Determinantal inequalities for $J$-accretive dissipative matrices

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Abstract. In this note we determine bounds for the determinant of the sum of two $J$-accretive dissipative matrices with prescribed spectra.

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1. Results

Consider the complex $n$-dimensional space $\mathbb{C}^n$ endowed with the indefinite inner product

$$\langle x, y \rangle_J = y^* J x, \quad x, y \in \mathbb{C}^n,$$

where $J = I_r \oplus -I_{n-r}$, and corresponding $J$-norm

$$\langle x, x \rangle_J = |x_1|^2 + \ldots + |x_r|^2 - |x_{r+1}|^2 - \ldots - |x_n|^2.$$

In the sequel we shall assume that $0 < r < n$, except where otherwise stated.

The $J$-adjoint of $A \in \mathbb{C}^{n \times n}$ is defined and denoted as

$$[A^\# x, x] = [x, Ax]$$

or, equivalently, $A^\# := J A^* J$, [4]. The matrix $A$ is said to be $J$-Hermitian if $A^\# = A$, and is $J$-positive definite (semi-definite) if $JA$ is positive definite (semi-definite). This kind of matrices appears on Quantum Physics and in Symplectic Geometry [10]. An arbitrary matrix $A \in \mathbb{C}^{n \times n}$ may be uniquely written in the form

$$A = \text{Re}^J A + i \text{Im}^J A,$$

where

$$\text{Re}^J A = (A + A^\#)/2, \quad \text{Im}^J A = (A - A^\#)/(2i)$$

are $J$-Hermitian. This is the so-called $J$-Cartesian decomposition of $A$. $J$-Hermitian matrices share properties with Hermitian matrices, but they also have important differences. For instance, they have real and complex eigenvalues, these occurring in
conjugate pairs. Nevertheless, the eigenvalues of a $J$-positive matrix are all real, being $r$ positive and $n - r$ negative, according to the $J$-norm of the associated eigenvectors being positive or negative. A matrix $A$ is said to be $J$-accretive (resp. $J$-dissipative) if $J \text{Re}^J A$ (resp. $J \text{Im}^J A$) is positive definite. If both matrices $J \text{Re}^J A$ and $J \text{Im}^J A$ are positive definite the matrix is said to be $J$-accretive dissipative. We are interested in obtaining determinantal inequalities for $J$-accretive dissipative matrices. Determinantal inequalities have deserved the attention of researchers, [2], [3], [5]-[9], [11].

Throughout, we shall be concerned with the set

$$ D^J(A, C) = \{ \det(A + VCV^\#) : V \in U(r, n - r) \}, $$

where $A, C \in \mathbb{C}^{n \times n}$ are $J$-unitarily diagonalizable with prescribed eigenvalues and $U(r, n - r)$ is the group of $J$-unitary transformations in $\mathbb{C}^n$ ($V$ is $J$-unitary if $VV^\# = I$), [12]. The so-called $J$-unitary group is connected, nevertheless it is not compact. As a consequence, $D^J(A, C)$ is connected. This set is invariant under the transformation $C \rightarrow UCU^\#$ for every $J$-unitary matrix $U$, and, for short, $D^J(A, C)$ is said to be $J$-unitarily invariant.

In the sequel we use the following notation. By $S_n$ we denote the symmetric group of degree $n$, and we shall also consider

$$ S_n^r = \{ \sigma \in S_n : \sigma(j) = j, j = r + 1, \ldots, n \}, $$

$$ \hat{S}_n^r = \{ \sigma \in S_n : \sigma(j) = j, j = 1, \ldots, r \}. $$

Let $\alpha_j, \gamma_j \in \mathbb{C}$, $j = 1, \ldots, n$ denote the eigenvalues of $A$ and $C$, respectively. The $r!(n-r)!$ points

$$ z_{\sigma} = z_{\xi \tau} = \prod_{j=1}^{r} (\alpha_j + \gamma_{\xi(j)}) \prod_{j=r+1}^{n} (\alpha_j + \gamma_{\tau(j)}), \xi \in S_n^r, \tau \in \hat{S}_n^r. $$

belong to $D^J(A, C)$.

The purpose of this note, which is in the continuation of [1], is to establish the following results.

**Theorem 1.1.** Let $J = I_r \oplus -I_{n-r}$, and $A$ and $C$ be $J$- positive matrices with prescribed real eigenvalues

$$ \alpha_1 \geq \ldots \geq \alpha_r > 0 > \alpha_{r+1} \geq \ldots \geq \alpha_n $$

and

$$ \gamma_1 \geq \ldots \geq \gamma_r > 0 > \gamma_{r+1} \geq \ldots \geq \gamma_n, $$

respectively. Then

$$ |\det(A + iC)| \geq \left( (\alpha_1^2 + \gamma_1^2) \ldots (\alpha_n^2 + \gamma_n^2) \right)^{1/2}. $$

**Corollary 1.2.** Let $J = I_r \oplus -I_{n-r}$, and $B$ be a $J$-accretive dissipative matrix. Assume that the eigenvalues of $\text{Re}^J B$ and $\text{Im}^J B$ satisfy (1.4) and (1.5), respectively. Then,

$$ |\det(B)| \geq \left( (\alpha_1^2 + \gamma_1^2) \ldots (\alpha_n^2 + \gamma_n^2) \right)^{1/2}. $$
Example 1.3. In order to illustrate the necessity of \( A \) and \( C \) to be \( J \)-positive matrices in Theorem 1.1, let \( A = \text{diag}(\alpha_1, \alpha_2) \), \( C = \text{diag}(\gamma_1, \gamma_2) \), with \( \alpha_1 = \gamma_1 = 1 \), \( \alpha_2 = 3/2 \), \( \gamma_2 = -2 \), and \( J = \text{diag}(1, -1) \). We find \((\alpha_2^2 + \gamma_2^2)(\alpha_1^2 + \gamma_1^2) = 27/2\). However, the minimum of \(|\det(A + iV BV^\#)|^2\), for \( V \) ranging over the \( J \)-unitary group, is \(49/4\).

Theorem 1.4. Let \( J = I_r \oplus -I_{n-r} \), and \( A \) and \( C \) be \( J \)-unitary matrices with prescribed eigenvalues
\[
\alpha_1, \ldots, \alpha_r, \alpha_{r+1}, \ldots, \alpha_n
\]
and
\[
\gamma_1, \ldots, \gamma_r, \gamma_{r+1}, \ldots, \gamma_n,
\]
respectively. Assume moreover that
\[
\frac{\Re \alpha_1}{2(1 + \Re \alpha_1)} \leq \cdots \leq \frac{\Re \alpha_r}{2(1 + \Re \alpha_r)} < 0 < \frac{\Re \alpha_{r+1}}{2(1 + \Re \alpha_{r+1})} \leq \cdots \leq \frac{\Re \alpha_n}{2(1 + \Re \alpha_n)} \tag{1.6}
\]
and
\[
\frac{\Im \gamma_1}{2(1 - \Re \gamma_1)} \leq \cdots \leq \frac{\Im \gamma_r}{2(1 - \Re \gamma_r)} < 0 < \frac{\Im \gamma_{r+1}}{2(1 - \Re \gamma_{r+1})} \leq \cdots \leq \frac{\Im \gamma_n}{2(1 - \Re \gamma_n)} \tag{1.7}
\]
Then
\[
D^J(A, C) = (\alpha_1 + \gamma_1)(\alpha_n + \gamma_n)[1, +\infty[.
\]

We shall present the proofs of the above results in the next section.

2. Proofs

Lemma 2.1. Let \( g: U(r, n-r) \to \mathbb{R} \) be the real valued function defined by
\[
g(U) = \det(I + A_0^{-1}UC_0JU^*JA_0^{-1}UC_0JU^*J),
\]
where \( A_0 = \text{diag}(\alpha_1, \ldots, \alpha_n) \), \( C_0 = \text{diag}(\gamma_1, \ldots, \gamma_n) \) and \( \alpha_i, \gamma_j \) satisfy (1.4) and (1.5). Then the set
\[
\{U \in U(r, n-r) : g(U) \leq a\},
\]
where
\[
a > \prod_{j=1}^n \left(1 + \frac{\gamma_j^2}{\alpha_j^2}\right),
\]
is compact.

Proof. Notice that \( JA_0 > 0 \), \( JC_0 > 0 \), so we may write
\[
g(U) = \det(I + WW^*WW^*),
\]
where
\[
W = (JA_0)^{-1/2}U(JC_0)^{1/2}.
\]
The condition \( g(U) \leq a \) implies that \( W \) is bounded, and is satisfied if we require that \( WW^* \leq \kappa I \), for \( \kappa > 0 \) such that \((1 + \kappa^2)^n \leq a\). Thus, also \( U \) is bounded. The result follows by Heine-Borel Theorem.
\[\Box\]
Proof of Theorem 1.1

Under the hypothesis, $A$ is nonsingular. Since the determinant is $J$-unitarily invariant and $C$ is $J$-unitarily diagonalizable, we may consider $C = \text{diag}(\gamma_1, \ldots, \gamma_n)$. We observe that

$$|\det(A + iC)|^2 = |\det((A + iC)(A - iC))| = \left(\prod_{i=1}^{n} \alpha_i\right)^2 \det((I + iA^{-1}C)(I - iA^{-1}C))$$

Clearly,

$$\det((I + iA^{-1}C)(I - iA^{-1}C)) = \det(I + A^{-1}CA^{-1}C).$$

The set of values attained by $|\det(A + iC)|^2$ is an unbounded connected subset of the positive real line. In order to prove the unboundedness, let us consider the $J$-unitary matrix $V$ obtained from the identity matrix $I$ through the replacement of the entries $(r, r)$, $(r+1, r+1)$ by $\cosh u$, and the replacement of the entries $(r, r+1)$, $(r+1, r)$ by $\sinh u$, $u \in R$. We may assume that $A_0 = \text{diag}(\alpha_1, \ldots, \alpha_n)$. A simple computation shows that

$$|\det(A_0 + iVCV^\#)|^2 = \prod_{j=1}^{n}(\alpha_j^2 + \gamma_j^2)$$

$$- 2(\alpha_r - \alpha_{r+1})(\gamma_r - \gamma_{r+1})(\alpha_{r+1}\gamma_r + \alpha_r\gamma_{r+1})(\sinh u)^2$$

$$+ (\alpha_r - \alpha_{r+1})^2(\gamma_r - \gamma_{r+1})^2(\sinh u)^4.$$ 

Thus, the set of values attained by $|\det(A_0 + iVCV^\#)|$ is given by

$$[(\alpha_1^2 + \gamma_1^2)^{1/2} \ldots (\alpha_n^2 + \gamma_n^2)^{1/2}, +\infty[.$$ 

As a consequence of Lemma 2.1, the set of values attained by $|\det(A + iC)|^2$ is closed and a half-ray in the positive real line. So, there exist matrices $A, C$ such that the endpoint of the half-ray is given by $|\det(A + iC)|^2$. Let us assume that the endpoint of this half-ray is attained at $|\det(A + iC)|^2$. We prove that $A$ commutes with $C$. Indeed, for $\epsilon \in R$ and an arbitrary $J$-Hermitian $X$, let us consider the $J$-unitary matrix given as

$$e^{iX} = i + i\epsilon X - \frac{\epsilon^2}{2}X^2 + \ldots.$$ 

We obtain by some computations

$$f(\epsilon) := \det(I + A^{-1}e^{-i\epsilon X}Ce^{i\epsilon X}A^{-1}e^{-i\epsilon X}Ce^{i\epsilon X})$$

$$= \det(I + A^{-1}CA^{-1}C - i\epsilon(A^{-1}[X, C]A^{-1}C + A^{-1}CA^{-1}[X, C])) + O(\epsilon^2)$$

$$= \det(I + A^{-1}CA^{-1}C)$$

$$\times \det(I - i\epsilon(A^{-1}CA^{-1})^{-1}(A^{-1}[X, C]A^{-1}C + A^{-1}CA^{-1}[X, C])) + O(\epsilon^2)$$

$$= \det(I + A^{-1}CA^{-1}C)$$

$$\times \exp(-i\epsilon \text{tr}((I + A^{-1}CA^{-1}C)^{-1}(A^{-1}[X, C]A^{-1}C + A^{-1}CA^{-1}[X, C]))) + O(\epsilon^2),$$
where $[X,Y] = XY - YX$ denotes the commutator of the matrices $X$ and $Y$. The function $f(\epsilon)$ attains its minimum at $\det(I + A^{-1}CA^{-1}C)$, if
\[
\frac{df}{d\epsilon}\bigg|_{\epsilon=0} = 0.
\]
Then we must have
\[
\text{tr} \left( (I + A^{-1}CA^{-1}C)^{-1}(A^{-1}[X,C]A^{-1}C + A^{-1}CA^{-1}[X,C]) \right) = 0,
\]
for every $J$-Hermitian $X$. That is
\[
[C, (A^{-1}C(I + A^{-1}CA^{-1}C)^{-1}A^{-1} + (I + A^{-1}CA^{-1}C)^{-1}A^{-1}CA^{-1}C)] = 0,
\]
and so, performing some computations, we find
\[
\begin{align*}
[C, (A^{-1}C(I + A^{-1}CA^{-1}C)^{-1}A^{-1}C + (I + A^{-1}CA^{-1}C)^{-1}A^{-1}CA^{-1}C)] &= 2 \left[ C, \frac{A^{-1}CA^{-1}C}{I + A^{-1}CA^{-1}C} \right] = 2 \left[ C, I - \frac{I}{I + A^{-1}CA^{-1}C} \right] \\
&= -2 \left[ C, \frac{I}{I + A^{-1}CA^{-1}C} \right] = \frac{2I}{I + (A^{-1}C)^2} [C, (A^{-1}C)^2] \frac{I}{I + (A^{-1}C)^2} = 0.
\end{align*}
\]
Thus
\[
[C, (A^{-1}C)^2] = 0.
\]
Assume that $C$, which is in diagonal form, has distinct eigenvalues. Then $(A^{-1}C)^2$ is a diagonal matrix as well as $((JA)^{-1}JC)^2$. Furthermore, $((JC)^{1/2}(JA)^{-1}(JC)^{1/2})^2$ is diagonal. Since $(JC)^{1/2}(JA)^{-1}(JC)^{1/2}$ is positive definite, it is also diagonal, and so are $(JA)^{-1}JC$ and $A^{-1}C$. Henceforth, $A$ is also a diagonal matrix and commutes with $C$. (If $C$ has multiple eigenvalues we can apply a perturbative technique and use a continuity argument).

For $\sigma \in S_n$, such that $\sigma(1), \ldots, \sigma(r) \leq r$, we have
\[
(\alpha_1^2 + \gamma_{\sigma(1)}^2) \ldots (\alpha_n^2 + \gamma_{\sigma(n)}^2) \geq (\alpha_1^2 + \gamma_1^2) \ldots (\alpha_n^2 + \gamma_n^2).
\]
Thus, the result follows. □

In the proof of Theorem 1.4, the following lemma is used (cf. [1, Theorem 1.1]).

**Lemma 2.2.** Let $B, D$ be $J$-positive matrices with eigenvalues satisfying
\[
\beta_1 \geq \ldots \geq \beta_r > 0 > \beta_{r+1} \geq \ldots \geq \beta_n,
\]
and
\[
\delta_1 \geq \ldots \geq \delta_r > 0 > \delta_{r+1} \geq \ldots \geq \delta_n.
\]
Then
\[
D^J(B,D) = \{(\beta_1 + \delta_1) \ldots (\beta_n + \delta_n) t : t \geq 1 \}.
\]

**Proof of Theorem 1.4**

Since, by hypothesis, $A, C$, are $J$-unitary matrices, considering convenient Möbius transformations, it follows that
\[
B = \frac{i}{2} A - I, \quad D = \frac{-i}{2} C + I \quad (2.1)
\]
are \( J \)-Hermitian matrices. Since
\[
B + D = -i(A + I)^{-1}(C + A)(C - I)^{-1},
\]
we obtain
\[
\det(B + D) = i^n \frac{\det(A + C)}{\prod_{j=1}^{n}(1 + \alpha_j)(1 - \gamma_j)}.
\]
Assume that the eigenvalues of \( B \) and \( D \) are
\[
\sigma(B) = \{\beta_1, \ldots, \beta_n\}, \quad \sigma(D) = \{\delta_1, \ldots, \delta_n\},
\]
respectively. From (2.1) we get,
\[
\beta_j = -\frac{\Im \alpha_j}{2(1 + \Re \alpha_j)}, \quad \delta_j = -\frac{\Im \gamma_j}{2(1 - \Re \gamma_j)}.
\]
From (1.6) and (1.7) we conclude that
\[
\beta_1 \geq \ldots \geq \beta_r > 0 > \beta_{r+1} \geq \ldots > \beta_n,
\]
and
\[
\delta_1 \geq \ldots \geq \delta_r > 0 > \delta_{r+1} \geq \ldots > \delta_n,
\]
so that the matrices \( B \) and \( D \) are \( J \)-positive. From Lemma 2.2 it follows that
\[
D^J(B, D) = (\beta_1 + \delta_1) \ldots (\beta_n + \delta_n)[1, +\infty[.
\]
Thus, \( D^J(A, C) \) is a half-line with endpoint at
\[
(\alpha_1 + \gamma_1) \ldots (\alpha_n + \gamma_n),
\]
or, more precisely,
\[
D^J(A, C) = \{(\alpha_1 + \gamma_1) \ldots (\alpha_n + \gamma_n) : t \geq 1\}. \quad \Box
\]

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**References**


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