(r,x(r),y(r))dW^l(r)\Big)ds, \quad t \in (t_1,t_2], \ a.e.\ \omega \in \Omega. \end{cases} \tag{3.5}$$

and

$$A_3 = x_1(t_1) + I_1(x_1(t_1)), \quad A_4 = x_1'(t_1^-) + I_1^1(x_1(t_1^-)),$$
$$B_3 = y_1(t_1) + \overline{I}_1(y_1(t_1)), \quad B_4 = y_1'(t_1^-) + \overline{I}_1^2(y_1(t_1^-)).$$

As in Step 1, we can prove that P^1 has at least one fixed point which is a solution to (3.4).

Step 3. We continue this process taking into account that

$$(x_m, y_m) := (x\big|_{(t_m,T]}, y\big|_{(t_m,T]})$$

is a solution to the problem

$$\begin{cases} (\phi(x'(t)))' &= f^1(t,x(t),y(t))dt + \sum_{l=1}^{\infty} \sigma_l^1(t,x(t)),y(t))dW^l(t), \ t \in (t_m,T], \\ (\phi(y'(t)))' &= f^2(t,x(t),y(t))dt + \sum_{l=1}^{\infty} \sigma_l^2(t,x(t)),y(t))dW^l(t), \ t \in ((t_m,T], \\ x(t_m^+) &= x_{m-1}(t_m^-) + I_m(x_0(t_{m-1}^-)), \\ x'(t_m^+) &= x'_{m-1}(t_m^-) + I_m^1(x_{m-1}(t_m^-)), \\ y(t_m^+) &= y_{m-1}(t_m^-) + \overline{I}_m(x_0(t_{m-1}^-)), \\ y'(t_m^+) &= y'_{m-1}(t_m^-) + \overline{I}_m^2(y_{m-1}(t_m^-)). \end{cases} \tag{3.6}$$

A solution (x, y) of problem (3.6) is ultimately defined by

$$(x(t),y(t)) = \begin{cases} (x_0(t),y_0(t)), & \text{if } t \in [0,t_1], \\ (x_1(t),y_1(t)), & \text{if } t \in (t_1,t_2], \\ \dots \\ (x_m(t),y_m(t)), & \text{if } t \in (t_m,T]. \end{cases}$$

Step 4. Now we show that the set

$$S_c = \{(x,y) \in PC \times PC : (x,y) \text{ is a solution of } (1.1)\}$$

is compact. Let $(x_n, y_n)_{n\in\mathbb{N}}$ be a sequence in S_c. We put

$$B = \{(x_n,y_n) : n \in \mathbb{N}\} \subseteq PC \times PC.$$

Then from earlier parts of the proof of this theorem, we conclude that B is bounded and equicontinuous and from the Ascoli-Arzela theorem, we can also conclude that B is compact.
Recall that $J_0 = [0, t_1]$ and $J_k = (t_k, t_{k+1}]$, $k = 1, \dots, m$. Hence:
• $(x_n, y_n)|_{J_0}$ has a subsequence

$$(x_{n_m}, y_{n_m})_{n_m \in \mathbb{N}} \subset S_{c_1} = \{(x, y) \in C_{t_0} \times C_{t_0} : (x, y) \text{ is a solution of } (3.1)\}$$

such that (x_{n_m}, y_{n_m}) converges to (x, y). Let

$$\begin{cases} z_0(t) &= A_0 + \displaystyle\int_0^t \phi^{-1}\Big(\phi(A_1) + \int_0^s f^1(r, x(r), y(r))dr + \\ & + \displaystyle\sum_{l=1}^{\infty} \int_0^s \sigma^1(r, x(r), y(r))dW^l(r)\Big)ds, \quad t \in [0, t_1], \ a.e.\ \omega \in \Omega. \\ \overline{z}_0(t) &= B_0 + \displaystyle\int_0^t \phi^{-1}\Big(\phi(B_1) + \int_0^s f^2(r, x(r), y(r))dr + \\ & + \displaystyle\sum_{l=1}^{\infty} \int_0^s \sigma^2(r, x(r), y(r))dW^l(r)\Big)ds, \quad t \in [0, t_1], \ a.e.\ \omega \in \Omega. \end{cases} \tag{3.7}$$

$$\begin{aligned} E\Big|x_{n_m}(t) - z_0(t)\Big|_X^2 \leq \int_0^t E\Big|\phi^{-1}\Big(\phi(A_1) + \int_0^s f^1(r, x_{n_m}(r), y_{n_m}(r))dr \\ + \sum_{l=1}^{\infty} \int_0^s \sigma_l^1(r, x_{n_m}(r), y_{n_m}(r))dW^l(r)\Big) \\ - \phi^{-1}\Big(\phi(A_1) + \int_0^s f^1(r, x(r), y(r))dr + \\ + \sum_{l=1}^{\infty} \int_0^s \sigma_l^1(r, x(r), y(r))dW^l(r)\Big)\Big|_X^2 ds, \end{aligned}$$

and

$$\begin{aligned} E\Big|y_{n_m}(t) - \overline{z}_0(t)\Big|_X^2 \leq \int_0^t E\Big|\phi^{-1}\Big(\phi(B_1) + \int_0^s f^1(r, x_{n_m}(r), y_{n_m}(r))dr \\ + \sum_{l=1}^{\infty} \int_0^s \sigma_l^1(r, x_{n_m}(r), y_{n_m}(r))dW^l(r)\Big) \\ - \phi^{-1}\Big(\phi(A_1) + \int_0^s f^1(r, x(r), y(r))dr + \\ + \sum_{l=1}^{\infty} \int_0^s \sigma_l^1(r, x(r), y(r))dW^l(r)\Big)\Big|_X^2 ds. \end{aligned}$$

As $n_m \to +\infty$, $(x_{n_m}, y_{n_m}) \to (z_0(t), \overline{z}_0(t))$, then

$$x(t) = A_0 + \int_0^t \phi^{-1}\Big(\phi(A_1) + \int_0^s f^1(r, x(r), y(r))dr + \sum_{l=1}^{\infty} \int_0^s \sigma_l^1(r, x(r), y(r))dW^l(r)\Big)ds,$$

and

$$y(t) = B_0 + \int_0^t \phi^{-1}\Big(\phi(B_1) + \int_0^s f^2(r, x(r), y(r))dr + \sum_{l=1}^{\infty} \int_0^s \sigma_l^2(r, x(r), y(r))dW^l(r)\Big)ds.$$

• $(x_n, y_n)|_{J_1}$ has a subsequence relabeled as $(x_{n_m}, y_{n_m}) \subset S_{c_2}$ converging to (x, y) in $C_1 \times C_1$ where

$$S_{c_2} = \{(x, y) \in C_1 \times C_1 : (x, y) \text{ is a solution of (3.4)}\}.$$

Let

$$z_1(t) = A_3 + \int_{t_1}^t \phi^{-1}\Big(\phi(A_4) + \int_{t_1}^s f^1(r, x(r), y(r))dr + \sum_{l=1}^{\infty} \int_{t_1}^s \sigma_l^1(r, x(r), y(r))dW^l(r)\Big)ds,$$

and

$$\overline{z}_1(t) = B_3 + \int_{t_1}^t \phi^{-1}\Big(\phi(B_4) + \int_{t_1}^s f^2(r, x(r), y(r))dr + \sum_{l=1}^{\infty} \int_{t_1}^s \sigma_l^2(r, x(r), y(r))dW^l(r)\Big)ds.$$

Then

$$E\Big|x_{n_m}(t) - z_1(t)\Big|^2 \leq \int_{t_1}^t E\Big|\phi^{-1}\Big(\phi(A_4) + \int_{t_1}^s f^1(r, x_{n_m}(r), y_{n_m}(r))dr + \sum_{l=1}^{\infty} \int_{t_1}^s \sigma_l^1(r, x_{n_m}(r), y_{n_m}(r))dW^l(r)\Big) - \phi^{-1}\Big(\phi(A_4) + \int_{t_1}^s f^1(r, x(r), y(r))dr + + \sum_{l=1}^{\infty} \int_{t_1}^s \sigma^1(r, x(r), y(r))dW^l(r)\Big)\Big|^2 ds,$$

and

$$E\Big|y_{n_m}(t)-\overline{z}_1(t)\Big|^2 \leq \int_{t_1}^{t} E\Big|\phi^{-1}\Big(\phi(B_4)+\int_{t_1}^{s} f^1(r,x_{n_m}(r),y_{n_m}(r))dr + \sum_{l=1}^{\infty}\int_{t_1}^{s}\sigma^1(r,x_{n_m}(r),y_{n_m}(r))dW^l(r)\Big) - \phi^{-1}\Big(\phi(A_4)+\int_{t_1}^{s} f^1(r,x(r),y(r))dr+ + \sum_{l=1}^{\infty}\int_{t_1}^{s}\sigma^1(r,x(r),y(r))dW^l(r)\Big)\Big|^2 ds.$$

As $n_m \to +\infty$, $(x_{n_m}(t), y_{n_m}(t)) \to (z_1(t), \overline{z}_1(t))$, and then

$$x(t) = A_3 + \int_{t_1}^{t}\phi^{-1}\Big(\phi(A_4)+\int_{t_1}^{s} f^1(r,x(r),y(r))dr + \sum_{l=1}^{\infty}\int_{t_1}^{s}\sigma_l^1(r,x(r),y(r))dW^l(r)\Big)ds,$$

and

$$y(t) = B_3 + \int_{t_1}^{t}\phi^{-1}\Big(\phi(B_4)+\int_{t_1}^{s} f^2(r,x(r),y(r))dr + \sum_{l=1}^{\infty}\int_{t_1}^{s}\sigma^2(r,x(r),y(r))dW^l(r)\Big)ds.$$

• We continue this process, and we conclude that $\{(x_n, y_n) \mid n \in \mathbb{N}\}$ has a subsequence converging to

$$z_m(t) = A_{m+2} + \int_{t_m}^{t}\phi^{-1}\Big(\phi(A_{m+3})+\int_{t_m}^{s} f^1(r,x(r),y(r))dr + \sum_{l=1}^{\infty}\int_{t_m}^{s}\sigma_l^1(r,x(r),y(r))dW^l(r)\Big)ds,$$

and

$$\overline{z}_m(t) = B_{m+2} + \int_{t_m}^{t}\phi^{-1}\Big(\phi(B_{m+3})+\int_{t_m}^{s} f^2(r,x(r),y(r))dr + \sum_{l=1}^{\infty}\int_{t_m}^{s}\sigma_l^2(r,x(r),y(r))dW^l(r)\Big)ds.$$

Hence S_c is compact. □

Next we replace $(H2)$ and $(H3)$ in Theorem 3.1 by

$(H3)'$ Then there exist a function $p_i \in L^1(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi_i : [0,\infty) \to [0,\infty)$for each $i = 1,2$ such that

$$E|f^i(t,x,y)|^2 \leq p_i(t)\psi_i(E(|x|^2 + |y|^2)),$$

and

$$E||\sigma^i(t,x,y)||^2 \leq p_i(t)\psi_i(E(|x|^2 + |y|^2)).$$

Theorem 3.2. *Under assumption* $(H3)'$, *problem* (1.1) *has at least one solution and the solution set is compact.*

Proof. As in the proof of Theorem 3.1 we can show that (1.1) has at least one solution by applying the nonlinear alternative of Leray-Schauder. We show only the estimation of a solution (x,y) of (1.1).

• For $t \in [0,t_1]$, we have

$$\begin{cases} x(t) &= A_0 + \displaystyle\int_0^t \phi^{-1}\Big(\phi(A_1) + \int_0^s f^1(r,x(r),y(r))dr + \\ & +\displaystyle\sum_{l=1}^{\infty}\int_0^s \sigma_l^1(r,x(r),y(r))dW^l(r)\Big)ds, \quad t \in [0,t_1], \ a.e. \ \omega \in \Omega. \\ \\ y(t) &= B_0 + \displaystyle\int_0^t \phi^{-1}\Big(\phi(B_1) + \int_0^s f^2(r,x(r),y(r))dr + \\ & +\displaystyle\sum_{l=1}^{\infty}\int_0^s \sigma_l^2(r,x(r),y(r))dW^l(r)\Big)ds, \quad t \in [0,t_1], \ a.e. \ \omega \in \Omega. \end{cases} \tag{3.8}$$

Then

$$E|x(t)|^2 \leq 2E|A_0|^2 + 2\int_0^t E\Big|\phi^{-1}\Big(\phi(A_1) + \int_0^s f^1(r,x(r),y(r))dr + \sum_{l=1}^{\infty}\int_0^s \sigma_l^1(r,x(r),y(r))dW^l(r)\Big)\Big|^2 ds.$$

Consider functions $\mu, \overline{\mu}$ defined on $t \in [0,t_1]$ by

$$\mu(t) = \sup\{E|x(s)|^2 \, : 0 \leq s \leq t\}, \qquad \overline{\mu}(t) = \sup\{E|y(s)|^2 \, : 0 \leq s \leq t\}.$$

From Lemma 2.4, we obtain

$$\begin{aligned} E\Big|\phi(A_1) &+ \int_0^s f^1(r,x(r),y(r))dr + \sum_{l=1}^{\infty}\int_0^s \sigma_l^1(r,x(r),y(r))dW^l(r)\Big|^2 \\ &\leq 3E|\phi(A_1)|_X^2 + 3\int_0^s p_1(r)\psi_1(E(|x(r)|^2 + |y(r)|^2))dr \\ &+3C_2\int_0^s p_1(r)\psi_1(E(|x(r)|_X^2 + |y(r)|^2))dr \\ &\leq 3E|\phi(A_1)|^2 + ||\overline{p}||_{L^1}\psi_1(\mu(s) + \overline{\mu}(s)), \end{aligned}$$

where $||\overline{p}||_{L^1} = (3t_1 + 3C_2)\|p_1\|_{L^1}$, and, consequently,

$$\mu(t) \leq 2E|A_0|^2 + \int_0^t \widehat{\psi}_1(\mu(s) + \overline{\mu}(s)), \quad t \in [0, t_1],$$

where $\widehat{\psi}_1 = (\phi^{-1} \circ \widetilde{\psi}_1)$ and $\widetilde{\psi}_1(u) = 3E|\phi(A_1)|^2 + ||\overline{p}_1||_{L^1}\psi_1(u)$. and similarly

$$\overline{\mu}(t) \leq 2E|B_0|^2 + \int_0^t \widehat{\psi}_2(\mu(s) + \overline{\mu}(s))ds, \quad t \in [0, t_1],$$

where $\widehat{\psi}_2 = (\phi^{-1} \circ \widetilde{\psi}_2)$ and $\widetilde{\psi}_2(u) = 3E|\phi(B_1)|^2 + ||p_2||_{L^1}\psi_1(u)$, combining $\mu(t)$ and $\overline{\mu}(t)$,

$$\mu(t) + \overline{\mu}(t) \leq 2E|A_0|^2 + 2E|B_0|^2 + \int_0^t \widehat{\psi}_1(\mu(s) + \overline{\mu}(s))ds + \int_0^t \widehat{\psi}_2(\mu(s) + \overline{\mu}(s))ds, \quad t \in [0, t_1].$$

Using the nonlinear Grönwall-Bihari inequality (Lemma 2.8), we infer the bound

$$\mu(t) + \overline{\mu}(t) \leq H^{-1}(t) \leq M_0.$$

Consequently, there exists a constant M_1 which only depends on t_1, t_2 such that

$$\sup_{t\in[0,t_1]} E|x(t)|^2 \leq M_0, \text{ and } \sup_{t\in[0,t_1]} E|y(t)|^2 \leq M_0,$$

where $H(t) = \displaystyle\int_{2E|A_0|^2_X + 2E|B_0|^2_X}^t \frac{d\tau}{(\phi^{-1} \circ \widetilde{\psi}_1(\tau) + \phi^{-1} \circ \widetilde{\psi}_2(\tau)}$.

• For $t \in (t_1, t_2]$, we have

$$\begin{cases} x(t) = A_3 + \displaystyle\int_{t_1}^t \phi^{-1}\Big(\phi(A_4) + \int_{t_1}^s f^1(r, x(r), y(r))dr + \\ \qquad + \displaystyle\sum_{l=1}^{\infty} \int_{t_1}^s \sigma_l^1(r, x(r), y(r))dW^l(r)\Big)ds, \quad t \in [0, t_1]. \\ y(t) = B_3 + \displaystyle\int_{t_1}^t \phi^{-1}\Big(\phi(B_4) + \int_{t_1}^s f^2(r, x(r), y(r))dr + \\ \qquad + \displaystyle\sum_{l=1}^{\infty} \int_{t_1}^s \sigma_l^2(r, x(r), y(r))dW^l(r)\Big)ds, \quad t \in [0, t_1]. \end{cases} \tag{3.9}$$

Then

$$E|x(t)|^2 \leq 2E|A_3|^2 + 2\int_{t_1}^t E\Big|\phi^{-1}\Big(\phi(A_4) + \int_{t_1}^s f^1(r, x(r), y(r))dr + \sum_{l=1}^{\infty} \int_{t_1}^s \sigma_l^1(r, x(r), y(r))dW^l(r)\Big)\Big|_X^2 ds.$$

Consider functions $\mu, \overline{\mu}$ defined on $t \in (t_1, t_2]$ by

$$\mu(t) = \sup\{E|x(s)|^2 : t_1 \leq s \leq t\}, \qquad \overline{\mu}(t) = \sup\{E|y(s)|^2 : t_1 \leq s \leq t\}.$$

$$E\Big|\phi(A_4)+\int_{t_1}^{s} f^1(r,x(r),y(r))dr+\sum_{l=1}^{\infty}\int_{t_1}^{s}\sigma_l^1(r,x(r),y(r))dW^l(r)\Big|_X^2$$

$$\le 3E|\phi(A_4)|^2+3\int_{t_1}^{s}p_1(r)\psi_1(E(|x(r)|^2+|y(r)|_X^2))dr$$

$$+3C_2\int_{t_1}^{s}p_1(r)\psi_1(E(|x(r)|_X^2+|y(r)|^2))dr$$

$$\le 3E|\phi(A_4)|^2+||\overline{p}||_{L^1}\psi_1(\mu(s)+\overline{\mu}(s)),$$

where $||\overline{p}||_{L^1}=(3t_2+3C_2)\|p_1\|_{L^1}$, and, consequently,

$$\mu(t)\le 2E|A_3|_X^2+\int_{t_1}^{t}\widehat{\psi}_1(\mu(s)+\overline{\mu}(s)),\quad t\in(t_1,t_2],$$

where $\widehat{\psi}_1=(\phi^{-1}\circ\widetilde{\psi}_1)$ and $\widetilde{\psi}_1(u)=3E|\phi(A_4)|_X^2+||\overline{p}_1||_{L^1}\psi_1(u)$. and similarly

$$\overline{\mu}(t)\le 2E|B_0|_X^2+\int_{t_1}^{t}\widehat{\psi}_2(\mu(s)+\overline{\mu}(s))ds,\quad t\in(t_1,t_2],$$

where $\widehat{\psi}_2=(\phi^{-1}\circ\widetilde{\psi}_2)$ and $\widetilde{\psi}_2(u)=3E|\phi(B_1)|_X^2+||p_2||_{L^1}\psi_1(u)$. Now, taking into account all the previous estimates we can write

$$\mu(t)+\overline{\mu}(t)\le 2E|A_3|^2+2E|B_3|^2+\int_{t_1}^{t}\widehat{\psi}_1(\mu(s)+\overline{\mu}(s))ds$$

$$+\int_{t_1}^{t}\widehat{\psi}_2(\mu(s)+\overline{\mu}(s))ds,\quad t\in(t_1,t_2],$$

By the nonlinear Grönwall-Bihari inequality (Lemma 2.8), we infer the bound

$$\mu(t)+\overline{\mu}(t)\le H^{-1}(t)\le M_1.$$

Consequently, there exists a constant M_1 which only depends on t_1,t_2 such that

$$\sup_{t\in(t_1,t_2]}E|x(t)|^2\ \le\ M_1,\ \text{and}\ \sup_{t\in(t_1,t_2]}E|y(t)|^2\le M_1.$$

where $H(t)=\displaystyle\int_{2E|A_3|^2+2E|B_3|^2}^{t}\frac{d\tau}{(\phi^{-1}\circ\widetilde{\psi}_1(\tau)+\phi^{-1}\circ\widetilde{\psi}_2(\tau)}$.

• For $t\in(t_m,T]$, we have

$$x(t)=A_{m+2}+\int_{t_m}^{t}\phi^{-1}\Big(\phi(A_{m+3})+\int_{t_m}^{s}f^1(r,x(r),y(r))dr+$$
$$\sum_{l=1}^{\infty}\int_{t_m}^{s}\sigma_l^1(r,x(r),y(r))dW^l(r)\Big)ds,$$

and

$$y(t)=B_{m+2}+\int_{t_m}^{t}\phi^{-1}\Big(\phi(B_{m+3})+\int_{t_m}^{s}f^2(r,x(r),y(r))dr$$
$$+\sum_{l=1}^{\infty}\int_{t_m}^{s}\sigma^2(r,x(r),y(r))dW^l(r)\Big)ds.$$

As in the pattern shown above, there exists $M_m > 0$ such that

$$\mu(t) + \overline{\mu}(t) \leq H^{-1}(t) \leq M_m.$$

Consequently, there exists a constant M_1 which only depends on t_m, T such that

$$\sup_{t\in(t_m,T]} E|x(t)|^2 \quad \leq \quad M_m, \text{ and } \sup_{t\in(t_m,T]} E|y(t)|^2 \leq M_m.$$

where $H(t) = \displaystyle\int_{2E|A_{m+2}|^2_X + 2E|B_{m+2}|^2_X}^{t} \frac{d\tau}{(\phi^{-1}\circ\widetilde{\psi}_1(\tau) + \phi^{-1}\circ\widetilde{\psi}_2(\tau)}$.

Hence

$$\|x\|_{PC} \quad \leq \max(M_0, M_1, \ldots, M_m) = M,$$

and

$$\|y\|_{PC} \quad \leq \max(M_0, M_1, \ldots, M_m) = M.$$

The proof is complete. □

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