Fekete-Szegö problem for a class of analytic functions defined by Carlson-Shaffer operator

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Abstract. In the present paper, authors study a Fekete-Szegö problem for a class of analytic functions defined by Carlson-Shaffer operator. Relevant connections of the results presented here with various known results are briefly indicated.

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1. Introduction

Let $A$ denote the class of functions $f$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1.1)$$

which are analytic in the open unit disk $U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}$ and $S$ denote

the subclass of $A$ that are univalent in $U$. Fekete and Szegö [10] proved a interesting

result that the estimate

$$|a_3 - \lambda a_2^2| \leq 1 + 2 \exp \left( \frac{-2\lambda}{1-\lambda} \right) \quad (1.2)$$

holds for any normalized univalent function $f(z)$ of the form (1.1) in the open unit
disk $U$ for $0 \leq \lambda \leq 1$. This inequality is sharp for each $\lambda$.

The coefficient functional

$$\phi_\lambda (f) = a_3 - \lambda a_2^2 = \frac{1}{6} \left( f'''(0) - \frac{3\lambda}{2} [f''(0)]^2 \right), \quad (1.3)$$

on normalized analytic functions $f$ in the unit disk represents various geometric quantities, for example, when $\lambda = 1$, $\phi_\lambda (f) = a_3 - a_2^2$, becomes $\frac{S_f(0)}{6}$, where $S_f$ denote

the Schwarzian derivative $(f''/f')' - (f''/f')^2/2$ of locally univalent functions $f$ in
The problem of maximising the absolute value of the functional $\phi_\lambda(f)$ is called the Fekete-Szegö problem.

The Fekete-Szegö problem is one of the interesting problems in Geometric Function Theory. This attracts many researchers (see the work of [1]-[5], [7]-[9], [12], [13], [16], [17], [20] and [3]) to study the Fekete-Szegö problem for the various classes of analytic univalent functions. Very recently, Bansal [4] introduced the class $R^\gamma_\tau(\phi)$ of functions in $f \in S$ for which

$$1 + \frac{1}{\tau} (f'(z) + \gamma zf''(z) - 1) \prec \phi(z), \quad z \in U$$

where $0 \leq \gamma < 1$, $\tau \in C \setminus \{0\}$, $\phi(z)$ is an analytic function with positive real part on $U$ with $\phi(0) = 0$, $\phi'(0) > 0$ which maps the unit like disk $U$ onto a starlike region with respect to 1 which is symmetric with respect to the real axis and $\prec$ denotes the subordination between analytic functions and studied the Fekete-Szegö problem for this class.

Now, by using the Carlson-Shaffer operator we introduce a new subclass $R^\gamma_\tau(\phi, a, c)$ for functions $f \in A$ and $0 \leq \gamma < 1$, $\tau \in C \setminus \{0\}$, $a, c \in C$, $\{c \neq 0, -1, -2,\ldots\}$ satisfying the condition

$$1 + \frac{1}{\tau} ((L(a, c)f(z))' + \gamma z(L(a, c)f(z))'' - 1) \prec \phi(z) \quad (z \in U) \quad (1.4)$$

where $\phi(z)$ is defined the same as above and $L(a, c)$ denotes the Carlson-Shaffer operator introduced in [6] and defined in the following way:

$$L(a, c)f(z) = f(z) \ast zh(a, c; z),$$

where

$$h(a, c; z) = 1 + \sum_{n=1}^{\infty} \frac{(a)^n}{(c)^n} z^n$$

$L(a, c)$ maps $A$ into itself. $L(c, c)$ is the identity and if $a \neq 0, -1, -2,\ldots$, then $L(a, c)$ has a continuous inverse $L(c, a)$ and is an one-to-one mapping of $A$ onto itself. $L(a, c)$ provides a convenient representation of differentiation and integration. If $g(z) = zf'(z)$, then $g = L(2, 1)f$ and $f = L(1, 2)g$. If we set

$$\phi(z) = \frac{1 + Az}{1 + Bz}, \quad (-1 \leq B < A \leq 1; z \in U),$$

in (1.4), we obtain

$$R^\gamma_\tau\left(\frac{1 + Az}{1 + Bz}, a, c\right) = R^\gamma_\tau(A, B, a, c)$$

$$= \left\{ f \in A : \left| \frac{(L(a, c)f(z))' + \gamma z(L(a, c)f(z))'' - 1}{\tau(A - B) - B ((L(a, c)f(z))' + \gamma z(L(a, c)f(z))'' - 1)} \right| < 1 \right\}$$

which is again a new class.

By specializing parameters in the subclass $R^\gamma_\tau(A, B, a, c)$ we obtain the following known subclasses studied earlier by various authors.

1. $R^\gamma_\tau(A, B, a, a) \equiv R^\gamma_\tau(A, B)$ studied by Bansal [4].
2. $R^\gamma_\tau(1 - 2\beta, -1, a, a) \equiv R^\gamma_\tau(\beta)$ for $0 \leq \beta < 1$, studied by Swaminathan [21].
3. \( R^\tau_\gamma (1 - 2\beta, -1, a, a) \equiv R^\tau_\gamma (\beta) \) for \( \tau = e^{i\pi \cos \eta}, 0 \leq \beta < 1 \), where \(-\pi/2 < \eta < \pi/2\) introduced by Ponnusamy and Rønning [19], (see also [18]).

4. \( R^\tau_1 (0, -1, a, a) \equiv R^\tau (\beta) \) for \( \tau = e^{i\pi \cos \eta} \) was considered in [14].

To prove our main result, we shall require the following lemma.

**Lemma 1.1.** (see [11], [15]). If \( p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots \) \((z \in U)\) is a function with positive real part, then for any complex number \( \mu \),

\[
|c_2 - \mu c_1^2| \leq 2 \max \{1, |2\mu - 1|\}
\]

and the result is sharp for the functions given by

\[
p(z) = \frac{1 + z^2}{1 - z^2}, \quad p(z) = \frac{1 + z}{1 - z} \quad (z \in U).
\]

**2. Main results**

Our main result is contained in the following theorem.

**Theorem 2.1.** Let \( \phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots \), where \( \phi(z) \in A \) with \( \phi'(0) > 0 \).

If \( f(z) \) given by (1.1) belongs to \( R^\tau_\gamma (\phi, a, c) \) \((0 \leq \gamma \leq 1, \tau \in C \setminus \{0\}, a, c \in C, \{c \neq 0, -1, -2, \ldots\}, z \in U)\), then for any complex number \( \mu \)

\[
|a_3 - \mu a_2^2| \leq \frac{B_1 |\tau| c(c+1)}{3a(a+1)(1+2\gamma)} \max \left\{ 1, \frac{B_2}{B_1} - \frac{3\mu\tau B_1 c(a+1)(1+2\gamma)}{4a(c+1)(1+\gamma)^2} \right\}.
\]

This result is sharp.

**Proof.** If \( f(z) \in R^\tau_\gamma (\phi, a, c) \), then there exists a Schwarz function \( w(z) \) analytic in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) in \( U \) such that

\[
1 + \frac{1}{\tau} \left( (L(a,c)f(z))' + \gamma z(L(a,c)f(z))'' - 1 \right) = \phi(w(z)), \quad (z \in U).
\]

Define the function \( p_1(z) \) by

\[
p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \ldots.
\]

Since \( w(z) \) is a Schwarz function, we see that \( \text{Re} \{p_1(z)\} > 0 \) and \( p_1(0) = 1 \).

Define the function \( p(z) \) by,

\[
p(z) = 1 + \frac{1}{\tau} \left( (L(a,c)f(z))' + \gamma z(L(a,c)f(z))'' - 1 \right) = 1 + b_1 z + b_2 z^2 + \ldots.
\]

In view of (2.2), (2.3), (2.4), we have

\[
p(z) = \phi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right) = \phi \left( \frac{c_1 z + c_2 z^2 + \ldots}{2 + c_1 z + c_2 z^2 + \ldots} \right)
\]

\[
= \phi \left( \frac{1}{2} c_1 z + \frac{1}{2} \left( c_2 - \frac{c_1^2}{2} \right) z^2 + \ldots \right)
\]

\[
= 1 + B_1 \frac{1}{2} c_1 z + B_1 \frac{1}{2} \left( c_2 - \frac{c_1^2}{2} \right) z^2 + B_2 \frac{1}{4} c_1^2 z^2 + \ldots
\]
Thus,
\[ b_1 = \frac{1}{2} B_1 c_1; \quad b_2 = \frac{1}{2} B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2. \]  
(2.6)

From (2.4), we obtain
\[ a_2 = \frac{\tau B_1 c_1 c}{4a(1+\gamma)}; \quad a_3 = \frac{\tau c(c+1)}{6a(a+1)(1+2\gamma)} \left[ B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{2} B_2 c_1^2 \right]. \]  
(2.7)

Therefore, we have
\[ a_3 - \mu a_2^2 = \frac{B_1 \tau c(c+1)}{6a(a+1)(1+2\gamma)} \left( c_2 - \nu c_1^2 \right) \]  
(2.8)

where
\[ \nu = \frac{1}{2} \left( 1 - \frac{B_2}{B_1} + \frac{3\tau \mu B_1 c(a+1)(1+2\gamma)}{4a(c+1)(1+\gamma)^2} \right). \]  
(2.9)

Our result now is followed by an application of Lemma 1.1. Also, by the application of Lemma 1.1 equality in (2.1) is obtained when
\[ p_1(z) = 1 + \frac{z^2}{1-z^2} \text{ or } p_1(z) = \frac{1+z}{1-z}. \]  
(2.10)

but
\[ p(z) = 1 + \frac{1}{\tau} \left( (L(a,c) f(z))' + \gamma z (L(a,c) f(z))' - 1 \right) = \phi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right). \]  
(2.11)

Putting value of \( p_1(z) \) we get the desired results. Thus the proof of Theorem 2.1 is established. \( \square \)

For the class \( R^\tau_\gamma (A, B, a, c) \),
\[ \phi(z) = \frac{1 + Az}{1 + Bz} = (1 + Az)(1 + Bz)^{-1} = 1 + (A - B)z - (AB - B^2)z^2 + \ldots \]  
(2.12)

Thus, putting \( B_1 = A - B \) and \( B_2 = -B(A - B) \) in Theorem 2.1, we get the following corollary.

**Corollary 2.2.** If \( f(z) \) given by (1.1) belongs to \( R^\tau_\gamma (A, B, a, c) \), then
\[ |a_3 - \mu a_2^2| \leq \frac{(A - B) |\tau| c(c+1)}{3a(a+1)(1+2\gamma)} \max \left\{ 1, \left| B + \frac{3\tau \mu c(a+1)(A - B)(1+2\gamma)}{4a(c+1)(1+\gamma)^2} \right| \right\}. \]  
(2.13)

If we put \( a = c \) in Theorem 2.1, then we obtain the following result of Bansal [4].

**Corollary 2.3.** Let \( \phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots \), where \( \phi(z) \in A \) with \( \phi'(0) > 0 \). If \( f(z) \) given by (1.1) belongs to \( R^\tau_\gamma (\phi)(0 \leq \gamma \leq 1, \tau \in C \setminus \{0\}, z \in U) \) then for any complex number \( \mu \)
\[ |a_3 - \mu a_2^2| \leq \frac{B_1 |\tau|}{3(1+2\gamma)} \max \left\{ 1, \left| \frac{B_2}{B_1} - \frac{3\mu \tau B_1(1+2\gamma)}{4(1+\gamma)^2} \right| \right\}. \]

This result is sharp.
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References


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