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Some properties of solutions to a planar system of nonlinear differential equations

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Abstract. In this paper we present for the solutions of a planar system of differential equations, extremal principle, Nicolescu-type and Butlewski-type separation theorems. Some applications and examples are given.

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1. Introduction

Let $F, G \in C([a,b] \times \mathbb{R}^3)$. We consider the following first order system of differential equation

$$\begin{cases} F(x,y,z,y') = 0 \\ G(x,y,z,z') = 0. \end{cases} \tag{1.1}$$

In this paper by a solution of the system (1.1) we understand a function $(y,z) \in C^1([a,b],\mathbb{R}^2)$ which satisfies (1.1).

For a function $u : [a,b] \to \mathbb{R}$ we denote by Z_u the zero set of u,

$$Z_u := \{x \in [a,b] |\ u(x) = 0\}.$$

Let us recall now some essential definitions and fundamental results.

Definition 1.1. A function $f : D \to \mathbb{R}$ ($D \subset \mathbb{R}^2$) is called homogeneous of degree n if $f(tu,tv) = t^n f(u,v)$, for each $(u,v) \in D$ and $t > 0$.

The linear case of (1.1) is the following system

$$\begin{cases} y' + p_1(x)y + q_1(x)z = 0 \\ z' + p_2(x)y + q_2(x)z = 0 \end{cases} \tag{1.2}$$

with $p_i, q_i \in C[a,b]$, $i = 1,2$.

For the system (1.2) the following properties of the solution are well known (see [9], [10], [11], [6], [2], [12]).

Theorem 1.2. *If* $(y,z) \neq 0$ *is a solution of* (1.2) *then we have:*

(i) $Z_y \cap Z_z = \emptyset$;
(ii) *if* $q_1(x) \neq 0,\ p_2(x) \neq 0,\ \forall x \in [a,b]$, *then the zeros of* y *and* z *are simple and isolated on* $[a,b]$.

Theorem 1.3. *(Nicolescu's theorem* [5]*) We suppose that* $q_1(x)p_2(x) < 0$, $\forall x \in [a,b]$. *If* $(y,z) \neq 0$ *is a solution of* (1.2)*, then the zeros of* y *and* z *separate each other on* $[a,b]$.

Theorem 1.4. *(Butlewski's theorem* [1]*) We suppose that* $p_i, q_i \in C[a,b]$, $i = 1,2$ *and* $q_1(x) \neq 0$ $(p_2(x) \neq 0)$, $\forall x \in [a,b]$. *If* (y_1, z_1) *and* (y_2, z_2) *are two linear independent solutions of* (1.2)*, then the zeros of* y_1 *and* y_2 $(z_1$ *and* $z_2)$ *separate each other on* $[a,b]$.

The aim of this paper is to extend the above results to the solutions of (1.1). For some results in this directions see [7], [8], [4] and [3].

The organization of this paper is as follows. In Section 2 we prove some extremal principles for nonlinear first order system of differential equations and in Section 3 we study some properties of the zeros of the components of the solutions for such systems and in the end we prove Nicolescu-type and Butlewski-type separation theorems, by using Tonelli's Lemmas. The results presented in this paper generalize the main results from [3].

2. Extremal Principles

We consider the system (1.1) with $F, G \in C([a,b] \times \mathbb{R}^3)$. We have the following extremal principle for the solutions of (1.1).

Theorem 2.1. *(Extremal principle) Let* $(y,z) \in C^1([a,b], \mathbb{R}^2)$ *be a solution of* (1.1) *and we suppose that:*

(i) $F(x,\cdot,\cdot,0)$ *and* $G(x,\cdot,\cdot,0)$ *are homogeneous for all* $x \in [a,b]$;
(ii) $F(x,y,\cdot,0)$ *and* $G(x,\cdot,z,0)$ *are increasing,* $\forall x \in [a,b]$;
(iii) $F(x,1,1,0) < 0$, $G(x,1,1,0) < 0$, $\forall x \in [a,b]$.

Then:

(a) *If there exists* $x_0 \in [a,b]$ *such that*

$$\max\left\{\max_{x\in[a,b]} y(x),\ \max_{x\in[a,b]} z(x)\right\} = \max\left\{y(x_0), z(x_0)\right\} > 0,$$

then $x_0 \in \{a,b\}$;

(b) *If there exists* $x_0 \in [a,b]$ *such that*

$$\min\left\{\max_{x\in[a,b]} y(x),\ \max_{x\in[a,b]} z(x)\right\} = \max\left\{y(x_0), z(x_0)\right\} < 0,$$

then $x_0 \in \{a,b\}$.

Proof. (a) We suppose that $x_0 \in]a,b[$. Let

$$\max\left\{\max_{x\in[a,b]} y(x),\ \max_{x\in[a,b]} z(x)\right\} = \max\{y(x_0), z(x_0)\} = y(x_0) > 0.$$

We shall show that this leads to a contradiction.
Since $y \in C^1[a,b]$ we have that $y'(x_0) = 0$. From (1.1) we have

$$F(x_0, y(x_0), z(x_0), 0) = 0.$$

Using (i) and (ii) we obtain

$$\begin{aligned} 0 &= F(x_0, y(x_0), z(x_0), 0) \le F(x_0, y(x_0), y(x_0), 0) \\ &= y(x_0)F(x_0, 1, 1, 0) < 0. \end{aligned}$$

So, $x_0 \in \{a,b\}$.
Now let $\max\left\{\max_{x\in[a,b]} y(x),\ \max_{x\in[a,b]} z(x)\right\} = \max\{y(x_0), z(x_0)\} = z(x_0) > 0.$
Since $z \in C^1[a,b]$ we have that $z'(x_0) = 0$. From (1.1) we have

$$G(x_0, y(x_0), z(x_0), 0) = 0.$$

Using (i) and (ii) we obtain

$$\begin{aligned} 0 &= G(x_0, y(x_0), z(x_0), 0) \le G(x_0, z(x_0), z(x_0), 0) \\ &= z(x_0)G(x_0, 1, 1, 0) < 0. \end{aligned}$$

So, $x_0 \in \{a,b\}$.

(b) Let $\min\left\{\max_{x\in[a,b]} y(x),\ \max_{x\in[a,b]} z(x)\right\} = \max\{y(x_0), z(x_0)\} = y(x_0) < 0.$
We suppose that $x_0 \in]a,b[$. Analogous, we prove that this leads to a contradiction. □

Corollary 2.2. *Let $(y,z) \in C^1([a,b], \mathbb{R}^2)$ be a solution of the following system*

$$\begin{cases} p_1 y + q_1 z - y' = 0 \\ p_2 y + q_2 z - z' = 0 \end{cases}$$

and we suppose that $p_2 > 0, q_1 > 0,\ p_1 + q_1 < 0$ and $p_2 + q_2 < 0$. Then:

(a) *If there exists $x_0 \in [a,b]$ such that*

$$\max\left\{\max_{x\in[a,b]} y(x),\ \max_{x\in[a,b]} z(x)\right\} = \max\{y(x_0), z(x_0)\} > 0,$$

then $x_0 \in \{a,b\}$;

(b) *If there exists $x_0 \in [a,b]$ such that*

$$\min\left\{\max_{x\in[a,b]} y(x),\ \max_{x\in[a,b]} z(x)\right\} = \max\{y(x_0), z(x_0)\} < 0,$$

then $x_0 \in \{a,b\}$.

Example 2.3. We consider on $[0,1]$ the system

$$\begin{cases} -2y + z - y' = 0 \\ y - 4z - z' = 0 \end{cases}$$

with initial conditions $y(0) = z(0) = 1$. We have

$$q_1 = p_2 = 1, \ p_1 + q_1 < 0, \ p_2 + q_2 < 0.$$

From Figure 1 one can see that the conditions of Corollary 2.2 hold.

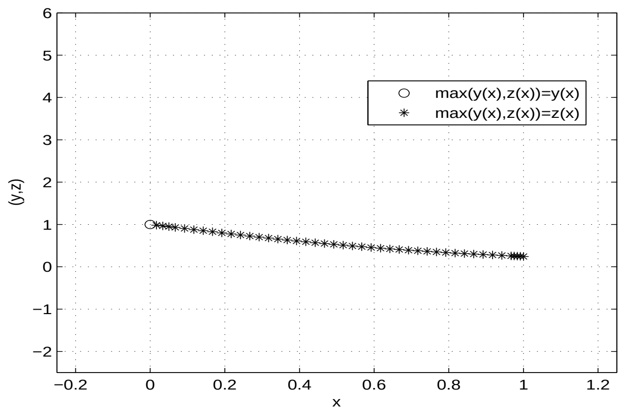

Figure 1. Plot of $\max\{y(x), z(x)\}$ as function of x

Theorem 2.4. *Let $(y,z) \in C^1([a,b], \mathbb{R}^2)$ be a solution of the following system*

$$\begin{cases} f(x,y,z) - y' = 0 \\ g(x,y,z) - z' = 0 \end{cases}$$

and we suppose that

(i) *f and g are homogeneous with respect to the last two arguments;*
(ii) *$f(x,y,\cdot)$ and $g(x,\cdot,z)$ are increasing, $\forall x \in [a,b]$;*
(iii) *$f(x,1,1) < 0$, $g(x,1,1) < 0$, $\forall x \in [a,b]$.*

Then:

(a) *If there exists $x_0 \in [a,b]$ such that*

$$\max\left\{\max_{x\in[a,b]} y(x), \max_{x\in[a,b]} z(x)\right\} = \max\{y(x_0), z(x_0)\} > 0,$$

then $x_0 \in \{a,b\}$;

(b) *If there exists $x_0 \in [a,b]$ such that*

$$\min\left\{\max_{x\in[a,b]} y(x), \max_{x\in[a,b]} z(x)\right\} = \max\{y(x_0), z(x_0)\} < 0,$$

then $x_0 \in \{a,b\}$.

Proof. The system satisfies the condition from Theorem 2.1. □

Corollary 2.5. *Let* $(y,z) \in C^1([a,b],\mathbb{R}^2)$ *be a solution of the following system*

$$\begin{cases} p_1y^3 + q_1z^3 - y' = 0 \\ p_2y^3 + q_2z^3 - z' = 0 \end{cases}$$

and we suppose that $p_2 > 0, q_1 > 0,\ p_1 + q_1 < 0$ *and* $p_2 + q_2 < 0$.

Then:

(a) *If there exists* $x_0 \in [a,b]$ *such that*

$$\max\left\{\max_{x\in[a,b]} y(x),\ \max_{x\in[a,b]} z(x)\right\} = \max\{y(x_0), z(x_0)\} > 0,$$

then $x_0 \in \{a,b\}$;

(b) *If there exists* $x_0 \in [a,b]$ *such that*

$$\min\left\{\max_{x\in[a,b]} y(x),\ \max_{x\in[a,b]} z(x)\right\} = \max\{y(x_0), z(x_0)\} < 0,$$

then $x_0 \in \{a,b\}$.

Example 2.6. We consider on $[0,1]$ the system

$$\begin{cases} -5y^3 + 2z^3 - y' = 0 \\ 2y^3 - 6z^3 - z' = 0 \end{cases}$$

with initial conditions $y(0) = z(0) = 1$. We have

$$q_1 = 2, p_2 = 2,\ p_1 + q_1 < 0,\ p_2 + q_2 < 0.$$

From Figure 2 one can see that the conditions of Corollary 2.5 hold.

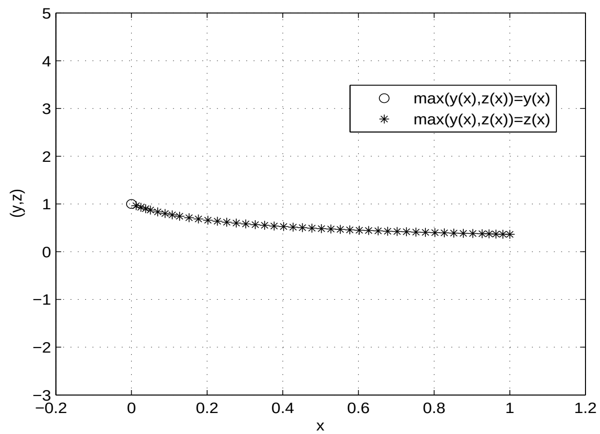

Figure 2. Plot of $\max\{y(x), z(x)\}$ as function of x

In the end of this section, we consider the following functional-differential system

$$\begin{cases} F(x,y,y(g),z,y') = 0 \\ G(x,y,z,z(h),z') = 0, \end{cases} \tag{2.1}$$

Theorem 2.7. *Let $(y,z)\in C^1([a,b],\mathbb{R}^2)$ be a solution of the system (2.1), where $g,h\in C[a,b]), g(x)\le x, h(x)\le x,\ a\le g(x)\le b,\ a\le h(x)\le b,\ \forall x\in[a,b]$ and we suppose that:*

(i) *$F(x,\cdot,\cdot,\cdot,0)$ and $G(x,\cdot,\cdot,\cdot,0)$ are homogeneous, for all $x\in[a,b]$;*
(ii) *$F(x,y,\cdot,\cdot,0)$ and $G(x,\cdot,z,\cdot,0)$ are increasing, $\forall x\in[a,b]$;*
(iii) *$F(x,1,1,1,0)<0,\ G(x,1,1,1,0)<0,\ \forall x\in[a,b]$.*

Then:

(a) *If there exists $x_0\in[a,b]$ such that*

$$\max\left\{\max_{x\in[a,b]}y(x),\ \max_{x\in[a,b]}z(x)\right\}=\max\left\{y(x_0),z(x_0)\right\}>0,$$

then $x_0\in\{a,b\}$;

(b) *If there exists $x_0\in[a,b]$ such that*

$$\min\left\{\max_{x\in[a,b]}y(x),\ \max_{x\in[a,b]}z(x)\right\}=\max\left\{y(x_0),z(x_0)\right\}<0,$$

then $x_0\in\{a,b\}$.

Proof. (a) We suppose that $x_0\in]a,b[$. Let

$$\max\left\{\max_{x\in[a,b]}y(x),\ \max_{x\in[a,b]}z(x)\right\}=\max\left\{y(x_0),z(x_0)\right\}=y(x_0)>0.$$

We shall show that this leads to a contradiction.

Since $y\in C^1[a,b]$ we have that $y'(x_0)=0$. From (1.1) we have

$$F(x_0,y(x_0),y(g(x_0)),z(x_0),0)=0.$$

Using (i) and (ii) we obtain

$$\begin{aligned}0&=F(x_0,y(x_0),y(g(x_0)),z(x_0),0)\le F(x_0,y(x_0),y(x_0),y(x_0),0)\\&=y(x_0)F(x_0,1,1,1,0)<0.\end{aligned}$$

So, $x_0\in\{a,b\}$.

Now let $\max\left\{\max_{x\in[a,b]}y(x),\ \max_{x\in[a,b]}z(x)\right\}=\max\left\{y(x_0),z(x_0)\right\}=z(x_0)>0.$

Since $z\in C^1[a,b]$ we have that $z'(x_0)=0$. From (1.1) we have

$$G(x_0,y(x_0),z(x_0),z(h(x_0)),0)=0.$$

Using (i) and (ii) we obtain

$$\begin{aligned}0&=G(x_0,y(x_0),z(x_0),z(h(x_0)),0)\le G(x_0,z(x_0),z(x_0),z(x_0),0)\\&=z(x_0)G(x_0,1,1,1,0)<0.\end{aligned}$$

So, $x_0\in\{a,b\}$.

(b) Let $\min\left\{\max_{x\in[a,b]}y(x),\ \max_{x\in[a,b]}z(x)\right\}=\max\left\{y(x_0),z(x_0)\right\}=y(x_0)<0.$

We suppose that $x_0\in]a,b[$. Analogous, we prove that this leads to a contradiction. $\square$

Corollary 2.8. *Let $(y, z) \in C^1([a, b], \mathbb{R}^2)$ be a solution of the following system*

$$\begin{cases} p_1 y + q_1 z - y' = 0 \\ p_2 y + q_2 z + q_3 z(h) - z' = 0 \end{cases}$$

and we suppose that $q_1 > 0$, $p_2 > 0$, $q_3 > 0$, $p_1 + q_1 < 0$ and $p_2 + q_2 + q_3 < 0$, $h(x) \leq x$, $a \leq h(x) \leq b$, $\forall x \in [a, b]$.

Then:

(a) *If there exists $x_0 \in [a, b]$ such that*

$$\max\left\{\max_{x\in[a,b]} y(x), \max_{x\in[a,b]} z(x)\right\} = \max\{y(x_0), z(x_0)\} > 0,$$

then $x_0 \in \{a, b\}$;

(b) *If there exists $x_0 \in [a, b]$ such that*

$$\min\left\{\max_{x\in[a,b]} y(x), \max_{x\in[a,b]} z(x)\right\} = \max\{y(x_0), z(x_0)\} < 0,$$

then $x_0 \in \{a, b\}$.

Example 2.9. We consider on $[0, 1]$ the system

$$\begin{cases} -4y^6(x) + z^6(x) - y'(x) = 0 \\ 3y(x) - 5z(x) + z(x^2) - z'(x) = 0 \end{cases}$$

with initial conditions $y(0) = z(0) = 1$. We have $q_1 = 1$, $p_2 = 3$, $q_3 = 1$, $p_1 + q_1 < 0$, $p_2 + q_2 + q_3 < 0$, $h(x) = x^2$. From Figure 3 one can see that the conditions of Theorem 2.7 hold.

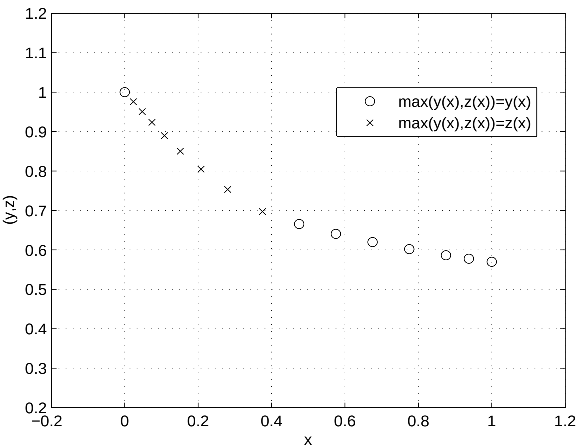

Figure 3. Plot of $\max\{y(x), z(x)\}$ as function of x

3. Zeros of the Components of the Solutions of the System (1.1)

Let us consider the following conditions on the system (1.1):

(C_1) $F(x,0,0,0)=0$, $G(x,0,0,0)=0$, $\forall x\in[a,b]$.

(C_2) If (y,z) is a solution of (1.1) and $y(x_0)=z(x_0)=0$ for some $x_0\in[a,b]$, then $y=z=0$.

(C_3) The Cauchy problem for (1.1) has at most a solution.

(C_4) Let (y,z) a solution of (1.1), then the problem

$$\begin{cases} F(x,y,z,y')=0\\ G(x,y,z,z')=0\\ y(x_0)=y_0, y'(x_0)=y_0' \end{cases}$$

where $x_0\in[a,b]$, $y_0,y_0'\in\mathbb{R}$ has at most a solution.

(C_5) Let (y,z) a solution of (1.1) then the problem

$$\begin{cases} F(x,y,z,y')=0\\ G(x,y,z,z')=0\\ z(x_0)=z_0, z'(x_0)=z_0' \end{cases}$$

where $x_0\in[a,b]$, $z_0,z_0'\in\mathbb{R}$ has at most a solution.

Remarks. (1) If $F(x,u,\cdot,w)=0$ has a solution in v, $\forall x_0\in[a,b]$, $u,w\in\mathbb{R}$, then (C_3) implies (C_4).

(2) If $F(x,\cdot,v,w)=0$ has a solution in u, $\forall x_0\in[a,b]$, $v,w\in\mathbb{R}$, then (C_3) implies (C_5).

(3) (C_1) and (C_3) imply (C_2).

(4) Let us consider the system

$$\begin{cases} p_1y+q_1z-y'=0\\ p_2y+q_2z-z'=0 \end{cases},\ p_i,q_i\in C[a,b], i=1,2.$$

In this case the conditions (C_1),(C_2) and (C_3) are satisfy.

If $q_1(x)\neq 0, \forall x_0\in[a,b]$, then the condition ($C_4$) is satisfied.

If $p_2(x)\neq 0, \forall x_0\in[a,b]$, then the condition ($C_5$) is satisfied.

(5) (C_4) and (C_5) imply (C_3).

In what follows we also need the next result (see [13], [9], [4]).

Lemma 3.1. *(Tonelli's Lemma; see* [13] *and* [7]*) Let $y_1,y_2\in C^1[a,b]$ be two functions that satisfy the following conditions:*

(i) $y_1(a)=0, y_1(b)=0$ *and* $y_1(x)>0, \forall x\in]a,b[$;

(ii) $y_2(x)>0\ ,\ \forall\ x\in[a,b]$.

Then there exists $\lambda>0$ and $x_0\in]a,b[$ such that:

$$y_2(x_0)=\lambda y_1(x_0) \text{ and } y_2'(x_0)=\lambda y_1'(x_0).$$

Next, we use another version of Tonelli's lemma.

Lemma 3.2. *(Tonelli's Lemma) Let $y_1,y_2\in C[a,b]$ be such that:*

(i) $y_1(a)=0, y_1(b)=0$ *and* $y_1(x)\neq 0, \forall x\in]a,b[$;

(ii) $y_2(x)\neq 0\ ,\ \forall\ x\in[a,b]$.

Then there exists $\lambda \in \mathbb{R}^*$ *and* $x_0 \in [a, b]$ *such that:*

$$y_2(x_0) = \lambda y_1(x_0) \text{ and } y_2'(x_0) = \lambda y_1'(x_0).$$

Remark 3.3. For Lemma 3.1 see also: [3], [4] and [9].

Our results are the following.

Theorem 3.4. *(Nicolescu-type separation theorem) For the system* (1.1)*, we suppose that:*

(i) F *and* G *are homogeneous with respect to the last three arguments;*
(ii) $F(x, y, z, \cdot)$ *and* $G(x, y, z, \cdot)$ *are increasing,* $\forall x \in [a, b]$;
(iii) $F(x, 1, \lambda, 1) \neq 0,\ G(x, \lambda, 1, 1) \neq 0$, *for all* $\lambda \in \mathbb{R}^*$.

Then, if (y, z) *is a solution of* (1.1)*, the zeros of* y *and* z *separate each other.*

Proof. We consider x_1 and x_2 two consecutive zeros of $y(x)$. We have to prove that $z(x)$ has at least one zero in the interval (x_1, x_2).

We suppose that $z(x) \neq 0,\ x \in [x_1, x_2]$. Applying Tonelli's Lemma 3.2 there exists $x_0 \in (x_1, x_2)$ and $\lambda \in \mathbb{R}^*$ such that

$$z(x_0) = \lambda y(x_0),\ z'(x_0) = \lambda y'(x_0).$$

From (1.1) we have

$$F(x_0, y(x_0), \lambda y(x_0), y'(x_0)) = 0.$$

We suppose that $y(x_0) \geq y'(x_0)$. Then

$$\begin{aligned} 0 &= F(x_0, y(x_0), \lambda y(x_0), y'(x_0)) \leq F(x_0, y(x_0), \lambda y(x_0), y(x_0)) \\ &= y(x_0) F(x_0, 1, \lambda, 1) < 0, \\ 0 &= G(x_0, y(x_0), \lambda y(x_0), \lambda y'(x_0)) \leq G(x_0, y(x_0), \lambda y(x_0), \lambda y(x_0)) \\ &= \lambda y(x_0) G(x_0, \tfrac{1}{\lambda}, 1, 1) < 0, \end{aligned}$$

so we have reached a contradiction.

If $y(x_0) \leq y'(x_0)$. Then

$$\begin{aligned} 0 &= F(x_0, y(x_0), \lambda y(x_0), y'(x_0)) \geq F(x_0, y(x_0), \lambda y(x_0), y(x_0)) \\ &= y(x_0) F(x_0, 1, \lambda, 1) > 0, \\ 0 &= G(x_0, y(x_0), \lambda y(x_0), \lambda y'(x_0)) \geq G(x_0, y(x_0), \lambda y(x_0), \lambda y(x_0)) \\ &= \lambda y(x_0) G(x_0, \tfrac{1}{\lambda}, 1, 1) > 0, \end{aligned}$$

so we have reached a contradiction. □

Theorem 3.5. *(Butlewski-type separation theorem) For the homogeneous system* (1.1)*, we suppose that it satisfies condition* (C_2)*. Then, if* (y_1, z_1) *and* (y_2, z_2) *are two linear independent solutions of* (1.1)*, then the zeros of* y_1 *and* y_2 $(z_1$ *and* $z_2)$ *separate each other on* $[a, b]$.

Proof. We consider x_1 and x_2 two consecutive zeros of $y_1(x)$. We have to prove that $y_2(x)$ has at least one zero in $[x_1, x_2]$.

We suppose that $y_2(x) \neq 0,\ x \in [x_1, x_2]$. Applying Tonelli's Lemma 3.2 there exists $x_0 \in (x_1, x_2)$ and $\lambda \in \mathbb{R}^*$ such that

$$y_2(x_0) = \lambda y_1(x_0),\ y_2'(x_0) = \lambda y_1'(x_0).$$

Taking into account (C_2) we have that $y_2(x) = \lambda y_1(x)$ and so we have reached a contradiction. □

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